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# ARITHMETIC OF POSITIVE CHARACTERISTIC $L$ -SERIES VALUES IN TATE ALGEBRAS

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WITH AN APPENDIX BY F. DEMESLAY<sup>1</sup>.

ABSTRACT. The second author has recently introduced a new class of  $L$ -series in the arithmetic theory of function fields over finite fields. We show that the value at one of these  $L$ -series encode arithmetic informations of certain Drinfeld modules defined over Tate algebras. This enables us to generalize Anderson's log-algebraicity Theorem and an analogue of Herbrand-Ribet Theorem recently obtained by Taelman.

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## 1. INTRODUCTION

We fix a finite field  $k$  with  $q$  elements; we denote by  $p$  its characteristic. We further set  $A = k[\theta]$  (polynomial ring with coefficients in  $k$  in an indeterminate  $\theta$ ) and  $K = k(\theta)$  (the field of fractions of  $A$ ). We also consider the field  $K_\infty = k((\theta^{-1}))$ , the completion of  $K$  with respect to the place at infinity; we write  $|\cdot|$  for the the absolute value of  $K_\infty$  normalized by setting  $|\theta| = q$ . We denote by  $\mathbb{C}_\infty$  the

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completion of a fixed algebraic closure of  $K_\infty$  and we denote by  $K^{ac}$  the algebraic closure of  $K$  in  $\mathbb{C}_\infty$ .

Carlitz [10] introduced the so-called *Carlitz zeta values*

$$\zeta_C(n) := \sum_{a \in A_+} a^{-n} \in K_\infty, \quad n > 0, \quad n \text{ integer},$$

as some analogues, up to a certain extent, of the classical zeta values

$$\zeta(n) = \sum_{k>0} k^{-n} \in \mathbb{R}$$

( $n > 1$  integer). In the definition of  $\zeta_C(n)$ ,  $A_+$  denotes the set on monic polynomials in  $A$  and provides a kind of substitute of the set of positive integers. The Carlitz zeta values offer interesting analogies with the classical zeta values. Let us look at the archimedean example of the divergent series

$$(1) \quad \zeta(1) = \sum_{k \geq 1} k^{-1} = \prod_p \left(1 - \frac{1}{p}\right)^{-1} = \infty$$

which we have developed as a divergent eulerian product (running over the prime numbers  $p$ ). For a commutative ring  $R$  and a functor  $G$  from  $R$ -algebras to  $R$ -modules, we denote by  $\text{Lie}(G)$  the functor from  $R$ -algebras to  $R$ -modules defined, for  $B$  an  $R$ -algebra, by:

$$\text{Lie}(G)(B) = \text{Ker}(G(B[\epsilon]/(\epsilon^2)) \rightarrow G(B)).$$

The local factor at  $p$  in (1) is

$$\left(1 - \frac{1}{p}\right)^{-1} = \frac{p}{p-1} = \frac{|\text{Lie}(\mathbb{G}_m)(\mathbb{Z}/p\mathbb{Z})|}{|\mathbb{G}_m(\mathbb{Z}/p\mathbb{Z})|},$$

where  $|\cdot|$  denotes here the cardinality of a set. The above cardinalities can also be seen as positive generators of *Fitting ideals* of finite  $\mathbb{Z}$ -modules (see §5.2).

In parallel, let  $C$  be the *Carlitz functor* from  $A$ -algebras to  $A$ -modules (see §2.1 for the background on the Carlitz module). Then, for  $P$  a *prime* of  $A$  (<sup>1</sup>) the module  $C(A/PA)$  is a finite  $A$ -module and one can easily prove (in different ways; read Goss, [12, Theorem 3.6.3], Taelman, [19, Proposition 1], see also Anderson and Thakur's paper [3, Proposition 1.2.1]) that  $P-1$  is the monic generator of the Fitting ideal of  $M$ . For a finitely generated and torsion  $A$ -module  $M$ ,  $[M]_A$  denotes the monic generator of its Fitting ideal. Then,

$$[C(A/PA)]_A = P-1$$

and

$$(2) \quad \zeta_C(1) = \prod_P \left(1 - \frac{1}{P}\right)^{-1} = \prod_P \frac{P}{P-1} = \prod_P \frac{[\text{Lie}(C)(A/PA)]_A}{[C(A/PA)]_A}.$$

The tensor powers of the Carlitz module functor introduced by Anderson and Thakur [3] provide a way to interpret the values  $\zeta_C(n)$  as well, and this can be viewed as one of the main sources of analogies between the theory of the Carlitz zeta values and the values of the Riemann zeta function at integers  $n \geq 2$ .

<sup>1</sup>A prime of  $A$  is an irreducible monic polynomial of  $A$ .

Carlitz proves that for all  $n > 0$  divisible by  $q - 1$ ,  $\zeta_C(n)$  is, up to a scalar factor of  $K^\times$ , proportional to  $\tilde{\pi}^n$ , where the quantity  $\tilde{\pi}$  is defined [12, definition 3.2.7] by

$$\tilde{\pi} = \sqrt[q-1]{\theta - \theta^q} \prod_{i \geq 1} \left( 1 - \frac{\theta^{q^i} - \theta}{\theta^{q^{i+1}} - \theta} \right) \in \theta \sqrt[q-1]{-\theta} (1 + \theta^{-1} k[[\theta^{-1}]])$$

unique up to multiplication by an element of  $k^\times$ , see Goss, [12, Chapter 3]. We consider the *Carlitz exponential*  $\exp_C$  (see §2.1 for the background about this function). Carlitz proved the formula

$$(3) \quad \exp_C(\zeta_C(1)) = 1.$$

Knowing that 1 belongs to the domain of convergence of the Carlitz logarithm  $\log_C$ , composition inverse of  $\exp_C$  (see §2.1.2) and comparing the absolute values of  $\log_C(1)$ ,  $\zeta_C(1)$ ,  $\tilde{\pi}$ , we see that the above formula is equivalent to

$$(4) \quad \zeta_C(1) = \log_C(1).$$

Taelman [19] recently exhibited an appropriate setting to interpret the above formula as an instance of the *class number formula*. His approach, involving determinants of Fredholm operators, also relies on the formula (2). He did this in the broader framework of *Drinfeld modules* defined over the ring of integers  $R$  of a finite extension  $L$  of  $K$ . Taelman associated, to such a Drinfeld module  $\phi$ , a finite  $A$ -module called the *class module* (of  $\phi$  over  $L$ ), and a finitely generated  $A$ -module called the *unit module* (of  $\phi$  over  $L$ ). An *L-series value*  $L(\phi/R)$  that he also defines is then equal to the product of the monic generator of the Fitting ideal of the class module times the regulator of the unit module (see Theorem 1 of loc. cit.).

In the case of  $\phi = C$  the Carlitz module, and  $L = K$ , the  $L$ -series value is equal to  $\zeta_C(1)$ , the class module is trivial, and the regulator of the unit module is  $\log_C(1)$ , the Carlitz logarithm of 1, yielding (3).

**1.1. L-series.** The  $L$ -series values of Taelman are typical elements of  $\mathbb{C}_\infty$ . We generalize the Carlitz theory in another direction, and introducing  $L$ -series values which are elements of *Tate algebras* of positive dimension. The “classical”  $L$ -series values are then just elements of zero-dimensional Tate algebras.

**1.1.1. The definition of L-series values.** The *Tate algebra*  $\mathbb{T}_s$  of dimension  $s$  is the completion of the polynomial algebra  $\mathbb{C}_\infty[t_1, \dots, t_s]$  for the *Gauss norm* (see §2.2) and we have  $\mathbb{T}_0 = \mathbb{C}_\infty$ . Observe that  $\tau$  extends to continuous homomorphism of  $k[t_1, \dots, t_s]$ -algebras  $\tau : \mathbb{T}_s \rightarrow \mathbb{T}_s$  and that  $k[t_1, \dots, t_s] = \{f \in \mathbb{T}_s, \tau(f) = f\}$ . We consider the maps

$$\rho_\alpha : A \rightarrow k[t_1, \dots, t_s]$$

defined by

$$\rho_\alpha(b) = \text{Res}_\theta(b, \alpha),$$

where  $\alpha$  is a polynomial of  $A[t_1, \dots, t_s] \setminus \{0\}$  and where  $\text{Res}_\theta(P, Q)$  denotes the resultant of two polynomials  $P, Q$  in  $\theta$  ([14, §IV.8]). The typical  $L$ -series value at  $n > 0$  that we handle in this paper is of the form

$$(5) \quad L(n, \alpha) = \sum_{b \in A_+} \text{Res}_\theta(b, \alpha) b^{-n} = \prod_P \left( 1 - \frac{\rho_\alpha(P)}{P^n} \right)^{-1}$$

(the product is taken over the primes  $P$  of  $A$ ). It is easy to see that this series converges in the Tate algebra  $\mathbb{T}_s$ ; see §4. If  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$ , then we recover

Goss abelian  $L$ -functions by specializing the variables  $t_i$  to elements in  $k^{\text{ac}}$ . In §10 we have included some remarks on the link existing between our  $L$ -series values and the global  $L$ -functions of Goss, Taguchi-Wan and Böckle-Pink (see [12, 7, 9, 22]). These remarks may be of help for the reader to orientate in the literature.

**1.2. Drinfeld modules.** We discuss of a generalization of Drinfeld modules as follows.

Classically, a Drinfeld module  $\phi$  of rank  $r$  is the datum of an injective  $k$ -algebra homomorphism

$$\phi : A \rightarrow \text{End}_{k\text{-lin}}(\mathbb{C}_\infty),$$

uniquely defined by the image of  $\theta$ , that is, the value  $\phi_\theta$  of  $\phi$  at  $\theta$ , which is of the form

$$(6) \quad \phi_\theta = \theta + \alpha_1\tau + \cdots + \alpha_r\tau^r,$$

where the *parameters*  $\alpha_1, \dots, \alpha_r$  are elements of  $\mathbb{C}_\infty$  with  $\alpha_r \neq 0$ . We use the  $k[t_1, \dots, t_s]$ -linear automorphism  $\tau$  of  $\mathbb{T}_s$  to define *Drinfeld modules of rank  $r$  over  $\mathbb{T}_s$* ; a Drinfeld module  $\phi$  of rank  $r$  over  $\mathbb{T}_s$  is an injective  $k[t_1, \dots, t_s]$ -algebra homomorphism

$$\phi : A[t_1, \dots, t_s] \rightarrow \text{End}_{k[t_1, \dots, t_s]\text{-lin.}}(\mathbb{T}_s),$$

with  $\phi_\theta$  as in (6) but where the parameters  $\alpha_1, \dots, \alpha_r$  are now allowed to be chosen in  $\mathbb{T}_s$  and, of course,  $\alpha_r \neq 0$ ; see §3.

When the rank of  $\phi$  is one and when the unique parameter  $\alpha_1$  is in  $A[t_1, \dots, t_s]$ , our construction of  $L$ -series values (5) with  $n = 1$  (see §4) extends the construction of Taelman. The reader can see §5.2 and, in particular, Proposition 5.11 where we show that the local factors in (5) are ratios of monic generators of Fitting ideals exactly as in (2).

We will write, all along this paper,  $L(n, \phi) = L(n, \alpha)$  where  $\phi$  is the Drinfeld module of rank one over  $\mathbb{T}_s$  and parameter  $\alpha \in A[t_1, \dots, t_s] \setminus \{0\}$ . With our definition of  $L$ -series values, we will cover already many  $L$ -series values studied by Goss, as well as in [15, 13, 16] and [4].

**1.3. The main results.** The  $L$ -series values that we study being elements of the Tate algebras  $\mathbb{T}_s$ , they have the double status of “numbers” and “functions”. As numbers, the indeterminates  $t_1, \dots, t_s$  are unspecified and the series  $L(n, \phi)$  are handled as elements of  $\mathbb{T}_s$ . As functions, the variables  $t_1, \dots, t_s$  can be specialized and the analytic properties of the functions  $L(n, \phi)$  can be used to obtain arithmetic information e.g. on Carlitz zeta values.

