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Estimating extreme quantiles under random truncation

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Abstract. The goal of this paper is to provide estimators of the tail index and extreme quantiles of a heavy-tailed random variable when the data is right-truncated. The weak consistency and asymptotic normality of the estimators are established and we illustrate the finite sample performance of our estimators on a simulation study.

AMS Subject Classifications: 62G05, 62G20, 62G30, 62G32.

Keywords: Heavy-tailed distribution, tail index, extreme quantile, consistency, asymptotic normality.

1 Introduction

Studying extreme events is relevant in numerous fields of statistical applications. One can think about hydrology, where one may want to estimate the maximum level reached by seawater along a coast over a given period, or to study extreme rainfall at a given location; in actuarial science, a pivotal problem for an insurance firm is to estimate the probability that a claim so large that it represents a threat to its solvency is filed. In this type of problem, the focus is not in the estimation of “central” parameters of the random variable of interest, such as its mean or median, but rather in the understanding of its behavior in its right tail.

A particular relevant case is when the random variable of interest Y is heavy-tailed, namely, when its survival function \bar{F} can be written $\bar{F}(y) = y^{-1/\gamma}L(y)$ for all $y > 0$; here, $\gamma > 0$ shall be referred to as the tail index and L is a slowly varying function at infinity, meaning that it satisfies $L(\lambda y)/L(\lambda) \rightarrow 1$ as $\lambda \rightarrow \infty$ for all $y > 0$. In this case, γ clearly drives the tail behavior of \bar{F} and its knowledge is necessary if, for instance, we are interested in the estimation of extreme quantiles of Y . The estimation of the tail index is thus one of the central topics in extreme value theory, which is why this problem has been extensively studied in the literature. Recent overviews on univariate tail index estimation can be found in the monographs of Beirlant *et al.* [1] and de Haan and Ferreira [6].

A further challenge rises when facing incomplete data. An example of such a situation is the estimation of (extreme) survival times based on a follow-up study of patients suffering from a given illness. If at the time the data is collected a patient is still alive, then his survival time is not available to the researcher, although it is known that the patient survived until the end of the study. This case is the archetypal example of right-censoring. Estimating the tail index in this situation is much more difficult than when having complete data, since information about the right tail of the variable of interest is

missing. In this setting, the estimation of the tail index and extreme quantiles has been considered only recently by Beirlant *et al.* [2] and Einmahl *et al.* [5].

In this paper, we consider the case when the data is right-truncated. In this framework, one observes the variable of interest if and only if it is less than or equal to a truncation variable T . This situation is different from right-censoring since nothing is known about Y in the case $Y > T$, which adds a further difficulty to the analysis of the right tail of Y . Truncated data may be collected in various cases, for instance when estimating incubation times for a given disease, see Lagakos *et al.* [15] and Kalbfleisch and Lawless ([12], [13]); when studying the luminosity of astronomical objects such as quasars, see Lynden-Bell [17] and Jackson [11]; when accounting for reporting lags in insurance data, also referred to as the incurred but not yet reported problem, see Kaminsky [14], Lawless [16] and Herbst [7]; or when considering warranty data, see Hu and Lawless ([9], [10]) and the monograph by Meeker and Escobar [18]. To the best of our knowledge, the estimation of the tail index and extreme quantiles in this context is, up to now, still an open question.

The outline of this paper is as follows. In Section 2, we give a precise definition of our model and define our estimators of the tail index and the extreme quantile of a truncated random variable. Some asymptotic properties of our estimators are stated in Section 3. The finite sample performance of the extreme quantile estimator is studied in Section 4. Preliminary results and proofs of the main results are deferred to Section 5, while proofs of the preliminary results are deferred to the Appendix.

2 Framework

Let $(Y_1, T_1), \dots, (Y_n, T_n)$ be n independent copies of a random pair $(Y, T) \in [y_0, \infty) \times [t_0, \infty)$, where Y and T are independent, $y_0 \geq 0$ and $y_0 \leq t_0$ are the left endpoints of Y and T . The right endpoints of Y and T are supposed to be infinite. The joint cumulative distribution function (cdf) of the random pair (Y, T) is then given for all $(y, t) \in \mathbb{R}^2$ by $H(y, t) := \mathbb{P}(Y \leq y, T \leq t) = F(y)G(t)$, where F and G are the cdfs of Y and T . The focus of this paper is on extreme quantiles of Y and, as a first step, on the estimation of the cdf F which is assumed to be continuous in what follows. Of course, because we only happen to record the Y_i and T_i such that $Y_i \leq T_i$, the classical nonparametric estimator of F cannot be used. However, the conditional cdfs of Y and T given $Y \leq T$, defined for all $(y, t) \in \mathbb{R}^2$ by

$$F^*(y) = \mathbb{P}(Y \leq y | Y \leq T) = \frac{1}{p} \int_{y_0}^y \bar{G}(z) dF(z) \quad \text{and} \quad G^*(t) = \mathbb{P}(T \leq t | Y \leq T) = \frac{1}{p} \int_{t_0}^t F(z) dG(z), \quad (1)$$

where $p := \mathbb{P}(Y \leq T)$, may be estimated nonparametrically: let

$$N := \sum_{i=1}^n \mathbb{I}_{\{Y_i \leq T_i\}}.$$

Notice that N is a binomial random variable with parameters n and p , which counts the (random) number of observed pairs (Y_i, T_i) such that $Y_i \leq T_i$. Such pairs shall be denoted in the sequel as (Y_i^*, T_i^*) , $1 \leq i \leq N$. It can be shown (see Lemma 2) that the conditional distribution of $\{(Y_i^*, T_i^*), i = 1, \dots, N\}$ given $N = q$ is equal to the distribution of q independent copies of a random vector (Y^*, T^*) with cdf H^* given by $H^*(y, t) = \mathbb{P}(Y \leq y, T \leq t | Y \leq T)$. The standard estimators of the conditional cdfs of Y^* and T^* are then

$$\hat{F}_N^*(y) = \frac{1}{N} \sum_{i=1}^N \mathbb{I}_{\{Y_i^* \leq y\}} \quad \text{and} \quad \hat{G}_N^*(t) = \frac{1}{N} \sum_{i=1}^N \mathbb{I}_{\{T_i^* \leq t\}}.$$

Note now that, to estimate F , it is enough to estimate the function Λ^F defined by, for all $y \geq y_0$,

$$\Lambda^F(y) = -\log F(y) = \int_y^\infty \frac{dF(z)}{F(z)}.$$

The following result shows that this quantity is in fact linked to F^* and G^* :

Proposition 1. *Let $C^* := F^* - G^*$. Then for all $z > \widehat{y}_0$, $C^*(z) > 0$ and for all $y \geq y_0$,*

$$\Lambda^F(y) = \int_y^\infty \frac{dF^*(z)}{C^*(z)}.$$

This result can be used to build an estimator of the function Λ^F and consequently of the cdf F . Remarking that C^* shall be estimated by $\widehat{C}_N^* = \widehat{F}_N^* - \widehat{G}_N^*$, we propose the following estimator for Λ^F : for all $y > y_0$,

$$\widehat{\Lambda}_N^F(y) = \frac{1}{N} \sum_{i=1}^N \frac{\mathbb{I}_{\{Y_i^* > y\}}}{\widehat{C}_N^*(Y_i^*)}.$$

The survival function \overline{F} and its associated quantile function $\alpha \mapsto q(\alpha) := \inf\{y \geq y_0 \mid \overline{F}(y) \leq \alpha\}$, which is the right-continuous inverse of \overline{F} , are then estimated by:

$$\widehat{F}_N(y) = \exp\left(-\widehat{\Lambda}_N^F(y)\right) \quad \text{and} \quad \widehat{q}_N(\alpha) = \inf\left\{y \geq y_0 \mid \widehat{F}_N(y) \leq \alpha\right\}$$

where we let $\widehat{F}_N = 1 - \widehat{F}_N^*$. The first aim of this paper is to study the asymptotic behavior of the estimator $\widehat{q}_N(\alpha_n)$ where $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. We shall tackle this problem in the following framework:

(M) The survival functions \overline{F} and \overline{G} are regularly varying functions at infinity with respective indices $-1/\gamma_F$ and $-1/\gamma_G$ with $0 < \gamma_F \leq \gamma_G$. In other words, for all $y, t > 0$, as $\lambda \rightarrow \infty$:

$$\overline{F}(\lambda y)/\overline{F}(\lambda) \rightarrow y^{-1/\gamma_F} \quad \text{and} \quad \overline{G}(\lambda t)/\overline{G}(\lambda) \rightarrow t^{-1/\gamma_G}.$$

Model **(M)** is a standard extreme-value model adapted to right-truncated data; see also [2] and [5] for closely related models when there is right-censoring. The cdfs of Y and T are thus heavy-tailed with respective tail indices γ_F and γ_G . The condition $\gamma_F \leq \gamma_G$ ensures that we have at our disposal enough observations pertaining to the right tail of Y . In this context, the extreme quantile $\widehat{q}_N(\alpha_n)$ is consistent if $\alpha_n \rightarrow 0$ slowly enough, see Theorem 2. To remove the restriction on the rate of convergence of α_n , we note that under model **(M)**, the quantile function q is regularly varying at 0 (see Corollary 1.2.10 p.23 in [6]), so that if $\beta_n < \alpha_n$ are two positive sequences tending to 0 such that $\beta_n/\alpha_n \rightarrow 0$ then

$$q(\beta_n) \approx q(\alpha_n) (\alpha_n/\beta_n)^{\gamma_F}. \quad (2)$$

To derive an estimator of an extreme quantile $q(\beta_n)$ from (2), we first need to build an estimator of γ_F . To this end, we introduce the class of estimators

$$\widehat{\gamma}_{N,F}(k_N, k'_N) = \frac{\widehat{\gamma}_{N,F^*}(k_N) \widehat{\gamma}_{N,G}(k'_N)}{\widehat{\gamma}_{N,G}(k'_N) - \widehat{\gamma}_{N,F^*}(k_N)} \quad (3)$$

where $\widehat{\gamma}_{N,F^*}(k_N)$ and $\widehat{\gamma}_{N,G}(k'_N)$ are the Hill-type estimators (see Hill [8])

$$\widehat{\gamma}_{N,F^*}(k_N) = \frac{1}{k_N} \sum_{i=1}^{k_N} \log \frac{Y_{N-i+1,N}^*}{Y_{N-k_N,N}^*} \quad \text{and} \quad \widehat{\gamma}_{N,G}(k'_N) = \frac{1}{k'_N} \sum_{i=1}^{k'_N} \log \frac{T_{N-i+1,N}^*}{T_{N-k'_N,N}^*}.$$

Here (k_n) and (k'_n) are sequences of integers which belong to $\{1, \dots, n-1\}$ for each n and $Y_{1,N}^* \leq \dots \leq Y_{N,N}^*$, $T_{1,N}^* \leq \dots \leq T_{N,N}^*$ are the order statistics deduced from the samples $(Y_i^*)_{1 \leq i \leq N}$, $(T_i^*)_{1 \leq i \leq N}$. Remark that $\overline{F^*}$ and $\overline{G^*}$ are heavy-tailed with respective tail indices $\gamma_{F^*} := \gamma_G/(\gamma_F + \gamma_G)$ and γ_G (see Lemma 4). As a consequence, $\widehat{\gamma}_{N,F^*}(k_N)$ and $\widehat{\gamma}_{N,G}(k'_N)$ shall be consistent estimators of γ_{F^*} and γ_G under mild conditions, which makes $\widehat{\gamma}_{N,F}(k_N, k'_N)$ a consistent estimator of γ_F , see Theorem 3.

Using (2) motivates the following Weissman-type estimator (see Weissman [22]) for a quantile having arbitrary order $\beta_n \rightarrow 0$:

$$\widehat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N) = \widehat{q}_N(\alpha_n) (\alpha_n / \beta_n)^{\widehat{\gamma}_{N,F}(k_N, k'_N)} \quad (4)$$

where $\alpha_n \rightarrow 0$ converges slowly enough.

