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To cite this version:

Charles-Michel Marle. Relativity, the Special Theory, explained to Children (from 7 to 107 years old). Albert Einstein Century International Conference, Jun 2005, Paris, France. CDRom p. 1124–1131. hal-00940573

HAL Id: hal-00940573

https://hal.science/hal-00940573v1

Submitted on 1 Feb 2014

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Relativity, the Special Theory, explained to Children (from 7 to 107 years old)

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15 July 2005

Abstract

The author thinks that the main ideas or Relativity Theory can be explained to children (around the age of 15 or 16) without complicated calculations, by using very simple arguments of affine geometry. The proposed approach is presented as a conversation between the author and one of his grand-children. Limited here to the Special Theory, it will be extended to the General Theory elsewhere, as sketched in conclusion.

For Agathe, Florent, Basile, Mathis, Gabrielle,
Morgane, Quitterie and my future other grand-children

1 Prologue

Maybe one day, one of my grand-children, at the age of 15 or 16, will ask me:

— Grand-father, could you explain what is Relativity Theory? My Physics teacher lectured about it, talking of rolling trains and of lightnings hitting the railroad, and I understood almost nothing!

This is the discussion I would like to have with her (or him).

— Do you know the theorem: the diagonals of a parallelogram meet at their middle point?

— Yes, I do! I even know that the converse is true: if the diagonals of a plane quadrilateral meet at their middle point, that quadrilateral is a parallelogram. And I believe that I know a proof!

— Good! You know all the stuff needed to understand the basic idea of Relativity theory! However, we must first think about Time and Space.

— Time and space seem to me very intuitive, and yet difficult to understand in deep!

— Many people feel the same. The true nature of Time and Space is mysterious. Let us say that together, Time and Space make the frame in which all physical phenomena take place, in which all material objects evolve, including our bodies. We should keep a

modest mind profile on such a subject. We cannot hope to understand all the mysteries of Time and Space. We should only try to understand some of their properties and to use them to describe physical phenomena. We should be ready to change the way we think about Time and Space, if some experimental evidence shows that we were wrong.

— But if we do not know what are Time and Space, how can we hope to understand some of their properties, and to be able to use them?

— By building mental pictures of Time and Space. Unfortunately we, poor limited human beings, cannot do better: we know the surrounding world only through our senses (enhanced by the measurement and observation instruments we have built) and our ability of reasoning. Our reasoning always apply to the mental pictures we have built of reality, not to reality itself.

Let me now indicate how the mental pictures of Time and Space used by scientists have evolved, mainly from Newton to Einstein.

2 Newton's and Leibniz's views about Time and Space

2.1 Newtonian Time

The great scientist Isaac Newton [2] (1642–1727) used, as mental picture of Time, a straight line $\mathcal{T}$, going to infinity on both sides, hence without beginning nor end, with no privileged origin. Each particular time, for example "now", or "three days ago at the sunset at Paris", corresponds to a particular element of that straight line.

Observe that Newton considered, without any discussion, that for each event happening in the universe, there was a corresponding well defined time (element of the straight line $\mathcal{T}$), the time at which that event happens.

— Where is that straight line $\mathcal{T}$? Is it drawn in some plane or in space?

— Nowhere! You should not think about the straight line of Time $\mathcal{T}$ as drawn in something of larger dimension. Newton considered Time as an abstract straight line, because successive events are linearly ordered, like points on a straight line. Don't forget that $\mathcal{T}$ is a mental picture of Time, not Time itself! However, that mental picture is much more than a confuse idea: it has very well defined mathematical properties. In modern language, we say that $\mathcal{T}$ is endowed with an *affine structure* and with an *orientation*.

— What is an affine structure? and what is its use?

— An affine structure allows us to compare two time intervals and to take their ratio, for example to say that one of these intervals is two times the other one. Newton considered the comparison of two time intervals as possible, even when they were many centuries or millenaries apart, and to take their ratio. In modern mathematical language, that property determines an *affine structure*.

