



**HAL**  
open science

# Representations of quasiprojective groups, Flat connections and Transversely projective foliations

Frank Loray, Frédéric Touzet, Jorge Vitorio Pereira

► **To cite this version:**

Frank Loray, Frédéric Touzet, Jorge Vitorio Pereira. Representations of quasiprojective groups, Flat connections and Transversely projective foliations. 2015. hal-00942861v2

**HAL Id: hal-00942861**

**<https://hal.science/hal-00942861v2>**

Preprint submitted on 16 Oct 2015 (v2), last revised 4 Jul 2016 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# REPRESENTATIONS OF QUASIPROJECTIVE GROUPS, FLAT CONNECTIONS AND TRANSVERSELY PROJECTIVE FOLIATIONS

FRANK LORAY, JORGE VITÓRIO PEREIRA, AND FRÉDÉRIC TOUZET

ABSTRACT. The main purpose of this paper is to provide a structure theorem for codimension one singular transversely projective foliations on projective manifolds. To reach our goal, we firstly extend Corlette-Simpson's classification of rank two representations of fundamental groups of quasiprojective manifolds by dropping the hypothesis of quasi-unipotency at infinity. Secondly we establish an analogue classification for rank two flat meromorphic connections. In particular, we prove that a rank two flat meromorphic connection with irregular singularities having non trivial Stokes projectively factors through a connection over a curve.

## CONTENTS

1.	Introduction	1
2.	Existence of fibrations	7
3.	Factorization of representations	9
4.	Riccati foliations	13
5.	Polar divisor and reduction of Riccati foliations	18
6.	Irregular divisor	22
7.	Structure	28
8.	Transversely projective foliations	31
9.	Examples	33
	Appendix A. Reduction of singularities of Riccati foliations	35
	References	36

## 1. INTRODUCTION

Let  $X$  be a smooth projective manifold over  $\mathbb{C}$ . A (holomorphic singular) codimension one foliation  $\mathcal{F}$  on  $X$  is defined by a non zero rational 1-form  $\omega$  satisfying Frobenius integrability condition  $\omega \wedge d\omega = 0$ . The foliation is said transversely projective if there are rational 1-forms  $\alpha, \beta$  on  $X$  such that the  $\mathfrak{sl}_2$ -connection defined on the trivial vector bundle  $X \times \mathbb{C}^2$  by

$$(1) \quad Z \mapsto \nabla Z = dZ + AZ \quad \text{with} \quad A = \begin{pmatrix} \alpha & \beta \\ \omega & -\alpha \end{pmatrix}$$

---

*Key words and phrases.* Foliation, Transverse Structure, Birational Geometry, Flat Connections, Irregular Singular Points, Stokes Matrices.

The first author is supported by CNRS, and the second author, by CNPq and FAPERJ.

is **flat**:  $dA + A \cdot A = 0$ . This definition, equivalent to [32], extends to the singular case the classical definition [16] for smooth foliations. Outside of the polar divisor of the matrix connection  $A$ , the foliation  $\mathcal{F}$  admits distinguished germs of first integrals taking values on  $\mathbb{P}^1$ , well defined up to left composition with elements of  $\text{Aut}(\mathbb{P}^1) = \text{PSL}_2(\mathbb{C})$ . Precisely, given a local basis of  $\nabla$ -horizontal sections  $B = (b_{ij}) \in \text{SL}_2(\mathcal{O}(U))$  on some open set  $U$ , i.e. satisfying  $dB + A \cdot B = 0$ , the ratio  $\varphi := b_{21}/b_{22}$  provides such a local first integral for  $\mathcal{F}$ ; changing to another basis  $B \cdot B_0$  will have the effect of composing  $\varphi$  with a Moebius transformation.

Transversely projective foliations play a singular role in the study of codimension one foliations. They are precisely those foliations whose Galois groupoid in the sense of Malgrange is small (see [9, 23]). They often occur as exceptions or counter examples [5, 18, 35] and played an important role in our study of foliations with numerically trivial canonical bundle [21]. For these foliations, one can define a monodromy representation by considering analytic continuation of distinguished germs of first integrals, making the transverse pseudo-group into a group (see [19]); it also coincides with the (projectivization of the) monodromy of the flat connection  $\nabla$ .

The goal of this paper is to provide a structure theorem for transversely projective foliations in the spirit of [7, 8] and what has been done recently in [14] for transversely affine foliations.

In fact, we mainly work with the connection  $\nabla$  defined by (1) up to birational bundle transformation. When it has at worst regular singularities,  $\nabla$  is characterized by its monodromy representation up to birational bundle transformations [15]. One of the main ingredients that goes into the proof of our structure theorem is an extension of Corlette-Simpson's classification of rank two representations of quasiprojective fundamental groups [11] which we now proceed to explain.

### 1.1. Rank-two representations of quasiprojective fundamental groups.

Let  $X^\circ$  be a quasiprojective manifold and consider  $X$  a projective compactification of  $X^\circ$  with boundary equal to a simple normal crossing divisor  $D$ . If  $D_i$  is an irreducible component of  $D$  then by a small loop around  $D_i$  we mean a loop  $\gamma : S^1 \rightarrow X^\circ$  that extends to a smooth map  $\bar{\gamma} : \bar{\mathbb{D}} \rightarrow X$  which intersects  $D$  transversely on a unique smooth point of  $D_i$ . A representation  $\rho : \pi_1(X^\circ, x) \rightarrow \text{SL}_2(\mathbb{C})$  is quasi-unipotent at infinity if for every irreducible component  $D_i$  of  $D$  and every small loop  $\gamma$  around  $D_i$ , the conjugacy class of  $\rho(\gamma)$  is quasi-unipotent (eigenvalues are roots of the unity).

A representation  $\rho : \pi_1(X^\circ, x) \rightarrow \text{SL}_2(\mathbb{C})$  projectively factors through an orbifold  $Y$  if there exists a morphism  $f : X^\circ \rightarrow Y$  and a representation  $\tilde{\rho} : \pi_1^{\text{orb}}(Y, f(x)) \rightarrow \text{PSL}_2(\mathbb{C})$  such that the diagram

$$\begin{array}{ccc} \pi_1(X^\circ, x) & \xrightarrow{\rho} & \text{SL}_2(\mathbb{C}) \\ f_* \downarrow & & \downarrow \text{proj} \\ \pi_1^{\text{orb}}(Y, f(x)) & \xrightarrow{\tilde{\rho}} & \text{PSL}_2(\mathbb{C}) \end{array}$$

is commutative.

A polydisk Shimura modular orbifold is a quotient  $\mathfrak{H}$  of a polydisk  $\mathbb{H}^n$  by a group of the form  $\mathcal{U}(P, \Phi)$  where  $P$  is a projective module of rank two over the ring of integers  $\mathcal{O}_L$  of a totally imaginary quadratic extension  $L$  of a totally real

number field  $F$ ;  $\Phi$  is a skew hermitian form on  $P_L = P \otimes_{\mathcal{O}_L} L$ ; and  $\mathcal{U}(P, \Phi)$  is the subgroup of the  $\Phi$ -unitary group  $\mathcal{U}(P_L, \Phi)$  consisting of elements which preserve  $P$ . This group acts naturally on  $\mathbb{H}^n$  where  $n$  is half the number of embeddings  $\sigma : L \rightarrow \mathbb{C}$  such that the quadratic form  $\sqrt{-1}\Phi(v, v)$  is indefinite. Note that there is one tautological representation

$$\pi_1^{orb}(\mathbb{H}^n/\mathcal{U}(P, \Phi)) \simeq \mathcal{U}(P, \Phi)/\{\pm \text{Id}\} \hookrightarrow \text{PSL}_2(L),$$

which induces for each embedding  $\sigma : L \rightarrow \mathbb{C}$  one tautological representation  $\pi_1^{orb}(\mathbb{H}^n/\mathcal{U}(P, \Phi)) \rightarrow \text{PSL}_2(\mathbb{C})$ . The quotients  $\mathbb{H}^n/\mathcal{U}(P, \Phi)$  are always quasiprojective orbifolds, and when  $[L : \mathbb{Q}] > 2n$  they are projective (i.e. proper/compact) orbifolds. The archetypical examples satisfying  $[L : \mathbb{Q}] = 2n$  are the Hilbert modular orbifolds, which are quasiprojective but not projective. We refer to [11] for a thorough discussion and point out that our definition of tautological representations differs slightly from loc. cit. as they consider polydisk Shimura modular stacks instead of orbifolds and consequently their representations take values in  $\text{SL}_2(\mathbb{C})$ . Here we are lead to consider representations with values in  $\text{PSL}_2(\mathbb{C})$  because  $\pm \text{Id} \in \mathcal{U}(P, \Phi)$  acts trivially on  $\mathbb{H}^n$ .

**Theorem 1.1** (Corlette-Simpson). *Suppose that  $X^\circ$  is a quasiprojective manifold and  $\rho : \pi_1(X^\circ, x) \rightarrow \text{SL}_2(\mathbb{C})$  is a Zariski dense representation which is quasi-unipotent at infinity. Then  $\rho$  projectively factors through*

- (1) a morphism  $f : X^\circ \rightarrow Y$  to an orbicurve  $Y$  (orbifold of dimension one); or
- (2) a morphism  $f : X^\circ \rightarrow \mathfrak{H}$  to a polydisk Shimura modular orbifold  $\mathfrak{H}$ .

*In the latter case, the representation actually projectively factors through one of the tautological representations of  $\mathfrak{H}$ .*

Although their hypothesis is natural, as representations coming from geometry (Gauss-Manin connections) are automatically quasi-unipotent at infinity, Corlette and Simpson asked in [11, Section 12.1] what happens if this assumption is dropped. Our first main result answers this question.

**Theorem A.** *Suppose that  $X^\circ$  is a quasiprojective manifold and  $\rho : \pi_1(X^\circ, x) \rightarrow \text{SL}_2(\mathbb{C})$  is a Zariski dense representation which is not quasi-unipotent at infinity. Then  $\rho$  projectively factors through a morphism  $f : X^\circ \rightarrow Y$  to an orbicurve  $Y$ .*

Our method to deal with representations which are not quasi-unipotent at infinity is considerably more elementary than the sophisticated arguments needed to deal with the quasi-unipotent case. The non quasi-unipotency allows us to prove the existence of effective divisors with topologically trivial normal bundle at the boundary. We then use Malcev's Theorem combined with a result of Totaro about the existence of fibrations to produce the factorization.

Combining Corlette-Simpson Theorem with Theorem A and factorization results for representations of quasiprojective fundamental groups on the affine group  $\text{Aff}(\mathbb{C})$ , see [2, 14] and references therein, we get the following corollary.

**Corollary B.** *Suppose that  $X^\circ$  is a quasiprojective manifold and  $\rho : \pi_1(X^\circ, x) \rightarrow \text{SL}_2(\mathbb{C})$  is a representation which is not virtually abelian. Then  $\rho$  projectively factors through*

- (1) a morphism  $f : X^\circ \rightarrow Y$  to an orbicurve  $Y$ ; or
- (2) a morphism  $f : X^\circ \rightarrow \mathfrak{H}$  to a polydisk Shimura modular orbifold  $\mathfrak{H}$  equipped with one of its tautological representations.

*In the second case, the representation is Zariski dense, quasi-unipotent at infinity.*

**1.2. Riccati foliations.** Since we are interested in  $\mathrm{PSL}_2$  rather than  $\mathrm{SL}_2$ , we will essentially work with the projective connection associated to (1), or more geometrically the foliation  $\mathcal{H}$  induced by  $\nabla$ -horizontal sections on the total space of the projective bundle  $X \times \mathbb{P}^1$ . In the affine chart  $\mathbb{P}(Z) = (z_1 : z_2) \sim (1 : z)$ ,  $\mathcal{H}$  is defined by the ‘‘Riccati’’ pfaffian equation

$$dz + \omega - 2\alpha z - \beta z^2 = 0.$$

More generally, a **Riccati foliation** over a projective manifold  $X$  consists of a pair  $(\pi : P \rightarrow X, \mathcal{H}) = (P, \mathcal{H})$  where  $\pi : P \rightarrow X$  is a locally trivial  $\mathbb{P}^1$ -fiber bundle in the Zariski topology (i.e.  $P$  is the total space of the projectivization  $\mathbb{P}(E)$  of a rank two vector bundle  $E$ ) and  $\mathcal{H}$  is a codimension one foliation on  $P$  which is transverse to a general fiber of  $\pi$ . If the context is clear, we will omit the  $\mathbb{P}^1$ -bundle  $P$  from the notation and call  $\mathcal{H}$  a Riccati foliation.

The foliation  $\mathcal{H}$  is defined by the projectivization of horizontal sections of a (non unique) flat meromorphic connection  $\nabla$  on  $E$ : it is the phase portrait of the projective connection  $\mathbb{P}(E, \nabla)$ . The connection  $\nabla$  is uniquely determined by  $\mathcal{H}$  and by its trace on  $\det(E)$ . We say that the Riccati foliation  $\mathcal{H}$  has regular singularities if it can be lifted to a meromorphic connection  $\nabla$  with regular singularities (see [15]), and **irregular** if not.

We will say that a Riccati foliation  $(P, \mathcal{H})$  over  $X$  factors through a projective manifold  $X'$  if there exists a Riccati foliation  $(\pi' : P' \rightarrow X', \mathcal{H}')$  over  $X'$ , and rational maps  $\phi : X \dashrightarrow X'$ ,  $\Phi : P \dashrightarrow P'$  such that  $\pi' \circ \Phi = \phi \circ \pi$ ,  $\Phi$  has degree one when restricted to a general fiber of  $P$ , and  $\mathcal{H} = \Phi^* \mathcal{H}'$ .

Our second main result describes the structure of Riccati foliations having irregular singularities.

**Theorem C.** *Suppose that  $X$  is a projective manifold, and  $(P, \mathcal{H})$  is a Riccati foliation over  $X$ . If  $\mathcal{H}$  is irregular then at least one of the following assertions holds true.*

- (1) *Maybe after passing to a two-fold covering, the Riccati foliation  $\mathcal{H}$  is defined by a closed rational 1-form.*
- (2) *The Riccati foliation  $(P, \mathcal{H})$  factors through a curve.*

The proof of Theorem C relies on Corollary B, on an infinitesimal criterion for the existence of fibrations due to Neeman (Theorem 2.3), and on the semi-local study of Riccati foliations at a neighborhood of irregular components of the polar divisor (Section 6).

**1.3. Transversely projective foliations.** Our main goal, the description of the structure of transversely projective foliations, is achieved by combining Corollary B and Theorem C. To this end, recall that for a polydisk Shimura modular orbifold  $\mathfrak{H}$  and any of its tautological representations  $\rho$ , Deligne extension of the associated local system provides a logarithmic flat connection; denote by  $(\mathfrak{H} \times_{\rho} \mathbb{P}^1, \mathcal{H}_{\rho})$  the induced Riccati foliation.

**Theorem D.** *Let  $\mathcal{F}$  be a codimension one transversely projective foliation on a projective manifold  $X$ . Then at least one of the following assertions holds true.*

- (1) *There exists a generically finite Galois morphism  $f : Y \rightarrow X$  such that  $f^* \mathcal{F}$  is defined by a closed rational 1-form.*

- (2) *There exists a rational dominant map  $f : X \dashrightarrow S$  to a ruled surface  $S$ , and a Riccati foliation  $\mathcal{H}$  on  $S$  such that  $\mathcal{F} = f^*\mathcal{H}$ .*
- (3) *There exists a polydisk Shimura modular orbifold  $\mathfrak{H}$  and a rational map  $f : X \dashrightarrow \mathfrak{H} \times_\rho \mathbb{P}^1$  towards one of its tautological Riccati foliations such that  $\mathcal{F} = f^*\mathcal{H}_\rho$ .*

**Remark 1.2.** In the third item, we note (see [30]) that, after blowing-up the ambient,  $\mathcal{H}_\rho$  (resp.  $\mathcal{F}$ ) is locally defined at singular points by a 1-form of the type

$$\sum_{i=1}^k \lambda_i \frac{dx_i}{x_i}, \quad 1 \leq k \leq n, \quad \lambda_i \in \mathbb{R}_{>0}$$

for local coordinates  $(x_1, \dots, x_n)$  (where  $n$  is the dimension of the ambient space). In particular, for instance for  $X$  a surface, if  $\mathcal{F}$  has (maybe after blowing-up) a hyperbolic singular point, a saddle-node or a non linearizable saddle, then we are in the first two items.

There are previous results on the subject [32] and on the neighboring subject of transversely affine foliations ([8], [7], [14]). With the exception of [14], all the other works impose strong restrictions on the nature of the singularities of the foliation. Our only hypothesis is the algebraicity of the ambient manifold.

Theorem D also answers a question left open in [14]. There, a similar classification for transversely affine foliations is established for foliations on projective manifolds with zero first Betti number. Theorem D gives the analogue classification for arbitrary projective manifolds, showing that the hypothesis on the first Betti number is not necessary.

**1.4. Flat meromorphic  $\mathfrak{sl}_2$ -connections.** A meromorphic rank 2 connection  $(E, \nabla)$  on a projective manifold  $X$  is the datum of a rank 2 vector bundle  $E$  equipped with a  $\mathbb{C}$ -linear morphism  $\nabla : E \rightarrow E \otimes \Omega_X^1(D)$  satisfying Leibniz rule

$$\nabla(f \cdot s) = f \cdot \nabla(s) + df \otimes s \quad \text{for any section } s \text{ and function } f.$$

Here  $D$  is the polar divisor of the connection  $\nabla$ . The connection  $\nabla$  is *flat* when the curvature vanishes, that is  $\nabla \cdot \nabla = 0$ , meaning that it has no local monodromy outside the support of  $D$ ; we can therefore define its monodromy representation. *Throughout the text, all connections are assumed to be flat.*

When  $\det(E) = \mathcal{O}_X$  and the trace connection  $\text{tr}(\nabla)$  is the trivial connection on  $\mathcal{O}_X$ , we say that  $(E, \nabla)$  is a  *$\mathfrak{sl}_2$ -connection*. In particular, its monodromy representation takes values into  $\text{SL}_2(\mathbb{C})$ .

We will say that any two connections  $(E, \nabla)$  and  $(E', \nabla')$  are *birationally equivalent* when there is a birational bundle transformation  $\phi : E \rightarrow E'$  that conjugates the two operators  $\nabla$  and  $\nabla'$ . Keep in mind that the polar divisor might be not the same for  $\nabla$  and  $\nabla'$ . The connection  $(E, \nabla)$  is called *regular* if it is birationally equivalent to a logarithmic connection (see [15]), in particular having only simple poles (i.e. with reduced polar divisor); it is called *irregular* if not.

We will say that  $(E, \nabla)$  and  $(E', \nabla')$  are *projectively equivalent* if the induced  $\mathbb{P}^1$ -bundles coincide  $\mathbb{P}E = \mathbb{P}E'$ , and if moreover  $\nabla$  and  $\nabla'$  induce the same projective connection  $\mathbb{P}\nabla = \mathbb{P}\nabla'$ ; equivalently, they induce the same Riccati foliation on the total space of  $\mathbb{P}E$ .

