Representations of quasiprojective groups, Flat connections and Transversely projective foliations

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REPRESENTATIONS OF QUASIPROJECTIVE GROUPS, FLAT CONNECTIONS AND TRANSVERSELY PROJECTIVE FOLIATIONS

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Abstract. The main purpose of this paper is to provide a structure theorem for codimension one singular transversely projective foliations on projective manifolds. To reach our goal, we firstly extend Corlette-Simpson's classification of rank two representations of fundamental groups of quasiprojective manifolds by dropping the hypothesis of quasi-unipotency at infinity. Secondly we establish an analogue classification for rank 2 flat meromorphic connections. In particular, we prove that a rank 2 flat meromorphic connection with irregular singularities having non trivial Stokes projectively factors through a connection over a curve.

Contents

1. Introduction

Let X be a smooth projective manifold over $\mathbb{C}$. A (holomorphic singular) codimension one foliation $\mathcal{F}$ on X is defined by a non zero rational 1-form ω satisfying Frobenius integrability condition $\omega \wedge d\omega = 0$. The foliation is transversely projective if there are rational 1-forms α, β on X such that the $\mathfrak{sl}_2$-connection on the rank 2 trivial vector bundle defined by

$$\nabla = d + \begin{pmatrix} \alpha & \beta \\ \omega & -\alpha \end{pmatrix} \tag{1}$$

is flat. This definition, due to Scardua [37], extends to the singular case the classical definition [18] for smooth foliations. Roughly speaking, a foliation is transversely

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projective if outside a boundary divisor, the foliation $\mathcal{F}$ admits distinguished germs of first integrals taking values on $\mathbb{P}^1$ well defined up to left composition with elements of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PSL}_2(\mathbb{C})$; and the Schwarzian derivative of these distinguished first integrals with respect to any rational vector field generically transverse to $\mathcal{F}$ extends meromorphically through the boundary divisor.

Transversely projective foliations play a singular role in the study of codimension one foliations. They are precisely those foliations whose Galois groupoïd in the sense of Malgrange is small (see [9]). They often occur as exceptions or counter examples [4, 21, 40] and played an important role in our study of foliations with numerically trivial canonical bundle [24]. For these foliations, one can define a monodromy representation by considering analytic continuation of distinguished germs of first integrals, making the transversal pseudo-group into a group (see [22]).

The goal of this paper is to provide a structure theorem for transversely projective foliations in the spirit of what has been done recently by Gaël Cousin and the second author for transversely affine foliations [16]. In fact, we mainly work with the connection (1) up to birational bundle transformation. When it has at worst regular singularities, the connection (1) is characterized by its monodromy representation up to birational bundle transformations [17]. One of the main ingredients that goes into the proof of our structure theorem is an extension of Corlette-Simpson's classification of rank two representations of quasiprojective fundamental groups [13] which we now proceed to explain after settling some necessary terminology.

1.1. **Rank-two representations of quasiprojective fundamental groups.** Let X° be a quasiprojective manifold and consider X a projective compactification of X° with boundary equal to a simple normal crossing divisor D. If D_i is an irreducible component of D then by a small loop around D_i we mean a loop $\gamma : S^1 \to X^\circ$ that extends to a smooth map $\overline{\gamma} : \overline{\mathbb{D}} \to X$ which intersects D transversely on an unique smooth point of D_i. A representation $\rho : \pi_1(X^\circ, x) \to \mathrm{SL}_2(\mathbb{C})$ is quasi-unipotent at infinity if for every irreducible component D_i of D and every small loop γ around D_i, the conjugacy class of $\rho(\gamma)$ is quasi-unipotent (eigenvalues are roots of the unity).

A representation $\rho : \pi_1(X^\circ, x) \to \mathrm{SL}_2(\mathbb{C})$ projectively factors through an orbifold Y if there exists a morphism $f : X^\circ \to Y$ and a representation $\tilde{\rho} : \pi_1^{\mathrm{orb}}(Y, f(x)) \to \mathrm{PSL}_2(\mathbb{C})$ such that the diagram

$$\begin{array}{ccc} \pi_1(X^\circ, x) & \xrightarrow{\ \rho\ } & \mathrm{SL}_2(\mathbb{C}) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \mathrm{proj}} \\ \pi_1^{\mathrm{orb}}(Y, f(x)) & \xrightarrow[\ \tilde{\rho}\ ]{} & \mathrm{PSL}_2(\mathbb{C}) \end{array}$$

is commutative.

A polydisk Shimura modular orbifold is a quotient $\mathfrak{H}$ of a polydisk $\mathbb{H}^n$ by a group of the form $\mathcal{U}(P, \Phi)$ where P is a projective module of rank two over the ring of integers $\mathcal{O}_L$ of a totally imaginary quadratic extension L of totally real number field F; Φ is a skew hermitian form on $P_L = P \otimes_{\mathcal{O}_L} L$; and $\mathcal{U}(P, \Phi)$ is the subgroup of the Φ-unitary group $\mathcal{U}(P_L, \Phi)$ consisting of elements which preserve P. This group acts naturally on $\mathbb{H}^n$ where n is half the number of embeddings $\sigma : L \to \mathbb{C}$ such that the quadratic form $\sqrt{-1}\Phi(v, v)$ is indefinite. Note that there is one tautological representation

$$\pi_1^{orb}(\mathbb{H}^n/\mathcal{U}(P,\Phi)) \simeq \mathcal{U}(P,\Phi)/\{\pm\,\mathrm{Id}\} \hookrightarrow PSL(2,L)\,,$$

which induces for each embedding $\sigma : L \to \mathbb{C}$ one tautological representation $\pi_1^{orb}(\mathbb{H}^n/\mathcal{U}(P,\Phi)) \to PSL(2,\mathbb{C})$. The quotients $\mathbb{H}^n/\mathcal{U}(P,\Phi)$ are always quasiprojective orbifolds, and when $[L:\mathbb{Q}] > 2n$ they are projective (i.e. proper/compact) orbifolds. The archetypical examples satisfying $[L:\mathbb{Q}] = 2n$ are the Hilbert modular orbifolds, which are quasiprojective but not projective. We refer to [13] for a thorough discussion and point out that our definition of tautological representations differs slightly from loc. cit. as they consider polydisk Shimura modular stacks instead of orbifolds and consequently their representations take values in $SL(2,\mathbb{C})$. Here we are forced to consider representations with values in $PSL(2,\mathbb{C})$ because $\pm\,\mathrm{Id} \in \mathcal{U}(P,\Phi)$ acts trivially on $\mathbb{H}^n$.

Theorem 1.1 (Corlette-Simpson)**.** *Suppose that X° is a quasiprojective manifold and $\rho : \pi_1(X^\circ, x) \to \mathrm{SL}_2(\mathbb{C})$ is a Zariski dense representation which is quasi-unipotent at infinity. Then ρ projectively factors through*

(1) *a morphism $f : X^\circ \to Y$ to an orbicurve Y (orbifold of dimension one); or*
(2) *a morphism $f : X^\circ \to \mathfrak{H}$ to a polydisk Shimura modular orbifold $\mathfrak{H}$.*

In the latter case, the representation actually projectively factors through of one of the tautological representations of $\mathfrak{H}$.

Although their hypothesis is natural, as representations coming from geometry (Gauss-Manin connections) are automatically quasi-unipotent at infinity, Corlette and Simpson asked in [13, Section 12.1] what happens if this assumption is dropped. Our first main result answers this question.

Theorem A. *Suppose that X° is a quasiprojective manifold and $\rho : \pi_1(X^\circ, x) \to \mathrm{SL}_2(\mathbb{C})$ is a Zariski dense representation which is not quasi-unipotent at infinity. Then ρ projectively factors through a morphism $f : X^\circ \to Y$ to an orbicurve Y.*

Our method to deal with representations which are not quasi-unipotent at infinity is considerably more elementary than the sophisticated arguments needed to deal with the quasi-unipotent case. The non quasi-unipotency allows us to prove the existence of effective divisors with topologically trivial normal bundle at the boundary. We then use Malcev's Theorem combined with a result of Totaro about the existence of fibrations to produce the factorization.

Combining Corlette-Simpson Theorem with Theorem A, and factorization results for representations of quasiprojective fundamental groups on the affine group $\mathrm{Aff}(\mathbb{C})$, see [2, 16] and references therein, we get the following corollary.

Corollary B. *Suppose that X° is a quasiprojective manifold and $\rho : \pi_1(X^\circ, x) \to \mathrm{SL}_2(\mathbb{C})$ is a representation which is not virtually abelian. Then ρ projectively factors through*

(1) *a morphism $f : X^\circ \to Y$ to an orbicurve Y; or*
(2) *a morphism $f : X^\circ \to \mathfrak{H}$ to a polydisk Shimura modular orbifold $\mathfrak{H}$.*

Moreover, in the second case the representation is Zariski dense, quasi-unipotent at infinity and projectively factors through one of the tautological representations of $\mathfrak{H}$.

1.2. **Riccati foliations.** For us, a **Riccati foliation** over a projective manifold X consists of a pair $(\pi : P \to X, \mathcal{H}) = (P, \mathcal{H})$ where $\pi : P \to X$ is a locally trivial $\mathbb{P}^1$ fiber bundle in the Zariski topology (i.e. P is the projectivization of the total space of a rank two vector bundle E) and $\mathcal{H}$ is a codimension one foliation on P which is transverse to a general fiber of π. If the context is clear, we will omit the $\mathbb{P}^1$-bundle P from the notation and call $\mathcal{H}$ a Riccati foliation.

The foliation $\mathcal{H}$ is defined by the projectivization of horizontal sections of a (non unique) flat meromorphic connection ∇ on E. The connection ∇ is uniquely determined by $\mathcal{H}$ and its trace on $\det(E)$. We say that the Riccati foliation $\mathcal{H}$ is **regular** if it lifts to a meromorphic connection ∇ with at worst regular singularities (see [17]), and **irregular** if not.

We will say that a Riccati foliation $(P, \mathcal{H})$ over X factors through a projective manifold X' if there exists a Riccati foliation $(\pi' : P' \to X', \mathcal{H}')$ over X', and rational maps $\phi : X \dashrightarrow X'$, $\Phi : P \dashrightarrow P'$ such that $\pi' \circ \Phi = \phi \circ \pi$, Φ has degree one when restricted to a general fiber of P, and $\mathcal{H} = \Phi^* \mathcal{H}'$.

Our second main result describes the structure of Riccati foliations having irregular singularities.

Theorem C. *Suppose that X is a projective manifold, and $(P, \mathcal{H})$ is a Riccati foliation over X. If $\mathcal{H}$ is irregular then at least one of the following assertions holds true.*

1. *There exists a generically finite Galois morphism $f : Y \to X$ such that $f^* \mathcal{H}$ is defined by a closed rational 1-form.*
2. *The Riccati foliation $(P, \mathcal{H})$ factors through a curve.*

The proof of Theorem C relies on Corollary B, on an infinitesimal criterion for the existence of fibrations due to Neeman (Theorem 2.3), and on the semi-local study of Riccati foliations at a neighborhood of irregular singularities carried out in Section 8.

1.3. **Transversely projective foliations.** Our main goal, the description of the structure of transversely projective foliations, is achieved by combining Corollary B and Theorem C.

Theorem D. *Let $\mathcal{F}$ be a codimension one transversely projective foliation on a projective manifold X. Then at least one of the following assertions holds true.*

1. *There exists a generically finite Galois morphism $f : Y \to X$ such that $f^* \mathcal{F}$ is defined by a closed rational 1-form.*
2. *There exists a rational map $f : X \dashrightarrow S$ to a ruled surface S, and a Riccati foliation $\mathcal{R}$ on S such that $\mathcal{F} = f^* \mathcal{R}$.*
3. *The transverse projective structure for $\mathcal{F}$ has at worst regular singularities, and the monodromy representation of $\mathcal{F}$ factors through one of the tautological representations of a polydisk Shimura modular orbifold $\mathfrak{H}$.*

There are previous results on the subject [37] and on the neighboring subject of transversely affine foliations ([7], [6], [16]). With the exception of [16], all the other works impose strong restrictions on the nature of the singularities of the foliation. Our only hypothesis is the algebraicity of the ambient manifold.

Theorem D also answers a question left open in [16]. There, a similar classification for transversely affine foliations is established for foliations on projective manifolds with zero first Betti number. Theorem D gives the analogue classification for arbitrary projective manifolds, showing that the hypothesis on the first Betti number is not necessary.

1.4. **Flat meromorphic $\mathfrak{sl}_2$-connections.** A meromorphic $\mathfrak{sl}_2$-connection (E, ∇) on a projective manifold X is the datum of a rank 2 vector bundle E equipped with a trace-free connection $\nabla : E \to E \otimes \Omega^1_X(D)$ where D is the polar divisor; in particular, $\det(E) = \mathcal{O}_X$. It is *flat* when the curvature vanishes, that is $\nabla \cdot \nabla = 0$, meaning that it has no local monodromy outside the support of D; we can therefore define its monodromy representation.

We will say that any two such connections (E, ∇) and (E', ∇') are *(birationally) equivalent* when there is a birational bundle transformation $\phi : E \to E'$ that conjugates the two operators ∇ and ∇'. Keep in mind that the polar divisor might be not the same for ∇ and ∇'. The connection (E, ∇) is called regular if it is birationally equivalent to a connection having only simple poles (i.e. with reduced polar divisor); it is called irregular if not. In order to state our result, we will say that (E, ∇) and (E', ∇') are *projectively equivalent* if, up to birational equivalence, the $\mathbb{P}^1$-bundles coincide, $\mathbb{P}E = \mathbb{P}E'$, and ∇, ∇' induce the same projective connection. Equivalently, there is a flat rank 1 logarithmic connection (L, δ) on X having monodromy in the center $\{\pm \mathrm{Id}\}$ of $\mathrm{SL}_2(\mathbb{C})$ such that (E, ∇) is (birationally) equivalent to $(L, \delta) \otimes (E', \nabla')$. In particular, (E, ∇) and (E', ∇') are birationally equivalent after pulling them back to the ramified two-fold cover $Y \to X$ determined by the monodromy representation of (L, δ).

A combination of Corollary B and Theorem C yields

Theorem E. *Let (E, ∇) be a flat meromorphic $\mathfrak{sl}_2$-connection on a projective manifold X. Then at least one of the following assertions holds true.*

(1) *There exists a generically finite Galois morphism $f : Y \to X$ such that $f^*(E, \nabla)$ is equivalent to one of the following connections defined on the trivial bundle:*

$$\nabla = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \quad \text{or} \quad d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$$

with ω a rational closed 1-form on X.

(2) *There exists a rational map $f : X \dashrightarrow C$ to a curve and a meromorphic connection (E_0, ∇_0) on C such that (E, ∇) is projectively equivalent to $f^*(E_0, \nabla_0)$.*

(3) *The $\mathfrak{sl}_2$-connection (E, ∇) has at worst regular singularities and there exists a rational map $f : X \dashrightarrow \mathfrak{H}$ which projectively factors the monodromy through one of the tautological representations of a polydisk Shimura modular orbifold $\mathfrak{H}$. In particular, the monodromy representaion of (E, ∇) is quasi-unipotent at infinity, rigid, and Zariski dense.*

In particular, when (E, ∇) is irregular, only the former two cases occur. As mentioned in the abstract, the connection projectively factors through a curve whenever it has non trivial Stokes.

1.5. **Structure of the paper.** The paper is divided in two parts, with the first independent of the second. In the first part, Sections 2 and 3, we recall some results on the existence of fibrations which will be used throughout the paper, and present the proof of Theorem A. In this first part we avoided using foliation theory aiming at a wider audience. The second part is organized as follows. Section 4 presents foundational results about transversely projective foliations and Riccati foliations, most of them borrowed from [23] and [16]. Section 5 recalls the definition of transversely projective structures in minimal form and describes the behaviour of the transverse projective structure over non-special points of the polar divisor. Section 6 describes the singularities of foliations on surfaces which admit a transversely projective structure following closely [3] and [40]. In Section 7 we combine the results of Sections 5 and 6 to establish a kind of normal form for Riccati foliations over surfaces very much in the spirit of Sabbah's good formal models. Section 8 analyzes the Riccati foliations over surfaces at a neighborhood of its irregular singularities, showing in particular the existence of flat coordinates defining the irregular divisor which will be essential to produce the fibration in the absence of rich monodromy. Finally, Section 9 contains the proofs of Theorems C, D, and E, as well as many examples underlining the sharpness of our results.

2. Existence of fibrations

In this section we collect some results about the existence of fibrations which will be used in the sequel. For the proof of Theorem A all we will need is Theorem 2.1 below, which is due to Totaro [38]. Toward the end of the paper (proof of Theorem C), we will make use of the two other results below.