For the mathematician, that property means that we can apply transforms to $\mathcal{T}$ by sliding it along itself, without contraction nor dilation, and that these transforms (called *translations*) do not change its properties.

For the physicist, it means that the physical laws remain the same at all times.

Another important property of Time: it always flows from past to future. To take it into account, we endow $\mathcal{T}$ with an *orientation*; it means that we consider the two directions (from past to future and from future to past) as different, not equivalent, for example by choosing the direction from past to future as preferred. We then say that $\mathcal{T}$ is *oriented*.

2.2 Newton's absolute space

— OK, I roughly agree with that mental picture, although it does not account for the main property of Time: it flows continuously and we cannot stop it! And what about Space?

— Newton identified Space with the three dimensional space of geometers, denoted by $\mathcal{E}$: the space in which there are various figures made of planes, straight lines, spheres, polyhedra, which obey the theorems developed in Euclidean geometry: Thales and Pythagoras theorems, the theorem which says that the diagonals of a parallelogram meet at their middle point, ...

2.3 The concept of Space-Time

Newton used Time and Space to describe the motion of every object A of the physical world as follows. That object occupies, at each time t (element of $\mathcal{T}$) for which it exists, a position A_t in Space $\mathcal{E}$. The motion of of A is described by its successive positions A_t when t varies in $\mathcal{T}$.

Let me introduce now a new concept, that of Space-Time [1], due to the German mathematician Hermann Minkowski (1864–1909). That concept was not used in Mechanics before the discovery of Special Relativity. That is very unfortunate, since its use makes much easier the understanding of the foundations of Classical Mechanics, as well as those of Relativistic Mechanics. Therefore I use it now, with the absolute Time and Space of Newton, although Newton himself did not use that concept.

Newton Space-Time is simply the product set $\mathcal{E} \times \mathcal{T}$, whose elements are pairs (called *events*) (x,t), made by a point x of $\mathcal{E}$ and a time t of $\mathcal{T}$.

— What is the use of that Space-Time?

— It is very convenient to describe motions. For example, the motion of a material particle a (a very small object whose position, at each time $t \in \mathcal{T}$, is considered as a point $a_t \in \mathcal{E}$), is described by a line in $\mathcal{E} \times \mathcal{T}$, made by the events (a_t,t), for all t in the interval of time during which a exists. That line is called the *world line* of a.

You will see on Figure 1 (where, for simplicity, Space is represented as a straight line, as if it were one-dimensional) the world lines of three particles, a, b and c.

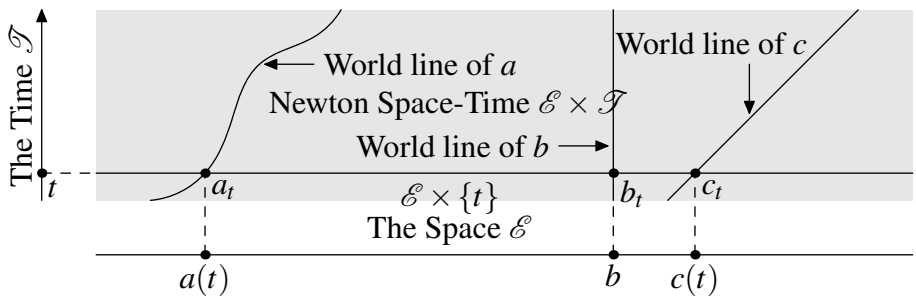

Figure 1: World lines in Newton Space-Time.

- The world line of b est parallel to the Time axis $\mathcal{T}$: that particle is at rest, il occupies a fixed position in the absolute Space $\mathcal{E}$.
- The world line of c is a slanting straight line. The trajectory of that particle in absolute Space $\mathcal{E}$ is a straight line and its velocity is constant.

- The world line of a is a curve, not a straight line. It means that the velocity of a changes with time.

2.4 Absolute rest and motion

For Newton, *rest* and *motion* were absolute concepts: a physical object is at rest if its position in Space does not change with time; otherwise, it is in motion.

— It seems very natural. Why should we change this view?