**1.3.1.  $L$ -series values as “numbers”.** Let  $\phi$  be a Drinfeld module of rank one over  $\mathbb{T}_s$  of parameter  $\alpha$  (that is,  $\phi_\theta = \theta + \alpha\tau$ ) in  $A_s := A[t_1, \dots, t_s]$ . We introduce, in §5.1, the *class module*  $H_\phi$  and the *unit module*  $U_\phi$  associated to  $\phi$ . This provides a generalization of the constructions of Taelman paper [20] in which the class module  $H_\phi$ , instead of being a vector space of finite dimension over  $k$ , it is a module of finite rank over  $k[t_1, \dots, t_s]$ . In the case of  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$ , the class modules are “universal” in the sense that they can be used to interpolate Taelman’s class modules (for the Carlitz module and for the cyclotomic extensions)

We set  $k_s = k(t_1, \dots, t_s)$  and  $R_s = k_s[\theta] = k_s \otimes_{k[t_1, \dots, t_s]} A_s$ . The  $k_s$ -vector space

$$V_\phi = k_s \otimes_{k[t_1, \dots, t_s]} H_\phi$$

is of finite dimension and endowed with the structure of  $R_s$ -module (Corollary 5.7). Let  $[V_\phi]_{R_s} \in R_s$  be the monic generator of its Fitting ideal. We will see (Proposition 5.4) that

$$k_s \otimes_{k[t_1, \dots, t_s]} U_\phi$$

is a free  $R_s$ -module of rank one, to which we can associate a *regulator*  $[R_s : k_s \otimes_{k[t_1, \dots, t_s]} U_\phi]_{R_s}$ . Then, the class number formula for the  $L$ -series value  $L(1, \phi)$  (Theorem 5.12) can be obtained (the notation will be made more precise later in this text):

$$L(1, \phi) = [V_\phi]_{R_s} [R_s : k_s \otimes_{k[t_1, \dots, t_s]} U_\phi]_{R_s}.$$

This result is deduced from Theorem 11.4 of the Appendix, by F. Demeslay. The originality of our approach is the use we make of the above class number formula.

The properties of the *exponential function*  $\exp_\phi$  (§3.1) strongly influence the properties of  $L(1, \phi)$ ,  $H_\phi$  and  $U_\phi$ . This depends on whether  $\exp_\phi$  is surjective as an endomorphism of  $\mathbb{T}_s$  or not, and on whether, in the affirmative case, its kernel has non-zero intersection with  $\mathbb{T}_s(K_\infty)$ , the completion of  $K_\infty[t_1, \dots, t_s]$  for the Gauss norm. Indeed, by the results obtained in §5, 6, and 7, for  $\phi$  a Drinfeld module of rank one defined over  $A_s$  (with  $A_s = A[t_1, \dots, t_s]$ ),  $\phi_\theta = \theta + \alpha\tau$  with  $\alpha \in A_s \setminus \{0\}$ , we have that  $\exp_\phi(L(1, \phi)) \in \mathbb{T}_s$  belongs to  $A_s$ . It is a torsion point for the structure of  $A_s$ -module induced by  $\phi$  if and only if the parameter  $-\alpha \in A_s$  is a monic polynomial in  $\theta$  of degree  $r \equiv 1 \pmod{q-1}$ . We will see that this latter condition is equivalent to the fact that the function  $\exp_\phi$  is surjective, and its kernel has non-trivial intersection with  $\mathbb{T}_s(K_\infty)$ . Furthermore by the results of §7.1.1 and Theorem 7.7, we have:

*If  $-\alpha$  is monic as a polynomial in  $\theta$ , of degree  $r \equiv 1 \pmod{q-1}$  with  $r \geq 2$ , then  $\exp_\phi(L(1, \phi)) = 0$ . In the special case of  $\phi$  of parameter  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$ , we have the formula*

$$L(1, \phi) = \frac{\tilde{\pi} \mathbb{B}_s}{\omega(t_1) \cdots \omega(t_s)},$$

where  $\mathbb{B}_s \in A_s$  is the monic generator of the Fitting ideal of the  $A_s$ -module  $H_\phi$  and  $\omega$  is the Anderson-Thakur function introduced in [3].

We have few explicit examples of polynomial  $\mathbb{B}_s$ . Here are some:

$$\begin{aligned} \mathbb{B}_q &= -1, \\ \mathbb{B}_{2q-1} &= \theta - \sum_{1 \leq i_1 < \cdots < i_{q-1} \leq 2q-1} \prod_{k=1}^{q-1} t_{i_k}. \end{aligned}$$

Also, by Lemma 7.1, we can set  $\mathbb{B}_1 = \frac{1}{\theta - t_1}$ .

As a consequence of the class number formula, we shall also mention the *log-algebraicity Theorem of Anderson*, in the case of the Carlitz module, see [2, Theorem 3 and Proposition 8 (I)] (so, surprisingly, the class number formula implies Anderson's log-algebraic theorem for the Carlitz module). In fact, we prove in §8 a result which can be interpreted as an *operator theoretic version*, thus a refinement of Anderson Proposition 8 (I) loc. cit. We introduce a class of formal series in

infinitely many indeterminates  $X_i, \tau(X_i), \dots, Z, \tau(Z), \dots$  ( $i = 1, \dots, r$ ) by setting:

$$\mathcal{L}_r(X_1, \dots, X_r; Z) = \sum_{d \geq 0} \left( \sum_{a \in A_{+,d}} C_a(X_1) \cdots C_a(X_r) a^{-1} \right) \tau^d(Z),$$

where  $A_{+,d}$  denotes the set of monic polynomials of degree  $d$  and  $C_a(X_i)$  denotes a certain polynomial in  $X_i, \tau(X_i), \dots$  obtained from the action of the Carlitz module evaluated at  $a$  on the indeterminate  $X_i$ ; for example,  $C_\theta(X_1) = \theta X_1 + \tau(X_1)$ . We have (Theorem 8.1):

$$\exp_C(\mathcal{L}_r(X_1, \dots, X_r; Z)) \in A[X_i, \tau(X_i), \dots, Z, \tau(Z), \dots, i = 1, \dots, r].$$

If we substitute, in the above result,  $X_1 = \dots = X_r = X$  and  $\tau^n(X) = X^{q^n}, \tau^n(Z) = Z^{q^n}$  for all  $n \geq 0$ , we recover Anderson's original result asserting that

$$\exp_C \left( \sum_{a \in A_+} Z^{q^{\deg_\theta(a)}} a^{-1} C_a(X)^r \right) \in A[X, Z].$$

**1.3.2.  $L$ -series values as "functions".** The evaluation of  $L$ -series values is the necessary step to deduce from the above results, arithmetic results on the values of Goss abelian  $L$ -series. One of the main novelties of our work is that we are able to study the isotypic components of Taelman's class modules in families. Let  $\chi$  be a Dirichlet character of type  $s$  such that  $s \equiv 1 \pmod{q-1}$  and conductor  $a \in A_+$  (see §9.1). Let us denote by  $k_a$  the subfield of  $\mathbb{C}_\infty$  obtained adjoining to  $k$  the roots of  $a$ , by  $K_a$  the  $a$ -th cyclotomic field and by  $\Delta_a$  the Galois group of  $K_a$  over  $K$ . We denote by  $H_a$  the Taelman class  $A$ -module associated to the Carlitz module and relative to the extension  $K_a/K$ . This is a finite  $k[\Delta_a]$ -module. Let  $e_\chi$  be the idempotent element of  $k_a[\Delta_a]$  associated to  $\chi$ . Then, the  $\chi$ -isotypic component

$$H_\chi = e_\chi(H_a \otimes_k k_a)$$

is a finite  $A[k_a]$ -module endowed with a suitable structure of  $k_a[\Delta_a]$ -module. The evaluation map  $\text{ev}_\chi$  is described in §9; it is obtained by substituting the variables  $t_i$  (for  $i = 1, \dots, s$ ) by appropriate roots of unity chosen among the roots of the conductor  $a$  in  $k^{ac}$ . By Corollary 9.3, the Fitting ideal of the  $A[k_a]$ -module  $H_\chi$  is generated by  $\text{ev}_\chi(\mathbb{B}_s)$ . A similar result is obtain when the type  $s$  satisfies  $s \not\equiv 1 \pmod{q-1}$ , see Theorem 9.6. In §9.5 we associate to our character  $\chi$  certain *generalized Bernoulli-Carlitz numbers* denoted by  $\text{BC}_{i,\chi^{-1}}$ . These are elements of the compositum  $K(k_a)$  of  $k_a$  and  $K$  in  $\mathbb{C}_\infty$ .

Let us write

$$\chi = \vartheta_P^N \tilde{\chi},$$

where  $P$  is a prime dividing the conductor  $a$  of  $\chi$  (so that  $a = Pb$  with  $P$  not dividing  $b$ ),  $\tilde{\chi}$  is a Dirichlet character of conductor  $b$ ,  $\vartheta_P$  is the Teichmüller character associated to  $P$ , and  $N$  is an integer between 0 and  $q^d - 2$ , with  $d$  the degree of  $P$ .

If  $\widehat{A}_P$  is the valuation ring of  $\widehat{K}_P$ , the valuation ring of the field  $\widehat{K}_P(k_a)$  is  $\widehat{A}_P[k_a]$ . We obtain a generalization of Herbrand-Ribet-Taelman Theorem of [20] (Theorem 9.15):

*Here  $N \geq 2$  if  $\tilde{\chi} = 1$ . The  $\widehat{A}_P[k_a][\Delta_a]$ -module  $e_\chi(H_a \otimes_A \widehat{A}_P[k_a])$  is non-trivial if and only if  $\text{BC}_{q^d - N, \tilde{\chi}^{-1}} \equiv 0 \pmod{P}$ .*

The original result of Taelman [20, Theorem 1] corresponds to the case in which  $\tilde{\chi}$  is the trivial character. See also [6, Theorem 8.14].

## 2. NOTATION AND BACKGROUND

The basic list of notation of this paper is the following:

- $k$ : a fixed finite field with  $q$  elements.
- $p$ : the characteristic of  $k$ .
- $\theta$ : an indeterminate over  $k$ .
- $A$ : the polynomial ring  $k[\theta]$ .
- $A_+$  the set of monic elements in  $A$ .
- For  $n \in \mathbb{N}$ ,  $A_{+,n}$  denotes the set of monic elements in  $A$  of degree  $n$ .
- $K = k(\theta)$ : the fraction field of  $A$ .
- $\infty$ : the unique place of  $K$  which is a pole of  $\theta$  and  $v_\infty$  is the discrete valuation on  $K$  corresponding to the place  $\infty$ . The valuation  $v_\infty$  is normalized such that  $v_\infty(\theta) = -1$ .
- $K_\infty = k((\theta^{-1}))$ : the completion of  $K$  at  $\infty$ .
- $\mathbb{C}_\infty$ : a fixed completion of an algebraic closure of  $K_\infty$ . The valuation on  $\mathbb{C}_\infty$  that extends  $v_\infty$  will still be denoted by  $v_\infty$ .
- $|\cdot|$ : the absolute value of  $\mathbb{C}_\infty$  defined by  $|\alpha| = q^{-v_\infty(\alpha)}$  for  $\alpha \in \mathbb{C}_\infty$ .
- $K^{\text{ac}}$ : the algebraic closure of  $K$  in  $\mathbb{C}_\infty$ .
- $k^{\text{ac}}$ : the algebraic closure of  $k$  in  $\mathbb{C}_\infty$ .

For  $s \in \mathbb{N}$ , let  $t_1, \dots, t_s$  be  $s$  indeterminates over  $\mathbb{C}_\infty$ . We set:

- $k_s$ : the field  $k(t_1, \dots, t_s)$ . Observe that  $k_0 = k$ .
- $A_s$ : the ring  $A[t_1, \dots, t_s]$ .
- $R_s$ : the ring  $k_s[\theta]$ . Observe that  $R_0 = A$ .
- $K_{s,\infty}$ : the field  $k_s((\theta^{-1}))$ .

**2.1. The Carlitz exponential.** The Carlitz exponential is the function

$$\exp_C : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$$

defined by

$$\exp_C(X) = \sum_{i \geq 0} \frac{X^{q^i}}{D_i}, \quad X \in \mathbb{C}_\infty,$$

where  $(D_i)_{i \geq 0}$  is the sequence of  $A$  defined by  $D_0 = 1$  and, for  $i \geq 1$ ,

$$D_i = (\theta^{q^i} - \theta)D_{i-1}^q.$$

This function,  $k$ -linear, is entire because  $|D_i| = q^{iq^i}$ . In particular,  $\exp_C$  is surjective.

