On interface transmission conditions for conservation laws with discontinuous flux of general shape

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ON INTERFACE TRANSMISSION CONDITIONS FOR CONSERVATION LAWS WITH DISCONTINUOUS FLUX OF GENERAL SHAPE

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Abstract. Conservation laws of the form $\partial_t u + \partial_x f(x; u) = 0$ with space-discontinuous flux $f(x; \cdot) = f_l(\cdot)\mathbf{1}_{x<0} + f_r(\cdot)\mathbf{1}_{x>0}$ were deeply investigated in the last ten years, with a particular emphasis in the case where the fluxes are "bell-shaped". In this paper, we introduce and exploit the idea of *transmission maps* for the interface condition at the discontinuity, leading to the well-posedness for the Cauchy problem with general shape of $f_{l,r}$. The design and the convergence of monotone Finite Volume schemes based on one-sided approximate Riemann solvers is then assessed. We conclude the paper by illustrating our approach by several examples coming from real-life applications.

Keywords: Hyperbolic conservation law; Discontinuous flux; Interface coupling; Boundary layer; Monotone finite volume scheme; Interface flux; Entropy solution; Well-posedness; Convergent scheme.

1. Presentation of the problem

Scalar conservation laws of the form

$$\partial_t u + \partial_x f(u; x) = 0 \tag{1.1}$$

with Lipschitz in u, piecewise regular and jump-discontinuous in x flux function f appear in applications such as sedimentation, two-phase flows in porous media, road or pedestrian traffic, and others. The equation should be considered subject to an initial condition and (in the case of a bounded domain) with some boundary conditions. Due to the finite speed of propagation typical for hyperbolic equations, and because the influence of boundary conditions is rather well studied (at least in the Dirichlet case, see [21]; see [25] and [16,17] for the Neumann case and for more general boundary conditions), we will focus on the Cauchy initial-value problem.

The case of x-independent or regular (Lipschitz continuous in x also) flux function f is a classical one; then, the central concept for equation (1.1) are Kruzhkov *entropy inequalities* and the associated notion of *entropy solution* ([57]). In particular, it is known that typical entropy solutions u are piecewise regular, and jumps in u obey some *admissibility conditions* deduced from the weak (distributional) formulation of equation (1.1) and also from the entropy inequalities.

The discontinuous-flux case we are interested in has been studied for more than twenty years (see in particular [50,68,4,56,67,2,3,19,20,49,45,14,62,1], see also the references of paper [14]), and its theory is still not complete although much progress in understanding the problem has been achieved. A basic goal of a mathematical theory for equation (1.1) is to formulate an adequate notion of admissible solution (generally based on some adapted entropy inequalities), and then to prove existence and uniqueness of solutions subject to an initial condition and (in the case of a bounded domain) with some boundary conditions. The convergence of numerical approximation is also often addressed.

Understanding of these problems can easily be reduced to understanding of the model case considered in most of the mathematical studies of the subject:

$$f(\cdot;x) = f_l(\cdot)\mathbf{1}_{x<0} + f_r(\cdot)\mathbf{1}_{x>0}. \tag{1.2}$$

Here, $f_{l,r}$ are x-independent nonlinearities, so that $x \mapsto f(u;x)$ experiences a jump across the hypersurface

$$\Sigma := \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ \mid x = 0\}$$

called *interface* in the sequel.

In what follows, we assume that there exist time-independent real values $\underline{u}_{l,r}$ and $\overline{u}_{l,r}$, such that $\underline{u}_{l,r} \leq \overline{u}_{l,r}$, and such that the piecewise constant functions

$$\underline{u}(x) = \underline{u}_l\mathbf{1}_{x<0} + \underline{u}_r\mathbf{1}_{x>0} \quad \text{ and } \quad \overline{u}(x) = \overline{u}_l\mathbf{1}_{x<0} + \overline{u}_r\mathbf{1}_{x>0} \tag{1.3}$$

are steady solutions to the problem (1.1) (we refer to [14, Prop. 3.20 and §6] to highlight the role and the importance of this kind of assumptions for the existence theory). In particular, the Rankine-Hugoniot condition across the interface Σ for the solutions $\underline{u}$ and $\overline{u}$ enforces the compatibility conditions

$$f_l(\underline{u}_l) = f_r(\underline{u}_r), \quad \text{ and } \quad f_l(\overline{u}_l) = f_r(\overline{u}_r). \tag{1.4}$$

Moreover, the flux functions $f_{l,r}$ are assumed to be Lipschitz continuous on the intervals $U_{l,r}$, where

$$U_l = [\underline{u}_l, \overline{u}_l], \quad \text{ and } \quad U_r = [\underline{u}_r, \overline{u}_r].$$

In what follows, we denote by $L_{l,r}$ the (smallest) Lipschitz constant for $f_{l,r}$.

More generally, the flux functions $f_{l,r}$ can be assumed time-dependent, which is needed, e.g., in the hyperbolic Buckley-Leverett model(see, e.g., [7]) or for modeling lights in traffic flows (see [41,11]). We refer to [11] for techniques specific for such time-dependent cases.

The initial datum $u_0 \in L^\infty(\mathbb{R})$ is assumed to be bounded between $\underline{u}$ and $\overline{u}$, i.e.

$$\underline{u}(x) \le u_0(x) \le \overline{u}(x), \qquad \text{for a.e. } x \in \mathbb{R}. \tag{1.5}$$

In what follows, we will focus on the Cauchy problem (1.1) with fluxes (1.2),(1.4) and initial datum (1.5).

Due to assumption (1.2), the main delicate point in order to construct a well-posedness theory for the above Cauchy problem is to fix a notion of admissibility of jumps in u across the interface Σ where $f(u;\cdot)$ experiences a jump. Indeed, it was established in [3] that there may exist infinitely many different, equally mathematically consistent ways to define admissibility of solutions at the interface; then, it was understood that different solution notions actually correspond to different physical models. This means in particular that some additional information, not contained in the nonlinearities $f_{l,r}$, may be needed in each concrete application in order to single out the solution relevant for this application.

The method we propose in this paper goes back to the idea introduced in [39] consisting in solving two scalar conservation laws on half-space coupled by an *ad hoc* transmission condition at the interface. With the notation

$$\Omega_l := \{x < 0\} \text{ and } \Omega_r := \{x > 0\},$$

we require $u|_{\Omega_{l,r}\times\mathbb{R}_+}$ to be the entropy solutions of

$$\begin{cases} \partial_t u + \partial_x f_{l,r}(u) = 0 & \text{in } \Omega_{l,r} \times \mathbb{R}_+, \\ u_{|t=0} = u_0 & \text{in } \Omega_{l,r}, \\ u_{|x=0} = \tilde{u}_{l,r} & \text{on } \Sigma. \end{cases} \tag{1.6}$$

In the problem (1.6), the functions $t \mapsto \tilde{u}_{l,r}(t) \in L^\infty(\mathbb{R}_+; U_{l,r})$ are unspecified for the moment. Recall that the notion of entropy solution to Cauchy-Dirichlet problems (1.6) first requires that

$$\begin{array}{l} u(\cdot,0) = u_0 \text{ and the functions } u|_{\Omega_{l,r}\times\mathbb{R}_+} \text{ are Kruzhkov entropy solutions of} \\ \text{conservation laws } \partial_t u + \partial_x f_{l,r}(u) = 0 \text{ in domains } \Omega_{l,r} \times \mathbb{R}_+, \text{ respectively.} \end{array} \tag{1.7}$$

Observe that, without loss of generality, one can assume that $u \in C([0,+\infty); L^1_{loc}(\mathbb{R}))$ (see, e.g., [34]). This gives pointwise strong sense to the initial condition in (1.7) and thus in (1.6). Furthermore, in order to write explicitly the boundary conditions in (1.6), in this introduction let us assume for a moment that we deal with solutions $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ of the local entropy formulations (1.7) that have strong interface traces[a]. More precisely, we work with u that admits

[a] This is true for any $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ solution of (1.7) provided $f_{l,r}$ have non-zero derivative on any nontrivial subinterval of $U_{l,r}$, see [65]. Without any additional assumption on the fluxes, this is true for any self-similar solution (which is important because this applies to solutions of

strong one-sided traces on the interface Σ:

$$u_l(t) = \lim_{h\to 0} \int_{-h}^{0} u(x,t)\mathrm{d}x, \quad u_r(t) = \lim_{h\to 0} \int_{0}^{h} u(x,t)\mathrm{d}x, \qquad \text{for a.e. } t \geq 0. \tag{1.8}$$

It is well known since the work of Bardos, Leroux and Nédélec [21] that the one-sided boundary condition $u_{l,r} = \tilde{u}_{l,r}$ formally imposed in (1.6) should be understood "up to a converged boundary layer". Namely, the boundary conditions in (1.6) are enforced in the sense of [21], which we express as follows (see [47] for a closely related approach in the system case). Defining by $G_{l,r} : U_l \times U_r \to \mathbb{R}$ the Godunov solvers associated to $f_{l,r}$, i.e.,

$$G_{l,r}(a,b) = \begin{cases} \min_{s\in[a,b]} f_{l,r}(s) & \text{if } a \leq b, \\ \max_{s\in[b,a]} f_{l,r}(s) & \text{if } a \geq b, \end{cases} \tag{1.9}$$

we require

$$f_l(u_l(t)) = G_l(u_l(t), \tilde{u}_l(t)), \quad G_r(\tilde{u}_r(t), u_r(t)) = f_r(u_r(t)) \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{1.10}$$

Now, in order to determine the couple of functions $t \mapsto (\tilde{u}_l(t), \tilde{u}_r(t))$ appearing as the boundary values in (1.6), we impose two *transmission conditions* across the interface Σ:

- **a)** the mass conservativity condition, i.e.,

$$f_l(u_l(t)) = f_r(u_r(t)), \qquad \text{for a.e. } t > 0;$$

- **b)** the additional condition that for a.e. $t > 0$, the couple $(\tilde{u}_l(t), \tilde{u}_r(t))$ belong to a given, model-dependent subset β of $U_l \times U_r$.

In this contribution, we focus on the case of a monotone transmission, in which case β is assumed to be a maximal monotone graph in $U_l \times U_r$.

Definition 1.1. *In the context of assumptions* (1.2),(1.4)*, a maximal monotone graph β in $U_l \times U_r$ is called* transmission map *for* (1.1).

We will show that different transmission maps used in **b)** may lead to different admissibility notions for the Cauchy problem (1.1),(1.2),(1.4),(1.5), which contains in particular the different solutions discovered in [3]. Let us stress that in concrete examples, the transmission map to be used in **b)** can be determined from some mesoscale information relevant for the underlying application context (see §4). As an example, we point out that the vanishing viscosity limits (see [45,13] and references therein) in the case $U_l = U_r$ can be obtained with $\beta = Id$, which was also the choice of [39]. Notice that due to condition **a)**, we focus on the case of conservative

the Riemann problem).

The simplifying assumption of existence of traces of u can be easily bypassed in the arguments we develop, keeping in mind that traces of $f_{l,r}(u)$ and those of the associated Godunov fluxes $G_{l,r}(u,\kappa)$ exist (see the arguments in [16,14] that make explicit the result contained in [65]).

coupling; yet, adaptation to non-conservative coupling is possible (see the example of § 4.6, which provides a new interpretation of the results of [18]).

We will show that the maximal monotonicity assumption on β guarantees that there exists a unique function $t \mapsto (\tilde{u}_l(t), \tilde{u}_r(t))$ such that one has

$$f_l(u_l) = G_l(u_l, \tilde{u}_l) = G_r(\tilde{u}_r, u_r) = f_r(u_r) \text{ with } (\tilde{u}_l, \tilde{u}_r) \in \beta \tag{1.11}$$

pointwise for a.e. $t \in \mathbb{R}_+$, where $u_{l,r}$ are the traces in the sense (1.8) of a solution u to the Cauchy-Dirichlet problems (1.6). Observe that condition (1.11) summarizes both transmission conditions **a)** and **b)**, as well as the form (1.10) of the boundary conditions in (1.6).

According to the above analysis, by admissible solution associated with a given transmission map β we will mean a function $u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ satisfying (1.7), such that its one-sided traces on the interface Σ fulfill the transmission condition (1.11). Let us explain how well-posedness for the above interpretation (1.7),(1.11) is established. First, we observe that if (1.11) holds, the one-sided traces $u_{l,r}$ of u have to take their values in the set

$$\mathcal{G}(\beta) = \{(u_l, u_r) \in U_l \times U_r \mid \text{ there exists } (\tilde{u}_l, \tilde{u}_r) \text{ such that (1.11) holds}\}. \tag{1.12}$$

Next, we interpret the set $\mathcal{G}(\beta)$ in (1.12) within the framework of the general theory of L^1-dissipative solvers for (1.1),(1.2),(1.5) developed in [14], by establishing that the solution is admissible in the above sense if and only if it is a $\mathcal{G}(\beta)$-entropy solution of (1.1),(1.2),(1.5) in the sense of [14]. This claim follows readily from the analysis of §2.2 that shows that $\mathcal{G}(\beta)$ is a *complete maximal L^1-dissipative germ* in the sense of [14]. Yet our results consist not only in a reduction to the setting of $\mathcal{G}(\beta)$-entropy solutions. Indeed, due to the generality of the setting of [14], the only generic convergent finite volume scheme is the Godunov scheme which makes appeal to a Godunov solver at the interface; in general, this Godunov solver can be quite intricate. In §2.2, following [37] we describe the interface Godunov solver obtained for a given transmission map β. Furthermore, in §3 we show that the structure (1.6),(1.11) of the particular solutions associated to transmission maps permits to design and analyze reasonably cheap convergent finite volume schemes for $\mathcal{G}(\beta)$-entropy solutions. The key ingredient here is a rather natural discretization of the β-dependent transmission condition (1.11). In the last section of the paper, namely §4, we extend the notion of *optimal entropy solution* introduced in [3] for bell-shaped fluxes to the case of general fluxes. We finally show how several problems coming from clarifier-thickener unit modeling, porous media flow modeling, traffic flow modeling or pedestrian flow modeling enter the transmission map framework we propose in this paper.

2. L^1-dissipative germs and transmission maps

We briefly recall the main definitions and results given in the work [14] of Karlsen, Risebro and the first author. Then we establish relations between L^1-dissipative germs and transmission maps, and analyze the Riemann problem for β-admissible solution. As a consequence, we infer well-posedness and convergence of the Godunov numerical scheme for β-admissible solutions of (1.1),(1.2).

2.1. *Germs formalism and general well-posedness results*

Consider problem (1.1),(1.2) with fluxes $f_{l,r}$ defined on the intervals[b] $U_{l,r} = [\underline{u}_{l,r}, \overline{u}_{l,r}]$. In this setting, the following definitions and results are given in [14].

Definition 2.1. *A subset $\mathcal{G}$ of $U_l \times U_r$ is called L^1-dissipative germ (L^1D germ, for short) if for all $(u_l, u_r) \in \mathcal{G}$ there holds $f_l(u_l) = f_r(u_r)$, and moreover, for all $(u_l, u_r), (v_l, v_r) \in \mathcal{G}$*

$$q_l(u_l, v_l) - q_r(u_r, v_r) \geq 0, \tag{2.1}$$

where

$$q_{l,r}(u, \kappa) := sign(u - \kappa)\big(f_{l,r}(u) - f_{l,r}(\kappa)\big).$$

The following particular properties of germs are of interest.

- *An L^1D germ is called* maximal, *if it is not a strict subset of some other L^1D germ.*
- *An L^1D germ $\mathcal{G}$ is called* definite, *if there exists a unique maximal L^1D germ $\widetilde{\mathcal{G}}$ such that $\mathcal{G}$ is a subset of $\widetilde{\mathcal{G}}$.*
- *An L^1D germ $\mathcal{G}$ is called* complete, *if for every Riemann data*

$$u_0(x) = u_l \mathbf{1}_{x<0} + u_r \mathbf{1}_{x>0} \ \text{ with } u_l \in U_l,\ u_r \in U_r$$

there exists a self-similar (i.e., $\frac{x}{t}$-dependent) solution u such that

a) $u_{|\Omega_l \times \mathbb{R}_+}$ is a Kruzhkov entropy solution of $\partial_t u + \partial_x f_l(u) = 0$ with $u(x,0) = u_l$, $x < 0$,
b) $u_{|\Omega_r \times \mathbb{R}_+}$ is a Kruzhkov entropy solution of $\partial_t u + \partial_x f_r(u) = 0$ with $u(x,0) = u_r$, $x > 0$,
c) the one-sided traces $\lim_{x\to 0^-} u(x,t) =: \gamma_l u$, $\lim_{x\to 0^+} u(x,t) =: \gamma_r u$ of u at the interface $\{x = 0\}$ verify $(\gamma_l u, \gamma_r u) \in \mathcal{G}$.

