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| Samy Skander Bahoura. On the mass of the exterior blow-up points.. 2014. hal-00940977

**HAL Id: hal-00940977**

**<https://hal.science/hal-00940977>**

Preprint submitted on 3 Feb 2014

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# ON THE MASS OF THE EXTERIOR BLOW-UP POINTS.

SAMY SKANDER BAHOURA

ABSTRACT. We consider the following problem on open set  $\Omega$  of  $\mathbb{R}^2$ :

$$\begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We assume that :

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

On the other hand, if we assume that  $V_i$   $s$ -holderian with  $1/2 < s \leq 1$ , then, each exterior blow-up point is simple. As application, we have a compactness result for the case when:

$$\int_{\Omega} V_i e^{u_i} dy \leq 40\pi - \epsilon, \quad \epsilon > 0$$

## 1. INTRODUCTION AND MAIN RESULTS

We set  $\Delta = \partial_{11} + \partial_{22}$  on open set  $\Omega$  of  $\mathbb{R}^2$  with a smooth boundary.

We consider the following problem on  $\Omega \subset \mathbb{R}^2$ :

$$(P) \begin{cases} -\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\ u_i = 0 & \text{in } \partial\Omega. \end{cases}$$

We assume that,

$$\int_{\Omega} e^{u_i} dy \leq C,$$

and,

$$0 \leq V_i \leq b < +\infty$$

The previous equation is called, the Prescribed Scalar Curvature equation, in relation with conformal change of metrics. The function  $V_i$  is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type were studied by many authors, see [5-8, 10-15]. We can see in [5], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on  $V$ , for example we suppose  $V_i \geq 0$  and  $V_i \in L^p(\Omega)$  or  $V_i e^{u_i} \in L^p(\Omega)$  with  $p \in [1, +\infty]$ .

Among other results, we can see in [5], the following important Theorem,

**Theorem A (Brezis-Merle [5]).** *If  $(u_i)_i$  and  $(V_i)_i$  are two sequences of functions relatively to the previous problem (P) with,  $0 < a \leq V_i \leq b < +\infty$ , then, for all compact set  $K$  of  $\Omega$ ,*

$$\sup_K u_i \leq c = c(a, b, m, K, \Omega) \quad \text{if} \quad \inf_{\Omega} u_i \geq m.$$

A simple consequence of this theorem is that, if we assume  $u_i = 0$  on  $\partial\Omega$  then, the sequence  $(u_i)_i$  is locally uniformly bounded. We can find in [5] an interior estimate if we assume  $a = 0$ , but we need an assumption on the integral of  $e^{u_i}$ , precisely, we have in [5]:

**Theorem B** (Brezis-Merle [5]). If  $(u_i)_i$  and  $(V_i)_i$  are two sequences of functions relatively to the previous problem (P) with,  $0 \leq V_i \leq b < +\infty$ , and,

$$\int_{\Omega} e^{u_i} dy \leq C,$$

then, for all compact set  $K$  of  $\Omega$ ,

$$\sup_K u_i \leq c = c(b, C, K, \Omega).$$

If, we assume  $V$  with more regularity, we can have another type of estimates,  $\sup + \inf$ . It was proved, by Shafrir, see [13], that, if  $(u_i)_i, (V_i)_i$  are two sequences of functions solutions of the previous equation without assumption on the boundary and,  $0 < a \leq V_i \leq b < +\infty$ , then we have the following interior estimate:

$$C \left( \frac{a}{b} \right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).$$

We can see in [7], an explicit value of  $C \left( \frac{a}{b} \right) = \sqrt{\frac{a}{b}}$ . In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [3]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose  $(V_i)_i$  uniformly Lipschitzian with  $A$  the Lipschitz constant, then,  $C(a/b) = 1$  and  $c = c(a, b, A, K, \Omega)$ , see Brézis-Li-Shafrir [4]. This result was extended for Hölderian sequences  $(V_i)_i$  by Chen-Lin, see [7]. Also, we can see in [10], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [11] explicit form,  $(8\pi m, m \in \mathbb{N}^*$  exactly), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [14] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

We have in [15]:

**Theorem C** (Wolansky.G.[15]). If  $(u_i)$  and  $(V_i)$  are two sequences of functions solutions of the problem (P) without the boundary condition, with,

$$0 \leq V_i \leq b < +\infty,$$

$$\|\nabla V_i\|_{L^\infty(\Omega)} \leq C_1,$$

$$\int_{\Omega} e^{u_i} dy \leq C_2,$$

and,

$$\sup_{\partial\Omega} u_i - \inf_{\partial\Omega} u_i \leq C_3,$$

the last condition replace the boundary condition.