The kernel of  $\exp_C$  is the  $A$ -module  $\tilde{\pi}A$ , where

$$\tilde{\pi} = \sqrt[q-1]{\theta - \theta^q} \prod_{i \geq 1} \left( 1 - \frac{\theta^{q^i} - \theta}{\theta^{q^{i+1}} - \theta} \right)$$

is the Carlitz period (see [12], Corollary 3.2.9). We have

$$|\tilde{\pi}| = q^{\frac{q}{q-1}}.$$



We can expand the function  $\exp_C$  in a convergent infinite product:

$$\exp_C(X) = X \prod_{a \in A \setminus \{0\}} \left(1 - \frac{X}{\tilde{\pi}a}\right), \quad X \in \mathbb{C}_\infty.$$

We observe that  $\exp_C$  induces an isometric  $k$ -automorphism of the disk

$$D_{\mathbb{C}_\infty}(0, q^{\frac{q}{q-1}}) = \{z \in \mathbb{C}_\infty; |z| < q^{\frac{q}{q-1}}\}.$$

2.1.1. *The Carlitz module.* The  $\mathbb{C}_\infty$ -algebra of the  $k$ -linear algebraic endomorphisms of  $\mathbb{G}_a(\mathbb{C}_\infty)$

$$\text{End}_{k\text{-lin.}}(\mathbb{G}_a(\mathbb{C}_\infty))$$

can be identified with the skew polynomial ring  $\mathbb{C}_\infty[\tau]$  whose elements are the finite sums  $\sum_{i \geq 0} c_i \tau^i$  with the  $c_i$ 's in  $\mathbb{C}_\infty$ , subject to the product rule defined by  $\tau x = x^q \tau$  for all  $x \in \mathbb{C}_\infty$ . If  $X \in \mathbb{C}_\infty$  and  $P = \sum_{i=0}^d P_i \tau^i$  is an element of  $\mathbb{C}_\infty[\tau]$ , the evaluation of  $P$  at  $X$  is defined by setting

$$P(X) = \sum_{i=0}^d P_i X^{q^i}.$$

For example, the evaluation of  $\tau$  at  $X$  is  $\tau(X) = X^q$ .

The *Carlitz module* is the unique  $k$ -algebra homomorphism

$$C : A \rightarrow \text{End}_k(\mathbb{G}_a(\mathbb{C}_\infty))$$

determined by

$$C_\theta = \theta + \tau.$$

If  $a \in A_{+,d}$ , we denote by  $C_a$  the evaluation of  $C$  at  $a$ . We have  $C_a = a_0 \tau^0 + a_1 \tau^1 + \dots + a_{d-1} \tau^{d-1} + \tau^d$  with  $a_0 = a$ , and if  $X \in \mathbb{C}_\infty$ , we will write, in particular,

$$C_a(X) = a_0 X + a_1 X^q + \dots + a_{d-1} X^{q^{d-1}} + X^{q^d}.$$

This endows  $\mathbb{C}_\infty$  with a structure of  $A$ -module that will be denoted by  $C(\mathbb{C}_\infty)$  and we have

$$C_a(\exp_C(X)) = \exp_C(aX)$$

for all  $a \in A$  and  $X \in \mathbb{C}_\infty$ . The Carlitz module  $C$  allows to make the exact sequence of  $k$ -vector spaces  $0 \rightarrow \tilde{\pi}A \rightarrow \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \rightarrow 0$  induced by  $\exp_C$  into an exact sequence of  $A$ -modules

$$0 \rightarrow \tilde{\pi}A \rightarrow \mathbb{C}_\infty \rightarrow C(\mathbb{C}_\infty) \rightarrow 0.$$

2.1.2. *The Carlitz logarithm.* The Carlitz logarithm is the rigid analytic function defined, for  $X \in \mathbb{C}_\infty$  such that  $|X| < q^{\frac{q}{q-1}}$ , by the convergent series

$$\log_C(X) = \sum_{i \geq 0} \frac{X^{q^i}}{l_i},$$

where  $(l_i)_{i \geq 0}$  is the sequence defined by  $l_0 = 1$  and, for  $i \geq 1$ ,

$$l_i = (\theta - \theta^{q^i}) l_{i-1}.$$

The convergence property is due to the fact that  $|l_i| = q^{q \frac{i-1}{q-1}}$ . We then have, for all  $X \in \mathbb{C}_\infty$  such that  $|X| < q^{\frac{q}{q-1}}$ ,

$$(7) \quad |X| = |\exp_C(X)| = |\log_C(X)|$$

and

$$(8) \quad \log_C(\exp_C(X)) = \exp_C(\log_C(X)) = X, \quad |X| < q^{\frac{q}{q-1}}.$$

2.1.3. *The Carlitz torsion.* We recall that a monic irreducible element in  $A_+$  is called a prime. For  $a \in A_+$ , we set

$$\lambda_a = \exp_C\left(\frac{\tilde{\pi}}{a}\right) \in \mathbb{C}_\infty.$$

The subfield of  $\mathbb{C}_\infty$

$$K_a = K(\lambda_a),$$

a finite extension of  $K$ , will be called the  $a$ -th *cyclotomic function field*. A reference for the basic theory of these fields is [18, Chapter 12]. Here, we recall that  $K_a/K$  is a finite abelian extension unramified outside  $a$  and  $\infty$ . Its Galois group

$$\Delta_a = \text{Gal}(K_a/K)$$

is isomorphic to the unit group

$$\left(\frac{A}{aA}\right)^\times.$$

If  $b \in A$  is prime to  $a$ , there exists  $\sigma_b \in \Delta_a$  such that:

$$\sigma_b(\lambda_a) = C_b(\lambda_a).$$

2.2. **Tate algebras.** Let  $s \geq 1$  be an integer and let  $t_1, \dots, t_s$  be  $s$  indeterminates over  $\mathbb{C}_\infty$ . If  $s = 1$ , we will also write  $t = t_1$ . Let  $L/K_\infty$  be a subextension of in  $\mathbb{C}_\infty/K_\infty$  such that  $L$  is complete with respect to  $v_\infty|_L$ . Let us consider a polynomial  $f \in L[t_1, \dots, t_s]$ , expanded as a finite sum

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} x_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}, \quad x_{i_1, \dots, i_s} \in L.$$

We set

$$v_\infty(f) = \inf\{v_\infty(x_{i_1, \dots, i_s}), i_1, \dots, i_s \in \mathbb{N}\}.$$

We then have, for  $f, g \in L[t_1, \dots, t_s]$ :

$$v_\infty(f + g) \geq \inf(v_\infty(f), v_\infty(g)).$$

Furthermore, we have

$$v_\infty(fg) = v_\infty(f) + v_\infty(g)$$

so that  $v_\infty$  is a valuation, called the *Gauss valuation*.

Let us set, for  $f \in L[t_1, \dots, t_s]$ ,  $\|f\| = q^{-v_\infty(f)}$  if  $f \neq 0$  and  $\|0\| = 0$ . We have  $\|f + g\| \leq \max\{\|f\|, \|g\|\}$ ,  $\|fg\| = \|f\|\|g\|$  and  $\|f\| = 0$  if and only if  $f = 0$ ; the function  $\|\cdot\|$  is an  $L$ -algebra norm on  $L[t_1, \dots, t_s]$  and an absolute value, called the *Gauss absolute value*.

We denote by  $\mathbb{T}_s(L)$  the completion of  $L[t_1, \dots, t_s]$  with respect to  $\|\cdot\|$ . When  $s = 1$ , we also write  $\mathbb{T}(L)$  for  $\mathbb{T}_1(L)$  and we adopt the convention that  $\mathbb{T}_0(L) = L$ . Equipped with the Gauss norm,  $\mathbb{T}_s(L)$  is an  $L$ -Banach algebra that can be identified with the set of formal series of  $f \in L[[t_1, \dots, t_s]]$  such that, writing

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} x_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}, \quad x_{i_1, \dots, i_s} \in L,$$

we have

$$\lim_{i_1 + \dots + i_s \rightarrow +\infty} x_{i_1, \dots, i_s} = 0.$$

The Gauss norm of  $f$  as above is then given by

$$\|f\| = \sup\{|x_{i_1, \dots, i_s}|, (i_1, \dots, i_s) \in \mathbb{N}^s\}.$$

When  $L = \mathbb{C}_\infty$ , we shall write  $\mathbb{T}_s, \mathbb{T}$  instead of  $\mathbb{T}_s(\mathbb{C}_\infty), \mathbb{T}_1(\mathbb{C}_\infty)$ . We refer the reader to [11, Chapter 3] for the basic properties of Tate algebras.

We denote by  $\mathfrak{o}_L$  the valuation ring of  $L$  (whose elements  $x$  are characterized by the fact that  $|x| \leq 1$ ). We denote by  $\mathfrak{m}_L$  the maximal ideal of  $\mathfrak{o}_L$  whose elements  $x$  are such that  $|x| < 1$ . Then there exists a field  $\overline{L} \subset L \cap k^{\text{ac}}$  such that we have an isomorphism of  $k$ -algebras  $\overline{L} \simeq \frac{\mathfrak{o}_L}{\mathfrak{m}_L}$ . We further denote by  $\mathfrak{o}_{\mathbb{T}_s(L)}$  the subring of elements  $f \in \mathbb{T}_s(L)$  such that  $\|f\| \leq 1$  and by  $\mathfrak{m}_{\mathbb{T}_s(L)}$  the prime ideal of  $\mathfrak{o}_{\mathbb{T}_s(L)}$  whose elements are the  $f$  such  $\|f\| < 1$ . Then, we have that

$$\overline{\mathbb{T}_s(L)} := \frac{\mathfrak{o}_{\mathbb{T}_s(L)}}{\mathfrak{m}_{\mathbb{T}_s(L)}} \simeq \overline{L}[t_1, \dots, t_s].$$

If  $L/K_\infty$  is a finite complete extension, let  $\pi_L$  be a uniformizer of  $L$ . Then, we have that  $L = \overline{L}((\pi_L))$ ,  $\mathfrak{o}_L = \overline{L}[[\pi_L]]$ . In particular:

$$\mathbb{T}_s(L) = \overline{L}[t_1, \dots, t_s]((\pi_L)).$$

2.2.1.  $k[t_1, \dots, t_s]$ -linear endomorphisms of  $\mathbb{T}_s$ . Let

$$\tau \in \text{End}_{k[t_1, \dots, t_s]\text{-lin.}}(\mathbb{T}_s)$$

be the  $k[t_1, \dots, t_s]$ -linear extension of the previous operator  $\tau$  defined as follows: for  $f \in \mathbb{T}_s$  with

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} x_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}, \quad x_{i_1, \dots, i_s} \in \mathbb{C}_\infty,$$

we set

$$\tau(f) = \sum_{i_1, \dots, i_s \in \mathbb{N}} x_{i_1, \dots, i_s}^q t_1^{i_1} \cdots t_s^{i_s}.$$

This is a  $k[t_1, \dots, t_s]$ -linear automorphism of  $\mathbb{T}_s$ . In fact,  $\tau$  is also an automorphism for the structure of  $k[t_1, \dots, t_s]$ -algebra of  $\mathbb{T}_s$ . If we set  $\mathbb{T}_s^{\tau=1} = \{f \in \mathbb{T}_s, \tau(f) = f\}$ , we have  $\mathbb{T}_s^{\tau=1} = k[t_1, \dots, t_s]$ . Observe that:

$$\|\tau^n(f)\| = \|f\|^{q^n}, \quad n \geq 0, \quad f \in \mathbb{T}_s.$$

With the action of  $\tau$  on  $\mathbb{T}_s$ , we have the non-commutative skew polynomial rings  $\mathbb{T}_s[\tau]$  and  $\mathbb{T}_s[[\tau]]$ . The latter is, as a set, constituted of the formal series  $\sum_{i \geq 0} f_i \tau^i$  with  $f_i \in \mathbb{T}_s$  for all  $i$ , and the elements of the former are the formal series whose sequences of coefficients are eventually zero. The commutation rule defining the product is given by

$$\tau f = \tau(f)\tau,$$

for  $f \in \mathbb{T}_s$ . Moreover, the ring  $\mathbb{T}_s[\tau]$  acts on  $\mathbb{T}_s$ : if  $P = \sum_{i=0}^d P_i \tau^i \in \mathbb{T}_s[\tau]$  and  $f \in \mathbb{T}_s$ , then we set

$$P(f) = \sum_{i=0}^d P_i \tau^i(f) \in \mathbb{T}_s.$$

3. DRINFELD  $A_s$ -MODULES OVER  $\mathbb{T}_s$ .

From now on, we will write  $\underline{t}_s$  for the set of variables  $\{t_1, \dots, t_s\}$ . If  $s = 0$ , this set is empty. If the value of  $s$  is clear from the context, we will more simply write  $\underline{t}$ . For example, we will write  $k[\underline{t}_s]$  or  $k[\underline{t}]$  instead of  $k[t_1, \dots, t_s]$  and  $k[\underline{t}_0] = k$ .