3 Main results

In this section, we examine the asymptotic properties of our estimator. We start by noting that if **(M)** holds and $n\overline{F}(y_n)\overline{G}(y_n) \rightarrow \infty$ then $\mathbb{P}(Y_{N,N}^* \leq y_n) \rightarrow 0$, see Lemma 5 for a proof. As a consequence, $\widehat{F}_N(y_n) > 0$ with arbitrarily large probability for n large enough. To establish the asymptotic normality of $\widehat{F}_N(y_n)$, we introduce the following additional condition:

$$\int_{y_0}^{\infty} \frac{dF(z)}{\overline{G}(z)} < \infty. \quad (5)$$

This assumption is classical in the study of the estimator of the cdf of a truncated random variable, see for instance Woodroffe [23] and Stute and Wang [21] for related hypotheses when there is left-truncation. Note that under model **(M)**, then (5) automatically holds if $\gamma_F < \gamma_G$, see Lemma 3.

Theorem 1. *Let $y_n \rightarrow \infty$. Assume that **(M)** and (5) hold, and that $v(y_n)\sqrt{n} \rightarrow \infty$. Let*

$$v(y) := \overline{F}(y) \left\{ \int_y^{\infty} \frac{dF(z)}{\overline{G}(z)} \right\}^{-1/2}.$$

Then

$$v(y_n)\sqrt{n} \left(\frac{\widehat{F}_N(y_n)}{\overline{F}(y_n)} - 1 \right) = \begin{cases} \xi_n & \text{if } \gamma_F < \gamma_G \\ \text{O}_{\mathbb{P}}(1) & \text{if } \gamma_F = \gamma_G \end{cases}$$

where ξ_n is a random variable which is asymptotically standard Gaussian distributed.

We now establish the asymptotic normality of $\widehat{q}_N(\alpha_n)$.

Theorem 2. *Let $\alpha_n \rightarrow 0$. Assume that F is a derivable function in a neighborhood of infinity such that $yF'(y)/\overline{F}(y) \rightarrow 1/\gamma_F$ as $y \rightarrow \infty$, that **(M)** and (5) hold, and that $v(q(\alpha_n))\sqrt{n} \rightarrow \infty$. Then*

$$v(q(\alpha_n))\sqrt{n} \left(\frac{\widehat{q}_N(\alpha_n)}{q(\alpha_n)} - 1 \right) = \begin{cases} \zeta_n & \text{if } \gamma_F < \gamma_G \\ \text{O}_{\mathbb{P}}(1) & \text{if } \gamma_F = \gamma_G \end{cases}$$

where ζ_n is a random variable which is asymptotically Gaussian centered with variance γ_F^2 .

Theorem 2 is a convergence result for the intermediate quantile estimator $\widehat{q}_N(\alpha_n)$, provided $\alpha_n \rightarrow 0$ slowly enough. To examine the asymptotic properties of the extreme quantile estimator (4), we start by proving a couple of results on the tail index estimator $\widehat{\gamma}_{N,F}(k_N, k'_N)$. To this end, we introduce a second-order condition on \overline{F} and \overline{G} :

(C) The survival functions \overline{F} and \overline{G} are derivable functions which satisfy the equations

$$F'(y) = \left[\frac{1}{\gamma_F} - \Delta_F(y) \right] \frac{\overline{F}(y)}{y} \quad \text{and} \quad G'(t) = \left[\frac{1}{\gamma_G} - \Delta_G(t) \right] \frac{\overline{G}(t)}{t}, \quad (6)$$

where Δ_F, Δ_G are bounded measurable functions having ultimately constant sign and converging to 0 at infinity, such that $|\Delta_F|, |\Delta_G|$ are ultimately monotonic regularly varying functions at infinity, with respective indices $\rho_F/\gamma_F \leq 0, \rho_G/\gamma_G \leq 0$.

From (6), we deduce that F' and G' are regularly varying functions with indices $-1 - 1/\gamma_F$ and $-1 - 1/\gamma_G$, respectively. Furthermore, if U_F, U_G are the left-continuous inverses of $1/\overline{F}$ and $1/\overline{G}$, condition **(C)** entails that U_F and U_G are derivable functions with

$$U'_F(\alpha) = [\gamma_F + \Delta_{U_F}(\alpha)] \frac{U_F(\alpha)}{\alpha} \quad \text{and} \quad U'_G(\beta) = [\gamma_G + \Delta_{U_G}(\beta)] \frac{U_G(\beta)}{\beta},$$

where $\Delta_{U_F}, \Delta_{U_G}$ are measurable functions tending to 0 at infinity, such that $|\Delta_F|, |\Delta_G|$ are regularly varying functions at infinity, with respective indices $\rho_F \leq 0$ and $\rho_G \leq 0$ (see the proof of Lemma 10).

We now recall the notation $s \vee t$ for the maximum of two real numbers s and t , we set

$$R_{F^*}(y) = |\Delta_F(y)| \vee |\Delta_G(y)| \quad \text{and} \quad R_{G^*}(t) = |\overline{F}(t)| \vee |\Delta_G(t)|$$

and we let U_{F^*}, U_{G^*} be the left-continuous inverses of $1/\overline{F^*}$ and $1/\overline{G^*}$. The following result examines the asymptotic properties of $\widehat{\gamma}_{N,F}(k_N, k'_N)$ to γ_F , where $s \wedge t$ denotes the minimum of s and t .

Theorem 3. *Let $(k_n), (k'_n)$ be such that $k_n, k'_n \rightarrow \infty$ and $k_n/n, k'_n/n \rightarrow 0$. Assume that **(M)** holds. Then we have $\widehat{\gamma}_{N,F}(k_N, k'_N) \xrightarrow{\mathbb{P}} \gamma_F$. Suppose moreover that **(C)** holds, that $\rho_F/\gamma_F \neq \rho_G/\gamma_G$ and $(\rho_F/\gamma_F) \vee (\rho_G/\gamma_G) \neq -1/\gamma_F$, and that $k_n R_{F^*}^2(U_{F^*}(n/k_n)) \vee k'_n R_{G^*}^2(U_{G^*}(n/k'_n)) \rightarrow 0$. Then if either $k_n/k'_n \rightarrow 0$ or $k'_n/k_n \rightarrow 0$, it holds that*

$$\sqrt{k_N \wedge k'_N} (\widehat{\gamma}_{N,F}(k_N, k'_N) - \gamma_F) \xrightarrow{d} \mathcal{N}(0, \sigma_F^2),$$

where σ_F^2 is equal to $p\gamma_F^2[1 + \gamma_F/\gamma_G]^3$ if $k_n/k'_n \rightarrow 0$ and $p\gamma_F^4/\gamma_G^2$ if $k'_n/k_n \rightarrow 0$. In the case $k_n/k'_n \rightarrow 1$, then we have

$$\sqrt{k_N} (\widehat{\gamma}_{N,F}(k_N, k'_N) - \gamma_F) = \mathcal{O}_{\mathbb{P}}(1).$$

Note that the rate of convergence in Theorem 3 is the random quantity $\sqrt{k_N \wedge k'_N}$; taking into account that $N/np = 1 + o_{\mathbb{P}}(n^{-1/4})$ (see Lemma 2), it follows that under the additional mild condition

$$\sup_{q,r \in I_n} \left| \frac{k_q \wedge k'_q}{k_r \wedge k'_r} - 1 \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{where } I_n = [np(1 - n^{-1/4}), np(1 + n^{-1/4})]$$

this rate becomes the nonrandom sequence $\sqrt{k_{\lfloor np \rfloor} \wedge k'_{\lfloor np \rfloor}}$, where $\lfloor \cdot \rfloor$ denotes the floor function. Note also that under condition **(C)**, Lemma 9 entails that there exist functions $\Delta_{F^*}, \Delta_{G^*}$ which converge to 0 at infinity such that

$$(F^*)'(y) = \left[\frac{1}{\gamma_{F^*}} - \Delta_{F^*}(y) \right] \frac{\overline{F^*}(y)}{y} \quad \text{and} \quad (G^*)'(t) = \left[\frac{1}{\gamma_{G^*}} - \Delta_{G^*}(t) \right] \frac{\overline{G^*}(t)}{t}.$$

Condition $k_n R_{F^*}^2(U_{F^*}(n/k_n)) \vee k'_n R_{G^*}^2(U_{G^*}(n/k'_n)) \rightarrow 0$ implies (see Lemma 9) the convergences $k_n \Delta_{F^*}^2(U_{F^*}(n/k_n)) \rightarrow 0$ and $k'_n \Delta_{G^*}^2(U_{G^*}(n/k'_n)) \rightarrow 0$ which are analogues of the condition classically used to prove the asymptotic normality of Hill estimator. They ensure that the bias of the estimator is negligible with respect to its standard deviation. The final result of this section gives some asymptotic properties of the estimator $\widehat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N)$:

Theorem 4. *Let $\alpha_n \rightarrow 0, \beta_n \rightarrow 0, k_n \wedge k'_n \rightarrow \infty$ and $(k_n/n \vee k'_n/n) \rightarrow 0$. Assume that **(M)**, (5) and **(C)** hold. Assume that $\rho_F/\gamma_F \neq \rho_G/\gamma_G$ and $(\rho_F/\gamma_F) \vee (\rho_G/\gamma_G) \neq -1/\gamma_F$, that $\beta_n/\alpha_n \rightarrow 0, v(q(\alpha_n))\sqrt{n} \rightarrow \infty, nv^2(q(\alpha_n))\Delta_F^2(q(\alpha_n)) \rightarrow 0, k_n R_{F^*}^2(U_{F^*}(n/k_n)) \vee k'_n R_{G^*}^2(U_{G^*}(n/k'_n)) \rightarrow 0,$*

$$(k_{\lfloor np \rfloor} \wedge k'_{\lfloor np \rfloor})/nv^2(q(\alpha_n)) \rightarrow 1 \quad \text{and} \quad \sup_{q,r \in I_n} \left| \frac{k_q \wedge k'_q}{k_r \wedge k'_r} - 1 \right| \rightarrow 0.$$

Then, if $\gamma_F < \gamma_G$ and either $k_n/k'_n \rightarrow 0$ or $k'_n/k_n \rightarrow 0$, it holds that

$$\frac{v(q(\alpha_n))\sqrt{n}}{\log(\alpha_n/\beta_n)} \left(\frac{\widehat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N)}{q(\beta_n)} - 1 \right) \xrightarrow{d} \mathcal{N}(0, \sigma_F^2).$$

In the case $k_n/k'_n \rightarrow 1$, or if $\gamma_F = \gamma_G$ and either $k_n/k'_n \rightarrow 0$ or $k'_n/k_n \rightarrow 0$, it holds that

$$\frac{v(q(\alpha_n))\sqrt{n}}{\log(\alpha_n/\beta_n)} \left(\frac{\widehat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N)}{q(\beta_n)} - 1 \right) = O_{\mathbb{P}}(1).$$

4 Simulation study

In this section, we examine the finite sample behavior of \widehat{q}_N and \widehat{q}_N^W . We shall use the following model:

$$\forall y, t > 0, \overline{F}(y) = (1 + y^{1/\delta})^{-\delta/\gamma_F} \quad \text{and} \quad \overline{G}(t) = (1 + t^{1/\delta})^{-\delta/\gamma_G},$$

where $\delta > 0$ and $0 < \gamma_F < \gamma_G$. Note that in this situation, the truncation probability is $1 - p$, with p given by $p = \gamma_G/(\gamma_F + \gamma_G)$. Besides, remark that for $\alpha \in (0, 1)$,

$$\widehat{q}_N(\alpha) = \sum_{i=1}^{N-1} Y_{i,N}^* \mathbb{I}_{\{\alpha \in [\Theta_{i+1}, \Theta_i]\}} + Y_{N,N}^* \mathbb{I}_{\{\alpha < \Theta_N\}},$$

where for $i = 1, \dots, N$,

$$\Theta_i = 1 - \exp \left(-\frac{1}{N} \sum_{j=i}^N \frac{1}{\widehat{C}_N^*(Y_{j,n}^*)} \right).$$