We provide a structure theorem for connections up to the combination of projective and birational equivalence. In order to settle our result, we note that any

meromorphic rank 2 connection is (projectively birationally) equivalent to a  $\mathfrak{sl}_2$ -connection; moreover, any two  $\mathfrak{sl}_2$ -connections  $(E, \nabla)$  and  $(E', \nabla')$  are equivalent if, and only if, there exists a flat rank 1 logarithmic connection  $(L, \delta)$  on  $X$  having monodromy into the binary group  $\{\pm 1\} \in \mathbb{G}_m$  such that  $(E, \nabla)$  is birationally equivalent to  $(L, \delta) \otimes (E', \nabla')$ . In particular,  $(E, \nabla)$  and  $(E', \nabla')$  are birationally equivalent after pulling them back to the ramified two-fold cover  $Y \rightarrow X$  determined by the monodromy representation of  $(L, \delta)$ . A combination of Corollary B and Theorem C yields:

**Theorem E.** *Let  $(E, \nabla)$  be a flat meromorphic rank 2 connection on a projective manifold  $X$ . Then at least one of the following assertions holds true.*

- (1) *There exists a generically finite Galois morphism  $f : Y \rightarrow X$  such that  $f^*(E, \nabla)$  is projectively birationally equivalent to one of the following connections defined on the trivial bundle:*

$$\nabla = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \quad \text{or} \quad d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$$

*with  $\omega$  a rational closed 1-form on  $X$ .*

- (2) *There exists a rational map  $f : X \dashrightarrow C$  to a curve and a meromorphic connection  $(E_0, \nabla_0)$  on  $C$  such that  $(E, \nabla)$  is projectively birationally equivalent to  $f^*(E_0, \nabla_0)$ .*
- (3) *The  $\mathfrak{sl}_2$ -connection  $(E, \nabla)$  has at worst regular singularities and there exists a rational map  $f : X \dashrightarrow \mathfrak{S}$  which projectively factors the monodromy through one of the tautological representations of a polydisk Shimura modular orbifold  $\mathfrak{S}$ . In particular, the monodromy representation of  $(E, \nabla)$  is quasi-unipotent at infinity, rigid, and Zariski dense.*

In particular, when  $(E, \nabla)$  is irregular, only the former two cases occur. As mentioned in the abstract, the connection projectively factors through a curve whenever it has non trivial Stokes.

**1.5. Structure of the paper.** The paper is divided in two parts, with the first independent of the second. In the first part, Sections 2 and 3, we recall some results on the existence of fibrations which will be used throughout the paper, and present the proof of Theorem A. In this first part we avoided using foliation theory aiming at a wider audience. The second part deals with the irregular case and is organized as follows. Section 4 presents foundational results about Riccati foliations, most of them borrowed from [20] and [14]. In particular, we show how to reduce the general problem to the surface case. Section 5 describes the local structure of a Riccati foliation along its polar divisor (on a surface); we state there, in Theorem 5.6, a reduction of singularities in the spirit of Sabbah's "good formal model" for meromorphic connections by using blowing-ups and ramified covers; the proof of this technical result is postponed in Section 8, using tools from the theory of transversely projective foliations, like classification of their reduced singular points following [4, 35]. Section 6 analyzes the structure of reduced Riccati foliations over surfaces at a neighborhood of a connected component its irregular divisor, showing in particular the existence of flat tranverse coordinates: it will be essential to produce the fibration in the absence of rich monodromy. Section 7 contains the proofs of Theorems C and E. Section 8 deals with transversely projective foliations, including the proof of Theorem D, and the fact that it is actually equivalent to

our structure result on  $\mathfrak{sl}_2$ -connections. Section 9 presents a series of examples underlining the sharpness of our results. Finally in an appendix we prove a result reminiscent of Sabbah's good formal models in the context of Riccati foliations.

## 2. EXISTENCE OF FIBRATIONS

In this section we collect three results about the existence of fibrations on projective manifolds which will be used in the sequel. For the proof of Theorem A all we will need is Theorem 2.1 below, which is due to Totaro [33]. Toward the end of the paper (proof of Theorem C), we will make use of the two other results below.

**Theorem 2.1.** *Let  $X$  be a projective manifold and  $D_1, D_2, \dots, D_r$ ,  $r \geq 3$ , be connected effective divisors which are pairwise disjoint and whose Chern classes lie in a line inside of  $H^2(X, \mathbb{R})$ . Then there exists a non constant morphism  $f : X \rightarrow C$  to a smooth curve  $C$  with connected fibers which maps the divisors  $D_i$  to points.*

The original proof studies the restriction map  $H^1(X, \mathbb{Q}) \rightarrow H^1(\hat{D}_1, \mathbb{Q})$ , where  $\hat{D}_1$  is the disjoint union of desingularizations of the irreducible components of  $D_1$ . When it is injective, this map leads to a divisor linearly equivalent to zero in the span of  $D_2, \dots, D_r$  which defines the fibration. Otherwise, the fibration is constructed as a quotient of the Albanese map of  $X$ . An alternative proof, based on properties of some auxiliary foliations is given in [28]. It goes as follows: given two divisors with proportional Chern classes, one constructs a logarithmic 1-form with poles on these divisors and purely imaginary periods. The induced foliation, although not by algebraic leaves in general, admits a non-constant real first integral. Comparison of the leaves of two of these foliations coming from three pairwise disjoint divisors with proportional Chern classes reveals that they are indeed the same foliation. The proportionality factor of the corresponding two logarithmic 1-forms gives, after Stein factorization, the sought fibration. For details, see respectively [33, 28].

In general, two disjoint divisors with same Chern classes are not fibers of a fibration. Indeed, if  $L$  is a non-torsion flat line-bundle over a projective curve  $C$  then the surface  $S = \mathbb{P}(L \oplus \mathcal{O}_C)$  admits two homologous disjoint curves, corresponding to the inclusions of  $L$  and  $\mathcal{O}_C$  in  $L \oplus \mathcal{O}_C$ , which are not fibers of a fibration. The point is that the normal bundle of these curves sections are  $L^*$  and  $L$ . Nevertheless, if the normal bundle of one of the effective divisors is torsion then we do have a fibration containing them as fibers.

**Theorem 2.2.** *Let  $X$  be a projective manifold and  $D_1, D_2$  be connected effective divisors which are pairwise disjoint and whose Chern classes lie in a line of  $H^2(X, \mathbb{R})$ . If  $\mathcal{O}_X(D_1)|_{D_1}$  is torsion then there exists a non constant map  $f : X \rightarrow C$  to a smooth curve  $C$  with connected fibers which maps the divisors  $D_i$  to points.*

A proof of this result is presented in Section 2.1 below. There is also an infinitesimal variant of Theorem 2.1 due to Neeman [27, Article 2, Theorem 5.1] where instead of three divisors we have only one divisor with torsion normal bundle and constraints on a infinitesimal neighborhood of order bigger than the order of the normal bundle. The formulation below is due to Totaro, see [34, paragraph before the proof of Lemma 4.1], and the proof we present is an adaptation of Neeman's proof. For an alternative proof see [3, Section 3].

**Theorem 2.3.** *Let  $X$  be a projective manifold and  $D$  be a connected effective divisor. Suppose that  $\mathcal{O}_X(D)|_D$  is torsion of order  $m$ . If*

$$\mathcal{O}_X(mD)|_{(m+1)D} \simeq \mathcal{O}_{(m+1)D}$$

*then there exists a nonconstant morphism  $f : X \rightarrow C$  to a smooth curve  $C$  with connected fibers which maps the divisor  $D$  to a point.*

*Proof.* Let  $I = \mathcal{O}_X(-D)$  be the defining ideal of  $D$ . Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(mD) & \longrightarrow & \mathcal{O}_X(mD) \otimes \frac{\mathcal{O}_X}{I^m} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \wr \downarrow \\ 0 & \longrightarrow & \mathcal{O}_D & \longrightarrow & \mathcal{O}_X(mD) \otimes \frac{\mathcal{O}_X}{I^{m+1}} & \longrightarrow & \mathcal{O}_X(mD) \otimes \frac{\mathcal{O}_X}{I^m} \longrightarrow 0 \end{array}$$

deduced from the standard exact sequence

$$0 \rightarrow \mathcal{O}_X(-mD) \rightarrow \mathcal{O}_X \rightarrow \frac{\mathcal{O}_X}{I^m} \rightarrow 0.$$

In cohomology we get the diagram

$$\begin{array}{ccccc} & & H^0(X, \mathcal{O}_X(mD) \otimes \frac{\mathcal{O}_X}{I^m}) & \longrightarrow & H^1(X, \mathcal{O}_X) \\ & & \wr \downarrow & & \downarrow \\ H^0((m+1)D, \mathcal{O}_X(mD) \otimes \frac{\mathcal{O}_X}{I^{m+1}}) & \longrightarrow & H^0(mD, \mathcal{O}_X(mD) \otimes \frac{\mathcal{O}_X}{I^m}) & \longrightarrow & H^1(D, \mathcal{O}_D) \end{array}$$

If  $\mathcal{O}_X(mD)|_{(m+1)D} \simeq \mathcal{O}_{(m+1)D}$  then  $\mathcal{O}_X(mD)|_{mD} \simeq \mathcal{O}_{mD}$  and

$$1 \in H^0(mD, \mathcal{O}_{mD}) = H^0(mD, \mathcal{O}_X(mD) \otimes \frac{\mathcal{O}_X}{I^m})$$

belongs to the image of the map in the lower left corner. Exactness of the bottom row implies that  $1 \in H^0(mD, \mathcal{O}_{mD})$  is mapped to zero in  $H^1(D, \mathcal{O}_D)$ .

According to the diagram, the morphism  $H^0(mD, \mathcal{O}_{mD}) \rightarrow H^1(D, \mathcal{O}_D)$  factors through  $H^1(X, \mathcal{O}_X) \rightarrow H^1(D, \mathcal{O}_D)$ .

If  $1 \in H^0(mD, \mathcal{O}_{mD})$  is mapped to zero in  $H^1(X, \mathcal{O}_X)$  then we deduce from the first row of the first diagram above that  $h^0(X, \mathcal{O}_X(mD)) \geq 2$ . Therefore  $mD$  moves in a pencil and we have the sought fibration.

If instead  $1 \in H^0(mD, \mathcal{O}_{mD})$  is mapped to a nonzero element in  $H^1(X, \mathcal{O}_X)$  then  $H^1(X, \mathcal{O}_X) \rightarrow H^1(D, \mathcal{O}_D)$  is not injective. Thus the same holds true for the map  $H^1(X, \mathcal{O}_X) \rightarrow \oplus H^1(D_i, \mathcal{O}_{D_i})$  where  $D_i$  are the irreducible components of  $D$ . It follows that  $\oplus \text{Alb}(D_i)$  (direct sum of the Albanese varieties) do not dominate  $\text{Alb}(X)$ . The morphism

$$X \longrightarrow \frac{\text{Alb}(X)}{\oplus \text{Alb}(D_i)}$$

contracts  $D$ , and is non constant. It follows (cf. [27] or [33]) that the image is a curve and we get the sought fibration as the Stein factorization of this morphism.  $\square$

**2.1. Proof of Theorem 2.2.** By assumption the effective divisors  $D_1$  and  $D_2$  have proportional Chern classes. Therefore, there exists non-zero positive integers  $a_1, a_2$  such that the line bundle  $\mathcal{L} = \mathcal{O}_X(a_1D_1 - a_2D_2)$  lies in  $\text{Pic}^0(X)$ . If  $\mathcal{L}$  is a torsion element of  $\text{Pic}^0(X)$  then there exists a rational function  $g : X \rightarrow \mathbb{P}^1$  with zero set supported on  $D_1$  and polar set supported on  $D_2$ . We can take  $f$  as the Stein factorization of  $g$ .

Suppose now that  $\mathcal{L}$  is not a torsion line-bundle. The restriction  $\mathcal{L}|_{D_1}$  of  $\mathcal{L}$  to  $D_1$  is isomorphic to  $\mathcal{O}_{D_1}(a_1D_1)$  and according to our assumptions is a torsion line-bundle. Hence for some integer  $m \neq 0$ ,  $\mathcal{L}^{\otimes m}$  is in the kernel of the restriction morphism  $\text{Pic}^0(X) \rightarrow \text{Pic}^0(D_1)$ . Since we are assuming that  $\mathcal{L}$  is not a torsion line-bundle, it follows that the restriction morphism  $H^1(X, \mathcal{O}_X) \rightarrow H^1(D_1, \mathcal{O}_{D_1})$  is not injective. We conclude as in the proof of Theorem 2.3.  $\square$

### 3. FACTORIZATION OF REPRESENTATIONS

**3.1. Criterion for factorization.** We apply Theorem 2.1 to establish a criterion for the factorization of representations of quasiprojective fundamental groups. In the statement below, we have implicitly fixed an ample divisor  $A$  in  $X$  and we consider the bilinear pairing defined in  $NS(X)$ , the Néron-Severi group of  $X$ , defined by  $(E, D) = E \cdot D \cdot A^{n-2}$  where  $n = \dim X$ . According to Hodge index Theorem this bilinear form has signature  $(1, \text{rank } NS(X) - 1)$ .

**Theorem 3.1.** *Let  $X$  be a projective manifold,  $D$  be a reduced simple normal crossing divisor in  $X$ , and  $\rho : \pi_1(X - D) \rightarrow G$  be a representation to a simple non-abelian linear algebraic group  $G$  with Zariski dense image. Suppose there exists  $E$ , a connected component of the support  $D$  with irreducible components  $E_1, \dots, E_k$ , and an (analytic) open subset  $U \subset X$  containing  $E$  such that the restriction of  $\rho$  to  $\pi_1(U - E)$  has solvable image. Then either the intersection matrix  $(E_i, E_j)$  is negative definite or the representation factors through an orbicurve.*

*Proof.* If the intersection matrix  $(E_i, E_j)$  is negative definite then there is nothing to prove. Throughout we will assume that  $(E_i, E_j)$  is not negative definite.

Let  $S$  be the Zariski closure of  $\rho(\pi_1(U - E))$  in  $G$ . Since  $G$  is simple and non-abelian while  $\rho(\pi_1(U - E))$  is solvable, we have that  $S \neq G$ . Let  $\mu$  be the derived length of  $S$  and choose an element  $\gamma$  of the  $(\mu + 1)$ -th derived group of  $\pi_1(X - D)$  such that  $\rho(\gamma) \neq \text{id}$ .

We apply Malcev's Theorem (any finitely generated subgroup of  $G$  is residually finite) [22] to obtain a morphism  $\varrho : \pi_1(X - D) \rightarrow \Gamma$  to a finite group  $\Gamma$  such that  $\varrho(\gamma) \neq \text{id}$ . This choice of  $\varrho$  implies that the derived length of  $\varrho(\pi_1(X - D))$  is at least  $\mu + 2$ . Since the derived length of  $S$  is  $\mu$ , it follows that  $c = [\varrho(\pi_1(X - D)) : \varrho(\pi_1(U - E))]$  is at least 3.

Let us now consider the covering  $p : X_\varrho \rightarrow X$ , ramified along  $D$ , determined by  $\varrho$ . Over  $X - D$ ,  $p|_{p^{-1}(X - D)}$  is the étale covering of  $X - D$  determined by  $\ker \varrho$ . In particular  $p^{-1}(X - D)$  is a smooth quasi-projective manifold. In general,  $X_\varrho$  is projective but not necessarily smooth. Notice that by construction  $p^{-1}(E)$  has at least  $c = [\varrho(\pi_1(X - D)) : \varrho(\pi_1(U - E))] \geq 3$  distinct connected components in  $X_\varrho$ .

To be able to apply Hodge index Theorem, we now proceed to *desingularize*  $X_\varrho$ . In order to keep track of the intersection matrix  $(E_i, E_j)$ , instead of applying Hironaka's Theorem we will construct a smooth variety  $Y$  together with a ramified covering  $\phi : Y \rightarrow X$  which factors through  $p$ .

To each irreducible component  $D_i$  of  $D$  let  $m_i$  be the order of  $\varrho(\gamma_i)$  on a short loop  $\gamma_i$  around  $D_i$ . Let  $\kappa : \tilde{X} \rightarrow X$  be a finite ramified covering of  $X$ , with  $\tilde{X}$  smooth, such that  $\kappa^*(D_i) = m_i \tilde{D}_i$  where  $\tilde{D}_i$  is a smooth and irreducible hypersurface and  $\tilde{D} = \sum \tilde{D}_i$  is a simple normal crossing divisor. For the existence of  $\kappa$  with the above properties, we can use Kawamata coverings (see [17, Proposition 4.1.12]). If we denote the ramification divisor of  $\kappa$  by  $R$  and its image under  $\kappa$  by  $\Delta$  then we can also assume that  $\Delta + D$  is a simple normal crossing divisor on  $X$ .

The composition  $\varrho \circ \kappa_* : \pi_1(\tilde{X} - \tilde{D}) \rightarrow \Gamma$  sends short loops around the irreducible components of  $\tilde{D}$  to the identity of  $\Gamma$  and therefore induces a representation  $\tilde{\varrho} : \pi_1(\tilde{X}) \rightarrow \Gamma$ . Let  $\pi : Y \rightarrow \tilde{X}$  be the étale covering of  $\tilde{X}$  determined by the kernel of  $\tilde{\varrho}$ . It is a projective manifold endowed with a ramified covering  $\phi = \kappa \circ \pi : Y \rightarrow X$ . Notice that  $\phi$  is the ramified covering associated to the representation  $\varrho \times \psi : \pi_1(X - (D + \Delta)) \rightarrow \Gamma \times \Lambda$ . Therefore  $\phi : Y \rightarrow X$  factors through  $p : X_\varrho \rightarrow X$  as wanted. Consequently  $\phi^{-1}(E)$  has (at least) as many connected components as  $p^{-1}(E)$ .

Let  $B$  be one of the connected components of  $\phi^{-1}(E)$  with irreducible components  $B_1, \dots, B_\ell$ . Consider the ample divisor  $A' := \phi^*A$  on  $Y$  and define  $(\cdot, \cdot)_Y$  using it. Notice that

$$(\phi^*C_1, \phi^*C_2)_Y = \phi^*C_1 \cdot \phi^*C_2 \cdot (\phi^*A)^{n-2} = \deg(\phi) \cdot (C_1, C_2)$$

for any divisors  $C_1, C_2$  in  $X$ . Since we are assuming that  $(E_i, E_j)$  is not negative definite, there exists an effective divisor  $F$  supported by  $B$  with  $(F, F)_Y \geq 0$ . In fact, there exists such divisor for each connected component of  $\phi^{-1}(E)$ , therefore at least three. Hence, we can produce  $F_1, F_2, F_3$  pairwise disjoint effective divisors with connected supports on  $Y$  satisfying  $(F_i, F_i)_Y \geq 0$  and  $(F_i, F_j)_Y = 0$ . Since the signature of the quadratic form  $(\cdot, \cdot)_Y$  is  $(1, \text{rank } NS(Y) - 1)$ , we deduce that all the three divisors have proportional Chern classes; moreover,  $(F_i, F_i)_Y = 0$ . Notice also that either  $F_i \cap B = \emptyset$  or the support of  $F_i$  coincides with  $B$ . Indeed, if  $F$  is an effective divisor with support contained in  $B$ , but not equal to  $B$ , then we can choose an irreducible component of  $B$ , say  $C$ , not contained in the support of  $F$  but intersecting it. Therefore  $(C, F)_Y > 0$  and, provided that  $(F, F)_Y \geq 0$ , for  $k$  large enough  $C + kF$  is an effective divisor with  $(C + kF, C + kF)_Y = (C, C)_Y + 2k(C, F)_Y + (F, F)_Y > 0$ . But, doing so, we would produce new disjoint  $F_i$ 's with  $(F_i, F_i)_Y > 0$ , contradicting Hodge index Theorem.

We can apply Theorem 2.1 to ensure the existence of a curve  $\Sigma$  and a nonconstant morphism with irreducible general fiber  $g : Y \rightarrow \Sigma$  such that the divisors  $F_1, F_2$ , and  $F_3$  are multiples of fibers of  $g$ . The morphism  $g$  is proper and open, thus all the other connected components of  $\phi^{-1}(D)$  are mapped by  $g$  to points. Let us denote by  $\tilde{\rho} := \rho \circ \phi_* : \pi_1(Y - \phi^*D) \rightarrow G$  the lifted representation. We want first to prove that  $\tilde{\rho}$  factors through  $g$ . Note that  $\tilde{\rho}$  also has Zariski dense image in  $G$ .