**Definition 3.1.** Let  $r \geq 1$  be an integer. A *Drinfeld  $A_s$ -module of rank  $r$  over  $\mathbb{T}_s$*  is a homomorphism of  $k[\underline{t}_s]$ -algebras

$$\phi : A_s \rightarrow \mathbb{T}_s[\tau]$$

defined by

$$\phi_\theta = \theta + \alpha_1 \tau + \dots + \alpha_r \tau^r,$$

for elements  $\alpha_1, \dots, \alpha_r \in \mathbb{T}_s$  with  $\alpha_r$  non-zero. The vector

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{T}_s^r$$

will be called the *parameter* of  $\phi$ . If  $r = 1$ , we identify the parameter with its unique entry  $\alpha_1$ .

Given a Drinfeld  $A_s$ -module  $\phi$  of rank  $r$  over  $\mathbb{T}_s$ , if  $M$  is a sub- $k[\underline{t}_s]$ -module of  $\mathbb{T}_s$  such that  $\phi_\theta(M) \subset M$ , we denote by  $\phi(M)$  the  $k[\underline{t}_s]$ -module  $M$  equipped with the  $A_s$ -module structure induced by  $\phi$ . In particular, we will often work in the module  $\phi(\mathbb{T}_s)$ .

If  $s \leq s'$  then we have the embedding  $\mathbb{T}_s \subset \mathbb{T}_{s'}$  induced by the inclusion  $k[\underline{t}_s] \subset k[t_1, \dots, t_s, t_{s+1}, \dots, t_{s'}]$ . Every Drinfeld  $A_s$ -module over  $\mathbb{T}_s$  can be extended in a natural way to a Drinfeld  $A_{s'}$ -module over  $\mathbb{T}_{s'}$  of the same rank, which will be denoted again by  $\phi$  for the sake of simplicity.

**Definition 3.2.** Let  $\phi, \phi'$  be two Drinfeld  $A_s$ -modules over  $\mathbb{T}_s$ . We will say that  $\phi$  is *isomorphic* to  $\phi'$  if there exists  $u \in \mathbb{T}_s^\times$  ( $\mathbb{T}_s^\times$  denotes the multiplicative group of the units of  $\mathbb{T}_s$ ) such that, in  $\mathbb{T}_s[\tau]$ :

$$\phi_\theta u = u \phi'_\theta.$$

If  $\phi$  and  $\phi'$  are isomorphic Drinfeld modules, they must have the same rank and we shall also write  $\phi \cong \phi'$ .

**Remark 3.3.** When two Drinfeld  $A_s$ -modules are isomorphic, it is understood that they are isomorphic *over  $\mathbb{T}_s$* .

Let  $\phi, \phi'$  be Drinfeld modules of rank  $r > 0$  over  $\mathbb{T}_s$  of respective parameters

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_r), \quad \underline{\alpha}' = (\alpha'_1, \dots, \alpha'_r) \in \mathbb{T}_s^r.$$

Then, the condition  $\phi \cong \phi'$  amounts to the existence of  $u \in \mathbb{T}_s^\times$  such that

$$\alpha_i \tau^i(u) = \alpha'_i u, \quad i = 1, \dots, r.$$

**Remark 3.4.** If  $s = 0$ , all the Drinfeld  $A$ -modules of rank one are isomorphic to the Carlitz module  $C$ . This is no longer true for Drinfeld  $A_s$ -modules of rank one if  $s \geq 1$ ; for example, the Drinfeld modules of rank 1 of parameters  $\alpha = 1$  (Carlitz module) and  $\alpha = t$  (both defined over  $\mathbb{T}_1 = \mathbb{T}$ ) are not isomorphic.

**Definition 3.5.** We will denote by  $C_s$  the Drinfeld module of rank one over  $\mathbb{T}_s$  with parameter

$$\alpha = (t_1 - \theta) \cdots (t_s - \theta).$$

Of course, if  $s = 0$ , we get  $C_0 = C$ , the Carlitz module. From now on, we will be focused on Drinfeld modules of rank 1.

**3.1. Exponential and logarithm.** Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one defined over  $\mathbb{T}_s$  with parameter  $\alpha \in \mathbb{T}_s$ . We also set  $\tau_\alpha = \alpha\tau$ . Explicitly, for any  $n \geq 0$ , we have

$$\tau_\alpha^n = \alpha\tau(\alpha) \cdots \tau^{n-1}(\alpha)\tau.$$

We will be particularly interested in the formal series of  $\mathbb{T}_s[[\tau]]$

$$\begin{aligned} \exp_\phi &= \sum_{n \geq 0} \frac{1}{D_n} \tau_\alpha^n, \\ \log_\phi &= \sum_{n \geq 0} \frac{1}{l_n} \tau_\alpha^n, \end{aligned}$$

respectively called the *exponential series* and the *logarithm series* associated to  $\phi$ .

It is easy to show that, in  $\mathbb{T}_s[[\tau]]$ , we have:

$$\exp_\phi \log_\phi = \log_\phi \exp_\phi = 1, \quad \exp_\phi \theta = \phi_\theta \exp_\phi.$$

A routine computation also shows the identities in  $\mathbb{T}_s[[\tau]]$ :

$$\phi_a \exp_\phi = \exp_\phi a, \quad \log_\phi \phi_a = a \log_\phi, \quad \text{for all } a \in A_s.$$

We observe that

$$\|D_n^{-1} \alpha \tau(\alpha) \cdots \tau^{n-1}(\alpha)\| = \|\alpha\| \frac{q^n - 1}{q - 1} q^{-nq^n}$$

so that for all  $f \in \mathbb{T}_s$ , the series

$$\exp_\phi(f) := \sum_{n \geq 0} \frac{\tau_\alpha^n(f)}{D_n} = \sum_{n \geq 0} \frac{\alpha \tau(\alpha) \cdots \tau^{n-1}(\alpha)}{D_n} \tau^n(f)$$

converges in  $\mathbb{T}_s$  <sup>(2)</sup>. The  $k[\underline{t}_s]$ -linear map

$$\exp_\phi : \mathbb{T}_s \rightarrow \mathbb{T}_s$$

defined by  $f \mapsto \exp_\phi(f)$  is called the *exponential function* of  $\phi$ . It is open and continuous, as the reader can easily check. Also, if  $\alpha \in \mathbb{T}_s(K_\infty)$ , then  $\exp_\phi$  induces a  $k[\underline{t}_s]$ -linear map  $\mathbb{T}_s(K_\infty) \rightarrow \mathbb{T}_s(K_\infty)$ .

If  $B$  is a normed ring with ultrametric norm  $\|\cdot\|$ , and if  $r \geq 0$ , we shall denote by  $D_B(0, r)$  (resp.  $\overline{D}_B(0, r)$ ) the set  $\{z \in B; \|z\| < r\}$  (resp.  $\{z \in B; \|z\| \leq r\}$ ). We notice that, for all  $r \geq 0$ , the sets  $D_{\mathbb{T}_s}(0, r)$  and  $\overline{D}_{\mathbb{T}_s}(0, r)$  are  $k[t_1, \dots, t_s]$ -submodules of  $\mathbb{T}_s$ . We observe that

$$\|l_n^{-1} \alpha \tau(\alpha) \cdots \tau^{n-1}(\alpha)\| = \|\alpha\| \frac{q^n - 1}{q - 1} q^{-q \frac{q^n - 1}{q - 1}}.$$

Let us set  $r = -v_\infty(\alpha)$ . For all  $f \in \mathbb{T}_s$  such that  $v_\infty(f) > \frac{r-q}{q-1}$  (that is,  $f \in D_{\mathbb{T}_s}(0, q^{\frac{q-r}{q-1}})$ ), the series

$$\log_\phi(f) := \sum_{n \geq 0} \frac{\tau_\alpha^n(f)}{l_n}$$

also converges in  $\mathbb{T}_s$ . The  $k[\underline{t}_s]$ -linear map

$$\log_\phi : D_{\mathbb{T}_s}(0, q^{\frac{q-r}{q-1}}) \rightarrow \mathbb{T}_s$$

---

<sup>2</sup>The reader is warned that we are using the same symbols to denote completely different entities. Indeed, at once,  $\exp_\phi$  denotes a formal series of  $\mathbb{T}_s[[\tau]]$  and a continuous endomorphism of  $\mathbb{T}_s$ . The same remark can be made for  $\log_\phi$ . This should not lead to confusion and contributes to easily manageable notation.

defined by  $f \mapsto \log_\phi(f)$  is called the *logarithm function* of  $\phi$ . We set

$$N_\alpha = D_{\mathbb{T}_s}(0, q^{\frac{q-r}{q-1}}).$$

As a consequence of the above discussion we have the next Lemma.

**Lemma 3.6.** *The functions  $\exp_\phi, \log_\phi$  induce isometric automorphisms of  $N_\alpha$  inverse of each other.*

3.1.1. *More about the modules  $N_\alpha$ .* Here we assume  $\alpha \in \mathbb{T}_s(K_\infty)$ . Because this will be needed in the computations of §5, we give some elementary properties of the module

$$N_\alpha(K_\infty) = D_{\mathbb{T}_s(K_\infty)}(0, q^{\frac{q-r}{q-1}}) = \{f \in \mathbb{T}_s(K_\infty), \|f\| < q^{\frac{q-r}{q-1}}\}.$$

Observe that we have a direct sum of  $k[\underline{t}_s]$ -modules:

$$\mathbb{T}_s(K_\infty) = A_s \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}.$$

Since for  $f \in \mathbb{T}_s(K_\infty)$  we have  $v_\infty(f) \in \mathbb{Z}$ , we can also write that  $N_\alpha(K_\infty) = \{f \in \mathbb{T}_s(K_\infty) : v_\infty(f) \geq \lfloor \frac{r-q}{q-1} \rfloor + 1\}$  ( $\lfloor x \rfloor$  denotes the biggest integer  $\leq x$ ). We denote by  $u(\alpha)$  the maximum of the lower integer part of  $\frac{r-q}{q-1}$  and zero:

$$u(\alpha) = \max \left\{ 0, \left\lfloor \frac{r-q}{q-1} \right\rfloor \right\}.$$

We notice that  $u(\alpha) > 0$  if and only if  $r \geq 2q - 1$ . The proof of the next Lemma is easy and left to the reader.

**Lemma 3.7.** *Let us assume that  $r \geq 2q - 1$ . Then, we have a direct sum of  $k[\underline{t}_s]$ -modules:*

$$\mathfrak{m}_{\mathbb{T}_s(K_\infty)} = N_\alpha(K_\infty) \oplus \theta^{-u(\alpha)} \langle 1, \dots, \theta^{u(\alpha)-1} \rangle_{k[\underline{t}_s]},$$

where  $\langle \dots \rangle_{k[\underline{t}_s]}$  denotes the  $k[\underline{t}_s]$ -span of a set of elements of  $\mathbb{T}_s$ .