We assume that (iii) holds in the theorem 3 of [5], then, in the sense of the distributions:

$$V_i e^{u_i} \rightarrow \sum_{j=0}^m 8\pi \delta_{x_j}.$$

in other words, we have:

$$\alpha_j = 8\pi, \quad j = 0 \dots m,$$

in (iii) of the theorem 3 of [5].

To understand the notations, it is interessant to take a look to a previous prints on arXiv, see [1] and [2].

Our main results are:

**Theorem 1.** Assume that,  $V_i$  is uniformly  $s$ -holderian with  $1/2 < s \leq 1$ , and that :

$$\max_{\Omega} u_i \rightarrow +\infty.$$

Then, each exterior blow-up point is simple.

There are  $m$  blow-ups points on the boundary (perhaps the same) such that:

$$\int_{B(x_i^j, \delta_i^j \epsilon^s)} V_i(x_i^j + \delta_i^j y) e^{u_i} \rightarrow 8\pi.$$

and,

$$\int_{\Omega} V_i e^{u_i} \rightarrow \int_{\Omega} V e^u + \sum_{j=1}^m 8\pi \delta_{x_j}.$$

and,

**Theorem 2.** Assume that,  $V_i$  is uniformly  $s$ -holderian with  $1/2 < s \leq 1$ , and,

$$\int_{B_1(0)} V_i e^{u_i} dy \leq 40\pi - \epsilon, \quad \epsilon > 0,$$

then we have:

$$\sup_{\Omega} u_i \leq c = c(b, C, A, s, \Omega).$$

where  $A$  is the holderian constant of  $V_i$ .

## 2. PROOF OF THE RESULT:

### Proof of the theorem 1:

Let's consider the following function on the ball of center 0 and radius 1/2; And let us consider  $\epsilon > 0$

$$v_i(y) = u_i(x_i + \delta_i y) + 2 \log \delta_i, \quad y \in B(0, 1/2)$$

This function is solution of the following equation:

$$-\Delta v_i = V_i(x_i + \delta_i y) e^{v_i}, \quad y \in B(0, 1/2)$$

The function  $v_i$  satisfy the following inequality (without loss of generality):

$$\sup_{\partial B(0, 1/4)} v_i - \inf_{\partial B(0, 1/4)} v_i \leq C,$$

Let us consider the following functions:

$$\begin{cases} -\Delta v_0^i = 0 & \text{in } B(0, 1/4) \\ v_0^i = u_i(x_i + \delta_i y) & \text{on } \partial B(0, 1/4). \end{cases}$$

By the elliptic estimates we have:

$$v_0^i \in C^2(\bar{B}(0, 1/4)).$$

We can write:

$$-\Delta(v_i - v_0^i) = V_i(x_i + \delta_i y) e^{v_0^i} e^{v_i - v_0^i} = K_1 K_2 e^{v_i - v_0^i},$$

With this notations, we have:

$$\|\nabla(v_i - v_0^i)\|_{L^q(B(0, \epsilon))} \leq C_q.$$

$$v_i - v_0^i \rightarrow G \text{ in } W_0^{1, q},$$

And, because, for  $\epsilon > 0$  small enough:

$$\|\nabla G\|_{L^q(B(0,\epsilon))} \leq \epsilon' \ll 1,$$

We have, for  $\epsilon > 0$  small enough:

$$\|\nabla(v_i - v_0^i)\|_{L^q(B(0,\epsilon))} \leq 2\epsilon' \ll 1.$$

and,

$$\|\nabla v_i\|_{L^q(B(0,\epsilon))} \leq 3\epsilon' \ll 1.$$

Set,

$$u = v_i - v_0^i, \quad z_1 = 0,$$

Then,

$$-\Delta u = K_1 K_2 e^u, \quad \text{in } B(0, 1/4),$$

and,

$$\text{osc}(u) = 0.$$

We use Woalnsky's theorem, see [15]. In fact  $K_2$  is a  $C^1$  function uniformly bounded and  $K_1$  is  $s$ -holderian with  $1/2 < s \leq 1$ . Because we take the logarithm in  $K$ , the part which contain  $K_2$  have similar proof as in this paper we use the Stokes formula. Only the case of  $K_1$   $s$ -holderian is difficult. For this and without loss of generality, we can assume the  $K = K_1 = V_i(x_i + \delta_i y)$ . We set:

$$\Delta \tilde{u} = \Delta v_i = \rho = -K e^{\tilde{u}} = -K_1 e^{v_i}$$

Let us consider the following term of Wolansky computations:

$$\int_{B^\epsilon} \text{div}((z - z_1)\rho) \log K + \int_{\partial B^\epsilon} \langle (z - z_1)|\nu \rangle \rho \log K,$$