Thus, the estimator \widehat{q}_N is fairly easy to compute. The estimator \widehat{q}_N^W is given for $\beta_n \rightarrow 0$ by

$$\widehat{q}_N^W(\beta_n | \alpha_n) = \widehat{q}_N(\alpha_n) (\alpha_n / \beta_n)^{\widehat{\gamma}_{n,F}(\alpha_n)},$$

where (α_n) is a sequence in $(0, 1)$ and $\widehat{\gamma}_{n,F}(\alpha_n)$ is the estimator of the tail-index γ_F defined in (3) with $k_n = k'_n = \lfloor n\alpha_n \rfloor$. Thus, the estimator \widehat{q}_N^W only depends on the choice of the parameter α_n . The aim of this section is to compare the performances of the estimators \widehat{q}_N and \widehat{q}_N^W for different values of γ_F and of the probability p . To this end, we generate $R = 1000$ samples of size $n = 200$ from the distributions F and G with $\delta \in \{1/3, 1\}$, $\gamma_F \in \{1/4, 1/2, 1\}$ and $p \in \{0.7, 0.8, 0.9, 0.95\}$. In each case and for a given α_n , we obtain R observations of the estimators \widehat{q}_N and \widehat{q}_N^W denoted by $\widehat{q}_N^{(r)}$ and $\widehat{q}_N^{W,(r)}(\cdot | \alpha_n)$ for $r = 1, \dots, R$. For each replication, the parameter α_n is taken as

$$\alpha_{opt}^{(r)} := \arg \min_{\alpha \in (0, 0.15]} \int_{0.07}^{0.15} \log^2 \left(\frac{\widehat{q}_N^{(r)}(\beta)}{\widehat{q}_N^{W,(r)}(\beta | \alpha)} \right) d\beta$$

The idea behind this choice is that for quantiles which are not too large, the estimators $\widehat{q}_N^{(r)}$ and $\widehat{q}_N^{W,(r)}(\cdot | \alpha_n)$ should be close if α_n is well chosen. Next, we compute the errors

$$E(\tilde{q}^{(r)}) := \int_0^{0.15} \log^2 \left(\frac{\tilde{q}^{(r)}(\beta)}{q(\beta)} \right) d\beta,$$

where $\tilde{q}^{(r)}$ is either $\widehat{q}_N^{(r)}$ or $\widehat{q}_N^{W,(r)}(\cdot | \alpha_{opt}^{(r)})$. The error E is a measure of the overall performance of a quantile estimator when estimating extreme quantiles. For $\theta = \{0.1, 0.5, 0.9\}$, let $r(\theta)$ (resp. $s(\theta)$) be the replication corresponding to the quantile of order θ of the set $\{E(\widehat{q}_N^{(r)}), r = 1, \dots, N\}$ (resp. $\{E(\widehat{q}_N^{W,(r)}), r = 1, \dots, N\}$). In Tables 1 and 2, the values of $E(\widehat{q}_N^{(r(\theta))})$ and $E(\widehat{q}_N^{W,(s(\theta))}(\cdot | \alpha_{opt}^{(s(\theta))}))$ are given in each situation. It appears that the estimator \widehat{q}_N^W performs better than \widehat{q}_N , which is no surprise

since the estimator \widehat{q}_N^W is a Weissman-type estimator specifically adapted to the estimation of extreme quantiles. Besides, we can see that the smaller is the probability p , the larger the bias of the estimators. This was expected since a smaller probability p means a greater number of observations missing in the right tail of Y . Finally, a comparison between Tables 1 and 2 shows that, contrary to the Weissman-type estimator, the empirical quantile estimator is not strongly impacted by the value of δ . This can be explained by the fact that the estimation of γ_F gets worse when γ_F increases.

5 Proofs

5.1 Preliminary results

The first result is a representation of $\widehat{\Lambda}_N^F(y_n) - \Lambda^F(y_n)$, which is the key to the proof of Theorem 1.

Lemma 1. *It holds that*

$$\left[\frac{\widehat{\Lambda}_N^F(y_n) - \Lambda^F(y_n)}{\overline{F}(y_n)} \right] = S_{n,1} + S_{n,2} + S_{n,3} - \frac{\Lambda^F(y_n)}{\overline{F}(y_n)} \mathbb{I}_{\{Y_{N,N}^* \leq y_n\}}$$

where

$$\begin{aligned} S_{n,1} &= \frac{1}{n} \sum_{i=1}^n \left(\frac{\mathbb{I}_{\{Y_i > y_n\}} \mathbb{I}_{\{Y_i \leq T_i\}}}{p \overline{F}(y_n) C^*(Y_i)} - \frac{\Lambda^F(y_n)}{\overline{F}(y_n)} \right) \mathbb{I}_{\{Y_{N,N}^* > y_n\}} \\ S_{n,2} &= \left(\frac{p}{N} - \frac{1}{n} \right) \left(\sum_{i=1}^n \frac{\mathbb{I}_{\{Y_i > y_n\}} \mathbb{I}_{\{Y_i \leq T_i\}}}{p \overline{F}(y_n) C^*(Y_i)} \right) \mathbb{I}_{\{Y_{N,N}^* > y_n\}} \\ \text{and } S_{n,3} &= \frac{1}{N \overline{F}(y_n)} \sum_{i=1}^N \mathbb{I}_{\{Y_i^* > y_n\}} \left\{ \frac{1}{\widehat{C}_{N,Y_i^*}^*} - \frac{1}{C^*(Y_i^*)} \right\} \mathbb{I}_{\{Y_{N,N}^* > y_n\}}. \end{aligned}$$

The second result gives an equivalent of the random variable N and the conditional distribution of $(Y_1^*, T_1^*), \dots, (Y_N^*, T_N^*)$ given N .

Lemma 2. *It holds that*

$$\left(\frac{np}{1-p} \right)^{1/2} \left(\frac{N}{np} - 1 \right) \xrightarrow{d} \mathcal{N}(0, 1).$$

Furthermore, the conditional distribution of $(Y_1^*, T_1^*), \dots, (Y_N^*, T_N^*)$ given $N = q > 0$ is equal to the distribution of q independent copies of a random vector (Y^*, T^*) with cdf H^* .

Lemma 3 is dedicated to the study of a kind of integrals that appear frequently in the proof of Theorem 1. For related results, see Proposition 1.5.9b and Theorem 1.6.5 in Bingham *et al.* [4].

Lemma 3. *Let φ and ψ be two regularly varying functions at infinity with respective indices $\alpha \in \mathbb{R}$ and $-\beta < 0$. Assume that ψ is right-continuous and nonincreasing on some interval $[A, \infty)$, $A \geq 0$.*

- *If $\alpha < \beta$, the function φ is integrable with respect to ψ on a neighborhood of infinity and*

$$\int_y^\infty \varphi(z) d\psi(z) = -\frac{\beta}{\beta - \alpha} \varphi(y) \psi(y) (1 + o(1)) \quad \text{as } y \rightarrow \infty.$$

- *If $\alpha = \beta$ and $\int_A^\infty \varphi(z) d\psi(z) < \infty$ then $y \mapsto \int_y^\infty \varphi(z) d\psi(z)$ is slowly varying at infinity and*

$$-\frac{1}{\varphi(y) \psi(y)} \int_y^\infty \varphi(z) d\psi(z) \rightarrow \infty \quad \text{as } y \rightarrow \infty.$$

Lemma 4 shows that the survival functions \overline{F}^* and \overline{G}^* are regularly varying at infinity.

Lemma 4. *Assume that (M) holds. Let $\overline{F}^* := 1 - F^*$ and $\overline{G}^* := 1 - G^*$ be the survival functions related to the cdfs F^* and G^* , respectively. Then as $y, t \rightarrow \infty$:*

$$\frac{\overline{F}^*(y)}{\overline{F}(y)\overline{G}(y)} \rightarrow \frac{1}{p} \frac{\gamma_G}{\gamma_F + \gamma_G} \quad \text{and} \quad \frac{\overline{G}^*(t)}{\overline{G}(t)} \rightarrow \frac{1}{p}.$$

Lemma 5 essentially implies that under the conditions of Theorem 1, the quantity $\widehat{\Lambda}_N^F(y_n)$ is nonzero with probability tending to one.

Lemma 5. *For all $y \geq y_0$, $\mathbb{P}(Y_{N,N}^* \leq y) = \left(1 - p\overline{F}^*(y)\right)^n$. Consequently, if (M) holds, $y_n \rightarrow \infty$ and $n\overline{F}(y_n)\overline{G}(y_n) \rightarrow \infty$ then $\mathbb{P}(Y_{N,N}^* \leq y_n) \rightarrow 0$.*

Lemma 6 below is similar to Lemma 1.2 in Stute [20].

Lemma 6. *For every λ such that $\lambda p > 1$, it holds that*

$$\mathbb{P}\left(\sup_{\substack{1 \leq i \leq N \\ Y_i^* > y_n}} \frac{C^*(Y_i^*)}{\widehat{C}_N^*(Y_i^*)} \geq \lambda\right) \leq \exp(-(1 - e^{-1})\lambda p F(y_n) + 2).$$

Especially, if $y_n \rightarrow \infty$ then

$$\sup_{\substack{1 \leq i \leq N \\ Y_i^* > y_n}} \frac{C^*(Y_i^*)}{\widehat{C}_N^*(Y_i^*)} = O_{\mathbb{P}}(1).$$

Lemma 7 is the last step in the proof of the second part of Theorem 2.

Lemma 7. *Let (X_n) be a sequence of positive real-valued random variables such that for every positive nonrandom sequence (δ_n) converging to 0, the random sequence $(\delta_n X_n)$ converges to 0 in probability. Then $X_n = O_{\mathbb{P}}(1)$.*

Lemma 8 is a second-order result on regularly varying functions which we shall use several times.

Lemma 8. *Let ψ be a regularly varying function at infinity with index $-\alpha < 0$ that may be written*

$$\forall y > 0, \quad \frac{\psi(y)}{y^{-\alpha}} = c \exp\left(\int_1^y \frac{\Delta(z)}{z} dz\right)$$

where c is a positive constant and Δ is a measurable function converging to 0 at infinity, having ultimately constant sign and such that $|\Delta|$ is regularly varying with index $\rho \leq 0$. Then it holds that

$$\frac{1}{\psi(y)} \int_y^\infty \frac{\psi(z)}{z} dz = \frac{1}{\alpha} + \frac{1}{\alpha(\alpha - \rho)} \Delta(y)(1 + o(1)) \quad \text{as } y \rightarrow \infty.$$

Lemma 9 examines the second-order properties of the regularly varying survival functions \overline{F}^* and \overline{G}^* .

Lemma 9. *Assume that (M) and (C) hold. If $\rho_F/\gamma_F \neq \rho_G/\gamma_G$ then, letting $\rho_{FG}/\gamma_{F^*} = (\rho_F/\gamma_F) \vee (\rho_G/\gamma_G)$ one has, as $y \rightarrow \infty$*

$$y \frac{d}{dy} \log \overline{F}^*(y) = -\frac{1}{\gamma_{F^*}} - \Delta_F(y) \frac{\rho_F(\gamma_F + \gamma_G)}{\gamma_F + (1 - \rho_F)\gamma_G} (1 + o(1)) + [\Delta_F(y) + \Delta_G(y)] \frac{1}{1 - \rho_{FG}} (1 + o(1))$$

and as $t \rightarrow \infty$

$$t \frac{d}{dt} \log \overline{G}^*(t) = -\frac{1}{\gamma_G} + \overline{F}(t) \frac{1}{\gamma_F + \gamma_G} (1 + o(1)) + \Delta_G(t)(1 + o(1)).$$

Epecially, if $\rho_{FG}/\gamma_{F^*} \neq -1/\gamma_F$, there exist positive constants c_{F^*} , c_{G^*} and bounded measurable functions Δ_{F^*} , Δ_{G^*} such that $|\Delta_{F^*}|$, $|\Delta_{G^*}|$ converge to 0 and are regularly varying which satisfy, for y , $t > 0$:

$$\overline{F^*}(y) = y^{-1/\gamma_{F^*}} \left[c_{F^*} \exp \left(\int_1^y \frac{\Delta_{F^*}(v)}{v} dv \right) \right] \quad \text{and} \quad \overline{G^*}(t) = t^{-1/\gamma_G} \left[c_{G^*} \exp \left(\int_1^t \frac{\Delta_{G^*}(v)}{v} dv \right) \right].$$

Lemma 10 is a key result in the analysis of the asymptotic properties of $\hat{\gamma}_N$.