Finally, given an L^1D germ $\mathcal{G}$, one defines its dual germ *$\mathcal{G}^*$ by setting*

$$\mathcal{G}^* = \Big\{ (v_l, v_r) \in U_l \times U_r \mid f_l(v_l) = f_r(v_r) \text{ and } \forall (u_l, u_r) \in \mathcal{G},\ (2.1) \text{ holds} \Big\}.$$

Another useful notion, that we do not put forward here because it is less fundamental, is the notion of closed germ (see [14]) used in Proposition 3.6 in § 3.2.

[b] It was supposed in [14] that $U_l = U_r$; yet the generalization to the case of different domains of definition of $f_{l,r}$ is immediate, bearing in mind that some compatibility conditions of the kind (1.4) are mandatory to develop an existence theory.

The following relations between the above properties of L^1D germs are relevant for the subsequent analysis.

Proposition 2.2. *The unique maximal extension of a definite L^1D germ $\mathcal{G}$ is its dual germ $\mathcal{G}^*$. In particular, an L^1D germ is maximal if and only if it coincides with its dual.*

Proposition 2.3. *A complete L^1D germ is maximal.*

Given a definite L^1D germ one defines $\mathcal{G}$-entropy solutions and infers uniqueness of the so defined solution; furthermore, given a complete L^1D germ one obtains well-posedness results and justifies convergence of the Godunov finite volume scheme.

Definition 2.4. *Let $\mathcal{G}$ be a definite L^1D germ. A function $u \in C(\mathbb{R}_+; L^1_{loc}(\mathbb{R}))$ such that $u|_{\Omega_l\times\mathbb{R}_+}$ takes values in U_l and $u|_{\Omega_r\times\mathbb{R}_+}$ takes values in U_r is called $\mathcal{G}$-entropy solution of equation (1.1),(1.2) with initial datum (1.5) if $u(\cdot,0)=u_0$, the Kruzhkov entropy inequalities are satisfied away from the interface:*

$$\forall\,\kappa \in U_{l,r}, \quad \partial_t|u(x,t)-\kappa| + \partial_x q_{l,r}(u(x,t),\kappa) \le 0 \quad \text{in } \mathcal{D}'(\Omega_{l,r}\times\mathbb{R}_+), \tag{2.2}$$

and the following adapted entropy inequalities *are satisfied*

$$\text{for all } (\kappa_l,\kappa_r)\in\mathcal{G},\ \text{setting } \kappa(x)=\kappa_l\mathbf{1}_{x<0}+\kappa_r\mathbf{1}_{x>0},\ \text{there holds}$$
$$\partial_t|u(x,t)-\kappa(x)| + \partial_x q(u(x,t),\kappa(x);x) \le 0 \quad \text{in } \mathcal{D}'(\mathbb{R}\times\mathbb{R}_+), \tag{2.3}$$

with the notation $q(\cdot,\cdot;x) = q_l(\cdot,\cdot)\mathbf{1}_{x<0} + q_r(\cdot,\cdot)\mathbf{1}_{x>0}$.

Let us recall the main results of the theory [14] on which our analysis will rely.

Theorem 2.5. *Under the assumptions of this section, given a definite L^1D germ $\mathcal{G}$ there exists at most one $\mathcal{G}$-entropy solution to the Cauchy problem (1.1),(1.2),(1.5) for every initial datum. Furthermore, if $\mathcal{G}$ is a complete L^1D germ, there exists a unique solution to every Cauchy problem and this solution can be obtained as the limit of approximations obtained by the time-explicit Godunov finite volume scheme under some appropriate CFL condition.*

For the details of the latter statement (description of the scheme, etc.) we refer to [14, §6], but also to §3 of the present paper.

Actually, the uniqueness proof in [14] does not rely on Definition 2.4 but on an equivalent definition of solution that makes appeal to strong boundary traces of suitably defined singular mappings. In the sequel, we will only need the following implication, contained in [14], which concerns solutions such that u itself possesses strong boundary traces.

Proposition 2.6. *Let $\mathcal{G}$ be a definite L^1D germ, and let $u \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ be such that $\underline{u} \le u \le \overline{u}$ a.e. in $\mathbb{R}\times\mathbb{R}_+$, and such that*

i. *the restriction $u_{|\Omega_{l,r}\times\mathbb{R}_+}$ of u to $\Omega_{l,r}\times\mathbb{R}_+$ is a Kruzhkov entropy solution, i.e. (2.2) holds;*

ii. *u admits strong one-sided traces $u_{l,r}\in L^\infty(\mathbb{R}_+;U_{l,r})$ on the interface Σ such that $(u_l(t),u_r(t))\in\mathcal{G}^*$ for a.e. $t>0$;*

then u is the unique $\mathcal{G}$-entropy solution.

In the above proposition, $\mathcal{G}^*$ is maximal due to the definiteness assumption of $\mathcal{G}$; as a matter of fact, $\mathcal{G}$- and $\mathcal{G}^*$-entropy solutions coincide. Let us stress that the main interest of the notion of definite L^1D germ (in which case $\mathcal{G}\subset\mathcal{G}^*$) as compared to the notion of maximal L^1D germ (in which case $\mathcal{G}=\mathcal{G}^*$) is to enforce the smallest possible number of constraints in (2.3). We refer to [28] and [14, §4.8] for one striking example where $\mathcal{G}$ can be chosen to be a singleton. Yet, in the present contribution we will not address the interesting question of finding the smallest definite subset of $\mathcal{G}(\beta)$ given in (1.12). We only prove that $\mathcal{G}(\beta)$ is a complete maximal L^1D germ, and we put forward a slightly smaller subset $\mathcal{G}^o(\beta)$ of "reachable states" that is a definite L^1D germ (see Proposition 3.6, cf. [14,13] for a particular example).

2.2. *Transmission maps yield maximal L^1-dissipative germs...*

We start by providing a convenient parametrization of a transmission map β. Denote $\underline{p}:=\underline{u}_l+\underline{u}_r$ and $\overline{p}:=\overline{u}_l+\overline{u}_r$. Given $\beta\subset[\underline{u}_l,\overline{u}_l]\times[\underline{u}_r,\overline{u}_r]$ a maximal monotone graph, the application

$$\Pi:\begin{cases}\beta\to[\underline{p},\overline{p}]\\(\tilde u_l,\tilde u_r)\mapsto\Pi(\tilde u_l,\tilde u_r)=\tilde u_l+\tilde u_r.\end{cases}$$

is a one-to-one mapping. The inverse of Π provides nondecreasing 1-Lipschitz continuous parametrizations of the coordinates of the elements of the graph β:

$$\begin{cases}\tilde u_{l,r}:[\underline{p},\overline{p}]\to[\underline{u}_{l,r},\overline{u}_{l,r}]\\p\mapsto\tilde u_{l,r}(p)\end{cases}\tag{2.4}$$

with $(\tilde u_l(p),\tilde u_r(p))\in\beta$ for all $p\in[\underline{p},\overline{p}]$ (i.e., $\Pi=(\tilde u_l^{-1},\tilde u_r^{-1})$).

In the next lemma, we define the interface numerical flux associated with the interface transmission condition (1.11). In Proposition 2.9 we will prove that this is exactly the Godunov flux for $\mathcal{G}(\beta)$-entropy Riemann solver for (1.1),(1.2).

Lemma 2.7. *For all $(u_l,u_r)\in U_l\times U_r$, there exists $p\in[\underline{p},\overline{p}]$ such that*

$$G_l(u_l,\tilde u_l(p))=G_r(\tilde u_r(p),u_r).$$

While p is not necessarily unique, the corresponding interface flux map $G_\beta(u_l,u_r)$

$$G_\beta:(u_l,u_r)\mapsto G_l(u_l,\tilde u_l(p))=G_r(\tilde u_r(p),u_r)\tag{2.5}$$

is uniquely defined. Moreover, G_β is Lipschitz continuous w.r.t. u_l and u_r, and

$$0\le\partial_{u_l}G_\beta(u_l,u_r)\le L_l,\qquad -L_r\le\partial_{u_r}G_\beta(u_l,u_r)\le 0.\tag{2.6}$$

Observe that our notation is slightly abusive: namely, $\mathcal{G}_{l,r}$ depend only on $f_{l,r}$, respectively, while $\mathcal{G}_\beta$ depends on β but also on f_l and f_r.

Proof. Define the functions $\Psi_{l,r} : [\underline{p}, \overline{p}] \to \mathbb{R}$ by

$$\Psi_l(p) = G_l(u_l, \tilde{u}_l(p)) \quad \text{and} \quad \Psi_r(p) = G_r(\tilde{u}_r(p), u_r), \qquad \forall p \in [\underline{p}, \overline{p}],$$

then in view of the monotonicity and of the Lipschitz continuity of $G_{l,r}$ and $p \mapsto \tilde{u}_{l,r}(p)$, one has

$$-L_l \le \Psi_l'(p) \le 0, \qquad 0 \le \Psi_r'(p) \le L_r, \qquad \text{for a.e. } p \in [\underline{p}, \overline{p}]. \tag{2.7}$$

Using once again the monotonicity of $G_{l,r}$, one obtains that

$$\Psi_l(\underline{p}) \ge f_{l,r}(\underline{u}_{l,r}) \ge \Psi_r(\underline{p}), \tag{2.8}$$

and

$$\Psi_l(\overline{p}) \le f_{l,r}(\overline{u}_{l,r}) \le \Psi_r(\overline{p}). \tag{2.9}$$

Therefore, the existence of $p \in [\underline{p}, \overline{p}]$ such that $\Psi_l(p) = \Psi_r(p)$ follows from (2.8)–(2.9), from the monotonicity (2.7) of $\Psi_{l,r}$ and from the intermediate value theorem. Moreover, if there exists $\hat{p} > p$ such that $\Psi_l(\hat{p}) = \Psi_r(\hat{p})$, then due to their monotonicity, both Ψ_l and Ψ_r are constant on $[p, \hat{p}]$, ensuring the uniqueness of the value $G_\beta(u_l, u_r)$ defined by (2.5).

Let us now show that (2.6) holds. Let $\hat{u}_l \in U_l$ be such that $\hat{u}_l \ge u_l$, and let $p, \hat{p}$ be given by

$$\begin{aligned} p &= \min\{\ \pi \mid G_l(u_l, \tilde{u}_l(\pi)) = G_r(\tilde{u}_r(\pi), u_r)\ \}, \\ \hat{p} &= \min\{\ \pi \mid G_l(\hat{u}_l, \tilde{u}_l(\pi)) = G_r(\tilde{u}_r(\pi), u_r)\ \}. \end{aligned}$$

Since $G_l(\hat{u}_l, \tilde{u}_l(p)) \ge G_l(u_l, \tilde{u}_l(p)) = G_r(\tilde{u}_r(p), u_r)$, then thanks to the monotonicity properties of $G_{l,r}$ and $\pi \mapsto \tilde{u}_{l,r}(\pi)$, we deduce that $\hat{p} \ge p$. As a consequence,

$$G_\beta(u_l, u_r) = G_r(\tilde{u}_r(p), u_r) \le G_r(\tilde{u}_r(\hat{p}), u_r) = G_\beta(\hat{u}_l, u_r),$$

ensuring that $\partial_{u_l} G_\beta(u_l, u_r) \ge 0$. Moreover, G_l is L_l-Lipschitz continuous w.r.t. its first variable, yielding

$$\begin{aligned} G_\beta(\hat{u}_l, u_r) - G_\beta(u_l, u_r) =& G_l(\hat{u}_l, \tilde{u}_l(\hat{p})) - G_l(u_l, \tilde{u}_l(p)) \\ \le& G_l(\hat{u}_l, \tilde{u}_l(p)) - G_l(u_l, \tilde{u}_l(p)) \le L_l(\hat{u}_l - u_l). \end{aligned}$$

This ensures that $\partial_{u_l} G_\beta(u_l, u_r) \le L_l$. Proving that $-L_r \le \partial_{u_r} G_\beta(u_l, u_r) \le 0$ is similar. $\square$

It is worth remarking that the set $\mathcal{G}(\beta) \subset U_l \times U_r$, defined by (1.12), can be rewritten under the form

$$\mathcal{G}(\beta) = \{(u_l, u_r) \mid f_l(u_l) = f_r(u_r) = G_\beta(u_l, u_r)\}. \tag{2.10}$$

Now, we examine $\mathcal{G}(\beta)$ in the light of Definition 2.1.

Proposition 2.8. *Consider equation (1.1) with fluxes (1.2),(1.4) and a transmission map β. The set $\mathcal{G}(\beta)$ is an L^1-dissipative germ in the sense of Definition 2.1.*

Proof. Given $(u_l, u_r) \in \mathcal{G}(\beta)$, equality $f_l(u_l) = f_r(u_r)$ holds by the definition of $\mathcal{G}(\beta)$. Further, let (u_l, u_r) and (v_l, v_r) be two elements of $\mathcal{G}(\beta)$; let us show that (2.1) holds. First of all, notice that

$$q_l(u_l, v_l) - q_r(u_r, v_r) = 0 \quad \text{ if } (u_l - v_l)(u_r - v_r) \geq 0,$$

so that we only have to consider the case $(u_l - v_l)(u_r - v_r) < 0$. Thanks to the symmetry of $q_{l,r}$ w.r.t. $u_{l,r}$ and $v_{l,r}$, we can assume without loss of generality that $u_l < v_l$ and $u_r > v_r$. Thanks to (2.10) and to the monotonicity of G_β stated in (2.6), we obtain that

$$f_{l,r}(u_{l,r}) = G_\beta(u_l, u_r) \leq G_\beta(v_l, v_r) = f_{l,r}(v_{l,r}).$$

Therefore, inequality (2.1) holds. □

Further, let us recall that the value of the Godunov numerical flux with arguments u_l and u_r for a hyperbolic conservation law is defined as the value of the flux at $x = 0$ in the solution of the Riemann problem with endpoints u_l and u_r.

Proposition 2.9.

(i) *The L^1-dissipative germ $\mathcal{G}(\beta)$ is complete; in particular, it is maximal.*

(ii) *The application $G_\beta : U_l \times U_r \to \mathbb{R}$ defined in (2.5) of Lemma 2.7 is the Godunov numerical flux at the interface corresponding to the $\mathcal{G}(\beta)$-entropy Riemann solver.*

Proof. Let $(u_l, u_r) \in U_l \times U_r$, then thanks to Lemma 2.7, there exists $(\tilde{u}_l, \tilde{u}_r) \in \beta$ such that

$$G_\beta(u_l, u_r) = G_l(u_l, \tilde{u}_l) = G_r(\tilde{u}_r, u_r). \tag{2.11}$$

Fix these values $\tilde{u}_{l,r}$ and define $v \in L^\infty(\Omega_{l,r} \times \mathbb{R}_+)$ as the solution of (1.6) with the Riemann datum $u_0 = u_l \mathbf{1}_{x<0} + u_r \mathbf{1}_{x>0}$ (in the sense of [21]), and denote by $v_{l,r}$ its one-sided traces of $\{x = 0\}$ (which exist and which are constant w.r.t. t, since v is self-similar). We claim that

$$(v_l, v_r) \in \mathcal{G}(\beta). \tag{2.12}$$

Indeed, $v_{l,r}$ are characterized by

$$v_l = \begin{cases} \min\{s \mid f_l(s) \leq f_l(w), \forall w \in [u_l, \tilde{u}_l]\} & \text{if } u_l \leq \tilde{u}_l, \\ \max\{s \mid f_l(s) \geq f_l(w), \forall w \in [\tilde{u}_l, u_l]\} & \text{if } \tilde{u}_l \leq u_l, \end{cases} \tag{2.13}$$

$$v_r = \begin{cases} \min\{s \mid f_r(s) \leq f_r(w), \forall w \in [\tilde{u}_r, u_r]\} & \text{if } \tilde{u}_r \leq u_r, \\ \max\{s \mid f_r(s) \geq f_r(w), \forall w \in [u_r, \tilde{u}_r]\} & \text{if } u_r \leq \tilde{u}_r. \end{cases} \tag{2.14}$$

The solution v of (1.6) satisfies the maximum principle on $\Omega_{l,r} \times \mathbb{R}_+$, ensuring that $v_{l,r} \in I(u_{l,r}, \tilde{u}_{l,r})$, where $I(a,b) = [\min(a,b), \max(a,b)]$. The Bardos-LeRoux-Nédélec condition [21] for $v_{l,r}$ then writes

$$f_l(v_l) = G_l(v_l, \tilde{u}_l), \qquad f_r(v_r) = G_r(\tilde{u}_r, v_r). \tag{2.15}$$

On the other hand, it follows from (2.13)–(2.14) and from the definition of Godunov fluxes $G_{l,r}$ that

$$G_l(u_l, \tilde{u}_l) = f_l(v_l) \quad \text{and} \quad f_r(v_r) = G_r(\tilde{u}_r, u_r). \tag{2.16}$$

In conclusion, it follows from (2.11), and (2.15),(2.16) that

$$G_\beta(u_l, u_r) = f_{l,r}(v_{l,r}) = G_l(v_l, \tilde{u}_l) = G_r(\tilde{u}_r, v_r),$$

which ensures (2.12). Now, the claims **(i)** and **(ii)** follow.

