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# ON A CLASS OF WEIGHTED GAUSS-TYPE ISOPERIMETRIC INEQUALITIES AND APPLICATIONS TO SYMMETRIZATION

MICHELE MARINI AND BERARDO RUFFINI

ABSTRACT. We solve a class of weighted isoperimetric problems of the form

$$\min \left\{ \int_{\partial E} w e^V dx : \int_E e^V dx = \text{constant} \right\}$$

where  $w$  and  $V$  are suitable functions on  $\mathbb{R}^N$ . As a consequence, we prove a comparison result for the solutions of degenerate elliptic equations.

## 1. INTRODUCTION

In the celebrated paper [9], G. Talenti established several comparison results between the solutions of the Poisson equation with Dirichlet boundary condition (with suitable data  $f$  and  $E$ ):

$$(1.1) \quad -\Delta u = f \text{ in } E, \quad u = 0 \text{ on } \partial E$$

and the solutions of the corresponding problem where  $f$  and  $E$  are replaced by their spherical rearrangements (see [8, Chapter 3] for the definition and main properties of spherical rearrangement). Precisely, he proves that if we denote by  $v$  the solution of the problem with symmetrized data, then the rearrangement  $u^*$  of the (unique) solution  $u$  of (1.1) is pointwisely bounded by  $v$ . Moreover he shows that the  $L^q$  norm of  $\nabla u$  is bounded, as well, by the  $L^q$  norm of  $\nabla v$ , for  $q \in (0, 2]$ . The proof of these facts basically relies on two ingredients: the Hardy-Littlewood-Sobolev inequality and the isoperimetric inequality (see [1] and [8] for comprehensive accounts on the subjects).

Later on, following such a scheme, many other works have been developed to prove analogous comparison results related to the solutions of PDEs involving different kind of operators, see for instance [2, 3, 6, 7] and the references therein. A recurring idea in these works is, roughly speaking, the following. The operator considered is usually linked to a sort of *weighted perimeter*. Thus initially it is necessary to solve a corresponding isoperimetric problem; then the desired comparison results can be obtained following the ideas contained in [9].

For example in [3] the authors consider a class of weighted perimeters of the form

$$P_w(E) = \int_{\partial E} w(|x|) d\mathcal{H}^{d-1}(x),$$

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where  $E$  is a Lipschitz set and  $w : \mathbb{R} \rightarrow [0, \infty)$  a non-negative function, and prove, under suitable convexity assumptions on the weight  $w$ , that the ball centred at the origin is the unique solution of the mixed isoperimetric problem

$$\min\{P_w(E) : |E| = \text{constant}\}$$

where  $|\cdot|$  denotes the  $d$ -dimensional Lebesgue measure. As a consequence they prove comparison results, analogous to those considered by Talenti in [9], for the solutions of

$$-\operatorname{div}(w^2 \nabla u) = f \text{ in } E, \quad u = 0 \text{ on } \partial E.$$

Recently in [4], L. Brasco, G. De Philippis and the second author proved a quantitative version of the weighted isoperimetric inequality considered in [2]. Their proof is achieved by means of a sort of *calibration technique*. One advantage of this technique is that it is adaptable to other kind of problems, as that of considering other kind of functions in the weighted perimeter (e.g. Wulff-type weights, see [5]), or that of considering different measured spaces, as  $\mathbb{R}^d$  provided with the Gauss measure.

In this paper we consider degenerate elliptic equations with Dirichlet boundary condition of the form

$$(1.2) \quad -\operatorname{div}(w^2 e^V \nabla u) = f e^V \text{ in } E, \quad u = 0 \text{ on } \partial E$$

where  $w$  and  $V$  are two given functions, and we aim to prove analogous comparison results as those in [9]. The particular form in which is written the measure  $e^V$  is due to the later applications, whose main examples are Gauss-type measures, that is  $V(x) = -c|x|^2$ . Bearing in mind this instance, we consider a class of mixed isoperimetric problems of the form

$$\min \left\{ P_{we^V}(E) : \int_E e^V = \text{constant} \right\}$$

and prove, by means of a calibration technique reminiscent of that developed in [4], that the solutions, under suitable assumptions on  $V$  and  $w$ , are half-spaces, see Proposition 3.1. Then, using a suitable concept of decreasing rearrangement related to the measures considered, we prove, in the Main Theorem in section 4, some comparison results between the solutions of (1.2) and the solutions of the same equation with rearranged data.