Lemma 10. *Let $\gamma > 0$ and Z be a random variable whose survival function \overline{H} is regularly varying with index $-1/\gamma < 0$. Assume that*

- $N := N(n)$ is a sequence of integer-valued random variables such that there exists a positive sequence (u_n) of integers tending to infinity with $N/u_n \xrightarrow{\mathbb{P}} \infty$;
- $\hat{\gamma}_N(k_N)$ is a random variable such that the distribution of $\hat{\gamma}_N(k_N)$ given $N = q$ is that of

$$\tilde{\gamma}_q(k_q) = \frac{1}{k_q} \sum_{i=1}^{k_q} \log \frac{Z_{q-i+1,q}}{Z_{q-k_q,q}}$$

where $Z_{1,q} \leq \dots \leq Z_{q,q}$ are the order statistics related to a sample of independent and identically distributed copies Z_1, \dots, Z_q of Z .

Then for every sequence (k_n) such that $k_n \rightarrow \infty$ and $k_n/n \rightarrow 0$, it holds that $\hat{\gamma}_N(k_N) \xrightarrow{\mathbb{P}} \gamma$. If moreover \overline{H} may be written

$$\forall t > 0, \quad \overline{H}(t) = t^{-1/\gamma} \left[c_H \exp \left(\int_1^t \frac{\Delta(v)}{v} dv \right) \right]$$

with c_H a positive constant and Δ a bounded measurable function such that $|\Delta|$ converges to 0 and is regularly varying at infinity, then provided $k_n \rightarrow \infty$, $k_n/n \rightarrow 0$ and $\sqrt{k_n} \Delta \circ U(n/k_n) \rightarrow 0$ where U is the left-continuous inverse of $1/\overline{H}$, it holds that $\sqrt{k_N} (\hat{\gamma}_N(k_N) - \gamma) \xrightarrow{d} \mathcal{N}(0, \gamma^2)$.

5.2 Proofs of the main results

Proof of Proposition 1. The proof can be found in [23]; we reproduce it here for the sake of completeness. Pick $z \geq 0$ and note that, given $\{Y \leq T\}$, $\{T \leq z\} \subset \{Y \leq z\}$. Therefore, for all $z > y_0$,

$$C^*(z) = F^*(z) - G^*(z) = \mathbb{P}(Y \leq z, T > z | Y \leq T) = p^{-1} F(z) \overline{G}(z) > 0. \quad (7)$$

Moreover, equation (1) entails $dF^*(y) = p^{-1} \overline{G}(y) dF(y)$ for all $y \geq y_0$. Using (7) then concludes the proof of Proposition 1. \blacksquare

Proof of Theorem 1. As a preliminary step, use Lemma 1 to write

$$\begin{aligned} v(y_n) \sqrt{n} \left[\frac{\widehat{\Lambda}_N^F(y_n) - \Lambda^F(y_n)}{\overline{F}(y_n)} \right] &= v(y_n) \sqrt{n} S_{n,1} + v(y_n) \sqrt{n} S_{n,2} + v(y_n) \sqrt{n} S_{n,3} \\ &\quad - v(y_n) \sqrt{n} \frac{\Lambda^F(y_n)}{\overline{F}(y_n)} \mathbb{I}_{\{Y_{N^*,N}^* \leq y_n\}}. \end{aligned}$$

Since by Lemma 3, $\sqrt{n \overline{F}(y_n) \overline{G}(y_n)} = \sqrt{\gamma_G / (\gamma_G - \gamma_F)} v(y_n) \sqrt{n} (1 + o(1)) \rightarrow \infty$, Lemma 5 entails:

$$v(y_n) \sqrt{n} \left[\frac{\widehat{\Lambda}_N^F(y_n) - \Lambda^F(y_n)}{\overline{F}(y_n)} \right] = v(y_n) \sqrt{n} S_{n,1} + v(y_n) \sqrt{n} S_{n,2} + v(y_n) \sqrt{n} S_{n,3} + o_{\mathbb{P}}(1).$$

Let us first focus on the term $S_{n,1}$ which we can rewrite as

$$S_{n,1} = \frac{\mathbb{I}_{\{Y_{N,N}^* > y_n\}}}{n} \sum_{i=1}^n W_{n,i} \quad \text{where} \quad W_{n,i} := \frac{\mathbb{I}_{\{Y_i > y_n\}} \mathbb{I}_{\{Y_i \leq T_i\}}}{p\bar{F}(y_n)C^*(Y_i)} - \frac{\Lambda^F(y_n)}{\bar{F}(y_n)}, \quad i = 1, \dots, n. \quad (8)$$

It is easy to check that the $W_{n,i}$ are independent, identically distributed and centered random variables. From (8) we get

$$\mathbb{E}(W_{n,1}^2) = \frac{1}{p^2 \bar{F}^2(y_n)} \mathbb{E} \left(\frac{\mathbb{I}_{\{Y > y_n\}} \mathbb{I}_{\{Y \leq T\}}}{(C^*(Y))^2} \right) - \left(\frac{\Lambda^F(y_n)}{\bar{F}(y_n)} \right)^2. \quad (9)$$

We start by writing

$$\Lambda^F(y_n) - \bar{F}(y_n) = \int_{y_n}^{\infty} \frac{\bar{F}(z)}{F(z)} dF(z) \leq \frac{\bar{F}^2(y_n)}{F(y_n)}.$$

Hence

$$\frac{\Lambda^F(y_n)}{\bar{F}(y_n)} = 1 + O(\bar{F}(y_n)) = 1 + o(1). \quad (10)$$

Besides

$$\frac{1}{p^2} \mathbb{E} \left(\frac{\mathbb{I}_{\{Y > y_n\}} \mathbb{I}_{\{Y \leq T\}}}{(C^*(Y))^2} \right) = \int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}(z)F^2(z)}.$$

Since F is nondecreasing and $F(y_n) \rightarrow 1$, we obtain

$$\frac{1}{p^2} \mathbb{E} \left(\frac{\mathbb{I}_{\{Y > y_n\}} \mathbb{I}_{\{Y \leq T\}}}{(C^*(Y))^2} \right) = \int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}(z)} (1 + o(1)). \quad (11)$$

Noting that $dF(z) = -d\bar{F}(z)$ and applying Lemma 3, we get:

$$\frac{1}{p^2 \bar{F}^2(y_n)} \mathbb{E} \left(\frac{\mathbb{I}_{\{Y > y_n\}} \mathbb{I}_{\{Y \leq T\}}}{(C^*(Y))^2} \right) = \frac{1}{\bar{F}^2(y_n)} \int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}(z)} (1 + o(1)) \rightarrow \infty. \quad (12)$$

Thus we get from (9), (10) and (12) that

$$\mathbb{E}(W_{n,1}^2) = \frac{1}{\bar{F}^2(y_n)} \int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}(z)} (1 + o(1)). \quad (13)$$

Since the $W_{n,i}$ are independent, identically distributed and centered random variables, we shall be able to use Lyapunov's central limit theorem (see e.g. Billingsley [3], p.312) provided it holds that for some $\delta > 0$

$$\exists \delta > 0, \quad n^{-\delta/2} \frac{\mathbb{E}|W_{n,1}|^{2+\delta}}{[\text{Var}(W_{n,1})]^{1+\delta/2}} \rightarrow 0.$$

Because $\gamma_F < \gamma_G$, one can pick $\delta > 0$ such that $(1 + \delta)/\gamma_G - 1/\gamma_F < 0$. Hölder's inequality then gives

$$|W_{n,1}|^{2+\delta} \leq 2^{1+\delta} \left[\left(\frac{\mathbb{I}_{\{Y_i > y_n\}} \mathbb{I}_{\{Y_i \leq T_i\}}}{p\bar{F}(y_n)C^*(Y_i)} \right)^{2+\delta} + \left(\frac{\Lambda^F(y_n)}{\bar{F}(y_n)} \right)^{2+\delta} \right].$$

Recall (10) to get

$$\mathbb{E}|W_{n,1}|^{2+\delta} \leq 2^{1+\delta} \left[\mathbb{E} \left(\frac{\mathbb{I}_{\{Y_i > y_n\}} \mathbb{I}_{\{Y_i \leq T_i\}}}{p\bar{F}(y_n)C^*(Y_i)} \right)^{2+\delta} + 1 + o(1) \right].$$

Besides

$$\mathbb{E} \left(\frac{\mathbb{I}_{\{Y_i > y_n\}} \mathbb{I}_{\{Y_i \leq T_i\}}}{p\bar{F}(y_n)C^*(Y_i)} \right)^{2+\delta} = \frac{1}{\bar{F}^{2+\delta}(y_n)} \int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}^{1+\delta}(z)F^{2+\delta}(z)}.$$

Since F is nondecreasing and $F(y_n) \rightarrow 1$, we obtain

$$\int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}^{1+\delta}(z)F^{2+\delta}(z)} = \int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}^{1+\delta}(z)}(1 + o(1)).$$

Applying Lemma 3, we obtain

$$\int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}^{1+\delta}(z)} = \frac{\gamma_G}{\gamma_G - (1 + \delta)\gamma_F} \frac{\bar{F}(y_n)}{\bar{G}^{1+\delta}(y_n)}(1 + o(1)).$$

From this we deduce that

$$\mathbb{E}|W_{n,1}|^{2+\delta} = O\left(\left[\frac{1}{\bar{F}(y_n)\bar{G}(y_n)}\right]^{1+\delta}\right). \quad (14)$$

Since from (13) and Lemma 3 we have

$$\text{Var}(W_{n,1}) = \mathbb{E}(W_{n,1}^2) = \frac{\gamma_G}{\gamma_G - \gamma_F} \frac{1}{\bar{F}(y_n)\bar{G}(y_n)}(1 + o(1)), \quad (15)$$

it follows from (14) and (15) that

$$n^{-\delta/2} \frac{\mathbb{E}|W_{n,1}|^{2+\delta}}{[\text{Var}(W_{n,1})]^{1+\delta/2}} = O\left(\left[\frac{1}{n\bar{F}(y_n)\bar{G}(y_n)}\right]^{\delta/2}\right) \rightarrow 0.$$

Lyapunov's theorem now entails $\sqrt{n}S_{n,1}/\sqrt{\text{Var}(W_{n,1})} \xrightarrow{d} \mathcal{N}(0, 1)$. Using (13) leads to

$$v(y_n)\sqrt{n}S_{n,1} \xrightarrow{d} \mathcal{N}(0, 1). \quad (16)$$

Let us now focus on the term $S_{n,2}$. From the previous results, it is now clear that

$$\frac{1}{n} \sum_{i=1}^n \frac{\mathbb{I}_{\{Y_i > y_n\}} \mathbb{I}_{\{Y_i \leq T_i\}}}{p\bar{F}(y_n)C^*(Y_i)} \mathbb{I}_{\{Y_{N,N}^* > y_n\}} \xrightarrow{\mathbb{P}} 1.$$

Since $np/N = 1 + O_{\mathbb{P}}(n^{-1/2})$ from Lemma 2, one has, using Lemma 5, that $S_{n,2} = O_{\mathbb{P}}(n^{-1/2})$. Using the convergence $v(y_n) \rightarrow 0$ it is now obvious that

$$S_{n,2} = o_{\mathbb{P}}([v(y_n)\sqrt{n}]^{-1}). \quad (17)$$

Let us now control $S_{n,3}$. Note that (see [23], pp.172–173):

$$\sqrt{n} \sup_{z \in \mathbb{R}} \left| \widehat{F}_N^*(z) - F^*(z) \right| = O_{\mathbb{P}}(1) \quad \text{and} \quad \sqrt{n} \sup_{z \in \mathbb{R}} \left| \widehat{G}_N^*(z) - G^*(z) \right| = O_{\mathbb{P}}(1).$$

Therefore

$$\sqrt{n} \sup_{1 \leq i \leq N} \left| \widehat{C}_N^*(Y_i^*) - C^*(Y_i^*) \right| \mathbb{I}_{\{Y_i^* > y_n\}} = O_{\mathbb{P}}(1).$$