Theorem 2.1. *Let X be a projective manifold and $D_1, D_2, \ldots, D_r$, $r \geq 3$, be connected effective divisors which are pairwise disjoint and whose Chern classes lie in a line inside of $H^2(X, \mathbb{R})$. Then there exists a non constant morphism $f : X \to C$ to a smooth curve C with connected fibers which maps the divisors D_i to points.*

The original proof studies the restriction map $H^1(X, \mathbb{Q}) \to H^1(\hat{D}_1, \mathbb{Q})$, where $\hat{D}_1$ is the disjoint union of desingularizations of the irreducible components of D_1. When it is injective, this map leads to a divisor linearly equivalent to zero in the span of $D_2, \ldots, D_r$ which defines the fibration. Otherwise, the fibration is constructed as a quotient of the Albanese map of X. An alternative proof, based on properties of some auxiliary foliations is given in [34]. It goes as follows: given two divisors with proportional Chern classes, one constructs a logarithmic 1-form with poles on these divisors and purely imaginary periods. The induced foliation, although not by algebraic leaves in general, admits a non-constant real first integral. Comparison of the leaves of two of these foliations coming from three pairwise disjoint divisors with proportional Chern classes reveals that they are indeed the same foliation. The proportionality factor of the corresponding two logarithmic 1-forms gives, after Stein factorization, the sought fibration. For details, see respectively [38, 34].

In general, two disjoint divisors with same Chern classes are not fibers of a fibration. Indeed, if L is a non-torsion line-bundle over a projective curve C then the surface $S = \mathbb{P}(L \oplus \mathcal{O}_C)$ admits two homologous disjoint curves given by the splitting which are not fibers of a fibration. The point is that the normal bundle of the sections are L and L^*. Nevertheless, if the normal bundle of one of the effective divisors is torsion then we do have a fibration containing them as fibers.

Theorem 2.2. *Let X be a projective manifold and D_1, D_2 be connected effective divisors which are pairwise disjoint and whose Chern classes lie in a line of $H^2(X, \mathbb{R})$. If $\mathcal{O}_X(D_1)_{|D_1}$ is torsion then there exists a non constant map $f : X \to C$ to a smooth curve C with connected fibers which maps the divisors D_i to points.*

Theorem 2.2 can be proved along the lines of both proofs of Theorem 2.1 discussed above, but it is also a consequence of an infinitesimal variant of Theorem 2.1 due to Neeman [33, Article 2, Theorem 5.1] where instead of three divisors we have only one divisor with torsion normal bundle and constraints on a infinitesimal neighborhood of order bigger than the order of the normal bundle. The formulation below is due to Totaro, see [39, paragraph before the proof of Lemma 4.1], but the proof is a straightforward adaptation of Neeman's proof.

Theorem 2.3. *Let X be a projective manifold and D be a connected effective divisor. Suppose that $\mathcal{O}_X(D)_{|D}$ is torsion of order m. If*

$$\mathcal{O}_X(mD)_{|(m+1)D} \simeq \mathcal{O}_{(m+1)D}$$

then there exists a nonconstant morphism $f : X \to C$ to a smooth curve C with connected fibers which maps the divisor D to a point.

Proof. Let $I = \mathcal{O}_X(-D)$ be the defining ideal of D. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(mD) & \longrightarrow & \mathcal{O}_X(mD)\otimes\frac{\mathcal{O}_X}{I^m} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \wr\downarrow & & \\
0 & \longrightarrow & \mathcal{O}_D & \longrightarrow & \mathcal{O}_X(mD)\otimes\frac{\mathcal{O}_X}{I^{m+1}} & \longrightarrow & \mathcal{O}_X(mD)\otimes\frac{\mathcal{O}_X}{I^m} & \longrightarrow & 0
\end{array}$$

deduced from the standard exact sequence

$$0 \to \mathcal{O}_X(-mD) \longrightarrow \mathcal{O}_X \longrightarrow \frac{\mathcal{O}_X}{I^m} \to 0\,.$$

In cohomology we get the diagram

$$\begin{array}{ccccc}
 & & H^0(X, \mathcal{O}_X(mD)\otimes\frac{\mathcal{O}_X}{I^m}) & \longrightarrow & H^1(X, \mathcal{O}_X) \\
 & & \wr\downarrow & & \downarrow \\
H^0((m+1)D, \mathcal{O}_X(mD)\otimes\frac{\mathcal{O}_X}{I^{m+1}}) & \longrightarrow & H^0(mD, \mathcal{O}_X(mD)\otimes\frac{\mathcal{O}_X}{I^m}) & \longrightarrow & H^1(D, \mathcal{O}_D)
\end{array}$$

If $\mathcal{O}_X(mD)_{|(m+1)D} \simeq \mathcal{O}_{(m+1)D}$ then $\mathcal{O}_X(mD)_{|mD} \simeq \mathcal{O}_{mD}$ and

$$1 \in H^0(mD, \mathcal{O}_{mD}) = H^0(mD, \mathcal{O}_X(mD)\otimes\frac{\mathcal{O}_X}{I^m})$$

belongs to the image of the map in the lower left corner. Exactness of the bottom row implies that $1 \in H^0(mD, \mathcal{O}_{mD})$ is mapped to zero in $H^1(D, \mathcal{O}_D)$.

According to the diagram, the morphism $H^0(mD, \mathcal{O}_{mD}) \to H^1(D, \mathcal{O}_D)$ factors through $H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$.

If $1 \in H^0(mD, \mathcal{O}_{mD})$ is mapped to zero in $H^1(X, \mathcal{O}_X)$ then we deduce from the first row of the first diagram above that $h^0(X, \mathcal{O}_X(mD)) \geq 2$. Therefore mD moves in a pencil and we have the sought fibration.

If instead $1 \in H^0(mD, \mathcal{O}_{mD})$ is mapped to a nonzero element in $H^1(X, \mathcal{O}_X)$ then $H^1(X, \mathcal{O}_X) \to H^1(D, \mathcal{O}_D)$ is not injective. Thus the same holds true for the map $H^1(X, \mathcal{O}_X) \to \oplus H^1(D_i, \mathcal{O}_{D_i})$ where D_i are the irreducible components of D.

It follows that $\oplus\,\mathrm{Alb}(D_i)$ (direct sum of the Albanese varieties) do not dominate $\mathrm{Alb}(X)$. The morphism

$$X \longrightarrow \frac{\mathrm{Alb}(X)}{\oplus\,\mathrm{Alb}(D_i)}$$

contracts D, and is non constant. It follows (cf. [33] or [38]) that the image is a curve and we get the sought fibration as the Stein factorization of this morphism. □

3. Factorization of representations

3.1. **Criterion for factorization.** We apply Theorem 2.1 to establish a criterion for the factorization of representations of quasiprojective fundamental groups. In the statement below, we have implicitly fixed an ample divisor A in X and we consider the bilinear pairing defined in the Néron-Severi group of X defined by $(E, D) = E \cdot D \cdot A^{n-2}$ where $n = \dim X$. According to Hodge index Theorem this bilinear form has signature $(1, n-1)$.

Theorem 3.1. *Let X be a projective manifold, D be a reduced simple normal crossing divisor in X, and $\rho : \pi_1(X - D) \to G$ be a representation to a simple non-abelian linear algebraic group G with Zariski dense image. Suppose there exists E, a connected component of the support D with irreducible components $E_1, \ldots, E_k$, and an open subset $U \subset X$ containing E such that the restriction of ρ to $\pi_1(U - E)$ has solvable image. Then either the intersection matrix (E_i, E_j) is negative definite or the representation factors through an orbicurve.*

Proof. Let H be the Zariski closure of $\rho(\pi_1(U - E))$ in G. Since G is simple and non-abelian, we have that $H \neq G$. Let μ be the derived length of H and choose an element γ of the μ-th derived group of $\pi_1(X - D)$ such that $\rho(\gamma) \neq \mathrm{id}$.

We can apply Malcev's Theorem (any finitely generated subgroup of G is residually finite) [25] to obtain a morphism $\varrho : \pi_1(X - D) \to \Gamma$ to a finite group Γ such that $\varrho(\gamma) \neq \mathrm{id}$ and $\varrho(\gamma^2) \neq \mathrm{id}$. Notice that $\varrho(\gamma)$ and its square do not belong to $\varrho(\pi_1(U - E))$, since otherwise we would have that $H^{(\mu)}$ is non-trivial, contrary to our assumptions.

Let Y° be the covering of $X - D$ determined by the kernel of ϱ. It is a quasiprojective variety which can be compactified to a smooth projective surface Y endowed with a morphism $\phi : Y \to X$ with restriction to Y° equal to the covering determined by $\ker \varrho$. To wit, the morphism ϕ is the resolution of singularities of the ramified covering determined by ϱ.

By construction, $\phi^{-1}(E)$ has $c = [\varrho(\pi_1(X - D)) : \varrho(\pi_1(U - E))] \geq 3$ distinct connected components. Let B be one of these connected components with irreducible components $B_1, \ldots, B_\ell$. Fix an ample divisor A' on Y such that $\phi_* A' = A$ and define $(\cdot, \cdot)_Y$ using it. The intersection matrix (E_i, E_j) is negative definite if and only if the same holds true for the intersection matrix $(B_i, B_j)_Y$. Therefore, if (E_i, E_j) is not negative definite, there exists an effective divisor F supported on B with $(F, F) \geq 0$. Of course, the same holds true for any of the other $c-1$ connected components of $\phi^{-1}(E)$. Thus, under the assumption that the matrix $(E_i \cdot E_j)$ is not negative definite, we can produce F_1, F_2, F_3 pairwise disjoint effective divisors on Y satisfying $(F_i, F_i) \geq 0$. If one of these effective divisors has strictly positive self-intersection we arrive at a contradiction with Hodge index Theorem. Thus we have 3 pairwise disjoint divisors with zero self-intersection, and consequently with Chern classes lying on the same line. We can apply Theorem 2.1 to ensure the existence of a nonconstant morphism with irreducible general fiber $g : Y \to \Sigma$ to a curve Σ such that the divisor F_1, F_2, and F_3 are multiples of fibers of g. The morphism g is proper and open, thus all the other connected components of $\phi^{-1}(D)$ are mapped by g to points. Notice also that the critical set of $\phi : Y \to X$ is mapped by g to a finite union of points of Σ. This implies that two general fibers of g are mapped to two disjoint and homologous hypersurfaces of X. Consequently, the fibration defined by g descends to a fibration on X, and we have a non constant morphism $f : X \to C$ mapping connected components of D to points.

Let now $U \subset X - D$ be a Zariski open subset such that the restriction of f to U is a smooth and proper fibration, thus locally trivial in the C^∞ category, over $C^o = f(U)$. Let also F be a fiber of $f_{|U}$ and H be the Zariski closure of the image under ρ of $\pi_1(F)$ in G, and consider the following diagram where we have used that $\pi_2(C^o) = 0$ since C^o is non-compact.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & & & & \\ & & \rho\uparrow & & \rho\uparrow & & & & \\ 1 & \longrightarrow & \pi_1(F) & \longrightarrow & \pi_1(U) & \xrightarrow{f_*} & \pi_1(C^o) & \longrightarrow & 1 \end{array}$$

From this, it is clear that $\pi_1(F)$ is normal in $\pi_1(U)$, and since normality is a (Zariski) closed condition, we deduce that H is a normal subgroup of G. Since G is simple, we conclude that H must be trivial, i.e. the restriction of ρ to U factors through a curve. Following the proof of [13, Lemma 3.5] we see that this suffices to obtain the factorization through an orbicurve. □

Remark 3.2. We can avoid the factorization through an orbicurve, and the use of the arguments in [13, Lemma 3.5], by instead restricting the factorization to a Zariski open subset of $X - D$. In the opposite direction, if we allow one dimensional Deligne-Mumford stacks with general point having non trivial stabilizer as targets of the factorization, then we can replace simple linear algebraic groups by quasi-simple linear algebraic groups in the statement, since our proof shows that for G quasi-simple, there exists a fibration such that the Zariski closure of the images of fundamental groups of fibers of f under ρ are finite.

3.2. **Rank-two representations at neighborhoods of divisors.**

Proposition 3.3. *Let X be a complex manifold, D a reduced and simple normal crossing divisor in X, and $\rho : \pi_1(X - D) \to \mathrm{SL}_2(\mathbb{C})$ a representation. Let E be a connected divisor with support contained in D such that for each irreducible component E_i of E and any short loop γ_i turning around E_i, the element $\rho(\gamma_i)$ does not lie in the center of $\mathrm{SL}_2(\mathbb{C})$, i.e. $\rho(\gamma_i)$ is distinct from $\pm\,\mathrm{Id}$. Then there exists an open subset $U \subset X$ containing E such that the restriction of ρ to $\pi_1(U - D)$ has solvable image.*

Proof. Let $E_1, \ldots, E_k$ be the irreducible components of E and $\gamma_1, \ldots, \gamma_k$ be short loops turning around them. We will denote the set of smooth points of D in E_i by E_i°, i.e. $E_i^\circ = E_i - \cup_{j \neq i}(E_j \cap E_i)$.

Suppose first that $\rho(\gamma_1)$ is unipotent. Since, by hypothesis, it is different from the identity, its action on $\mathbb{C}^2$ leaves invariant a one-dimensional subspace L. If U_1 is a small tubular neighborhood of E_1 and $U_1^\circ = U_1 - D$ then U_1° has the homotopy type of a S^1-bundle over E_1° and therefore the subgroup generated by γ_1 in $\pi_1(U_1^\circ)$

is normal. It follows that every $\gamma \in \pi_1(U_1^\circ)$ also leaves L invariant. It follows that the rank two local system induced by ρ admits a unique rank one local subsystem determined by L on U_1°.

To analyze what happens at a non-empty intersection $E_1 \cap E_j$, we can assume that both γ_1 and γ_j have base points near $E_1 \cap E_j$. Thus γ_1 commutes with γ_j, both $\rho(\gamma_1)$ and $\rho(\gamma_j)$ are unipotent, and they both leave L invariant. Thus the rank one local subsystem determined by L on U_1° extends to a rank one local subsystem on $U_1^\circ \cup U_j^\circ$. Repeating the argument above for the other irreducible components $E_2, \ldots, E_k$, w e deduce the existence of a neighborhood U of E such that $\rho(\pi_1(U - D))$ is contained in a Borel subgroup of $\mathrm{SL}_2(\mathbb{C})$.

Similarly if $\rho(\gamma_1)$ is semi-simple, then the same holds true for every γ_i. Moreover, the representation now leaves invariant the union of two linear subspaces L_1 and L_2 (but does not necessarily leave invariant any of the two). We deduce that the image of ρ restricted to a neighborhood of E minus D is contained in an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{C}^*$. □

Remark 3.4. The proof above is very similar to the proof of [13, Lemma 4.5].

Proposition 3.5. *Let X, D, E, and ρ be as in Proposition 3.3. Assume also that every short loop γ turning around any irreducible component of $D - E$ which intersects E has monodromy in the center of* $\mathrm{SL}_2(\mathbb{C})$. *If the intersection matrix of E is negative definite then the restriction of ρ to $\pi_1(U - D)$ is quasi-unipotent at the irreducible components of E.*

Proof. Let (F, ∇) be a rank two vector bundle over U with a flat logarithmic connection whose monodromy is given by ρ (see [17]). Since the monodromy is solvable, around each point of $U - D$ we have one or two subbundles of F which are left invariant by ∇. Modulo passing to a double covering of $U - D$ if necessary, we can assume that (F, ∇) is reducible, i.e. we have a subbundle $F_1 \subset F$ and a logarithmic connection ∇_1 on F_1 such that $\nabla_1 = \nabla_{|F_1}$. The monodromy of ∇_1 on a loop γ around irreducible components of E equal to one of the eigenvalues of $\rho(\gamma)$, say λ_γ. If γ_i is a short loop around an irreducible component E_i of E then the residue of ∇_1 along E_i satisfies

$$\exp(2\pi i \,\mathrm{Res}_{E_1}(\nabla_1)) = \lambda_{\gamma_1} \,.$$

By the residue formula we can write

$$c_1(F_1) = \sum \mathrm{Res}_{E_1}(\nabla_1) E_i + \sum \mathrm{Res}_{D_j}(\nabla_1) D_j$$

where $D_1, \ldots, D_s$ are the irreducible components of D intersecting E but not contained in it. Since the eigenvalues around D_j are ± 1, we have that Res_{D_j} is a half-integer. Since $c_1(F_1)$ lies in $H^2(U, \mathbb{Z})$, we have that for every k

$$(c_1(F_1), E_k) = \sum \mathrm{Res}_{E_i}(\nabla_1)(E_i, E_k) + \sum \mathrm{Res}_{D_j}(\nabla_1)(D_j, E_k)$$

is an integer. Therefore the vector $v = (\mathrm{Res}_{E_1}(\nabla_1), \ldots, \mathrm{Res}_{E_k}(\nabla_1))^T$ satisfies a linear equation of the form $A \cdot v = b$ with $A = (E_i, E_j)$ and $2b \in \mathbb{Z}^k$. Since A has integral coefficients and is negative definite, it follows that v is a rational vector. Therefore the restriction of ρ to $U - D$ is quasi-unipotent at the irreducible components of E. □

Remark 3.6. The proof above is reminiscent of Mumford's computation [32] of the homology of the plumbing of a contractible divisor on a smooth surface.