— Because nothing is at rest in the Universe! The Earth rotates around its axis and around the Sun, which rotates around the center of our Galaxy. And there are billions of galaxies in the Universe, all moving with respect to the others! For these reasons, Newton's concept of an absolute Space was criticized very early, notably by his contemporary, the great mathematician and philosopher Gottfried Wilhelm Leibniz (1647–1716).

2.5 Reference frames

— But without knowing what is at rest in the Universe, how Newton managed to study the motions of the planets?

— To study the motion of a body A, Newton, and after him almost all scientists up to now, used a *reference frame*. It means that he used another body R which remained approximately rigid during the motion he wanted to study, and he made as if that body was at rest. Then he could study the *relative motion* of A with respect to R.

Assuming that Newton's absolute Space $\mathcal{E}$ exists, we recover the description of absolute motion of A by choosing, for R, a body at rest in $\mathcal{E}$. The corresponding reference frame is called the *absolute fixed frame*.

The body R used to determine a reference frame can be, for example,

- the Earth (if we want to study the motion of a falling apple),
- the trihedron made by the straight lines which join the center of the Sun to three distant stars (if we want to study the motions of the planets in the solar system).

2.6 Galilean frames and Leibniz Space-Time

All reference frames are not equivalent. A *Galilean frame* [1], also called an *inertial frame*, is a reference frame in which the *principe of inertia* holds true. That principle, first formulated for absolute motions in Newton's absolute space $\mathcal{E}$, says that the (absolute) motion of a free particle takes place on a straight line, at a constant speed. But, as shown by Newton himself, that principe remains true for the *relative motion* of a free particle with respect to some particular reference frames, the *Galilean frames*.

More exactly, let us assume that the principle of inertia holds true for the relative motion of free particles with respect to the reference frame defined by the rigid body R_1. What happens for the relative motion of these free particles with respect to another reference frame, defined by another rigid body R_2? It is easy to see that the principle of inertia still holds true *if and only if* the relative motion of R_2 with respect to R_1 is a motion by translation at a constant speed.

[1] In memory of Galileo Galilei, (1564–1642), the founder of modern Physics.

The absolute frame, if it exists, therefore appears as a Galilean frame among an infinite number of other Gallilean frames, that no measurement founded on mechanical properties can distinguish from the others. For this reason, several scientists, following Leibniz, doubted about its existence.

Leibniz accepted Newton's concept of an absolute Time, but not that of an absolute Space. His views were not successful during his life, probably because at that time nobody saw how to cast them in a mathematically rigorous setting. Now we can do that; let me explain how.

We will consider that at each time $t \in \mathcal{T}$, there exists a *Space at time t*, denoted by $\mathcal{E}_t$, whose properties are those of the three-dimensional Euclidean space of geometers. We must consider that the Spaces $\mathcal{E}_{t_1}$ and $\mathcal{E}_{t_2}$, at two different times t_1 and t_2, $t_1 \neq t_2$, have no common element. Leibniz Space-Time, which will be denoted by $\mathcal{U}$ (for Universe), is the disjoint union of all the Spaces $\mathcal{E}_t$ for all times $t \in \mathcal{T}$. So, according to Leibniz views, we still have a Space-Time, but no more an absolute space ! The next picture shows,

- on the left side, Newton Space-Time $\mathcal{E} \times \mathcal{T}$, with the two projections $p_1 : \mathcal{E} \times \mathcal{T} \to \mathcal{E}$ and $p_2 : \mathcal{E} \times \mathcal{T} \to \mathcal{T}$;
- on the right side, Leibniz Space-Time $\mathcal{U}$, endowed with only one natural projection onto absolute Time $\mathcal{T}$, still denoted by $p_2 : \mathcal{U} \to \mathcal{T}$; the horizontal lines represent the Spaces $\mathcal{E}_t = p_2^{-1}(t)$, for various values of $t \in \mathcal{T}$.