(i) Because the states $u_{l,r} \in U_{l,r}$ in the Riemann problem are arbitrary, property (2.12) of traces of the solution to the Riemann problem means in particular that the germ $\mathcal{G}(\beta)$ is complete in the sense of Definition 2.1. Therefore, $\mathcal{G}(\beta)$ is maximal thanks to Proposition 2.3.

(ii) As a consequence, it follows from Proposition 2.6 that v is the unique $\mathcal{G}(\beta)$-entropy solution to the Riemann problem (1.1),(1.2) with initial datum $u_0 := u_l \mathbf{1}_{x<0} + u_r \mathbf{1}_{x>0}$. Because the flux $f_{l,r}(v_{l,r})$ across $\{x = 0\}$ is given by $G_\beta(u_l, u_r)$, this means that (2.5) is the precise expression of the Godunov numerical flux for the Riemann problem set up at the interface. □

The interest of Proposition 2.9 is two-fold. First, after parametrizing β in a monotone way (for example as suggested in (2.4)), the formula (2.5) provides an expression of G_β depending on the one-sided Godunov solvers $G_{l,r}$. Extension of this idea will provide a generic construction for monotone schemes (developed in §3) for approximating the $\mathcal{G}(\beta)$-entropy solution to the problem. On the other hand, together with Theorem 2.5, it ensures the existence and the uniqueness of the $\mathcal{G}(\beta)$-entropy solution to the Cauchy problem (1.1),(1.2),(1.5) as stressed in the following statement.

Corollary 2.10. *Consider equation* (1.1) *with fluxes* (1.2),(1.4) *and a transmission map* β*. There exists a unique* $\mathcal{G}(\beta)$*-entropy solution to the problem* (1.1),(1.2),(1.5) *in the sense of Definition 2.4.*

Remark 2.11. According to Proposition 2.6, whenever strong boundary traces $u_{l,r}$ in the sense (1.8) of a $\mathcal{G}(\beta)$-entropy solution of (1.1),(1.2),(1.5) exist (this is the case, e.g., under the non-linearity assumption on $f_{l,r}$), the unique $\mathcal{G}(\beta)$-entropy solution of Corollary 2.10 is also the unique solution in the sense of local Kruzhkov entropy inequalities away from the interface, see (1.7), and the interface transmission condition encoded by the transmission map β, see (1.11).

Thus, we have reduced the study of well-posedness for admissible solutions originating from the transmission map approach to general results on L^1-dissipative solvers for conservation laws...

2.3. *...but some maximal L^1D germs do not result from a transmission map*

Indeed, we now present an explicit counter-example. In order to build an L^1-dissipative germ that does not correspond to any transmission map β, we utilize the notion of *connection* initially introduced by Adimurthi *et al.* [3] (see also [28]) in the case where $f_{l,r}$ are bell-shaped.

Definition 2.12. *A couple $(c_l, c_r) \in U_l \times U_r$ is said to be*

- *a* strict connection *if*

$$f_l(c_l) = f_r(c_r), \qquad \textit{and} \qquad \begin{cases} G_l(c_l, \tilde{c}_l) = f_l(c_l) \Leftrightarrow c_l = \tilde{c}_l, \\ G_r(\tilde{c}_r, c_r) = f_r(c_r) \Leftrightarrow c_r = \tilde{c}_r; \end{cases} \tag{2.17}$$

- *a* connection *if, for all $\epsilon > 0$, there exists a strict connection $(c_l^\epsilon, c_r^\epsilon)$ such that*

$$|c_l - c_l^\epsilon| + |c_r - c_r^\epsilon| \leq \epsilon.$$

We denote by $\mathfrak{C} \subset U_l \times U_r$ the set of all the strict connections, and by $\overline{\mathfrak{C}} \subset U_l \times U_r$ the set of all the connections.

It is worth noticing that if $f_{l,r}$ belong to $C^1(U_{l,r}; \mathbb{R})$, a sufficient condition for (c_l, c_r) to be a strict connection is that

$$f_l(c_l) = f_r(c_r), \quad f_l'(c_l) < 0, \quad \text{and} \quad f_r'(c_r) > 0. \tag{2.18}$$

Therefore, in the particular case of bell-shaped fluxes (cf. §2.4), our definition coincides with the one introduced by Adimurthi *et al.* [3] and exploited by Bürger *et al.* [28].

Let us state the following lemma, that claims that all the connections that are in the germ $\mathcal{G}(\beta)$ belong to the graph β itself.

Lemma 2.13. *Let $(c_l, c_r) \in \mathfrak{C} \cap \mathcal{G}(\beta)$, then $(c_l, c_r) \in \beta$.*

Proof. Let $(c_l, c_r) \in \mathfrak{C} \cap \mathcal{G}(\beta)$. Since $(c_l, c_r) \in \mathcal{G}(\beta)$, there exists $(\tilde{c}_l, \tilde{c}_r) \in \beta$ such that

$$G_l(c_l, \tilde{c}_l) = f_l(c_l) = f_r(c_r) = G_r(\tilde{c}_r, c_r).$$

Since $(c_l, c_r) \in \mathfrak{C}$, one has

$$G_l(c_l, \tilde{c}_l) = f_l(c_l) \Leftrightarrow c_l = \tilde{c}_l, \qquad G_r(\tilde{c}_r, c_r) = f_r(c_r) \Leftrightarrow c_r = \tilde{c}_r,$$

ensuring that $(c_l, c_r) = (\tilde{c}_l, \tilde{c}_r) \in \beta$. ◻

In view of Lemma 2.13, we propose to build a maximal L^1D germ $\mathcal{G}^*$ such that $\mathfrak{C}\cap\mathcal{G}^*$ contains two elements (c_l,c_r) and $(\hat{c}_l,\hat{c}_r)$ with $c_l<\hat{c}_l$ and $c_r>\hat{c}_r$. Then, since no monotone graph β can contain both (c_l,c_r) and $(\hat{c}_l,\hat{c}_r)$, the germ $\mathcal{G}^*$ can not correspond to any maximal monotone graph β. Further, as a consequence of Zorn's Lemma, any germ $\mathcal{G}$ admits a (non necessarily unique) maximal extension; hence we can restrict our study to the building of a germ $\mathcal{G}$ containing two connections (c_l,c_r) and $(\hat{c}_l,\hat{c}_r)$ such that $c_l<\hat{c}_l$ and $c_r>\hat{c}_r$.

In the counter-example we present now, $\underline{u}_{l,r}=0$, $\overline{u}_{l,r}=2\pi$, and

$$f_l(u)=-\sin(u),\quad f_r(u)=+\sin(u).$$

Set $c_l\in(0,\pi/2)$, $c_r=2\pi-c_l$, $\hat{c}_l=c_r$ and $\hat{c}_r=c_l$, then both (c_l,c_r) and $(\hat{c}_l,\hat{c}_r)$ belong to $\mathfrak{C}$ since the sufficient condition (2.18) holds for each of the couples. Moreover, property

$$q_l(c_l,\hat{c}_l)\geq q_r(c_r,\hat{c}_r),$$

is easily checked, so that $\mathcal{G}:=\{(c_l,c_r),(\hat{c}_l,\hat{c}_r)\}$ is an L^1D germ. This provides the required counterexample: no L^1D germ of the form $\mathcal{G}(\beta)$ can contain $\mathcal{G}$.

2.4. *The case of bell-shaped fluxes*

In this section, we assume in addition to assumptions (1.2),(1.4) that the fluxes $f_{l,r}$ are bell-shaped, in the sense that there exist $\sigma_{l,r}\in U_{l,r}$ such that

$$f'_{l,r}(s)(\sigma_{l,r}-s)>0\quad\text{for a.e. } s\in U_{l,r}. \tag{2.19}$$

In particular, $s\mapsto f_{l,r}(s)$ admit a unique maximum at $s=\sigma_{l,r}$. In the bell-shaped case, the set $\overline{\mathfrak{C}}$ of the connections introduced in Definition 2.12 reduces to a portion of a strictly decreasing graph in $U_l\times U_r$, and (2.18) becomes the necessary and sufficient condition for (c_l,c_r) to be a strict connection. We refer to [7] or [8] for graphical illustrations of the set $\overline{\mathfrak{C}}$ (called $\mathcal{U}$ is these works).

In the bell-shaped case, it is possible to classify all definite L^1D germs. Indeed, it is shown in [14, §4.8] that

$$\begin{array}{l}\text{every maximal } L^1D \text{ germ contains one and only one connection } (c_l,c_r)\\ \text{i.e., a singleton } \{(c_l,c_r)\} \text{ is a definite } L^1D \text{ germ } \Leftrightarrow (c_l,c_r)\in\overline{\mathfrak{C}}.\end{array} \tag{2.20}$$

In our preceding work [7], we also underlined the interest of the parametrization of the set $\overline{\mathfrak{C}}$ of connections (c_l,c_r) by the *flux limitation level* F in the interval

$$I_F:=\Big[\max\{f_{l,r}(0),f_{l,r}(1)\}\,,\,\min\{\max_{U_l}f_l,\max_{U_r}f_r\}\Big],$$

in the spirit of the idea introduced in [41]. Indeed, given $F\in I_F$, there exists a unique $(c_l,c_r)\in\overline{\mathfrak{C}}$ such that $f_{l,r}(c_{l,r})=F$. Moreover, if $\mathcal{G}$ is a definite L^1D germ that contains $(c_l,c_r)\in\overline{\mathfrak{C}}$, one easily checks that

$$f_{l,r}(u_{l,r})\leq F=f_{l,r}(c_{l,r})\in I_F\ \text{ for all } (u_l,u_r)\in\mathcal{G}^*. \tag{2.21}$$

In this case, the flux limitation constraint "$\lim_{x\to 0} f(u(t,x);x) \le F$" is fulfilled for every $\mathcal{G}$-entropy solution u of (1.1),(1.2) (see [7], cf. [41,11] for details).

To sum up, in the preceding works three equivalent points of view for admissibility of solutions to (1.1),(1.2) in the bell-shaped case (2.19) were put forward:

1. Fix a definite L^1-dissipative germ $\mathcal{G}$. Consider $\mathcal{G}$-entropy solutions of [14].
2. Fix a connection $(c_l, c_r) \in \overline{\mathfrak{C}}$. Use the definitions of [3] and [28].
3. Fix a constraint level $F \in I_F$ in (2.21). Proceed as in [11] (see also [7]).

In this paper, we advocate for a yet another approach to (1.1),(1.2), which is equivalent to **1.–3.** in the bell-shaped case:

4. Fix a transmission map β. Consider solutions admissible in the sense of local Kruzhkov entropy conditions (1.7) and the interface transmission condition (1.11).

Indeed, since $\mathfrak{C}$ reduces to a portion of a decreasing graph in $U_l \times U_r$, its intersection with any given maximal monotone graph β contains at most one singleton (c_l, c_r). The following alternative holds.

• If $\beta \cap \mathfrak{C} = \{(c_l, c_r)\} \neq \emptyset$, then it is evident that $(c_l, c_r) \in \mathcal{G}(\beta)$. Due to (2.20), this identifies the L^1D germ $\mathcal{G}(\beta)$ with the unique maximal extension $\{(c_l, c_r)\}^*$ of the L^1D germ $\{(c_l, c_r)\}$. In this case the $\mathcal{G}(\beta)$-entropy solution coincides with the $\{(c_l, c_r)\}$-entropy solution introduced in [3] and [28].

• If $\beta \cap \mathfrak{C} = \emptyset$, then reproducing the analysis performed in [8, §1.3] permits to prove that the unique non-strict connection $(c_l^{\rm opt}, c_r^{\rm opt}) \in \overline{\mathfrak{C}} \setminus \mathfrak{C}$ belongs to $\mathcal{G}(\beta)$. According to (2.20), $\{(c_l^{\rm opt}, c_r^{\rm opt})\}$ is a definite germ, therefore $\mathcal{G}(\beta) = \{(c_l^{\rm opt}, c_r^{\rm opt})\}^*$ and, following the terminology introduced in [3], the $\mathcal{G}(\beta)$-entropy solution coincides with the so-called *optimal entropy solution*.

Remark 2.14. The transmission map approach **4.** developed in this paper leads, as for the approaches **1–3**, to a complete description of the set of the L^1-dissipative germs in the bell-shaped case. We demonstrate in this paper that it can be applied directly for the general non-bell-shaped case, while the approach by connections **2.** becomes quite technical in the non-bell-shaped case (we refer in particular to the PhD thesis of S. Mishra [61], where this problem has been studied thoroughly). In the sequel, we will also extend the notion of optimal entropy solution to the non-bell-shaped case.

3. Design and study of numerical scheme

While the convergence study of numerical schemes performed in [14] for general L^1-dissipative germs was restricted to abstract Godunov schemes, we will take advantage of the particular structure of the germs generated *via* a transmission map for designing convergent numerical schemes based on any monotone approximate

Riemann solvers associated to $f_{l,r}$. This approach was suggested by the authors in [9] (see also [6]) in the framework of multiphase flows in porous media.

3.1. *Building monotone finite volume schemes*

Definition 3.1. *A monotone approximate Riemann solver associated to a L-Lipschitz continuous nonlinearity $f : [\underline{u}, \overline{u}] \to \mathbb{R}$ consists in an application $F : [\underline{u}, \overline{u}]^2 \to \mathbb{R}$ such that*

a) *F is* consistent *w.r.t. f: $F(u,u) = f(u)$, $\quad \forall u \in [\underline{u}, \overline{u}]$.*
b) *F is* monotone *and* Lipschitz continuous *w.r.t. its two variables, namely, there exists $M > 0$ such that*

$$0 \le \partial_a F(a,b) \le M, \qquad -M \le \partial_b F(a,b) \le 0 \quad \textit{for a.e. } (a,b) \in [\underline{u}, \overline{u}]^2.$$

While the exact Riemann (or Godunov) solver

$$G(a,b) = \begin{cases} \min_{s \in [a,b]} f(s) & \text{if } a \le b \\ \max_{s \in [b,a]} f(s) & \text{if } a \ge b \end{cases}$$

is a particular monotone approximate Riemann solver, numerous other examples are encountered in practice, such as

- the Lax-Friedrichs solver F_{LF} (under the CFL condition $\frac{\Delta t}{\Delta x} \le \frac{1}{L}$):
$$F_{\mathrm{LF}}(a,b) = \frac{f(a)+f(b)}{2} + \frac{\Delta x}{2\Delta t}(a-b);$$
- the Rusanov solver F_{Rus}:
$$F_{\mathrm{Rus}}(a,b) = \frac{f(a)+f(b)}{2} + \frac{L(a-b)}{2};$$
- the Engquist-Osher solver F_{EO}:
$$F_{\mathrm{EO}}(a,b) = \frac{f(a)+f(b)}{2} - \frac{1}{2}\int_a^b |f'(s)| \mathrm{d}s.$$

Let us also mention two important application-based monotone finite volume schemes:

- the so-called phase-by-phase upstream scheme studied in [66,23] for approximating immiscible incompressible flows in porous media;
- the so-called Hilliges-Weidlich scheme [54] for approximating the car density prescribed by first-order LWR models [60,64].

Given a couple of monotone approximate Riemann solvers $F_{l,r}$ we can define as in (2.5), the corresponding interface flux map $F_\beta : U_l \times U_r \to \mathbb{R}$. Indeed, we have the following lemma whose proof is the same as the one of Lemma 2.7, the Godunov solvers $G_{l,r}$ being replaced by $F_{l,r}$.

Lemma 3.2. *Given monotone approximate Riemann solvers $F_{l,r}$ associated to $f_{l,r}$, as in Definition (3.1), for all $(u_l, u_r) \in U_l \times U_r$, there exists a unique value $F_\beta(u_l, u_r)$ satisfying*

$$F_\beta(u_l, u_r) := F_l(u_l, \tilde{u}_l) = F_r(\tilde{u}_r, u_r) \qquad \text{for some } (\tilde{u}_l, \tilde{u}_r) \in \beta. \tag{3.1}$$

Moreover, the so defined map $F_\beta : U_l \times U_r \to \mathbb{R}$ is Lipschitz continuous, and for a.e. $(u_l, u_r) \in U_l \times U_r$, one has

$$0 \le \partial_{u_l} F_\beta(u_l, u_r) \le M_l, \qquad -M_r \le \partial_{u_r} F_\beta(u_l, u_r) \le 0,$$

$M_{l,r}$ being the best Lipschitz constants of $F_{l,r}$ respectively.