First, we write:

$$\int_{B^\epsilon} \text{div}((z - z_1)\rho) \log K = 2 \int_{B^\epsilon} \rho \log K + \int_{B^\epsilon} \langle (z - z_1)|\nabla \rho \rangle \log K$$

which we can write as:

$$- \int_{B^\epsilon} \text{div}((z - z_1)\rho) \log K = 2 \int_{B^\epsilon} K \log K e^u + \int_{B^\epsilon} \langle (z - z_1)|\nabla u \rangle K \log K e^u + \int_{B^\epsilon} \langle (z - z_1)|(\nabla K) \log K \rangle e^u,$$

We can write:

$$\nabla(K(\log K) - K) = (\nabla K)(\log K)$$

Thus, and by integration by part we have:

$$\begin{aligned} & \int_{B^\epsilon} \langle (z - z_1)|(\nabla K) \log K \rangle e^u = \int_{B^\epsilon} \langle (z - z_1)|(\nabla(K \log K - K)) \rangle e^u = \\ & = \int_{\partial B^\epsilon} \langle (z - z_1)|\nu \rangle (K \log K - K) e^u - 2 \int_{B^\epsilon} (K \log K - K) e^u - \int_{B^\epsilon} \langle (z - z_1)|\nabla u \rangle (K \log K - K) e^u \end{aligned}$$

Thus,

$$\begin{aligned} & - \left( \int_{B^\epsilon} \text{div}((z - z_1)\rho) \log K + \int_{\partial B^\epsilon} \langle (z - z_1)|\nu \rangle \rho \log K \right) = \\ & = - \int_{\partial B^\epsilon} \langle (z - z_1)|\nu \rangle K e^u + \int_{B^\epsilon} \langle (z - z_1)|\nabla u \rangle K e^u + 2 \int_{B^\epsilon} K e^u \end{aligned}$$

But, we can write the following,

$$\int_{B^\epsilon} \langle (z-z_1)|\nabla u \rangle K e^u = \int_{B^\epsilon} \langle (z-z_1)|\nabla u \rangle (K-K(z_1))e^u + K(z_1) \int_{B^\epsilon} \langle (z-z_1)|\nabla u \rangle e^u,$$

and, after integration by parts:

$$K(z_1) \int_{B^\epsilon} \langle (z-z_1)|\nabla u \rangle e^u = K(z_1) \int_{\partial B^\epsilon} \langle (z-z_1)|\nu \rangle e^u - 2K(z_1) \int_{B^\epsilon} e^u,$$

Finally, we have, for the Wolansky term:

$$\begin{aligned} & \int_{B^\epsilon} \operatorname{div}((z-z_1)\rho) \log K + \int_{\partial B^\epsilon} (\langle (z-z_1)|\nu \rangle \rho) \log K = \\ & = \int_{B^\epsilon} \langle (z-z_1)|\nabla u \rangle (K-K(z_1))e^u + \left( 2 \int_{B^\epsilon} (K-K(z_1))e^u \right) + \\ & \quad + \left( \int_{\partial B^\epsilon} \langle (z-z_1)|\nu \rangle (K(z_1)-K)e^u \right) \end{aligned}$$

But, we have soon that if  $K$  is  $s$ -holderian with  $1 \geq s > 1/2$ , around each exterior blow-up we have, the following estimate:

$$\begin{aligned} & \int_{B^\epsilon} \langle (z-z_1)|\nabla u \rangle (K-K(z_1))e^u = \\ & = \int_{B(0,\epsilon)} \langle (y-z_1)|\nabla v_i \rangle (V_i(x_i+\delta_i y) - V_i(x_i))e^{v_i} dy = \\ & = \int_{B(x_i,\delta_i\epsilon)} \langle (x-x_i)|\nabla u_i \rangle (V_i(x) - V_i(x_i))e^{u_i} dy = o(1)M_\epsilon \\ & = o(1) \int_{B(x_i,\delta_i\epsilon)} V_i e^{u_i} = o(1) \int_{B^\epsilon} K e^u, \end{aligned}$$