## 2. PRELIMINARIES ON REARRANGEMENT INEQUALITIES

In this section we introduce the main definitions and properties about the concept of symmetrization and rearrangement we shall make use of.

Let  $\mu$  be a finite Radon measure on  $\mathbb{R}^d$ , a *right rearrangement* with respect to  $\mu$  is defined, for any Borel set  $A$ , as

$$R_A^\mu = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 > t_A\},$$

where  $t_A = \inf \{t : \mu(A) = \mu(\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 > t\})\}$ . Notice that if  $d\mu = f dx$ , for some positive and measurable function  $f$ , then the value of  $t$  is uniquely determined. Given a non-negative Borel function  $f : \mathbb{R}^d \rightarrow$

$[0, +\infty)$ , we call *right increasing rearrangement* of  $f$  the function  $f^{*\mu}$  given by

$$f^{*\mu}(x) = \int_0^{+\infty} \chi_{R_{\{f>t\}}^\mu}(x) dt$$

where  $\chi_A$  is the characteristic function of the set  $A$ . As an aside we notice that the right increasing rearrangement of the characteristic function of a Borel set  $A$  coincides with the characteristic function of  $R_A^\mu$ . Clearly  $f^{*\mu}$  is non-negative, increasing with respect to the first variable  $x_1$ , and constant on the sets  $\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 = t\}$ , for  $t \in \mathbb{R}$ . Moreover  $f$  and  $f^{*\mu}$  share the same distribution function:

$$\mu_f(t) := \mu(\{f > t\}) = \mu(\{f^{*\mu} > t\}) = \mu_{f^{*\mu}}(t).$$

We furthermore define  $f^{*\mu} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  as the smallest decreasing function satisfying  $f^{*\mu}(\mu_f(t)) \geq t$ ; in other words

$$f^{*\mu}(s) = \inf\{t > 0 : \mu_f(t) < s\}.$$

It is useful to bear in mind that  $\{s : f^{*\mu}(s) > t\} = [0, \mu_f(t)]$  so that by the Layer-Cake Representation Theorem (see for instance [8]) we have

$$(2.3) \quad \int_0^{\mu(\{x_1 > t\})} f^{*\mu}(s) ds = \int_t^\infty \mu_f(s) ds = \int_{\{x_1 > t\}} f^{*\mu}(x) dx.$$

We conclude this section by proving the *Hardy-Littlewood* rearrangement inequality related to the right symmetrization.

**Lemma 2.1** (Hardy-Littlewood rearrangement inequality). *Let  $f$  and  $g$  be non-negative Borel functions from  $\mathbb{R}^d$  to  $\mathbb{R}$ . Then for any non-negative Borel measure  $\mu$  we have*

$$\int_{\mathbb{R}^d} f g d\mu \leq \int_{\mathbb{R}^d} f^{*\mu} g^{*\mu} d\mu.$$

*Proof.* We have

$$\begin{aligned} \int_{\mathbb{R}^d} f g d\mu &= \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \chi_{\{f>t\}}(x) \chi_{\{g>s\}}(x) dt ds d\mu(x) \\ &= \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} \chi_{\{f>t\} \cap \{g>s\}}(x) d\mu(x) dt ds \\ &= \int_0^\infty \int_0^\infty \mu(\{f > t\} \cap \{g > s\}) dt ds \\ &\leq \int_0^\infty \int_0^\infty \min(\mu(\{f > t\}), \mu(\{g > s\})) dt ds \\ &= \int_0^\infty \int_0^\infty \min(\mu(\{f^{*\mu} > t\}), \mu(\{g^{*\mu} > s\})) dt ds \\ &= \int_0^\infty \int_0^\infty \mu(\{f^{*\mu} > t\} \cap \{g^{*\mu} > s\}) dt ds = \int_{\mathbb{R}^d} f^{*\mu} g^{*\mu} d\mu \end{aligned}$$

where we used the fact that  $\{f^{*\mu} > t\}$  and  $\{g^{*\mu} > s\}$  are hyper-spaces of the form  $\{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 > r\}$  for some  $r \in \mathbb{R}$  and so

$$\min(\mu(\{f^{*\mu} > t\}), \mu(\{g^{*\mu} > s\})) = \mu(\{f^{*\mu} > t\} \cap \{g^{*\mu} > s\}).$$