We denote by  $M_\alpha$  the module

$$(9) \quad M_\alpha = \theta^{-u(\alpha)} \langle 1, \dots, \theta^{u(\alpha)-1} \rangle_{k[\underline{t}_s]},$$

(by convention, for  $r < 2q - 1$ , we have  $M_\alpha = (0)$ ). Then, if  $r \geq 2q - 1$ ,

$$(10) \quad \mathbb{T}_s(K_\infty) = A_s \oplus M_\alpha \oplus N_\alpha(K_\infty).$$

3.2. **An example of  $\exp_\phi$  injective and not surjective.** We shall consider here the case of  $\alpha = t \in \mathbb{T}$  and describe some properties of the associated exponential function  $\exp_\phi$ , given by

$$\exp_\phi = \sum_{i \geq 0} \frac{t^i}{D_i} \tau^i.$$

This map  $\exp_\phi : \mathbb{T} \rightarrow \mathbb{T}$  is obviously injective. Moreover, it is not surjective. To see this, let us extend  $\exp_\phi$  to  $\mathbb{C}_\infty[[t]]$  by  $k[t]$ -linearity. We have:

**Lemma 3.8.** *Let  $y$  be an element of  $\mathbb{C}_\infty$ . There exists a unique formal series  $x = \sum_{i \geq 0} x_n t^n \in \mathbb{C}_\infty[[t]]$  such that  $\exp_\phi(x) = y$ . Furthermore, let  $\epsilon \in \mathbb{R}$  be such that*

$$|y| = q^{\frac{q-\epsilon}{q-1}}.$$

Then, for all  $n \geq 0$ ,

$$|x_n| = q^{\frac{q-q^n \epsilon}{q-1}}.$$

In particular,  $x \in \mathbb{T}$  if and only if  $|y| < q^{\frac{q}{q-1}}$ .

*Proof.* Let  $x = \sum_{i \geq 0} x_i t^i \in \mathbb{C}_\infty[[t]]$  such that  $\exp_\phi(x) = y$ . Then:

$$x_0 = y,$$

and for  $n \geq 1$  :

$$x_n = -(x_0^{q^n} D_n^{-1} + x_1^{q^{n-1}} D_{n-1}^{-1} + \cdots + x_{n-1}^q D_1^{-1}).$$

One can then prove that  $|x_n| = q^{\frac{q-q^n}{q-1}\epsilon}$  by induction on  $n$ .  $\square$

**3.3. Entire operators.** Let

$$f = \sum_{i_1, \dots, i_s} f_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}$$

be an element of  $\mathbb{T}_s$  (the coefficients  $f_{i_1, \dots, i_s}$  lie in  $\mathbb{C}_\infty$ ). We say that  $f$  is an *entire function* if

$$\lim_{i_1 + \dots + i_s \rightarrow +\infty} \frac{v_\infty(f_{i_1, \dots, i_s})}{i_1 + \dots + i_s} = +\infty.$$

The subset  $\mathbb{E}_s$  of entire functions of  $\mathbb{T}_s$  is a subring containing the subring of polynomials  $\mathbb{C}_\infty[\underline{t}_s]$ . Observe that  $\tau(\mathbb{E}_s) \subset \mathbb{E}_s$ .

Let us consider a sequence of entire functions  $(F_n)_{n \geq 0}$  and an operator

$$F = \sum_{n \geq 0} F_n \tau^n \in \mathbb{E}_s[[\tau]].$$

We say that  $F$  is an *entire operator* if  $\lim_{n \rightarrow \infty} v_\infty(F_n)q^{-n} = +\infty$ . In particular, for all  $f \in \mathbb{T}_s$ ,  $F(f) = \sum_{n \geq 0} F_n \tau^n(f)$  converges in  $\mathbb{T}_s$ .

**Lemma 3.9.** *Let  $F = \sum_{n \geq 0} F_n \tau^n$  be an entire operator. Then:  $F(\mathbb{E}_s) \subset \mathbb{E}_s$ .*

*Proof.* With  $\underline{i}$  we shall denote here a multi-index  $(i_1, \dots, i_s)$  whose entries are non-negative integers. We denote by  $|\underline{i}|$  the integer  $i_1 + \dots + i_s$ , and if  $\underline{i}, \underline{j}$  are such multi-indices, then  $\underline{i} + \underline{j}$  denotes their component-wise sum. We also write  $\underline{t}^{\underline{i}}$  for the monomial  $t_1^{i_1} \cdots t_s^{i_s}$ . Hence, we have  $f = \sum_{\underline{j}} f_{\underline{j}} \underline{t}^{\underline{j}}$ . We expand each entire function  $F_n$  in series

$$F_n = \sum_{\underline{i}} F_{n, \underline{i}} \underline{t}^{\underline{i}},$$

where by hypothesis,  $\lim_{|\underline{i}| \rightarrow +\infty} \frac{v_\infty(F_{n, \underline{i}})}{|\underline{i}|} = +\infty$ . Now, we verify easily that  $F(f) = \sum_{\underline{k}} c_{\underline{k}} \underline{t}^{\underline{k}} \in \mathbb{T}_s$  where

$$c_{\underline{k}} = \sum_{\underline{i} + \underline{j} = \underline{k}} \sum_{n \geq 0} F_{n, \underline{i}} f_{\underline{j}}^{q^n}.$$

Since

$$v_\infty(c_{\underline{k}}) \geq \inf_{\underline{i} + \underline{j} = \underline{k}, n \geq 0} (v_\infty(F_{n, \underline{i}}) + q^n v_\infty(f_{\underline{j}})),$$

and since  $\lim_{n \rightarrow \infty} v_\infty(F_n)q^{-n} = \lim_{|\underline{i}| \rightarrow +\infty} v_\infty(f_{\underline{i}})|\underline{i}|^{-1} = \lim_{|\underline{j}| \rightarrow +\infty} v_\infty(f_{n_0, \underline{j}})|\underline{j}|^{-1} = +\infty$  for all  $n_0 \in \mathbb{N}$ , we get  $\lim_{|\underline{k}| \rightarrow +\infty} v_\infty(c_{\underline{k}})|\underline{k}|^{-1} = +\infty$  and thus  $F(f) \in \mathbb{E}_s$ .  $\square$

Let  $\alpha$  be an element of  $\mathbb{E}_s$ . Then,  $\alpha \tau(\alpha) \cdots \tau^{i-1}(\alpha)$  is also entire for all  $i$  and

$$\lim_{i \rightarrow \infty} v_\infty(\alpha \tau(\alpha) \cdots \tau^{i-1}(\alpha) D_i^{-1}) q^{-i} = +\infty.$$

Therefore we deduce from the latter Lemma the following Proposition which will be of some help later on in this paper:

**Proposition 3.10.** *Let  $\phi$  be a Drinfeld module of rank one over  $\mathbb{T}_s$  and let  $\alpha$  be its parameter. Let us assume that  $\alpha \in \mathbb{E}_s$ . Then:*

$$\exp_\phi(\mathbb{E}_s) \subset \mathbb{E}_s.$$

#### 4. L-SERIES VALUES

In this section we consider a Drinfeld  $A_s$ -module  $\phi$  of rank one over  $\mathbb{T}_s$  with parameter  $\alpha \in A_s^*$  <sup>(3)</sup>. We are going to associate to such a parameter  $\alpha$  an *L-series value*. Although the hypotheses can be relaxed, the analysis of this particular case will be enough for the purpose of this article.

**4.1. Definition of L-series values.** In a fixed algebraic closure  $k_s^{ac}$  of  $k_s$  we can find elements  $x_1, \dots, x_r$  and  $\beta \in k[\underline{t}_s]^*$  so that, in  $k_s^{ac}[\theta]$ ,

$$(11) \quad \alpha = \beta(x_1 - \theta) \cdots (x_r - \theta).$$

We define

$$\rho_\alpha : A \rightarrow k_s^{ac}$$

by  $\rho_\alpha(0) = 0$  and

$$\rho_\alpha(a) = \beta^{\deg_\theta(a)} a(x_1) \cdots a(x_r), \quad a \in A^*.$$

An alternative way to write it is:

$$\rho_\alpha(a) = \text{Res}_\theta(a, \alpha) \in k[\underline{t}_s],$$

where  $\text{Res}_\theta(P, Q)$  denotes the resultant of two polynomials in the indeterminate  $\theta$  <sup>(4)</sup>. In particular, with  $P$  a prime of  $A$ ,  $\rho_\alpha(P) = 0$  if and only if  $P$  divides  $\alpha$  in  $A_s$ .

If  $a, b \in A$ , then  $\rho_\alpha(ab) = \rho_\alpha(a)\rho_\alpha(b)$  and if  $\alpha_1, \alpha_2$  are polynomials of  $A_s$ , then

$$\rho_{\alpha_1\alpha_2}(a) = \rho_{\alpha_1}(a)\rho_{\alpha_2}(a), \quad a \in A.$$

**Definition 4.1.** Let  $\phi$  be the Drinfeld  $A_s$ -module of rank one of parameter  $\alpha$ , let  $n \geq 1$  be an integer. The *L-series value of order  $n$  associated to  $\phi$*  is the unit of norm one of  $\mathbb{T}_s(K_\infty)$  defined by

$$L(n, \phi) = \sum_{a \in A_+} \frac{\rho_\alpha(a)}{a^n} = \prod_{\mathfrak{p} \text{ prime of } A} \left(1 - \frac{\rho_\alpha(\mathfrak{p})}{\mathfrak{p}^n}\right)^{-1}.$$

By [4, Lemma 4], we also have that  $L(-n, \phi) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \rho_\alpha(a) a^n$  converges in  $\mathbb{T}_s$  for  $n \geq 0$  and is in fact in  $A_s$ .

**Remark 4.2.** We can also associate *L-series values*  $L(n, \phi)$  to Drinfeld  $R_s$ -modules of rank one defined over  $R_s$  (we recall that  $R_s = k_s[\theta]$ ). In the sequel, we will also work with such modules and *L-series values*, but the most interesting examples discussed here will arise from the case of  $A_s$ -modules defined over  $A_s$ .

The value  $L(1, \phi)$  will be one of the main objects of interest of the present paper.

#### 4.2. Examples.

<sup>3</sup>For an integral domain  $R$ ,  $R^*$  denotes the multiplicative monoid whose elements are the non-zero elements of  $R$ .

<sup>4</sup>We recall that if  $P = P_0\theta^d + P_1\theta^{d-1} + \cdots + P_d$  and  $Q = Q_0\theta^r + Q_1\theta^{r-1} + \cdots + Q_r$  are polynomials with roots respectively  $\zeta_i$  and  $x_j$ , then, for the resultant  $\text{Res}_\theta(P, Q)$ , we have the identity  $\text{Res}_\theta(P, Q) = P_0^r \prod_i Q(\zeta_i) = (-1)^{dr} Q_0^d \prod_j P(x_j)$ .



4.2.1. *Examples with  $s = 0$ .* If  $s = 0$  and  $\alpha = 1$ , we have  $\phi = C$  and

$$L(n, \phi) = L(n, C) = \zeta_C(n),$$

where  $\zeta_C(n)$  is, for  $n > 0$ , the *Carlitz zeta value*

$$\zeta_C(n) = \sum_{a \in A^+} a^{-n} \in 1 + \theta^{-1}k[[\theta^{-1}]].$$

If  $\alpha \in A^*$ , then we can write  $\alpha = \beta \mathfrak{p}_1^{\nu_1} \cdots \mathfrak{p}_m^{\nu_m}$  with  $\beta \in k^\times$ , for primes  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  of respective degrees  $d_1, \dots, d_m$  so that  $\sum_i d_i \nu_i = r = \deg_\theta(\alpha)$ , we have, for  $a \in A^+$ ,

$$\rho_\alpha(a) = \beta^{\deg_\theta(a)} \prod_{i=1}^m \prod_{j=1}^{d_i} a(\zeta_{i,j})^{\nu_i},$$

where  $\zeta_{i,1}, \dots, \zeta_{i,d_i}$  are the zeros of  $\mathfrak{p}_i$  in  $k^{ac}$  for all  $i$ . This implies, in the case  $\beta = 1$  (that is,  $\alpha \in A^+$ ), that the series  $L(n, \phi)$  is the special value of a Dirichlet  $L$ -series:

$$L(n, \phi) = \sum_{a \in A_+} a^{-n} \prod_{i=1}^m \prod_{j=1}^{d_i} a(\zeta_{i,j})^{\nu_i} \in K_\infty.$$

4.2.2. *Case of  $\alpha = t$ .* It is understood here that  $s = 1$  so that we are in  $\mathbb{T} = \mathbb{T}_1$ . This case directly refers to the example of Drinfeld module  $\phi$  treated in 3.2. We have then

$$L(n, \phi) = \sum_{d \geq 0} t^d \sum_{a \in A_{+,d}} a^{-n} \in \mathbb{T} \cap K[[t]]$$

if  $n > 0$ . It is easy to see that

$$L(1, \phi) = \sum_{i \geq 0} t^i \ell_i^{-1} = \log_\phi(1) \in \mathbb{T}.$$

We have that the series

$$L(-j, \phi) := \sum_{d \geq 0} t^d \sum_{a \in A_{+,d}} a^j$$

defines an element of  $A[t]$  for  $j \geq 0$ , and

$$L(-j, \phi) = z(t^{-1}, -j)$$

where the function  $z$  is defined as in Goss' book [12, Remark 8.12.1]. In loc. cit. Goss computes recursively the polynomial  $z(t^{-1}, -j) \in A[t]$  for all  $j \geq 0$ .