Thus,

$$\int_{B^\epsilon} \operatorname{div}((z-z_1)\rho) \log K + \int_{\partial B^\epsilon} (\langle (z-z_1)|\nu \rangle \rho) \log K = o(1)M_\epsilon = o(1) \int_{B^\epsilon} K e^u$$

We argue by contradiction and we suppose that we have around the exterior blow-up point 2 or 3 blow-up points, for example. We prove, as in a previous paper, that, the last quantity tends to 0. But according to Wolansky paper, see [15]:

$$\int_{B^\epsilon} V_i(x_i+\delta_i y)e^{v_i} \rightarrow 8\pi.$$

Around each exterior blow-up points, there is one blow-up point.

Consider the following quantity:

$$B_i = \int_{B(x_i,\delta_i\epsilon)} \langle (x-x_i)|\nabla u_i \rangle (V_i(x) - V_i(x_i))e^{u_i} dy.$$

Suppose that, we have  $m > 0$  interior blow-up points. Consider the blow-up point  $t_i^k$  and the associated set  $\Omega_k$  defined as the set of the points nearest  $t_i^k$  we use step by step triangles which are nearest  $x_i$  and we take the mediatrices of those triangles.

$$\Omega_k = \{x \in B(x_i, \delta_i\epsilon), |x - t_i^k| \leq |x - t_i^j|, j \neq k\},$$

we write:

$$B_i = \sum_{k=1}^m \int_{\Omega_k} \langle (x-x_i)|\nabla u_i \rangle (V_i(x) - V_i(x_i))e^{u_i} dy.$$

We set,

$$B_i^k = \int_{\Omega_k} \langle (x-x_i)|\nabla u_i \rangle (V_i(x) - V_i(x_i))e^{u_i} dy,$$

We divide this integral in 4 integrals:

$$\begin{aligned}
B_i^k &= \int_{\Omega_k} \langle (x-t_i^k)|\nabla u_i \rangle (V_i(x)-V_i(x_i))e^{u_i} dy + \int_{\Omega_k} \langle (t_i^k-x_i)|\nabla u_i \rangle (V_i(x)-V_i(x_i))e^{u_i} dy = \\
&= \int_{\Omega_k} \langle (x-t_i^k)|\nabla u_i \rangle (V_i(x)-V_i(t_i^k))e^{u_i} dy + \int_{\Omega_k} \langle (x-t_i^k)|\nabla u_i \rangle (V_i(t_i^k)-V_i(x_i))e^{u_i} dy + \\
&+ \int_{\Omega_k} \langle (t_i^k-x_i)|\nabla u_i \rangle (V_i(x)-V_i(t_i^k))e^{u_i} dy + \int_{\Omega_k} \langle (t_i^k-x_i)|\nabla u_i \rangle (V_i(t_i^k)-V_i(x_i))e^{u_i} dy,
\end{aligned}$$

We set:

$$\begin{aligned}
A_1 &= \int_{\Omega_k} \langle (x-t_i^k)|\nabla u_i \rangle (V_i(x)-V_i(t_i^k))e^{u_i} dy, \\
A_2 &= \int_{\Omega_k} \langle (x-t_i^k)|\nabla u_i \rangle (V_i(t_i^k)-V_i(x_i))e^{u_i} dy, \\
A_3 &= \int_{\Omega_k} \langle (t_i^k-x_i)|\nabla u_i \rangle (V_i(x)-V_i(t_i^k))e^{u_i} dy, \\
A_4 &= \int_{\Omega_k} \langle (t_i^k-x_i)|\nabla u_i \rangle (V_i(t_i^k)-V_i(x_i))e^{u_i} dy.
\end{aligned}$$