□

**Remark.** Setting  $g = \chi_A$  in Lemma 2.1 and thanks to (2.3) we get

$$(2.4) \quad \int_A f \, dx \leq \int_{R_A^\mu} f^{*\mu}(x) \, dx = \int_0^{\mu(A)} f^{*\mu}(s) \, ds.$$

### 3. A CLASS OF WEIGHTED ISOPERIMETRIC INEQUALITIES

Given a measurable function  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  we denote by  $\mu[V]$  the absolutely continuous measure whose density equals  $e^V$ , that is, for any measurable set  $E \subset \mathbb{R}^d$

$$\mu[V](E) = \int_E e^{V(x)} \, dx;$$

in what follows with the scope of simplifying the notation, and if there is no risk of confusion, we will drop the dependence of  $V$ , writing  $\mu$  instead of  $\mu[V]$ . Moreover we will often adopt the notation  $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1}$  and denote by  $R_A$  instead of  $R_A^{\mu[V]}$  the right rearrangement of  $A$  with respect to the measure  $\mu[V]$ . Given a Borel *weight* function  $w : \mathbb{R} \rightarrow [0, +\infty]$  we define, for any open set  $A$  with Lipschitz boundary, the following concept of *weighted perimeter*:

$$P_{w,V}(A) = \int_{\partial A} w(x_1) e^{V(x)} \, d\mathcal{H}^{d-1}(x).$$

In the following proposition we show that, under suitable conditions on  $w$  and  $V$ , the half-spaces of the form  $\{(x_1, x') : x_1 > t\}$  are the only minimizers of the weighted perimeter among the sets of fixed volume with respect to the measure  $\mu[V]$ .

**Proposition 3.1.** *Let  $A \subset \mathbb{R}^d$  be a Lipschitz set. Suppose that  $w : \mathbb{R} \rightarrow \mathbb{R}^+$  and  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  are  $C^1$ -regular functions satisfying the following assumptions.*

- (i)  $\mu(A)$ ,  $P_{w,V}(A)$ ,  $\mu(R_A)$  and  $P_{w,V}(R_A)$  are finite quantities;
- (ii) the function  $\partial_1 V(x)$  depends only on  $x_1$  and  $g(x) := -w'(x_1) - w(x_1)\partial_1 V(x)$  is a decreasing function on the real line.

Then

$$P_{w,V}(A) \geq P_{w,V}(R_A).$$

*Proof.* Let  $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$  and consider the vector field  $-e_1 w(x_1) e^{V(x)}$ . Its divergence is given by

$$\operatorname{div}(-e_1 w(x_1) e^{V(x)}) = (-w'(x_1) - w(x_1)\partial_1 V(x)) e^{V(x)} = g(x) e^{V(x)}.$$

By an application of the Divergence Theorem we have

$$(3.5) \quad \begin{aligned} \int_A g(x) \, d\mu(x) &= \int_A \operatorname{div}(-e_1 w(x_1) e^{V(x)}) \, dx \\ &= \int_{\partial A} w(x_1) e^{V(x)} \langle \nu_A(x), -e_1 \rangle \, d\mathcal{H}^{d-1}(x) \\ &\leq \int_{\partial A} w(x_1) e^{V(x)} \, d\mathcal{H}^{d-1}(x) = P_{w,V}(A), \end{aligned}$$

where  $\nu_A(x)$  is the outer unit normal to  $\partial A$  at  $x$ . Let  $t_A$  be a real number such that the right half-space  $R_A = \{(x_1, x') : x_1 \geq t_A\}$  satisfies  $\mu(R_A) = \mu(A)$ .

Then, since the outer normal of  $R_A$  is the constant vector field  $-e_1$ , the inequality in (3.5) turns into an equality if we replace  $A$  with  $R_A$ . Hence, by condition (i), we get

$$P_{w,V}(A) - P_{w,V}(R_A) \geq \int_A g(x) d\mu(x) - \int_{R_A} g(x) d\mu(x).$$

Since, by definition,  $\mu(A) = \mu(R_A)$  again by condition (i) we obtain  $\mu(A \setminus R_A) = \mu(R_A \setminus A)$ . Thus

$$\begin{aligned} \int_A g(x) d\mu(x) - \int_{R_A} g(x) d\mu(x) &= \int_{A \setminus R_A} g(x) d\mu(x) - \int_{R_A \setminus A} g(x) d\mu(x) \\ &= \int_{A \setminus R_A} (g(x) - g(t_A e_1)) d\mu(x) - \int_{R_A \setminus A} (g(x) - g(t_A e_1)) d\mu(x). \end{aligned}$$