Hence the equality

$$S_{n,3} = O_{\mathbb{P}}\left(\frac{n^{-1/2}}{N\bar{F}(y_n)} \sum_{i=1}^N \frac{\mathbb{I}_{\{Y_i^* > y_n\}}}{\widehat{C}_N^*(Y_i^*)C^*(Y_i^*)} \mathbb{I}_{\{Y_{N,N}^* > y_n\}}\right).$$

Apply Lemmas 2 and 6 to obtain

$$\sqrt{n}S_{n,3} = O_{\mathbb{P}}\left(\frac{1}{n\bar{F}(y_n)} \sum_{i=1}^N \frac{\mathbb{I}_{\{Y_i^* > y_n\}}}{(C^*(Y_i^*))^2} \mathbb{I}_{\{Y_{N,N}^* > y_n\}}\right).$$

Since

$$\sum_{i=1}^N \frac{\mathbb{I}_{\{Y_i^* > y_n\}}}{(C^*(Y_i^*))^2} = \sum_{i=1}^n \frac{\mathbb{I}_{\{Y_i > y_n\}} \mathbb{I}_{\{Y_i \leq T_i\}}}{(C^*(Y_i))^2},$$

it follows from (11) that

$$\mathbb{E} \left(\frac{1}{n\bar{F}(y_n)} \sum_{i=1}^N \frac{\mathbb{I}_{\{Y_i^* > y_n\}}}{(C^*(Y_i^*))^2} \right) = \frac{p^2}{\bar{F}(y_n)} \int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}(z)} (1 + o(1)).$$

Especially

$$\mathbb{E} \left(v(y_n) \left\{ \frac{1}{n\bar{F}(y_n)} \sum_{i=1}^N \frac{\mathbb{I}_{\{Y_i^* > y_n\}}}{(C^*(Y_i^*))^2} \right\} \right) = p^2 \sqrt{\int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}(z)}} (1 + o(1)).$$

Since the integral in the right-hand side converges to 0, this yields

$$v(y_n) \left\{ \frac{1}{n\bar{F}(y_n)} \sum_{i=1}^N \frac{\mathbb{I}_{\{Y_i^* > y_n\}}}{(C^*(Y_i^*))^2} \right\} = o_{\mathbb{P}}(1).$$

We then obtain

$$v(y_n) \sqrt{n} S_{n,3} = O_{\mathbb{P}} \left(v(y_n) \left\{ \frac{1}{n\bar{F}(y_n)} \sum_{i=1}^N \frac{\mathbb{I}_{\{Y_i^* > y_n\}}}{(C^*(Y_i^*))^2} \right\} \mathbb{I}_{\{Y_{N,N}^* \geq y_n\}} \right) = o_{\mathbb{P}}(1). \quad (18)$$

Use finally (16), (17) and (18) together to get

$$v(y_n) \sqrt{n} \left[\frac{\widehat{\Lambda}_N^F(y_n) - \Lambda^F(y_n)}{\bar{F}(y_n)} \right] \xrightarrow{d} \mathcal{N}(0, 1).$$

Using the delta-method completes the proof of the first part of Theorem 1.

We now turn to the proof of the second part of the result. We shall show that

$$v(y_n) \sqrt{n} \left[\frac{\widehat{\Lambda}_N^F(y_n) - \Lambda^F(y_n)}{\bar{F}(y_n)} \right] = O_{\mathbb{P}}(1).$$

The desired statement is then a consequence of this equality and of the mean value theorem applied to the exponential function. Since by Lemma 3

$$\sqrt{n\bar{F}(y_n)\bar{G}(y_n)} = v(y_n) \sqrt{n} \left\{ \frac{\sqrt{\bar{F}(y_n)\bar{G}(y_n)}}{v(y_n)} \right\} (1 + o(1)) \rightarrow \infty$$

we can start with the same decomposition as in the proof of Theorem 1:

$$v(y_n) \sqrt{n} \left[\frac{\widehat{\Lambda}_N^F(y_n) - \Lambda^F(y_n)}{\bar{F}(y_n)} \right] = v(y_n) \sqrt{n} S_{n,1} + v(y_n) \sqrt{n} S_{n,2} + v(y_n) \sqrt{n} S_{n,3} + o_{\mathbb{P}}(1).$$

To prove that $S_{n,1} = O_{\mathbb{P}}((v(y_n) \sqrt{n})^{-1})$, it is sufficient to show that

$$\text{Var} \left(\frac{1}{n} \sum_{i=1}^n W_{n,i} \right) = \frac{1}{n} \mathbb{E}(W_{n,1}^2) = O((v(y_n) \sqrt{n})^{-2}).$$

Using condition (5), Lemma 3 entails that the function v is regularly varying with index $-1/\gamma_F < 0$. This yields

$$\bar{F}^2(y_n) \left\{ \int_{y_n}^{\infty} \frac{dF(z)}{\bar{G}(z)} \right\}^{-1} \rightarrow 0.$$

Apply now (9), (10) and (11) to obtain

$$\frac{1}{n}\mathbb{E}(W_{n,1}^2) = O((v(y_n)\sqrt{n})^{-2}) \rightarrow 0.$$

Now, mimicking the end of the proof of the first part of the result, we easily get that $v(y_n)\sqrt{n}S_{n,2} = o_{\mathbb{P}}(1)$ and $v(y_n)\sqrt{n}S_{n,3} = o_{\mathbb{P}}(1)$, which concludes the proof of Theorem 1. \blacksquare

Proof of Theorem 2. Let $\sigma_n = q(\alpha_n)/[v(q(\alpha_n))\sqrt{n}]$. We shall prove that for every $z \in \mathbb{R}$,

$$\Phi_n(z) := \mathbb{P}(\sigma_n^{-1}(\widehat{q}_N(\alpha_n) - q(\alpha_n)) \leq z) \rightarrow \Phi(z),$$

where Φ is the cdf of a $\mathcal{N}(0, \gamma_F^2)$ distribution. Let us introduce the sequence $\vartheta_n := \gamma_F v(q(\alpha_n))\sqrt{n}/\alpha_n$. It is easy to check that $\Phi_n(z) = \mathbb{P}(W_n \leq z_n)$, where

$$W_n = \vartheta_n \left(\widehat{F}_N(q(\alpha_n) + \sigma_n z) - \overline{F}(q(\alpha_n) + \sigma_n z) \right) \text{ and } z_n = \vartheta_n (\alpha_n - \overline{F}(q(\alpha_n) + \sigma_n z)).$$

Let us first focus on the nonrandom term z_n . Since F is a derivable function, there exists $\theta_n \in (0, 1)$ such that $\alpha_n - \overline{F}(q(\alpha_n) + \sigma_n z) = \sigma_n z F'(q(\alpha_n) + \theta_n \sigma_n z)$. Since $\sigma_n/q(\alpha_n) \rightarrow 0$ as n goes to infinity, we may use the convergence $y\overline{F}(y)/F'(y) \rightarrow 1/\gamma_F$ to get $F'(q(\alpha_n) + \theta_n \sigma_n z) = \gamma_F^{-1} \alpha_n/q(\alpha_n)(1 + o(1))$. Hence the equality

$$z_n = \vartheta_n \sigma_n z \frac{1}{\gamma_F} \frac{\alpha_n}{q(\alpha_n)} (1 + o(1)) = z(1 + o(1)). \quad (19)$$

We now consider the random term W_n . One has

$$W_n = \frac{\vartheta_n \overline{F}(q(\alpha_n) + \sigma_n z)}{v(q(\alpha_n) + \sigma_n z)\sqrt{n}} Z_n,$$

where

$$Z_n = v(q(\alpha_n) + \sigma_n z)\sqrt{n} \left(\frac{\widehat{F}_N(q(\alpha_n) + \sigma_n z)}{\overline{F}(q(\alpha_n) + \sigma_n z)} - 1 \right).$$

Note that from model **(M)**, $\overline{F}(q(\alpha_n) + \sigma_n z) = \alpha_n(1 + o(1))$. Moreover, it is a consequence of Lemma 3 that the function v is regularly varying, so that $v(q(\alpha_n) + \sigma_n z) = v(q(\alpha_n))(1 + o(1))$. Consequently

$$\frac{\vartheta_n \overline{F}(q(\alpha_n) + \sigma_n z)}{v(q(\alpha_n) + \sigma_n z)\sqrt{n}} = \gamma_F(1 + o(1)).$$

Apply then Theorem 1 to obtain that $Z_n \xrightarrow{d} \mathcal{N}(0, 1)$, which concludes the proof of the first statement. To prove the second one, we start by showing that if (ε_n) is an arbitrary nonrandom positive sequence tending to 0 at infinity such that $\varepsilon_n v(q(\alpha_n))\sqrt{n} = \varepsilon_n q(\alpha_n)/\sigma_n \rightarrow \infty$, we have

$$\varepsilon_n \sigma_n^{-1} |\widehat{q}_N(\alpha_n) - q(\alpha_n)| \xrightarrow{\mathbb{P}} 0. \quad (20)$$

Pick then an arbitrary $z > 0$. We shall show that $\varphi_n(z) := \mathbb{P}(\varepsilon_n \sigma_n^{-1} |\widehat{q}_N(\alpha_n) - q(\alpha_n)| > z) \rightarrow 0$. With ϑ_n as above, it is easy to check that $\varphi_n(z) = \mathbb{P}(W_{n,+} > z_{n,+}) + \mathbb{P}(W_{n,-} < z_{n,-})$, where

$$W_{n,\pm} = \vartheta_n \left(\widehat{F}_N(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z) - \overline{F}(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z) \right) \text{ and } z_{n,\pm} = \vartheta_n (\alpha_n - \overline{F}(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z)).$$

Let us first focus on the nonrandom term $z_{n,\pm}$. Mimicking the arguments leading to (19) in the proof of the first part of Theorem 2, we get that

$$z_{n,\pm} = \pm \vartheta_n \varepsilon_n^{-1} \sigma_n z \frac{1}{\gamma_F} \frac{\alpha_n}{q(\alpha_n)} (1 + o(1)) = \pm \varepsilon_n^{-1} z (1 + o(1)). \quad (21)$$

We now consider the random term $W_{n,\pm}$. One has

$$W_{n,\pm} = \frac{\vartheta_n \bar{F}(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z)}{v(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z) \sqrt{n}} Z_{n,\pm},$$

where

$$Z_{n,\pm} = v(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z) \sqrt{n} \left(\frac{\widehat{F}_N(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z)}{\bar{F}(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z)} - 1 \right).$$

Since \bar{F} and v are regularly varying, we have $\bar{F}(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z) = \alpha_n(1+o(1))$ and $v(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z) = v(q(\alpha_n))(1+o(1))$. Hence the equality

$$\frac{\vartheta_n \bar{F}(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z)}{v(q(\alpha_n) \pm \varepsilon_n^{-1} \sigma_n z) \sqrt{n}} = \gamma_F(1+o(1)).$$

On the other hand, the second part of Theorem 1 implies that $\varepsilon_n Z_{n,\pm} = o_{\mathbb{P}}(1)$, so that using (21) we obtain for n large enough

$$\varphi_n(z) = \mathbb{P}(W_{n,+} > z_{n,+}) + \mathbb{P}(W_{n,-} < z_{n,-}) \leq \mathbb{P}(\varepsilon_n Z_{n,+} > z/2) + \mathbb{P}(\varepsilon_n Z_{n,-} < -z/2) \rightarrow 0$$

and the proof of (20) is complete. Note now that if (ε_n) is an arbitrary nonrandom positive sequence, because of the inequality $\varepsilon_n \leq \varepsilon_n \vee \{v(q(\alpha_n))\sqrt{n}\}^{-1/2}$ it can easily be seen that in fact (20) holds for every positive sequence (ε_n) ; applying Lemma 7 completes the proof of Theorem 2. ■

Proof of Theorem 3. Recall the notation $I_n = \mathbb{N} \cap [(1-n^{-1/4})np, (1+n^{-1/4})np]$ and use Lemma 2 to get $\mathbb{P}(N \in I_n) \rightarrow 1$. The consistency statement is thus an immediate consequence of Lemmas 4 and 10.