4.2.3. *Case in which  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$ .* In this case we have  $\phi = C_s$  with  $C_s$  the Drinfeld modules of Definition 3.5 and we recover the functions

$$L(\chi_{t_1} \cdots \chi_{t_s}, n) = \sum_{a \in A_+} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{a^n} \in \mathbb{T}_s(K_\infty)^\times$$

studied in [4] and where for all  $a \in A$ ,  $\chi_{t_i}(a) = a(t_i)$ . The case  $s = 1$  and  $\alpha = t - \theta$  yields the functions  $L(\chi_t, n)$  of [15].

4.2.4. *A further example.* We shall also trace a connection with the Goss zeta functions, especially the functions considered by Goss in [13], see also [4, Section 2.1]. We recall, from Section 2.1 of loc. cit. the definition of the  $L$ -series  $L(\chi_{t_1} \cdots \chi_{t_s}, x, y)$ , with  $(x, y)$  in the topological group  $\mathbb{C}_\infty^\times \times \mathbb{Z}_p$  denoted by  $\mathbb{S}_\infty$  there:

$$L(\chi_{t_1} \cdots \chi_{t_s}, x, y) = \sum_{k \geq 0} x^{-k} \sum_{a \in A_{k,+}} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \langle a \rangle^{-y},$$

where  $\langle a \rangle$  is the 1-unit  $a/\theta^{\deg_\theta(a)}$ , and its  $p$ -adic exponentiation by  $-y$  is well defined. For fixed  $(x, y) \in \mathbb{S}_\infty$ , the above series is a well defined unit element of  $\mathbb{T}_s$ . Thanks to [4, Corollary 8], we know that for any choice of  $n \geq 1$ , the above series in  $\mathbb{T}_s$  also defines an entire function  $\mathbb{C}_\infty^s \rightarrow \mathbb{C}_\infty$ .

We have, for  $\beta \in k^\times$ :

$$\begin{aligned} L(\chi_{t_1} \cdots \chi_{t_s}, \beta^{-1}\theta^n, n) &= \sum_{k \geq 0} \beta^k \theta^{-kn} \sum_{a \in A_{k,+}} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \theta^{kn} a^{-n} \\ &= \sum_{k \geq 0} \beta^k \sum_{a \in A_{k,+}} \chi_{t_1}(a) \cdots \chi_{t_s}(a) a^{-n}. \end{aligned}$$

This equals  $L(n, C_s)$  if  $\beta = 1$ .

## 5. THE CLASS NUMBER FORMULA

We introduce the *class module* and the *unit module* associated to a given Drinfeld module of rank one of parameter  $\alpha \in A_s^*$ . We then give a *class number formula* which relates  $L(1, \phi)$  to these objects.

5.1. **Class and unit modules.** Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one with parameter  $\alpha \in A_s^*$ . Recall that  $r = -v_\infty(\alpha) \in \mathbb{N}$ . The definitions below are inspired by Taelman's work [20, 21].

5.1.1. *The class module.* We define the class module  $H_\phi$  as the quotient of  $A_s$ -modules:

$$H_\phi := \frac{\phi(\mathbb{T}_s(K_\infty))}{\exp_\phi(\mathbb{T}_s(K_\infty)) + \phi(A_s)}.$$

5.1.2. *The unit module.* It is the  $A_s$ -submodule of  $\mathbb{T}_s(K_\infty)$  defined by:

$$U_\phi := \{f \in \mathbb{T}_s(K_\infty); \exp_\phi(f) \in A_s\}.$$

5.1.3. *Computation of  $H_\phi$  when  $r < 2q - 1$ .* We use here the computations of §3.1.1. We have

$$\mathbb{T}_s(K_\infty) = A_s \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}.$$

If  $r < 2q - 1$ , then  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)} \subset N_\alpha$  and  $U_\phi \cap \mathfrak{m}_{\mathbb{T}_s(K_\infty)} = (0)$ . The function  $\exp_\phi$  induces an injective morphism

$$\frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}} \rightarrow \frac{\mathbb{T}_s(K_\infty)}{A_s \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}}$$

the cokernel of which is  $H_\phi$ . But the module on the right is zero; we deduce that  $H_\phi = (0)$  and that  $U_\phi$  is non-trivial.

**Lemma 5.1.** *Let us suppose that  $r \geq 2q - 1$ , so that  $u(\alpha) > 0$ . The exponential function  $\exp_\phi : \mathbb{T}_s(K_\infty) \rightarrow \mathbb{T}_s(K_\infty)$  induces an injective homomorphism of  $k[\underline{t}_s]$ -modules*

$$\frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus N_\alpha(K_\infty)} \rightarrow \frac{\mathbb{T}_s(K_\infty)}{A_s \oplus N_\alpha(K_\infty)}$$

whose cokernel is  $H_\phi$ .

*Proof.* This is plain by the fact that  $\exp_\phi$  restricted to  $N_\alpha$  is an isometric isomorphism and the fact that  $\exp_\phi^{-1}(A_s) \cap \mathbb{T}_s(K_\infty) = U_\phi$ .  $\square$

**Corollary 5.2.** *For all Drinfeld modules  $\phi$  as above,  $H_\phi$  is a finitely generated  $k[\underline{t}_s]$ -module of rank  $\leq u(\alpha)$ .*

**Remark 5.3.** When  $r \geq 2q - 1$  we have constructed a short exact sequence of  $A_s$ -modules

$$(12) \quad 0 \rightarrow \frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus N_\alpha(K_\infty)} \rightarrow \frac{\phi(\mathbb{T}_s(K_\infty))}{\phi(A_s) \oplus N_\alpha(K_\infty)} \rightarrow H_\phi \rightarrow 0.$$

On the other hand, there is an isomorphism of  $k[\underline{t}_s]$ -modules between  $M_\alpha$  (the module defined in (9)) and

$$\frac{\mathbb{T}_s(K_\infty)}{A_s \oplus N_\alpha(K_\infty)}.$$

Therefore the finitely generated  $k[\underline{t}_s]$ -modules

$$\frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus N_\alpha(K_\infty)}, \quad H_\phi$$

have finite ranks and their ranks add up to  $u(\alpha)$ , which is the rank of  $M_\alpha$ . This tells us in particular that  $U_\phi$  is non-zero.

Observe that  $\mathbb{T}_s(K_\infty) \subset K_{s,\infty}$  and that  $\tau$  extends to a continuous morphism of  $k_s$ -algebras still denoted by  $\tau : K_{s,\infty} \rightarrow K_{s,\infty}$ . If  $M \subset \mathbb{T}_s(K_\infty)$  is a  $k[\underline{t}_s]$ -module, we denote by  $k_s M \subset K_{s,\infty}$  the  $k_s$ -module generated by  $M$ .

**Proposition 5.4.** *The following properties hold.*

- (1) *The  $R_s$ -module  $k_s U_\phi$  is free of rank one.*
- (2) *We have:*

$$k_s U_\phi = \{f \in K_{s,\infty}, \exp_\phi(f) \in R_s\}.$$

*Proof.*

(1). We assume that  $r \geq 2q - 1$ . Since  $U_\phi$  is non-trivial, there exists an element  $f \in k_s U_\phi$  with  $\|f\| > 0$  minimal. Indeed, by the fact that  $\exp_\phi$  induces an isometric isomorphism of  $N_\alpha$ ,  $k_s U_\phi$  is discrete, that is,  $k_s U_\phi \cap \mathfrak{m}_{K_{s,\infty}}^n = (0)$  for  $n$  big enough.

Let  $g$  be another element of  $k_s U_\phi$ . Then, since  $K_{s,\infty} = R_s \oplus \mathfrak{m}_{K_{s,\infty}}$ , there exists a polynomial  $h$  of  $R_s$  such that  $g = hf + b$  where  $b \in K_{s,\infty}$  is such that  $\|b\| < \|f\|$ . Since  $k_s U_\phi$  is an  $R_s$ -module, we get  $b \in k_s U_\phi$  so that  $b = 0$ . This means that  $k_s U_\phi$  is free of rank one. The proof for  $r < 2q - 1$  is similar and left to the reader.

(2). Observe that  $k_s \mathbb{T}_s(K_\infty)$  is dense in  $K_{s,\infty}$ , thus:

$$K_{s,\infty} = k_s \mathbb{T}_s(K_\infty) + \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}.$$

It is clear that  $k_s U_\phi \subset \{f \in K_{s,\infty}, \exp_\phi(f) \in R_s\}$ . Now, let  $f \in K_{s,\infty}$  be such that  $\exp_\phi(f) \in R_s$ . We can write  $f$  as a sum  $g + h$ , where  $g \in k_s \mathbb{T}_s(K_\infty)$  and  $h \in \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}$ . We get:

$$\exp_\phi(h) = \exp_\phi(f) - \exp_\phi(g) \in k_s \mathbb{T}_s(K_\infty).$$

This implies that:

$$\exp_\phi(h) \in k_s \mathfrak{m}_{\mathbb{T}_s(K_\infty)}^{u(\alpha)+1} = \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1} \cap k_s \mathbb{T}_s(K_\infty).$$

Therefore  $h \in k_s \mathfrak{m}_{\mathbb{T}_s(K_\infty)}^{u(\alpha)+1}$  and thus  $f \in k_s \mathbb{T}_s(K_\infty)$ . We conclude that  $f \in k_s U_\phi$ .  $\square$

**Corollary 5.5.** *The  $A_s$ -module  $U_\phi$  is free of rank one.*

*Proof.* By Proposition 5.4, we have  $k_s U_\phi = f R_s$ , with  $f \in \mathbb{T}_s(K_\infty)$ . Without loss of generality, we can also suppose that if  $h$  divides  $f$  in  $\mathbb{T}_s(K_\infty)$ , with  $h \in k[\underline{t}_s]$ , then  $h \in k^\times$ . Clearly,  $U_\phi \supset f A_s$ . Let us consider now an element  $g$  of  $U_\phi$ . We have that  $g \in f R_s$ , and we can write  $g = a f / \delta$  where  $a \in A_s$  and  $\delta \in k[\underline{t}_s]^*$ . This means that  $\delta$  divides  $a f$  in  $\mathbb{T}_s(K_\infty)$  which is a unique factorization domain. So  $\delta$  must divide  $a$  in  $A_s$  and we get  $g \in f A_s$ .  $\square$

**Remark 5.6.** Let's consider the following  $R_s$ -module:

$$V_\phi = \frac{\phi(K_{s,\infty})}{\phi(R_s) + \exp_\phi(K_{s,\infty})}.$$

Just as in the proof of the second part of Proposition 5.4, we see in fact that for all  $n \geq 1$ ,

$$K_{s,\infty} = k_s \mathbb{T}_s(K_\infty) + \mathfrak{m}_{K_{s,\infty}}^n.$$

For all  $n$  big enough,  $\exp_\phi$  induces an isometric automorphism of  $\mathfrak{m}_{K_{s,\infty}}^n$  (for instance, it suffices to take  $n \geq u(\alpha) + 1$ ). Therefore, for such a choice of  $n$ , we have the isomorphism of  $k_s$ -vector spaces

$$V_\phi = \frac{k_s \mathbb{T}_s(K_\infty) + \mathfrak{m}_{K_{s,\infty}}^n}{R_s + \exp_\phi(k_s \mathbb{T}_s(K_\infty)) + \mathfrak{m}_{K_{s,\infty}}^n} \cong \frac{k_s \mathbb{T}_s(K_\infty)}{R_s + \exp_\phi(k_s \mathbb{T}_s(K_\infty))}.$$

This implies that we have an isomorphism of  $R_s$ -modules:

$$(13) \quad V_\phi \simeq H_\phi \otimes_{k[\underline{t}_s]} k_s.$$

Note that  $\dim_{k_s} H_\phi \otimes_{k[\underline{t}_s]} k_s \leq u(\alpha)$  by Corollary 5.2.