Since every  $x \in A \setminus R_A$  (respectively  $x \in R_A \setminus A$ ) satisfies  $\langle x, e_1 \rangle < t_A$  (respectively  $\langle x, e_1 \rangle > t_A$ ), by condition (ii) we deduce

$$\begin{aligned} P_{w,V}(A) - P_{w,V}(R_A) &\geq \int_{A \setminus R_A} |g(x) - g(t_A e_1)| d\mu(x) + \int_{R_A \setminus A} |g(x) - g(t_A e_1)| d\mu(x) \\ &= \int_{A \Delta R_A} |g(x) - g(t_A e_1)| d\mu \geq 0, \end{aligned}$$

where  $A \Delta R_A = (A \setminus R_A) \cup (R_A \setminus A)$  stands for the symmetric difference between  $A$  and  $R_A$ . This concludes the proof.  $\square$

Condition (ii) in Proposition 3.1 is satisfied by functions of the form

$$V(x_1, x') = V_1(x_1) + V_2(x'),$$

where  $V_1 \in C^2(\mathbb{R})$  and, together with the weight  $w \geq 0$  respect the ordinary differential inequality

$$(3.6) \quad w'' + V_1'' w + V_1' w' \geq 0.$$

To get non-trivial instances of functions which fulfil inequality (3.6) together with the integrability property (i) of Proposition 3.1 it is worth restricting our attention to the half-space

$$\Omega = \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{d-1} : x_1 > 0\}.$$

In this case we get as an immediate corollary of Proposition 3.1 that the solution of the problem

$$(3.7) \quad \min \left\{ P_{w,V}(A) : \chi_A \in L^1(\Omega, w e^V), \partial A \text{ Lipschitz}, \int_A e^V = c \right\}$$

is given by  $R_c = \{x_1 \geq t_c\}$  where  $t_c$  is such that  $\int_{R_c} e^V = c$ . We remark, moreover, that the requirement for the sets of having Lipschitz boundary in (3.7) can be easily removed by means of an approximation argument, that is,  $R_c$  is the unique minimizer of

$$(3.8) \quad \min \left\{ P_{w,V}(A) : \chi_A \in L^1(\Omega, w e^V), \int_A e^V = c \right\}.$$

A class of examples of particular interest is given by  $V(x) = -c|x|^2$  for  $c > 0$ , that is  $e^V dx$  corresponds to the (rescaled) Gauss measure. In this case instances of functions  $w$  which satisfy the hypotheses of Proposition

3.1 are given by  $w(t) = t^{-\alpha}$  with  $\alpha \geq 1$  or  $w(t) = e^{-t}$ . We recall that, as pointed out in the Introduction, similar problems are studied, for instance, in [6] where it is considered the case  $w(t) = t^k$ , with  $k > 0$ .

#### 4. MAIN RESULT

In this section we define  $d\mu = e^V dx$ ,  $R_E = \{x_1 > t_E\}$  where  $t_E \in \mathbb{R}$  is such that  $\mu(R_E) = \mu(E)$  and  $f^* = f^{*\mu}$  the right-symmetrized of a function  $f$  with respect to  $\mu$ .

**Main Theorem.** *Suppose that the set  $E \subset \Omega = \{(x_1, x') : x_1 > 0\}$  and the functions  $w : [0, +\infty] \rightarrow [0, +\infty]$  and  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfy the hypotheses of Proposition 3.1. Consider the two problems*

$$(4.9) \quad \begin{cases} -\operatorname{div}(w^2 e^V \nabla u) = f e^V & \text{in } E \\ u = 0 & \text{on } \partial E \end{cases}$$

and

$$(4.10) \quad \begin{cases} -\operatorname{div}(w^2 e^V \nabla v) = f^* e^V & \text{in } R_E \\ v = 0 & \text{on } \partial R_E \end{cases}$$

where  $f \in L^2(\Omega, \mu)$ . Then the problem (4.10) has as solution the one variable function  $v(z)$  given by

$$(4.11) \quad v((z, z')) = v(z) = \int_{\mu(\{x_1 \geq z\})}^{\mu(R_E)} \frac{1}{h^2(s)} \left( \int_0^s f^*(\xi) d\xi \right) ds,$$

where

$$(4.12) \quad h(m) = w(\Phi^{-1}(m)) \int_{\mathbb{R}^{d-1}} \mu(\Phi^{-1}(m), x') dx',$$

being  $\Phi(t) = \mu(\{x_1 > t\})$ . Moreover, for any solution  $u$  of the problem (4.9), we have

$$(4.13) \quad u^*(x) \leq v^*(x),$$

and, for any  $q \in (0, 2]$ ,

$$(4.14) \quad \int_E |\nabla u|^q w^q d\mu \leq \int_{R_E} |\nabla v|^q w^q d\mu$$