Let  $U \subset Y - \phi^*D$  be a Zariski open subset such that the restriction of  $g$  to  $U$  is a smooth and proper fibration, thus locally trivial in the  $C^\infty$  category, over  $\Sigma^\circ = g(U)$ . Let also  $F$  be a fiber of  $g|_U$  and  $H$  be the Zariski closure in  $G$  of  $\tilde{\rho}(\pi_1(F))$ , and consider the following diagram where we have used that  $\pi_2(\Sigma^\circ) = 0$  since  $\Sigma^\circ$  is a non-compact curve.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & H & \longrightarrow & G & & \\
 & & \tilde{\rho} \uparrow & & \tilde{\rho} \uparrow & & \\
 1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(U) & \xrightarrow{g_*} & \pi_1(\Sigma^\circ) \longrightarrow 1
 \end{array}$$

From this, it is clear that  $\pi_1(F)$  is normal in  $\pi_1(U)$ , and since normality is a (Zariski) closed condition, we deduce that  $H$  is a normal subgroup of  $G$ . Since  $G$  is simple, we conclude that  $H$  must be trivial, i.e. the restriction of  $\tilde{\rho}$  to  $U$  factors through the curve  $\Sigma^\circ$ . Following the proofs of [11, Lemma 3.5] we see that this suffices to obtain the factorization through an orbicurve. Still denote by  $g : Y \rightarrow \Sigma$  the factorization morphism.

We will now prove that the morphism  $g : Y \rightarrow \Sigma$  descends to a morphism  $f : X \rightarrow C$  that factors the initial representation  $\rho$ . Here, we follow the argument of [11, Lemma 3.6]. By Stein factorization Theorem, we can assume that  $g : Y \rightarrow \Sigma$  has connected fibers. Assume also  $\phi : Y \rightarrow X$  is Galois, with Galois group  $\Gamma$ ; if not, replace  $Y$  by some finite Galois covering  $\tilde{\phi} : \tilde{Y} \xrightarrow{\tilde{\rho}} Y \xrightarrow{\phi} X$  (choose a finite index normal subgroup of  $\pi_1(X)$  that contains the subgroup defining  $\phi$ ). We note that  $Y$  might become singular, but this does not matter in what follows. For any  $\gamma \in \Gamma$ , we want to prove that  $\gamma$  permutes  $g$ -fibers, i.e. for a generic fiber  $F$  of  $g$ , then  $g \circ \gamma(F)$  is a point.

Aiming at a contradiction assume  $g \circ \gamma(F)$  is not a point. Then  $g|_{\gamma(F)} : \gamma(F) \rightarrow \Sigma$  is surjective and, since the representation  $\tilde{\rho}$  factors through the curve  $\Sigma$ , it follows that the representation  $\tilde{\rho}$  has Zariski dense image in restriction to  $\gamma(F)$ . Strictly speaking, in order to define the restriction  $\tilde{\rho}$  to  $\gamma(F)$ , we have to move the base point of the fundamental group (that we have omitted so far) to put it into  $\gamma(F)$  and the restriction depends on the way we do this, but different choices lead to conjugated subgroups and the property of being Zariski dense is invariant by conjugation. Once this has been done, the image  $H$  of  $\tilde{\rho}|_{\gamma(F)}$  must be of finite index in the image of the factorizing representation  $\pi_1(\Sigma) \rightarrow G$ . By hypothesis  $G$  is simple and therefore  $H$  is Zariski dense. On the other hand, since  $\tilde{\rho}$  comes from a representation on  $X = Y/\Gamma$ , it follows that  $\tilde{\rho}$  must be trivial in restriction to  $\gamma(F)$ , since it is in restriction to  $F$ . This contradiction shows that  $\Gamma$ , the Galois group of  $\phi$ , permutes the fibers of  $g : Y \rightarrow \Sigma$ , and thus permutes the points of  $\Sigma$  (connectedness of fibers). We get a morphism from  $X$  to the orbicurve  $C = \Sigma/\Gamma$  through which  $\rho$  factors.  $\square$

**Remark 3.2.** We can avoid the factorization through an orbicurve by instead restricting the factorization to a Zariski open subset of  $X - D$ . In the opposite direction, if we allow one dimensional Deligne-Mumford stacks with general point having non trivial stabilizer as targets of the factorization, then we can replace simple linear algebraic groups by quasi-simple linear algebraic groups in the statement, since our proof shows that for  $G$  quasi-simple, there exists a fibration such that the Zariski closure of the images of fundamental groups of fibers of  $f$  under  $\rho$  are finite.

### 3.2. Rank-two representations at neighborhoods of divisors.

**Proposition 3.3.** *Let  $X$  be a complex manifold,  $D$  a reduced and simple normal crossing divisor in  $X$ , and  $\rho : \pi_1(X - D) \rightarrow \mathrm{SL}_2(\mathbb{C})$  a representation. Let  $E$  be a connected divisor with support contained in  $D$  such that for each irreducible component  $E_i$  of  $E$  and any short loop  $\gamma_i$  turning around  $E_i$ , the element  $\rho(\gamma_i)$  does*

not lie in the center of  $\mathrm{SL}_2(\mathbb{C})$ , i.e.  $\rho(\gamma_i)$  is distinct from  $\pm \mathrm{Id}$ . Then there exists an open subset  $U \subset X$  containing  $E$  such that the restriction of  $\rho$  to  $\pi_1(U - D)$  has solvable image.

*Proof.* Let  $E_1, \dots, E_k$  be the irreducible components of  $E$  and  $\gamma_1, \dots, \gamma_k$  be short loops turning around them. We will denote the set of smooth points of  $D$  in  $E_i$  by  $E_i^\circ$ , i.e.  $E_i^\circ = E_i - \cup_{j \neq i} (E_j \cap E_i)$ .

Suppose first that  $\rho(\gamma_1)$  is unipotent. Since, by hypothesis, it is different from the identity, its action on  $\mathbb{C}^2$  leaves invariant a one-dimensional subspace  $L$ . If  $U_1$  is a small tubular neighborhood of  $E_1$  and  $U_1^\circ = U_1 - D$  then  $U_1^\circ$  has the homotopy type of a  $S^1$ -bundle over  $E_1^\circ$  and therefore the subgroup generated by  $\gamma_1$  in  $\pi_1(U_1^\circ)$  is normal. It follows that every  $\gamma \in \pi_1(U_1^\circ)$  also leaves  $L$  invariant. It follows that the rank two local system induced by  $\rho$  admits a unique rank one local subsystem determined by  $L$  on  $U_1^\circ$ .

To analyze what happens at a non-empty intersection  $E_1 \cap E_j$ , we can assume that both  $\gamma_1$  and  $\gamma_j$  have base points near  $E_1 \cap E_j$ . Thus  $\gamma_1$  commutes with  $\gamma_j$ , both  $\rho(\gamma_1)$  and  $\rho(\gamma_j)$  are unipotent, and they both leave  $L$  invariant. Thus the rank one local subsystem determined by  $L$  on  $U_1^\circ$  extends to a rank one local subsystem on  $U_1^\circ \cup U_j^\circ$ . Repeating the argument above for the other irreducible components  $E_2, \dots, E_k$ , we deduce the existence of a neighborhood  $U$  of  $E$  such that  $\rho(\pi_1(U - D))$  is contained in a Borel subgroup of  $\mathrm{SL}_2(\mathbb{C})$ .

Similarly if  $\rho(\gamma_1)$  is semi-simple, then the same holds true for every  $\gamma_i$ . Moreover, the representation now leaves invariant the union of two linear subspaces  $L_1$  and  $L_2$  (but does not necessarily leave invariant any of the two). We deduce that the image of  $\rho$  restricted to a neighborhood of  $E$  minus  $D$  is contained in an extension of  $\mathbb{Z}/2\mathbb{Z}$  by  $\mathbb{C}^*$ .  $\square$

**Remark 3.4.** The proof above is very similar to the proof of [11, Lemma 4.5].

**Proposition 3.5.** *Let  $X, D, E, U$  and  $\rho$  be as in Proposition 3.3. Assume also that every short loop  $\gamma$  turning around any irreducible component of  $D - E$  which intersects  $E$  has monodromy in the center of  $\mathrm{SL}_2(\mathbb{C})$ . If the intersection matrix of  $E$  is negative definite then the restriction of  $\rho$  to  $\pi_1(U - D)$  is quasi-unipotent at the irreducible components of  $E$ .*

*Proof.* Let  $(F, \nabla)$  be a rank two vector bundle over  $U$  with a flat logarithmic connection whose monodromy is given by  $\rho$  (see [15]). Since the monodromy is solvable, around each point of  $U - D$  we have one or two subbundles of  $F$  which are left invariant by  $\nabla$ . Modulo passing to a double covering of  $U - D$  if necessary, we can assume that  $(F, \nabla)$  is reducible, i.e. we have a subbundle  $F_1 \subset F$  and a logarithmic connection  $\nabla_1$  on  $F_1$  such that  $\nabla_1 = \nabla|_{F_1}$ . The monodromy of  $\nabla_1$  on a loop  $\gamma$  around irreducible components of  $E$  equal to one of the eigenvalues of  $\rho(\gamma)$ , say  $\lambda_\gamma$ . If  $\gamma_i$  is a short loop around an irreducible component  $E_i$  of  $E$  then the residue of  $\nabla_1$  along  $E_i$  satisfies

$$\exp(2\pi i \mathrm{Res}_{E_i}(\nabla_1)) = \lambda_{\gamma_i}.$$

By the residue formula we can write

$$c_1(F_1) = \sum \mathrm{Res}_{E_i}(\nabla_1)E_i + \sum \mathrm{Res}_{D_j}(\nabla_1)D_j$$

where  $D_1, \dots, D_s$  are the irreducible components of  $D$  intersecting  $E$  but not contained in it. Since the eigenvalues around  $D_j$  are  $\pm 1$ , we have that  $\mathrm{Res}_{D_j}$  is a

half-integer. Since  $c_1(F_1)$  lies in  $H^2(U, \mathbb{Z})$ , we have that for every  $k$

$$(c_1(F_1), E_k) = \sum \text{Res}_{E_i}(\nabla_1)(E_i, E_k) + \sum \text{Res}_{D_j}(\nabla_1)(D_j, E_k)$$

is an integer. Therefore the vector  $v = (\text{Res}_{E_1}(\nabla_1), \dots, \text{Res}_{E_k}(\nabla_1))^t$  satisfies a linear equation of the form  $A \cdot v = b$  with  $A = (E_i, E_j)$  and  $2b \in \mathbb{Z}^k$ . Since  $A$  has integral coefficients and is negative definite, it follows that  $v$  is a rational vector. Therefore the restriction of  $\rho$  to  $U - D$  is quasi-unipotent at the irreducible components of  $E$ .  $\square$

**Remark 3.6.** The proof above is reminiscent of Mumford's computation [26] of the homology of the plumbing of a contractible divisor on a smooth surface.

**3.3. Proof of Theorem A.** Let  $\rho : \pi(X - D) \rightarrow \text{SL}_2(\mathbb{C})$  be a Zariski dense representation which is not quasi-unipotent at infinity. Let  $E$  be a connected divisor with support contained in  $|D|$  such that  $\rho(\gamma) \neq \pm \text{Id}$  for every small loop around a irreducible component of  $E$ , and  $\rho(\gamma)$  is not quasi-unipotent for at least one small loop. If  $E$  is maximal with respect to these properties, Proposition 3.3 implies that the restriction of the projectivization of  $\rho$  to a neighborhood of  $E$  is solvable, and Proposition 3.5 implies that the intersection matrix of  $E$  is indefinite. Since  $\text{PSL}_2(\mathbb{C})$  is a simple group we can apply Theorem 3.1 to factorize the projectivization of  $\rho$  through an orbicurve. Theorem A follows.  $\square$

#### 4. RICCATI FOLIATIONS

Here we recall basic definitions and properties of Riccati foliations, and provide some reduction lemmata. Throughout  $X$  will be a projective manifold.

**4.1. Projective connections and Riccati foliations.** Let  $E$  be a rank 2 vector bundle on  $X$  and  $\nabla : E \rightarrow E \otimes \Omega_X^1(D)$  be a meromorphic connection on  $E$  with (effective) polar divisor  $D$ : the operator  $\nabla$  is  $\mathbb{C}$ -linear and satisfying Leibniz rule

$$\nabla(f \cdot s) = f \cdot \nabla(s) + df \otimes s \quad \text{for any local section } s \text{ and function } f.$$

In any local trivialization  $Z : E \rightarrow \mathbb{C}^2$ , the connection is defined by

$$Z \mapsto \nabla Z = dZ + AZ \quad \text{with} \quad Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \alpha & \beta \\ \omega & -\alpha \end{pmatrix}$$

where  $A$  is a matrix of meromorphic 1-forms (sections of  $\Omega_X^1(D)$ ).

The operator  $\nabla$ , being linear, commutes with the fiberwise action of  $\mathbb{G}_m = \mathbb{C}^*$  on  $E$  and induces a projective connection on the  $\mathbb{P}^1$ -bundle  $\mathbb{P}(E)$ . In local trivialization above, after setting  $(1 : z) = (z_1 : z_2)$ , the differential equation  $\nabla Z = 0$  takes the form  $\Omega = 0$  for the Riccati 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2$$

with  $(\omega_0, \omega_1, \omega_2) = (\omega, \delta - \alpha, -\beta)$ . Any two connections  $(E, \nabla)$  and  $(E', \nabla')$  induce the same projective connection if, and only if there is a rank 1 meromorphic connection  $(L, \zeta)$  such that  $(E', \nabla') = (L, \zeta) \otimes (E, \nabla)$ . The connection  $(E, \nabla)$  is actually determined by the (independent) data of the projective connection and the trace connection  $(\det(E), \text{tr}(\nabla))$  (defined in local trivialization by  $d + \alpha + \beta$ ).

The distribution  $\Omega = 0$  defines a (singular codimension one) foliation  $\mathcal{H}$  on the total space  $P$  of the  $\mathbb{P}^1$ -bundle if, and only if it satisfies Frobenius integrability condition  $\Omega \wedge d\Omega = 0$ , which is equivalent to

$$(2) \quad \begin{cases} d\omega_0 = \omega_0 \wedge \omega_1 \\ d\omega_1 = 2\omega_0 \wedge \omega_2 \\ d\omega_2 = \omega_1 \wedge \omega_2 \end{cases}$$

The flatness condition  $\nabla \cdot \nabla = 0$  for the linear connection, which writes  $dA + A \cdot A = 0$  in local trivialization, is equivalent to (2) (flatness of the projective connection) and the flatness  $d(\alpha + \delta) = 0$  for the trace connection. In this case,  $\nabla$ -horizontal sections project onto the leaves of  $\mathcal{H}$ .

We say that  $(P \rightarrow X, \mathcal{H})$  is a Riccati foliation since over a general point of  $X$ , the  $\mathbb{P}^1$ -fiber is tranverse to  $\mathcal{H}$ .

**Remark 4.1.** A codimension one foliation  $\mathcal{H}$  on (the total space of a)  $\mathbb{P}^1$ -bundle  $P \rightarrow X$  is a Riccati foliation if and only if it is tranverse to the generic  $\mathbb{P}^1$ -fiber. Indeed, in local trivialization,  $\mathcal{H}$  must be defined by a 1-form  $P(x, z)dz + \tilde{\Omega}(x, z)$  polynomial in  $z$ . One easily checks that transversality, for finite  $z$ , implies that  $P$  does not depend on the variable  $z$ . On the other hand, transversality near  $z = \infty$  implies that  $\tilde{\Omega}$  has degree at most 2 in  $z$ .

One could define Riccati foliations on arbitrary  $\mathbb{P}^1$ -bundles, not necessarily of the form  $\mathbb{P}(E)$ . However, in this paper, *all Riccati foliations are defined on projectivized vector bundles*  $\mathbb{P}(E)$ . In particular, they are birationally equivalent to the trivial bundle  $X \times \mathbb{P}^1$  where all above formula makes sense globally; moreover, they come from the  $\mathfrak{sl}_2$ -connection on  $X \times \mathbb{C}^2$  obtained by setting  $\alpha + \delta = 0$ .

The (effective) polar divisor  $(\mathcal{H})_\infty$  of the Riccati foliation is defined as the direct image under  $\pi : P \rightarrow X$  of the tangency divisor between  $\mathcal{H}$  and the vertical foliation defined by the fibers of  $\pi$ . It corresponds to the polar divisor of  $\Omega$  in local trivialization. We have  $(\nabla)_\infty \geq (\mathcal{H})_\infty$  since the trace  $\text{tr}(\nabla)$  may have more, or higher order poles; however, in the  $\mathfrak{sl}_2$ -case, the two divisors coincide.

The *monodromy representation* of the Riccati foliation  $(P \rightarrow X, \mathcal{H})$  is the representation

$$\rho_{\mathcal{H}} : \pi_1(X \setminus |(\mathcal{H})_\infty|) \longrightarrow \text{PSL}_2(\mathbb{C})$$

defined by lifting paths on  $X \setminus |(\mathcal{H})_\infty|$  to the leaves of  $\mathcal{H}$ . We note that  $\rho_{\mathcal{H}}$  is just the projectivization of the linear monodromy of  $\nabla$ .

**4.2. Riccati defined by a closed 1-form.** Let us start with two criteria.

**Lemma 4.2.** *Let  $(P, \mathcal{H})$  be a Riccati foliation over a projective manifold  $X$ . If there exists a birational map  $\Phi : P \dashrightarrow P$  distinct from the identity such that  $\Phi^*\mathcal{H} = \mathcal{H}$  and  $\pi \circ \Phi = \pi$  then there exists a generically finite morphism of degree at most two  $f : Y \rightarrow X$  such that  $f^*(P, \mathcal{H})$  is defined by a closed rational 1-form.*

*Proof.* Since  $\Phi$  commutes with projection  $\pi : P \rightarrow X$  it follows that over a general fiber of  $\pi$ ,  $\Phi$  is an automorphism. Let  $F = \{z \in P - \text{indet}(\Phi) \mid \Phi(z) = z\}$  be the set of fixed points of  $\Phi$ . Since we are dealing with a family of automorphism of  $\mathbb{P}^1$ , the projection of  $Z$  to  $X$  is generically finite of degree one or two. Assume first that the degree is one. Then we can birationally trivialize  $P$  in such a way that  $F$  becomes the section at infinity and  $\Phi$  is of the form  $z \mapsto z + \tau$  for some  $\tau \in \mathbb{C}(X)$ .

Let  $\Omega = dz + \omega_0 + \omega_1 z + \omega_2 z^2$  be a rational form defining  $\mathcal{H}$ . The invariance of  $\mathcal{H}$  under  $\Phi$  reads as

$$\Omega \wedge \Phi^* \Omega = 0 \iff \omega_2 = 0 \text{ and } \omega_1 = -d \log \tau \iff d(\tau \Omega) = 0.$$

If the degree of  $\pi|_Z$  is two then after replacing  $X$  by (the resolution of) a ramified double covering we can assume that  $P$  is trivial and that  $\Phi$  is given by  $z \mapsto \lambda(x)z$ . The invariance of  $\mathcal{H}$  under  $\Phi$  reads as

$$\Omega \wedge \Phi^* \Omega = 0 \iff \omega_0 = \omega_2 = d\lambda = 0 \iff d(z^{-1}\Omega) = 0.$$

This concludes the proof of the lemma.  $\square$

**Lemma 4.3.** *Let  $(P, \mathcal{H})$  be a Riccati foliation over a complex manifold  $X$ . Let  $H \subset P$  be a  $\mathcal{H}$ -invariant (maybe singular) hypersurface which intersects the generic fiber of  $P \rightarrow X$  at  $2 \leq n < \infty$  distinct points.*

- *If  $n \geq 3$ , then  $\mathcal{H}$  has a (non constant) meromorphic first integral.*
- *If  $n = 2$ , maybe passing to a two-fold cover  $X' \rightarrow X$ , then  $\mathcal{H}$  is defined after a convenient bundle trivialization by  $\Omega = \frac{dz}{z} + \omega$  with  $\omega$  a closed 1-form.*

*In particular, the monodromy of  $\mathcal{H}$  is virtually abelian.*

Since  $H$  is  $\mathcal{H}$ -invariant, the intersection set between  $H$  and a fiber must be (globally) invariant by the monodromy group computed on the same fiber. For  $n = 2$ , this implies that the monodromy is dihedral, and for  $n > 2$ , that it is finite. The conclusion of the Lemma is much stronger in case  $\mathcal{H}$  is irregular, since  $\mathcal{H}$  could have no monodromy but transcendental leaves in that case.