For  $A_1$  and  $A_2$  we use the fact that in  $\Omega_k$  we have:

$$u_i(x) + 2 \log |x - t_i^k| \leq C,$$

to conclude that for  $0 < s \leq 1$ :

$$A_1 = A_2 = o(1),$$

we have integrals of the form:

$$A'_1 = \int_{\Omega_k} |\nabla u_i| e^{(1/2-s/2)u_i} dy = o(1),$$

and,

$$A'_2 = \int_{\Omega_k} |\nabla u_i| e^{(1/2-s/4)u_i} dy = o(1).$$

For  $A_3$  we use the previous fact and the sup + inf inequality to conclude that for  $1/2 < s \leq 1$ :

$$A_3 = o(1)$$

because we have an integral of the form:

$$A'_3 = \int_{\Omega_k} |\nabla u_i| e^{(3/4-s/2)u_i} dy = o(1).$$

For  $A_4$  we use integration by part to have:

$$A_4 = \int_{\partial\Omega_k} \langle (t_i^k-x_i)|\nu \rangle (V_i(t_i^k)-V_i(x_i))e^{u_i} dy.$$

But, the boundary of  $\Omega_k$  is the union of parts of mediatrices of segments linked to  $t_i^k$ . Let's consider a point  $t_i^j$  linked to  $t_i^k$  and denote  $D_{i,j,k}$  the mediatrice of the segment  $(t_i^j, t_i^k)$ , which is in the boundary of  $\Omega_k$ . Note that this mediatrice is in the boundary of  $\Omega_j$  and the same decomposition for  $\Omega_j$  gives us the following term:

$$A'_4 = - \int_{D_{i,j,k}} \langle (t_i^j-x_i)|\nu \rangle (V_i(t_i^j)-V_i(x_i))e^{u_i} dy.$$

Thus, we have to estimate the sum of the 2 following terms:

$$A_5 = \int_{D_{i,j,k}} \langle (t_i^k - x_i) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy.$$

and,

$$A_6 = A'_4 = - \int_{D_{i,j,k}} \langle (t_i^j - x_i) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy.$$

We can write them as follows:

$$A_5 = \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy + \int_{D_{i,j,k}} \langle (t_i^k - x) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy.$$

and,

$$A_6 = - \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy - \int_{D_{i,j,k}} \langle (t_i^j - x) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy.$$

We can write:

$$\begin{aligned} & \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy - \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy = \\ & = \int_{D_{i,j,k}} \langle (x - x_i) | \nu \rangle (V_i(t_i^k) - V_i(t_i^j)) e^{u_i} dy = o(1), \end{aligned}$$

for  $1/2 < s \leq 1$ . Because, we do integration on the mediatrice of  $(t_i^j, t_i^k)$ ,  $|x - t_i^j| = |x - t_i^k|$ , and:

$$|V_i(t_i^k) - V_i(t_i^j)| \leq 2A|x - t_i^k|^s$$

$$u_i(x) + 2 \log |x - t_i^k| \leq C,$$

and,

$$|x - x_i| \leq \delta_i \epsilon,$$

To estimate the integral of the following term:

$$e^{(3/4-s/2)u_i} \leq Cr^{(-3/2+s)},$$

which is integrable and tends to 0, for  $1/2 < s \leq 1$ , because we are on the ball  $B(x_i, \delta_i \epsilon)$ .

In other part, for the term:

$$\int_{D_{i,j,k}} \langle (t_i^k - x) | \nu \rangle (V_i(t_i^k) - V_i(x_i)) e^{u_i} dy - \int_{D_{i,j,k}} \langle (t_i^j - x) | \nu \rangle (V_i(t_i^j) - V_i(x_i)) e^{u_i} dy.$$

We use the fact that, on  $D_{i,j,k}$ :

$$|x - t_i^j| = |x - t_i^k|,$$

$$u_i(x) + 2 \log |x - t_i^k| \leq C,$$

$$|V_i(t_i^k) - V_i(x_i)| \leq 2A|x_i - t_i^k|^s \leq \delta_i^s,$$

and,

$$|V_i(t_i^j) - V_i(x_i)| \leq 2A|x_i - t_i^j|^s \leq \delta_i^s,$$

To estimate the integral of the following term:

$$e^{(1/2-s/4)u_i} \leq Cr^{(-1+s/2)},$$

which is integrable and tends to 0, because we are on the ball  $B(x_i, \delta_i \epsilon)$ .