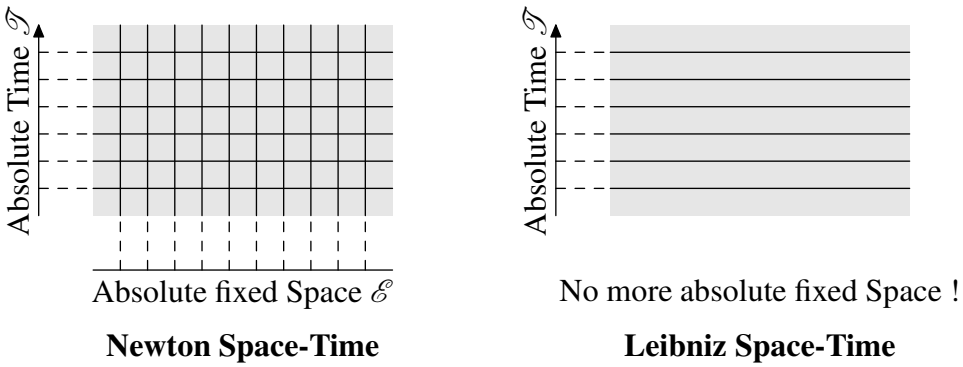

Figure 2: Newton and Leibniz Space-Time.

— But how do you put together the Spaces at various times $\mathcal{E}_t$ to make Leibniz Space-Time $\mathcal{U}$? Are they stacked in an arbitrary way?

— Of course no! Leibniz Space-Time $\mathcal{U}$ is a 4-dimensional affine space, fibered (*via* an affine map) over Time $\mathcal{T}$, which is itself a 1-dimensional affine space. Its fibres, the Spaces $\mathcal{E}_t$ at various times $t \in \mathcal{T}$, are 3-dimensional Euclidean spaces. The affine structure of $\mathcal{U}$ is determined by the *principle of inertia* of which we have already spoken. That principle can be formulated in a way which does not use reference frames, by saying:

The world line of any free particle is a straight line.

So formulated, the principle of inertia can be applied to Newton Space-Time $\mathcal{E} \times \mathcal{T}$ and to Leibniz Space-Time $\mathcal{U}$ as well. More, it *determines* the affine structure of $\mathcal{U}$, since one can easily show that the affine structure for which it holds true, if any, is unique. A physical law, the *principle of inertia*, is so embedded in the geometry of Leibniz Space-Time $\mathcal{U}$.

By using a reference frame R, one can split Leibniz Space-Time into a product of two factors: a space $\mathcal{E}_R$, fixed with respect to that frame, and the absolute Time $\mathcal{T}$. But of course, the space $\mathcal{E}_R$ depends on the choice of the reference frame R. For that reason, it seems that before 1905, not many scientists were aware of the fact that by dropping Newton's absolute Space $\mathcal{E}$, they already had completely changed the conceptual setting in which motions are described:

- according to Newton, absolute Space $\mathcal{E}$ and absolute Time $\mathcal{T}$ were directly related to reality, while Space-Time $\mathcal{E} \times \mathcal{T}$ was no more than a mathematical object, not very interesting (he did not use it) and not directly related to reality;
- but according to Leibniz's views, when expressed as done above, it is Space-Time $\mathcal{U}$ which is directly related to reality, as well as absolute Time $\mathcal{T}$; absolute Space $\mathcal{E}$ no more exists.

3 Relativity

Einstein [1] was led to drop Leibniz Space-Time when trying to reconcile the theories used in two different parts of Physics: Mechanics on one hand, Electromagnetism and Optics on the other hand.

According to the theory built by the great Scotch physicist James Clerk Maxwell (1831–1879), electromagnetic phenomena propagate in vacuum as waves, with the same velocity in all directions, independently of the motion of the source of these phenomena. Maxwell soon understood that light was an elecromagnetic wave, and lots of experimental results confirmed his views.