This yields:

**Corollary 5.7.** *The  $R_s$ -module  $V_\phi$  is a finite dimensional  $k_s$ -vector space of dimension at most  $u(\alpha)$ .*

**5.2. Local factors of the L-series values.** Let  $R$  be a unitary commutative ring. Let  $M$  be a finitely generated  $R$ -module. As *Fitting ideal* of  $M$  we mean the *initial Fitting ideal* as defined in [14, Chapter XIX]. By Chapter XIX, Corollary 2.9 of loc. cit., if  $M$  is a finite direct sum of cyclic modules,

$$M = \bigoplus_{i=1}^n \frac{R}{\mathfrak{a}_i}, \quad \mathfrak{a}_i \text{ ideal of } R,$$

then

$$\text{Fitt}_R(M) = \mathfrak{a}_1 \cdots \mathfrak{a}_n.$$

Let  $\theta$  be an indeterminate over a field  $F$ , write  $R = F[\theta]$ , and let us consider an  $R$ -module  $M$  which also is an  $F$ -vector space of finite dimension. Let  $e_\theta$  be the endomorphism of  $M$  induced by the multiplication by  $\theta$ . Then, we write

$$[M]_R = \det_R(Z - e_\theta|_M)|_{Z=\theta} \in R$$

for the characteristic polynomial of  $e_\theta$ , where the indeterminate  $Z$  is replaced with  $\theta$ . This is a monic polynomial in  $\theta$  of  $R = F[\theta]$  and its important property is that it is the monic generator of  $\text{Fitt}_R(M)$ .

Let  $\alpha$  be an element of  $R_s^*$  and let us consider the Drinfeld  $R_s$ -module of rank one and parameter  $\alpha$ , that is, the injective homomorphism of  $k_s$ -algebras

$$\phi : R_s \rightarrow \text{End}_{k_s\text{-lin.}}(K_{s,\infty})$$

given by  $\phi_\theta = \theta + \alpha\tau$ . For all  $a \in A$ , the resultant  $\rho_\alpha(a) = \text{Res}_\theta(a, \alpha)$  is a well defined element of  $k_s$  making the series (and the corresponding eulerian product)

$$L(n, \phi) = \sum_{a \in A_+} \rho_\alpha(a) a^{-n} = \prod_P \left(1 - \frac{\rho_\alpha(P)}{P^n}\right)^{-1}, \quad n > 0$$

convergent in  $K_{s,\infty}$ .

**Lemma 5.8.** *Let  $P$  be a prime of  $A$  of degree  $d$ . Then, the following congruence holds:*

$$\phi_P \equiv \rho_\alpha(P)\tau^d \pmod{PR_s[\tau]}.$$

*Proof.* Let  $a \in A_+$  of degree  $d$ . We expand in  $R_s[\tau]$ :

$$\phi_a = \sum_{i=0}^d (a)_i \tau^i,$$

where it is easy to see that  $(a)_0 = a$ ,  $(a)_i \in R_s$  for  $i = 0, \dots, d$ , and

$$(a)_d = \alpha\tau(\alpha) \cdots \tau^{d-1}(\alpha).$$

From the relation  $\phi_a \phi_\theta = \phi_\theta \phi_a$  we get, by induction on  $i = 1, \dots, d-1$ ,

$$(a)_i = \frac{\alpha\tau((a)_{i-1}) - \tau^{i-1}(\alpha)(a)_{i-1}}{\theta^{q^i} - \theta}.$$

If  $P$  is a prime of degree  $d$ , since  $P$  does not divide  $\theta^{q^i} - \theta$  if  $i < d$ , we get for  $i = 0, \dots, d-1$ ,  $(P)_i \equiv 0 \pmod{PR_s}$ . This implies that

$$\phi_P \equiv \alpha\tau(\alpha) \cdots \tau^{d-1}(\alpha)\tau^d \pmod{PR_s[\tau]}.$$

Now, we observe that, if  $\zeta_1, \dots, \zeta_d$  are the roots of  $P$  in  $k^{ac}$  and if  $\zeta$  is one of these roots,

$$\begin{aligned} \rho_\alpha(P) &= \text{Res}_\theta(P, \alpha) \\ &= \prod_{j=1}^d \alpha|_{\theta=\zeta_j} \\ &= \alpha\tau(\alpha) \cdots \tau^{d-1}(\alpha)|_{\theta=\zeta} \\ &\equiv \alpha\tau(\alpha) \cdots \tau^{d-1}(\alpha) \pmod{PR_s}. \end{aligned}$$

□

**Lemma 5.9.** *Let  $L$  be a field and let  $\sigma \in \text{Aut}(L)$ . Set  $F = \{x \in L, \sigma(x) = x\}$ . Let  $r \geq 0$  be an integer strictly less than the order of  $\sigma$ , let  $a_0, \dots, a_{r-1}$  be  $r$  elements in  $L$ . Then*

$$V = \left\{ x \in L; \sigma^r(x) + \sum_{i=0}^{r-1} a_i \sigma^i(x) = 0 \right\}$$

is an  $F$ -vector space of dimension not exceeding  $r$ .

*Proof.* A sketch of proof will be enough as this is essentially well known, see [17, §1.2]. Let  $n \geq 1$  be an integer and let  $\mathcal{A}$  be a matrix with entries in  $L$ . Let  $v_1, \dots, v_r$  be vectors of  $L^n$  such that  $\sigma(v_i) = \mathcal{A}v_i, i = 1, \dots, r$ . Then by the proof of [17, Lemma 1.7], if the vectors  $v_1, \dots, v_r$  are linearly dependent over  $L$ , they are also linearly dependent over  $F$ . This implies that the  $F$ -vector space  $W = \{v \in L^n, \sigma(v) = \mathcal{A}v\}$  satisfies:

$$\dim_F(W) \leq n.$$

Let  $\mathcal{A}$  be the companion matrix of the equation

$$\sigma^r(x) + \sum_{i=0}^{r-1} a_i \sigma^i(x) = 0$$

(see [17, p. 8]). Let  $W = \{v \in L^r, \sigma(v) = \mathcal{A}v\}$ . Then the map  $V \mapsto W$ ,

$$x \mapsto {}^t(x, \sigma(x), \dots, \sigma^{r-1}(x))$$

(the sign  ${}^t$  means transposition) is an isomorphism of  $F$ -vector spaces.  $\square$

**Lemma 5.10.** *Let  $\phi$  be a Drinfeld  $R_s$ -module of rank one over  $K_{s,\infty}$  with parameter  $\alpha \in R_s^*$ . Let  $P$  be a prime of  $A$  of degree  $d$ . Then, we have an isomorphism of  $R_s$ -modules:*

$$\phi \left( \frac{R_s}{PR_s} \right) \cong \frac{R_s}{(P - \rho_\alpha(P))R_s}.$$

*Proof.* By Lemma 5.8, we have:

$$(P - \rho_\alpha(P)) \cdot \phi \left( \frac{R_s}{PR_s} \right) = (0).$$

We set  $L = R_s/PR_s$ . Then  $\tau$  induces an automorphism of  $L$  and  $\{x \in L, \tau(x) = x\} = k_s$ , so that  $\tau \in \text{Gal}(L/k_s)$  is of order  $d$ . Also,  $\phi$  induces a morphism of  $k_s$ -algebras  $\phi : R_s \rightarrow L[\tau]$ . For  $a$  in  $R_s$  we set  $\text{Ker}(\phi_a) = \{x \in L; \phi_a(x) = 0\}$ . We write  $b = P - \rho_\alpha(P)$ . We notice that  $d = \dim_{k_s}(L) = \deg_\theta(b)$  and  $L = \text{Ker}(\phi_b)$ . We have, by Lemma 5.9, for all  $a \in R_s$ ,  $\dim_{k_s}(\text{Ker}(\phi_a)) \leq \deg_\theta(a)$ . This implies (see for example [12, proof of Theorem 6.3.2]) that we have an isomorphism of  $R_s$ -modules  $\text{Ker}(\phi_b) \cong R_s/bR_s$ .  $\square$

Let  $P$  be a prime of  $A$ . By Lemma 5.10, we have  $[\phi(\frac{R_s}{PR_s})]_{R_s} = P - \rho_\alpha(P)$ . We now introduce an infinite product running over the primes  $P$  of  $A$ :

$$(14) \quad \mathcal{L}(\phi/R_s) = \prod_P \frac{[\frac{R_s}{PR_s}]_{R_s}}{[\phi(\frac{R_s}{PR_s})]_{R_s}}.$$

**Proposition 5.11.** *Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one of parameter  $\alpha \in A_s^*$ . The product  $\mathcal{L}(\phi/R_s)$  in (14) converges in  $K_{s,\infty}$  and we have*

$$\mathcal{L}(\phi/R_s) = L(1, \phi).$$

*Proof.* We have, by Lemma 5.10,

$$\frac{[\frac{R_s}{PR_s}]_{R_s}}{[\phi(\frac{R_s}{PR_s})]_{R_s}} = \frac{P}{P - \rho_\alpha(P)} = \left(1 - \frac{\rho_\alpha(P)}{P}\right)^{-1}.$$

The factors of the infinite product defining  $\mathcal{L}(\phi/R_s)$  agree with the eulerian factors of  $L(1, \phi)$ . Since the product  $L(1, \phi)$  converges in  $K_{s, \infty}$ , this implies that the product  $\mathcal{L}(\phi/R_s)$  converges to  $L(1, \phi)$  in  $K_{s, \infty}$ .  $\square$

**5.3. The class number formula.** An element

$$f = \sum_{i \geq i_0} f_i \theta^{-i} \in K_{s, \infty}, \quad f_i \in k_s$$

is *monic* if the leading coefficient  $f_{i_0}$  is equal to one. We shall write

$$[R_s : k_s U_\phi]_{R_s}$$

for the unique monic element  $f$  in  $K_{s, \infty}$  such that  $k_s U_\phi = f R_s$ , the existence of which is guaranteed by the first part of Proposition 5.4.

We recall from Corollary 5.7 that the  $R_s$ -module  $V_\phi$  is a  $k_s$ -vector space of finite dimension  $\leq u(\alpha)$ . The next Theorem directly follows from Theorem 11.4, proved in the appendix by Florent Demeslay, by means of Proposition 5.11.

**Theorem 5.12** (The class number formula). *Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one of parameter  $\alpha \in A_s^*$ . The following equality holds in  $K_{s, \infty}$ :*

$$L(1, \phi) = [V_\phi]_{R_s} [R_s : k_s U_\phi]_{R_s}.$$

Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$  with parameter  $\alpha \in A_s^*$ . The following Corollary to the class number formula will be crucial:

**Corollary 5.13.** *We have:*

$$\exp_\phi(L(1, \phi)) \in A_s.$$

*Proof.* By definition,  $[R_s : k_s U_\phi]_{R_s} R_s = k_s U_\phi$  and obviously,  $[V_\phi]_{R_s} \in R_s$ . Thus:

$$L(1, \phi) \in k_s U_\phi.$$

By Part (2) of Proposition 5.4,  $\exp_\phi(L(1, \phi)) \in R_s$ . At once, by construction,  $L(1, \phi) \in \mathbb{T}_s(K_\infty)$  so that  $\exp_\phi(L(1, \phi)) \in \mathbb{T}_s(K_\infty)$ . But then,

$$\exp_\phi(L(1, \phi)) \in R_s \cap \mathbb{T}_s = A_s.$$

$\square$

**Remark 5.14.** If  $\alpha$  is as in (11) and  $0 \leq r \leq q-1$  ( $r = -v_\infty(\alpha)$ ) we have that  $L(1, \phi) \in N_\alpha$  and  $L(1, \phi) - 1 \in \mathfrak{m}_{\mathbb{T}_s(K_\infty)}$  (see §3.1.1). Since  $\exp_\phi$  is an isometric automorphism of  $N_\alpha$  we also get  $\exp_\phi(L(1, \phi)) - 1 \in \mathfrak{m}_{\mathbb{T}_s(K_\infty)}$  but  $R_s \cap \mathfrak{m}_{\mathbb{T}_s(K_\infty)} = (0)$ . We have obtained the identity

$$(15) \quad \exp_\phi(L(1, \phi)) = 1.$$

This can be rewritten as

$$(16) \quad L(1, \phi) = \log_\phi(1)$$

because  $1 \in N_\alpha$  thanks again to the hypothesis on  $r$ . Similar formulas have been observed by Perkins in [16].