Thus,

$$B_i = o(1),$$

Proof of the theorem 2:

Next, we use the formulation of the case of three blow-up points, see [2]. Because the blow-ups points are simple, we can consider the following function:

$$v_i(\theta) = u_i(x_i + r_i\theta) - u_i(x_i),$$

where  $r_i$  is such that:

$$r_i = e^{-u_i(x_i)/2},$$

$$\int_{B^\epsilon} V_i(x_i + \delta_i y) e^{v_i} \rightarrow 8\pi.$$

$$\begin{aligned} u_i(x_i + r_i\theta) &= \int_{\Omega} G(x_i + r_i\theta, y) V_i(y) e^{u_i(y)} dx = \\ &= \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i\epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy = \end{aligned}$$

We write,  $y = x_i + r_i\tilde{\theta}$ , with  $|\tilde{\theta}| \leq 2\frac{\delta_i}{r_i}\epsilon'$ ,

$$\begin{aligned} u_i(x_i + r_i\theta) &= \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i\tilde{\theta})(x_i + r_i\tilde{\theta})|}{r_i|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + \\ &\quad + \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy \end{aligned}$$

$$u_i(x_i) = \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy$$

Hence,

$$\begin{aligned} u_i(x_i) &= \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|1 - \bar{x}_i(x_i + r_i\tilde{\theta})|}{r_i|\tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + \\ &\quad + \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy \end{aligned}$$

We look to the difference,

$$v_i(\theta) = u_i(x_i + r_i\theta) - u_i(x_i) = \int_{B(0, 2\frac{\delta_i}{r_i}\epsilon')} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy + h_1 + h_2,$$

where,

$$h_1(\theta) = \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy - \int_{\Omega - B(x_i, 2\delta_i\epsilon')} G(x_i, y) V_i e^{u_i(y)} dy,$$

and,

$$h_2(\theta) = \int_{B(0, 2\delta_i\epsilon')} \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i\tilde{\theta})y|}{|1 - \bar{x}_i y|} V_i e^{u_i(y)} dy.$$

Remark that,  $h_1$  and  $h_2$  are two harmonic functions, uniformly bounded.

According to the maximum principle, the harmonic function  $G(x_i + r_i\theta, \cdot)$  on  $\Omega - B(x_i, 2\delta_i\epsilon')$  take its maximum on the boundary of  $B(x_i, 2\delta_i\epsilon')$ , we can compute this maximum:

$$G(x_i + r_i\theta, y_i) = \frac{1}{2\pi} \log \frac{|1 - (\bar{x}_i + r_i\tilde{\theta})y_i|}{|x_i + r_i\theta - y_i|} \simeq \frac{1}{2\pi} \log \frac{(|1 + |x_i||\delta_i - \delta_i(3\epsilon' + o(1))|}{\delta_i\epsilon'} \leq C_{\epsilon'} < +\infty$$

with  $y_i = x_i + 2\delta_i\theta_i\epsilon'$ ,  $|\theta_i| = 1$ , and  $|r_i\theta| \leq \delta_i\epsilon'$ .

We can remark, for  $|\theta| \leq \frac{\delta_i\epsilon'}{r_i}$ , that  $v_i$  is such that:

$$v_i = h_1 + h_2 + \int_{B(0, 2\frac{\delta_i\epsilon'}{r_i})} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy,$$

$$v_i = h_1 + h_2 + \int_{B(0, 2\frac{\delta_i\epsilon'}{r_i})} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \tilde{\theta}|} V_i(x_i + r_i\tilde{\theta}) e^{v_i(\tilde{\theta})} d\tilde{\theta},$$

with  $h_1$  and  $h_2$ , the two uniformly bounded harmonic functions.

**Remark:** In the case of 2 or 3 or 4 blow-up points, and if we consider the half ball, we have supplementary terms, around the 2 other blow-up terms. Note that the Green function of the half ball is quasi-similar to the one of the unit ball and our computations are the same if we consider the half ball.

By the asymptotic estimates of Cheng-Lin, we can see that, we have the following uniform estimates at infinity. We have, after considering the half ball and its Green function, the following estimates:

$\forall \epsilon > 0, \epsilon' > 0 \exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+, i_{\epsilon, \epsilon'} \in \mathbb{N}$  and  $C_{\epsilon, \epsilon'} > 0$ , such that, for  $i \geq i_{\epsilon, \epsilon'}$  and  $k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i\epsilon'}{r_i}$ ,

$$(-4 - \epsilon) \log |\theta| - C_{\epsilon, \epsilon'} \leq v_i(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, \epsilon'},$$

and,

$$\begin{aligned} \partial_j v_i &\simeq \partial_j u_0(\theta) \pm \frac{\epsilon}{|\theta|} + C \left( \frac{r_i}{\delta_i} \right)^2 |\theta| + m \times \left( \frac{r_i}{\delta_i} \right) + \\ &\quad + \sum_{k=2}^m C_1 \left( \frac{r_i}{d(x_i, x_i^k)} \right), \end{aligned}$$