3.1 The luminiferous ether, a short lived hypothesis

In Leibniz Space-Time (as well as in Newton Space-Time) *relative velocities behave additively*. In that setting, it is with respect to *at most one particular reference frame* that light can propagate with the same velocity in all directions. Physicists introduced a new hypothesis: electromagnetic waves were considered as vibrations of an hypothetic, very subtle, but highly rigid medium called the *luminiferous ether*, everywhere present in space, even inside solid bodies. They thought that it was with respect to the ether's reference frame that light propagates at the same velocity in all directions. This new hypothesis amounts to come back to Newton's absolute Space identified with the ether. There were even physicists who introduced additional complications, by assuming that the ether, partially drawn by the motion of moving bodies, could deform with time!

— But if the luminiferous ether really exists, accurate measurements of the velocity of light in all directions should allow the determination of the Earth's relative velocity with respect to the ether!

— Good remark! These measurements were made several times, notably by Albert Abraham Michelson (1852–1931) and Edward Williams Morley (1838–1923), between 1880 et 1887. No relative velocity of the Earth with respect to the luminiferous ether could be detected.

These results remained not understood until 1905, despite many attempts. The most interesting of these attempts was that due to Hendrik Anton Lorentz (1853–1928) and

George Francis FitzGerald (1851–1901). Independently, they proposed the following hypothesis: when a rigid body, for example a rule or the arm of an interferometer, is moving with respect to the luminiferous ether, that body contracts slightly in the direction of its relative displacement.

— So that is the famous relativistic contraction my teacher spoke about!

— No! Not at all! Lorentz and FitzGerald considered that contraction as a true physical effect of the relative motion of a body with respect to the ether. This assumption is now completely abandoned, together with the luminiferous ether! The relativistic contraction of lengths and dilation of times has nothing to do with it: rather than a real phenomenon, it is only an appearance, like the following effect of perspective. Imagine that you look at a 20 centimeters rule, from a distance of, say two meters from its center. That rule looks shorter when it is not perpendicular to the straight line which joins your eye to its center than when it is. It may even seem to be reduced to a point when it lies along that straight line. As we will soon see, the relativistic contraction of lengths and dilation of times has a similar origin.

3.2 Minkowski Space-Time

Einstein was the first [2] to understand (in 1905) that the results of Michelson and Morley experiments could be explained by a deep change of the properties ascribed to Space and Time. At that time, his idea appeared as truly revolutionary. But now it may appear as rather natural, if we think along the following lines:

When we dropped Newton Space-Time in favour of Leibniz Space-Time, we recognized that there is no absolute Space, but that Space depends on the choice of a reference frame. Maybe Time too is no more absolute than Space, and depends on the choice of a reference frame!

— But if we drop absolute Time, which properties are left to our Space-Time?

— In 1905, Einstein implicitly considered that Space-Time still was a 4-dimensional affine space, which will be called *Minkowski Space-Time* and will be denoted by $\mathcal{M}$. He implicitly considered too that *translations* of $\mathcal{M}$ leave its properties unchanged, and he assumed that the *principe of inertia* still holds true in $\mathcal{M}$ when expressed without the use of reference frames:

The world line of any free particle is a straight line.

He also kept the notion of a *Galilean frame*. In $\mathcal{M}$, a Galilean frame is determined by a direction of straight line (not any straight line, a *time-like* straight line, as we will see below). Given a Galilean frame R, Minkowski Space-Time $\mathcal{M}$ can be split into a product $\mathcal{E}_R \times \mathcal{T}_R$ of a three-dimensional Space $\mathcal{E}_R$ and a one-dimensional Time $\mathcal{T}_R$, which both depend on R. Let me recall that in Leibniz Space-Time $\mathcal{U}$, a Galilean frame R allowed us to split $\mathcal{U}$ into a product $\mathcal{E}_R \times \mathcal{T}$ of a three-dimensional Space $\mathcal{E}_R$, which depended on R, and the one-dimensional absolute Time $\mathcal{T}$, which did not depend on R. That is the main difference between Leibniz's and Einstein's views about Space and Time.

Under these hypotheses, the properties of Space-Time follow from two principles:

- the *Principle of Relativity:* all physical laws have the same expression in all Galilean frames;

[2] The great French mathematician Jules Henri Poincaré (1854–1912) has, almost simultaneously and independently, presented very similar ideas [3], without explicitly recommending to drop the concept of an absolute Time.