To prove the second and third statement, write

$$\widehat{\gamma}_{n,F}(k_N, k'_N) - \gamma_F = \frac{\widehat{\gamma}_{N,F^*}(k_N) \widehat{\gamma}_{N,G}(k'_N)}{\widehat{\gamma}_{N,G}(k'_N) - \widehat{\gamma}_{N,F^*}(k_N)} - \frac{\gamma_{F^*} \gamma_G}{\gamma_G - \gamma_{F^*}}.$$

It is then straightforward to obtain the equality

$$\begin{aligned} \widehat{\gamma}_{n,F}(k_N, k'_N) - \gamma_F &= \frac{1}{\gamma_G - \gamma_{F^*}} \left[\frac{\widehat{\gamma}_{N,F^*}(k_N) \widehat{\gamma}_{N,G}(k'_N)}{\widehat{\gamma}_{N,G}(k'_N) - \widehat{\gamma}_{N,F^*}(k_N)} + \gamma_G \right] (\widehat{\gamma}_{N,F^*}(k_N) - \gamma_{F^*}) \\ &+ \frac{\widehat{\gamma}_{N,F^*}(k_N)}{\gamma_G - \gamma_{F^*}} \left[1 - \frac{\widehat{\gamma}_{N,G}(k'_N)}{\widehat{\gamma}_{N,G}(k'_N) - \widehat{\gamma}_{N,F^*}(k_N)} \right] (\widehat{\gamma}_{N,G}(k'_N) - \gamma_G) \end{aligned} \quad (22)$$

Since $\widehat{\gamma}_{N,F^*}(k_N) \xrightarrow{\mathbb{P}} \gamma_{F^*}$ and $\widehat{\gamma}_{N,G}(k'_N) \xrightarrow{\mathbb{P}} \gamma_G$, one has

$$\frac{1}{\gamma_G - \gamma_{F^*}} \left[\frac{\widehat{\gamma}_{N,F^*}(k_N) \widehat{\gamma}_{N,G}(k'_N)}{\widehat{\gamma}_{N,G}(k'_N) - \widehat{\gamma}_{N,F^*}(k_N)} + \gamma_G \right] = \left(1 + \frac{\gamma_F}{\gamma_G} \right)^2 + o_{\mathbb{P}}(\widehat{\gamma}_{N,F^*}(k_N) - \gamma_{F^*}) + o_{\mathbb{P}}(\widehat{\gamma}_{N,G}(k'_N) - \gamma_G)$$

and

$$\begin{aligned} \frac{\widehat{\gamma}_{N,F^*}(k_N)}{\gamma_G - \gamma_{F^*}} \left[1 - \frac{\widehat{\gamma}_{N,G}(k'_N)}{\widehat{\gamma}_{N,G}(k'_N) - \widehat{\gamma}_{N,F^*}(k_N)} \right] &= - \left(1 + \frac{\gamma_F}{\gamma_G} \right)^2 \left\{ \frac{\gamma_F}{\gamma_F + \gamma_G} \right\}^2 \\ &+ o_{\mathbb{P}}(\widehat{\gamma}_{N,F^*}(k_N) - \gamma_{F^*}) + o_{\mathbb{P}}(\widehat{\gamma}_{N,G}(k'_N) - \gamma_G). \end{aligned}$$

Plugging these last two equalities into (22) we get

$$\begin{aligned} \widehat{\gamma}_{n,F}(k_N, k'_N) - \gamma_F &= \left(1 + \frac{\gamma_F}{\gamma_G} \right)^2 \left[\widehat{\gamma}_{N,F^*}(k_N) - \gamma_{F^*} - \frac{\gamma_F^2}{(\gamma_F + \gamma_G)^2} (\widehat{\gamma}_{N,G}(k'_N) - \gamma_G) \right] \\ &+ o_{\mathbb{P}}(\widehat{\gamma}_{N,F^*}(k_N) - \gamma_{F^*}) + o_{\mathbb{P}}(\widehat{\gamma}_{N,G}(k'_N) - \gamma_G). \end{aligned}$$

The result now follows from applying Lemmas 9 and 10. ■

Proof of Theorem 4. From condition **(C)**, note that we may write

$$\forall y > 0, \bar{F}(y) = y^{-1/\gamma_F} L_F(y) \quad \text{with} \quad L_F(y) = c_F \exp\left(\int_1^y \frac{\Delta_F(v)}{v} dv\right)$$

where c_F is a positive constant. Especially, the function L_F is a normalized slowly varying function, see [4]. Further, since q is the inverse function of \bar{F} , it satisfies the equation

$$\forall \alpha \in (0, 1), \quad q(\alpha) = \alpha^{-\gamma_F} L_F^{\gamma_F}(q(\alpha)). \quad (23)$$

Use now (23) to write

$$\begin{aligned} \log\left(\frac{\hat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N)}{q(\beta_n)}\right) &= \log\left(\frac{\hat{q}_N(\alpha_n)}{q(\alpha_n)}\right) + (\hat{\gamma}_{N,F}(k_N, k'_N) - \gamma_F) \log(\alpha_n/\beta_n) \\ &\quad - \gamma_F \log\left(\frac{L_F(q(\beta_n))}{L_F(q(\alpha_n))}\right). \end{aligned} \quad (24)$$

Note that since $|\Delta_F|$ is ultimately monotonic, one has for n large enough

$$\left|\log\left(\frac{L_F(q(\beta_n))}{L_F(q(\alpha_n))}\right)\right| = \left|\int_{q(\alpha_n)}^{q(\beta_n)} \frac{\Delta_F(t)}{t} dt\right| \leq |\Delta_F(q(\alpha_n))| \log\left(\frac{q(\beta_n)}{q(\alpha_n)}\right).$$

Using again (23) and letting

$$\varepsilon_n := \frac{1}{\log(\alpha_n/\beta_n)} \left|\log\left(\frac{L_F(q(\beta_n))}{L_F(q(\alpha_n))}\right)\right|$$

we obtain for n large enough $\varepsilon_n \leq \gamma_F |\Delta_F(q(\alpha_n))| (1 + \varepsilon_n)$ which entails

$$\varepsilon_n \leq \frac{\gamma_F |\Delta_F(q(\alpha_n))|}{1 - \gamma_F |\Delta_F(q(\alpha_n))|} = O(|\Delta_F(q(\alpha_n))|).$$

Recalling (24) we get

$$\begin{aligned} \log\left(\frac{\hat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N)}{q(\beta_n)}\right) &= \log\left(\frac{\hat{q}_N(\alpha_n)}{q(\alpha_n)}\right) + (\hat{\gamma}_{N,F}(k_N, k'_N) - \gamma_F) \log(\alpha_n/\beta_n) \\ &\quad + O(|\Delta_F(q(\alpha_n))|). \end{aligned}$$

Therefore, since $\log(\alpha_n/\beta_n) \rightarrow \infty$, applying Theorems 2 and 3 together with Slutsky's lemma yields

$$\frac{v(q(\alpha_n))\sqrt{n}}{\log(\alpha_n/\beta_n)} \log\left(\frac{\hat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N)}{q(\beta_n)}\right) \xrightarrow{d} \mathcal{N}(0, \sigma_F^2)$$

if either $k_n/k'_n \rightarrow 0$ or $k'_n/k_n \rightarrow 0$, and

$$\frac{v(q(\alpha_n))\sqrt{n}}{\log(\alpha_n/\beta_n)} \log\left(\frac{\hat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N)}{q(\beta_n)}\right) = O_{\mathbb{P}}(1)$$

otherwise. Using the delta-method ends the proof of the first statement.

To prove the second statement, recall from the proof of the first statement that

$$\begin{aligned} \log\left(\frac{\hat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N)}{q(\beta_n)}\right) &= \log\left(\frac{\hat{q}_N(\alpha_n)}{q(\alpha_n)}\right) + (\hat{\gamma}_{N,F}(k_N, k'_N) - \gamma_F) \log(\alpha_n/\beta_n) \\ &\quad + O(|\Delta_F(q(\alpha_n))|). \end{aligned}$$

and use Theorems 2 and 3 to get

$$\frac{v(q(\alpha_n))\sqrt{n}}{\log(\alpha_n/\beta_n)} \log\left(\frac{\hat{q}_N^W(\beta_n | \alpha_n, k_N, k'_N)}{q(\beta_n)}\right) = O_{\mathbb{P}}(1).$$

Using the delta-method ends the proof of Theorem 4. ■

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Appendix - Proofs of preliminary results

Proof of Lemma 1. The proof of this result is straightforward and is thus omitted. ■

Proof of Lemma 2. The first part of the result is a straightforward consequence of Paul Lévy’s central limit theorem. Let now $(A_i)_{i \geq 1}$ be arbitrary Borel subsets of $[y_0, \infty) \times [t_0, \infty)$. If $q \geq 1$,

$$\begin{aligned}
& \mathbb{P}((Y_1^*, T_1^*) \in A_1, \dots, (Y_N^*, T_N^*) \in A_N, N = q) \\
&= \binom{n}{q} \mathbb{P}((Y_i, T_i) \in A_i, Y_i \leq T_i, Y_j > T_j, i = 1, \dots, q, j = q + 1, \dots, n) \\
&= \left[\binom{n}{q} \prod_{i=1}^q \mathbb{P}(Y_i \leq T_i) \prod_{j=q+1}^n \mathbb{P}(Y_j > T_j) \right] \prod_{i=1}^q \mathbb{P}((Y_i, T_i) \in A_i | Y_i \leq T_i) \\
&= \mathbb{P}(N = q) \prod_{i=1}^q \mathbb{P}((Y^*, T^*) \in A_i),
\end{aligned}$$

which concludes the proof. ■

Proof of Lemma 3. We start by the case $\alpha < \beta$. Let $2\delta = \beta - \alpha > 0$, take y so large that $y^{-\alpha-\delta}\varphi(y) \leq 1$ and write for $Y > y$:

$$\int_y^Y \varphi(z) d\psi(z) = y^{-\alpha-\delta}\varphi(y) \left[\int_y^Y \left[\frac{z^{-\alpha-\delta}\varphi(z)}{y^{-\alpha-\delta}\varphi(y)} - \left(\frac{z}{y}\right)^{-\delta} \right] z^{\alpha+\delta} d\psi(z) + y^\delta \int_y^Y z^\alpha d\psi(z) \right]. \quad (25)$$

Clearly, since ψ is nonincreasing,

$$\left| \int_y^Y \left[\frac{z^{-\alpha-\delta}\varphi(z)}{y^{-\alpha-\delta}\varphi(y)} - \left(\frac{z}{y}\right)^{-\delta} \right] z^{\alpha+\delta} d\psi(z) \right| \leq -\sup_{\lambda \geq 1} \left| \frac{(\lambda y)^{-\alpha-\delta}\varphi(\lambda y)}{y^{-\alpha-\delta}\varphi(y)} - \lambda^{-\delta} \right| \int_y^Y z^{\alpha+\delta} d\psi(z). \quad (26)$$

A uniform convergence result for the function $y \mapsto y^{-\alpha-\delta}\varphi(y)$ (see e.g. Theorem 1.5.2 in [4]) entails

$$\sup_{\lambda \geq 1} \left| \frac{(\lambda y)^{-\alpha-\delta}\varphi(\lambda y)}{y^{-\alpha-\delta}\varphi(y)} - \lambda^{-\delta} \right| \rightarrow 0 \text{ as } y \rightarrow \infty.$$

Besides, Theorem 1.6.5 in [4] entails for all $\theta < \beta$

$$\int_y^\infty z^\theta d\psi(z) = -\frac{\beta}{\beta - \theta} y^\theta \psi(y) (1 + o(1)) \text{ as } y \rightarrow \infty. \quad (27)$$

Consequently, the expression on the left-hand side of (26) has a finite limit as $Y \rightarrow \infty$ and we may write

$$\left| \int_y^\infty \left[\frac{z^{-\alpha-\delta}\varphi(z)}{y^{-\alpha-\delta}\varphi(y)} - \left(\frac{z}{y}\right)^{-\delta} \right] z^{\alpha+\delta} d\psi(z) \right| = o(y^{\alpha+\delta}\psi(y)) \text{ as } y \rightarrow \infty. \quad (28)$$

A use of (27) entails

$$\int_y^\infty z^\alpha d\psi(z) = -\frac{\beta}{\beta - \alpha} y^\alpha \psi(y) (1 + o(1)) \quad \text{as } y \rightarrow \infty. \quad (29)$$

Collecting (25), (28) and (29) concludes the proof of the first statement.