*Proof.* Maybe passing to a finite cover  $X' \rightarrow X$ , we can assume that  $H$  splits into  $n$  meromorphic sections. For  $n \geq 3$  one can send three of them to  $z = 0, 1, \infty$  and observe that their  $\mathcal{H}$ -invariance implies  $\omega_0 = \omega_1 = \omega_2 = 0$ . Therefore,  $z$  is a first integral and all leaves have algebraic closure. For  $n = 2$  one can send the two sections to  $z = 0$  and  $z = \infty$  and we get that the Riccati 1-form defining  $\mathcal{H}$  takes the form  $dz + \omega z$ ; Frobenius integrability implies  $d\omega = 0$ .  $\square$

Let us now describe Riccati foliations defined by closed 1-forms.

**Proposition 4.4.** *Let  $(P, \mathcal{H})$  be a Riccati foliation over a complex manifold  $X$ . If  $\mathcal{H}$  is defined by a closed 1-form  $\Omega$  on  $P$ , then:*

- *either  $\mathcal{H}$  has a (non constant) meromorphic first integral,*
- *or after a convenient bimeromorphic bundle trivialization, we have*

$$(3) \quad \Omega = c \left( \frac{dz}{z} + \omega \right) \quad \text{or} \quad \Omega = dz + \omega$$

*with  $\omega$  a meromorphic closed 1-form on  $X$  and  $c \in \mathbb{C}^*$ .*

*Proof.* Since  $\Omega$  is closed, the support  $H$  of the divisor  $\text{div}(\Omega)$  is  $\mathcal{H}$ -invariant. The hypersurface  $H$  intersects the generic fiber of  $P \rightarrow X$  in  $n \geq 1$  distinct points (1-forms have non trivial divisor on  $\mathbb{P}^1$ ). If  $n \geq 3$ , then  $\mathcal{H}$  has a first integral.

If  $n = 2$ , then  $H$  splits into the union of two meromorphic sections,  $H_0$  and  $H_\infty$  say; indeed, either they are zero and polar locus of  $\Omega$ , or they are both simple pole, but with opposite residue. After trivialization of  $P$  sending them to  $z = 0$  and  $z = \infty$  respectively, we get that  $\mathcal{H}$  is defined by a closed 1-form  $\Omega' := \frac{dz}{z} + \omega$ ; but  $\Omega = f \cdot \Omega'$  is closed as well, which implies that  $f$  is a first integral. We are thus in one of the two items depending on  $f$  is constant or not.

Finally, when  $n = 1$ , we can assume  $H = \{z = \infty\}$  and we get that  $\Omega = f(dz + \omega_0 + z\omega_1 + z^2\omega_2)$  restrict to the generic fiber as  $c \cdot dz$ , with  $c \in \mathbb{C}^*$ . This means that  $f$  does not depend on  $z$  and, by gauge transformation  $z \mapsto cz$ , we can assume  $f = 1$ . The 1-form  $\Omega$  is closed if, and only if,  $d\omega_0 = \omega_1 = \omega_2 = 0$ .  $\square$

**4.3. Generically finite morphisms and factorizations.** In order to prove Theorems C and D, it will be useful to blow-up the manifold  $X$  and pass to a finite cover in order to simplify the foliation. At the end, we will need the following descent lemma to come back to a conclusion on  $X$ .

**Proposition 4.5.** *Let  $(P, \mathcal{H})$  be a Riccati foliation over  $X$  and assume that  $f^*(P, \mathcal{H})$  is not defined by a closed rational 1-form for any dominant morphism  $f : Y \rightarrow X$ . If the pull-back  $\varphi^*(P, \mathcal{H})$  via generically finite morphism  $\varphi : \tilde{X} \rightarrow X$  factors through a curve, then the same holds true for  $(P, \mathcal{H})$ .*

*Proof.* It is very similar to the proof of [11, Lemma 3.6]. Maybe composing by a generically finite morphism, we can assume that

- (1)  $\varphi$  is Galois in the sense that there is a finite group  $G$  of birational transformations acting on  $\tilde{X}$  and acting transitively on a general fiber;
- (2) there is a morphism  $\tilde{f} : \tilde{X} \rightarrow \tilde{C}$  with connected fibers and a Riccati foliation  $(\tilde{P}_0, \tilde{\mathcal{H}}_0)$  over  $\tilde{C}$  such that its pull-back on  $\tilde{X}$  is birationally equivalent to  $(\tilde{P}, \tilde{\mathcal{H}}) := \varphi^*(P, \mathcal{H})$ .

In consequence, for any  $g \in G$ ,  $g^*(\tilde{P}, \tilde{\mathcal{H}})$  is birationally equivalent to  $(\tilde{P}, \tilde{\mathcal{H}})$  and the Riccati foliation  $(\tilde{P}, \tilde{\mathcal{H}})$  factors through  $\tilde{f} \circ g : \tilde{X} \dashrightarrow \tilde{C}$ . Consider a general fiber  $Z$  of  $\tilde{f}$ . The Riccati foliation  $(\tilde{P}, \tilde{\mathcal{H}})|_Z$  restricted to  $Z$  is birationally equivalent to the trivial Riccati foliation on  $Z$ , and thus admits a rational first integral. If the map  $\tilde{f} \circ g$  were dominant in restriction to  $Z$ , this would imply that  $(\tilde{P}_0, \tilde{\mathcal{H}}_0)$  also has a rational first integral, and the same for  $(\tilde{P}, \tilde{\mathcal{H}})$  and  $(P, \mathcal{H})$ , contradiction. Thus  $g$  (and  $G$ ) must permute general fibers of  $\tilde{f}$  and acts on  $\tilde{C}$ . Moreover,  $g^*(\tilde{P}_0, \tilde{\mathcal{H}}_0)$  is birational to  $(\tilde{P}_0, \tilde{\mathcal{H}}_0)$  for all  $g \in G$ . Lemma 4.2 implies that over each  $g : \tilde{X} \dashrightarrow \tilde{X}$  there exists a unique birational map  $\hat{g} : \tilde{P} \dashrightarrow \tilde{P}$  such that  $\hat{g}^*\tilde{\mathcal{H}} = \tilde{\mathcal{H}}$ , and a similar statement holds true for the action of  $G$  on  $\tilde{C}$ . Therefore, we get an action of  $G$  on the diagram

$$\begin{array}{ccc} \tilde{P} & \longrightarrow & \tilde{P}_0 \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{C} \end{array}$$

which preserves the Riccati foliations. Passing to the quotient, we get a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & P_0 \\ \downarrow & & \vdots \\ X & \xrightarrow{f} & C \end{array}$$

where  $P_0 \dashrightarrow C$  has  $\mathbb{P}^1$  as a general fiber. Moreover, the quotient foliation  $\mathcal{H}_0$  on  $P_0$  is transverse to the general fiber and thus of Riccati type.  $\square$

**4.4. Monodromy and factorization.** We say that a Riccati foliation  $(P, \mathcal{H})$  has regular singularities when it can be locally induced by a flat meromorphic linear connection having regular singularities in the sense of [15, Chapter II]. Like in the linear case, any two Riccati foliations  $(P, \mathcal{H})$  and  $(P', \mathcal{H}')$  having regular singularities have the same monodromy if and only if there exists a birational bundle map  $\phi : P \dashrightarrow P'$  such that  $\phi^*\mathcal{H}' = \mathcal{H}$ , see [14, Lemma 2.13]. In particular, if the monodromy factors through a curve, then so does the Riccati foliation. The next proposition, borrowed from [14, Proposition 2.14], tells us what remains true in the irregular case.

**Proposition 4.6.** *Let  $(P, \mathcal{H})$  be a Riccati foliation over  $X$ . Suppose there exists a morphism  $f : X \rightarrow C$  with connected fibers such that the polar divisor  $(\mathcal{P})_\infty$  of  $\mathcal{P}$  intersects the general fiber of  $f$  at most on regular singularities; assume moreover that the monodromy representation  $\rho$  of  $\mathcal{P}$  factors through  $f$ , i.e. there exists a divisor  $F$  supported on finitely many fibers of  $f$  and a representation  $\rho_0$  from the fundamental group of  $C_0 = f(X - |(\mathcal{P})_\infty + F|)$  to  $\mathrm{PSL}_2(\mathbb{C})$  fitting in the diagram below.*

$$\begin{array}{ccc} \pi_1(X - |(\mathcal{P})_\infty + F|) & \xrightarrow{\rho} & \mathrm{PSL}_2(\mathbb{C}) \\ f_* \downarrow & \nearrow \rho_0 & \\ \pi_1(C_0) & & \end{array}$$

*Then  $(P, \mathcal{H})$  factors through  $f : X \rightarrow C$ .*

**Remark 4.7.** For a Riccati or a linear connection, to have regular singularities is a local property which, if satisfied at some point of the polar divisor, remains true all along the irreducible component. The support of the polar divisor splits into regular and irregular components. The assumption in Proposition 4.6 that the general fiber of  $f$  intersect only regular singularities just means that the fiber intersects only regular components of the polar locus. Equivalently, the connection (or Riccati foliation) restricts to the fiber as a regular singular connection (resp. Riccati foliation).

**4.5. Reduction to the two-dimensional case.** The proposition below allows us to reduce our study of Riccati foliations over arbitrary projective manifolds to study of Riccati foliation over projective surfaces.

**Proposition 4.8.** *Let  $(P, \mathcal{H})$  be a Riccati foliation over a projective manifold  $X$  and assume it has no rational first integral. If the restriction of  $(P, \mathcal{H})$  to a sufficiently general surface  $Z \subset X$  factors through a curve, then the same holds true for  $(P, \mathcal{H})$  over  $X$ .*

*Proof.* Denote by  $\pi : P \rightarrow X$  the natural projection. Let  $(P_Z, \mathcal{H}_Z)$  be the restriction of  $(P, \mathcal{H})$  to  $Z \subset X$ , i.e.  $P_Z = P|_{\pi^{-1}(Z)}$  and  $\mathcal{H}_Z = \mathcal{H}|_{\pi^{-1}(Z)}$ . By assumption, we get a rational bundle map  $\phi : P_Z \dashrightarrow P_0$  such that  $\phi^*\mathcal{H}_0 = \mathcal{H}_Z$ . This shows that  $\mathcal{H}_Z$  contains a foliation by algebraic curves.

Applying the same argument for different choices of  $Z$ , we obtain that through a general point  $p \in P$  the leaf of  $\mathcal{H}$  through  $p$  contains an algebraic subvariety  $A_p$ . It is known that the germ of leaf at a general point contains an unique germ of algebraic subvariety maximal with respect to inclusion which turns out to be irreducible, see the proof of [21, Lemma 2.4]. In our setup, we can thus assume

that  $A_p$  has codimension at most two, since otherwise we would be able to enlarge  $A_p$  by choosing an appropriate  $Z$  and applying the above argument.

It follows from [21, Lemma 2.4] that  $\mathcal{H}$  contains a foliation  $\mathcal{G}$  with all leaves algebraic and having codimension at most two, and that  $\mathcal{H}$  is the pull-back of a foliation on a curve or a surface. In the former case, we get a rational first integral for  $\mathcal{H}$ , contradiction. Hence the codimension of  $\mathcal{G}$  must be equal to two. Let now  $L$  be a general leaf of  $\mathcal{G}$  and consider its projection to  $X$ . Since  $\mathcal{H}$  is tranverse to the general of fiber of  $\pi$ , this projection is generically étale and  $\pi(L)$  is (a Zariski open subset) of a divisor  $D_L$  on  $X$ . The construction method of  $\mathcal{G}$  makes clear that  $\pi^{-1}(D_L)$  is invariant by  $\mathcal{G}$ , and also that the restriction of  $\mathcal{H}$  to  $\pi^{-1}(D_L)$  is birationally equivalent to the foliation on a trivial  $\mathbb{P}^1$ -bundle over  $D_L$  given by the natural projection to  $\mathbb{P}^1$ . This is sufficient to show that  $\mathcal{H}$  is the pull-back of a Riccati foliation on a  $\mathbb{P}^1$ -bundle over a curve.  $\square$

## 5. POLAR DIVISOR AND REDUCTION OF RICCATI FOLIATIONS

In the next two sections, we will restrict our attention to Riccati foliations over projective surfaces. Proposition 4.8 allows to transfer the conclusions to Riccati foliations over arbitrary projective manifolds. The general case will come back into play only at the proofs of Theorems C and D.

Let  $(P, \mathcal{H})$  be a Riccati foliation over a projective surface  $S$ . In this section, we review the local structure of  $(P, \mathcal{H})$  over a neighborhood of a general point of  $(\mathcal{H})_\infty$ . Maybe blowing-up  $S$  and passing to a ramified cover, we arrive to a list of nice local models that either are defined by a first integral, or factor through a curve (a local version of Theorem C). Let us first recall the one dimensional classification following the description of [6, Chapter 4, Section 1].

**5.1. A review of the one dimensional case.** Here, we follow the description of [6, Chapter 4, Section 1] which corresponds to Levelt-Turritin normalization.

**Proposition 5.1.** *Let  $(P \rightarrow \Delta, \mathcal{H})$  be a Riccati foliation over a disc  $0 \in \Delta \in \mathbb{C}$  with a pole at 0. Maybe after passing to the two-fold cover*

$$\Delta \rightarrow \Delta ; x \mapsto x^2,$$

*after bimeromorphic bundle trivialization and change of  $x$  coordinate, the Riccati 1-form  $\Omega$  defining the foliation fits into one of the following types:*

- $\Omega = dz$  (trivial case)
- $\mathbf{Log}^{\mathbb{G}_m}$ :  $\frac{\Omega}{z} = \frac{dz}{z} + \lambda \frac{dx}{x}$ , with  $\lambda \in \mathbb{C} \setminus \mathbb{Z}$ ;
- $\mathbf{Log}^{\mathbb{G}_a}$ :  $\Omega = dz + \frac{dx}{x}$ ;
- **Irreg**:  $\Omega = dz + b(x)z^2 dx + \left(\frac{dx}{x^{k+1}} + \lambda \frac{dx}{x}\right) z + c(x)dx$  with  $b, c$  holomorphic,  $k \in \mathbb{Z}_{\geq 1}$  and  $\lambda \in \mathbb{C}$ .

In cases  $\mathbf{Log}^{\mathbb{G}_m}$  and **Irreg**,  $\lambda$  is unique modulo addition by an integer, i.e. is unique in  $\mathbb{C}/\mathbb{Z}$ ; in case  $\lambda \in \mathbb{Z}$ , we choose  $\lambda = 0$  corresponding to the trivial connection. In the irregular case,  $k$  is also unique and is called **irregularity index** (or Poincaré-Katz rank) of the singular point (see [1]); the order of pole cannot decrease by birational gauge transformation.

**Remark 5.2.** If we allow only birational bundle modification and change of coordinate  $x$ , i.e. no ramification  $x \mapsto x^2$ , then we have to add the irregular ramified case (called nilpotent in [6]) in the list of Proposition 5.1

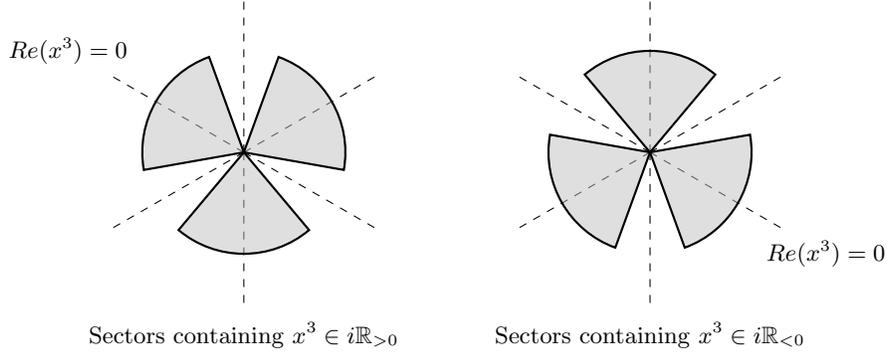


FIGURE 1.

- **Irreg<sup>ram</sup>**:  $\Omega = dz + \left( \frac{z^2-x}{4x^k} - \frac{1}{2} \right) \frac{dx}{x}$  + holomorphic with  $k \in \mathbb{Z}_{\geq 1}$ .

After ramification  $x = \tilde{x}^2$ , we get a singularity of irregular (unramified) type **Irreg** with irregularity index  $\tilde{k} = 2k - 1$ ; we say that this ramified singularity has irregularity index  $\frac{2k-1}{2}$ . We will get rid of this kind of singular points by using ramified coverings.

**5.2. Irregular singular points and Stokes matrices.** For details on what follows, see [24, Chapitre VI] or [19, Section 5]. In the normal form **Irreg**, coefficients  $b(x)$  and  $c(x)$  can be chosen with arbitrary large vanishing order at  $x = 0$ ; in fact, these coefficients can be killed by formal (generally divergent) gauge transformation and we arrive at the formal normal form

$$(4) \quad \frac{\Omega}{z} = \frac{dz}{z} + \frac{dx}{x^{k+1}} + \lambda \frac{dx}{x}$$

In the setting of  $\mathfrak{sl}_2$ -connections, this normal form writes

$$d + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \left( \frac{dx}{x^{k+1}} + \lambda \frac{dx}{x} \right).$$

In general this last normalization is not convergent. Nevertheless, there are  $2k$  closed sectors covering a neighborhood of 0 in the  $x$ -variable such that the differential equation is holomorphically conjugate to the normal form over the interior of the sector, and the conjugation extends continuously to the boundary. Each of the sectors contains exactly one of the arcs  $\{x \in \mathbb{D}_\varepsilon | x^k \in i\mathbb{R}\} - \{0\}$ . Over each of these sectors there are only two solutions with well defined limit when  $x$  approaches zero which correspond to the solutions  $\{z = 0\}$  and  $\{z = \infty\}$  for the normal form. When we change from a sector intersecting  $x^k \in i\mathbb{R}_{>0}$  to a sector intersecting  $x^k \in i\mathbb{R}_{<0}$  in the counter clockwise direction then we can continuously extend the solution corresponding to  $\{z = +\infty\}$  but the same does not hold true for the solution corresponding to  $\{z = 0\}$ . Similarly, when changing from sectors intersecting  $x^k \in i\mathbb{R}_{<0}$  to sectors with points in  $x^k \in i\mathbb{R}_{>0}$  in the counter clockwise direction we can extend continuously the solution corresponding to  $\{z = 0\}$  but not the one corresponding to  $\{z = \infty\}$ .