3.3. **Proof of Theorem A.** Let $\rho : \pi(X - D) \to \mathrm{SL}_2(\mathbb{C})$ be a Zariski dense representation which is not quasi-unipotent at infinity. Let E be a connected divisor with support contained in $|D|$ such that $\rho(\gamma) \neq \pm\,\mathrm{Id}$ for every small loop around a irreducible component of E, and $\rho(\gamma)$ is not quasi-unipotent for at least one small loop. If E is maximal with respect to these properties, Proposition 3.3 implies that the restriction of the projectivization of ρ to a neighborhood of E is solvable, and Proposition 3.5 implies that the intersection matrix of E is indefinite. Since $\mathrm{PSL}_2(\mathbb{C})$ is a simple group we can apply Theorem 3.1 to factorize the projectivization of ρ through an orbicurve. Theorem A follows. □

4. Transversely projective foliations and Riccati foliations

Here we recall basic definitions and properties of transversely projective foliations and Riccati foliations, and provide some reduction lemmata. Throughout X will be a projective manifold.

4.1. **Transversely projective foliations.** A transversely projective foliation $\mathcal{F}$ on X is defined by a triple $\mathcal{P} = (\omega_0, \omega_1, \omega_2)$ of rational 1-forms on X satisfying

$$\begin{cases} d\omega_0 = \ \ \omega_0 \wedge \omega_1 \\ d\omega_1 = 2\omega_0 \wedge \omega_2 \\ d\omega_2 = \ \ \omega_1 \wedge \omega_2 \end{cases} \tag{2}$$

with $\omega_0 \neq 0$. The first equality implies Frobenius integrability $\omega_0 \wedge d\omega_0 = 0$, and therefore the kernel of ω_0 defines the codimension one foliation $\mathcal{F}$ on X. The extra data of ω_1 and ω_2 allows to define projective coordinates transverse to the leaves in the following way. The system (2) is equivalent to the Frobenius integrability $\Omega \wedge d\Omega = 0$ of the Riccati 1-form

$$\Omega = dz + \omega_0 + z\omega_1 + z^2\omega_2 \tag{3}$$

which defines a codimension one foliation $\mathcal{H}$ on the trivial $\mathbb{P}^1$-bundle $X \times \mathbb{P}^1_z$ and $\mathcal{F}$ is just the restriction of $\mathcal{H}$ to the zero section, i.e. $\mathcal{F} = \mathcal{H}|_{z=0}$. Over a general point of X, the $\mathbb{P}^1$-fiber is transversal to $\mathcal{H}$. This is a particular case of a Riccati foliation (see Section 4.5). By selecting 3 distinct germs of leaves of $\mathcal{H}$ and sending them to $\tilde{z} = 0, 1, \infty$, we define a new local trivialization $\tilde{z}$ of the bundle in which the Riccati foliation $\mathcal{H}$ is defined by $d\tilde{z} = 0$. This germ of map $\tilde{z}$ restricted to $z = 0$ is thus a first integral for $\mathcal{F}$ and is uniquely defined by Ω up to left composition with elements of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PSL}_2(\mathbb{C})$ (the choice of 3 leaves). This can be done outside of the polar divisor $(\mathcal{P})_\infty$ of the structure, defined by the poles of the ω_i's. More precisely, $(\mathcal{P})_\infty$ is the direct image under $\pi : X \times \mathbb{P}^1 \to X$ of the tangency divisor between $\mathcal{H}$ and the $\mathbb{P}^1$-fibration π.

Another projective triple $\mathcal{P}' = (\omega_0', \omega_1', \omega_2')$ define the same transversely projective foliation, i.e.the same foliation $\mathcal{F}$ with the same collection of local first integrals at a general point of X if, and only if, there are rational functions a, b on X such that $a \not\equiv 0$ and

$$\begin{cases} \omega_0' = a\omega_0 \\ \omega_1' = \omega_1 - \frac{da}{a} + 2b\omega_0 \\ \omega_2' = \frac{1}{a}\left(\omega_2 + b\omega_1 + b^2\omega_0 - db\right) \end{cases} \tag{4}$$

(see [37]). This exactly means that the Riccati foliation $\mathcal{H}'$ defined by $\Omega' = d\tilde{z} + \omega_0' + \tilde{z}\omega_1' + \tilde{z}^2\omega_2'$ is derived by the birational bundle map $\frac{1}{z} = a\frac{1}{\tilde{z}} + b$. Note that polar divisors may be different.

Alternatively, we can say that a foliation $\mathcal{F}$ on X is transversely projective if there exists a triple $\mathcal{P} = (P, \mathcal{H}, \sigma)$ satisfying

(1) $(P, \mathcal{H})$ is a Riccati foliation over X; and
(2) $\sigma : X \dashrightarrow P$ is a rational section generically transverse to $\mathcal{H}$ such that $\mathcal{F} = \sigma^*\mathcal{H}$.

In this point of view, the polar divisor $(\mathcal{P})_\infty$ of $\mathcal{P}$ is defined as the direct image under $\pi : P \to X$ of the tangency divisor of $\mathcal{H}$ and the vertical foliation defined by the fibers of π. Notice that the section σ is not taken into account in this definition, and we will also call the same divisor as the polar divisor of the Riccati foliation $\mathcal{H}$ and denote it by $(\mathcal{H})_\infty$. The triple $\mathcal{P} = (P, \mathcal{H}, \sigma)$ is called a transverse projective structure for $\mathcal{F}$.

Any two such triples $\mathcal{P} = (P, \mathcal{H}, \sigma)$ and $\mathcal{P}' = (P', \mathcal{H}', \sigma')$ define the same transversely projective foliation when there exists a birational bundle transformation $\phi : P \dashrightarrow P'$ satisfying $\phi^*\mathcal{H}' = \mathcal{H}$, and $\phi \circ \sigma = \sigma'$. In this case we say that $\mathcal{P}$ and $\mathcal{P}'$ are birationally equivalent. When X is projective, a projective triple $\mathcal{P}$ is always birationally equivalent to another one with $P = X \times \mathbb{P}^1_z$ the trivial bundle and σ equal to the zero section $z = 0$ (see [23, Remark 2.1]): this alternative definition is not more general than the former one (triple of 1-forms). The advantage of this more geometric point of view is that we can look for a representative $\mathcal{P} = (P, \mathcal{H}, \sigma)$ in the equivalence class minimizing the polar divisor (see [23]). This will be essential later.

We can also define transversely affine structure by requiring moreover $\omega_2 = 0$ in the first definition, or that some rational section of $P \to X$ is invariant under the Riccati foliation $\mathcal{H}$ for the second definition. In this case, local first integrals can be chosen uniquely up to left composition by affine transformations. In a similar way, one can further reduce the structure to a transversely euclidean foliation by requiring $\omega_1 = \omega_2 = 0$ in the first definition.

4.2. **Monodromy representation.** The monodromy representation of a projective structure $\mathcal{P} = (\pi : P \to X, \mathcal{H}, \sigma : X \dashrightarrow P)$ is the representation

$$\rho_{\mathcal{H}} : \pi_1(X \setminus |(\mathcal{P})_\infty|) \longrightarrow \mathrm{PSL}_2(\mathbb{C})$$

defined by analytic continuation of a given local first integral of the structure. Equivalently, it is the monodromy of the projective connection, i.e. obtained by lifting paths on $X \setminus |(\mathcal{P})_\infty|$ to the leaves of $\mathcal{H}$. When the projective connection is induced by a linear one (E, ∇), then $\rho_{\mathcal{H}}$ is just the projectivization of the linear monodromy of ∇. Notice that the $\rho_{\mathcal{H}}$ does not depend on σ.

Remark 4.1. In general the monodromy representation $\rho : \pi_1(X - (\mathcal{P})_\infty) \to \mathrm{PSL}_2(\mathbb{C})$ of a transversely projective structure $\mathcal{P}$ for a foliation $\mathcal{F}$ does not lift to a representation with values in $\mathrm{SL}_2(\mathbb{C})$. Nevertheless, after trivializing P as before, if we restrict ρ to the fundamental group of the Zariski open subset of X where the above 1-forms ω_0, ω_1, and ω_2 are holomorphic, then this restriction lifts to $\mathrm{SL}_2(\mathbb{C})$ as the monodromy of the flat $\mathfrak{sl}_2$-connection ∇ lifting $\mathcal{H}$. Note that during the trivialization of the bundle, we might have added new components for the polar locus with central monodromy $\pm\,\mathrm{Id}$.

4.3. **Pull-back of Riccati foliations.** In the introduction we have defined when a Riccati foliation over X factors through a projective manifold X'. Similarly we will say that a Riccati foliation $(P, \mathcal{H})$ over X factors through a morphism $\phi : X \dashrightarrow X'$

if there exists a Riccati foliation $(\pi' : P' \to X', \mathcal{H}')$ over X', and a rational map $\Phi : P \dashrightarrow P'$ such that $\pi' \circ \Phi = \phi \circ \pi$, Φ has degree one when restricted to a general fiber of P, and $\mathcal{H} = \Phi^*\mathcal{H}'$. In this case, we will say $(P, \mathcal{H})$ is a pull-back of $(P', \mathcal{H}')$ under ϕ, and we will write $(P, \mathcal{H}) = \phi^*(P', \mathcal{H}')$. Notice that there is an abuse of notation being made since $(P, \mathcal{H})$ is not completely determined by ϕ as it depends on the choices of P and of the rational map $\Phi : P \dashrightarrow P'$, but the birational gauge equivalence class of $\phi^*(P', \mathcal{H}')$ is independent of the choices involved. Also, if $\phi : X \dashrightarrow X'$ is rational map with image not contained in the polar divisor of $\mathcal{H}'$ then we can find a lift $\Phi : X \times \mathbb{P}^1 \dashrightarrow P'$ of ϕ to the trivial $\mathbb{P}^1$-bundle over X having degree one over the general fiber and we set $\phi^*(P', \mathcal{H}')$ as the birational equivalence class of $(X \times \mathbb{P}^1, \Phi^* P')$.

Remark 4.2. If $\phi : X \to X'$ is a morphism and $(P', \mathcal{H}')$ is a Riccati foliation over X' then we can unambiguously set $P = \phi^* P'$ as the fiber product $X \times_{X'} P'$ formed through the morphisms (ϕ, π') and $\Phi : P \to P'$ defined by the second projection. In order to have a Riccati foliation on P, we only have to assume that the image of ϕ is not contained in the polar divisor of $(P', \mathcal{H}')$. If $\phi : X \dashrightarrow X'$ is only rational then the pull-back of P' is not always defined since $\phi^*_{|X-\mathrm{indet}(\phi)} P'$ may not extend to a $\mathbb{P}^1$-bundle over the whole X. Nevertheless, $\phi^*_{|X-\mathrm{indet}(\phi)} P'$ is birationally gauge equivalent to $\mathbb{P}^1$-bundle defined on the whole X. This is why we defined $\phi^*(P', \mathcal{H}')$ only up to birational equivalence.

4.4. **Uniqueness of transversely projective structures.** If $(\omega_0, \omega_1, \omega_2)$ and $(\omega_0', \omega_1', \omega_2')$ are projective triples for a given foliation $\mathcal{F}$, then the latter triple is always birationally equivalent to a unique one of the form $(\omega_0, \omega_1, \omega_2'')$ by a change of the form (4). The corresponding two transversely projective structures for $\mathcal{F}$ are the same (i.e. define the same collection of local first integrals) if, and only if, $\omega_2'' = \omega_2$. The foliations $\mathcal{F}$ which admit several non equivalent transversely projective structures are very particular and fit in one of the classes listed below (cf [37]).

- **Dihedral.** The foliation $\mathcal{F}$ admits a triple of the special form $(\omega_0, \omega_1, \omega_2) = (\omega_0, \frac{1}{2}\frac{df}{f}, 0)$ for some rational function f. It is transversely affine, and defined by a closed 1-form on a two-fold covering of X. Its monodromy takes values in the additive dihedral group $\{z \mapsto \pm z + c,\ c \in \mathbb{C}\}$. It admits the 1-parameter family of non equivalent projective structures $(\omega_0, \frac{1}{2}\frac{df}{f}, \lambda f \omega_0)$ with $\lambda \in \mathbb{C}^*$, whose monodromy is in the multiplicative dihedral group $\left\{z \mapsto \mu z^{\pm 1},\ \mu \in \mathbb{C}^*\right\}$. If f is not a square, then only $\lambda = 0$ gives a transversely affine structure.
- **Closed 1-form.** The foliation $\mathcal{F}$ is defined by a closed rational 1-form ω_0. It is a subcase of the dihedral, when f is a square. The 1-parameter family of (non equivalent) projective triples is given by $(\omega_0, 0, \lambda\omega_0)$ where $\lambda \in \mathbb{C}$. The monodromy (Galois group) is additive when $\lambda = 0$, and the Riccati is gauge equivalent to $dz + \omega_0$, or multiplicative when $\lambda \neq 0$, and the Riccati is gauge equivalent to $\frac{dz}{z} + \tilde{\lambda}\omega_0$ with $\tilde{\lambda} = -4\lambda$.
- **Fibration.** The foliation $\mathcal{F}$ has a rational first integral $f : X \to \mathbb{P}^1$, i.e. it is defined by $\omega_0 = df$. Then we can pull-back on X any rational projective structure on $\mathbb{P}^1$ (or on the target of the Stein factorization of f). We have the following (non equivalent) triples $(\omega_0, \omega_1, \omega_2) = (df, 0, g df)$ with any function g satisfying $df \wedge dg = 0$. There is a huge freedom in this case.

Remark 4.3. Assume that $\mathcal{F}$ is a transversely projective foliation. Let $\mathcal{P} = (P, \mathcal{H}, \sigma)$ be a projective structure for $\mathcal{F}$. If $\mathcal{H}$ is defined by a closed rational 1-form then the same holds true for $\mathcal{F}$. In particular, to prove that $\mathcal{F}$ is defined by a closed rational 1-form up to a finite cover it suffices to do the same for $\mathcal{H}$.

The converse is almost true: if $\mathcal{F}$ is defined by a closed 1-form ω_0 and does not admit a rational first integral, then $(P, \mathcal{H})$ is birationally gauge-equivalent to the trivial bundle equipped with the Riccati foliation defined by the closed 1-form

$$\frac{dz}{z} + \lambda\omega_0, \ \lambda \in \mathbb{C}^*, \quad \text{or } dz + \omega_0. \tag{5}$$

In particular, if a Riccati foliation $\mathcal{H}$ is defined by a closed 1-form, then either it admits a rational first integral, or it is birationally gauge-equivalent to one of the normal forms (5).

The non-uniqueness of transversely projective structures for foliations is related to the existence of non-trivial birational maps preserving Riccati foliations $(P, \mathcal{H})$ as the lemma below shows.

Lemma 4.4. *Let $(P, \mathcal{H})$ be a Riccati foliation over a projective manifold X. If there exists a birational map $\Phi : P \dashrightarrow P$ distinct from the identity such that $\Phi^*\mathcal{H} = \mathcal{H}$ and $\pi \circ \Phi = \pi$ then there exists a generically finite morphism of degree at most two $f : Y \to X$ such that $f^*(P, \mathcal{H})$ is defined by a closed rational 1-form.*

Proof. Since Φ commutes with projection $\pi : P \to X$ it follows that over a general fiber of π, Φ is an automorphism. Let $F = \{z \in P - \text{indet}(\Phi) | \Phi(z) = z\}$ be the set of fixed points of Φ. Since we are dealing with a family of automorphism of $\mathbb{P}^1$, the projection of Z to X is generically finite of degree one or two. Assume first that the degree is one. Then we can birationally trivialize P in such a way that F becomes the section at infinity and Φ is of the form $z \mapsto z + \tau$ for some $\tau \in \mathbb{C}(X)$. Let $\Omega = dz + \omega_0 + \omega_1 z + \omega_2 z^2$ be a rational form defining $\mathcal{H}$. The invariance of $\mathcal{H}$ under Φ reads as

$$\Omega \wedge \Phi^*\Omega = 0 \iff \omega_2 = 0 \text{ and } \omega_1 = -d\log\tau \iff d(\tau\Omega) = 0\,.$$

If the degree of $\pi_{|Z}$ is two then after replacing X by (the resolution of) a ramified double covering we can assume that P is trivial and that Φ is given by $z \mapsto \lambda(x)z$. The invariance of $\mathcal{H}$ under Φ reads as

$$\Omega \wedge \Phi^*\Omega = 0 \iff \omega_0 = \omega_2 = d\lambda = 0 \iff d(z^{-1}\Omega) = 0\,.$$

This concludes the proof of the lemma. □

4.5. **Generalized Riccati foliations.** Let X be a projective manifold and $f : X \to Y$ be a morphism with general fiber isomorphic to $\mathbb{P}^1$. We are not assuming that the dimension of fibers is constant nor the existence of a rational section. Let $\mathcal{F}$ be a foliation transverse to the general fiber of f, i.e. $\mathcal{F}$ is a generalized Riccati foliation with respect to π. The foliation admits a natural projective structure. To construct it, form the fiber product $X \times_f X$ and consider the two natural projections $\pi_1, \pi_2 : X \times_f X \to X$. Notice that π_1 admits the diagonal section $\sigma_1 : X \to X \times_f X$: if $x \in X$ then $\sigma_1(x) = (x, x) \in X \times_f X$. Therefore $X \times_f X$ is birationally equivalent to a $\mathbb{P}^1$-bundle P over X also carrying a rational section $\sigma : X \dashrightarrow P$. Now, if we set $\mathcal{H}$ as the foliation of P induced by $\pi_2^*\mathcal{F}$ under the birational equivalence between P and $X \times_f X$, we see that $\mathcal{H}$ is transverse to the general fiber of $\pi : P \to X$ and $\mathcal{F} = \sigma^*\mathcal{H}$.