*Proof.* Let us suppose for the moment that the function  $v$  given in (4.11) is a solution for the problem (4.10). To prove (4.13) and (4.14) we start noticing that for every test function  $\phi(x)$  we have

$$(4.15) \quad \int_{\Omega} \langle \nabla u, \nabla \phi \rangle w^2 d\mu = \int_{\Omega} f \phi d\mu;$$

if we set

$$\phi_h(x) = \begin{cases} \operatorname{sign}(u) & \text{if } |u(x)| > t + h \\ \frac{u(x) - t \operatorname{sign} u(x)}{h} & \text{if } |u(x)| \in [t, t + h) \\ 0 & \text{if } |u(x)| < t, \end{cases}$$

where  $0 \leq t < \operatorname{ess\,sup}|u|$  and  $h > 0$ , then (4.15) turns into

$$\frac{1}{h} \int_{\{|u| \in [t, t+h)\}} \langle \nabla u, \nabla u \rangle w^2 d\mu = \frac{1}{h} \int_{\{|u| \in [t, t+h)\}} f \left( u - t \frac{u}{|u|} \right) d\mu + \int_{\{|u| > t+h\}} f \operatorname{sign}(u) d\mu.$$

Taking the limit for  $h \rightarrow 0$ , we get

$$(4.16) \quad -\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u|^2 w^2 d\mu = \int_{\{|u|>t\}} f d\mu.$$

Let us analyze the left-hand side of equation (4.16). We claim that it holds true the following inequality:

$$(4.17) \quad -\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u|^2 w^2 d\mu \geq \frac{\left(-\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u| w d\mu\right)^2}{-\mu'_u(t)},$$

where  $\mu_u(t)$  is the distribution function of  $u$  introduced in the first section. Indeed thanks to the Co-Area Formula and the Hölder inequality we get

$$\begin{aligned} -\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u| w d\mu &= \int_{\{|u|=t\}} w d\mu = \int_{\{|u|=t\}} \frac{|\nabla u|^{1/2}}{|\nabla u|^{1/2}} w d\mu \\ &\leq \left( \int_{\{|u|=t\}} |\nabla u| w^2 d\mu \right)^{1/2} \left( \int_{\{|u|=t\}} \frac{1}{|\nabla u|} d\mu \right)^{1/2} \\ &= \left( -\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u|^2 w^2 d\mu \right)^{1/2} \left( \int_{\{|u|=t\}} \frac{1}{|\nabla u|} d\mu \right)^{1/2}, \end{aligned}$$

and

$$-\mu'_u(t) = -\frac{d}{dt} \int_{\{|u|>t\}} e^V \frac{|\nabla u|}{|\nabla u|} = \int_{\{|u|=t\}} \frac{e^V}{|\nabla u|}.$$

Thus (4.17) follows. Thanks to Proposition 3.1 and (3.8) we have

$$(4.18) \quad -\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u| w d\mu = \int_{\{|u|=t\}} w d\mu = P_\mu(\{|u| > t\}) \geq P_\mu(\{u^* > t\}).$$

Let us define

$$(4.19) \quad \Phi(t) = \mu(\{x_1 > t\}).$$

We recall that the weight function  $w$  is constant on the boundary of the super level sets of  $u^*$ , then the perimeter of  $\{u^* > t\}$  can be written as

$$P_\mu(\{u^* > t\}) = w(\tau) \int_{\mathbb{R}^{d-1}} \mu(\tau, x') dx'.$$

Moreover  $\tau \in \mathbb{R}$  satisfies  $\mu_{u^*}(t) = \Phi(\tau)$  that is  $\tau = \Phi^{-1}(\mu_{u^*}(t))$  (notice that  $\Phi$  is a strictly decreasing function and thus invertible) so that we can write the previous formula as

$$(4.20) \quad P_\mu(\{u^* > t\}) = w(\Phi^{-1}(\mu_{u^*}(t))) \int_{\mathbb{R}^{d-1}} \mu(\Phi^{-1}(\mu_{u^*}(t)), x') dx' := h(\mu_{u^*}(t)).$$