The obstructions to glue continuously the two distinguished solutions over adjacent sectors are the only obstructions to analytically conjugate the differential

equation to its normal form. These obstructions are codified by the Stokes matrices (matrix point of view is more convenient here):

$$\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & b_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_k & 1 \end{pmatrix}$$

(well-defined up to simultaneous conjugacy by a diagonal matrix). Precisely,  $b_i$ 's (resp.  $c_i$ 's) are responsible for the divergence of the central manifold at  $z = \infty$  (resp.  $z = 0$ ). In other terms, the  $b_i$ 's (resp. the  $c_i$ 's) are the obstructions to kill the coefficient  $b(x)$  (resp.  $c(x)$ ). The monodromy around  $x = 0$  is given by multiplying this sequence of Stokes matrices (in this cyclic order) with the formal monodromy

$$\begin{pmatrix} e^{-i\pi\lambda} & 0 \\ 0 & e^{i\pi\lambda} \end{pmatrix}$$

(on the left or the right, as this does not matter up to diagonal conjugacy).

If  $x = \varphi(\tilde{x})$  is a change of coordinate compatible with normal form **Irreg**, then the linear part  $\varphi'(0)$  is a  $k^{\text{th}}$  root of unity; it permutes the sectors, and consequently induces a cyclic permutation of (indices of) Stokes matrices. All these classical results can be found in [24, Chapitre VI.2]. Furthermore, one also finds there explicit examples of Riccati foliations with non trivial Stokes matrices. The most famous of them is undoubtedly Euler's equation that can be interpreted as a Riccati foliation over  $\mathbb{P}^1$  defined by the rational 1-form

$$dz - \frac{(z-x)}{x^2} dx.$$

The fiber over  $\{x = 0\}$  is irregular unramified with non-trivial Stokes, since the weak separatrix through  $(0, 0)$  is divergent.

**5.3. Local factorization at a generic point of the polar divisor.** After reviewing the situation in dimension one we now move back to the two dimensional case. Let  $(\pi : P \rightarrow S, \mathcal{H})$  be a Riccati foliation over a surface  $S$  and let  $H \subset S$  be an irreducible component of the polar divisor  $(\mathcal{H})_\infty$ . The hyperplane  $\pi^{-1}(H)$  can be  $\mathcal{H}$ -invariant or not. In fact, the following assertions are equivalent (see [20, Lemma 4.1])

- $H$  is not  $\mathcal{H}$ -invariant,
- the polar order of  $d\Omega$  along  $H$  is strictly greater than the polar order of  $\Omega$ .

Moreover, in this case, if  $H$  is smooth (e.g. if  $(\mathcal{H})_\infty$  is simple normal crossing), then we can make a bundle modification over  $H$  such that  $H$  is no more a polar component of  $\mathcal{H}$ .

The singular set of  $\mathcal{H}$  is located over the polar divisor, and in restriction to  $\pi^{-1}(H)$ , it consists in a curve  $\Gamma$  intersecting the generic fiber  $\pi^{-1}(p)$ ,  $p \in \mathcal{H}$  in 1 or 2 points. Vertical irreducible components  $\pi^{-1}(p)$  of  $\Gamma$  occur over singular points of  $(\mathcal{H})_\infty$  and so-called *turning points* for the connection. Let us call them *special points*. The remaining (non vertical) part of  $\Gamma$  can consist in one or two sections, or also of an irreducible double section. When  $H$  is  $\mathcal{H}$ -invariant, then we have (see [20, Lemma 4.1])

**Lemma 5.3.** *Let  $H$  be an irreducible component of the polar divisor of  $(P, \mathcal{H})$ . If  $\pi^{-1}(H)$  is  $\mathcal{H}$ -invariant then the foliation  $\mathcal{H}$  locally factors through a curve along  $H$  minus its set of special points.*

The proof is given in Section 8.1. By local factorization we mean the following. At any non special point  $p \in H$ , and for sufficiently small disk  $\Delta \subset S$  tranverse to  $H$  at  $p$ , there exists a neighborhood  $U \subset S$  of  $p$  and a a submersion  $f : U \rightarrow \Delta$  with  $f^{-1}(p) = H \cap U$  such that the Riccati  $(P, \mathcal{H})|_U$  over the neighborhood  $U$  is biholomorphically equivalent to the pull-back of its restriction to  $\Delta$  through a fibre bundle isomorphism. In other words, the Riccati foliation is locally a product of a Riccati foliation over a disk by a disk (or a polydisk in higher dimension). In this situation, the isomorphism class of the Riccati foliation  $(P, \mathcal{H})|_\Delta$  is called *the transverse type* of  $(P, \mathcal{H})$  along the component  $H$ .

**Remark 5.4.** For a linear  $\mathfrak{sl}_2$ -connection  $(E, \nabla)$ , the corresponding Riccati foliation obtained by projectivization satisfies the assumption of Lemma 5.3 at a non special point  $p \in H$  of the polar divisor if, and only if, the matrix connection  $A$  of  $\nabla$  in any local trivialization of  $E$  satisfies  $(dA)_\infty \leq (A)_\infty$ . In the case of simple poles, this just means that the pole is actually logarithmic.

**Corollary 5.5** ([20]). *Let  $(\pi : P \rightarrow S, \mathcal{H})$  be a Riccati foliation over a projective surface  $S$ . Up to birational bundle modification, we can assume that all irreducible components  $H$  of  $(\mathcal{H})_\infty$  satisfy assumption of Lemma 5.3 and the Riccati foliation locally factors through a curve at the neighborhood of any non special point  $p \in S$ .*

We now explain how special points can be simplified by blowing-up.

**5.4. Reduction of singularities for Riccati foliations over surfaces.** The following result will be proved in Appendix A.

**Theorem 5.6.** *Let  $(\pi : P \rightarrow S, \mathcal{H})$  be a Riccati foliation over a projective surface  $S$ , with  $P$  a birationally trivial bundle. There exists a generically finite morphism  $\phi : \tilde{S} \rightarrow S$  (with  $\tilde{S}$  smooth) and a birational bundle modification of the pull-back bundle  $\phi^*P$  such that the pull-back Riccati foliation has normal crossing divisor  $D$ , and is locally defined over any point  $p \in |D|$  by a Riccati 1-form having one of the following types:*

- (1) **Log<sup>G<sub>m</sub></sup>**:  $\frac{\Omega}{z} = \frac{dz}{z} + \lambda_x \frac{dx}{x}$ , or  $\frac{\Omega}{z} = \frac{dz}{z} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$ ;
- (2) **Log<sup>G<sub>a</sub></sup>**:  $\Omega = dz + \frac{dx}{x}$ , or  $\Omega = dz + \frac{dx}{x} + \lambda \frac{dy}{y}$ ;
- (3) **Irreg<sup>0</sup>**:  $\frac{\Omega}{z} = \frac{dz}{z} + \frac{df}{f^{k+1}} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$ , where  $f = x$  or  $xy$ ;
- (4) **Irreg**:  $\Omega = dz + \left( \frac{df}{f^{k+1}} + \lambda \frac{df}{f} \right) z + (b(f)z^2 + c(f)) df$ , where  $f = x$  or  $xy$ .

In all cases, we have  $\lambda \in \mathbb{C}^*$ ,  $\lambda_x, \lambda_y \in \mathbb{C} \setminus \mathbb{Z}$ ,  $b, c$  holomorphic and  $k \in \mathbb{Z}_{\geq 1}$ .

The first part of the proof has to be compared with Sabbah's result in [31]. He defines, for linear meromorphic connections, a notion of good formal model at a point of a normal crossing divisor; this allows him to define Stokes matrices. Bad formal models occur in codimension 2; they correspond to special points of Section 5.3. When the base manifold  $X$  is a surface (and up to rank 5 connection), Sabbah proved that, maybe blowing-up  $X$ , we can assume that all points of the polar divisor have good formal model; this was generalized by André in [1] for any rank.

In Appendix ??, for the surface and rank 2 case, we provide a proof using an auxiliary (transversely projective) foliation, Seidenberg's resolution of singular points and the classification of reduced singular points of transversely projective foliations by Berthier and the third author in [4, 35]. We use ramified coverings to get rid of ramified irregular singular points.

**5.5. Closed 1-form and Stokes matrices.** Let  $(P, \mathcal{H})$  be a reduced Riccati foliation on a complex surface  $S$ . Let  $p \in S$  be a point of the support of the irregular divisor  $I$ . The local model for  $\mathcal{H}$  is either (3), or (4) in Theorem 5.6. In the first case,  $\mathcal{H}$  is defined by the closed 1-form  $\frac{\Omega}{z}$  of the model. In the second case, the coefficients  $b(f), c(f)$  can be killed by a formal bundle transformation  $\hat{z} = \hat{\phi}(z)$ , i.e. with  $\hat{\phi} \in \text{PGL}_2(\mathbb{C}[[f]])$ . By this way, we arrive at the normal form

$$\hat{\Omega} = \frac{d\hat{z}}{\hat{z}} + \left( \frac{\lambda}{f} + \frac{1}{f^{k+1}} \right) df$$

which is closed. From this formal model, one easily check that  $\hat{z} = 0$  and  $\hat{z} = \infty$  are the only formal sections on the bundle that are  $\mathcal{H}$ -invariant (i.e. in restriction to which  $\hat{\Omega}$  identically vanish). Therefore,  $\Omega$  has exactly two  $\mathcal{H}$ -invariant formal sections of  $P$  at  $p$ . The obstruction to the convergence of these two formal sections is given by non diagonal coefficients of Stokes matrices.

**Lemma 5.7.** *At a sufficiently small neighborhood  $V_p$  of  $p$ , the following assertions are equivalent:*

- $\mathcal{H}|_{V_p}$  is defined by a meromorphic closed 1-form on the restriction  $P|_{V_p}$ ,
- there are two analytic sections  $V_p \rightarrow P$  of the bundle that are  $\mathcal{H}$ -invariant,
- Stokes matrices are all trivial at  $p$ .

*The two analytic  $\mathcal{H}$ -invariant sections induce the formal sections at  $p$ .*

*Proof.* If  $\mathcal{H}|_{V_p}$  is defined by a meromorphic closed 1-form  $\Omega$ , then it follows from Proposition 4.4 that  $\Omega = c \cdot \left( \frac{dz}{z} + \omega \right)$  after convenient bundle trivialization, with  $\omega$  a meromorphic closed 1-form on  $X$  and  $c \in \mathbb{C}^*$ . Indeed,  $\mathcal{H}$  cannot have a meromorphic first integral over  $p$  since it is irregular, and it cannot be defined by  $dz + \omega$  since it has two  $\mathcal{H}$ -invariant formal sections. Therefore,  $z = 0$  and  $z = \infty$  are two  $\mathcal{H}$ -invariant sections which must coincide with the two formal ones at  $p$ . Conversely, if  $\mathcal{H}$  has two analytic sections, then it is defined by a closed 1-form (see Lemma 4.3 and its proof).  $\square$

**Remark 5.8.** At the neighborhood of an irregular singular point  $p$ , a Riccati foliation  $\mathcal{H}$  has no other local (formal) multi-section than the two formal sections discussed above. It has no meromorphic (formal) first integral and the unique formal 1-form defining  $\mathcal{H}$  is, up to a scalar constant, the 1-form  $\hat{\Omega}$  above.

## 6. IRREGULAR DIVISOR

Throughout this section,  $(P, \mathcal{H})$  is a **reduced** Riccati foliation over a projective surface  $S$ , i.e.  $\mathcal{H}$  is as in the conclusion of Theorem 5.6. We study the irregular part of the polar divisor of  $\mathcal{H}$ . Precisely, since  $\mathcal{H}$  is reduced, we have  $(\mathcal{H})_\infty = \sum_i (1 + k_i) D_i$  where  $k_i$  is the irregularity index of  $\mathcal{H}$  along  $D_i$ ; in particular,  $k_i = 0$  precisely if  $\mathcal{H}$  is logarithmic at the generic point of  $D_i$ . Then we can decompose

$$(\mathcal{H})_\infty = (\mathcal{H})_{\infty, red} + I$$

where  $(\mathcal{H})_{\infty, red}$  is the reduced divisor and  $I := \sum_i k_i D_i$  denotes the **irregular divisor**.

We note that the irregularity index  $k_i$  must be constant along each connected component of  $I$ , due to the local model of  $\mathcal{H}$  at intersection points  $p \in D_i \cap D_j$  (see (4) of Theorem 5.6).

**6.1. Self-intersection.** We start by noticing that the irregular part of the polar divisor of a reduced Riccati foliation has zero self-intersection.

**Proposition 6.1.** *Let  $(P, \mathcal{H})$  be reduced Riccati foliation on a projective surface  $S$ . Let  $D$  be a connected component of the irregular divisor  $I$  of  $\mathcal{H}$ ; then  $D^2 = 0$ .*

*Proof.* Although this is mainly a consequence of Proposition 6.2, we give a direct and short proof using Camacho-Sad formula (see [6, Chap.2, sec.2]). Consider an auxiliary foliation  $\mathcal{F}$  on  $S$  defined as the pull-back of  $\mathcal{H}$  by a general section  $\sigma : S \dashrightarrow P$ . Since  $D = kD_{red}$ , then it is enough to check that  $D_{red} = \sum_i D_i$  has zero self-intersection. If  $p \in \text{sing}(\mathcal{F}) \cap |D|$ , then we can check from the list of models of Theorem 5.6 that we have two possibilities:

- $p$  belongs to the smooth part of  $|D|$  and is a saddle-node for  $\mathcal{F}$ , with strong separatrix contained in  $|D|$ ; consequently, the Camacho-Sad index of  $\mathcal{F}$  along  $D$  is zero.
- $p$  is an intersection point of two components  $D_i$  and  $D_j$  and is a saddle for  $\mathcal{F}$ ; moreover, Camacho-Sad index is  $-1$  for each separatrix.

Camacho-Sad formula yields

$$D_i \cdot D_i = - \sum_{j \neq i} D_i \cdot D_j$$

and we get

$$D \cdot D = \sum_i D_i \cdot D_i + \sum_{i \neq j} D_i \cdot D_j = 0$$

as wanted. □

**6.2. Flat coordinates.** It turns out that the irregular divisor has not only zero self-intersection, but it also has torsion normal bundle. Indeed something even stronger holds true as proved in the proposition below. Before to state it, we note that the singular set  $\text{sing}(\mathcal{H})$  over the irregular divisor  $I$  consists of finitely many  $\mathbb{P}^1$ -fibers, located over intersection points of the polar divisor, and a smooth curve  $Z \subset \pi^{-1}(I)$  on which  $\pi$  induces a 2-fold étale covering  $\pi|_Z : Z \rightarrow I$ . If we restrict this picture to a connected component  $D$  of  $I$ , the curve  $Z|_D$  may **split** as a union of two sections, say  $Z_0$  and  $Z_\infty$ , or be irreducible (i.e. unsplit).

**Proposition 6.2.** *Let  $(P, \mathcal{H})$  be a reduced Riccati foliation on a projective surface  $S$ . Let  $D = k \cdot D_{red}$  be a connected component of the irregular divisor  $I$ . Then, the normal bundle of  $D_{red}$  is torsion of order  $r$  dividing  $2k$ . Assume now that the non vertical part  $Z \subset \text{sing}(\mathcal{H})$  of the singular set splits over  $D$ . In this case, torsion  $r$  actually divides  $k$ , and moreover*

$$\mathcal{O}_S(rD_{red})|_{kD_{red}} \simeq \mathcal{O}_{kD_{red}}.$$

*Proof.* Take an open covering of a neighborhood of  $D$  by sufficiently small open sets  $V_i$  on which  $\mathcal{H}$  is defined by models of Theorem 5.6, for convenient analytic coordinates  $x_i, y_i$ ; the restriction  $|D| \cap V_i$  is defined by  $\{f_i = 0\}$  with  $f_i = x_i$  or  $f_i = x_i y_i$ . We can assume that intersections  $V_i \cap V_j$  do not contain intersection points of the polar divisor  $(\mathcal{H})_\infty$ . We want to prove that, on  $V_i \cap V_j$ , we have

$$f_i = a_{ij} f_j + b_{ij} f_j^{k+1}$$

with constant  $a_{ij}^{2k} = 1$  and  $b_{ij}$  holomorphic; moreover, in the split case, we can modify  $f_i$  so that all  $a_{ij}^k = 1$ . This will prove the Proposition.

We are going to work at the formal completion of  $V$  along  $D$ . Over  $V_i \cap V_j$ , we can choose formal coordinates  $z_i$  and  $z_j$  such that the foliation  $\mathcal{H}$  is defined by the closed formal 1-forms

$$\Omega_i = \frac{dz_i}{z_i} + \left( \lambda \frac{df_i}{f_i} + \frac{df_i}{f_i^{k+1}} \right) \quad \text{and} \quad \Omega_j = \frac{dz_j}{z_j} + \left( \lambda \frac{df_j}{f_j} + \frac{df_j}{f_j^{k+1}} \right).$$

For instance, if  $\mathcal{H}$  has model (3) on  $U_i$ , with  $f_i = x_i y_i$ , and if  $U_j$  intersects  $x_i = 0$  and not  $y_i = 0$ , then we have

$$\frac{dz}{z} + \frac{df_i}{f_i^{k+1}} + \lambda_x \frac{dx_i}{x_i} + \lambda_y \frac{dy_i}{y_i} = \left( \frac{dz}{z} + (\lambda_y - \lambda_x) \frac{dy_i}{y_i} \right) + \frac{df_i}{f_i^{k+1}} + \lambda_x \frac{df_i}{f_i}$$

and we get the normal form  $\Omega_i$  by setting  $z_i = z \cdot \exp((\lambda_y - \lambda_x) \log(y_i))$  for a convenient determination of the logarithm. We are obviously assuming that the  $V_i$ 's have simply connected intersections, we omit details.

We claim that  $\Omega_i = \pm \Omega_j$  over  $V_i \cap V_j$ . Indeed, since  $\Omega_i$  and  $\Omega_j$  define the same foliation, they must be proportional: we have  $\Omega_i = g_{ij} \Omega_j$  for some (formal) function  $g_{ij}$ . From closedness condition for these 1-forms, we deduce the  $g_{ij}$  must be a first integral for  $\mathcal{H}$ ; therefore,  $g_{ij}$  is constant. Actually  $g_{ij} = \pm 1$  since residues of non vertical poles  $z_i, z_j = 0, \infty$  are already fixed to  $\pm 1$ .

If  $\lambda \neq 0$ , then  $\Omega_i = \Omega_j$  because of the sign of the residue over  $D$ . If  $\lambda = 0$ , we note that  $(x_i, z_i) \mapsto (ax_i, 1/z_i)$  is an automorphism of  $\mathcal{H}$  when  $a^k = -1$ , that permutes the two non vertical poles; it conjugates  $\Omega_i$  to  $-\Omega_i$  and change the sign of the residues along non vertical poles. In the split case, the non vertical polar locus splits into two disjoint components so that we can fix *a priori* the sign of the residue.

From now on, we assume that we are in the split case, i.e. that  $\Omega_i = \Omega_j$  on intersections  $V_i \cap V_j$ . Then, we must have  $z_i = g_{ij}(x_j, y_j) z_j$  for some non vanishing function  $g_{ij}$  on  $V_i \cap V_j$ . On the other hand, set  $f_i = f_{ij}(x_j, y_j) f_j$  for some non vanishing function  $f_{ij}$ . Then we get

$$0 = \Omega_i - \Omega_j = \frac{dg_{ij}}{g_{ij}} + \lambda \frac{df_{ij}}{f_{ij}} + \frac{1}{f_j^k} \frac{df_{ij}}{f_{ij}^{k+1}} + \left( \frac{1}{f_{ij}^k} - 1 \right) \frac{df_j}{f_j^{k+1}}$$

We deduce that

$$g_{ij} = c_{ij} \cdot \exp \left( - \int \left( \lambda \frac{df_{ij}}{f_{ij}} + \frac{1}{f_j^k} \frac{df_{ij}}{f_{ij}^{k+1}} + \left( \frac{1}{f_{ij}^k} - 1 \right) \frac{df_j}{f_j^{k+1}} \right) \right)$$

for some constant  $c_{ij} \in \mathbb{C}^*$ .