Notice that we have not used that the fibration f admits a rational section, as it may not. In [23, Section 3.1] the above argument is used to endow $\mathcal{F}$ with a transversely projective structure when f is a $\mathbb{P}^1$-bundle. Even if the fiber product is already smooth in that case, and there is no need to modify it to obtain P, there is no essential difference between the two cases.

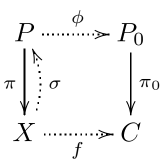
4.6. **Generically finite morphisms and factorizations.** Let now $\mathcal{F}$ be a codimension one foliation on X. We will say that $\mathcal{F}$ is the pull-back of a Riccati foliation over a curve if there is a rational dominant map $\phi : X \dashrightarrow P_0$ to a ruled surface $\pi : P_0 \to C$ and a Riccati foliation $\mathcal{H}_0$ on P_0 such that $\mathcal{F} = \phi^*\mathcal{H}_0$.

Proposition 4.5. *Let $\mathcal{F}$ be transversely projective foliation with projective triple $\mathcal{P} = (P, \mathcal{H}, \sigma)$. If $(P, \mathcal{H})$ factors through a curve, then one of the following holds true.*

(1) *$\mathcal{F}$ is pull-back of a Riccati foliation over a curve,*
(2) *$\mathcal{F}$ has a rational first integral.*

Conversely, if $\mathcal{F}$ admits a unique projective structure (i.e. does not belong to the list of Section 4.4) and is pull-back of a Riccati foliation over a curve, then $(P, \mathcal{H})$ factors through a curve.

Proof. For the first assertion, the factorization yields a commutative diagram of rational maps



Now the foliation $\mathcal{F}$ is defined by $(\phi \circ \sigma)^*\mathcal{H}_0$; if $\phi \circ \sigma$ is dominant, then we are in the former or latter case depending if $\phi \circ \sigma$ is dominant or not.

Conversely, assume that we have a rational dominant map $\phi : X \dashrightarrow P_0$ onto a ruled surface $\pi_0 : P_0 \to C$ and a Riccati foliation $\mathcal{H}_0$ on P_0 such that $\mathcal{F} = \phi^*\mathcal{H}_0$. From Section 4.5, we know that $\mathcal{H}_0$ is transversely projective with triple $(\tilde{P}_0, \tilde{\mathcal{H}}_0, \tilde{\sigma}_0)$ where $(\tilde{P}_0, \tilde{\mathcal{H}}_0) = \pi_0^*(P_0, \mathcal{H}_0)$ and $\tilde{\sigma}_0$ is the diagonal section. We deduce the following projective triple for $\mathcal{F}$: take the pull-back $(\tilde{P}, \tilde{\mathcal{H}}) := \phi^*(\tilde{P}_0, \tilde{\mathcal{H}}_0)$ together with the pulled-back section. By construction, $(\tilde{P}, \tilde{\mathcal{H}})$ is just the pull-back of the Riccati foliation $(P_0, \mathcal{H}_0)$ by $\pi_0 \circ \phi : X \dashrightarrow C$. By assumption, $\mathcal{F}$ has a unique projective structure and therefore the initial Riccati foliation $(P, \mathcal{H})$ is birationally equivalent to $(\tilde{P}, \tilde{\mathcal{H}})$, and thus factors through a curve. □

In order to prove Theorems C and D, it will be useful to blow-up the manifold X and pass to a finite cover in order to simplify the foliation. At the end, we will need the following descent lemma to come back to a conclusion on X.

Proposition 4.6. *Let $(P, \mathcal{H})$ be a Riccati foliation over X and $\varphi : \tilde{X} \to X$ be a generically finite morphism. If the pull-back $\varphi^*(P, \mathcal{H})$ factors through a curve, then the same holds true for $(P, \mathcal{H})$ provided that $f^*(P, \mathcal{H})$ is not defined by a closed rational 1-form for any dominant morphism $f : Y \to X$.*

Proof. It is very similar to the proof of [13, Lemma 3.6]. Maybe composing by a generically finite morphism, we can assume that

(1) φ is Galois in the sense that there is a finite group G of birational transformations acting on $\tilde{X}$ and acting transitively on a general fiber;
(2) there is a morphism $\tilde{f} : \tilde{X} \to \tilde{C}$ with connected fibers and a Riccati foliation $(\tilde{P}_0, \tilde{\mathcal{H}}_0)$ over $\tilde{C}$ such that its pull-back on $\tilde{X}$ is birationally equivalent to $(\tilde{P}, \tilde{\mathcal{H}}) := \varphi^*(P, \mathcal{H})$.

By consequence, for any $g \in G$, $g^*(\tilde{P}, \tilde{\mathcal{H}})$ is birationally equivalent to $(\tilde{P}, \tilde{\mathcal{H}})$ and the Riccati foliation $(\tilde{P}, \tilde{\mathcal{H}})$ factors through $\tilde{f} \circ g : \tilde{X} \dashrightarrow \tilde{C}$. Consider a general fiber Z of $\tilde{f}$. The Riccati foliation $(\tilde{P}, \tilde{\mathcal{H}})|_Z$ restricted to Z is birationally equivalent to the trivial Riccati foliation on Z, and thus admits a rational first integral. If the map $\tilde{f} \circ g$ were dominant in restriction to Z, this would imply that $(\tilde{P}_0, \tilde{\mathcal{H}}_0)$ also has a rational first integral, and the same for $(\tilde{P}, \tilde{\mathcal{H}})$ and $(P, \mathcal{H})$, contradiction. Thus g (and G) must permute general fibers of $\tilde{f}$ and acts on $\tilde{C}$. Moreover, $g^*(\tilde{P}_0, \tilde{\mathcal{H}}_0)$ is birational to $(\tilde{P}_0, \tilde{\mathcal{H}}_0)$ for all $g \in G$. Lemma 4.4 implies that over each $g : \tilde{X} \dashrightarrow \tilde{X}$ there exists a unique birational map $\hat{g} : \tilde{P} \dashrightarrow \tilde{P}$ such that $\hat{g}^*\tilde{\mathcal{H}} = \tilde{\mathcal{H}}$, and a similar statement holds true for the action of G on $\tilde{C}$. Therefore, we get an action of G on the diagram

$$\begin{array}{ccc} \tilde{P} & \longrightarrow & \tilde{P}_0 \\ \downarrow & & \downarrow \\ \tilde{X} & \xrightarrow[\tilde{f}]{} & \tilde{C} \end{array}$$

which preserves the Riccati foliations. Passing to the quotient, we get a commutative diagram

$$\begin{array}{ccc} P & \longrightarrow & P_0 \\ \downarrow & & \vdots \\ X & \xrightarrow[f]{} & C \end{array}$$

where $P_0 \dashrightarrow C$ has $\mathbb{P}^1$ as a general fiber. Moreover, the quotient foliation $\mathcal{H}_0$ on P_0 is transversal to the general fiber and thus of Riccati type. □

Remark 4.7. A similar statement can be proved for transversely projective foliations: if $\mathcal{F}$ is such a foliation on X and if $\pi^*\mathcal{F}$ is pull-back of a Riccati foliation over a curve, then the same holds true for $\mathcal{F}$ provided it has a unique transverse structure. The proof is the same as the proof of [16, Proposition 2.15]. It is also a corollary of Propositions 4.5 and 4.6.

4.7. **Monodromy and factorization.** We say that a Riccati foliation $(P, \mathcal{H})$ has regular singularities when it can be locally induced by a flat meromorphic linear connection having regular singularities in the sense of [17, Chapter II]. Like in the linear case, any two regular singular Riccati foliations $(P, \mathcal{H})$ and $(P', \mathcal{H}')$ have the same monodromy if and only if there exists a birational bundle map $\phi : P \dashrightarrow P'$ such that $\phi^*\mathcal{H}' = \mathcal{H}$, see [16, Lemma 2.13]. In particular, if the monodromy factors through a curve, then so does the Riccati foliation. The next proposition, borrowed from [16, Proposition 2.14], tells us what remains true in the irregular case.

Proposition 4.8. *Let $(P, \mathcal{H})$ be a Riccati foliation over X. Suppose there exists a morphism $f : X \to C$ with connected fibers such that the polar divisor $(\mathcal{P})_\infty$ of $\mathcal{P}$ intersects the general fiber of f at most on regular singularities; assume moreover* *that the monodromy representation ρ of $\mathcal{P}$ factors through f, i.e. there exists a divisor F supported on finitely many fibers of f and a representation ρ_0 from the fundamental group of $C_0 = f(X - |(\mathcal{P})_\infty + F|)$ to $\mathrm{PSL}(2,\mathbb{C})$ fitting in the diagram below.*

$$\begin{array}{ccc} \pi_1(X - |(\mathcal{P})_\infty + F|) & \xrightarrow{\ \rho\ } & \mathrm{PSL}_2(\mathbb{C}) \\ {\scriptstyle f_*}\downarrow & \nearrow_{\rho_0} & \\ \pi_1(C_0) & & \end{array}$$

Then $(P,\mathcal{H})$ factors through $f : X \to C$.

4.8. **Reduction to the two-dimensional case.** The proposition below allow us to reduce our study of Riccati foliations over arbitrary projective manifolds to study of Riccati foliation over projective surfaces.

Proposition 4.9. *Let $(P,\mathcal{H})$ be a Riccati foliation over a projective manifold X and assume it has no rational first integral. If the restriction of $(P,\mathcal{H})$ to any general surface $Z \subset X$ factors through a curve, then the same holds true for $(P,\mathcal{H})$ over X.*

Proof. Denote by $\pi : P \to X$ the natural projection. Let $(P_Z,\mathcal{H}_Z)$ be the restriction of $(P,\mathcal{H})$ to $Z \subset X$, i.e. $P_Z = P_{|\pi^{-1}(Z)}$ and $\mathcal{H}_Z = \mathcal{H}_{|\pi^{-1}(Z)}$. By assumption, we get a rational bundle map $\phi : P_Z \dashrightarrow P_0$ such that $\phi^*\mathcal{H}_0 = \mathcal{H}_Z$. This shows that $\mathcal{H}_Z$ contains a foliation by algebraic curves.

Applying the same argument for different choices of Z, we obtain that through a general point $p \in P$ the leaf of $\mathcal{H}$ through p contains an algebraic subvariety A_p. It is known that the germ of leaf at a general point contains an unique germ of algebraic subvariety maximal with respect to inclusion which turns out to be irreducible, see the proof of [24, Lemma 2.4]. In our setup, we can thus assume that A_p has codimension at most two, since otherwise we would be able to enlarge A_p by choosing an appropriate Z and applying the above argument.

It follows from [24, Lemma 2.4] that $\mathcal{H}$ contains a foliation $\mathcal{G}$ with all leaves algebraic and having codimension at most two, and that $\mathcal{H}$ is the pull-back of a foliation on a curve or a surface. In the former case, we get a rational first integral for $\mathcal{H}$, contradiction. Hence the codimension of $\mathcal{G}$ must be equal to two. Let now L be a general leaf of $\mathcal{G}$ and consider its projection to X. Since $\mathcal{H}$ is transversal to the general of fiber of π, this projection is generically étale and $\pi(L)$ is (a Zariski open subset) of a divisor D_L on X. The construction method of $\mathcal{G}$ makes clear that $\pi^{-1}(D_L)$ is invariant by $\mathcal{G}$, and also that the restriction of $\mathcal{H}$ to $\pi^{-1}(D_L)$ is birationally equivalent to the foliation on a trivial $\mathbb{P}^1$-bundle over D_L given by the natural projection to $\mathbb{P}^1$. This is sufficient to show that $\mathcal{H}$ is the pull-back of a Riccati foliation on a $\mathbb{P}^1$-bundle over a curve. □

5. Polar divisor and Riccati foliations in minimal form

In the next three sections we will restrict our attention to Riccati foliations over projective surfaces. Although it might be not strictly necessary, this will allow us to use the current knowledge on singularities of foliations on surfaces to simplify many arguments. Proposition 4.9 allows to transfer the conclusions to Riccati foliations over arbitrary projective manifolds. The general case will come back into play only at the proofs of Theorems C and D.

Let $(P, \mathcal{H})$ be a Riccati foliation over a projective surface S. In this section, we review the local structure of $(P, \mathcal{H})$ over a neighborhood of a general point of $(\mathcal{H})_\infty$..

5.1. **Local factorization.** Let $(\pi : P \to S, \mathcal{H})$ be a Riccati foliation over a projective surface S. Let $H \subset S$ be an irreducible component of the polar divisor $(\mathcal{H})_\infty$. The special points of $\mathcal{H}$ are the points $x \in H$ such that $\pi^{-1}(x)$ is contained in the singular set of $\mathcal{H}$. Since $\operatorname{cod}\operatorname{sing}(\mathcal{H}) \geq 2$, we note that the set of special points is a finite subset of $H \subset S$. We paraphrase below (part of) the statement of [23, Lemma 4.1].

Lemma 5.1. *Let H be an irreducible component of the polar divisor of $(P, \mathcal{H})$. If $\pi^{-1}(H)$ is $\mathcal{H}$-invariant then the foliation $\mathcal{H}$ locally factors through a curve along H minus its set of special points.*

By local factorization we mean the following. At any non special point $p \in H$, and for sufficiently small disk $\Delta \subset S$ transversal to H at p, there exists a neighborhood $U \subset S$ of p and a a submersion $f : U \to \Delta$ with $f^{-1}(p) = H \cap U$ such that the Riccati $(P, \mathcal{H})|_U$ over the neighborhood U is biholomorphically equivalent to the pull-back of its restriction to Δ through a fibre bundle isomorphism. In other words, the Riccati foliation is locally a product of a Riccati foliation over a disk by a disk (or a polydisk in higher dimension).

By the transversal type of the Riccati foliation $(P, \mathcal{H})$ along an irreducible component H of its polar locus, we mean the isomorphism class of the Riccati foliation $(P, \mathcal{H})|_\Delta$ in restriction to a small transversal disk at a non special point $p \in H$.

5.2. **Minimal form and regular/irregular singular points.** Let H be an irreducible component of the polar divisor $(\mathcal{H})_\infty$ invariant by $\mathcal{H}$. At a neighborhood of a non-special point of H, and up to bimeromorphic bundle transformations, the foliation $\mathcal{H}$ is defined by

$$\Omega = dz + \left(1 + \frac{\phi(x)}{2} z^2\right) dx \tag{6}$$

for a convenient local coordinate x, with $\phi(x)$ meromorphic with pole along $H = \{x = 0\}$. In fact, the local coordinate x can be taken as the pull-back under $f : U \to \Delta$ given by Lemma 5.1 of any coordinate function on Δ.

Remark 5.2. Let $\mathcal{F}$ be a transversely projective foliation and H an irreducible component of the polar locus of the structure $\mathcal{P} = (P, \mathcal{H}, \sigma)$. Let $p \in H$ be a point where $\mathcal{F}$ is regular and consider a local primitive first integral f for $\mathcal{F}$ at p. Then $x = f$ is a factorizing coordinate for the Riccati foliation $\mathcal{H}$. Moreover, after bimeromorphic bundle transformation, the projective structure is given by the normalized equation (6) together with the section $z = \infty$, and $\phi := \{\varphi, x\}$ is just the Schwarzian derivative of any projective coordinate of the transverse structure $\mathcal{P}$ (defined only locally, outside of H, but its Schwartzian derivative extends meromorphicaly through H).

We can rewrite (6) in the form

$$\Omega = dz + \left(1 + \frac{\tilde{\phi}(x)}{x^{2k+2}} z^2\right) dx$$

with $\tilde{\phi}$ holomorphic, not factor of x^2 (i.e. k minimal), and $k \geq 0$ is an integer. This model for transversely projective structure never minimize the order of poles.

A further bimeromorphic conjugation allows us to obtain a bimeromorphically equivalent projective structure with $\tilde{\mathcal{H}}$ defined by

$$\tilde{\Omega} = dz + \left(1 + (k+1)x^k z + \tilde{\phi}(x)z^2\right)\frac{dx}{x^{k+1}}\,. \tag{7}$$

The pole is **regular** when $k = 0$, and **irregular** when $k > 0$. In the irregular case, it is said to be **ramified** (or nilpotent, see [5]) when $\tilde{\phi}(0) = 0$; and **unramified** when $\tilde{\phi}(0) \neq 0$. In the ramified case, the singular point becomes unramified after a (local) ramified cover $x = \tilde{x}^2$. Except in the resonant regular case when the pole is apparent (with no monodromy) and can be killed by an additional bimeromorphic bundle modification, the order of poles is minimized by the normal form (7) in all other cases.