Plugging (4.18) in (4.17), and recalling (4.20) we get that

$$(4.21) \quad -\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u|^2 w^2 d\mu \geq \frac{h(\mu_{u^*}(t))^2}{-\mu'_{u^*}(t)}.$$



We pass now to estimate the right-hand side of (4.16): equation (2.4) with  $A = \{|u| > t\}$  turns into

$$(4.22) \quad \int_{\{|u|>t\}} f \, d\mu \leq \int_{\{|u^*>t\}} f^* \, d\mu = \int_0^{\mu_{u^*}(t)} f^*(s) \, ds.$$

Combining (4.22) and (4.21) we get

$$(4.23) \quad \frac{\left(\int_0^{\mu_{u^*}(t)} f^*(s) \, ds\right) \mu'_{u^*}(t)}{h^2(\mu_{u^*}(t))} \leq -1.$$

Reasoning analogously for the function  $v$ , we easily see that, since  $v$  is constant on every set  $\{x_1 = t\}$  and since  $v = v^*$ , (4.23) holds for  $v$  as an equality. Consider now the real function

$$F(r) = \frac{\int_0^r f(s) \, ds}{h(r)^2},$$

and let  $G$  be a primitive of  $F$ . Since  $F \geq 0$ , we have that  $G$  is increasing. Moreover by our previous analysis we have that

$$F(\mu_{u^*}(t)) \mu'_{u^*}(t) \leq -1 = F(\mu_v(t)) \mu'_v(t)$$

so that  $G(\mu_{u^*}(t)) \leq G(\mu_v(t))$ . This implies that  $\mu_{u^*}(t) \leq \mu_v(t)$  for any  $t$  and so that  $u^* \leq v^*$ , since  $u^*$  and  $v$  depends only on  $x_1$  and are increasing .

We pass now to the proof of (4.14). Using the Co-Area Formula and the Hölder inequality and reasoning as before we obtain, for  $0 < q \leq 2$ ,

$$\begin{aligned} -\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u|^q w^q \, d\mu &= \int_{\{|u|=t\}} (|\nabla u|^{\frac{q}{2}} w^q) |\nabla u|^{\frac{q}{2}-1} \, d\mu \\ &\leq \left( \int_{\{|u|=t\}} |\nabla u| w^2 \, d\mu \right)^{q/2} \left( \int_{\{|u|=t\}} |\nabla u|^{-1} \, d\mu \right)^{1-q/2} \\ &= \left( -\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u|^2 w^2 \, d\mu \right)^{q/2} (-\mu'_u(t))^{1-q/2}. \end{aligned}$$

Recalling (4.16) and (4.22) we have

$$-\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u|^2 w^2 \, d\mu \leq \int_0^{\mu_{u^*}(t)} f^*(s) \, ds,$$

thus

$$(4.24) \quad -\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u|^q w^q \, d\mu \leq \left( \int_0^{\mu_{u^*}(t)} f^*(s) \, ds \right)^{q/2} (-\mu'_u(t))^{1-q/2}.$$

Combining (4.24) and (4.23) we finally get

$$-\frac{d}{dt} \int_{\{|u|>t\}} |\nabla u|^q w^q \, d\mu \leq (-\mu'_{u^*}(t)) \left( h(\mu_{u^*}(t))^{-1} \int_0^{\mu_{u^*}(t)} f^*(s) \, ds \right)^q.$$

By integrating on both side between 0 and  $+\infty$ , we get

$$\int_E |\nabla u|^q w^q \, d\mu \leq \int_0^\infty (-\mu'_{u^*}(t)) \left( h(\mu_{u^*}(t))^{-1} \int_0^{\mu_{u^*}(t)} f^*(s) \, ds \right)^q \, dt.$$

We perform the change of variables  $r = \mu_{u^*}(t)$ , so that the above equation turns into

$$\int_E |\nabla u|^q w^q d\mu \leq \int_0^{\mu(E)} \left( h(r)^{-1} \int_0^r f^*(s) ds \right)^q dr.$$

By a straightforward inspection of those steps we notice that  $v$  satisfies

$$\int_{R_E} |\nabla v|^q w^q d\mu = \int_0^\infty (-\mu'_v(t)) \left( h(\mu_v(t))^{-1} \int_0^{\mu_v(t)} f^*(s) ds \right)^q dt;$$

By performing the change of variables  $r = \mu_v(t)$  we find

$$\int_{R_E} |\nabla v|^q w^q d\mu = \int_0^{\mu(R_E)} \left( h(r)^{-1} \int_0^r f^*(s) ds \right)^q dr.$$

Since  $\mu(E) = \mu(R_E)$  we get the desired result.