In the case, we have:

$$\frac{d(x_i, x_i^k)}{\delta_i} \rightarrow +\infty \text{ for } k = 2 \dots m,$$

We have after using the previous term of the Pohozaev identity, for  $1/2 < s \leq 1$ :

$$\begin{aligned} o(1) &= J'_i = m' + \sum_{k=1}^m C_k o(1), \\ 0 &= \lim_{\epsilon'} \lim_{\epsilon} \lim_i J'_i = m', \end{aligned}$$

which contradict the fact that  $m' > 0$ .

here,

$$J_i = B_i = \int_{B(x_i, \delta_i\epsilon')} \langle x_1^i |\nabla(u_i - u) \rangle (V_i - V_i(x_i)) e^{u_i} dy.$$

We use the previous formulation around each blow-up point.

If, for  $x_i^j$ , we have:

$$\frac{d(x_i^j, x_i^k)}{\delta_i^j} \rightarrow +\infty \text{ for } k \neq j, k = 1 \dots m,$$

We use the previous formulation around this blow-up point. We consider the following quantity:

$$J_i^j = B_i^j = \int_{B(x_i^j, \delta_i^j\epsilon')} \langle x_1^{i,j} |\nabla(u_i - u) \rangle (V_i - V_i(x_i^j)) e^{u_i} dy.$$

with,

$$x_1^{i,j} = (\delta_i^j, 0),$$

In this case, we set:

$$v_i^j(\theta) = u_i(x_i^j + r_i^j \theta) - u_i(x_i^j),$$

where  $r_i^j$  is such that:

$$r_i^j = e^{-u_i(x_i^j)/2},$$

$$\int_{B(x_i^j, \delta_i^j \epsilon')} V_i(x_i^j + \delta_i^j y) e^{v_i} \rightarrow 8\pi.$$

We have, after considering the half ball and its Green function, the following estimates:

$$\forall \epsilon > 0, \epsilon' > 0 \exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+, i_{\epsilon, \epsilon'} \in \mathbb{N} \text{ and } C_{\epsilon, \epsilon'} > 0, \text{ such that, for } i \geq i_{\epsilon, \epsilon'} \text{ and } k_{\epsilon, \epsilon'} \leq |\theta| \leq \frac{\delta_i^j \epsilon'}{r_i^j},$$

$$(-4 - \epsilon) \log |\theta| - C_{\epsilon, \epsilon'} \leq v_i^j(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, \epsilon'},$$

and,

$$\begin{aligned} \partial_k v_i^j &\simeq \partial_k u_0^j(\theta) \pm \frac{\epsilon}{|\theta|} + C \left( \frac{r_i^j}{\delta_i^j} \right)^2 |\theta| + m \times \left( \frac{r_i^j}{\delta_i^j} \right) + \\ &+ \sum_{l \neq j}^m C_1 \left( \frac{r_i^j}{d(x_i^j, x_i^l)} \right), \end{aligned}$$

We have after using the previous term of the Pohozaev identity, for  $1/2 < s \leq 1$ :

$$o(1) = J_i^j = B_i^j = m' + \sum_{l \neq j}^m C_l o(1),$$

$$0 = \lim_{\epsilon'} \lim_{\epsilon} \lim_i J_i^j = m',$$

which contradict the fact that  $m' > 0$ .

If, for  $x_i^j$ , we have:

$$\frac{d(x_i^j, x_i^k)}{\delta_i^j} \leq C_{j,k} \text{ for some } k = k_j \neq j, 1 \leq k \leq m,$$

All the distances  $d(x_i^j, x_i^k)$  are comparable with some  $\delta_i^j$ . This means that we can use the Pohozaev identity directly. We can do this for example, for 4 blow-ups points.

We have many cases:

Case 1: the blow-up points are "equivalents", it seems that we have the same radius for the blow-up points.

Case 2: 3 points are "equivalents" and another blow-up point linked to the 3 blow-up points. We apply the Pohozaev identity directly with central point which link the 3 blow-up to the last.

Case 3: 2 pair of blow-up points separated.

Case 3.1: the 2 pair are linked: we apply the Pohozaev identity.

Case 3.2: the two pair are separated. It is the case of two separated blow-up points, see [1]

#### ACKNOWLEDGEMENT.

The author is grateful to Professor G. Wolansky who has communicated him the private communication.

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