Since the base manifold S is a surface, there exists a birational bundle modification of the Riccati foliation $(P, \mathcal{H})$ minimizing the polar divisor in the following sense: $(\mathcal{H})_\infty \leq (\mathcal{H}')_\infty$ for any Riccati foliation $(P', \mathcal{H}')$ birationally equivalent to $(P, \mathcal{H})$. Such a birational model is obtained as follows. For each component H of the polar locus with transversal type which is not in the normal form (7), the singular points of $\mathcal{H}$ over H consist in (some fibers and) a section $s : H \to P$; one can decrease the multiplicity of the pole by performing an elementary modification of the bundle with indeterminacy locus containing the curve $s(P)$ (see [23, proof of Theorem 1]).

We can decompose the polar divisor of $\mathcal{H}$ as

$$(\mathcal{H})_\infty = R + I$$

where R is a reduced divisor with support equal to $(\mathcal{H})_\infty$ and, consequently, $I = (\mathcal{H})_\infty - (\mathcal{H})_{\infty,red}$. If the polar divisor of $(P, \mathcal{H})$ is minimal then I has an intrinsic meaning and will be called the **irregular part of the polar divisor** of $(P, \mathcal{H})$. Notice that $I = 0$ if and only if $(P, \mathcal{H})$ is regular.

We will now proceed to describe the behaviour/normal forms of $\mathcal{H}$ over non-special points of its polar divisor. Due to the existence of the local factorization over non-special points, this is equivalent to the description of Riccati foliations over a disk as is done in [5, Chapter 4, Section 1]. Throughout we assume that the polar divisor is minimal.

5.3. **Regular non resonant.** In a convenient local trivialization of the bundle, the singularity is defined by the 1-form

$$\Omega = \frac{dz}{z} + \lambda \frac{dx}{x}, \quad \text{with } \lambda \in \mathbb{C} \setminus \mathbb{Z}.$$

The monodromy around $\{x = 0\}$ is given by $z \mapsto \exp(2\pi i \lambda)z$. Note that when $\lambda \in \mathbb{Z}$, then the pole is apparent and thus not minimal, what we have excluded.

5.4. **Regular resonant.** This kind of singularity is also known as Poincaré-Dulac singularity. In a convenient bundle trivialization, it is defined by the Riccati 1-form

$$\Omega = dz + (nz + x^n)\frac{dx}{x}, \quad \text{with } n \in \mathbb{N}.$$

After birational bundle modification, we can moreover assume $n = 0$. The monodromy around $\{x = 0\}$ is a translation $z \mapsto z + 2\pi i$.

5.5. **Irregular unramified.** For details on what follows, see [28] or [22]. There are two singular points for the Riccati foliation, each of them being a saddle-node with a formal central manifold transversal to the invariant fiber $\{x = 0\}$. By a polynomial gauge transformation, these central manifolds can be sent to $z = 0$ and $z = \infty$ up to arbitrary large order which means that ω_0 and ω_2 are holomorphic, vanishing at arbitrary large order. Then, a polynomial change of the variable x normalizes the principal part of the 1-form ω_1 to $\frac{dx}{x^{k+1}} + \lambda\frac{dx}{x}$ (usually, for connections, this is not allowed and we get more formal invariants than only k and λ). The integer k is the irregularity index (or Poincaré-Katz rank) of the corresponding connection (see [1]) and in this case coincides with the multiplicity of $\{x = 0\}$ in the irregular divisor.

In a convenient local trivialization of the bundle, we get the following preliminary normal form

$$dz + b(x)z^2dx + \left(\frac{dx}{x^{k+1}} + \lambda\frac{dx}{x}\right)z + c(x)dx \tag{8}$$

with b, c holomorphic, vanishing at arbitrary large order. Finally, coefficients $b(x)$ and $c(x)$ can be killed by formal (generally divergent) gauge transformation and we finally arrive at the formal normal form

$$\Omega = \frac{dz}{z} + \frac{dx}{x^{k+1}} + \lambda\frac{dx}{x} \tag{9}$$

In general this last normalization is not convergent. Nevertheless, there are $2k$ closed sectors covering a neighborhood of 0 in the x-variable such that the differential equation is holomorphically conjugate to the normal form over the interior of the sector, and the conjugation extends continuously to the boundary. Each of the sectors contains exactly one of the arcs $\{x \in \mathbb{D}_\varepsilon | x^k \in i\mathbb{R}\} - \{0\}$. Over each of these sectors there are only two solutions with well defined limit when x approaches zero which correspond to the solutions $\{z = 0\}$ and $\{z = \infty\}$ for the normal form. When we change from a sector intersecting $x^k \in i\mathbb{R}_{>0}$ to a sector intersecting $x^k \in i\mathbb{R}_{<0}$ in the counter clockwise direction then we can continuously extend the solution corresponding to $\{z = +\infty\}$ but the same does not hold true for the solution corresponding to $\{z = 0\}$. Similarly, when changing from sectors intersecting $x^k \in i\mathbb{R}_{<0}$ to sectors with points in $x^k \in i\mathbb{R}_{>0}$ in the counter clockwise direction we can extend continuously the solution corresponding to $\{z = 0\}$ but not the one corresponding to $\{z = \infty\}$.

The obstructions to glue continuously the two distinguished solutions over adjacent sectors are the only obstructions to analytically conjugate the differential equation to its normal form. These obstructions are codified by the Stokes matrices (matrix point of view is more convenient here):

$$\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & b_k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c_k & 1 \end{pmatrix}$$

(well-defined up to cyclic permutation and simultaneous conjugacy by a diagonal matrix). Precisely, b_i's (resp. c_i's) are responsible for the divergence of the central manifold at $z = \infty$ (resp. $z = 0$). In other terms, the b_i's (resp. the c_i's) are the obstructions to kill the coefficient $b(x)$ (resp. $c(x)$). The monodromy around $x = 0$ is given by multiplying this sequence of Stokes matrices (in this cyclic order) with

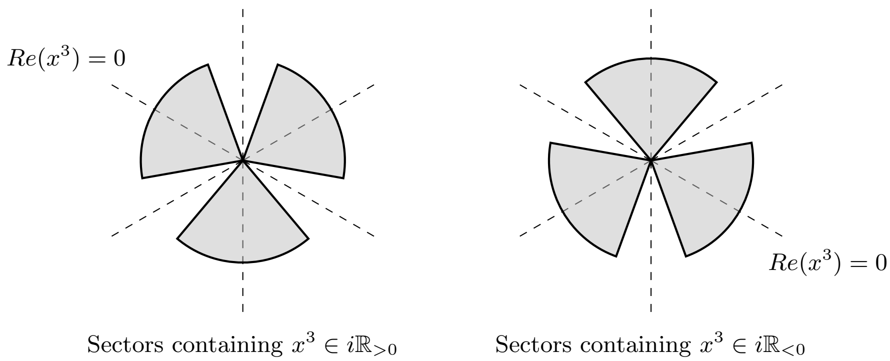


FIGURE 1.

the formal monodromy

$$\begin{pmatrix} e^{-i\pi\lambda} & 0 \\ 0 & e^{i\pi\lambda} \end{pmatrix}$$

(on the left or the right, as this does not matter up to diagonal conjugacy).

5.6. **Irregular ramified.** In the terminology of [5] this singularity is called nilpotent. It is a ramified version of the previous type. Going back to the preliminary normal form (7) $dz+(1+(k+1)x^k z+\tilde{\phi}(x)z^2)\frac{dx}{x^{k+1}}$ (with $\tilde{\phi}$ having a simple zero) we can set $x = \tilde{x}^2$ and $z = \tilde{z}/\tilde{x}$ and get an irregular unramified pole with irregularity index $\tilde{k} = 2k-1$. The irregularity index of the ramified initial singular point is $\kappa := \frac{\tilde{k}}{2} = k - \frac{1}{2}$, and the multiplicity of $\{x = 0\}$ in the irregular divisor is k.

Conversely, if we start with an unramified preliminary normal form $\tilde{\Omega} = d\tilde{z} + (b(\tilde{x})\tilde{z}^2 + \left(\frac{d\tilde{x}}{\tilde{x}^{\tilde{k}+1}} + \lambda\frac{d\tilde{x}}{\tilde{x}}\right)\tilde{z} + c(\tilde{x}))d\tilde{x}$, then commutation to $(\tilde{x}, \tilde{z}) \mapsto (-\tilde{x}, \frac{1}{\tilde{z}})$ gives that $\lambda = 0$ and $b(-\tilde{x}) = -c(\tilde{x})$. Then we can write $b(\tilde{x}) = B(\tilde{x}^2)+\tilde{x}C(\tilde{x}^2)$ (and thus $c(\tilde{x}) = -B(\tilde{x}^2) + \tilde{x}C(\tilde{x}^2)$) and express the Riccati equation in terms of invariant variables $x = \tilde{x}^2$ and $z = \tilde{x}\frac{\tilde{z}-1}{\tilde{z}+1}$:

$$\Omega = dz + \left(B(x)dx - \frac{1}{2}\frac{dx}{x}\right)z + \left(\frac{z^2 - x}{4x^k}\frac{dx}{x}\right) + C(x)\left(\frac{z^2 + x}{2}\frac{dx}{x}\right).$$

The Stokes of $\tilde{\Omega}$ are very special: an antidiagonal matrix conjugates the first half of matrices (an odd number) to the second half of matrices; after diagonal normalization, this gives the list of equalities $b_i = c_{i+k-1}$ for all $i = 1, \ldots, 2k-1$ (subscripts are modulo $2k-1$). When $b(x), c(x) \equiv 0$, Stokes matrices of $\tilde{\Omega}$ are trivial, and $\frac{\tilde{\Omega}}{\tilde{z}}$ is a closed 1-form; however, the quotient Riccati foliation is of dihedral type: the integrating factor f such that $d(f\Omega) = 0$ is given by $f = \frac{\tilde{x}}{z^2-x}$ (well-defined only on the 2-fold cover).

Remark 5.3. An irregular pole of order 2 (i.e. reduced component of the irregular divisor) with trivial monodromy has trivial Stokes. Indeed, in the unramified case, the SL_2-monodromy is given by

$$\begin{pmatrix} e^{-i\pi\lambda} & 0 \\ 0 & e^{i\pi\lambda} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} e^{-i\pi\lambda}(1+bc) & e^{-i\pi\lambda}b \\ e^{i\pi\lambda}c & e^{i\pi\lambda} \end{pmatrix}$$

which is $\pm I$ (projectively the identity) if, and only if, $\lambda \in \mathbb{Z}$ and $b = c = 0$. In the ramified case, it is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix}$$

which is never $\pm I$.

6. Singularities of transversely projective foliations

In this section we will review the structure of singularities of foliations on surfaces, and in particular possible transversely affine or projective structures that these might carry, following [3, 40].

For foliations on surfaces, we have Seidenberg's reduction of singularities (see [5, Chapter 1], for instance). A germ of singular foliation $\mathcal{F}$ on a surface, defined by a holomorphic vector fields v with isolated zeroes, is reduced when the linear part of v is non nilpotent and, when both eigenvalues are non-zero, we ask moreover that their quotient is not a positive rational number. Given $\mathcal{F}$ on a complex surface X, Seidenberg's Theorem tells us that after a locally finite sequence of blow-ups we obtain a foliation, birationally equivalent to $\mathcal{F}$, which has only reduced singular points. In this section, we review the classification of reduced singularities of foliations which carry a transversely projective structure following [3, 40].

6.1. **Linearizable reduced singularities.** These are the simplest kind of reduced singularities. By definition, they are locally analytically equivalent to foliations defined by a closed rational 1-form $\omega_0 = \frac{dy}{y} + \lambda \frac{dx}{x}$ with $\lambda \in \mathbb{C}^* \setminus \mathbb{Q}_{\leq 0}$ (corresponding to the diagonal vector field $v = x\frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y}$). They admit a holomorphic first integral if and only if $\lambda \in \mathbb{Q}_{>0}$.

If we do not have a holomorphic first integral then $\mathcal{F}$, the foliation defined by ω_0, admits a one parameter family of transversely projective structures (see Section 4.4). The corresponding Riccati foliations $\mathcal{H}$ on $(\mathbb{C}^2, 0) \times \mathbb{P}^1_z$ are given by the family

$$\Omega = \frac{dz}{z} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}, \quad \text{with } \frac{\lambda_x}{\lambda_y} = \lambda$$

and

$$\Omega = dz + \frac{dy}{y} + \lambda \frac{dx}{x}$$

and the section σ is given by $z = 1$, say. In the former generic case, the monodromy is linear, depending on (λ_x, λ_y). In the last case, the monodromy is euclidean and both branches $x = 0$ and $y = 0$ are logarithmic resonant poles (Poincaré-Dulac type). We note that for λ_x or $\lambda_y \in \mathbb{Z}$, the Riccati foliation has an apparent singular point and is thus not in minimal form.

When $\lambda = \frac{p}{q}$, the foliation admits the holomorphic first integral $f = x^p y^q$, with $\gcd(p, q) = 1$. A projective structure in this case is bimeromorphically gauge equivalent to a unique projective structure with triple of 1-forms satisfying $(\omega_0, \omega_1, \omega_2) = (df, 0, \phi(f)df)$, where ϕ is any meromorphic function of one variable. Branches $x = 0$ and $y = 0$ are either both irregular, or both logarithmic. In the logarithmic case, we have the local model

$$\Omega = \frac{dz}{z} + c\left(p\frac{dx}{x} + q\frac{dy}{y}\right), \quad \text{or} \quad dz + \left(p\frac{dx}{x} + q\frac{dy}{y}\right).$$

In the irregular case, starting with $\phi(f)$ having a pole of order $2\kappa+2$ (κ is the irregularity index, which can be integer or half integer), we get an irregular pole on $x=0$ with irregularity index $\kappa_x = p\kappa$, and multiplicity $\lfloor p\kappa + 1/2 \rfloor$ on the irregular divisor (here, $\lfloor \cdot \rfloor$ denotes the integral part). Notice that these saddles can be at the intersection of ramified and unramified irregular poles for the projective structure: if $f = xy^2$ and $\kappa = \frac{1}{2}$, then it is ramified with $\kappa_x = \frac{1}{2}$ along $x = 0$, and unramified with $\kappa_y = 1$ along $y = 0$.

6.2. **Diophantine saddles.** If a germ of foliation is defined in local coordinates by $v = x\frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial y} + \cdots$ with $\lambda \notin \mathbb{Q}$, then, thanks to Poincaré, the foliation can be linearized by a holomorphic change on coordinates, provided that $\lambda \notin \mathbb{R}_{<0}$. In the latter case, when we have a saddle, then the linearization of $\mathcal{F}$ is a subtle issue, related to Diophantine properties of λ. Nevertheless, when $\mathcal{F}$ admits a projective structure then also in this case we have that $\mathcal{F}$ is linearizable, [40, Theorem II.3.1].

6.3. **Saddle-nodes.** They are non linear analogues of unramified irregular singular points (see [28] and [22]). They are formally equivalent to the normal form

$$\omega_0 = \frac{dy}{y} + \frac{dx}{x^{k+1}} + \lambda \frac{dx}{x}.$$

Obstruction to analytic normalization are given by non linear Stokes data, which is a collection:

$$\varphi_1(z)\ ,\ z + c_1\ ,\ \ldots\ ,\ \varphi_k(z)\ ,\ z + c_k$$

of k germs of one-variable parabolic diffeomorphisms $\varphi_i(z) = z + \cdots$ and k translations. This is well-defined up to a cyclic permutation and simultaneous conjugacy by a linear map $z \mapsto cz$. Together with formal invariants k, λ, the Stokes data provide a complete set of analytic invariants (for biholomorphic local conjugacy): they are a complete set of coordinates on the (infinite dimensional) moduli space of saddle-nodes (see [28]). An important subclass is given by Riccati singular points over unramified irregular poles that can be written in the form

$$\Omega = dy + \left(a(x)y^2 + (\frac{1}{x^{k+1}} + \frac{\lambda}{x} + b(x))y + c(x) \right) dx$$

at the neighborhood of $(x, y) = (0, 0)$, with a, b, c holomorphic. In this case, non linear Stokes take the very special form $\varphi_i(z) = \frac{z}{1+b_i z}$; they are just projectivization of corresponding linear Stokes. Riccati saddle-nodes form a finite dimensional family (of infinite codimension in the total moduli space).

Translation constants c_i are obstructions to the convergence of the central manifold: one can kill the coefficient $c(x)$ by holomorphic change of coordinate if, and only if, all $c_i = 0$. In this case, the local holonomy of the central manifold $y = 0$ around $x = 0$ is given by

$$e^{2i\pi\lambda}\varphi_1 \circ \cdots \circ \varphi_k.$$

Classification of transversely affine or projective saddle-nodes has been done in [3, Proposition 5.5] and [40, Theorem II.4.2]. Here is the list.