We are left to prove that the function  $v$  given by (4.11) is a solution of problem (4.10). We start by noticing that equation (4.23) suggests how to derive (4.11): indeed, as we pointed out, any solution  $v$  of (4.10) such that  $v = v^*$  satisfies

$$\frac{\int_0^{\mu_v(t)} f^*(s) ds}{h^2(\mu_v(t))} \mu'_v(t) = -1.$$

By integrating both sides between 0 and  $r$  we obtain

$$\int_0^r \frac{\int_0^{\mu_v(t)} f^*(s) ds}{h^2(\mu_v(t))} \mu'_v(t) dt = -r.$$

so that, by performing the change of variables  $m = \mu_v(t)$ , we get

$$\int_{\mu_v(r)}^{\mu(R_E)} \frac{\int_0^m f^*(s) ds}{h^2(m)} dm = r$$

which is equivalent to

$$v(z, z') = \int_{\mu\{x_1 > z\}}^{\mu(R_E)} \frac{\int_0^m f^*(s) ds}{h^2(m)} dm,$$

that is (4.11). Notice that  $v$  is strictly decreasing and belongs to  $C_{\text{loc}}^{1,1}(R_E)$ . Indeed, recalling (4.12) one can explicitly compute

$$\nabla v(z, z') = e_1 \frac{\partial v}{\partial z}(z, z') = -e_1 \frac{\int_0^{\mu\{x_1 > z\}} f^*(s) ds}{w^2(z) \int_{R^{d-1}} e^{V(z, x')} dx'},$$

where  $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$ . Since  $f^*$  is a decreasing and locally integrable function, then  $f^* \in L_{\text{loc}}^\infty(\mathbb{R})$ ; thus, being  $z \mapsto \mu(\{x_1 > z\})$   $C^1$ -regular, we get that  $\int_0^{\mu\{x_1 > z\}} f^*(s) ds$  is a locally Lipschitz function. Moreover the denominator is locally Lipschitz as well, and locally bounded away from zero. Hence we have that  $\nabla v$  is locally Lipschitz. Thus, recalling that  $\partial_1 V$  depends only on the first variable  $x_1$  it is possible to explicitly compute the divergence of  $w^2 \nabla v e^V$  and check that it satisfies (4.10). This concludes the proof of the theorem.  $\square$

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## REFERENCES

- [1] L. AMBROSIO, N. FUSCO, D. PALLARA: *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.
- [2] M.F. BETTA, F. BROCK, A. MERCALDO, M. R. POSTERARO: *A comparison result related to Gauss measure*, C. R. Acad. Sci. Paris, **334** (2002), 451–456.
- [3] M.F. BETTA, BROCK, MERCALDO, POSTERARO, *A weighted isoperimetric inequality and applications to symmetrization*, J. of Inequal. Appl., **4** (1999), 215–240.
- [4] L. BRASCO, G. DE PHILIPPIS, B. RUFFINI: *Spectral optimization for the Stekloff-Laplacian: the stability issue*, J. Funct. Anal., **262** (2012), 4675–4710.
- [5] L. BRASCO, G. FRANZINA: *An anisotropic eigenvalue problem of Stekloff type and weighted Wulff inequalities*, Nonlinear Differential Equations and Applications, **6** (2013), 1795–1830.
- [6] F. BROCK, F. CHIACCHIO, A. MERCALDO: *A class of degenerate elliptic equations and a Dido’s problem with respect to a measure*, J. Math. Anal. Appl., **348** (2008), 356–365.
- [7] F. BROCK, F. CHIACCHIO, A. MERCALDO: *Weighted isoperimetric inequalities in cones and applications*, Nonlinear Analysis: Theory, Methods and Applications, **75** (2012), 5737–5755.
- [8] E.H. LIEB, M. LOSS: *Analysis*, American Mathematical Society, 1997.
- [9] G. TALENTI: *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **4** (1976), 697–718.

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