Let us now describe the scheme. For the ease of reading, we restrict our attention to the case of uniform discretization of $\mathbb{R} \times \mathbb{R}_+$ described by a space step $\Delta x > 0$ and a time step $\Delta t > 0$. We denote by $\{x_j = j\Delta x \mid j \in \mathbb{Z}\}$ the set of the edges (that are just breakpoints since we are in 1D), so that the interface is located at the edge x_0. For all $j \in \mathbb{Z}$, we initialize the scheme by setting, e.g.,

$$u^0_{j+1/2} = \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u_0(x)\mathrm{d}x, \qquad \forall j \in \mathbb{Z}. \tag{3.2}$$

Then, the quantities $(u^{n+1}_{j+1/2})_{j\in\mathbb{Z}}$ are deduced from $(u^n_{j+1/2})_{j\in\mathbb{Z}}$ by the explicit finite volume scheme

$$\frac{u^{n+1}_{j+1/2} - u^n_{j+1/2}}{\Delta t}\Delta x + F_{j+1}(u^n_{j+1/2}, u^n_{j+3/2}) - F_j(u^n_{j-1/2}, u^n_{j+1/2}) = 0, \tag{3.3}$$

where

$$F_j(a, b) = \begin{cases} F_{l,r}(a, b) & \text{if } x_j \in \Omega_{l,r}, \\ F_\beta(a, b) & \text{if } j = 0. \end{cases} \tag{3.4}$$

The discrete solution u_h is then defined almost everywhere by

$$u_h(x, t) = u^{n+1}_{j+1/2} \qquad \text{if } (x, t) \in (x_j, x_{j+1}) \times (n\Delta t, (n+1)\Delta t]. \tag{3.5}$$

The scheme (3.3) can be rewritten

$$u^{n+1}_{j+1/2} = H_{j+1/2}(u^n_{j+1/2}, u^n_{j-1/2}, u^n_{j+3/2}), \quad \forall j \in \mathbb{Z}, \forall n \in \mathbb{N},$$

where the map $H_{j+1/2}$ is nondecreasing w.r.t. each of its arguments under the CFL condition

$$\Delta t \le \frac{\Delta x}{2\max(M_l, M_r)}, \tag{3.6}$$

thanks to the monotonicity and the Lipschitz continuity of $F_{l,r}$ (cf. Definition 3.1) and of F_β (cf. Lemma 3.2).

3.2. *Exactly preserved and reachable steady states*

We will say that the scheme (3.3) based on solvers $F_{l,r}$ *preserves exactly* a state $(\kappa_l, \kappa_r) \in \mathcal{G}(\beta)$ if the discrete function

$$\kappa_h(x) = \sum_{j\in\mathbb{Z}} \kappa_{j+1/2} \mathbf{1}_{(x_j,x_{j+1})}(x) = \begin{cases} \kappa_l & \text{if } x < 0, \\ \kappa_r & \text{if } x > 0, \end{cases} \tag{3.7}$$

is a steady solution to the scheme (3.3). As it appears in the expression (2.10) of $\mathcal{G}(\beta)$, the Godunov scheme for $\mathcal{G}(\beta)$-entropy solutions of (1.1),(1.2) (whose interface flux is precisely G_β, according to Proposition 2.9(ii)) preserves exactly the whole maximal L^1D germ $\mathcal{G}(\beta)$. The exact preservation property is the main ingredient in the proof of convergence of this scheme to the $\mathcal{G}(\beta)$-entropy solution of (1.1),(1.2) (see Theorem 2.5 and [14]). This property is lost when considering general monotone solvers of the form (3.4). Nevertheless, whatever be the choice of $F_{l,r}$, a particular subset $\mathcal{G}^\flat(\beta)$ of $\mathcal{G}(\beta)$ is preserved exactly by the approximated interface Riemann solver F_β. Indeed, we have

Lemma 3.3. *Denote by* $\mathcal{G}^\flat(\beta) = \{\, (\kappa_l, \kappa_r) \in \beta \mid f_l(\kappa_l) = f_r(\kappa_r) \,\}$, *then*

$$F_\beta(\kappa_l, \kappa_r) = f_{l,r}(\kappa_{l,r}), \quad \forall (\kappa_l, \kappa_r) \in \mathcal{G}^\flat(\beta).$$

As a consequence of Lemma 3.3 and of the consistency of the numerical fluxes $F_{l,r}$ in $\Omega_{l,r}$, given $(\kappa_l, \kappa_r) \in \mathcal{G}^\flat(\beta)$, the discrete function (3.7) is an obvious steady solution to the scheme (3.3). Notice that, in particular, strict connections in the sense of Definition 2.12 that belong to the germ $\mathcal{G}(\beta)$ also belong to the set $\mathcal{G}^\flat(\beta)$ of exactly preserved states. Also observe that the extremal states $(\underline{u}_l, \underline{u}_r)$ and $(\overline{u}_l, \overline{u}_r)$ belong to $\mathcal{G}^\flat(\beta)$.

Proof of Lemma 3.3. Let $(\kappa_l, \kappa_r) \in \mathcal{G}^\flat(\beta)$, then since $(\kappa_l, \kappa_r) \in \beta$, one has

$$F_\beta(\kappa_l, \kappa_r) = F_l(\kappa_l, \kappa_l) = F_r(\kappa_r, \kappa_r) = f_{l,r}(\kappa_{l,r})$$

thanks to the consistency of $F_{l,r}$ with $f_{l,r}$. □

Unfortunately, the knowledge of the family $\mathcal{G}^\flat(\beta)$ of exactly preserved states is not sufficient to guarantee the convergence of the scheme. Therefore, we have to pay more attention to more complicated steady solutions of (3.3). To this end, we introduce (following [31]) the set of the so-called *reachable steady states*. The name "reachable" stems from the fact, proved in Lemmas 3.7, 3.8 below, that these states can be obtained as limits of nontrivial numerical profiles solving the scheme (3.3), whatever be the choice of solvers $F_{l,r}$.

Definition 3.4. *An element* $(\kappa_l, \kappa_r) \in U_l \times U_r$ *is said to be a* reachable state *if* $f_l(\kappa_l) = f_r(\kappa_r)$ *and if there exists* $(\tilde{\kappa}_l, \tilde{\kappa}_r) \in \beta$ *such that*

$$\begin{cases} f_l(\kappa_l) < f_l(s), & \forall s \in (\kappa_l, \tilde{\kappa}_l] \ \text{ if } \kappa_l \le \tilde{\kappa}_l, \\ f_l(\kappa_l) > f_l(s), & \forall s \in [\tilde{\kappa}_l, \kappa_l) \ \text{ if } \kappa_l \ge \tilde{\kappa}_l, \end{cases} \tag{3.8}$$

and

$$\begin{cases} f_r(\kappa_r) > f_r(s), & \forall s \in (\kappa_r, \tilde{\kappa}_r] \ \textit{if } \kappa_r \leq \tilde{\kappa}_r, \\ f_r(\kappa_r) < f_r(s), & \forall s \in [\tilde{\kappa}_r, \kappa_r) \ \textit{if } \kappa_r \geq \tilde{\kappa}_r. \end{cases} \tag{3.9}$$

We denote by $\mathcal{G}^o(\beta)$ the subset of $U_l \times U_r$ containing all the reachable states.

Observe that exactly preserved states are reachable, more precisely, we have

Lemma 3.5. *One has $\mathcal{G}^\flat(\beta) \subset \mathcal{G}^o(\beta) \subset \mathcal{G}(\beta)$.*

Proof. First remark that $\mathcal{G}^\flat(\beta) \subset \mathcal{G}^o(\beta)$ since one can choose $\tilde{\kappa}_{l,r} = \kappa_{l,r}$, so that the conditions in (3.8)–(3.9) are trivially satisfied. Now, let $(\kappa_l, \kappa_r) \in \mathcal{G}^o(\beta)$, and let $(\tilde{\kappa}_l, \tilde{\kappa}_r) \in \beta$ such that (3.8)–(3.9) hold, then, in view of the definition (1.9) of the one-sided Godunov solvers, one has

$$f_l(\kappa_l) = G_l(\kappa_l, \tilde{\kappa}_l), \qquad f_r(\kappa_r) = G_r(\tilde{\kappa}_r, \kappa_r).$$

Since $f_l(\kappa_l) = f_r(\kappa_r)$, we obtain that (κ_l, κ_r) belongs to $\mathcal{G}(\beta)$. □

The germ $\mathcal{G}^o(\beta)$ has three important properties that we now prove. Firstly, this is a definite germ.

Proposition 3.6. *The subset $\mathcal{G}^o(\beta)$ of $\mathcal{G}(\beta)$ is a definite L^1-dissipative germ, and $\mathcal{G}(\beta)$ is the unique maximal L^1-dissipative germ containing $\mathcal{G}^o(\beta)$.*

Proof. To prove this proposition, we can make appeal to the notion of closed germ developed in [14]. By the closure of an L^1D germ $\mathcal{G}$ we understand the smallest closed germ containing $\mathcal{G}$.

Let $(\kappa_l, \kappa_r) \in \mathcal{G}(\beta)$, then there exists $(\tilde{\kappa}_l, \tilde{\kappa}_r) \in \beta$ such that

$$f_{l,r}(\kappa_{l,r}) = G_l(\kappa_l, \tilde{\kappa}_l) = G_r(\tilde{\kappa}_r, \kappa_r). \tag{3.10}$$

Setting

$$\kappa^o_{l,r} = \begin{cases} \max\{s \in [\kappa_{l,r}, \tilde{\kappa}_{l,r}] \mid f_{l,r}(s) = f_{l,r}(\kappa_{l,r})\} & \text{if } \kappa_{l,r} \leq \tilde{\kappa}_{l,r}, \\ \min\{s \in [\tilde{\kappa}_{l,r}, \kappa_{l,r}] \mid f_{l,r}(s) = f_{l,r}(\kappa_{l,r})\} & \text{if } \kappa_{l,r} \geq \tilde{\kappa}_{l,r}, \end{cases} \tag{3.11}$$

it follows from the definition (1.9) of $G_{l,r}$ and the Definition 3.4 of $\mathcal{G}^o(\beta)$ that $(\kappa^o_l, \kappa^o_r) \in \mathcal{G}^o(\beta)$ and that

$$G_l(\kappa_l, \kappa^o_l) = f_{l,r}(\kappa_{l,r}) = f_{l,r}(\kappa^o_{l,r}) = G_r(\kappa^o_r, \kappa_r). \tag{3.12}$$

From (3.12), we see that the states κ_l and κ^o_l (respectively, κ_r and κ^o_r) can be connected by a zero-speed Kruzhkov-admissible shock of the conservation law $\partial_t u + \partial_x f_l(u) = 0$ (resp., of $\partial_t u + \partial_x f_r(u) = 0$), which means exactly that (κ_l, κ_r) belongs to the closure of $\mathcal{G}^o(\beta)$. Further, it is proved in [14] that every maximal extension of a germ contains its closure; thus, every maximal extension of $\mathcal{G}^o(\beta)$ contains $\mathcal{G}(\beta)$. But since we know that $\mathcal{G}(\beta)$ is maximal itself, this means that $\mathcal{G}(\beta)$ it the unique maximal extension of $\mathcal{G}^o(\beta)$; in other words, $\mathcal{G}^o(\beta)$ is a definite L^1D germ. □

Secondly, for $(u_l, u_r) \in \mathcal{G}^o(\beta)$ the states u_l (at $-\infty$) and u_r (at $+\infty$) can be connected by numerical profiles of the finite volume scheme we consider in this section, as explained in the following lemma:

Lemma 3.7. *Let $(\kappa_l, \kappa_r) \in \mathcal{G}^o(\beta)$, and let $(\tilde\kappa_l, \tilde\kappa_r) \in \beta$ such that (3.8)–(3.9) hold, then there exists $(\kappa_{j+1/2})_{j\in\mathbb{Z}}$ such that*

i) *the function $\kappa_h = \sum_{j\in\mathbb{Z}} \kappa_{j+1/2}\mathbf{1}_{(x_j,x_{j+1})}$ is a steady solution to the scheme (3.3), more precisely, there holds*

$$\begin{cases} f_l(\kappa_l) = F_l(\kappa_{j-1/2}, \kappa_{j+1/2}) = F_l(\kappa_{-1/2}, \tilde\kappa_l), & \forall j < 0, \\ f_r(\kappa_r) = F_r(\kappa_{j-1/2}, \kappa_{j+1/2}) = F_l(\tilde\kappa_r, \kappa_{1/2}), & \forall j > 0, \end{cases} \tag{3.13}$$

ii) *the sequences $(\kappa_{j+1/2})_{j\ge 0}$ and $(\kappa_{j-1/2})_{j\le 0}$ are monotone and*

$$\lim_{j\to-\infty} \kappa_{j-1/2} = \kappa_l, \qquad \lim_{j\to+\infty} \kappa_{j+1/2} = \kappa_r.$$

Proof. Let us focus on what occurs in Ω_r, the situation in Ω_l being similar. In the case where $\tilde\kappa_r = \kappa_r$, then one can choose $\kappa_{j+1/2} = \kappa_r$ for all $j \ge 0$. Assume now that $\tilde\kappa_r \ne \kappa_r$, say $\tilde\kappa_r > \kappa_r$ for the sake of being definite. Then due to (3.9), due to the consistency and to the monotonicity of F_r, one has

$$F_r(\tilde\kappa_r, \tilde\kappa_r) \le f_r(\kappa_r) \le F_r(\tilde\kappa_r, \kappa_r).$$

Therefore, since F_r is continuous w.r.t. its second argument, there exists $\kappa_{1/2} \in [\kappa_r, \tilde\kappa_r)$ such that $F_r(\tilde\kappa_r, \kappa_{1/2}) = f_r(\kappa_r)$. If $\kappa_{1/2} = \kappa_r$ (this is the case if $F_r \equiv G_r$), then one can choose $\kappa_{j+1/2} = \kappa_r$ for all $j \ge 0$.

Assume that for $j \ge 0$, we have built $\{\kappa_{1/2}, \dots, \kappa_{j+1/2}\}$ such that

$$\tilde\kappa_r > \kappa_{1/2} > \dots > \kappa_{j-1/2} > \kappa_{j+1/2} > \kappa_r,$$

and

$$f_r(\kappa_r) = F_r(\tilde\kappa_r, \kappa_{1/2}) = F_r(\kappa_{i+1/2}, \kappa_{i+3/2}), \qquad \forall i \in \{0, \dots, j-1\}.$$

Using again (3.9) as well as the properties of F_r, one gets that

$$f_r(\kappa_{j+1/2}) = F_r(\kappa_{j+1/2}, \kappa_{j+1/2}) > f_r(\kappa_r) \ge F_r(\kappa_{j+1/2}, \kappa_r),$$

ensuring the existence of $\kappa_{j+3/2} \in [\kappa_r, \tilde\kappa_r)$ such that $F_r(\kappa_{j+1/2}, \kappa_{j+3/2}) = f_r(\kappa_r)$. If $\kappa_{j+3/2} = \kappa_r$, then one can choose $\kappa_{i+3/2} = \kappa_r$ for all $i \ge j$.