- **Euclidean.** Stokes are trivial and the foliation is defined by the closed meromorphic 1-form ω_0 above. The projective structure is given either by $dz + \omega_0$, i.e.

$$\Omega = d\tilde{z} + \frac{dy}{y} + \lambda \frac{dx}{x}, \quad \tilde{z} = z - \frac{1}{kx^k},$$

(both branches are of Poincaré-Dulac type, i.e. logarithmic resonant) or by

$$\Omega = \frac{dz}{z} + \frac{dx}{x^{k+1}} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}, \quad \lambda_x, \lambda_y \in \mathbb{C} \setminus \mathbb{Z}.$$

The latter case is mixing unramified irregular and logarithmic branches.

- **Purely projective.** The foliation $\mathcal{F}$ does not admit an affine structure. In this case (see [40]) $\varphi_i(z) = \frac{z}{1+b_i z}$ are parabolic Moebius transformations and the foliation $\mathcal{F}$ is of Riccati type; the unique projective structure writes (in convenient local coordinates) as

$$\Omega = dz + \left(b(x)z^2 + (\frac{1}{x^{k+1}} + \frac{\lambda}{x})z + c(x) \right) dx$$

  with b, c holomorphic, vanishing with arbitrary large order at $x = 0$, and the foliation $\mathcal{F}$ is defined by the section $z = y$. In particular, this point is not special with respect to the Riccati foliation or connection, only the section σ is special, intersecting the singular locus of the Riccati foliation $\mathcal{H}$.
- **Affine with central manifold.** Here (see [3, Proposition 5.5]), all $c_i = 0$ and $\varphi_i(z) = \frac{z}{(1+b_i z^\nu)^{1/\nu}}$ are parabolic Moebius transformations in the ramified variable $w = z^\nu$, $\nu \in \mathbb{Z}_{>0}$; all such singularity can be obtained as an affine Riccati foliation:

$$\Omega = dz + \left(b(x)z^2 + (\frac{1}{x^{k+1}} + \frac{\lambda}{x})z \right) dx$$

  with a ramified section $z = y^\nu$ yielding a Bernoulli saddle-node

$$pdy + b(x)y^{\nu+1}dx + (\frac{1}{x^{k+1}} + \frac{\lambda}{x})ydx.$$

- **Affine without central manifold.** This is actually similar to the purely projective one:

$$\Omega = dz + \left((\frac{1}{x^{k+1}} + \frac{\lambda}{x})z + c(x) \right) dx$$

  with unramified section $z = y$.

Finally, except in the euclidean case with trivial Stokes, we see that the saddle-node is the trace of the singular set of the Riccati foliation on the section, but is never a special point of the Riccati foliation $\mathcal{H}$.

Another important remark is that the projective structure is unique as soon as we have non trivial Stokes matrices. Indeed, there is a unique (up to scalar) formal meromorphic closed 1-form defining the foliation (otherwise we would get a formal meromorphic first integral which would imply the existence of a convergent one); and Stokes matrices are responsible for the divergence of this 1-form. Divergence cannot be killed by ramified covers.

6.4. **Resonant saddles.** Their classification is similar to saddle-nodes case (see [29]) and their projective/affine structures classified in [3, 40]. Such a singular point is formally equivalent to the singular foliation defined by the closed 1-form

$$\omega_0 = \frac{dy}{y} + \frac{df}{f^{k+1}} + \lambda \frac{df}{f}, \quad f = x^p y^q$$

or equivalently by the holomorphic 1-form

$$xyf^k\omega_0 = pydx + qxdy + f^k(p\lambda ydx + q(\lambda+1)xdy).$$

Projective structures of this latter one are given by

$$d\tilde{z} + \left(\frac{dy}{y} + \lambda\frac{df}{f}\right), \quad \tilde{z} = z - \frac{1}{kf^k},$$

(both branches are of Poincaré-Dulac type, i.e. logarithmic resonant) or by

$$\frac{dz}{z} + \frac{df}{f^{k+1}} + \lambda_x\frac{df}{f} + \lambda_y\frac{dy}{y} \quad \text{with } \frac{\lambda_x}{\lambda_y} = \lambda$$

which is irregular on both branches with irregular divisor with multiplicities $k_x = pk$ along $x = 0$ and $k_y = qk$ along $y = 0$ (like for irregular projective structures for the linearizable saddle $df = 0$). Note that residues along $x = 0$ and $y = 0$ do not depend on the eigenvalues (p, q) of the foliation (in the linearizable case we have $\frac{\lambda_x}{p} = \frac{\lambda_y}{q}$).

Obstructions to the convergence of the normalization are again given by non linear Stokes data, which is a collection:

$$\varphi_1(z)\ ,\ \psi_1(z)\ ,\ \ldots\ ,\ \varphi_k(z)\ ,\ \psi_k(z)$$

of k germs of one-variable parabolic diffeomorphisms $\varphi_i(z) = z + \sum_{i\geq 2} b_i z^i$ at $z = 0$, and k germs of one-variable parabolic diffeomorphisms $\psi_i(z) = z + \sum_{i\leq 0} c_i z^i$ at $z = \infty$. This is well-defined up to a cyclic permutation and simultaneous conjugacy by a linear map $z \mapsto az$. Again, together with formal invariants p, q, k, λ, these Stokes data provide a complete set of analytic invariants, i.e. a one-to-one parametrization of the moduli space of resonant saddles (see [29]). According to [40, Proposition II.4.3], a resonant saddle is transversely projective if, and only if, all Stokes are ramified Moebius transformations for the same ramification index $\nu \in \mathbb{Z}_{>0}$:

$$\varphi_i(z) = \frac{z}{(1 + b_i z^\nu)^{1/\nu}} \quad \text{and} \quad \psi_i(z) = (z^\nu + c_i)^{1/\nu};$$

in this case, the projective structure is given (in convenient local coordinates) by

$$\Omega = dz + \left(b(f)z^2 + (\frac{1}{f^{k+1}} + \frac{\lambda}{f})z + c(f)\right)df, \quad f = x^p y^q,$$

with section $z = y^\nu$; this gives the following family of saddles

$$(pydx + qxdy)\left(1 + \lambda f^k + f^{k+1}(b(f)y^\nu + \frac{c(f)}{y^{-\nu}})\right) + \nu f^k xdy$$

(where $c(f)$ is assumed to vanish at sufficiently large order so that $\frac{c(f)}{y^{-\nu}}$ is holomorphic). From [3], the projective structure is actually affine if, and only if, either all φ_i, or all ψ_i are trivial (the identity), which is equivalent to say that either $b(x) \equiv 0$, or $c(x) \equiv 0$ respectively. On $\{x = 0\}$ (resp. $\{y = 0\}$), the pole of the projective structure is irregular unramified with irregularity index $k_x = pk$ (resp. $k_y = qk$) and appears in the irregular locus I with multiplicity k_x (resp. k_y). In fact, the Riccati/connection is the same as in the linearizable saddle case since a general section cut-out the Riccati foliation as a linearizable saddle singularity (e.g. $z = 1$). Clearly, there are non trivial Stokes along $x = 0$ if, and only if, there are non trivial Stokes along $y = 0$.

7. Riccati foliations over surfaces

From the results of the previous two sections we will deduce below the existence of privileged birational models for Riccati foliations over projective surfaces.

Sabbah proved a similar result in [36]. He defines a notion of good formal model at a point of a normal crossing divisor allowing to define Stokes matrices, proved that this was the case for a flat connection outside of a codimension two set, and conjectured that this can be achieved along all the irregular divisor after blowing-up the base manifold. This has been proved in the surface case in [36] (up to rank 5) and by Y. André in [1] (in any rank); this has been generalized to arbitrary dimension independently by Kedlaya and Mochizuki (only projective case) [19, 31]. Here, in the surface and rank 2 case, we prove it using an auxiliary foliation and the results of [3, 40].

Theorem 7.1. *Given a $\mathbb{P}^1$-bundle $\pi : P \to X$ over a projective surface X, and a foliation $\mathcal{H}$ transverse to the general fiber of π then after blowing-up the surface and applying a birational bundle transformations on the bundle P, we can assume that the polar divisor of $\mathcal{H}$ has at worst normal crossings, minimal order of poles and local models along the smooth part of the divisor as in Section 5. Moreover, at a neighborhood of each normal crossing, we have one of the following models for $\mathcal{H}$:*

(1) $\frac{dz}{z} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$ *where* $\lambda_x, \lambda_y \in \mathbb{C} \setminus \mathbb{Z}$*;*
(2) $dz + \frac{dy}{y} + \lambda \frac{dx}{x}$ *where* $\lambda \in \mathbb{C}^*$*;*
(3) $\frac{dz}{z} + \frac{dx}{x^{k+1}} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$ *where* $\lambda_x, \lambda_y \in \mathbb{C} \setminus \mathbb{Z}$*;*
(4) $\frac{dz}{z} + \frac{df}{f^{k+1}} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y}$ *with* $f = x^p y^q$ *(a pull-back of the previous case);*
(5) $dz + \left(z^2 \frac{df}{f^{k+1}} - (k+1) z \frac{df}{f} + \phi(f) \frac{df}{f^{k+1}} \right)$ *with* ϕ *holomorphic,* $f = x^p y^q$ *and* $\phi(0) \neq 0$ *(unramified case);*
(6) $dz + (z^2 + x^{\epsilon_1} y^{\epsilon_2} \phi(f)) \frac{df}{x^{\tilde{p}} y^{\tilde{q}} f^k} - \left(\tilde{p} \frac{dx}{x} + \tilde{q} \frac{dy}{y} + k \frac{df}{f} \right)$ *with* f *and* ϕ *as the previous case,* $\epsilon_1, \epsilon_2 = 0, 1$ *and* $(p, q) = (2\tilde{p} - \epsilon_1, 2\tilde{q} - \epsilon_2)$ *(ramified case).*

If besides blowing-up the surface we also allow taking ramified Galois coverings of X then we can moreover assume that $p = q = 1$ in items (4) and (5), and that the ramified case (6) does not occur.

Proof. Start by choosing a rational section $\sigma : X \dashrightarrow P$ which is not invariant by $\mathcal{H}$. Consider the transversely projective foliation $\mathcal{F} = \sigma^* \mathcal{H}$ and apply Seidenberg's Theorem to obtain a composition of blow-ups $r : \tilde{X} \to X$ such that $\tilde{\mathcal{F}} = \pi^* \mathcal{F}$ is a reduced foliation. The pull-back of $(P, \mathcal{H}, \sigma)$ under r, $(\tilde{P}, \tilde{\mathcal{H}}, \tilde{\sigma})$ is a transversely projective structure for $\tilde{\mathcal{F}}$.

Apply the main result of [23] to obtain a projective structure for $\tilde{\mathcal{F}}$ birationally equivalent to $(\tilde{P}, \tilde{\mathcal{H}}, \tilde{\sigma})$ and in minimal form. The local models of the Riccati foliation over $\tilde{X}$ along the divisor are exactly as in the list of Section 5 at smooth points, and in the above list at corners. Note that (6) is deduced from (5) in the ramified case where ϕ has a zero of order one by performing an additional bundle transformation to reach minimal model. This proves the first part of the result.

To prove the second part (where we allow ramified coverings), we first collect all the irreducible components of the polar divisor which are irregular and ramified. Let D be the reduced divisor with the very same support and construct a Kawamata covering ([20, Proposition 4.1.12]) $f : Y \to \tilde{X}$ such that $f^* D = 2D$. The resulting Riccati foliation over Y no longer has ramified irregular poles. Finally, if I the irregular divisor is a multiple of a reduced divisor then $p = q$ in items (4) and (5). Otherwise we can construct a Kawamata covering $g : Z \to Y$ such that the pull-back of $\mathcal{H}$ under $f \circ g$ will have this property. □

8. Irregular divisor

Throughout the section, $(P, \mathcal{H})$ is a **reduced** Riccati foliation over a projective surface S, i.e. $\mathcal{H}$ is as in the conclusion of Theorem 7.1: it is in minimal form and with local models over the singularities of the polar divisor as in (1), (2), (3), (4), (5), or (6) of Theorem 7.1. We study the irregular divisor of $\mathcal{H}$.

8.1. **Self-intersection.** We start by noticing that the irregular part of the polar divisor of a reduced Riccati foliation has zero self-intersection.

Proposition 8.1. *Let $(P, \mathcal{H})$ be reduced Riccati foliation on a projective surface S. If I is the irregular part of the polar divisor of $\mathcal{H}$, then $I^2 = 0$.*

Proof. Although this is mainly a consequence of Proposition 8.2, we give a direct and short proof using an auxiliary foliation $\mathcal{F}$ on S defined as the pull-back of $\mathcal{H}$ by a general section $\sigma : S \dashrightarrow P$. If $p \in \mathrm{sing}(\mathcal{F}) \cap |I|$ is a saddle-node then only the strong separatrix through p is contained in $|I|$. Moreover, the Camacho-Sad index of $\mathcal{F}$ along this strong separatrix is zero.

Write I as $\sum k_i I_i$ where I_i are irreducible hypersurfaces and k_i are strictly positive integers. If $p \in \mathrm{sing}(\mathcal{F}) \cap |I|$ is a saddle then both separatrices are contained in $|I|$. If $A \subset I_i$ and $B \subset I_j$ are the separatrices of $\mathcal{F}$ through p then, see §6.4,

$$CS(\mathcal{F}, A, p) = -\frac{k_j}{k_i} \quad \text{and } CS(\mathcal{F}, B, p) = -\frac{k_i}{k_j}\,.$$

Using Camacho-Sad index theorem we obtain that

$$\begin{aligned} I^2 &= \sum_i \kappa_i^2 I_i^2 + \sum_{i \neq j} (\kappa_i \cdot \kappa_j) I_i \cdot I_j \\ &= \sum_i \sum_{j \neq i} -(\kappa_i \cdot \kappa_j) I_i \cdot I_j + \sum_{i \neq j} (\kappa_i \cdot \kappa_j) I_i \cdot I_j = 0 \end{aligned}$$

as wanted. □

8.2. **Flat coordinates.** It turns out that the irregular divisor has not only zero self-intersection, but it also has torsion normal bundle. Indeed something even stronger holds true as proved in the proposition below. Not all the hypothesis are really necessary, but the statement in this form is all that we need for the applications we have in mind.

Proposition 8.2. *Let $(P, \mathcal{H})$ be a reduced Riccati foliation on a projective surface S. Let D be a connected component of the irregular divisor I and assume*

1. *the support of D is also a connected component of the suppoort of the polar divisor $(\mathcal{H})_\infty$ (i.e. D intersects no regular component);*
2. *D is a multiple of a reduced divisor, i.e. $D = k \cdot D_{red}$;*
3. *the set*
$$\overline{\mathrm{sing}(\mathcal{H}) \cap \pi^{-1}(D - \mathrm{sing}(D))}$$
has two distinct connected components.

Then the normal bundle of D_{red} is torsion of order r with r dividing k. Moreover, $\mathcal{O}_S(rD_{red})_{|kD_{red}} \simeq \mathcal{O}_{kD_{red}}$.

Proof. Let $\pi : P \to S$ be the projection of P to S and take an open covering of a neighborhood of D by sufficiently small open sets U_i. We are going to work at the formal completion of S along D.

Over each U_i choose (analytic) coordinates x_i, y_i in U_i and (formal) coordinate z_i in $\pi^{-1}(U_i)$ such that

- the restriction $|D| \cap U_i$ is defined by $\{f_i = 0\}$ with $f_i = x_i$ or $f_i = x_i y_i$;
- the restriction $\mathcal{H}_{|U_i}$ is defined by the closed formal 1-form

$$\Omega_i = \frac{dz_i}{z_i} + \left(\lambda \frac{df_i}{f_i} + \frac{df_i}{f_i^{k+1}}\right).$$

Over intersections $U_i \cap U_j$ we have $f_i(x_i, y_i) = f_{ij}(x_j, y_j) \cdot f_j(x_j, y_j)$, and we can write

$$f_{ij}(x_j, y_j) = (\beta_0 + \beta_1 f_j + \beta_2 f_j^2 + \cdots),$$

where β_l are formal power series in $\mathbb{C}[[x_j, y_j]]$. The β_l are not canonically determined but their restrictions to $\{f_j = 0\}$ are. Since Ω_i and Ω_j define the same foliation over $U_i \cap U_j$ we have that $z_i = g_{ij}(x_j, y_j) \cdot z_j^{\varepsilon}$ with $\varepsilon = \pm 1$ (the poles must be preserved since the foliation does not admit a (formal) meromorphic first integral at the intersections). Our hypothesis on the singularities of $\mathcal{H}$ over I allow us to take $\varepsilon = +1$. Thus assume that $z_i = g_{ij}(x_j, y_j) \cdot z_j$. Expanding Ω_i we obtain

$$0 = \Omega_i - \Omega_j = \left(\frac{dg_{ij}}{g_{ij}} + \lambda \frac{df_{ij}}{f_{ij}} + \frac{1}{f_j^k} \frac{df_{ij}}{f_{ij}^{k+1}} + \left(\frac{1}{f_{ij}^k} - 1\right) \frac{df_j}{f_j^{k+1}}\right)$$

We deduce that

$$g_{ij} = C_{ij} \cdot \exp\left(-\int \left(\lambda \frac{df_{ij}}{f_{ij}} + \frac{1}{f_j^k} \frac{df_{ij}}{f_{ij}^{k+1}} + \left(\frac{1}{f_{ij}^k} - 1\right) \frac{df_j}{f_j^{k+1}}\right)\right)$$

for some constant $C_{ij} \in \mathbb{C}^*$.