Notice that the integrand is the sum of the regular 1-form,  $\lambda \frac{df_{ij}}{f_{ij}}$ , with

$$\frac{1}{f_j^k} \frac{df_{ij}}{f_{ij}^{k+1}} + \left( \frac{1}{f_{ij}^k} - 1 \right) \frac{df_j}{f_j^{k+1}} = \frac{1}{f_{ij}^{k+1}} \underbrace{\left( \frac{df_{ij}}{f_j^k} + (f_{ij} - f_{ij}^{k+1}) \frac{df_j}{f_j^{k+1}} \right)}_{\Theta}.$$

Since  $g_{ij}$  has no essential singularities we deduce that  $\Theta$ , the 1-form over braces, cannot have poles.

Notice that the residue of  $f_j^k \Theta$  along  $\{f_j = 0\}$  is nothing but  $f_{ij} - f_{ij}^{k+1} \pmod{f_j}$ . Therefore  $f_{ij}^k = 1 \pmod{f_j}$ . It follows that the normal bundle of  $D$  is torsion of order  $r$  for some  $r$  dividing  $k$ .

Now, let us write  $f_{ij} = a_{ij}(1 + b_{ij}f_j^n)$  for some  $n > 0$  and  $b_{ij}$  holomorphic (and  $a_{ij}^k = 1$ ). Then we get

$$\Theta = \frac{(n-k)a_{ij}b_{ij}f_j^n + o(f_j^n)}{f_j^{k+1}}.$$

If  $n < k$ , then  $b_{ij}$  must vanish along  $D$  and we can set  $f_{ij} = a_{ij}(1 + \tilde{b}_{ij}f_j^{n+1})$ . By induction, we arrive at  $n = k$ . This establishes the proposition.  $\square$

**Remark 6.3.** In the system of “transverse coordinates”  $f_i : V_i \rightarrow \mathbb{C}$  constructed along the proof, the Riccati foliation  $\mathcal{H}|_{V_i}$  is also locally defined by

$$\Omega_i = \frac{dz_i}{z_i} + \left( b_i(f_i)z_i + \lambda \frac{1}{f_i} + \frac{1}{f_i^{k+1}} + \frac{c_i(f_i)t}{z_i} \right) df_i$$

for holomorphic bundle coordinates  $z_i$ , where  $b_i, c_i$  are holomorphic functions of  $f_i$ . We can assume the functions  $b_i, c_i$  vanishing at arbitrary high order along  $|D|$ . Recall from Section 5.2 that the obstruction to make them just zero is given by the non triviality of Stokes matrices. On  $V_i \cap V_j$  we get

$$\begin{cases} f_i = \alpha_{ij}f_j + \beta_{ij}f_j^{k+1} \\ z_i = \gamma_{ij}z_j \end{cases}$$

with  $\alpha_{ij}^k = 1$  and  $\beta_{ij} \in \mathcal{O}_{V_i \cap V_j}$ ,  $\gamma_{ij} \in \mathcal{O}_{V_i \cap V_j}^*$  satisfying  $\gamma_{ij} \exp(\beta_{ij})|_{|D|} \equiv 1$ .

**Remark 6.4.** Since transition functions  $\{f_{ij} \in \mathcal{O}_S^*(V_i \cap V_j)\}$  are equal to a  $k^{\text{th}}$ -root of the unity when restricted to  $|D|$ , we have an induced representation  $\psi : \pi_1(|D|) \rightarrow \mathbb{C}^*$  taking values in values in the group of  $k^{\text{th}}$ -roots of the unity, which describe how the differentials  $df_i$  change when we follow closed paths along  $D$ .

**6.3. Smooth fibration.** We have just proved that a connected component  $D$  of the irregular divisor  $I$  have local equations  $f_i^k : V_i \rightarrow \mathbb{C}$  that patch together up to order  $k$ , where  $k$  is the irregularity index. In the smooth  $C^\infty$  setting, we can modify these local equations so that they can patch together to define a fibration on  $V$  having  $D$  as a singular fiber and which is smooth elsewhere.

**Lemma 6.5.** *Let  $(P, \mathcal{H})$  be a reduced Riccati foliation on a projective surface  $S$ . Let  $D$  be a connected component of the irregular divisor  $I$  like in Proposition 6.2. Then there exists a  $C^\infty$  map  $f : V \rightarrow \mathbb{D}$  defined on an analytic neighborhood  $V$  of  $|D|$  such that:*

- $f$  induces a  $C^\infty$  locally trivial fibration over  $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ ,
- $f$  coincides with local equations  $f_i^k : V_i \rightarrow \mathbb{C}$  of Remark 6.3 up to order  $k$ .

*Proof.* The transition functions  $\{f_{ij} \in \mathcal{O}_V^*(V_i \cap V_j)\}$  of the proof of Proposition 6.2 define an element in  $H^1(V, \mathcal{O}_V^*)$  which corresponds to  $\mathcal{O}_V(D)$ . Let  $\mathcal{A}_V$  and  $\mathcal{A}_V^*$  denote, respectively, the sheaves of  $C^\infty$  functions and of  $C^\infty$  invertible functions. Notice that  $H^1(V, \mathcal{A}_V^*)$  is isomorphic to  $H^2(V, \mathbb{Z})$ , as  $\mathcal{A}_V$  has no cohomology in positive degree ( $\mathcal{A}_V$  is a fine sheaf). Notice also that the restriction morphism from  $H^2(V, \mathbb{Z})$  to  $H^2(|D|, \mathbb{Z})$  is an isomorphism since the inclusion of  $|D|$  in  $V$  is a homotopy equivalence.

Since  $f_{ij}^k|_{|D| \cap V_i} = 1$ , the element  $\{f_{ij}^k\} \in H^1(V, \mathcal{O}_V^*)$  maps to the trivial element of  $H^1(V, \mathcal{A}_V^*) \simeq H^2(V, \mathbb{Z}) \simeq H^2(|D|, \mathbb{Z})$ . Thus we can find non-vanishing  $C^\infty$ -functions  $\{g_i \in \mathcal{A}_V^*(V_i)\}$  such that  $f_{ij}^k = \frac{g_i}{g_j}$ . Moreover, we can choose the functions

$g_i$  satisfying  $g_i|_{|D| \cap V_i} = 1$ . Our assumptions imply that we can further assume that the function  $f_{ij}^k$  are constant equal to one when restricted to  $D$ .

Therefore we can define a  $C^\infty$  function  $f : V \rightarrow \mathbb{C}$  by the formulas  $f|_{V_i} = (f_i)^k / g_i$ . The function  $f$  clearly satisfies  $f^{-1}(0) = |D|$  set-theoretically. We claim that, after perhaps shrinking  $V$ , the critical set of  $f$  is contained in  $|D|$ . At a neighborhood of a smooth point of  $|D|$  the function  $f$  is a power of a submersion. At a neighborhood of a singular point of  $H$ , the function  $f$  is of the form  $h(x, \bar{x}, y, \bar{y})xy$  and therefore  $df = xydh + h(ydx + xdy)$ . Since this expression has an isolated singularity at zero (the singular point of  $|D|$ ) it follows that the critical set of  $f$  is indeed contained in  $D$ . Replacing  $V$  by  $f^{-1}(\mathbb{D}_\varepsilon)$  for a sufficiently small  $\varepsilon$  we have just proved the existence of a  $C^\infty$  proper map  $f : V \rightarrow \mathbb{D}_\varepsilon$  from  $V$  to the disk of radius  $\varepsilon$  which maps  $|D|$  to the origin in  $\mathbb{D}_\varepsilon$  and when restricted to  $V - |D|$  becomes a locally trivial fibration over  $\mathbb{D}_\varepsilon^*$ . Moreover, the construction of  $f$  respects the representation  $\psi$ , in the sense that  $R|_F : F \rightarrow |D|$ , the restriction of the deformation retract  $R : V \rightarrow |D|$  to a general fiber  $F$  of  $f$ , has monodromy given by  $\psi$ .  $\square$

**6.4. Closed 1-form and Riccati.** Here follow a semi-local version of Lemma 5.7

**Proposition 6.6.** *Let  $D$  be a connected component of the irregular divisor of  $(P, \mathcal{H})$  and  $V$  be a sufficiently small neighborhood of  $|D|$ . Maybe after passing to a étale two-fold cover  $V' \rightarrow V$ , the following assertions are equivalent:*

- $\mathcal{H}|_V$  is defined by a meromorphic closed 1-form on the restriction  $P|_V$ ,
- there are two analytic sections  $V \rightarrow P$  of the bundle that are  $\mathcal{H}$ -invariant,
- at some point  $p \in |D|$  (in fact any), the local model for  $\mathcal{H}$  is of type **Irreg<sup>0</sup>**, or of type **Irreg** with trivial Stokes (see Theorem 5.6).

*Proof.* Like in the beginning of Section 6.2, let us consider the non vertical component  $Z$  of  $\text{sing}(\mathcal{H})$  over  $D$ : the projection  $\pi$  induces a two-fold étale cover  $Z \xrightarrow{2:1} |D|$ . The monodromy of  $\pi|_Z$  is a representation  $\pi_1(|(\mathcal{H})_\infty|) \rightarrow \mathbb{Z}/2\mathbb{Z}$  and since  $|(\mathcal{H})_\infty|$  and  $S$  have the same homotopy type, it induces a representation  $\pi_1(S) \rightarrow \mathbb{Z}/2\mathbb{Z}$ . After passing to the étale covering determined by this representation we can assume  $Z$  has two distinct irreducible components.

If the local model for  $\mathcal{H}$  has trivial Stokes at  $p$  (which is automatic for type (1)), then it has two invariant analytic sections. By using local trivializations of  $\mathcal{H}$  given in Section 5.3 (see also Remark 6.3), we see that this property propagates all along  $|D|$ . By the way, we get a two-fold section of the bundle; since they are locally attached to the two irreducible components of  $Z$ , we get actually two distinct global  $\mathcal{H}$ -invariant sections. It follows from Lemma 4.3 that  $\mathcal{H}$  is defined by a meromorphic closed 1-form. Finally, it follows from Lemma 5.7 that, if defined by a closed 1-form, then Stokes are trivial at any point  $p \in |D|$ .  $\square$

**Remark 6.7.** When the formal invariant is not trivial,  $\lambda \neq 0$ , then the two-fold cover  $V' \rightarrow V$  is not necessary in Proposition 6.6.

**Corollary 6.8.** *Let  $(P, \mathcal{H})$  be an irregular Riccati foliation on a connected complex surface  $S$ . If there exists a generically finite morphism  $f : \tilde{S} \rightarrow S$  such that  $f^*(P, \mathcal{H})$  is defined by a closed rational 1-form, then there exists another one  $f' : \tilde{S}' \rightarrow S$  of degree at most two with the same property.*

*Proof.* The pull-back  $f^*(P, \mathcal{H})$  must be also irregular. At the neighborhood of a smooth point  $p$  of the irregular divisor, the closed 1-form defining  $f^*(P, \mathcal{H})$  must coincide with the formal model  $\frac{d\hat{z}}{\hat{z}} + \left(\frac{\lambda}{x} + \frac{1}{x^{k+1}}\right) dx$ . In particular, the invariant sections  $\hat{z} = 0, \infty$  extend as an invariant double-section of the bundle. After pushing-forward by  $f$ , we get an invariant multi-section of  $P \rightarrow S$  for  $\mathcal{H}$ , which intersects a generic fiber at least twice. We conclude applying Lemma 4.3: by irregularity,  $\mathcal{H}$  cannot have non constant meromorphic first integral and we are again in the case of a double section.  $\square$

### 6.5. Monodromy around the irregular divisor.

**Proposition 6.9.** *Let  $(P, \mathcal{H})$  be a reduced Riccati foliation on a complex surface  $S$ . Let  $D$  be a connected component of the irregular divisor  $I$  of the form  $D = kD_{red}$ . Then, there is a neighborhood  $V$  of  $|D|$  in  $S$  in restriction to which the monodromy of  $(P, \mathcal{H})$  is virtually abelian. More precisely, maybe after passing to an étale covering  $\tilde{V} \rightarrow V$  of degree two, at least one of the following assertions holds true.*

- (1) *The Riccati foliation  $\mathcal{H}$  is defined by a closed meromorphic 1-form over  $V$ .*
- (2) *The supports of  $(\mathcal{H})_\infty$  and  $D$  coincide on  $V$ , and there exists a  $C^\infty$ -fibration  $f : V - D \rightarrow \mathbb{D}^*$  which factors the monodromy.*

*Proof.* Let  $V$  be the connected component of a union of sufficiently small tubular neighborhood of the irreducible component of  $(\mathcal{H})_\infty$  containing  $D$ , and let  $R : V \rightarrow |D|$  be a deformation retract. Since the statement is local, from now on  $\mathcal{H}$  will be seen as a Riccati foliation defined over  $V$ . Like in the first part of the proof of Proposition 6.6, maybe passing to an étale covering  $\tilde{V} \rightarrow V$  of degree two, we may assume that the singular locus  $\text{sing}(\mathcal{H})$  consists in two disjoint sections  $D \rightarrow P|_D$ , and fibers over singular points of  $D$ .

If  $(\mathcal{H})_\infty$  and  $D$  have distinct support in  $V$ , this means that the model at the intersection point  $p$  is (3) in Theorem 5.6. Following Proposition 6.6, we are in case (1) of the statement.

Let us now analyze the case where  $(\mathcal{H})_\infty$  and  $D$  have the same support. In this case,  $\mathcal{P}$  satisfy all the assumptions of Proposition 6.2. If all the Stokes matrices along  $D$  are trivial, then we can conclude as in the previous case, since we do have two meromorphic sections on a neighborhood of  $|D|$ .

Let us finally assume that Stokes matrices are non trivial at a smooth point  $x_0 \in D$ . Let  $\Sigma$  be a germ of curve transverse to  $D$  at  $x_0$ . Let  $\pi : P \rightarrow V$  be the natural projection. The restriction of  $\mathcal{H}$  to  $\pi^{-1}(\Sigma)$ , is a Riccati foliation over  $\Sigma$  with an invariant fiber  $\{x_0\} \times \mathbb{P}^1$  having two saddle-nodes over it. As explained in Section 5.2, if  $k$  is the order of  $D$  at  $x_0$  then there are  $2k$  closed sectors on  $\Sigma$ , such that over the interior of each of them, the Riccati foliation is analytically conjugated to  $\frac{dz}{z} + \frac{dx}{x^{k+1}} + \lambda \frac{dx}{x}$  and the conjugation extends continuously to the boundary. Over each of these sectors there are exactly two leaves with distinguished topological behavior: the closure of each of these distinguished leaves intersect the central fiber  $\{x_0\} \times \mathbb{P}^1$  at a unique point.

Let  $\mathcal{S}_\Sigma$  be the interior of one of these sectors. The local triviality of  $\mathcal{H}$  along the smooth part of  $D$ , and the local normal form of  $\mathcal{H}$  at the singularities of  $D$ , allow us to construct an open set  $\mathcal{S} \subset V - |D|$  which extends  $\mathcal{S}_\Sigma$  and over which we never lose sight of the two distinguished leaves of the restriction of  $\mathcal{H}$  to  $\pi^{-1}(\mathcal{S}_\Sigma)$ . In particular, the restriction of the monodromy representation of  $\mathcal{P}$  to  $\mathcal{S}$  has its image (up to conjugation) contained in  $\mathbb{C}^*$  as  $\mathcal{H}$  over  $\mathcal{S}$  has two distinguished leaves

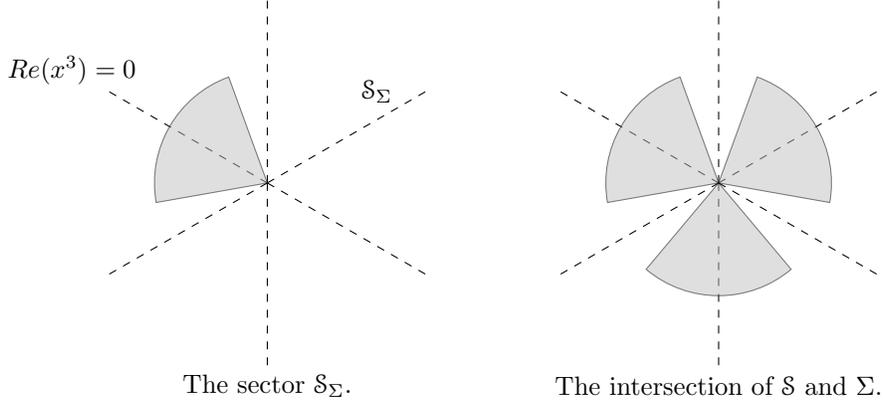


FIGURE 2.

which are not permuted because of the assumption on  $\text{sing}(\mathcal{H})$ . We can choose  $\mathcal{S}_\Sigma$  sufficiently small in such a way that the intersection of the resulting open set  $\mathcal{S}$  and the initial transversal  $\Sigma$  has  $r = \#\psi(\pi_1(|D|))$  distinct connected components, where  $\psi : \pi_1(|D|) \rightarrow \mathbb{C}^*$  is the unitary representation defined in Remark 6.4.

By construction the set  $\mathcal{S}$  has the same homotopy type as a general fiber of  $f_{|V-|D|} : V - |D| \rightarrow \mathbb{D}^*$ . Therefore  $\pi_1(\mathcal{S})$  is a normal subgroup of  $\pi_1(V - |D|)$ . If we do this construction choosing base points at two adjacent sectors with non trivial Stokes transition matrix, we conclude that indeed the monodromy of  $\mathcal{P}$  on  $\mathcal{S}$  is trivial since a nontrivial Stokes matrix at one hand conjugates the corresponding monodromy representations, and at the other hand it does not respect the fixed points of the two monodromy representations. We conclude that in the presence of a nontrivial Stokes matrix the monodromy factors through  $f_* : \pi_1(V - |D|) \rightarrow \pi_1(\mathbb{D}^*)$  as wanted.  $\square$

**Lemma 6.10.** *Let  $(P, \mathcal{H}), D, k$ , and  $r$  be as in the statement of Proposition 6.9. Assume that  $\mathcal{H}$  is not given by a closed meromorphic 1-form at the neighborhood  $V$  of  $D$ . If  $r = k$ , i.e. if  $\mathcal{O}_S(jD_{red})|_{kD_{red}}$  is not isomorphic to  $\mathcal{O}_{kD_{red}}$  for every  $j$  satisfying  $0 < j < k$  and  $\mathcal{O}_S(kD_{red})|_{kD_{red}} \simeq \mathcal{O}_{kD_{red}}$  then the monodromy of  $(P, \mathcal{H})$  is non-trivial.*

*Proof.* Notation as in the proof of Proposition 6.9. Let  $\gamma \in |D|$  a loop such that  $\psi(\gamma) = \exp(2\pi i/k)$  for the unitary representation  $\psi : \pi_1(|D|) \rightarrow \mathbb{C}^*$  defined in Remark 6.4. We can lift  $\gamma$  to an open path  $\tilde{\gamma}$  in a region  $\mathcal{S}$  such that the initial and final points lie in two consecutive small sectors. Moreover, since  $\mathcal{H}$  is not given by a closed meromorphic 1-form we can assume that at least one of the two distinguished leaves over the initial sector is not a distinguished leaf for the final sector. Closing the path  $\tilde{\gamma}$  using an arc on the transversal  $\Sigma$ , we obtain a path in  $V - |D|$  with non-trivial monodromy.  $\square$

## 7. STRUCTURE

**7.1. Proof of Theorem C.** Let  $(P, \mathcal{H})$  be a Riccati foliation over the projective manifold  $X$  with irregular singular points. Assuming that it is not defined by a

closed 1-form after a finite covering, our aim is to prove that it factors through a curve.

According to Proposition 4.8, it suffices to prove the factorization of  $(P, \mathcal{H})$  in restriction to a general surface  $S \subset X$ . Furthermore, Proposition 4.5 allows us to assume that the singularities of  $(P, \mathcal{H})$  are as in the conclusion of Theorem 5.6.