- the *Principle of Constancy of the velocity of light:* the modulus of the velocity of light is an universal constant, which depends neither on the Galilean frame with respect to which it is calculated, nor on the motion of the source of that light.

— You said that a direction of straight line was enough to determine a Galilean frame. But how is that possible, since we no more have an absolute Time?

— That determination will follow from the pinciple of constancy of the velocity of light. Let us call *light lines* the straight lines in $\mathcal{M}$ which are possible world lines of light signals. Given an event $A \in \mathcal{M}$, the light lines through A make a 3-dimensional cone, the *light cone with apex* A; the two layers of that cone are called *the past half-cone* and *the future half-cone* with apex A. Since it is assumed that translations leave unchanged the properties of Space-Time, the light cone with another event B as apex is deduced from the light cone with apex A by the translation which maps A onto B.

Apart from light lines, there are two other kinds of straight lines in $\mathcal{M}$:

- *time-like straight lines*, which lie *inside* the light cone with any one of their elements as apex;

- and *space-like straight lines*, which lie *outside* the light cone with any of their element as apex.

I can now explain how the direction of a time-like straight line $\mathcal{A}$ determines a Galilean frame R. That frame is such that the rigid bodies at rest in it are those whose all material points have, as world lines, straight lines parallel to $\mathcal{A}$. The straight lines parallel to $\mathcal{A}$ will be called the *isochorous lines* [3] of the reference frame R; each of these lines is a set of events which all happen at the same place in the Space $\mathcal{E}_R$ of our frame R. For each event $M \in \mathcal{M}$, the set of all other events which occur at the same time as M, for the Time $\mathcal{T}_R$ of our Galilean frame R, will be called the *isochronous subspace* through M, for the Galilean frame R. It is a 3-dimensional affine subspace $\mathcal{E}_{R,M}$ of $\mathcal{M}$ containing the event M, and the other isochronous subspaces for R are all the 3-dimensional subspaces of $\mathcal{M}$ parallel to $\mathcal{E}_{R,M}$. They are determined by the property: the length covered by a light signal, calculated in the reference frame R, during a given time interval, also evaluated in that reference frame, *is the same in any two opposite directions.*

In a schematic 2-dimensional Space-Time (or in a plane section containing $\mathcal{A}$ of the "true" 4-dimensional Space-Time), the direction of isochronous subspaces is easily obtained as shown on the left part of Figure 3: we take the two light lines $\mathcal{L}^g$ and $\mathcal{L}^d$ through an event $A \in \mathcal{A}$ (the red lines on that figure); we take another event $A_1 \in A$, for example in the future of A, and we build the parallelogram $AA_1^gA_2A_1^d$ with two sides supported by $\mathcal{L}^g$ and $\mathcal{L}^d$, with A as one of its apices and A_1 as center. The isochronous subspaces are all the straight lines parallel to the space-like diagonal $A_1^gA_1^d$ of that parallelogram. Three of these lines are drawn (in blue) on Figure 3, $\mathcal{E}_{R,A}$, $\mathcal{E}_{R,A_1}$ and $\mathcal{E}_{R,A_2}$.

— Why?

— A light signal starting from A covers, during the time interval between events A and A_1, the lengths $A_1A_1^g$ towards the left and $A_1A_1^d$ towards the right. These lengths are equal because $A_1^gA_1^d$ is the diagonal of a parallelogram whose center is A_1.

[3] The word *isochorous*, already used in Thermodynamics, refers here to a set of events which all occur at the same spatial location at various times, in similarity with the word *isochronous* which refers to a set of events which all occur simultaneously in time at various spatial locations.