Notice that the integrand is the sum of the regular 1-form, $\lambda \frac{df_{ij}}{f_{ij}}$, with

$$\frac{1}{f_j^k} \frac{df_{ij}}{f_{ij}^{k+1}} + \left(\frac{1}{f_{ij}^k} - 1\right) \frac{df_j}{f_j^{k+1}} = \frac{1}{f_{ij}^{k+1}} \underbrace{\left(\frac{df_{ij}}{f_j^k} + \left(f_{ij} - f_{ij}^{k+1}\right) \frac{df_j}{f_j^{k+1}}\right)}_{\Theta}.$$

Since g_{ij} has no essential singularities we deduce that Θ, the 1-form over braces, cannot have poles of order ≤ 2.

Notice that the residue of $f_j^k \Theta$ along $\{f_j = 0\}$ is nothing but $\beta_0 - \beta_0^{k+1} \mod f_j$. Therefore $\beta_0^k = 1 \mod f_j$. It follows that the normal bundle of I is torsion of order r for some r dividing k.

Similarly, the residue of $f_j^{k-1}\Theta$ is $2\beta_1 - (k+1)\beta_0^k \beta_1 = -(k-1)\beta_1 \mod f_j$. If $k > 1$ then $\beta_1 = 0 \mod f_j$. More generally, if $1 \leq \ell < k$ and $\beta_1 = \ldots = \beta_{\ell-1} = 0 \mod f_j$ then the residue of $f_j^{k-\ell}\Theta$ is

$$\beta_\ell(\ell + 1) - (k+1)\beta_0^k \beta_\ell = (\ell - k)\beta_\ell \mod f_j$$

Proceeding inductively we deduce that $\beta_\ell = 0$ for every $1 \leq \ell < k$. This establishes the proposition. □

8.3. **Monodromy around the irregular divisor.**

Proposition 8.3. *Let $(P, \mathcal{H})$ be a reduced Riccati foliation on a projective surface S. Let D be a connected component of the irregular divisor I of the form $D = kD_{red}$. Then, there is a neighborhood V of $|D|$ in S in restriction to which the monodromy of $(P, \mathcal{H})$ is virtually abelian and, perhaps passing to a étale covering $\tilde{V} \to V$ of degree two, at least one of the following assertions holds true.*

1. *The Riccati foliation $\mathcal{H}$ is defined by closed meromorphic 1-form over V.*
2. *The supports of $(\mathcal{H})_\infty$ and D coincide, and there exists a C^∞-fibration $f : V - D \to \mathbb{D}^*$ which factors the monodromy.*

Proof. Let V be the connected component of a union of sufficiently small tubular neighborhoods of the irreducible components of $(\mathcal{H})_{infty}$ containing D, and let $R : V \to |D|$ be a deformation retract. Since the statement is local, from now one $\mathcal{H}$ will be seen as a Riccati foliation defined over V.

Let us first look at the set

$$Z = \overline{\mathrm{sing}(\mathcal{H}) \cap \pi^{-1}(|(\mathcal{H})_\infty| - \mathrm{sing}(|(\mathcal{H})_\infty|))}.$$

The projection $\pi_{|Z} : Z \to |(\mathcal{H})_\infty)|$ is étale of degree two. The monodromy of $\pi_{|Z}$ is a representation $\pi_1(|(\mathcal{H})_\infty|) \to \mathbb{Z}/2\mathbb{Z}$ and since $|(\mathcal{H})_\infty|$ and S have the same homotopy type, it induces a representation $\pi_1(S) \to \mathbb{Z}/2\mathbb{Z}$. After passing to the étale covering determined by this representation we can assume Z has two distinct irreducible components.

Assume first that $(\mathcal{H})_\infty$ and D have distinct support. Let p be a point in the intersection of an irregular and a regular component of $(\mathcal{H})_\infty$. Since $(P, \mathcal{H})$ is reduced, it follows from Theorem 7.1 that $\mathcal{H}$ is defined by

$$\Omega = \frac{dz}{z} + \frac{dx}{x^k + 1} + \lambda_x \frac{dx}{x} + \lambda_y \frac{dy}{y} \, ,$$

with $\lambda_x, \lambda_y \in \mathbb{C} - \mathbb{Z}$ and $k > 0 \in \mathbb{Z}$. In particular, the sections $\{z = 0\}$ and $\{z = \infty\}$ are $\mathcal{H}$-invariant. The local product structure of $\mathcal{H}$ over general points of the polar divisor implies that at a neighborhood of D we still have two sections or one two-valued section invariant by $\mathcal{H}$. The existence of a two-valued irreducible section is impossible since Z has two distinct irreducible components. Now we can construct a bimeromorphic gauge transformation over a neighborhood V of D sending these two sections to $\{z = 0\}$ and $\{z = \infty\}$. It is now clear that $\mathcal{H}$ is defined by a closed meromorphic 1-form.

Let us now analyze the case where $(\mathcal{H})_\infty$ and D have the same support. In this case, $\mathcal{P}$ satisfy all the assumptions of Proposition 8.2. If all the Stokes matrices along D are trivial then we can conclude as in the previous case, since we do have two meromorphic sections on a neighborhood of $|D|$.

Let $\mathcal{V} = \{V_i\}$ be a sufficiently fine open covering of V. Let $f_i \in \mathcal{O}_S(V_i)$ be holomorphic functions defining D_{red} such that $\mathcal{H}_{|V_i}$ is defined by

$$\Omega_i = \frac{dz_i}{z_i} + \left(\frac{\alpha_i}{z_i} + \lambda \frac{df_i}{f_i} + \frac{df_i}{f_i^{k+1}} + z_i \gamma_i \right)$$

with α_i and γ_i vanishing on $|D|$ up to some fixed high order. The proof of Proposition 8.2 shows that the transition functions $\{f_{ij} \in \mathcal{O}_V^*(V_i \cap V_j)\}$ are equal to a r^{th}-root of the unity when restricted to $|D|$. Therefore we have an induced representation $\psi : \pi_1(|D|) \to \mathbb{C}^*$ taking values in values in the group of r^{th}-roots of the unity, which describe how the differentials df_i change when we follow closed paths on D.

The transition functions $\{f_{ij} \in \mathcal{O}_V^*(V_i \cap V_j)\}$ define an element in $H^1(V, \mathcal{O}_V^*)$ which corresponds to $\mathcal{O}_V(D)$. Let $\mathcal{A}_V$ and $\mathcal{A}_V^*$ denote, respectively, the sheaves of C^∞ functions and of C^∞ invertible functions. Notice that $H^1(V, \mathcal{A}_V^*)$ is isomorphic to $H^2(V, \mathbb{Z})$, as $\mathcal{A}_V$ has no cohomology in positive degree ($\mathcal{A}_V$ is a fine sheaf); and the restriction morphism from $H^2(V, \mathbb{Z})$ to $H^2(|D|, \mathbb{Z})$ is an isomorphism since the inclusion of $|D|$ in V is a homotopy equivalence.

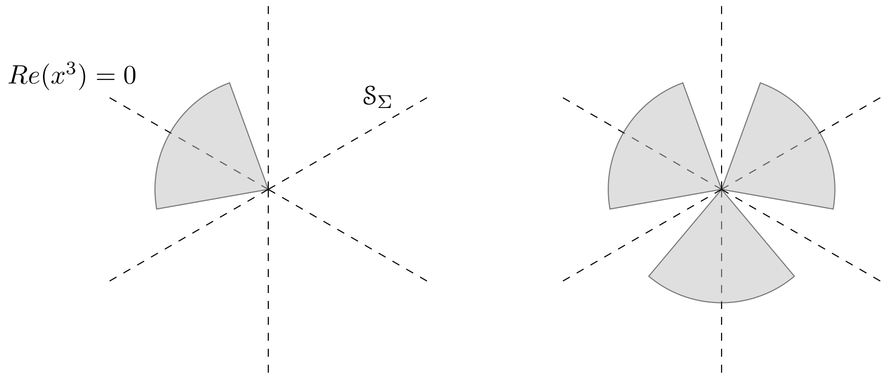
Since $f^r_{ij}{}_{||D|\cap V_i} = 1$, the element $\{f^r_{ij}\} \in H^1(V, \mathcal{O}_V^*)$ maps to the trivial element of $H^1(V, \mathcal{A}_V^*) \simeq H^2(V, \mathbb{Z}) \simeq H^2(|D|, \mathbb{Z})$. Thus we can find non-vanishing C^∞-functions $\{g_i \in \mathcal{A}_V^*(V_i)\}$ such that $f^r_{ij} = \frac{g_i}{g_j}$. Moreover, we can choose the functions g_i satisfying $g_{i||D|\cap V_i} = 1$. Our assumptions imply that we can further assume that the function f^r_{ij} are constant equal to one when restricted to D.

Therefore we can define a C^∞ function $f : V \to \mathbb{C}$ by the formulae $f_{|V_i} = (f_i)^r/g_i$. The function f clearly satisfies $f^{-1}(0) = |D|$ set-theoretically. We claim that, after perhaps shrinking V, the critical set of f is contained in $|D|$. At a neighborhood of a smooth point of $|D|$ the function f is a power of a submersion. At a neighborhood of a singular point of H, the function f is of the form $h(x, \overline{x}, y, \overline{y})xy$ and therefore $df = xydh + h(ydx + xdy)$. Since this expression has an isolated singularity at zero (the singular point of $|D|$) it follows that the critical set of f is indeed contained in D. Replacing V by $f^{-1}(\mathbb{D}_\varepsilon)$ for a sufficiently small ε we have just proved the existence of a C^∞ proper map $f : V \to \mathbb{D}$ from V to the disk of radius ε which maps $|D|$ to the origin in $\mathbb{D}_\varepsilon$ and when restricted to $V - |D|$ becomes a locally trivial fibration over $\mathbb{D}^*$. Moreover, the construction of f respects the representation ψ, in the sense that $R_{|F} : F \to |D|$, the restriction of the deformation retract $R : V \to |D|$ to a general fiber F of f, has monodromy given by ψ.

Let Σ be a germ of curve transversal to D at a smooth point $x_0 \in D$. Let $\pi : P \to V$ be the natural projection. The restriction of $\mathcal{H}$ to $\pi^{-1}(\Sigma)$, is a Riccati foliation over Σ with an invariant fiber $\{x_0\} \times \mathbb{P}^1$ having two saddle-nodes over it. As explained in Section 5.5, if k is the order of D at x_0 then there are $2k$ closed sectors on Σ, such that over the interior of each of them, the Riccati foliation is analytic conjugated to $\frac{dz}{z} + \frac{dx}{x^{k+1}} + \lambda \frac{dx}{x}$ and the conjugation extends continuously to the boundary. Over each of these sectors there are exactly two leaves with distinguished topological behavior: the closure of each of these distinguished leaves intersect the central fiber $\{x_0\} \times \mathbb{P}^1$ at a unique point.

Let $\mathcal{S}_\Sigma$ be the interior of one of these sectors. The local triviality of $\mathcal{H}$ along the smooth part of D, and the local normal form of $\mathcal{H}$ at the singularities of D, allow us to construct an open set $\mathcal{S} \subset V - |D|$ which extends $\mathcal{S}_\Sigma$ and over which we never lose sight of the two distinguished leaves of the restriction of $\mathcal{H}$ to $\pi^{-1}(\mathcal{S}_\Sigma)$. In particular, the restriction of the monodromy representation of $\mathcal{P}$ to $\mathcal{S}$ has its image (up to conjugation) contained in $\mathbb{C}^*$ as $\mathcal{H}$ over $\mathcal{S}$ has two distinguished leaves which are not permuted because of the assumption on $\mathrm{sing}(\mathcal{H})$. We can choose $\mathcal{S}_\Sigma$ sufficiently small in such a way that the intersection of the resulting open set $\mathcal{S}$ and the initial transversal Σ has $r = \#\psi(\pi_1(|D|))$ distinct connected components.

By construction the set $\mathcal{S}$ has the same homotopy type as a general fiber of $f_{|V-|D|} : V - |D| \to \mathbb{D}^*$. Therefore $\pi_1(\mathcal{S})$ is a normal subgroup of $\pi_1(V - |D|)$. If we do this construction choosing base points at two adjacent sectors with non



The sector $\mathcal{S}_\Sigma$. The intersection of $\mathcal{S}$ and Σ.

FIGURE 2.

trivial Stokes transition matrix we conclude that indeed the monodromy of $\mathcal{P}$ on $\mathcal{S}$ is trivial since a nontrivial Stokes matrix at one hand conjugates the corresponding monodromy representations, and at the other hand it does not respect the fixed points of the two monodromy representations. We conclude that in the presence of a nontrivial Stokes matrix the monodromy factors through $f_* : \pi_1(V - |D|) \to \pi_1(\mathbb{D}^*)$ as wanted. □

Lemma 8.4. *Let $(P, \mathcal{H}), D, k$, and r be as in the statement of Proposition 8.3. Assume that $\mathcal{H}$ is not given by a closed meromorphic 1-form at the neighborhood V of D. If $r = k$, i.e. if $\mathcal{O}_S(jD_{red})_{|kD_{red}}$ is not isomorphic to $\mathcal{O}_{kD_{red}}$ for every j satisfying $0 < j < k$ and $\mathcal{O}_S(kD_{red})_{|kD_{red}} \simeq \mathcal{O}_{kD_{red}}$ then the monodromy of $(P, \mathcal{H})$ is non-trivial.*

Proof. Notation as in the proof of Proposition 8.3. Let $\gamma \in |D|$ a loop such that $\psi(\gamma) = \exp(2\pi i/k)$. We can lift γ to an open path $\tilde{\gamma}$ in a region $\mathcal{S}$ such that the initial and final points lie in two consecutive small sectors. Moreover, since $\mathcal{H}$ is not given by a closed meromorphic 1-form we can assume that at least one of the two distinguished leaves over the initial sector is not a distinguished leaf for the final sector. Closing the path $\tilde{\gamma}$ using an arc on the transversal Σ, we obtain a path in $V - |D|$ with non-trivial monodromy. □

9. STRUCTURE

9.1. **Proof of Theorem C.** Let $(P, \mathcal{H})$ be a Riccati foliation over the projective manifold X with irregular singular points. Assuming that it is not defined by a closed 1-form after a finite covering, our aim is to prove that it factors through a curve.

According to Proposition 4.9, is suffices to prove the factorisation of $(P, \mathcal{H})$ in restriction to a general surface $S \subset X$. We can therefore apply the reduction process of Sections 4.8 and 6: after passing to a finite cover, blowing-up and applying birational bundle transformation, we can assume that the Riccati foliation $(P, \mathcal{H})$ satisfies all the conclusions of Theorem 7.1. To wit,

(1) the polar locus $(\mathcal{H})_\infty$ is minimal;

(2) we have local models for $(P,\mathcal{H})$ (in particular locally factorizing through a disk all along $(\mathcal{H})_\infty$); and
(3) all irregular poles are unramified with the same irregularity k.

According to Proposition 4.6 it is enough to prove the factorization after this reduction process.

Let D be a connected component of the irregular divisor I: we have $D = kD_{red}$. It follows from Proposition 8.1 that $D \cdot D = 0$, and from Proposition 8.3 that the monodromy of $(P,\mathcal{H})$ is virtually abelian at the neighborhood V of D.

Assume first that the global monodromy of $(P,\mathcal{H})$ on S is not virtually abelian, in particular strictly larger than the local monodromy around D

$$\rho(\pi_1(S - |(\mathcal{H})_\infty|)) \neq \rho(\pi_1(V - D)).$$

Then arguments involved in the proof of Theorem 3.1 show that D is the fiber of a fibration $f : S \to C$ (we use extra topology in $S - V$ to construct a ramified cover with several disjoint copies of D and then apply Theorem 2.2). Like in the proof of Theorem 3.1, the monodromy of a general fiber of f is a normal subgroup of the global monodromy group and is therefore trivial. Thus, the monodromy representation factors through f. But the general fiber of f does not intersect the irregular divisor and we can apply Proposition 4.8 to conclude that the Riccati foliation factors as well.

We can now assume that the global monodromy is virtually abelian, and after passing to a finite covering, that it is torsion free. Thus the global monodromy either abelian and infinite, or trivial.