It follows from the above construction (that can be carried out in a similar way if $\tilde\kappa_r < \kappa_r$) that the sequence $(\kappa_{j+1/2})_{j\ge 0}$ is monotone and bounded between κ_r and $\tilde\kappa_r$, so that it converges towards some $\ell_r \in [\min\{\kappa_r, \tilde\kappa_r\}, \max\{\kappa_r, \tilde\kappa_r\}]$ as $j \to \infty$. Passing to the limit $j \to \infty$ in (3.13) yields $f_r(\ell_r) = F_r(\ell_r, \ell_r) = f_r(\kappa_r)$, and thus $\ell_r = \kappa_r$ in view of (3.9). $\square$

And finally, the profiles constructed in Lemma 3.7 are convergent:

Lemma 3.8. *Let $(\kappa_l, \kappa_r) \in \mathcal{G}^o(\beta)$, let $(\kappa_{j+1/2})_{j\in\mathbb{Z}}$ be as in Lemma 3.7, and let $\kappa_h = \sum_{j\in\mathbb{Z}} \kappa_{j+1/2}\mathbf{1}_{(x_j,x_{j+1})}$, then*

$$\kappa_h \to \kappa_l \mathbf{1}_{x<0} + \kappa_r \mathbf{1}_{x>0} \qquad \text{in } L^1_{\rm loc}(\mathbb{R}) \text{ as } \Delta x \to 0.$$

Proof. Let $\epsilon > 0$, then since $\kappa_{j+1/2} \to \kappa_{l,r}$ as $j \to \pm\infty$, there exists $j_0(\epsilon) \geq 0$ such that

$$|j| \geq j_0(\epsilon) \implies |\kappa_{j+1/2} - \kappa_r| \leq \epsilon \text{ and } |\kappa_{-j-1/2} - \kappa_l| \leq \epsilon.$$

Let $(\Delta x_n)_{n\geq 1}$ be a sequence of space steps tending to 0 as $n \to \infty$, and let $x \in \mathbb{R} \setminus \bigcup_{n\geq 1} \Delta x_n \mathbb{Z}$, then there exists $n_0(x, \epsilon)$ such that

$$n \geq n_0(x, \epsilon) \implies |x| > j_0(\epsilon)\Delta x_n,$$

yielding $|\kappa_h(x) - \kappa_l \mathbf{1}_{x<0}(x) - \kappa_r \mathbf{1}_{x>0}(x)| \leq \epsilon$. Hence we get the almost everywhere convergence of κ_h towards $\kappa_l \mathbf{1}_{x<0} + \kappa_r \mathbf{1}_{x>0}$. Since $\underline{u} \leq \kappa_h \leq \overline{u}$, the convergence in $L^1_{\rm loc}(\mathbb{R})$ follows from the dominated convergence theorem. □

3.3. *Monotonicity estimates and compactness*

The monotonicity of $H_{j+1/2}$ under the CFL condition (3.6) has important consequences. In particular, let $v_0 \in L^\infty(\mathbb{R})$ with $\underline{u} \leq v_0 \leq \overline{u}$ a.e. in $\mathbb{R}$ be such that $u_0 - v_0 \in L^1(\mathbb{R})$, let

$$v^0_{j+1/2} = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} v_0(x)\mathrm{d}x, \qquad \forall j \in \mathbb{Z}, \tag{3.14}$$

and let $\left(v^{n+1}_{j+1/2}\right)_{j\in\mathbb{Z}, n\geq 0}$ be given by the scheme

$$v^{n+1}_{j+1/2} = H_{j+1/2}(v^n_{j+1/2}, v^n_{j-1/2}, v^n_{j+3/2}), \quad \forall j \in \mathbb{Z}, \forall n \in \mathbb{N}, \tag{3.15}$$

then it follows from classical arguments (see e.g. [43,44]) that for all $n \geq 0$, one has

$$\sum_{j\in\mathbb{Z}} (u^{n+1}_{j+1/2} - v^{n+1}_{j+1/2})^\pm \Delta x \leq \sum_{j\in\mathbb{Z}} (u^n_{j+1/2} - v^n_{j+1/2})^\pm \Delta x \leq \int_{\mathbb{R}} (u_0 - v_0)^\pm \mathrm{d}x. \tag{3.16}$$

Lemma 3.9. *Let $u_0 \in L^\infty(\mathbb{R})$ with $\underline{u} \leq u_0 \leq \overline{u}$ a.e. in $\mathbb{R}$, and let u_h be the discrete solution defined by (3.2)–(3.5), then, under the CFL condition (3.6), one has*

$$\underline{u}(x) \leq u_h(x, t) \leq \overline{u}(x) \qquad \text{for a.e. } (x, t) \in \mathbb{R} \times \mathbb{R}_+.$$

Proof. Since $(\underline{u}_l, \underline{u}_r)$ and $(\overline{u}_l, \overline{u}_r)$ belong to the subset $\mathcal{G}^\flat(\beta)$ of $\mathcal{G}(\beta)$ defined in Lemma 3.3, the steady states $\underline{u}$ and $\overline{u}$ are exactly preserved by the numerical scheme (3.3). Therefore we can choose $v^n_{j+1/2} = \underline{u}(x_{j+1/2})$ and $v^n_{j+1/2} = \overline{u}(x_{j+1/2})$ for all $j \in \mathbb{Z}$ and all $n \geq 0$ in (3.16), which permits us to conclude. □

Further, due to the consistency of $F_{l,r}$ with $f_{l,r}$, one has

$$\kappa = H_{j+1/2}(\kappa, \kappa, \kappa), \quad \forall j \in \mathbb{Z} \setminus \{-1, 0\}, \forall n \in \mathbb{N}, \forall \kappa \in [\underline{u}(x_{j+1/2}), \overline{u}(x_{j+1/2})].$$

Therefore, with the notation

$$a \top b = \max(a, b) \text{ and } a \bot b = \min(a, b),$$

standard monotonicity arguments (see e.g. [48]) provide that, for all $j \notin \{-1, 0\}$, for all $n \geq 0$ and for all $\kappa \in [\underline{u}(x_{j+1/2}), \overline{u}(x_{j+1/2})]$, one has

$$\begin{aligned}\frac{|u^{n+1}_{j+1/2} - \kappa| - |u^n_{j+1/2} - \kappa|}{\Delta t}\Delta x \\ + F_{j+1}(u^n_{j+1/2} \top \kappa, u^n_{j+3/2} \top \kappa) - F_{j+1}(u^n_{j+1/2} \bot \kappa, u^n_{j+3/2} \bot \kappa) \\ - F_j(u^n_{j-1/2} \top \kappa, u^n_{j+1/2} \top \kappa) + F_j(u^n_{j-1/2} \bot \kappa, u^n_{j+1/2} \bot \kappa) \leq 0. \quad (3.17)\end{aligned}$$

Similarly, let $(\kappa_l, \kappa_r) \in \mathcal{G}^o(\beta)$, and let $\left(\kappa_{j+1/2}\right)_{j\in\mathbb{Z}}$ be as in Lemma 3.7, then for all $j \in \mathbb{Z}$ and for all $n \geq 0$, one has

$$\frac{|u^{n+1}_{j+1/2} - \kappa_{j+1/2}| - |u^n_{j+1/2} - \kappa_{j+1/2}|}{\Delta t}\Delta x + Q^n_{j+1} - Q^n_j \leq 0, \qquad (3.18)$$

where we have set, for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\begin{aligned}Q^n_j =& F_j(u^n_{j-1/2} \top \kappa_{j-1/2}, u^n_{j+1/2} \top \kappa_{j+1/2}) \qquad (3.19)\\ &- F_j(u^n_{j-1/2} \bot \kappa_{j-1/2}, u^n_{j+1/2} \bot \kappa_{j+1/2})\end{aligned}$$

(recall that for F_j, we use F_l if $j < 0$, F_β if $j = 0$ and F_r if $j > 0$).

While (3.17)–(3.18) play an important role in identifying the limit u as $\Delta x, \Delta t \to 0$ of u_h as the unique $\mathcal{G}(\beta)$-entropy solution (as this will appear in §3.4), compactness properties on the family $(u_h)_{\Delta x, \Delta t}$ will be needed to ensure the existence of this aforementioned limit u of u_h. To this end, the so-called $\mathrm{BV}_{\mathrm{loc}}$ strategy proposed in [26,27] can be mimicked, leading to the following Proposition.

Proposition 3.10. *Assume additionally that $u_0 \in \mathrm{BV}(\mathbb{R})$, then, under the CFL condition (3.6), there exists $C > 0$ depending only on $M_{l,r}$ and u_0 (but neither on Δx nor Δt) such that*

$$\int_{\mathbb{R}} |u_h(x, t+\tau) - u_h(x)| \mathrm{d}x \leq C(\tau + \Delta t), \qquad \text{for a.e. } t > 0 \text{ and all } \tau > 0. \quad (3.20)$$

Moreover, for all $\eta > 0$ and all $T > 0$, there exists C_η depending on T, η, $M_{l,r}$ and u_0 (but neither on Δx nor on Δt) such that, for all $\xi \in (-\eta, \eta)$ and $\Delta x \in (0, \eta)$, one has

$$\int_0^T \int_{\mathbb{R}\setminus(-3\eta, 3\eta)} |u_h(x+\xi, t) - u_h(x, t)| \mathrm{d}x\mathrm{d}t \leq C_\eta(|\xi| + \Delta x). \qquad (3.21)$$

3.4. *Convergence of the scheme*

As a consequence of Lemma 3.9 and Proposition 3.10, if $u_0 \in \mathrm{BV}(\mathbb{R})$, there exists a function $u \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ with $\underline{u} \le u \le \overline{u}$ a.e. in $\mathbb{R}\times\mathbb{R}_+$ such that, up to an unlabeled subsequence,

$$u_h \to u \qquad \text{a.e. in } \mathbb{R}\times\mathbb{R}_+ \text{ as } \Delta x, \Delta t \to 0. \tag{3.22}$$

Our goal is now to identify the limit u of u_h as the unique $\mathcal{G}(\beta)$-entropy solution of the problem (1.1),(1.2),(1.5) in the sense of Definition 2.4. To this end, let us first check that the local Kruzhkov entropy inequalities (2.2) hold.

It remains to prove adapted entropy inequalities of Definition 2.4.

Proposition 3.11. *Let $u_0 \in \mathrm{BV}(\mathbb{R})$, and let u be given by (3.22), then, for all $\kappa \in [\underline{u}_{l,r}, \overline{u}_{l,r}]$ and all $\phi \in \mathcal{D}^+(\Omega_{l,r}\times\mathbb{R}_+)$, one has*

$$\iint_{\Omega_{l,r}\times\mathbb{R}_+} |u-\kappa|\partial_t\phi \,\mathrm{d}x\mathrm{d}t \\ + \int_{\Omega_{l,r}} |u_0-\kappa|\phi(\cdot,0)\,\mathrm{d}x + \iint_{\Omega_{l,r}\times\mathbb{R}_+} q_{l,r}(u,\kappa)\partial_x\phi\,\mathrm{d}x\mathrm{d}t \ge 0. \tag{3.23}$$

Proof. Since $\mathrm{supp}(\phi)$ is compact in $\Omega_{l,r}\times\mathbb{R}_+$, there exists $T>0$ and $\epsilon>0$ such that $\mathrm{supp}(\phi) \subset \{|x|\ge\epsilon\}\times[0,T]$. Therefore, thanks to Proposition 3.10, u_h belongs to $\mathrm{BV}(\mathrm{supp}(\phi))$ as soon as Δx is small enough. Proving that the limit u of discrete solutions u_h obtained *via* monotone finite volume schemes satisfies Kruzhkov entropy inequalities and the initial condition (1.5) is then classical (see e.g. [51,48]). □

Proposition 3.12. *Let $u_0 \in \mathrm{BV}(\mathbb{R})$, let u be given by (3.22), then, for all $(\kappa_l,\kappa_r) \in \mathcal{G}^o(\beta)$, property (2.3) holds true.*

Proof. First, it is easy to check that the space $\{\phi\in\mathcal{D}(\mathbb{R}) \mid \partial_x\phi \in \mathcal{D}(\mathbb{R}^*)\}$ of test functions with vanishing near $x=0$ derivative is dense in $\mathcal{D}(\mathbb{R})$ for the $W^{1,1}(\mathbb{R})$ topology, so that, since $\mathcal{D}(\mathbb{R})\otimes\mathcal{D}(\mathbb{R}_+)$ is dense in $W^{1,1}(\mathbb{R}\times\mathbb{R}_+)$, the set

$$\mathcal{T}^+ = \{\phi \in \mathcal{D}^+(\mathbb{R}\times\mathbb{R}_+) \mid \partial_x\phi \in \mathcal{D}(\mathbb{R}^*\times\mathbb{R}_+)\}$$

is dense in $\mathcal{D}^+(\mathbb{R}\times\mathbb{R}_+)$ for the $W^{1,1}(\mathbb{R}\times\mathbb{R}_+)$ topology. Therefore, since u,κ and $q_{l,r}(u,\kappa_{l,r})$ belong to $L^\infty(\mathbb{R}\times\mathbb{R}_+)$, it is sufficient to prove (2.3) with test function $\phi\in\mathcal{T}^+$.

Let $\phi\in\mathcal{T}^+$, then $\mathrm{supp}(\partial_x\phi)\cap((-\eta,\eta)\times\mathbb{R}) = \emptyset$ for some $\eta>0$. Since we are interested in the limit $\Delta x,\Delta t\to 0$, we can assume that $\Delta x<\eta$. Denote by $\phi^n_{j+1/2} = \phi(x_{j+1/2},t^n)$ for all $j\in\mathbb{Z}$ and all $n\ge 0$, then remark that

$$\phi^n_{j+1/2} = \phi^n_{j-1/2} \qquad \text{if } |j| \le \frac{\eta}{\Delta x} - \frac{1}{2}. \tag{3.24}$$

Multiplying the inequality (3.18) by $\Delta t\phi^{n+1}_{j+1/2}$ provides after classical reorganizations (see, e.g., [48]) that

$$\mathcal{A}_h + \mathcal{B}_h + \mathcal{C}_h \le 0, \tag{3.25}$$

with

$$\begin{aligned}
\mathcal{A}_h &= \sum_{n\ge 0}\Delta t\sum_{j\in\mathbb{Z}} |u^n_{j+1/2} - \kappa_{j+1/2}| \frac{\phi^n_{j+1/2} - \phi^{n+1}_{j+1/2}}{\Delta t}\Delta x,\\
\mathcal{B}_h &= \sum_{j\in\mathbb{Z}} |u^0_{j+1/2} - \kappa^0_{j+1/2}|\phi^0_{j+1/2}\Delta x,\\
\mathcal{C}_h &= \sum_{n\ge 0}\Delta t\sum_{j\in\mathbb{Z}} Q^n_j(\phi^{n+1}_{j-1/2} - \phi^{n+1}_{j+1/2}),
\end{aligned}$$

where Q^n_j was defined by (3.19). It follows from classical arguments that

$$\lim_{\Delta x,\Delta t\to 0}\mathcal{A}_h = -\iint_{\mathbb{R}\times\mathbb{R}_+} |u(x,t) - \kappa(x)|\partial_t\phi(x,t)\mathrm{d}x\mathrm{d}t, \tag{3.26}$$

and that

$$\lim_{\Delta x\to 0}\mathcal{B}_h = -\int_{\mathbb{R}} |u_0(x) - \kappa(x)|\phi(x,0)\mathrm{d}x. \tag{3.27}$$

Let us focus on the term $\mathcal{C}_h$. Thanks to (3.24), one has

$$\mathcal{C}_h = \sum_{n\ge 0}\Delta t\sum_{|j|\ge\eta/\Delta x} Q^n_j(\phi^{n+1}_{j-1/2} - \phi^{n+1}_{j+1/2}),$$

while thanks to the Lipschitz continuity and the consistency of the numerical fluxes F_j, one has

$$\begin{aligned}
&\left|\frac{1}{2}\left(q_l(u^n_{j+1/2},\kappa_l) + q_l(u^n_{j-1/2},\kappa_l)\right) - Q^n_j\right|\\
&\qquad \le M_l\left(|\kappa_{j+1/2} + \kappa_{j-1/2} - 2\kappa_l| + |u^n_{j+1/2} - u^n_{j-1/2}|\right) \qquad \text{if } j<0,
\end{aligned}\tag{3.28}$$

and

$$\begin{aligned}
&\left|\frac{1}{2}\left(q_r(u^n_{j+1/2},\kappa_r) + q_r(u^n_{j-1/2},\kappa_r)\right) - Q^n_j\right|\\
&\qquad \le M_r\left(|\kappa_{j+1/2} + \kappa_{j-1/2} - 2\kappa_r| + |u^n_{j+1/2} - u^n_{j-1/2}|\right) \qquad \text{if } j>0.
\end{aligned}\tag{3.29}$$

Moreover, since $\kappa_{j+1/2}$ tends to $\kappa_{l,r}$ as $|j|\to\infty$, then, for all $\epsilon>0$, there exists $\alpha>0$ such that

$$0<\Delta x<\alpha \quad\Longrightarrow\quad \begin{cases} |\kappa_{j+1/2} + \kappa_{j-1/2} - 2\kappa_l| \le \epsilon & \text{if } j < -\frac{\eta}{\Delta x},\\ |\kappa_{j+1/2} + \kappa_{j-1/2} - 2\kappa_r| \le \epsilon & \text{if } j > \frac{\eta}{\Delta x}.\end{cases}$$

Therefore, with the notation

$$\delta_x \phi_h(x,t) = \frac{\phi_{j+3/2}^{n+1} - \phi_{j-1/2}^{n+1}}{2\Delta x} \quad \text{if } (x,t) \in (x_j, x_{j+1/2}) \times (n\Delta t, (n+1)\Delta t],$$

one obtains that

$$\begin{aligned} &\left| \mathcal{C}_h + \sum_{m\in\{l,r\}} \iint_{\Omega_m \times \mathbb{R}_+} q_m(u_h, \kappa_m)\delta_x \phi_h \mathrm{d}x\mathrm{d}t \right| \\ &\qquad \le \max(M_{l,r}) \left(\epsilon \sum_{n\ge 0} \Delta t \sum_{j\in\mathbb{Z}} |\phi_{j+1/2}^{n+1} - \phi_{j-1/2}^{n+1}| \right. \\ &\qquad\quad \left. + \|\partial_x \phi\|_\infty \int_{\mathbb{R}_+} \int_{|x|\ge \eta - \Delta x/2} |u_h(x+\Delta x, t) - u_h(x,t)| \mathrm{d}x\mathrm{d}t \right). \end{aligned} \tag{3.30}$$

Thanks to (3.22) and Proposition 3.10, passing to the limit $\Delta x, \Delta t \to 0$ in (3.30) provides

$$\left| \lim_{\Delta x, \Delta t \to 0} \mathcal{C}_h + \sum_{m\in\{l,r\}} \iint_{\Omega_m \times \mathbb{R}_+} q_m(u, \kappa_m) \partial_x \phi \mathrm{d}x\mathrm{d}t \right| \le C\epsilon$$

for all $\epsilon > 0$, hence

$$\lim_{\Delta x, \Delta t \to 0} \mathcal{C}_h = - \sum_{m\in\{l,r\}} \iint_{\Omega_m \times \mathbb{R}_+} q_m(u, \kappa_m) \partial_x \phi \mathrm{d}x\mathrm{d}t. \tag{3.31}$$

Taking (3.26)–(3.27) and (3.31) into account in (3.25) provides that (2.3) holds with the chosen couples (κ_l, κ_r). $\square$

By virtue of Proposition 3.6, $\mathcal{G}^o(\beta)$ is a definite germ; therefore the information (2.3) with $(\kappa_l, \kappa_r) \in \mathcal{G}^o(\beta)$ only is sufficient to characterize the $\mathcal{G}(\beta)$-entropy solutions. To sum up, we have

Corollary 3.13. *Let $u_0 \in \mathrm{BV}(\mathbb{R})$, let u be given by (3.22), then (2.3) holds for all $(\kappa_l, \kappa_r) \in \mathcal{G}(\beta)$.*

We have now all the necessary tools to prove the main theorem of the section. We strengthen the CFL condition (3.6) into

$$\frac{2\max(M_{l,r})\Delta t}{\Delta x} \in [\alpha, 1] \quad \text{for some } \alpha > 0, \tag{3.32}$$

prohibiting the fact that $\Delta t/\Delta x \to 0$ when $\Delta t, \Delta x \to 0$. This ensures in particular the finite speed of propagation for the discrete solution, more precisely

$$\int_{|x|\le R} |u_h(x,t) - v_h(x,t)| \mathrm{d}x \le \int_{|x|\le R + \mathcal{M}t + h} |u_h(x,0) - v_h(x,0)| \mathrm{d}x, \qquad \forall t > 0, \tag{3.33}$$

where $\mathcal{M} = \frac{2\max(M_{l,r})}{\alpha}$. Since $M_{l,r} \le L_{l,r}$ necessarily, then $\mathcal{M} \ge \max(L_{l,r})$.