We turn to the case $\alpha = \beta$. Pick $M \in \mathbb{R}$ and $\mu > 0$ such that $\alpha \log \mu \geq 2M$. Since ψ is nonincreasing,

$$-\frac{1}{\varphi(y)\psi(y)} \int_y^\infty \varphi(z) d\psi(z) \geq -\frac{1}{\varphi(y)\psi(y)} \int_y^{\mu y} \varphi(z) d\psi(z). \quad (30)$$

The expression on the right-hand side of (30) can be rewritten as

$$-\frac{1}{\varphi(y)\psi(y)} \int_y^{\mu y} \varphi(z) d\psi(z) = -\frac{1}{\psi(y)} \left[\int_y^{\mu y} \left[\frac{\varphi(z)}{\varphi(y)} - \left(\frac{z}{y} \right)^\alpha \right] d\psi(z) + y^{-\alpha} \int_y^{\mu y} z^\alpha d\psi(z) \right]. \quad (31)$$

Once again, since ψ is nonincreasing,

$$\frac{1}{\psi(y)} \left| \int_y^{\mu y} \left[\frac{\varphi(z)}{\varphi(y)} - \left(\frac{z}{y} \right)^\alpha \right] d\psi(z) \right| \leq \sup_{1 \leq \lambda \leq \mu} \left| \frac{\varphi(\lambda y)}{\varphi(y)} - \lambda^\alpha \right| \left| 1 - \frac{\psi(\mu y)}{\psi(y)} \right|.$$

A uniform convergence result for the function φ on compact sets (see e.g. Theorem 1.5.2 in [4]) and the convergence $\psi(\mu y)/\psi(y) \rightarrow \mu^{-\beta}$ as $y \rightarrow \infty$ entail

$$\frac{1}{\psi(y)} \left| \int_y^{\mu y} \left[\frac{\varphi(z)}{\varphi(y)} - \left(\frac{z}{y} \right)^\alpha \right] d\psi(z) \right| \rightarrow 0 \quad \text{as } y \rightarrow \infty. \quad (32)$$

Further, an integration by parts yields

$$-\frac{y^{-\alpha}}{\psi(y)} \int_y^{\mu y} z^\alpha d\psi(z) = \left[1 - \frac{(\mu y)^\alpha \psi(\mu y)}{y^\alpha \psi(y)} \right] + \alpha \int_y^{\mu y} \left[\frac{z^\alpha \psi(z)}{y^\alpha \psi(y)} - 1 \right] \frac{dz}{z} + \alpha \log \mu.$$

Since $y \mapsto y^\alpha \psi(y)$ is slowly varying at infinity, Theorem 1.5.2 in [4] therefore gives

$$-\frac{y^{-\alpha}}{\psi(y)} \int_y^{\mu y} z^\alpha d\psi(z) \rightarrow \alpha \log \mu \quad \text{as } y \rightarrow \infty. \quad (33)$$

Collect then (30), (31), (32) and (33) to get for y large enough

$$-\frac{1}{\varphi(y)\psi(y)} \int_y^\infty \varphi(z) d\psi(z) \geq -\frac{1}{\varphi(y)\psi(y)} \int_y^{\mu y} \varphi(z) d\psi(z) = \alpha \log(\mu) (1 + o(1)) \geq \frac{1}{2} \alpha \log \mu \geq M \quad (34)$$

which proves that the left-hand side of this inequality tends to infinity as $y \rightarrow \infty$. We conclude the proof by noting that from (34), for all $\lambda > 0$,

$$\frac{\int_{\lambda y}^\infty \varphi(z) d\psi(z)}{\int_y^\infty \varphi(z) d\psi(z)} - 1 = -\frac{\int_y^{\lambda y} \varphi(z) d\psi(z)}{\int_y^\infty \varphi(z) d\psi(z)} = -\frac{\varphi(y)\psi(y)}{\int_y^\infty \varphi(z) d\psi(z)} \alpha \log(\lambda) (1 + o(1)) \rightarrow 0 \quad \text{as } y \rightarrow \infty$$

which is what we wanted to show. ■

Proof of Lemma 4. Pick $y, t > 0$ and recall from (1) that

$$\overline{F^*}(y) = \frac{1}{p} \int_y^\infty \overline{G}(z) dF(z) \quad \text{and} \quad \overline{G^*}(t) = \frac{1}{p} \int_t^\infty F(z) dG(z).$$

Note first that $F(t) \rightarrow 1$ as $t \rightarrow \infty$; besides, since $dF(z) = -d\overline{F}(z)$, applying Lemma 3 entails

$$\overline{F^*}(y) = \frac{1}{p} \frac{\gamma_G}{\gamma_F + \gamma_G} \overline{F}(y) \overline{G}(y) (1 + o(1)) \quad \text{and} \quad \overline{G^*}(t) = \frac{1}{p} \overline{G}(t) (1 + o(1))$$

as $y, t \rightarrow \infty$, which is Lemma 4. ■

Proof of Lemma 5. Given $N = q$, the random variables Y_1^*, \dots, Y_q^* are independent and identically distributed with cdf F^* . Therefore for all $y \geq y_0$,

$$\mathbb{P}(Y_{N,N}^* \leq y) = \mathbb{P}(N = 0) + \sum_{q=1}^n \mathbb{P}(Y_{N,N}^* \leq y | N = q) \mathbb{P}(N = q) = \sum_{q=0}^n C_n^q [pF^*(y)]^q (1-p)^{n-q}$$

which yields the first part of the result. Use now Lemma 4 to obtain

$$n \log(1 - p\bar{F}^*(y_n)) = -pn\bar{F}^*(y_n)(1 + o(1)) = -\frac{\gamma_G}{\gamma_F + \gamma_G} n\bar{F}(y_n)\bar{G}(y_n)(1 + o(1)).$$

Especially, $\mathbb{P}(Y_{N,N}^* \leq y_n) \rightarrow 0$, which concludes the proof. \blacksquare

Proof of Lemma 6. Pick λ such that $\lambda p > 1$. A proof similar to that of Lemma 1.2 in [20] yields

$$\mathbb{P}\left(\sup_{\substack{1 \leq i \leq N \\ Y_i^* > y_n}} \frac{C^*(Y_i^*)}{\bar{C}_N^*(Y_i^*)} \geq \lambda\right) \leq \exp((- \lambda p + \log(\lambda p) + 1)F(y_n) + 1). \quad (35)$$

Moreover, since $\log(x)/x \leq e^{-1}$ for all $x > 1$,

$$(- \lambda p + \log(\lambda p) + 1)F(y_n) + 1 = \lambda p \left[-1 + \frac{\log(\lambda p)}{\lambda p}\right] + F(y_n) + 1 \leq \lambda p[-1 + e^{-1}] + 2. \quad (36)$$

Collecting (35) and (36) completes the proof. \blacksquare

Proof of Lemma 7. Assume that the sequence (X_n) is not stochastically bounded, that is:

$$\exists \varepsilon > 0, \forall M > 0, \exists q \in \mathbb{N}, \mathbb{P}(|X_q| > M) > \varepsilon.$$

This entails that there exists a sequence of integers (q_k) , which we may choose to be increasing, such that $\forall k \in \mathbb{N} \setminus \{0\}$, $\mathbb{P}(|X_{q_k}| > k) > \varepsilon$. Let then (δ_n) be the sequence defined as $\delta_n = 1/k$ if $q_k \leq n < q_{k+1}$. The sequence (δ_n) converges to 0, and it holds that $\forall k \in \mathbb{N} \setminus \{0\}$, $\mathbb{P}(|\delta_{q_k} X_{q_k}| > 1) > \varepsilon$ which shows that $(\delta_n X_n)$ does not converge in probability to 0: Lemma 7 is proven. \blacksquare

Proof of Lemma 8. Start by using the change of variables $z = yu$ to obtain

$$\frac{1}{\psi(y)} \int_y^\infty \frac{\psi(z)}{z} dz - \frac{1}{\alpha} = \int_1^\infty u^{-\alpha} \left[\exp\left(\int_y^{yu} \frac{\Delta(v)}{v} dv\right) - 1 \right] \frac{du}{u} \quad (37)$$

where for every $y > 0$, $\varphi_y : (1, \infty) \rightarrow \mathbb{R}$ is the function defined by

$$\forall u \geq 1, \varphi_y(u) = \exp\left(\int_y^{yu} \frac{\Delta(v)}{v} dv\right) - 1.$$

Since we are only interested in the behavior of Δ at infinity, we may assume without loss of generality that Δ has constant sign. Define for all $u \geq 1$,

$$D_\rho(u) = \int_1^u t^{\rho-1} dt.$$

Since

$$\int_y^{yu} \frac{\Delta(v)}{v} dv = \Delta(y) \int_1^u \frac{\Delta(yw)}{\Delta(y)} \frac{dw}{w}$$

a uniform convergence result (see Theorem 1.5.2 in [4]) yields

$$\forall u \geq 1, \varphi_y(u) = \Delta(y) D_\rho(u) (1 + o(1)) \text{ as } y \rightarrow \infty. \quad (38)$$

Further, Theorem 1.5.6 in [4] entails that there exists $y_1 > 0$ such that

$$\forall w \geq 1, \forall y \geq y_1, \left| \frac{\Delta(yw)}{\Delta(y)} \right| \leq 2w^{\rho+\alpha/4}$$

and we may choose y_1 so large that for every $y \geq y_1$, $|\Delta(y)| < \alpha/2$. It is then clear that for $y \geq y_1$:

$$\forall u \geq 1, \frac{1}{\Delta(y)} \left| \int_y^{yu} \frac{\Delta(v)}{v} dv \right| \leq 2D_{\rho+\alpha/4}(u).$$

The inequality $|e^t - 1| \leq |t|e^{|t|}$, valid for every $t \in \mathbb{R}$, now entails

$$\forall u \geq 1, u^{-\alpha-1} \sup_{y \geq y_1} \left| \frac{\varphi_y(u)}{\Delta(y)} \right| \leq 2u^{-1-\alpha/2} D_{\rho+\alpha/4}(u). \quad (39)$$

Because the right-hand side of this inequality defines an integrable function on $[1, \infty)$, we may collect (37), (38), (39) and apply the dominated convergence theorem to obtain

$$\frac{1}{\psi(y)} \int_y^\infty \frac{\psi(z)}{z} dz = \frac{1}{\alpha} + \Delta(y) \int_1^\infty u^{-\alpha} D_\rho(u) \frac{du}{u} (1 + o(1)) \text{ as } y \rightarrow \infty.$$

The result is then a straightforward consequence of this equality. ■

Proof of Lemma 9. Note that

$$y \frac{d}{dy} \log \overline{F^*}(y) = -\frac{yF'(y)\overline{G}(y)}{p\overline{F^*}(y)} \text{ and } t \frac{d}{dt} \log \overline{G^*}(t) = -\frac{tF(t)G'(t)}{p\overline{G^*}(t)}.$$

We start by controlling $\overline{F^*}(y)$. Using (6) leads to

$$\frac{p\overline{F^*}(y)}{yF'(y)\overline{G}(y)} = \frac{1}{1 - \gamma_F \Delta_F(y)} [I_1(y) + I_2(y)]$$

with

$$I_1(y) = \int_y^\infty \frac{\overline{F}(z)\overline{G}(z)}{\overline{F}(y)\overline{G}(y)} \frac{dz}{z} \text{ and } I_2(y) = -\gamma_F \Delta_F(y) \int_y^\infty \frac{\Delta_F(z)\overline{F}(z)\overline{G}(z)}{\Delta_F(y)\overline{F}(y)\overline{G}(y)} \frac{dz}{z}.$$

Lemma 8 leads to

$$I_1(y) = \gamma_{F^*} + [\Delta_F(y) + \Delta_G(y)] \frac{\gamma_{F^*}^2}{1 - \rho_{FG}} (1 + o(1)) \text{ as } y \rightarrow \infty. \quad (40)$$

Further, using Theorem 0.6.(a) in [19], we obtain

$$I_2(y) = -\gamma_F \Delta_F(y) \left[\frac{1}{\gamma_{F^*}} - \frac{\rho_F}{\gamma_F} \right]^{-1} (1 + o(1)) \text{ as } y \rightarrow \infty. \quad (41)$$

Collect now (40), (41) and use a Taylor expansion to obtain

$$\frac{yF'(y)\overline{G}(y)}{p\overline{F^*}(y)} = \gamma_{F^*} \left[1 - \Delta_F(y) \frac{\rho_F \gamma_{F^*} (\gamma_F + \gamma_G)}{\gamma_F + (1 - \rho_F) \gamma_G} (1 + o(1)) + [\Delta_F(y) + \Delta_G(y)] \frac{\gamma_{F^*}}{1 - \rho_{FG}} (1 + o(1)) \right]$$

as $y \rightarrow \infty$. As a consequence, we get

$$y \frac{d}{dy} \log \overline{F^*}(y) = -\frac{1}{\gamma_{F^*}} - \Delta_F(y) \frac{\rho_F (\gamma_F + \gamma_G)}{\gamma_F + (1 - \rho_F) \gamma_G} (1 + o(1)) + [\Delta_F(y) + \Delta_G(y)] \frac{1}{1 - \rho_{FG}} (1 + o(1))$$

as $y \rightarrow \infty$, which completes the proof of the second statement. We now turn to the control of $\overline{G^*}(t)$.