Suppose first that the monodromy is abelian and infinite. We claim that D is the fiber of a fibration. Indeed, if $\rho(\pi_1(S-|(\mathcal{H})_\infty|)) \neq \rho(\pi_1(V-D))$, then we can proceed like above and construct a ramified cover containing at least two disjoint copies of D. The existence of the sought fibration follows from Theorem 2.2. If $\rho(\pi_1(S-|(\mathcal{H})_\infty|)) = \rho(\pi_1(V-D))$, then let us analyze the two possible outcomes of of Proposition 8.3. In case (1) of Proposition 8.3, there are no Stokes and $\mathcal{H}$ is defined by a closed 1-form Ω near D; since Ω is invariant under the local/global monodromy, it extends everywhere on X outside the irregular divisor. Having assumed that $\mathcal{H}$ is not globally defined by a closed rational 1-form, there should be another connected component for the irregular divisor and the fibration comes from Theorem 2.2. In conclusion (2) of Proposition 8.3, we have non-trivial Stokes matrices and the monodromy near D comes from a small loop around D (via a C^∞-fibration). According to [2, Proposition 3.6] there exists a torsion character $\tau \in H^1(S,\mathbb{C}^*)$, a closed logarithmic 1-form $\eta \in H^0(S,\Omega^1_S(\log|(\mathcal{H})_\infty)|)$, and $\delta \in H^1(S,\mathbb{R})$ such that

$$\rho(\gamma) = \tau(\gamma) \cdot \int_\gamma (\eta + i\delta)\,.$$

Since the monodromy around D is infinite by assumption it follows that η has non-zero residues along D. Thus the residue theorem applied to η implies the existence of another divisor with support in $|(\mathcal{H})_\infty| - |D|$ with Chern class proportional to the Chern class of D. We can again conclude using Theorem 2.2.

We have just proved, in the case monodromy is abelian and infinite, that D is fiber of a fibration $f : S \to C$. It remains to show that the Riccati foliation $\mathcal{H}$ factors. When the monodromy is trivial along a generic fiber of f, this clearly follows from Proposition 4.8. If not, we have infinite monodromy along fibers and, by looking at a fiber close to D, we must be in case (1) of Proposition 8.3; in particular, the monodromy takes values in $\mathbb{C}^*$, i.e. has two fixed points. Over each fiber, we have exactly two sections of the $\mathbb{P}^1$-bundle P that are invariant by the Riccati foliation $\mathcal{H}$. The sections are tangent to the Riccati foliation $\mathcal{H}$ and also to the pull-back to P of the foliation defined by the fibration. We obtain two curves in the Hilbert scheme of P. Since tangency to foliations define closed subscheme of the Hilbert scheme, these two curves have Zariski closure of dimension one. Thus they spread two surfaces, sections of P, which are invariant by $\mathcal{H}$. After birational bundle transformation, we can assume $P = S \times \mathbb{P}^1_z$ (the trivial bundle) with the two sections $\{z = 0\}$ and $\{z = \infty\}$. Then the Riccati 1-form $\Omega = dz + \alpha z^2 + \beta z + \gamma$ defining $\mathcal{H}$ satisfies $\alpha = \gamma = 0$ and integrability condition shows that $\frac{\Omega}{z}$ is closed, contradiction.

Finally, it remains to consider the case where the monodromy of the Riccati foliation $\mathcal{H}$ is trivial. Like in the previous case, if $\mathcal{H}$ is defined by a closed 1-form at the neighborhood of D, then it extends as a global 1-form except if there is another irregular polar component, in which case we get a fibration by Theorem 2.2 and $\mathcal{H}$ factors by Proposition 4.8. On the other hand, if we are in case (2) of Proposition 8.3, then Lemma 8.4 implies that $r < k$, i.e. the order of the normal bundle of D_{red} is strictly smaller than the multiplicity of the irregular divisor. The existence of a fibration with a fiber supported on $|D|$ follows from Theorem 2.3, and the Riccati foliation factors by Proposition 4.8. □

We point out that in the presence of Stokes phenomena, an irregular Riccati foliation is never defined by a closed rational 1-form. There are many explicit examples in the literature and the most famous of them is undoubtly Euler's equation. It can be interpreted as a Riccati foliation over $\mathbb{P}^1$ defined by the rational 1-form

$$dz - \frac{(z-x)}{x^2}dx.$$

The fiber over $\{x = 0\}$ is irregular unramified with non-trivial Stokes since the weak separatrix through $(0, 0)$ is divergent. Let us register also, a simple example which shows there are irregular Riccati foliations which are defined by closed rational 1-forms but which do not factor though curves.

Example 9.1. Let $X = A \times B$ be the product of an abelian variety A and a smooth projective curve B. Let α be a non-zero holomorphic 1-form on A and $\beta = df$ be the differential of a rational function on B. Consider the trivial $\mathbb{P}^1$-bundle $P = X \times \mathbb{P}^1$ over X, and on it the family of Riccati foliations $\mathcal{H}_\lambda$, $\lambda \in \mathbb{C}$, defined by

$$dz + \omega(1 + \lambda z^2), \text{ where } \omega = \alpha + \beta.$$

The Riccati foliation $\mathcal{H}_0$ is regular, and for $\lambda \in \mathbb{C}^*$ the Riccati foliation $\mathcal{H}_\lambda$ is irregular with irregular divisor equal to the divisor of poles of the pull-back of f to X. The monodromy of $\mathcal{H}_\lambda$, for every $\lambda \in \mathbb{C}$, factors through the natural projection $A \times B \to A$ and is non-trivial. Thus, for any $\lambda \neq 0$, $\mathcal{H}_\lambda$ is an example of irregular Riccati foliation which do not factor through a curve (even if its irregular divisor is supported on fibers of a fibration).

The existence of the fibration with fibers containing the support of the irregular divisor in the example above is just a coincidence, and not a general phenomenon. Let Y be a projective manifold and consider a representation $\rho : \pi_1(Y) \to (\mathbb{C}, +)$. It determines a cohomology class $[\rho] \in H^1(Y, \mathbb{C})$. If its image under the natural morphism $H^1(Y,\mathbb{C}) \to H^1(Y,\mathcal{O}_Y)$ is non zero then it determines a non trivial extension $0 \to \mathcal{O}_Y \to E \to \mathcal{O}_Y \to 0$, endowed with a flat connection with monodromy given by ρ. The projectivization $X = \mathbb{P}(E)$ is a $\mathbb{P}^1$-bundle over Y with a Riccati foliation defined by a closed rational 1-form ω with polar divisor equal to 2Δ, where Δ is image of the unique section $Y \to \mathbb{P}(E)$. If P is the trivial $\mathbb{P}^1$-bundle P over X then we have a family of Riccati foliations $\mathcal{H}_\lambda$, $\lambda \in \mathbb{C}$, on it defined by

$$dz + \omega(1 + \lambda z^2).$$

As in the example above, the Riccati foliation $\mathcal{H}_\lambda$ is irregular for $\lambda \neq 0$, does not factor through a curve, and its irregular divisor is not a fiber of a fibration. If we take Y equal to an elliptic curve, then $X - |\Delta|$ is nothing but Serre's example of Stein quasi-projective surface which is not affine.

9.2. **Proof of Theorem D.** Let $\mathcal{F}$ be a transversely projective foliation on a projective manifold X with transverse projective structure $\mathcal{P} = (P, \mathcal{H}, \sigma)$. If $\mathcal{P}$ is regular then the monodromy representation determines $\mathcal{H}$ up to birational bundle transformations. If the monodromy is virtually abelian, then after replacing X by a (resolution of) a ramified Galois covering we get that $\mathcal{H}$ is defined by a closed rational 1-form and the same holds for $\mathcal{F}$. If the monodromy is not virtually abelian then Corollary B combined with Propositions 4.8 and 4.6, implies that $\mathcal{F}$ is the pull-back of a Riccati foliation over a curve or that $\mathcal{H}$ comes from one of the tautological $\mathbb{P}SL(2,\mathbb{C})$-representations on a polydisk Shimura modular orbifold. This proves the Theorem in the regular case. The irregular case follows from Theorem C. □

We will now present some examples which show that the conclusions of Theorem D are sharp.

There are many examples of foliations defined by closed rational 1-forms which are not pull-backs of foliations on lower dimensional manifolds. Perhaps the simplest examples are linear foliations on simple abelian varieties.

Lemma 9.2. *Let A be a simple abelian variety. Let $\omega \in H^0(A,\Omega^1_A)$ be a nonzero 1-form. If $i : Z \to A$ is a morphism from a projective manifold then $i^*\omega = 0$ if and only if the image $i(Z)$ reduces to a point.*

Proof. If A is simple then the same holds true for $\mathrm{Alb}(A)$. Let $i_* : \mathrm{Alb}(Z) \to \mathrm{Alb}(A)$ the natural morphism induced by i. Since $i^*\omega = 0$ for a nonzero $\omega \in H^0(A,\Omega^1_A)$, i_* is not a surjective morphism. But since $\mathrm{Alb}(A)$ is simple, the image of i_* must be a point. This clearly implies that $i(Z)$ is also a point as wanted. □

Example 9.3. Let A be a simple abelian variety and X a submanifold of A. If $\mathcal{F}$ is a foliation on X such that $\mathcal{F}$ is the pull-back of some foliation $\mathcal{G}$ under a rational map $f : X \dashrightarrow Y$ on some lower dimensional manifold Y then the leaves of $\mathcal{F}$ contain subvarieties coming from the closures of fibers of f. Lemma 9.2 implies that every foliation on X defined by the restriction of 1-forms on A to X is not a pull-back from a lower dimensional manifold.

The simpleness of A is not really essential here, we can instead assume that 1-form ω is sufficiently general. For instance if the leaves of the foliation defined by ω on A are all biholomorphic to $\mathbb{C}^{n-1}$ then they cannot contain compact subvarieties by the maximal principle and the induced foliation on A (or on any submanifold $X \subset A$) is not the pull-back of some foliation on a lower dimensional manifold.

In a similar vein, we can consider foliations on $\mathbb{P}^n$ (or submanifolds of $\mathbb{P}^n$) defined by logarithmic 1-forms with poles on $n+1$ hyperplanes in general position.

For sufficiently general residues, the leaves not contained in the union of the $n+1$ hyperplanes are also biholomorphic to $\mathbb{C}^{n-1}$ and have no positive dimensional subvarieties contained in them.

All these foliations admit a 1-parameter family of pairwise non-equivalent transversely projective structures.

Example 9.4. Let $n \geq 2$ and let X be the quotient of $\mathbb{H}^n$ by cocompact torsion free irreducible lattice $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})^n$. The natural projections $\mathbb{H}^n \to \mathbb{H}$ define n codimension one smooth foliations on X which are transversely projective (indeed transversely hyperbolic). Contrary to what have been state by the second author and Mendes in [30, Theorem 1], countably many leaves of these foliations may have nontrivial topology (with fundamental groups isomorphic to isotropy groups of the action of Γ on the corresponding one dimensional factor of $\mathbb{H}^n$), but the very general leaf is biholomorphic to $\mathbb{H}^{n-1}$. Again the maximal principle tell us that the general leaf cannot contain positive dimensional subvarieties, and consequently the foliations are not pull-backs from lower dimensional manifolds.

Again the assumptions on Γ can be considerably weakened. All we have to ask is that Γ is an irreducible lattice of $\mathrm{PSL}(2,\mathbb{R})^n$ for some $n \geq 2$. Notice that the rigidity theorem of Margulis (resp. the classification of representations by Corlette and Simpson) implies that all these lattices are commensurable to (resp. conjugated to a subgroup of) arithmetic lattices of the form $\mathcal{U}(P,\Phi)/\pm \mathrm{Id}$ for some totally imaginary quadratic extension L of a totally real number field F, some rank two projective $\mathcal{O}_L$-module P and some skew Hermitian form Φ. Besides the n representations coming from the n projections $\pi_1^{orb}(X) \simeq \Gamma \subset \mathrm{PSL}(2,\mathbb{R})^n \to \mathrm{PSL}(2,\mathbb{R})$, we also have $[L:\mathbb{Q}]-2n$ representations of $\pi_1^{orb}(X)$ with values in $\mathrm{PSL}(2,\mathbb{C})$ which do not factor through lower dimensional projective manifolds. The associated $\mathbb{P}^1$-bundles are birationally trivial (since the underlying representation is a Galois conjugate of the representations coming from the transversely projective foliations on X defined by the submersions $\mathbb{H}^n \to \mathbb{H}$), and by taking a rational section we can produce further examples of transversely projective foliations on X which do not factor. Although the underlying representations are Galois conjugate to the representations in $PSL(2,\mathbb{R})$, the topology of the Riccati foliations over X associated to embeddings $\sigma : L \to \mathbb{C}$ for which $\sqrt{-1}\Phi$ is definite is quite different. In the former case the Riccati foliation leaves invariant two open subsets, corresponding to the complement of $\mathbb{P}^1(\mathbb{R}) \simeq S^1$ in $\mathbb{P}^1$, while in the latter case the Riccati foliation is quasi-minimal: all the leaves not contained in $\pi^{-1}((\mathcal{H})_\infty)$ are dense in the corresponding $\mathbb{P}^1$-bundle.

Explicit examples of foliations on $\mathbb{P}^2$ defined by the submersions $\mathbb{H}^2 \to \mathbb{H}$ with $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})^2$ and Γ isomorphic to $\mathrm{PSL}(2,\mathcal{O}_K)$ (and certain subgroups) have been determined by the second author and Mendes in [30] ($K=\mathbb{Q}(\sqrt{5})$) using the work of Hirzebruch on the description of these surfaces, and by Cousin in [14], see also [15], ($K=\mathbb{Q}(\sqrt{3})$) using an algebraic solution of a Painlevé VI equation.

Example 9.5. The degree of the generically finite morphism in Assertion (1) of Theorem D cannot be bounded, even if we restrict to transversely projective foliations on a rational surface. Let $C_d = \{x^d + y^d + z^d = 0\} \subset \mathbb{P}^2$ be the Fermat curve of degree $d \geq 3$. On $S_d = C_d \times C_d$ consider the action of $\mathbb{Z}/d\mathbb{Z}$ given by $\varphi(x,y) = (\xi_d x, \xi_d y)$ where ξ_d is a primitive d-th root of the unity. Let $\omega \in H^0(S_d, \Omega^1_{S_d})$ be a general holomorphic 1-form satisfying $\varphi^*\omega = \xi_d\omega$. The induced foliation is invariant by the action of $\mathbb{Z}/d\mathbb{Z}$, but the 1-form ω is not. The quotient of S_d by $\mathbb{Z}/d\mathbb{Z}$ is a rational surface R and the foliation induced by ω on R is transversely projective (indeed transversely affine). The monodromy group is an extension of the group of d-th of unities by an infinite subgroup of $(\mathbb{C},+)$. Explicit equations for birational models of these foliations on $\mathbb{P}^2$ can be found in [35, Example 3.1].

9.3. **Proof of Theorem E.** Let (E,∇) be a flat meromorphic $\mathfrak{sl}_2$-connection on X. In the case (E,∇) is regular, then the conclusion of Theorem E directly follows from Corollary B. Let us assume (E,∇) irregular.

Let us consider $P := \mathbb{P}E$ the $\mathbb{P}^1$-bundle associate to E; horizontal section of ∇ induce a Riccati foliation $\mathcal{H}$ on $\pi : P \to X$ which is irregular by assumption. We can apply Theorem C to the projective connection/Riccati foliation $(P,\mathcal{H})$ and discuss on two possible conclusions.

Assume first that $\mathcal{H}$ is defined by a closed 1-form (maybe passing to a finite covering of X). After birational bundle transformation, we can assume $P_0 = X\times\mathbb{P}^1$ and $\mathcal{H}_0$ defined by

$$\Omega_0 = \frac{dz}{z} + 2\omega \quad \text{or} \quad \Omega_0 = dz+\omega$$

with ω a closed 1-form on X (see Remark 4.3). These Riccati foliations are induced by those explicit connections of Corollary E (1):

$$\nabla_0 = d + \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} \quad \text{or} \quad \nabla_0 = d + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}$$

on the trivial bundle $E_0 := \mathcal{O}_X \oplus \mathcal{O}_X$. There is a birational bundle trivialization $E \dashrightarrow E_0$ making commutative the diagramm

$$\begin{array}{ccc} E & \overset{\phi}{\dashrightarrow} & E_0 \\ \downarrow & & \downarrow \\ P & \dashrightarrow & P_0 \end{array}$$

Obviously, $\phi_*\nabla$ is projectively equivalent to one of the models ∇_0: $\phi_*\nabla = \nabla_0 \otimes \zeta$ with $(\mathcal{O}_X,\zeta)$ a flat rank one connection over X, birationally equivalent to the trivial connection by construction. This means that one can write $\zeta = d + \frac{df}{f}$ and, maybe tensoring by the logarithmic connection $(\mathcal{O}_X, d+\frac{1}{2}\frac{df}{f})$ (whose square has trivial monodromy), we get equality $\phi_*\nabla = \nabla_0$. Note that, passing to the 2-fold cover defined by $z^2 = f$, the connection ∇ is birationally gauge equivalent to ∇_0 (without tensoring).

Assume now that $(P,\mathcal{H})$ is birationally gauge equivalent to the pull-back $f^*(P_0,\mathcal{H}_0)$ of a Riccati foliation over a curve, $f : X \dashrightarrow C$ with $P_0 = C\times\mathbb{P}^1$. Denote by ∇_0 the unique $\mathfrak{sl}_2$-connection on the trivial bundle E_0 over C inducing the projective connection $(P_0,\mathcal{H}_0)$. Then (E,∇) is birationally equivalent to $f^*(E_0,\nabla_0)\otimes(\mathcal{O}_X,\zeta)$ with ζ logarithmic rank one connection having monodromy in the center of $\mathrm{SL}(2,\mathbb{C})$. □

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