Theorem 3.14. *Let $u_0 \in L^\infty(\mathbb{R})$ be such that (1.5) holds, then, under the CFL condition (3.32), the discrete solution u_h defined by (3.2)–(3.5) converges strongly in $L^1_{\rm loc}(\mathbb{R}\times\mathbb{R}_+)$ towards the unique $\mathcal{G}(\beta)$-entropy solution of (1.1),(1.2),(1.5) in the sense of Definition 2.4.*

Proof. Assume first that $u_0 \in \mathrm{BV}(\mathbb{R})$, then we have proved in Corollary 3.13 that, under the CFL condition (3.6), any limit value u as $\Delta x, \Delta t \to 0$ of the sequentially compact in $L^1_{\rm loc}(\mathbb{R}\times\mathbb{R}_+)$ family $(u_h)_{\Delta x,\Delta t}$ is a $\mathcal{G}(\beta)$-entropy solution, which is unique, ensuring the strong convergence in $L^1_{\rm loc}(\mathbb{R}\times\mathbb{R}_+)$ of the whole sequences.

Let K be a compact subset of $\mathbb{R}\times\mathbb{R}_+$, then there exists R such that

$$K \subset C_R := \bigcup_{t\ge 0} \left\{x \in \mathbb{R} \mid |x| < (R - \mathcal{M}t)^+\right\} \times \{t\}.$$

Now, if $u_0 \in L^\infty(\mathbb{R})$, then for all $\epsilon > 0$, there exists $u_0^\epsilon \in \mathrm{BV}(\mathbb{R})$ such that

$$\int_{-R}^{R} |u_0(x) - u_0^\epsilon(x)| \mathrm{d}x \le \epsilon.$$

Denote by u_h the discrete solution corresponding to the initial datum u_0, u_h^ϵ the one corresponding to u_0^ϵ, by u the unique $\mathcal{G}(\beta)$-entropy solution corresponding to u_0, and by u^ϵ the one corresponding to u_0^ϵ, then

$$\begin{aligned}\iint_K |u_h - u| \mathrm{d}x\mathrm{d}t &\le \iint_{C_R} |u_h - u| \mathrm{d}x\mathrm{d}t \\ &\le \iint_{C_R} |u_h - u_h^\epsilon| \mathrm{d}x\mathrm{d}t + \iint_{C_R} |u_h^\epsilon - u^\epsilon| \mathrm{d}x\mathrm{d}t + \iint_{C_R} |u^\epsilon - u| \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.34}$$

Thanks to (3.33), one has

$$\iint_{C_R} |u_h - u_h^\epsilon| \mathrm{d}x\mathrm{d}t \le \frac{R}{\mathcal{M}} \int_{-R}^{R} |u_h(x,0) - u_h^\epsilon(x,0)| \mathrm{d}x \le \frac{R}{\mathcal{M}}\epsilon,$$

while, thanks to the L^1-contraction at the continuous level proved in [14], one has

$$\iint_{C_R} |u - u^\epsilon| \mathrm{d}x\mathrm{d}t \le \frac{R}{\mathcal{M}} \int_{-R}^{R} |u_0 - u_0^\epsilon| \mathrm{d}x \le \frac{R}{\mathcal{M}}\epsilon.$$

Therefore, (3.34) yields

$$\iint_K |u_h - u| \mathrm{d}x\mathrm{d}t \le 2\frac{R}{\mathcal{M}}\epsilon + \iint_{C_R} |u_h^\epsilon - u^\epsilon| \mathrm{d}x\mathrm{d}t.$$

It follows from the strong convergence in $L^1_{\rm loc}(\mathbb{R}\times\mathbb{R}_+)$ of the discrete solution corresponding to initial data in $\mathrm{BV}(\mathbb{R})$ that

$$\lim_{\Delta x, \Delta t \to 0} \iint_{C_R} |u_h^\epsilon - u^\epsilon| \mathrm{d}x\mathrm{d}t = 0,$$

ensuring by the way that

$$\liminf_{\Delta x, \Delta t \to 0} \iint_K |u_h - u| \mathrm{d}x\mathrm{d}t \le 2\frac{R}{\mathcal{M}}\epsilon, \qquad \forall \epsilon > 0,$$

thus the convergence in $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)$ of u_h towards the unique $\mathcal{G}(\beta)$-entropy solution u associated to u_0. □

4. Examples and applications

In this section, we aim to connect our work to existing results related to scalar conservation laws with discontinuous flux function, embedding them in our framework. But first, we focus on monotonicity properties w.r.t. the graph β in §4.1, leading to the introduction of the extremal graphs $\beta_{\min}$ and $\beta_{\max}$.

4.1. *Order relation for transmission maps, comparison of fluxes*

Let us define a partial order on the set of transmission maps of Definition 1.1.

Definition 4.1. *Let $\beta, \hat\beta \subset U_l \times U_r$ be two maximal monotone graphs, then one says that $\beta \succeq \hat\beta$ if, for all $s_l \in U_l$, for all $s_r, \hat s_r \in U_r$ such that $(s_l, s_r) \in \beta$ and $(s_l, \hat s_r) \in \hat\beta$, then*

$$s_r \le \hat s_r \quad \Longrightarrow \quad (s_l, \sigma_r) \in \beta \cap \hat\beta \text{ for all } \sigma_r \in [s_r, \hat s_r]. \tag{4.1}$$

Notice that the above property is equivalent to having $\beta_o(s) \ge \hat\beta_o(s)$ for a.e. $s \in U_l$, where β_o,$\hat\beta_o$ are arbitrarily fixed, everywhere defined single-valued sections of the graphs β,$\hat\beta$, respectively.

Proposition 4.2. *Let $\beta \succeq \hat\beta$ be two maximal monotone graphs of $U_l \times U_r$, then*

$$F_\beta(u_l, u_r) \ge F_{\hat\beta}(u_l, u_r), \quad \forall (u_l, u_r) \in U_l \times U_r,$$

where F_β and $F_{\hat\beta}$ are obtained from one-sided monotone fluxes $F_{l,r}$ through (3.1) for the graphs β and $\hat\beta$ respectively.

Proof. Let $(u_l, u_r) \in U_l \times U_r$, and let $(s_l, s_r) \in \beta$ such that

$$F_\beta(u_l, u_r) = F_l(u_l, s_l) = F_r(s_r, u_r).$$

Consider an auxiliary value $\widetilde{s}_r$ such that $(s_l, \widetilde{s}_r) \in \hat\beta$ (such a $\widetilde{s}_r$ always exists thanks to the maximality of the graph $\hat\beta$). If $\widetilde{s}_r \ge s_r$, it follows from (4.1) that $(s_l, s_r) \in \hat\beta$, so that $F_{\hat\beta}(u_l, u_r) = F_\beta(u_l, u_r)$ and the claim of the lemma holds true. Assume now that $\widetilde{s}_r < s_r$, then, due to the monotonicity of F_r, one has

$$F_l(u_l, s_l) \ge F_r(\widetilde{s}_r, u_r). \tag{4.2}$$

If (4.2) is in fact an equality, then $F_{\hat\beta}(u_l, u_r) = F_\beta(u_l, u_r)$ again. Assume finally that the inequality (4.2) is strict. Let $(\hat s_l, \hat s_r) \in \hat\beta$ be the couple defining $F_{\hat\beta}(u_l, u_r)$: namely, $F_l(u_l, \hat s_l) = F_r(\hat s_r, u_r)$. Using the monotone parametrization (2.4) of $\hat\beta$, consider the function $\Psi : p \in [\underline{p}, \overline{p}] \mapsto F_l(u_l, \tilde{u}_l(p)) - F_r(\tilde{u}_r(p), u_r)$. For $p := s_l + \widetilde{s}_r$, from $s_{l,r} = \tilde{u}_{l,r}(p)$ we find

$$\Psi(p) = F_l(u_l, s_l) - F_r(\widetilde{s}_r, u_r) > 0,$$

while for $\hat{p} := \hat{s}_l + \hat{s}_r$ we have $\Psi(\hat{p}) = 0$ by the definition of $\hat{s}_{l,r}$. The function Ψ being non-increasing, this implies that $\hat{p} > p$, thus that $\hat{s}_l \geq s_l$, and finally,

$$F_{\hat{\beta}}(u_l, u_r) := F_l(u_l, \hat{s}_l) \leq F_l(u_l, s_l) = F_\beta(u_l, u_r)$$

due to the monotonicity of F_l. This ends the proof. □

Despite the set of transmission maps is only partially ordered by "$\succeq$", it admits the maximal element $\beta_{\max}$ defined by

$$(u_l, u_r) \in \beta_{\max} \quad \text{iff } (u_l - \underline{u}_l)(u_r - \overline{u}_r) = 0, \tag{4.3}$$

and the minimal element $\beta_{\min}$ defined by

$$(u_l, u_r) \in \beta_{\min} \quad \text{iff } (u_l - \overline{u}_l)(u_r - \underline{u}_r) = 0. \tag{4.4}$$

It follows directly from Proposition 4.2 that for all transmission map β, one has

$$F_{\beta_{\min}}(u_l, u_r) \leq F_\beta(u_l, u_r) \leq F_{\beta_{\max}}(u_l, u_r), \qquad \forall (u_l, u_r) \in U_l \times U_r.$$

Analogous property holds true for any complete maximal L^1D germ $\mathcal{G}$ in the sense of Definition 2.1. To be precise, we restrict our attention to the exact Godunov solver $G_{\mathcal{G}}$ comparing it to the Godunov solvers $G_{\beta_{\min}}$ and $G_{\beta_{\max}}$ (for general germs, the Godunov flux $G_{\mathcal{G}}$ is the only generic consistent interface flux; it is defined via the exact Riemann solver by $G_{\mathcal{G}}(u_l, u_r) := f_l(\gamma_l u) = f_r(\gamma_r u)$, where $\gamma_{l,r} u$ are the interface traces of the self-similar solution $u = u(x/t)$ of the Riemann problem with data $u_0(x) = u_l \mathbf{1}_{x<0} + u_r \mathbf{1}_{x>0}$ as described in Definition 2.1 a)-c)).

Proposition 4.3. *Let $\mathcal{G}$ be a complete maximal L^1-dissipative germ in the sense of Definition 2.1, then denote by $G_{\mathcal{G}}$ the corresponding Godunov solver, and by $G_{\beta_{\min}}$ and $G_{\beta_{\max}}$ the Godunov solvers corresponding to the maximal L^1-dissipative germs $\mathcal{G}(\beta_{\min})$ and $\mathcal{G}(\beta_{\max})$ respectively, then*

$$G_{\beta_{\min}}(u_l, u_r) \leq G_{\mathcal{G}}(u_l, u_r) \leq G_{\beta_{\max}}(u_l, u_r), \qquad \forall (u_l, u_r) \in U_l \times U_r.$$

Proof. Let us prove that $G_{\mathcal{G}}(u_l, u_r) \leq G_{\beta_{\max}}(u_l, u_r)$, the proof of the other inequality being similar. First, given $(u_l, u_r) \in U_l \times U_r$, by the definition of $G_{\mathcal{G}}$ we find that there exist $(s_l, s_r) \in \mathcal{G}$ (given by $s_{l,r} = \gamma_{l,r} u$ in Definition 2.1 a)-c) of the Riemann solver) such that

$$G_l(u_l, s_l) = f_l(s_l) = G_{\mathcal{G}}(u_l, u_r) = f_r(s_r) = G_r(s_r, u_r).$$

On the other hand, it is easy to check that the maximal Godunov solver $G_{\beta_{\max}}$ is defined by

$$G_{\beta_{\max}}(u_l, u_r) = \begin{cases} G_l(u_l, \underline{u}_l) & \text{if } G_l(u_l, \underline{u}_l) - G_r(\overline{u}_r, u_r) \leq 0, \\ G_r(\overline{u}_r, u_r) & \text{otherwise.} \end{cases} \tag{4.5}$$

Thanks to the monotonicity of $G_{l,r}$, one has

$$G_l(u_l, \underline{u}_l) \geq G_l(u_l, s_l) \quad \text{and} \quad G_r(\overline{u}_r, u_r) \geq G_r(s_r, u_r),$$

ensuring that $G_{\beta_{\max}}(u_l, u_r) \geq G_{\mathcal{G}}(u_l, u_r)$ for every $(u_l, u_r) \in U_l \times U_r$. □

Based on the above considerations, we extend the notion of *optimal entropy solution* first introduced by Adimurthi *et al.* in [3] in the case of bell-shaped flux functions. Since, on the contrary to this particular case, the flux functions we consider have a general shape, the notion of optimal entropy solution can be extended in two ways, leading to the so-called *minimal* and *maximal entropy solutions*.

Definition 4.4. *A function $u \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ such that $\underline{u} \leq u \leq \overline{u}$ a.e. in $\mathbb{R}\times\mathbb{R}_+$ is said to be the* maximal *(resp.* minimal*) entropy solution if u is the unique $\mathcal{G}(\beta_{\max})$-entropy solution (resp. $\mathcal{G}(\beta_{\min})$-entropy solution).*

Notice that even in the bell-shaped case, both notion are of great interest. For example, both of them appear as asymptotic limits for two-phase flows in porous media when the capillary pressure depends only on space [31,32,33]. The maximal entropy solution also appears naturally in the modeling of traffic flows (see §4.4).

4.2. *The vanishing viscosity solution*

A classical example consists in considering the case where a weak solution u of (1.1) is judged admissible if and only if u is the vanishing viscosity limit $\epsilon \to 0$, i.e., $u = \lim_{\epsilon\to 0^+} u^\epsilon$ with

$$\partial_t u^\epsilon + \partial_x f(u^\epsilon; x) = \epsilon \partial_{xx} u^\epsilon.$$

We assume that the flux f is of the form prescribed by (1.2), requiring by the way that $U_l = U_r$. This limit appears in the modeling of clarifier-thickener units for water treatmentand it has been extensively studied in the literature (see in particular [56,67,45,5,13,62,15]). The corresponding germ $\mathcal{G}_{VV}$ and its definite part $\mathcal{G}^o_{VV}$ consisting in couples of states that can be connected by vanishing viscosity profiles are described in [14,13].

Since for all $\epsilon > 0$, u^ϵ is continuous w.r.t. space variable across $\{x = 0\}$, one could expect its limit u to be "as continuous as possible" across the interface, i.e., the jump across the interface has to be minimized (we refer to [15] to a deeper discussion on this issue). It is proved in [45] that the one-sided traces $u_{l,r}$ of the solution (when they exist) fulfill the so-called Γ-condition, namely

$$\text{there exists } s \text{ such that } G_l(u_l, s) = G_r(s, u_r). \tag{4.6}$$

In view of the analysis carried out above, this turns out to correspond to the transmission map $\mathrm{Id} = \{(s, s) \mid s \in U_{l,r}\}$, i.e., the "vanishing viscosity" germ $\mathcal{G}_{VV}$ of [14,13] coincides with $\mathcal{G}(\beta)$ for the choice $\beta = \mathrm{Id}$. Notice that also G^o_{VV} coincides with the set of reachable states $\mathcal{G}^o(\mathrm{Id})$. In this relation, let us observe that reachable states can be connected by numerical profiles described in § 3.2, and also by vanishing viscosity profiles, see [13].