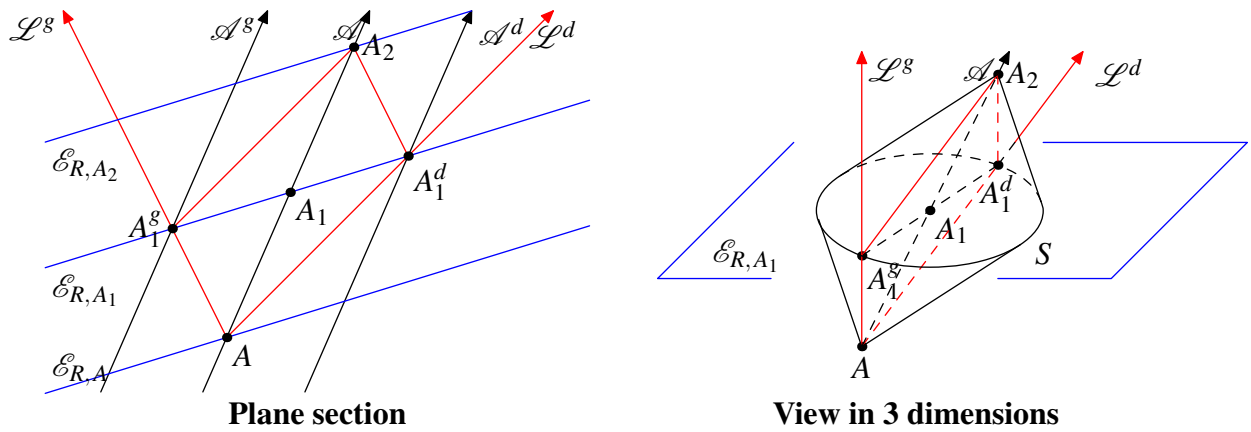

Figure 3: Construction of Space and Time relative to a Galilean frame

— What for the "true" 4-dimensional Minkowski Space-Time $\mathcal{M}$? And what are the Space $\mathcal{E}_R$ and the Time $\mathcal{T}_R$ of our reference frame R?

— It is the same, as shown on the right side of Figure 3. Take the event A_2 on the light line $\mathcal{A}$ such that A_1 is the middle point of AA_2. Consider the future light half-cone with apex A and the past light half-cone with apex A_2. Their intersection is a 2-dimensional sphere S. The unique affine hyperplane $\mathcal{E}_{R,A_1}$ which contains S is an isochronous subspace for the Galilean frame determined by the direction of $\mathcal{A}$ (in blue on Figure 3). The other isochronous subspaces for that Galilean frame are all the hyperplanes parallel to $\mathcal{E}_{R,A_1}$. The Space $\mathcal{E}_R$ is the set of all the isochorous lines, *i.e* the set of all straight lines parallel to $\mathcal{A}$, and the Time $\mathcal{T}_R$ the set of all isochronous subspaces. Minkowski Space-Time $\mathcal{M}$ splits into the product $\mathcal{E}_R \times \mathcal{T}_R$, or in other words can be identified with that product, because a pair made by an isochorous line and an isochronous subspace determine a unique element of $\mathcal{M}$, the event at which they meet.

— What happens if you change your Galilean frame?

— Of course, as for Galilean frames in Leibniz Space-Time, the direction of isochorous lines (the straight world lines of points at rest with respect to the chosen Galilean frame) is changed. Moreover, contrary to what happened in Leibniz Space-Time, the direction of isochronous subspaces is also changed! Therefore, the chronological order of two events can be different when it is appreciated in two different Galilean frames!

3.3 Metric properties of Minkowski Space-Time

Up to now, we have compared the lengths of two straight line segments in $\mathcal{M}$ only when they were supported by parallel straight lines. That was allowed by the *affine structure* of $\mathcal{M}$. We need more, because the spectral lines of atoms allow us to build clocks and to compare time intervals measured in two different Galilean frames.