Let  $D$  be a connected component of the irregular divisor  $I$ : we have  $D = kD_{red}$ . It follows from Proposition 6.1 that  $D \cdot D = 0$ , and from Proposition 6.9 that the monodromy of  $(P, \mathcal{H})$  is virtually abelian at the neighborhood  $V$  of  $D$ .

*First case: the global monodromy of  $(P, \mathcal{H})$  on  $S$  is not virtually abelian.* In particular, it is strictly larger than the local monodromy around  $D$ :

$$\rho(\pi_1(S - |(\mathcal{H})_\infty|)) \neq \rho(\pi_1(V - D)).$$

Then arguments used in the proof of Theorem 3.1 show that  $D$  is the fiber of a fibration  $f : S \rightarrow C$  (we use extra topology in  $S - V$  to construct a ramified cover with several disjoint copies of  $D$  and then apply Theorem 2.2). Like in the proof of Theorem 3.1, the monodromy of a general fiber of  $f$  is a normal subgroup of the global monodromy group and is therefore trivial. Thus, the monodromy representation factors through  $f$ . But the general fiber of  $f$  does not intersect the irregular divisor and we can apply Proposition 4.6 to conclude that the Riccati foliation factors as well.

We can now assume that the global monodromy is virtually abelian, and after passing to a finite covering, that it is torsion free. Thus the global monodromy either is cyclic abelian and infinite, or trivial.

*Second case: the global monodromy of  $(P, \mathcal{H})$  on  $S$  is cyclic abelian and infinite.*

In conclusion (2) of Proposition 6.9, the monodromy factors, around  $D$ , through the  $C^\infty$ -fibration  $f : V - D \rightarrow \mathbb{D}^*$ : the monodromy is infinite cyclic. If the representation takes values in  $\mathbb{G}_a$  we compose a suitable multiple of it with an exponential so that we can assume that it takes values in  $\mathbb{G}_m$  and is still infinite. Consider the Deligne logarithmic flat connection  $(L, \nabla)$  realizing this representation. By construction, the residue of  $\nabla$  along an irreducible component  $E_i$  of the polar divisor takes the form  $k_i\lambda + n_i$ , where  $\lambda$  is an irrational number (the monodromy is infinite) and  $k_i, n_i$  are integers. The residue formula gives

$$0 = \sum_i \text{Res}_{E_i} c_1(E_i) + c_1(L) = \left( \sum_i k_i c_1(E_i) \right) \lambda + \left( \sum_i n_i c_1(E_i) + c_1(L) \right).$$

Since  $\lambda$  is irrational, each term in parenthesis is zero and we get in particular

$$0 = \sum_i k_i c_1(E_i) = c_1(D) - c_1(E)$$

where  $D$  is our connected component of the irregular divisor, and  $E$  is a divisor disjoint from  $D$  and contained in  $|(\mathcal{H})_\infty|$ . After splitting  $E = E_+ - E_-$  with  $E_+$  and  $E_-$  effective, we know that  $E_+$  and  $E_-$  have non positive self-intersection by Hodge index Theorem. On the other hand, the equality

$$0 = D \cdot D = E \cdot E = E_+ \cdot E_+ - 2E_+ \cdot E_- + E_- \cdot E_-$$

forces each term, *a priori*  $\leq 0$ , to be zero. Finally, again by Hodge index Theorem,  $E_+$  and  $E_-$  must be disjoint and have Chern classes proportional to the Chern class of  $D$ . We can apply Theorem 2.3 to produce a fibration.

We have just proved, in the case monodromy is abelian and infinite, that  $D$  is fiber of a holomorphic fibration  $f : S \rightarrow C$ . It remains to show that the Riccati foliation  $\mathcal{H}$  factors. When the monodromy is trivial along a generic fiber of  $f$ , this clearly follows from Proposition 4.6. If not, we have infinite monodromy along fibers and, by looking at a fiber close to  $D$ , we must be in case (1) of Proposition 6.9. In particular, the monodromy takes values in  $\mathbb{C}^*$ , i.e. has two fixed points. Over each fiber, we have exactly two sections of the  $\mathbb{P}^1$ -bundle  $P$  that are invariant by the Riccati foliation  $\mathcal{H}$ . The sections are tangent to the Riccati foliation  $\mathcal{H}$  and also to the pull-back to  $P$  of the foliation defined by the fibration. We obtain two curves in the Hilbert scheme of  $P$ . Since tangency to foliations define closed subscheme of the Hilbert scheme, these two curves have Zariski closure of dimension one (recall that over each fiber of  $f$  we have exactly two sections of  $P$ ). Thus they spread two surfaces, sections of  $P$ , which are invariant by  $\mathcal{H}$ . After birational bundle transformation, we can assume  $P = S \times \mathbb{P}_z^1$  (the trivial bundle) with the two sections  $\{z = 0\}$  and  $\{z = \infty\}$ . Then the Riccati 1-form  $\Omega = dz + \alpha z^2 + \beta z + \gamma$  defining  $\mathcal{H}$  satisfies  $\alpha = \gamma = 0$  and integrability condition shows that  $\frac{\Omega}{z}$  is closed, contradiction.

*Third case: the monodromy of the Riccati foliation  $\mathcal{H}$  is trivial.* Like in the previous case, if  $\mathcal{H}$  is defined by a closed 1-form at the neighborhood of  $D$ , then it extends as a global 1-form except if there is another irregular polar component, in which case we get a fibration by Theorem 2.2 and  $\mathcal{H}$  factors by Proposition 4.6. On the other hand, if we are in case (2) of Proposition 6.9, then Lemma 6.10 implies that  $r < k$ , i.e. the order of the normal bundle of  $D_{red}$  is strictly smaller than the multiplicity of the irregular divisor. The existence of a fibration with a fiber supported on  $|D|$  follows from Theorem 2.3, and the Riccati foliation factors according to Proposition 4.6.  $\square$

**7.2. Proof of Theorem E.** Let  $(E, \nabla)$  be a flat meromorphic  $\mathfrak{sl}_2$ -connection on  $X$ . In the case  $(E, \nabla)$  is regular, then the conclusion of Theorem E directly follows from Corollary B. Let us assume  $(E, \nabla)$  irregular.

Let us consider  $P := \mathbb{P}E$  the  $\mathbb{P}^1$ -bundle associated to  $E$ ; horizontal section of  $\nabla$  induce a Riccati foliation  $\mathcal{H}$  on  $\pi : P \rightarrow X$  which is irregular by assumption. We can apply Theorem C to the projective connection/Riccati foliation  $(P, \mathcal{H})$  and discuss the two possible conclusions.

Assume first that  $\mathcal{H}$  is defined by a closed 1-form (maybe passing to a finite covering of  $X$ ). After birational bundle transformation, we can assume  $P_0 = X \times \mathbb{P}^1$  and  $\mathcal{H}_0$  defined by

$$\Omega_0 = \frac{dz}{z} + 2\omega \quad \text{or} \quad \Omega_0 = dz + \omega$$

with  $\omega$  a closed 1-form on  $X$  (see Proposition 4.4). These Riccati foliations are induced by those explicit connections of Corollary E (1):

$$\nabla_0 = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \quad \text{or} \quad \nabla_0 = d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$$

on the trivial bundle  $E_0 := \mathcal{O}_X \oplus \mathcal{O}_X$ . There is a birational bundle trivialization  $E \dashrightarrow E_0$  making commutative the diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E_0 \\ \downarrow & & \downarrow \\ P & \xrightarrow{\quad} & P_0 \end{array}$$

Obviously,  $\phi_*\nabla$  is projectively equivalent to one of the models  $\nabla_0$ :  $\phi_*\nabla = \nabla_0 \otimes \zeta$  with  $(\mathcal{O}_X, \zeta)$  a flat rank one connection over  $X$ , birationally equivalent to the trivial connection by construction. This means that one can write  $\zeta = d + \frac{df}{f}$  and, maybe tensoring by the logarithmic connection  $(\mathcal{O}_X, d + \frac{1}{2}\frac{df}{f})$  (whose square has trivial monodromy), we get equality  $\phi_*\nabla = \nabla_0$ . Note that, passing to the 2-fold covering defined by  $z^2 = f$ , the connection  $\nabla$  is birationally gauge equivalent to  $\nabla_0$  (without tensoring).

Assume now that  $(P, \mathcal{H})$  is birationally gauge equivalent to the pull-back  $f^*(P_0, \mathcal{H}_0)$  of a Riccati foliation over a curve,  $f : X \dashrightarrow C$  with  $P_0 = C \times \mathbb{P}^1$ . Denote by  $\nabla_0$  the unique  $\mathfrak{sl}_2$ -connection on the trivial bundle  $E_0$  over  $C$  inducing the projective connection  $(P_0, \mathcal{H}_0)$ . Then  $(E, \nabla)$  is birationally equivalent to  $f^*(E_0, \nabla_0) \otimes (\mathcal{O}_X, \zeta)$  with  $\zeta$  logarithmic rank one connection having monodromy in the center of  $\mathrm{SL}_2(\mathbb{C})$ .  $\square$

## 8. TRANSVERSELY PROJECTIVE FOLIATIONS

**8.1. Basic Lemma.** Let  $\mathcal{F}$  be a transversely projective foliation on a projective manifold  $X$  defined by a triple  $(\omega_0, \omega_1, \omega_2)$ . The Riccati foliation  $\mathcal{H}$  defined on  $X \times \mathbb{P}^1$  by

$$dz + \omega_0 + z\omega_1 + z^2\omega_2 = 0$$

is integrable (2)

$$\begin{cases} d\omega_0 = \omega_0 \wedge \omega_1 \\ d\omega_1 = 2\omega_0 \wedge \omega_2 \\ d\omega_2 = \omega_1 \wedge \omega_2 \end{cases}$$

and the foliation  $\mathcal{F}$  is defined by restricting  $\mathcal{H}$  to the section  $\sigma : X \rightarrow P$  given by  $z = 0$ . Another projective triple  $(\omega'_0, \omega'_1, \omega'_2)$  defines the same transversely projective foliation, i.e. the same foliation  $\mathcal{F}$  with the same collection of local first integrals at a general point of  $X$  if, and only if, there are rational functions  $a, b$  on  $X$  such that  $a \neq 0$  and

$$(5) \quad \begin{cases} \omega'_0 = a\omega_0 \\ \omega'_1 = \omega_1 - \frac{da}{a} + 2b\omega_0 \\ \omega'_2 = \frac{1}{a}(\omega_2 + b\omega_1 + b^2\omega_0 - db) \end{cases}$$

(see [32]). This exactly means that the Riccati foliation  $\mathcal{H}'$  defined by  $\Omega' = d\tilde{z} + \omega'_0 + \tilde{z}\omega'_1 + \tilde{z}^2\omega'_2$  is derived by gauge transformation  $\frac{1}{\tilde{z}} = a\frac{1}{z} + b$ . The following is well known (cf [32, Chap. II, Prop. 2.1, p.193] and [10, Lemma 2.20]).

**Lemma 8.1.** *If  $\mathcal{F}$  is defined by a closed 1-form  $\omega$ ,  $d\omega = 0$ , then up to gauge transformation (5), the Riccati foliation  $\mathcal{H}$  is defined by*

$$\Omega = dz + \omega(1 + \phi z^2) \quad \text{where } \phi \text{ is a first integral for } \mathcal{F}.$$

*Proof.* One can first write  $\omega = a\omega_0$  for some rational function  $a$  and use gauge transformation (5) to set  $(\omega_0, \omega_1, \omega_2) = (\omega, 0, \omega'_2)$ . Indeed, once  $\omega_0 = \omega$  is closed, we deduce from (2) that  $\omega_0 \wedge \omega_1 = 0$ , and therefore  $\omega_1 = b\omega_0$  for some rational function  $b$ ; consequently,  $\omega_1$  can be killed by gauge transformation. Finally, integrability condition gives  $d\omega'_2 = \omega \wedge \omega'_2 = 0$ , which means that  $\omega'_2 = f\omega$  and  $d(f\omega) = 0$  for some rational function  $f$ . The latter condition, together with closedness of  $\omega$ , gives  $df \wedge \omega = 0$ , i.e.  $f$  is a first integral for  $\mathcal{F}$ .  $\square$

**Corollary 8.2.** *If  $\mathcal{F}$  is defined by a closed 1-form, then  $\mathcal{H}$  is also defined by a closed 1-form, or factors through a curve. If  $\mathcal{F}$  admits a rational first integral, then  $\mathcal{H}$  factors through a curve.*

*Proof.* If  $\mathcal{F}$  admits a rational first integral  $f$ , then after resolution of indeterminacy points of  $f$  by blowing-ups  $\tilde{X} \rightarrow X$ , Stein Factorization gives a fibration  $\tilde{f} : \tilde{X} \rightarrow C$  over a curve with connected fibers coinciding generically with leaves of  $\tilde{\mathcal{F}}$ . Applying Lemma 8.1 to  $\omega = df$ , we get that  $\phi = \phi(f)$  and  $\mathcal{H}$  actually factors through  $\tilde{f}$ . If  $\mathcal{F}$  is now defined by a closed 1-form  $\omega$ , but does not admit a rational first integral, then applying Lemma 8.1, we get that  $\phi = c \in \mathbb{C}$  is a constant, and  $\mathcal{H}$  is defined by  $\frac{dz}{1+cz^2} + \omega$  which is closed.  $\square$

**Remark 8.3.** Statements similar to Lemma 6.10 and Corollary 8.2 hold with the very same proofs in the local setting, replacing rational functions and 1-forms by their meromorphic analogues.

*Proof of Lemma 5.3.* At a generic point  $p \in \pi^{-1}(H)$ , the foliation  $\mathcal{H}$  is smooth and vertical: let  $p_0 = \pi(p)$  be the projection and  $\sigma : (X, p_0) \rightarrow P$  be a germ of section transverse to  $\mathcal{H}$ . The induced transversely projective foliation is regular and therefore admits a holomorphic first integral. By the local version of Corollary 8.2,  $\mathcal{H}$  locally factors through a curve.  $\square$

**8.2. Proof of Theorem D.** Under previous notations, we now apply to  $(P, \mathcal{H})$  our results on projective connections (Corollary B and Theorem C, or equivalently Theorem E). There are three cases.

**First case.** There is a generically finite morphism  $f : Y \rightarrow X$  such that  $f^*\mathcal{H}$  is defined by a closed 1-form  $\Omega$ . The pull-back  $f^*\mathcal{F}$  on  $Y$  is still defined by restricting the Riccati foliation to the pull-back section  $\tilde{\sigma} : Y \rightarrow f^*P$ ; it is therefore also defined by restricting the closed 1-form  $\Omega$  to the section.

**Second case.** There is a map  $f : X \rightarrow C$  to a curve and a Riccati  $\mathcal{H}_0$  on  $P_0 = C \times \mathbb{P}^1$  such that  $(P, \mathcal{H})$  is equivalent to  $f^*(P_0, \mathcal{H}_0)$  by birational bundle transformation. We thus deduce a map  $\Phi : P \dashrightarrow P_0$  such that  $\mathcal{H} = \Phi^*\mathcal{H}_0$ . Consider the composition  $\Phi \circ \sigma : X \rightarrow P_0$ . Then either it is dominant and  $\mathcal{F}$  is the corresponding pull-back of  $\mathcal{H}_0$ , or the image is a curve and fibers force the leaves of  $\mathcal{F}$  to be algebraic.

**Third case.** There exists a rational map  $f : X \dashrightarrow \mathfrak{H}$  to a polydisk Shimura modular orbifold such that  $(P, \mathcal{H})$  is equivalent to the pull-back of one the tautological Riccati foliations  $(P_\rho := \mathfrak{H} \times_\rho \mathbb{P}^1, \mathcal{H}_\rho)$  by birational bundle transformation. Again, we deduce a map  $\Phi : P \dashrightarrow P_\rho$  such that  $\mathcal{H} = \Phi^*\mathcal{H}_\rho$ , and considering  $\Phi \circ \sigma : X \rightarrow P_\rho$ , we can conclude as before.  $\square$

**8.3. Theorem D implies Theorem E.** Although we have followed the other direction, it is interesting to notice that our results on projective (or  $\mathfrak{sl}_2$ ) connections

and transversely projective foliations are actually equivalent. Indeed, given say a Riccati foliation  $(\pi : P \rightarrow S, \mathcal{H})$ , with  $P$  birationally trivial, we can take a general rational section  $\sigma : S \rightarrow P$  and consider the induced transversely projective foliation  $\mathcal{F} := \sigma^*\mathcal{H}$ . Applying Theorem D to  $\mathcal{F}$  gives the following possibilities.

**First case.** The foliation  $\mathcal{F}$  has a rational first integral. By Corollary 8.2, we deduce that  $\mathcal{H}$  factors through a curve.

**Second case.** There is a generically finite morphism  $f : Y \rightarrow X$  such that  $f^*\mathcal{F}$  is defined by a closed 1-form  $\omega$ , but  $\mathcal{F}$  does not admit a rational first integral. By Corollary 8.2, we deduce that  $f^*\mathcal{H}$  is also defined by a closed 1-form.

**Third case.** Maybe after blowing-up  $S$ , there is a morphism  $f : S \rightarrow P_0 := C \times \mathbb{P}^1$  and a Riccati foliation  $\mathcal{H}_0$  on  $P_0$  such that  $\mathcal{F} = f^*\mathcal{H}_0$ . Then, considering now the pull-back  $\phi^*\mathcal{H}_0$  the fiber product:

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P_0 \\ \pi \downarrow & & \downarrow \pi_0 \\ X & \xrightarrow{f} & C \end{array}$$

we get another transversely projective structure for  $\mathcal{F}$ . If  $\mathcal{F}$  is not defined by a closed 1-form up to finite cover, then its projective structure is unique, and  $\mathcal{H}$  is birational to  $\mathcal{H}_0$ , thus pull-back from a curve.

**Fourth case.** There exists a polydisk Shimura modular orbifold  $\mathfrak{H}$  and a rational map  $f : X \dashrightarrow P_\rho := \mathfrak{H} \times_\rho \mathbb{P}^1$  towards one of its tautological Riccati foliations such that  $\mathcal{F} = f^*\mathcal{H}_\rho$ . Like in the previous case, we can prove that  $\mathcal{F}$  is defined by a closed 1-form after a finite cover (and go back to the first two cases), or  $\mathcal{H}$  is birationally equivalent to the pull-back of  $(P_\rho, \mathcal{H}_\rho)$  by  $f$ .  $\square$

## 9. EXAMPLES

We will now present some examples which show that our results are sharp.

**Example 9.1.** Let  $Y$  be a projective manifold and consider a representation  $\rho : \pi_1(Y) \rightarrow (\mathbb{C}, +)$ . It determines a cohomology class  $[\rho] \in H^1(Y, \mathbb{C})$ . If its image under the natural morphism  $H^1(Y, \mathbb{C}) \rightarrow H^1(Y, \mathcal{O}_Y)$  is non zero then it determines a non trivial extension  $0 \rightarrow \mathcal{O}_Y \rightarrow E \rightarrow \mathcal{O}_Y \rightarrow 0$ , endowed with a flat connection with monodromy given by

$$\begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}.$$

The projectivization  $X = \mathbb{P}(E)$  is a  $\mathbb{P}^1$ -bundle over  $Y$  with a Riccati foliation defined by a closed rational 1-form  $\omega$  with polar divisor equal to  $2\Delta$ , where  $\Delta$  is image of the unique section  $Y \rightarrow \mathbb{P}(E)$ . If  $P$  is the trivial  $\mathbb{P}^1$ -bundle  $P$  over  $X$  then we have a family of Riccati foliations  $\mathcal{H}_\lambda$ ,  $\lambda \in \mathbb{C}$ , on it defined by

$$dz + \omega(1 + \lambda z^2).$$

The Riccati foliation  $\mathcal{H}_\lambda$  is irregular for  $\lambda \neq 0$ , does not factor through a curve, and its irregular divisor is not a fiber of a fibration. If we take  $Y$  equal to an elliptic curve, then  $X - |\Delta|$  is nothing but Serre's example of Stein quasi-projective surface which is not affine.