4.3. *The vanishing capillarity solution for porous media flows*

The immiscible incompressible flow of oil and water in porous media are often modeled by the so-called Darcy-Muskat equations

$$\phi(x)\partial_t s_\alpha - \partial_x \left(\eta_\alpha(s_\alpha;x)K(x)(\partial_x p_\alpha - \rho_\alpha g)\right) = 0, \qquad \alpha \in \{o,w\}, \tag{4.7}$$

where the subscript o (resp. w) stand for oil (resp. water), $\phi \in (0,1)$ is the porosity of the medium, $s_\alpha \in [0,1]$ is the saturation of the phase α, η_α its mobility (which is nondecreasing w.r.t. s_α), p_α its pressure and ρ_α its density. $K(x)$ denotes the intrinsic permeability of the porous medium, and g the gravity.

A first natural assumption consists in requiring that the pore volume is filled by the two fluids, i.e.

$$s_o + s_w = 1 \qquad \Longrightarrow\ s := s_o, \quad s_w = 1 - s;$$

Moreover, there might exist irreducible saturations $0 \leq \underline{s}(x) \leq \overline{s}(x) \leq 1$ such that $\eta_o(s) = 0$ if $s \leq \underline{s}$ and $\eta_w(s) = 0$ if $s \leq 1 - \overline{s}$.

Following [29,35], the relevant way to prescribe a (static) capillary pressure relation in the one-dimensional case (see [24,36] in the multidimensional context) consists in requiring

$$p_o - p_w \in \pi(s;x),$$

where $\pi(\cdot;x) \subset [\underline{s}(x), \overline{s}(x)] \times \overline{\mathbb{R}}$ is some maximal monotone graph. This implies the existence of a maximal monotone graph $\pi^{-1} \subset \overline{\mathbb{R}} \times [\underline{s}, \overline{s}]$ such that

$$s \in \pi^{-1}(p;x), \tag{4.8}$$

where $p = p_o - p_w$ denotes the capillary pressure.

Standard reformulation of the problem (see e.g. [8,9]) allow to reduce formally the system to the following degenerate parabolic equation

$$\phi(x)\partial_t s + \partial_x f(s;x) = \partial_x \left(\lambda(s;x)\partial_x \pi(s;x)\right)$$

where $\lambda(\underline{s}(x);x) = \lambda(\overline{s}(x);x) = 0$.

Therefore, the quantity to be "as continuous as possible" is now the capillary pressure, and not the saturation. Assume that f is of the form (1.2), and that

$$\phi(x) = \phi_l \mathbf{1}_{x<0} + \phi_r \mathbf{1}_{x>0}, \quad \lambda(s;x) = \lambda_l(s)\mathbf{1}_{x<0} + \lambda_r(s)\mathbf{1}_{x>0},$$

and

$$\pi(s;x) = \pi_l(s)\mathbf{1}_{x<0} + \pi_r(s)\mathbf{1}_{x>0},$$

then the scaling $x := x/\epsilon$ and $t := t/\epsilon$ yields

$$\phi(x)\partial_t s^\epsilon + \partial_x f(s^\epsilon;x) = \epsilon\partial_x \left(\lambda(s^\epsilon;x)\partial_x \pi(s^\epsilon;x)\right). \tag{4.9}$$

If $s = \lim_{\epsilon\to 0^+} s^\epsilon$ with s^ϵ satisfying (4.9), we say that s is a vanishing capillarity solution to the limit equation $\phi(x)\partial_t s + \partial_x f(s;x) = 0$.

Remark 4.5. The limit problem can be rewritten under the form (1.1),(1.2) provided the unknown s is changed into $u := \phi s$ (then also $f(\cdot;x)$ is changed into $\widetilde{f}(\cdot;x) := f(\frac{\cdot}{\phi(x)};x)$, cf. [8, Remark 2], and the intervals $U_{l,r}$ of values of the new unknown u are taken to be $[\phi_{l,r}\underline{s}_{l,r}, \phi_{l,r}\overline{s}_{l,r}])$ and the whole theory developed here and in preceding contributions can be applied. Considering transmission maps in new variables, as maximal monotone graphs in $U_l \times U_r$, amounts to consider maximal monotone graphs in $[\underline{s}_l, \overline{s}_l] \times [\underline{s}_r, \overline{s}_r]$ for the original unknown s.

As proved in [8] (see also [6,9]), the vanishing capillarity solution s, corresponding to the limit as $\epsilon \to 0$ of s^ϵ, can be seen as the $\mathcal{G}(\beta)$-entropy solution corresponding to the graph

$$\beta = \left\{(\pi_l^{-1}(p), \pi_r^{-1}(p)) \mid p \in \overline{\mathbb{R}}\right\} \subset [\underline{s}_l, \overline{s}_l] \times [\underline{s}_r, \overline{s}_r].$$

As an important by-product, the vanishing capillarity solution is highly sensitive to the choice of capillary pressure graphs $\pi_{l,r}$. Therefore, any convergent numerical method for approximation of vanishing capillarity solutions must let these capillary pressure graphs appear, as it is the case in the method we propose here and in [9] in the particular case of the phase-by-phase upstream scheme. We refer to [6] for construction of the scheme, based on these ideas, suitable for approximation of the multi-dimensional hyperbolic-elliptic Buckley-Leverett problem.

4.4. *Transmission maps for traffic flows*

In the context of traffic flow modeling, the density of cars ρ on highways can be modeled by a continuous transport equation

$$\partial_t \rho + \partial_x(\rho v) = 0.$$

Following Lighthill & Whitham [60] and Richards [64], the speed v depends in a non-increasing way on the density and vanishes as the road is saturated (traffic jam), i.e.

$$v = v(\rho), \qquad \partial_\rho v \le 0, \qquad v(\overline{\rho}) = 0 \text{ for some } \overline{\rho} > 0.$$

We assume the resulting flux function $\rho \mapsto f(\rho) = \rho v(\rho)$ to be a Lipschitz continuous bell-shaped function, i.e.

$$\text{there exists } \sigma \in (0,\overline{\rho}) \text{ such that } f'(\rho)(\rho - \sigma) < 0 \text{ for a.e. } \rho \text{ in } (0,\overline{\rho})$$

the problem reduces to the scalar conservation law

$$\partial_t \rho + \partial_x f(\rho) = 0.$$

We then define the monotone functions ϕ and ψ by

$$\phi(s) = \int_0^s (f')^+(a)\mathrm{d}a, \qquad \psi(s) = \int_s^{\overline{\rho}} (f')^-(a)\mathrm{d}a, \qquad \forall s \in [0,\overline{\rho}],$$

so that

$$f(\rho) = \min\left(\phi(\rho), \psi(\rho)\right), \qquad \forall \rho \in [0,\overline{\rho}]. \tag{4.10}$$

Across a discontinuity (ρ_l, ρ_r), the flux is given by

$$G(\rho_l, \rho_r) = \min(\phi(\rho_l), \psi(\rho_r)), \tag{4.11}$$

in order to optimize the flux across the discontinuity. Since f is bell-shaped, it appears that $G(\rho_l, \rho_r)$ coincides with the usual Godunov solver corresponding to the usual Kruzhkov entropy solution.

In the case of discontinuous road parameters, like for instance when the number of lines is changing, the maximal density $\overline{\rho}$ can depend on x, as well as the speed distribution $\rho \mapsto v(\rho; x)$. Assume that

$$\overline{\rho}(x) = \overline{\rho}_l \mathbf{1}_{x>0}(x) + \overline{\rho}_r \mathbf{1}_{x>0}(x), \qquad v(\rho; x) = v_l(\rho)\mathbf{1}_{x<0}(x) + v_r(\rho)\mathbf{1}_{x>0}(x),$$

so that the flux f is of the form (1.2), and

$$\phi(\rho; x) = \phi_l \mathbf{1}_{x>0}(x) + \phi_r \mathbf{1}_{x>0}(x), \qquad \psi(\rho; x) = \psi_l(\rho)\mathbf{1}_{x<0}(x) + \psi_r(\rho)\mathbf{1}_{x>0}(x).$$

It is then expected from (4.11) that the Godunov solver at the interface is given by

$$G_{\text{int}}(\rho_l, \rho_r) = \min(\phi_l(\rho_l), \psi_r(\rho_r)), \qquad \forall (\rho_l, \rho_r) \in [0, \overline{\rho}_l] \times [0, \overline{\rho}_r]. \tag{4.12}$$

The bell-shaped property of the fluxes $f_{l,r}$ implies that

$$G_{\text{int}}(\rho_l, \rho_r) = G_{\beta_{\max}}(\rho_l, \rho_r), \qquad \forall (\rho_l, \rho_r) \in [0, \overline{\rho}_l] \times [0, \overline{\rho}_r],$$

where, following (4.3), the maximal graph $\beta_{\max} \subset [0, \overline{\rho}_l] \times [0, \overline{\rho}_r]$ is defined by

$$(\rho_l, \rho_r) \in \beta_{\max} \;\text{ if } \rho_l(\rho_r - \overline{\rho}_r) = 0.$$

In relation with the result of Proposition 4.3, one can interpret this result by saying that the choice of the Godunov flux (4.12) corresponds to modeling a rational behavior of drivers: namely, the model prescribes a maximal possible flux at the sites where the number of lines on the road changes.

4.5. *Locally constrained conservation law with general flux*

In this section, we consider the particular case $f = f_l = f_r$, where $f : [0, R] \to \mathbb{R}$ satisfies the following properties:

$$f(\rho) \geq 0 \text{ for all } \rho \in [0, R], \qquad f(0) = f(1) = 0.$$

We consider weak solutions of

$$\partial_t \rho + \partial_x f(\rho) = 0, \qquad \rho_{|t=0} = \rho_0 \in L^\infty(\mathbb{R}; [0, R]) \tag{4.13}$$

such that u satisfies entropy inequalities away from the interface, i.e.

$$\forall \kappa \in [0, R], \quad \partial_t |\rho - \kappa| - \partial_x q(\rho, \kappa) \leq 0 \text{ in } \mathcal{D}'(\mathbb{R}^* \times \mathbb{R}_+),$$

where $q(\rho, \kappa) = \operatorname{sign}(\rho - \kappa)(f(\rho) - f(\kappa))$. Following [41], we modify the admissibility conditions at the point $\{x = 0\}$, by requiring that the flux $f(\rho)_{|x=0}$ is constrained to remain lower or equal to a prescribed value $\overline{F} \in \big[0, \max_{s \in [0,R]} f(s)\big]$, i.e.

$$f(\rho)_{|x=0}(t) \leq \overline{F} \quad \text{for a.e. } t \in \mathbb{R}_+. \tag{4.14}$$

In general, the constraint can be time-dependent, to model, e.g., road lights [41] or non-local "panic" effects in crowd dynamics [10], but here we restrict our attention to the case of a constant $\overline{F}$ suitable, e.g., for modeling of toll gates.

If the constraint is not saturated on a time interval (t_1, t_2), we expect the solution ρ to fulfill the Kruzhkov entropy inequalities on the whole domain:

$$\forall \kappa \in [0, R], \quad \partial_t |\rho - \kappa| - \partial_x q(\rho, \kappa) \leq 0 \text{ in } \mathcal{D}'(\mathbb{R} \times (t_1, t_2)).$$

Otherwise, non-classical shocks at $\{x = 0\}$ appear, as described by the Riemann solver constructed in [41] under the assumption of bell-shaped flux function f. Both situations (the unsaturated and the saturated constraint) can be accounted for via the suitable notion of entropy solution proposed in [41], leading to well-posedness for the Cauchy problem. The link with the theory of L^1-dissipative germs was made in [11], where the convergence of monotone Finite Volume scheme was established. Error estimates for the Godunov approximation of the solution were proposed in [37].

Let us interpret the results of [41,11,37] for the bell-shaped case (and also those of [38] for a non-bell-shaped case[c], see below) in terms of a well-chosen transmission map. We have not identified a choice of β that would stem from modeling assumptions, but we can suggest an optimal choice for a graph that constrains the flux in the weakest possible sense. Indeed, recalling the result of Proposition 4.2, a larger graph (in the sense of the partial order relation $\succeq$) allows for a larger interface flux. Further, the unconstrained situation corresponds to the identity graph Id (indeed, the classical solution is nothing but the vanishing viscosity solution corresponding to $\beta = \mathrm{Id}$, see § 4.2). Finally, from the description of the Riemann solver in [41] and analysis of [11], the unique[d] connection $(\hat{\rho}_F, \check{\rho}_F)$ at the flux level F belongs to the associated germ. Because a simple way to require that the connection $(\hat{\rho}_F, \check{\rho}_F)$ belongs to the underlying germ is to ask that $(\hat{\rho}_F, \check{\rho}_F) \in \beta$ (see Lemma 2.13), we can look for a graph β that lies "as close as possible" below the identity graph Id and that contains $(\hat{\rho}_F, \check{\rho}_F) \in \mathfrak{C}$, the connection at flux level F. Therefore, the transmission map β corresponding to entropy solutions of (4.13),(4.14) in the sense of [41,11] can be chosen as follows:

$$\beta := \Big\{ (c, c) \,|\, c \in [0, 1] \setminus [\check{\rho}, \hat{\rho}] \Big\} \cup \Big\{ (c, \check{\rho}) \,|\, c \in [\check{\rho}, \hat{\rho}] \Big\} \cup \Big\{ (\hat{\rho}, c) \,|\, c \in [\check{\rho}, \hat{\rho}] \Big\}.$$

All the aforementioned works on constrained traffic flows focused on the case of bell-shaped f. More recently, Chalons *et al.* [38] considered the case of non-bell-shaped (but yet not general) fluxes for modeling pedestrian flows (cf. [52,53] for

[c] To be specific, in [38] two different notions of solution were considered: the one based on the classical Kruzhkov solutions away from the interface, and the one based on the non-classical Riemann solver of Colombo and Rosini [42]. The present paper deals exclusively with the first (classical) interpretation.

[d] Due to the bell-shape assumption on f, given $F \in [0, \max f)$, the only element of $\mathfrak{C}$ with flux equal to F is the couple $(\hat{\rho}_F, \check{\rho}_F)$ defined by $\check{\rho}_F < \hat{\rho}_F$, $f(\check{\rho}_F) = F = f(\hat{\rho}_F)$. This couple always lies below the diagonal of $U_l \times U_r = [0, 1]^2$.

empirical evidence of relevancy of non-bell-shaped flux models). Contrarily to the bell-shaped case, here the set of connections at level F:

$$\mathfrak{C}_F = \{(\hat{\rho}, \check{\rho}) \in \mathfrak{C} \mid f(\hat{\rho}) = F = f(\check{\rho})\}$$

is not reduced to a singleton. Among these connections at the flux level F, only appear as non-classical shocks in the construction of the Riemann solver those that lie below the diagonal of $[0,1]^2$, i.e., those with $\hat{\rho} \geq \check{\rho}$ (see [38, Rem. 1]). Therefore the recipe proposed hereabove for the bell-shaped case still applies in the case of fluxes considered in [38] and it yields the same admissible solutions: namely, one can take for transmission map β, the projection of the identity graph Id on the set of maximal monotone graphs containing all the elements $(\hat{\rho}, \check{\rho})$ of $\mathfrak{C}_F$ such that $\hat{\rho} \geq \check{\rho}$ (see Figure 1 for an example). To sum up, the L^1-dissipative germ exhibited in [38] coincides with the germ $\mathcal{G}(\beta)$. Therefore, the admissible solution considered in [38] coincides with the $\mathcal{G}(\beta)$-entropy solution prescribed by the transmission map β obtained as the projection of the identity on the set of the graphs containing the connections located below the identity.

This fact permits to use the transmission-map-based finite volume scheme we propose in § 3 for approximating admissible solutions in the sense of [38]. Let us provide an illustration for a particular case inspired from [42,38]. Define the function

$$f(\rho) = \max\left(\frac{\rho(7-\rho)}{6}, \frac{3(\rho-6)(2\rho-21)}{20(\rho-12)}\right), \qquad \forall \rho \in [0, 10.5].$$

At the interface $\{x = 0\}$, we impose that

$$f(\rho)_{|x=0}(t) \leq 0.3, \qquad \text{for a.e. } t \geq 0,$$

leading to the configuration illustrated by Figure 1. This allows to determine the 4 elements of $\mathfrak{C}_F$, three of them being below the identity graph, from what we deduce the expression of the graph β to be used as transmission map. Let us expose some numerical results provided by the scheme analyzed in §3. The computations are performed thanks to the Godunov and Rusanov schemes with hybridization of the interface at $\{x = 0\}$. The first initial data we considered is

$$\rho_0(x) = \begin{cases} 5.3 & \text{if } x \in (-0.8, -0.5), \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

First, on Figure 2, we plot in the (x,t)-plane the solution obtained via the Godunov scheme with 300 space cells. One can in particular see an undercompressive wave leaving the interface.