Let AA_1 and AB_1 be two straight line segments supported by two different time-like straight lines $\mathcal{A}$ and $\mathcal{B}$, which meet at the event A. Let $R_\mathcal{A}$ and $R_\mathcal{B}$ be the Galilean frames determined by the directions of $\mathcal{A}$ and $\mathcal{B}$, respectively. We assume that the time intervals corresponding to AA_1 measured in $R_\mathcal{A}$, and to AB_1 measured in $R_\mathcal{B}$, are the same. Let B' be the event at which the time-like straight line $\mathcal{B}$ meets the isochronous subspace $\mathcal{E}_{R_\mathcal{A},A_1}$ containing A_1 of the Galilean frame $R_\mathcal{A}$ (figure 4). Since the events A_1 and B' are

synchronous for $R_{\mathcal{A}}$, the time interval corresponding to AB_1 appears longer than the time interval corresponding to AA_1 when both are observed in the reference frame $R_{\mathcal{A}}$, by the ratio $\dfrac{AB_1}{AB'}$. That ratio is the *ratio of dilation of times* of the Galilean frame of $R_{\mathcal{B}}$, when observed in the Galilean frame $R_{\mathcal{A}}$. Similarly, $\dfrac{AA_1}{AA'}$ is the ratio of dilation of times of the Galilean frame $R_{\mathcal{A}}$ when observed in the Galilean frame $R_{\mathcal{B}}$. According to the Principle of Relativity, these two Galilean frames must play the same role with respect to the other, which implies the equality $\dfrac{AA_1}{AA'} = \dfrac{AB_1}{AB'}$. By a well known property of hyperbolae, that equality holds *if and only if A_1 and B_1 lie on the same arc of hyperbola which has the light lines $\mathcal{L}^d$ and $\mathcal{L}^g$ (which meet at A and are contained in the two-dimensional plane which contains $\mathcal{A}$ and $\mathcal{B}$) as asymptotes*. Or more generally, on the same hyperboloid with the light cone of A as asymptotic cone.

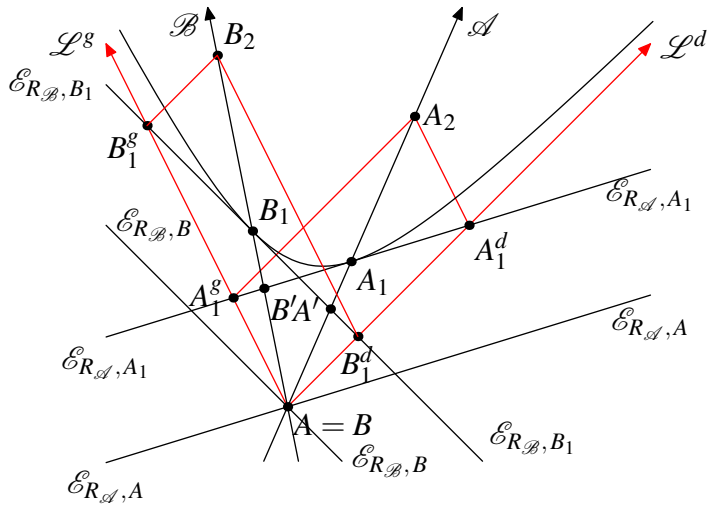

Figure 4: Comparison of times.

The comparison of lengths on two non-parallel space-like straight lines is similar to the comparison of time intervals. Let AA^d and AB^d be two segments supported by two space-like straight lines which meet at the event A. They are of equal length *if and only if A^d and B^d lie on the same hyperboloid with the light cone of A as asymptotic cone*.

4 Conclusion

The comparison of time intervals and lengths presented above allows a very natural introduction of the pseudo-Euclidean metric of Minkowski Space-Time. The construction of isochronous subspaces in two different Galilean frames, as presented above, leads to the formulas for Lorentz transformations with a minimum of calculations. The pictures we have presented allow a very easy explanation of the apparent contraction of lengths and dilation of times associated to a change of Galilean frames and a very simple explanation, without complicated calculations, of the (improperly called) paradox of Langevin's twins.

By explaining that the affine structure of Space-Time should be questioned, a smooth transition towards General Relativity, suitable from children from 8 to 108 years old, seems possible.

Acknowledgements. The author thanks the team "Analyse algébrique" of the "Institut de Mathématiques de Jussieu" and his University for taking in charge his registration fee at this International Conference.

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