Using (6) entails

$$\frac{p\overline{G^*}(t)}{tF(t)G'(t)} = J_1(t) + \frac{1}{1 - \gamma_G \Delta_G(t)} [J_2(t) + J_3(t)]$$

with

$$J_1(t) = \int_t^\infty \left[\frac{F(z)}{F(t)} - 1 \right] \frac{zG'(z) dz}{tG'(t) z}, \quad J_2(t) = \int_t^\infty \frac{\overline{G}(z) dz}{\overline{G}(t) z} \quad \text{and} \quad J_3(t) = -\gamma_G \Delta_G(t) \int_t^\infty \frac{\Delta_G(z) \overline{G}(z) dz}{\Delta_G(t) \overline{G}(t) z}.$$

To control J_1 , note that

$$J_1(t) = \frac{\overline{F}(t)}{F(t)} \int_t^\infty \left[1 - \frac{\overline{F}(z)}{\overline{F}(t)} \right] \frac{zG'(z) dz}{tG'(t) z};$$

we may then apply Theorem 0.6.(a) in [19] to obtain

$$J_1(t) = \overline{F}(t) \frac{\gamma_G^2}{\gamma_F + \gamma_G} (1 + o(1)) \quad \text{as } t \rightarrow \infty. \quad (42)$$

The function J_2 is controlled by using Lemma 8:

$$J_2(t) = \gamma_G + \Delta_G(t) \frac{\gamma_G^2}{1 - \rho_G} (1 + o(1)) \quad \text{as } t \rightarrow \infty. \quad (43)$$

Finally, using once again Theorem 0.6.(a) in [19] yields

$$J_3(t) = -\Delta_G(t) \frac{\gamma_G^2}{1 - \rho_G} (1 + o(1)) \quad \text{as } t \rightarrow \infty. \quad (44)$$

Collecting (42), (43) and (44) we obtain

$$\frac{p\overline{G}^*(t)}{tF(t)G'(t)} = \gamma_G \left[1 + \overline{F}(t) \frac{\gamma_G}{\gamma_F + \gamma_G} (1 + o(1)) + \gamma_G \Delta_G(t) (1 + o(1)) \right] \quad \text{as } t \rightarrow \infty.$$

Hence the equality

$$t \frac{d}{dt} \log \overline{G}^*(t) = -\frac{1}{\gamma_G} + \overline{F}(t) \frac{1}{\gamma_F + \gamma_G} (1 + o(1)) + \Delta_G(t) (1 + o(1)) \quad \text{as } t \rightarrow \infty$$

which completes the proof. ■

Proof of Lemma 10. We start by noting that if $I_n = \mathbb{N} \cap [u_n, \infty)$ then $\mathbb{P}(N \in I_n) \rightarrow 1$. To prove the consistency statement, we write for all $t > 0$, $\mathbb{P}(|\widehat{\gamma}_N(k_N) - \gamma| > t) = r_{1,n} + r_{2,n}$ with

$$r_{1,n} = \mathbb{P}(|\widehat{\gamma}_N(k_N) - \gamma| > t, N \notin I_n) \quad \text{and} \quad r_{2,n} = \sum_{q \in I_n} \mathbb{P}(|\widehat{\gamma}_N(k_N) - \gamma| > t | N = q) \mathbb{P}(N = q).$$

It is clear that $r_{1,n} \leq \mathbb{P}(N \notin I_n) \rightarrow 0$. Besides

$$r_{2,n} \leq \sup_{q \in I_n} \mathbb{P}(|\widehat{\gamma}_q(k_q) - \gamma| > t) \rightarrow 0$$

since $\widetilde{\gamma}_n(k_n) \xrightarrow{\mathbb{P}} \gamma$, see Theorem 3.2.2 p.70 in [6]. To show the asymptotic normality of $\widehat{\gamma}_N(k_N)$, we let φ be the cumulative distribution function of a centered Gaussian distribution with variance γ^2 and we write for all $t \in \mathbb{R}$, $\mathbb{P}(\sqrt{k_N}(\widehat{\gamma}_N(k_N) - \gamma) \leq t) - \varphi(t) = r_{3,n} + r_{4,n}$ with

$$\begin{aligned} r_{3,n} &= \mathbb{P}(\sqrt{k_N}(\widehat{\gamma}_N(k_N) - \gamma) \leq t, N \notin I_n) - \varphi(t) \mathbb{P}(N \notin I_n) \\ \text{and } r_{4,n} &= \sum_{q \in I_n} \left[\mathbb{P}(\sqrt{k_N}(\widehat{\gamma}_N(k_N) - \gamma) \leq t | N = q) - \varphi(t) \right] \mathbb{P}(N = q). \end{aligned}$$

Again, clearly $r_{3,n} \leq 2\mathbb{P}(N \notin I_n) \rightarrow 0$. Moreover

$$r_{4,n} \leq \sup_{q \in I_n} \left| \mathbb{P}(\sqrt{k_q}(\widetilde{\gamma}_q(k_q) - \gamma) \leq t) - \varphi(t) \right|$$

so that it is enough to show that $\tilde{\gamma}_n(k_n)$ converges in distribution to a centered Gaussian distribution with variance γ^2 . To this end first note that

$$\forall t > 0, \frac{1}{\overline{H}(t)} = t^{1/\gamma} c_H^{-1} \exp\left(\int_1^t -\frac{\Delta(v)}{v} dv\right) \Rightarrow \forall y \geq 1, \log y = \frac{1}{\gamma} \log U(y) - \log c_H - \int_1^{U(y)} \frac{\Delta(v)}{v} dv.$$

It is then straightforward that $\forall y \geq 1, U(y)/(yU'(y)) = \gamma^{-1} - \Delta \circ U(y)$ which entails that $\forall y \geq 1,$

$$U(y) = y^{1/\gamma} \left\{ c \exp\left(\int_1^y \frac{\delta(v)}{v} dv\right) \right\}$$

where c is a positive constant and $\delta = -\Delta \circ U$ is a bounded measurable function converging to 0 at infinity. Consequently, U (resp. $|\delta|$) is regularly varying at infinity with index γ (resp. with index ρ). Since (Z_1, \dots, Z_n) has the same distribution as $(U(X_1), \dots, U(X_n))$ where (X_1, \dots, X_n) is a sample of independent copies of standard Pareto random variables, we may then write $\tilde{\gamma}_n(k_n) \stackrel{d}{=} S_{n,1} + S_{n,2}$ with

$$S_{n,1} = \frac{\gamma}{k_n} \sum_{i=1}^{k_n} [\log X_{n-i+1,n} - \log X_{n-k_n,n}] \text{ and } S_{n,2} = \frac{1}{k_n} \sum_{i=1}^{k_n} \int_{X_{n-k_n,n}}^{X_{n-i+1,n}} \frac{\delta(v)}{v} dv.$$

Rényi's representation and Paul Lévy's central limit theorem entail

$$\sqrt{k_n}(S_{n,1} - \gamma) \stackrel{d}{=} \gamma \sqrt{k_n} \left[\frac{1}{k_n} \sum_{i=1}^{k_n} \log(X_i) - 1 \right] \xrightarrow{d} \mathcal{N}(0, \gamma^2).$$

Moreover, since $(n/k_n)X_{n-k_n,n} \xrightarrow{\mathbb{P}} 1$ (see Corollary 2.2.2 p.41 in [6]) and $|\delta|$ is regularly varying, we get $\sqrt{k_n}|S_{n,2}| \leq \sqrt{k_n}|\delta(X_{n-k_n,n})|S_{n,1} = \sqrt{k_n}|\delta(n/k_n)|(1 + o_{\mathbb{P}}(1)) \xrightarrow{\mathbb{P}} 0$. Using Slutsky's lemma completes the proof of Lemma 10. ■

Estimator $\hat{q}_N^{(r(\theta))}$												
p	0.7			0.8			0.9			0.95		
θ	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\gamma_F = 1/4$	0.08	0.10	0.16	0.07	0.08	0.12	0.06	0.07	0.10	0.05	0.06	0.08
$\gamma_F = 1/2$	0.31	0.38	0.60	0.26	0.31	0.45	0.23	0.27	0.36	0.21	0.25	0.32
$\gamma_F = 1$	1.22	1.53	2.27	1.05	1.28	1.82	0.91	1.08	1.49	0.85	0.99	1.29

Estimator $\hat{q}_N^{W,(s(\theta))}(\cdot \alpha_{opt}^{(s(\theta))})$												
p	0.7			0.8			0.9			0.95		
θ	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\gamma_F = 1/4$	0.004	0.03	0.22	0.003	0.02	0.10	0.002	0.01	0.06	0.002	0.01	0.04
$\gamma_F = 1/2$	0.01	0.10	0.50	0.007	0.05	0.27	0.004	0.03	0.16	0.004	0.03	0.12
$\gamma_F = 1$	0.04	0.39	1.71	0.03	0.25	1.15	0.02	0.13	0.61	0.01	0.09	0.39

Table 1: Errors associated with the estimators $\hat{q}_N^{(r(\theta))}$ and $\hat{q}_N^{W,(s(\theta))}(\cdot|\alpha_{opt}^{(s(\theta))})$ for $\delta = 1/3$.

Estimator $\hat{q}_N^{(r(\theta))}$												
p	0.7			0.8			0.9			0.95		
θ	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\gamma_F = 1/4$	0.11	0.17	0.36	0.09	0.12	0.25	0.07	0.1	0.16	0.07	0.09	0.13
$\gamma_F = 1/2$	0.34	0.46	0.85	0.28	0.36	0.55	0.24	0.29	0.43	0.22	0.27	0.37
$\gamma_F = 1$	1.22	1.55	2.46	1.05	1.29	1.9	0.92	1.09	1.46	0.86	1.01	1.33

Estimator $\hat{q}_N^{W,(s(\theta))}(\cdot \alpha_{opt}^{(s(\theta))})$												
p	0.7			0.8			0.9			0.95		
θ	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9	0.1	0.5	0.9
$\gamma_F = 1/4$	0.05	0.22	2.84	0.04	0.17	1.00	0.03	0.12	0.49	0.03	0.10	0.30
$\gamma_F = 1/2$	0.04	0.24	2.43	0.03	0.14	0.85	0.02	0.09	0.42	0.02	0.07	0.27
$\gamma_F = 1$	0.05	0.46	2.65	0.03	0.25	1.42	0.02	0.15	0.66	0.02	0.11	0.53

Table 2: Errors associated with the estimators $\hat{q}_N^{(r(\theta))}$ and $\hat{q}_N^{W,(s(\theta))}(\cdot|\alpha_{opt}^{(s(\theta))})$ for $\delta = 1$.