We have also plotted on Figure 3 the solutions at $t = 2$ computed with or without constraint and with the Godunov and Rusanov schemes. For this test, we only used 100 cells so that the Rusanov scheme is not yet converged, and one can see a difference between the solution obtained by the Rusanov scheme and the one obtained by the Godunov scheme.

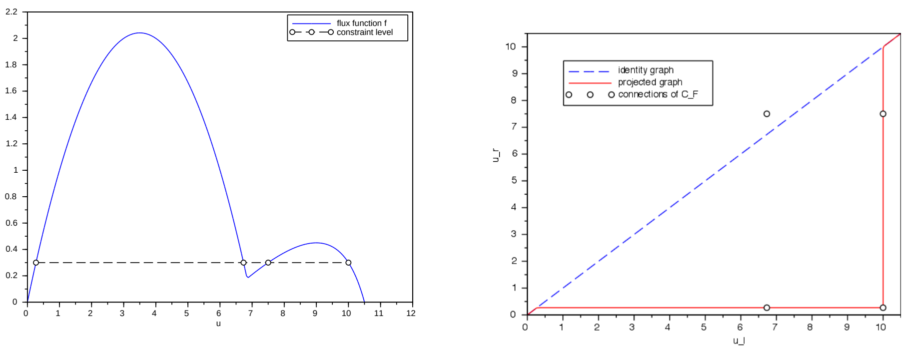

Fig. 1. Example of §4.5. The flux function f, represented in the left figure, is not a bell-shaped function. Therefore, the set $\mathfrak{C}_F$ contains 4 singletons represented in the right figure by black circles. The graph β (red solid line) is defined as the projection of the identity (blue dashed line) on the set of the maximal monotone graphs containing the strict connections of $\mathfrak{C}_F$ located below the graph Id.

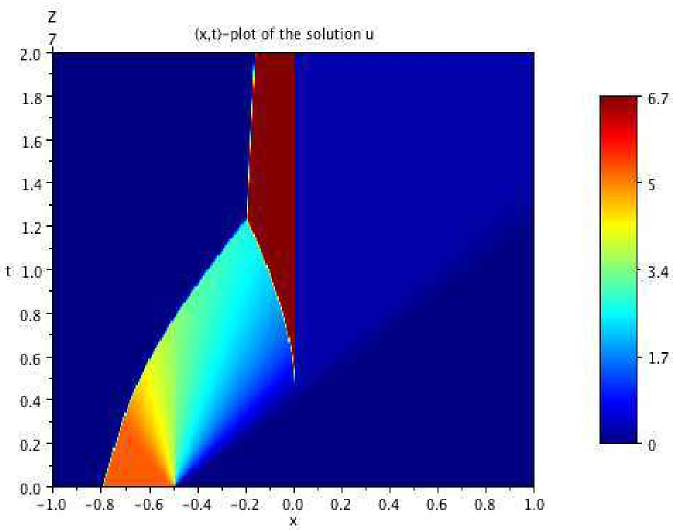

Fig. 2. Example of §4.5. Plot of the solution ρ corresponding to the initial data (4.15) in the (x,t)-plane. One can see two waves leaving the interface, leading to a so-called undercompressive discontinuity, that is usually forbidden in the Kruzhkov theory.

The second initial data we consider is

$$\rho_0(x) = 6.8 \quad \text{for all } x \in \mathbb{R}. \tag{4.16}$$

The couple $(6.8, 6.8)$ belongs to $\mathcal{G}(\beta)$, hence the exact solution is constant, and it

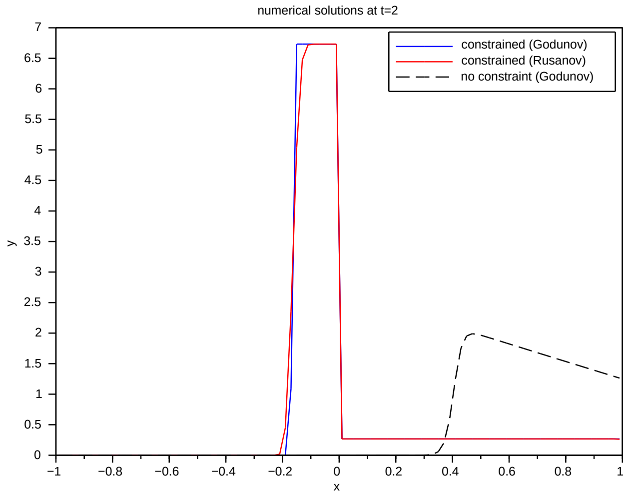

Fig. 3. Example of §4.5. Despite the Rusanov scheme is more diffusive than the Godunov scheme, its numerical diffusion is well-balanced to preserve the connections of $\mathcal{G}(\beta)$ (cf. Lemma 3.3), so that the undercompressive waves are accurately solved.

is exactly computed thanks to the Godunov scheme. But since $(6.8, 6.8) \in \mathcal{G}^o(\beta) \setminus \mathcal{G}^\flat(\beta)$, the solution is not exactly computed by the Rusanov scheme. Indeed, a numerical viscous layer appears nearby the interface Σ (cf. Figure 4).

4.6. *Example of non-conservative coupling using transmission map and additional dissipation information*

To conclude, we have chosen to present briefly an example where a transmission map is combined with a non-conservative coupling. The example refers to the problem

$$\partial_t u + \partial_x \frac{u^2}{2} = -\lambda u \delta_0(x) \tag{4.17}$$

considered in [18]; the motivation comes from particle-in-Burgers model proposed in [58], see also [12]. In this case, $f_l(u) = f_r(u) = \frac{u^2}{2}$, $U_{l,r} = \mathbb{R}$ (condition (1.4) being dropped); $\lambda > 0$ is a parameter. The action of the singular right-hand side in (4.17) is dissipative, that's why the problem can be treated within a slight generalization of the "germs theory" recalled in Section 2.1. The transmission map approach of the preceding sections can be adapted so that to include problem (4.17), as a representative of large a class of problems with non-conservative interface coupling.

Let us recall the established well-posedness theory for (4.17). First, the right-hand side of (4.17) has to be given a precise sense, because it has the form of a non-conservative product (cf. [46]). The interpretation used in [18] goes back to the

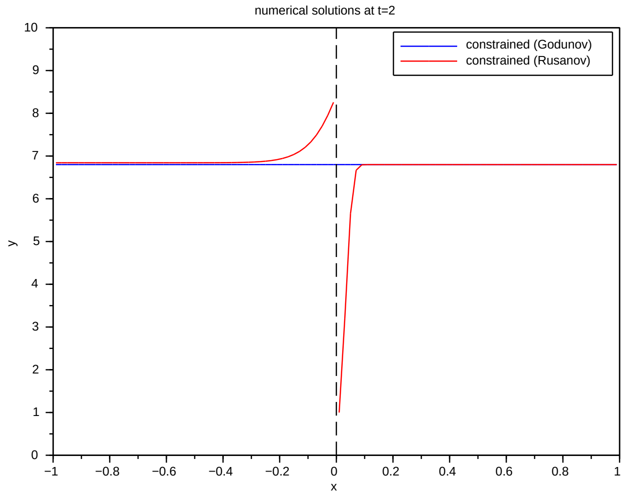

Fig. 4. Example of §4.5. Numerical solutions for 100 celles corresponding to the initial data (4.16) provided by the Godunov (blue) and Rusanov (red) schemes. In the case of the Rusanov scheme, numerical profiles are observed, as suggested by the study carried out in §3.2.

analysis of the particle-in-Burgers model of Lagoutière, Seguin and Takahashi [58] where the Dirac measure δ_0 is seen as the (weak-$*$, in the sense of measures) limit of some monotone profiles $\delta_\epsilon \in C_c^\infty(\mathbb{R})$. This singular limit approach leads to the identification of the subset

$$\mathcal{G}_\lambda = \Big\{(u_l, u_r) \,\big|\, u_l - u_r = \lambda \ \text{ or } \ \Big(u_r \le 0 \le u_l \text{ and } |u_l + u_r| \le \lambda\Big)\Big\} \tag{4.18}$$

which plays the role of an L^1-dissipative germ (in this case, Definition 2.1 should be relaxed because the Rankine-Hugoniot condition $f_l(u_l) = f_r(u_r)$ is not relevant for the non-conservative equation (4.17)). In [18], it is shown that there exists a unique $\mathcal{G}_\lambda$-entropy solution to the Cauchy problem for (4.17) with given L^∞ datum; the solution is defined in the same way as in Definition 2.4 and in Proposition 2.6.

We claim that this unique $\mathcal{G}_\lambda$-entropy solution is the unique function that fulfills (1.7) and verifies the following non-conservative analogue of condition (1.11):

the one-sided traces $u_{l,r}(t)$ of u in the sense (1.8) fulfill for a.e. $t > 0$

$$\exists p \in \mathbb{R} \ \text{ such that}^{\text{e}} \qquad \begin{cases} f_l(u_l) = G_l(u_l, \frac{p+\lambda}{2}), \\ G_r(\frac{p-\lambda}{2}, u_r) - G_l(u_l, \frac{p+\lambda}{2}) + \frac{\lambda}{2} p = 0, \\ G_r(\frac{p-1}{2}, u_r) = f_r(u_r). \end{cases} \tag{4.19}$$

Condition (4.19) corresponds to the transmission map $\beta_\lambda := \mathrm{Id} - \lambda$ parametrized,

[e] Here, we keep the notation $G_{l,r}$ for Godunov fluxes for the sake of subsequent generalizations; but in the case of problem (4.17), we have $f_l = f_r$ so that $G_l = G_r$.

as in § 2.2, by $p = \tilde{u}_l + \tilde{u}_r$. But in addition, the monotone non-decreasing function[f] $\psi_\lambda := \frac{\lambda}{2}\,\mathrm{Id}$ enters the flux balance which can be written under the form

$$\begin{array}{l}\Psi(p) = 0, \text{ where } \Psi : p \mapsto G_r(\tilde{u}_r(p), u_r) - G_l(u_l, \tilde{u}_l(p)) + \psi_\lambda(p), \\ \text{with } \beta_\lambda \text{ parametrized by } p \in \mathbb{R} \mapsto (\tilde{u}_l(p), \tilde{u}_r(p)) \in \beta_\lambda,\ p = \tilde{u}_l(p) + \tilde{u}_r(p).\end{array} \tag{4.20}$$

The transmission condition (4.19) characterizes $\mathcal{G}_\lambda$-entropy solutions due to the following lemma, which can be proved thanks to a simple case-by-case study using the expression of the Godunov flux $G_l = G_r$ corresponding to $f_{l,r} : u \mapsto \frac{u^2}{2}$.

Lemma 4.6. *Denote by $\mathcal{G}(\beta_\lambda, \psi_\lambda)$ the subset of $\mathbb{R}^2$ consisting of all couples $(u_l, u_r) \in \mathbb{R}^2$ that fulfill (4.19). Then $\mathcal{G}(\beta_\lambda, \psi_\lambda)$ coincides with $\mathcal{G}_\lambda$ given by (4.18).*

From now on, we will drop the subscript λ in the notation for monotone graphs $\beta_\lambda, \psi_\lambda$ and for the corresponding set $\mathcal{G}(\beta_\lambda, \psi_\lambda) \subset \mathbb{R}^2$ defined as in Lemma 4.6; this set is the non-conservative analogue of L^1D germs. Indeed, the precise choice $\beta_\lambda = \mathrm{Id} - \lambda$ and $\psi_\lambda = \frac{\lambda}{2}\,\mathrm{Id}$ is only needed in order to specify our analysis to the example (4.17) of non-conservative interface coupling.

To conclude, we briefly investigate finite volume approximation of $\mathcal{G}(\beta, \psi)$-entropy solutions. First, the monotonicity of $\Psi : \mathbb{R} \mapsto \mathbb{R}$ defined in (4.20) and the fact that $\Psi(\pm\infty) = \pm\infty$ ensure that for every couple $(u_l, u_r) \in \mathbb{R}^2$ there exist unique values $F^-_{\beta,\psi}(u_l, u_r)$ and $F^+_{\beta,\psi}(u_l, u_r)$ equal to $G_l(u_l, \tilde{u}_l(p))$ and to $G_r(\tilde{u}_r(p), u_r)$, respectively, for which $\Psi(p) = 0$. This defines, respectively, the left and the right numerical fluxes $F^-_{\beta,\psi}(\cdot,\cdot)$ and $F^+_{\beta,\psi}(\cdot,\cdot)$ at the interface. The resulting finite volume scheme takes the same form as (3.3) in Section 3, with the exception of the equations for $j = -1$ and $j = 0$ that become

$$\begin{array}{l}\frac{u^{n+1}_{-1/2} - u^n_{-1/2}}{\Delta t}\Delta x + F^-_{\beta,\psi}(u^n_{-1/2}, u^n_{1/2}) - G_l(u^n_{-3/2}, u^n_{-1/2}) = 0, \\ \frac{u^{n+1}_{1/2} - u^n_{1/2}}{\Delta t}\Delta x + G_r(u^n_{1/2}, u^n_{3/2}) - F^+_{\beta,\psi}(u^n_{-1/2}, u^n_{1/2}) = 0.\end{array} \tag{4.21}$$

As for the conservative case, it is not difficult to prove the convergence of this scheme to the unique $\mathcal{G}(\beta, \psi)$-entropy solution; moreover, as in Proposition 2.9, one sees that the resulting scheme is the Godunov scheme.

One can also replace the Godunov fluxes $G_{l,r}$ in (4.20) (and in the corresponding definition of one-sided interface numerical fluxes $F^\pm_{\beta,\psi}$) with any other monotone, Lipschitz, consistent with $f_{l,r}$ numerical fluxes $F_{l,r}$; this means that we define

$$\begin{array}{l}F^-_{\beta,\psi}(a, b) := F_l(a, \tilde{u}_l(p)) \text{ and } F^+_{\beta,\psi}(a, b) := F_r(\tilde{u}_r(p), b) \text{ where } p \text{ solves} \\ \begin{cases} F_r(\tilde{u}_r(p), b) - F_l(a, \tilde{u}_l(p)) + \psi(p) = 0, \\ (\tilde{u}_l(p), \tilde{u}_r(p)) \in \beta, \quad p = \tilde{u}_l(p) + \tilde{u}_r(p).\end{cases}\end{array} \tag{4.22}$$

In the case of problem (4.17), for $F_{l,r}$ one chooses a single numerical flux F consistent with $f_l = f_r : u \mapsto \frac{u^2}{2}$. The resulting finite volume scheme using equations

[f] More generally, one could ask that ψ be a maximal monotone graph relating the canonical parameter p of graph β to the amount of flux dissipation corresponding to interface couple of states $(\tilde{u}_l(p), \tilde{u}_r(p))$

(4.21) near the interface is slightly different from the scheme for $\mathcal{G}_\lambda$-entropy solutions designed in [18]. Our scheme has the disadvantage of including one non-linear equation to be solved at every time step, while the scheme of [18] is fully explicit. At the same time we expect more robust stability properties for our scheme; in particular, the condition

$$\partial_a(\partial_a F(a,b) + \partial_b F(a,b)) \geq 0 \text{ and } \partial_b(\partial_a F(a,b) + \partial_b F(a,b)) \geq 0 \tag{4.23}$$

imposed in the analysis of the scheme of [18] is not needed for our new scheme. Indeed, we have the following easy observation directly related to the discrete L^1 contraction property of [18, Prop. 8].

Lemma 4.7. *Consider the numerical fluxes defined by* (4.22)*. Then for all real numbers* $a \leq A$ *and* $b \leq B$*, there holds* $G^+ - G^- \leq 0$*, where*

$$G^\pm := F^\pm_{\beta,\psi}(A,B) - F^\pm_{\beta,\psi}(a,b). \tag{4.24}$$

Proof. The result follows readily from the definition (4.24) and the fact that $F_{l,r}(\cdot,b)$ and $-F_{l,r}(a,\cdot)$ are non-decreasing maps while β and ψ are monotone graphs. $\square$

Lemma 4.7 can be used in the place of the argument of [18, proof of Prop. 8, p. 1956] where (4.23) was exploited. Henceforth, we easily adapt the convergence proof to the numerical scheme for problem (4.17) using discretization (4.21) near the interface; the one-sided interface numerical fluxes (4.22) are defined thanks to the transmission map approach advocated in the present paper.

From the above example, it is clear that discontinuous-flux problems with conservative coupling by a transmission map β can be seen as a particular case of non-conservative ones, where β prescribes the "desired trace values" and an additional non-decreasing function (or, more generally, maximal monotone graph) ψ prescribes the "desired interface dissipation"; the choice $\psi \equiv 0$ corresponds to the conservative case.

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