**Example 9.2.** Let  $n \geq 2$  and let  $X$  be the quotient of  $\mathbb{H}^n$  by cocompact torsion-free irreducible lattice  $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})^n$ . The natural projections  $\mathbb{H}^n \rightarrow \mathbb{H}$  define  $n$

codimension one smooth foliations on  $X$  which are transversely projective (indeed transversely hyperbolic). Contrary to what have been stated by the second author and Mendes in [25, Theorem 1], countably many leaves of these foliations may have nontrivial topology (with fundamental groups isomorphic to isotropy groups of the action of  $\Gamma$  on the corresponding one dimensional factor of  $\mathbb{H}^n$ ), but the very general leaf is biholomorphic to  $\mathbb{H}^{n-1}$ . The maximal principle tells us that the general leaf cannot contain positive dimensional subvarieties, and consequently the foliations are not pull-backs from lower dimensional manifolds.

Again the assumptions on  $\Gamma$  can be considerably weakened. All we have to ask is that  $\Gamma$  is an irreducible lattice of  $\mathrm{PSL}_2(\mathbb{R})^n$  for some  $n \geq 2$ . Notice that the rigidity theorem of Margulis (resp. the classification of representations by Corlette and Simpson) implies that all these lattices are commensurable to (resp. conjugated to a subgroup of) arithmetic lattices of the form  $\mathcal{U}(P, \Phi)/\pm \mathrm{Id}$  for some totally imaginary quadratic extension  $L$  of a totally real number field  $F$ , some rank two projective  $\mathcal{O}_L$ -module  $P$  and some skew Hermitian form  $\Phi$ . Besides the  $n$  representations coming from the  $n$  projections  $\pi_1^{orb}(X) \simeq \Gamma \subset \mathrm{PSL}_2(\mathbb{R})^n \rightarrow \mathrm{PSL}_2(\mathbb{R})$ , we also have  $[L : \mathbb{Q}] - 2n$  representations of  $\pi_1^{orb}(X)$  with values in  $\mathrm{PSL}_2(\mathbb{C})$  which do not factor through lower dimensional projective manifolds. The associated  $\mathbb{P}^1$ -bundles are birationally trivial (since the underlying representation is a Galois conjugate of the representations coming from the transversely projective foliations on  $X$  defined by the submersions  $\mathbb{H}^n \rightarrow \mathbb{H}$ ), and by taking a rational section we can produce further examples of transversely projective foliations on  $X$  which do not factor. Although the underlying representations are Galois conjugate to the representations in  $\mathrm{PSL}_2(\mathbb{R})$ , the topology of the Riccati foliations over  $X$  associated to embeddings  $\sigma : L \rightarrow \mathbb{C}$  for which  $\sqrt{-1}\Phi$  is definite is quite different. In the former case the Riccati foliation leaves invariant two open subsets, corresponding to the complement of  $\mathbb{P}^1(\mathbb{R}) \simeq S^1$  in  $\mathbb{P}^1$ , while in the latter case the Riccati foliation is quasi-minimal: all the leaves not contained in  $\pi^{-1}((\mathcal{H})_\infty)$  are dense in the corresponding  $\mathbb{P}^1$ -bundle.

Explicit examples of foliations on  $\mathbb{P}^2$  defined by the submersions  $\mathbb{H}^2 \rightarrow \mathbb{H}$  with  $\Gamma \subset \mathrm{PSL}_2(2, \mathbb{R})^2$  and  $\Gamma$  isomorphic to  $\mathrm{PSL}_2(\mathcal{O}_K)$  (and certain subgroups) have been determined by the second author and Mendes in [25] ( $K = \mathbb{Q}(\sqrt{5})$ ) using the work of Hirzebruch on the description of these surfaces, and by Cousin in [12], see also [13], ( $K = \mathbb{Q}(\sqrt{3})$ ) using an algebraic solution of Painlevé VI equation.

**Example 9.3.** The degree of the generically finite morphism in Assertion (1) of Theorem D cannot be bounded, even if we restrict to transversely projective foliations on a rational surface. Let  $C_d = \{x^d + y^d + z^d = 0\} \subset \mathbb{P}^2$  be the Fermat curve of degree  $d \geq 3$ . On  $S_d = C_d \times C_d$  consider the action of  $\mathbb{Z}/d\mathbb{Z}$  given by  $\varphi(x, y) = (\xi_d x, \xi_d y)$  where  $\xi_d$  is a primitive  $d$ -th root of the unity. Let  $\omega \in H^0(S_d, \Omega_{S_d}^1)$  be a general holomorphic 1-form satisfying  $\varphi^* \omega = \xi_d \omega$ . The induced foliation is invariant by the action of  $\mathbb{Z}/d\mathbb{Z}$ , but the 1-form  $\omega$  is not. The quotient of  $S_d$  by  $\mathbb{Z}/d\mathbb{Z}$  is a rational surface  $R$  and the foliation induced by  $\omega$  on  $R$  is transversely projective (indeed transversely affine). The monodromy group is an extension of the group of  $d$ -th of unities by an infinite subgroup of  $(\mathbb{C}, +)$ . Explicit equations for birational models of these foliations on  $\mathbb{P}^2$  can be found in [29, Example 3.1].

APPENDIX A. REDUCTION OF SINGULARITIES OF RICCATI FOLIATIONS

This appendix is devoted to the proof of Theorem 5.6.

Let  $(\pi : P \rightarrow S, \mathcal{H})$  be a Riccati foliation over a projective surface  $S$ . One can first blow-up  $S$  until we get a simple normal crossing polar divisor  $(\mathcal{H})_\infty$ , and then apply elementary transformations over irreducible components of  $(\mathcal{H})_\infty$  until we minimize order of poles on each component. This is explained in [20]. Then all components of poles satisfy assumptions of Lemma 5.3 and, outside of special  $\pi$ -fibers, the Riccati foliation locally factors into one dimensional models of Proposition 5.1 or Remark 5.2. In particular, we can define the irregularity divisor as  $I = \sum_i k_i D_i$  where  $D_i$  run over irreducible components of  $(\mathcal{H})_\infty$ , and  $k_i \in \frac{1}{2}\mathbb{Z}_{\geq 0}$  is the irregularity index.

To get rid of special points, we use the fact that  $\pi : P \rightarrow S$  is birationally trivial and choose a rational section  $\sigma : S \dashrightarrow P$  which is not invariant by  $\mathcal{H}$ . Consider the transversely projective foliation  $\mathcal{F} = \sigma^*\mathcal{H}$  and apply Seidenberg's Theorem: maybe blowing-up  $S$ , we can now assume that  $\mathcal{F}$  has only reduced singular points. Following the classification of Berthier and the third author [4, 35], reduced singular points of transversely projective foliations fall into one of the following types:

- (1)  $\mathcal{F}$  admits the holomorphic first integral:
  - **First integral:**  $\omega = d(x^p y^q)$  with  $p, q \in \mathbb{Z}_{>0}$ ;
- (2)  $\mathcal{F}$  is defined by a closed 1-form  $\omega$  (but without first integral):
  - **Linear:**  $\omega = \frac{dx}{x} + \lambda \frac{dy}{y}$  with  $\lambda \in \mathbb{C} - \mathbb{Q}$ ;
  - **Saddle-node:**  $\omega = \frac{dy}{y} + \frac{dx}{x^{k+1}} + \lambda \frac{dx}{x}$  with  $\lambda \in \mathbb{C}$ ;
  - **Resonant saddle:**  $\omega = \frac{dy}{y} + \frac{df}{f^{k+1}} + \lambda \frac{df}{f}$  with  $f = x^p y^q$  and  $\lambda \in \mathbb{C}$ ;
- (3)  $\mathcal{F}$  is the pull-back of a singular point of a Riccati foliation by a ramified cover:
  - **Riccati saddle-node:**  $\omega = dy + (b(x)y^2 + (\frac{1}{x^{k+1}} + \frac{\lambda}{x})y + c(x)) dx$ ;
  - **Bernoulli saddle-node:**  $\omega = dz + (b(x)z^2 + (\frac{1}{x^{k+1}} + \frac{\lambda}{x})z) dx$  with  $z = y^\nu$ ;
  - **Resonant saddle:**  $\omega = dz + (b(f)z^2 + (\frac{1}{f^{k+1}} + \frac{\lambda}{f})z + c(f)) df$  with  $f = x^p y^q$  and  $z = y^\nu$ ;

In case (1), the local version of Corollary 8.2 tells us that  $\mathcal{H}$  factors through  $f$ , and we can reduce it to the models given by Proposition 5.1.

In case (2), where  $\mathcal{F}$  is defined by a closed 1-form  $\omega$ , we can similarly reduce  $\mathcal{H}$  to either  $\Omega = dz + \omega$ , or  $\frac{dz}{z} + \lambda \omega$  with  $\lambda \in \mathbb{C}^*$ .

Finally, in case (3), we again see that  $\mathcal{H}$  factors through  $f = x$  or  $f = x^p y^q$ , reducing to the model  $\Omega = dz + (b(f)z^2 + (\frac{1}{f^{k+1}} + \frac{\lambda}{f})z + c(f)) df$  and  $\mathcal{F}$  is defined by  $z = y$  or  $z = y^\nu$ .

We summarize all the possibilities studied above in the next result.

**Theorem A.1.** *Let  $(\pi : P \rightarrow S, \mathcal{H})$  be a Riccati foliation over a projective surface  $S$ , with  $P$  a birationally trivial bundle. There exists a birational morphism  $\phi : \tilde{S} \rightarrow S$  (with  $\tilde{S}$  smooth) and a birational bundle modification of the pull-back bundle  $\phi^*P$  such that the pull-back Riccati foliation has normal crossing divisor  $D$ , and is locally defined over any point  $p \in |D|$  by a Riccati 1-form having one of the following types:*

- (1) **Log<sup>G<sub>m</sub></sup>:**  $\frac{\Omega}{z} = \frac{dz}{z} + \lambda_x \frac{dx}{x}$ , or  $\frac{\Omega}{z} = \frac{dz}{z} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$ ;
- (2) **Log<sup>G<sub>a</sub></sup>:**  $\Omega = dz + \frac{dx}{x}$ , or  $\Omega = dz + \frac{dx}{x} + \lambda \frac{dy}{y}$ ;

- (3) **Irreg**<sup>0</sup>:  $\frac{\Omega}{z} = \frac{dz}{z} + \frac{df}{f^{k+1}} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$ , where  $f = x$  or  $x^p y^q$ ;
- (4) **Irreg**:  $\Omega = dz + \left( \frac{df}{f^{k+1}} + \lambda \frac{df}{f} \right) z + (b(f)z^2 + c(f)) df$ , where  $f = x$  or  $x^p y^q$ ;
- (5) **Irreg**<sup>ram</sup>:  $\Omega = dz + (z^2 + x^{\epsilon_1} y^{\epsilon_2} \phi(f)) \frac{df}{x^p y^q f^k} - \left( \tilde{p} \frac{dx}{x} + \tilde{q} \frac{dy}{y} + k \frac{df}{f} \right)$  with  $f$  and  $\phi$  holomorphic,  $\phi(0) \neq 0$ ,  $\epsilon_1, \epsilon_2 = 0, 1$  and  $(p, q) = (2\tilde{p} - \epsilon_1, 2\tilde{q} - \epsilon_2)$ .

We note that the meromorphic gauge reduction process of Proposition 5.1 which is *a priori* local can be done globally along irreducible components of the polar divisor. Indeed, it can be checked on the proof of Proposition 5.1 that when the polar divisor is not minimal along an irreducible component  $D_i$ , then the non vertical part of the singular set of  $\mathcal{H}$  is a (smooth) section; after an elementary transformation along it (blowing-up the section and then contracting the strict transform of  $\pi^{-1}(D_i)$ ), the polar order decreases. After a finite number of steps, the polar order is minimal, and all local models reduce to those of Proposition 5.1 by local biholomorphic bundle trivialization. For details, see the proof of the main result of [20].

The last step towards Theorem 5.6 consists in passing to use a ramified covering in order to kill ramifications. Going back to the irregular divisor  $I = \sum_i k_i D_i$ , we can choose positive integers  $m_i$  such that  $m_i k_i = k$  for some fixed positive integer  $k$  (a common multiple of all  $k_i$ 's). We now consider a Kawamata covering ([17, Proposition 4.1.12]): there exists a ramified cover  $f : \tilde{S} \rightarrow S$  with  $S$  smooth such that  $f^* D_i = m_i \tilde{D}_i$  for some smooth reduced divisors  $\tilde{D}_i$  on  $\tilde{S}$ , and  $\sum_i \tilde{D}_i$  has simple normal crossings. One easily check from models of Theorem A.1 that they are stable under ramified covers in variables  $x$  or  $y$ , and that ramifying at order  $m_i$  along  $D_i$  multiply the irregularity index by  $m_i$ . Finally, after covering, the irregularity index is  $k$  all along the irregular divisor. In particular, there are no more ramified components, and we have no more to consider the model **Irreg**<sup>ram</sup> of Theorem A.1.  $\square$

## REFERENCES

1. Y. ANDRÉ, *Structure des connexions méromorphes formelles de plusieurs variables et semi-continuité de l'irrégularité*, Invent. Math. 170 (2007), no. 1, 147-198.
2. E. A. BAROLO, J. I. COGOLLUDO-AGUSTIN, AND D. MATEI, *Characteristic varieties of quasiprojective manifolds and orbifolds*, Geometry & Topology **17** (2013), no. 1, 273-309.
3. B. CLAUDON, F. LORAY, J. V. PEREIRA, F. TOUZET, *Compact leaves of codimension one holomorphic foliations on projective manifolds*. Preprint (2015).
4. M. BERTHIER, F. TOUZET, *Sur l'intégration des équations différentielles holomorphes réduites en dimension deux*, Bol. Soc. Brasil. Mat. (N.S.), **30** (1999), no.3, 247-286.
5. M. BRUNELLA, *Minimal models of foliated algebraic surfaces*, Bull. Soc. Math. France **127** (1999), no. 2, 289-305.
6. M. BRUNELLA, *Birational Geometry of Foliations*. Publicações Matemáticas do IMPA. Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, 2004.
7. D. CERVEAU AND P. SAD, *Liouvillian integration and Bernoulli foliations*, Trans. Amer. Math. Soc. **350** (1998), no. 8, 3065-3081.
8. C. CAMACHO AND B. A. SCÁRDUA, *Holomorphic foliations with Liouvillian first integrals*, Ergodic Theory Dynam. Systems **21** (2001), no. 3, 717-756.
9. G. CASALE, *Feuilletages singuliers de codimension un, groupeïde de Galois et intégrales premières*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 3, 735-779.
10. D. CERVEAU, A. LINS NETO, F. LORAY, J. V. PEREIRA, F. TOUZET, *Complex Codimension one singular foliations and Godbillon-Vey Sequences*. Moscow Math. Jour. **7** (2007) p.21-54.
11. K. CORLETTE, C. SIMPSON, *On the classification of rank-two representations of quasiprojective fundamental groups*. Compos. Math. **144** (2008), no. 5, 1271-1331.

12. G. COUSIN, *Un exemple de feuilletage modulaire déduit d'une solution algébrique de l'équation de Painlevé VI*. Ann. Inst. Fourier **64** (2014), 699-737.
13. G. COUSIN, *Connexions plates logarithmiques de rang deux sur le plan projectif complexe*. Phd Thesis IRMAR (2011). <http://tel.archives-ouvertes.fr/tel-00779098>.
14. G. COUSIN, J. V. PEREIRA, *Transversely affine foliations on projective manifolds*. Math. Res. Letters **21** (2014), 985-1014.
15. P. DELIGNE, *Équations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, Vol. 163, Springer-Verlag, Berlin, 1970.
16. C. GODBILLON, *Feuilletages. Études géométriques*. With a preface by G. Reeb. Progress in Mathematics, 98. Birkhäuser Verlag, Basel, 1991.
17. R. LAZARSFELD, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, **48**. Springer-Verlag, Berlin, 2004. xviii+387 pp.
18. A. LINS NETO, *Some examples for the Poincaré and Painlevé problems*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 2, 231-266.
19. F. LORAY, *Pseudo-groupe d'une singularité de feuilletage holomorphe en dimension deux*. HAL : hal-00016434, version 1. <http://hal.archives-ouvertes.fr/ccsd-00016434>.
20. F. LORAY AND J.V. PEREIRA, *Transversely projective foliations on surfaces: existence of normal forms and prescription of the monodromy*. Intern. Jour. Math. **18** (2007) p.723-747.
21. F. LORAY, J. V. PEREIRA, F. TOUZET, *Singular foliations with trivial canonical class*. Preprint. arXiv:1107.1538 (2011).
22. A. I. MAL'CEV, *On the faithful representation of infinite groups by matrices*, Mat. Sb., 8(50) (1940), pp. 405-422.
23. B. MALGRANGE, *On nonlinear differential Galois theory*. Chinese Ann. Math. Ser. B **23** (2002) p. 219-226.
24. J. MARTINET AND J.-P. RAMIS, *Problèmes de modules pour des équations différentielles non linéaires du premier ordre*. Inst. Hautes Études Sci. Publ. Math. **55** (1982), p.63-164.
25. L.G. MENDES AND J. V. PEREIRA, *Hilbert modular foliations on the projective plane*. Comment. Math. Helv. **80** (2005), no. 2, 243-291.
26. D. MUMFORD *The topology of normal singularities of an algebraic surface and a criterion for simplicity*. Inst. Hautes Études Sci. Publ. Math. No. **9** (1961), 5-22.
27. A. NEEMAN, *Ueda theory: theorems and problems*. Memoirs of the AMS **81** (415) (1989).
28. J. V. PEREIRA, *Fibrations, divisors and transcendental leaves. With an appendix by Laurent Meersseman*. J. Algebraic Geom. **15** (2006), no. 1, 87-110.
29. J. V. PEREIRA, P. SAD, *Rigidity of fibrations*. Astérisque No. **323** (2009), 291-299.
30. E. ROUSSEAU, F. TOUZET *Curves in Hilbert modular varieties*. arXiv:1501.03261
31. C. SABBAH, *Équations différentielles à points singuliers irréguliers et phénomènes de Stokes en dimension 2* Astérisque No. 263 (2000), viii+190 pp.
32. B. SCARDUA, *Transversely affine and transversely projective holomorphic foliations*, Ann. Sci. École Norm. Sup. (4) 30 (1997), no. 2, 169-204.
33. B. TOTARO, *The topology of smooth divisors and the arithmetic of abelian varieties* (Dedicated to William Fulton on the occasion of his 60th birthday), Michigan Math. J. 48 (2000), 611-624.
34. B. TOTARO, *Moving codimension-one subvarieties over finite fields*. Amer. J. Math. 131 (2009), no. 6, 1815-1833.
35. F. TOUZET, *Sur les feuilletages holomorphes transversalement projectifs*, Ann. Inst. Fourier (Grenoble), 53 (2003), no.3, 815-846.

IRMAR, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE  
*E-mail address:* [frank.loray@univ-rennes1.fr](mailto:frank.loray@univ-rennes1.fr)

IMPA, ESTRADA DONA CASTORINA, 110, HORTO, RIO DE JANEIRO, BRASIL  
*E-mail address:* [jvp@impa.br](mailto:jvp@impa.br)

IRMAR, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE  
*E-mail address:* [frederic.touzet@univ-rennes1.fr](mailto:frederic.touzet@univ-rennes1.fr)