5.3.1. *The circular unit module.* This is the sub- $A_s$ -module

$$U_\phi^c = L(1, \phi)A_s \subset U_\phi.$$

Since  $L(1, \phi)$  is a unit of norm one in  $\mathbb{T}_s(K_\infty)$ , we have the decomposition

$$\mathbb{T}_s(K_\infty) = U_\phi^c \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}.$$

Let  $f$  be in  $U_\phi$ . We can then write, according to the above decomposition,

$$(17) \quad f = aL(1, \phi) + \mu, \quad a \in A_s, \quad \mu \in \mathfrak{m}_{\mathbb{T}_s(K_\infty)}.$$

If  $1 \leq r < 2q - 1$ , Since  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)} \subset N_\alpha$ , we have the decomposition:

$$\exp_\phi(\mathbb{T}_s(K_\infty)) = \exp_\phi(U_\phi^c \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}) = \exp_\phi(U_\phi^c) \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}$$

which implies that  $\mu = 0$  in (17) (by the fact that  $\exp_\phi$  induces an isometric isomorphism of  $N_\alpha$ , hence on  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)}$ ). This yields that  $\exp_\phi(U_\phi) = \exp_\phi(U_\phi^c)$ . We have obtained:

**Proposition 5.15.** *If  $r < 2q - 1$  then  $H_\phi = (0)$  and  $\exp_\phi(U_\phi) = \exp_\phi(U_\phi^c)$ .*

**Remark 5.16.** If the function  $\exp_\phi$  is injective we get, with the hypotheses of the above Proposition,  $U_\phi = U_\phi^c$ . With the study of *uniformizable* Drinfeld modules of rank one of §6, we will be able to say when this occurs.

**Proposition 5.17.** *Let us suppose that  $r \geq 2q - 1$ . Then  $\frac{U_\phi}{U_\phi^c}$  and  $H_\phi$  are finitely generated  $k[\underline{t}_s]$ -modules of equal rank at most  $u(\alpha)$ .*

*Proof.* We have that  $u(\alpha) \geq 1$ . By Lemma 3.7, we have a direct sum of  $k[\underline{t}_s]$ -modules:

$$\mathbb{T}_s(K_\infty) = \theta^{-u(\alpha)}U_\phi^c \oplus N_\alpha(K_\infty).$$

Then, we have an exact sequence of  $k[\underline{t}_s]$ -modules:

$$(18) \quad 0 \rightarrow \frac{U_\phi}{U_\phi^c} \rightarrow \frac{\theta^{-u(\alpha)}U_\phi^c \oplus N_\alpha(K_\infty)}{U_\phi^c \oplus N_\alpha(K_\infty)} \rightarrow \frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus N_\alpha(K_\infty)} \rightarrow 0.$$

Observe that the  $k[\underline{t}_s]$ -module in the middle is free of rank  $u(\alpha)$ . Thus  $\frac{U_\phi}{U_\phi^c}$  is a finitely generated torsion-free  $k[\underline{t}_s]$ -module of rank not bigger than  $u(\alpha)$ .

Recall from Remark 5.3 that if  $r \geq 2q - 1$ , then  $\exp_\phi$  induces an exact sequence of finitely generated  $k[\underline{t}_s]$ -modules:

$$0 \rightarrow \frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus N_\alpha(K_\infty)} \rightarrow \frac{\mathbb{T}_s(K_\infty)}{A_s \oplus N_\alpha(K_\infty)} \rightarrow H_\phi \rightarrow 0.$$

Since there is an isomorphism of  $k[\underline{t}_s]$ -modules

$$\frac{\theta^{-u(\alpha)}U_\phi^c \oplus N_\alpha(K_\infty)}{U_\phi^c \oplus N_\alpha(K_\infty)} \cong \frac{\mathbb{T}_s(K_\infty)}{A_s \oplus N_\alpha(K_\infty)}$$

the modules  $\frac{U_\phi}{U_\phi^c}$  and  $H_\phi$  have the same rank over  $k[\underline{t}_s]$ . □

We deduce, from Theorem 5.12:

**Corollary 5.18.** *We have*

$$[H_\phi \otimes_{k[\underline{t}_s]} k_s]_{R_s} = [k_s U_\phi : k_s U_\phi^c]_{R_s}.$$



*Proof.* We have:

$$[R_s : k_s U_\phi]_{R_s} = [R_s : R_s L(1, \phi)]_{R_s} \left[ \frac{k_s U_\phi}{R_s L(1, \phi)} \right]_{R_s}^{-1}.$$

Then, by Theorem 5.12, we obtain:

$$\left[ \frac{k_s U_\phi}{R_s L(1, \phi)} \right]_{R_s} = [V_\phi]_{R_s}.$$

It remains to apply isomorphism (13).  $\square$

**Remark 5.19.** Observe that, by Corollary 5.5 and since  $L(1, \phi) \in \mathbb{T}_s(K_\infty)^\times$ , the  $k[\underline{t}_s]$ -module  $\frac{U_\phi}{U_\phi^c}$  is free. Therefore, by Corollary 5.18, we have:

$$[H_\phi \otimes_{k[\underline{t}_s]} k_s]_{R_s} \in A_s \cap \mathbb{T}_s(K_\infty)^\times.$$

To proceed further, we need a precise characterization of the  $A_s$ -modules whose exponential function is injective. This is in fact closely related to the non-surjectivity of  $\exp_\phi$  and will be investigated in the next Section.

## 6. UNIFORMIZABLE DRINFELD MODULES OF RANK ONE.

In this Section we consider general Drinfeld modules of rank one defined over  $\mathbb{T}_s$ . If  $F \in \mathbb{T}_s[\tau]$ , we set

$$\mathbb{T}_s^{F=1} = \{g \in \mathbb{T}_s, F(g) = g\}.$$

This is a  $k[\underline{t}_s]$ -submodule of  $\mathbb{T}_s$ . Observe that if  $\alpha = 1$  (case in which  $\phi = C$ , the Carlitz module), then  $\exp_C : \mathbb{T}_s \rightarrow \mathbb{T}_s$  is a surjective homomorphism of  $k[\underline{t}_s]$ -modules. Furthermore

$$\text{Ker}(\exp_C) = \tilde{\pi}A_s,$$

and

$$\mathbb{T}_s^{\tau=1} = k[\underline{t}_s].$$

We are going to study a class of Drinfeld  $A_s$ -modules of rank one defined over  $\mathbb{T}_s$  which have similar properties.

**Definition 6.1.** Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$ . We say that  $\phi$  is *uniformizable* if  $\exp_\phi$  is surjective on  $\mathbb{T}_s$ .

**Proposition 6.2.** *Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$  and let  $\alpha \in \mathbb{T}_s \setminus \{0\}$  be its parameter. The following conditions are equivalent.*

- (1)  $\phi$  is uniformizable,
- (2)  $\mathbb{T}_s^{\tau_\alpha=1} \neq (0)$ ,
- (3)  $\alpha \in \mathbb{T}_s^\times$ ,
- (4)  $\phi$  is isomorphic to the Carlitz module.

*Proof.* We begin by proving that (3) implies (4). Since  $\alpha \in \mathbb{T}_s^\times$ , there exists  $x \in \mathbb{C}_\infty^\times$  such that  $v_\infty(\alpha - x) > v_\infty(\alpha)$ . Observe that:

$$v_\infty \left( \frac{\tau^i(\alpha)}{x^{q^i}} - 1 \right) \geq q^i (v_\infty(\alpha - x) - v_\infty(\alpha)).$$

Thus the product  $\prod_{i \geq 0} \left( \frac{x^{q^i}}{\tau^i(\alpha)} \right)$  converges in  $\mathbb{T}_s^\times$ . Now let us choose an element  $\gamma \in \mathbb{C}_\infty^\times$  such that:

$$\gamma^{q-1} = x.$$

We set:

$$(19) \quad \omega_\alpha = \gamma \prod_{i \geq 0} \left( \frac{x^{q^i}}{\tau^i(\alpha)} \right) \in \mathbb{T}_s^\times.$$

At first sight, these functions depend on the choice of  $x$  but it is easy to show that they are defined up to a scalar factor of  $k^\times$ . We also notice that, for  $\alpha_1, \alpha_2 \in \mathbb{T}_s^\times$ ,

$$\omega_{\alpha_1 \alpha_2} \in k^\times \omega_{\alpha_1} \omega_{\alpha_2}, \quad \alpha_1, \alpha_2 \in \mathbb{T}_s^\times.$$

Then:

$$(20) \quad \tau(\omega_\alpha) = \alpha \omega_\alpha.$$

This implies that, in  $\mathbb{T}_s[[\tau]]$ , we have:

$$C_\theta \omega_\alpha = \omega_\alpha \phi_\theta,$$

that is,  $\phi$  and  $C$  are isomorphic.

In fact, it is also easy to show that (4) implies (3). Indeed, assuming that the Drinfeld module of rank one  $\phi$  is isomorphic to  $C$ , we see directly that the parameter  $\alpha$  of  $\phi$  must satisfy  $\tau(u)/u = \alpha$  for a unit  $u$  of  $\mathbb{T}_s$ , but this implies that  $\alpha$  is a unit as well.

Next, we prove that (4) implies (1). By hypothesis, there exists  $\vartheta \in \mathbb{T}_s^\times$  such that, in  $\mathbb{T}_s[[\tau]]$ :

$$\vartheta \tau_\alpha = \tau \vartheta$$

We get in  $\mathbb{T}_s[[\tau]]$ :

$$\exp_\phi = \vartheta^{-1} \exp_C \vartheta.$$

Since  $\exp_C$  is surjective on  $\mathbb{T}_s$ , we obtain that  $\exp_\phi$  is also surjective.

We prove that (1) implies (2). Let us then suppose that  $\exp_\phi$  is surjective. The map  $\mathbb{T}_s \rightarrow \mathbb{T}_s$  defined by  $f \mapsto \phi_\theta(f)$  is surjective. Explicitly, for all  $f \in \mathbb{T}_s$  there exists  $g \in \mathbb{T}_s$  such that:

$$\alpha \tau(g) + \theta g = f.$$

Therefore:

$$\alpha \tau \left( \frac{g}{\lambda_\theta} \right) - \frac{g}{\lambda_\theta} = \frac{f}{\lambda_\theta^q} = -\frac{f}{\lambda_\theta \theta}$$

(we recall that we have set  $\lambda_\theta = \exp_C(\tilde{\pi}/\theta)$ ). This implies that the map  $\tau_\alpha - 1$  is surjective on  $\mathbb{T}_s$ .

Hence, we can assume, without loss of generality, that  $\|\alpha\| = 1$ . Let us suppose, by contradiction, that  $\mathbb{T}_s^{\tau_\alpha=1} = (0)$ . Then, the map  $f \mapsto \tau_\alpha(f) - f$  is an isomorphism of  $k[\underline{t}_s]$ -modules which satisfies  $\|\alpha \tau(f) - f\| < 1$  if and only if  $\|f\| < 1$  and  $\|\alpha \tau(f) - f\| > 1$  if and only if  $\|f\| > 1$ . In particular, this map induces an automorphism of the  $k[\underline{t}_s]$ -module  $\{f \in \mathbb{T}_s : \|f\| = 1\}$ .

Reducing modulo  $\mathfrak{m}_{\mathbb{T}_s}$ , the above map induces the  $k[\underline{t}_s]$ -linear endomorphism of  $k^{ac}[\underline{t}_s]$  given by  $\bar{f} \mapsto \bar{\alpha} \tau(\bar{f}) - \bar{f}$  where  $\bar{\alpha} \neq 0$  is the image of  $\alpha$  in  $k^{ac}[\underline{t}_s]$  and  $\bar{f} \in k^{ac}[\underline{t}_s]$ . One can easily verify that this endomorphism is not an automorphism.

This constitutes a contradiction with the assumption that  $\mathbb{T}_s^{\tau_\alpha=1} = (0)$ .

We finally prove that (2) implies (3). Let  $g$  be a non-zero element of  $\mathbb{T}_s^{\tau_\alpha=1}$ . By  $\alpha \tau(g) = g$  we deduce that  $\alpha \tau(\alpha) \cdots \tau^{n-1}(\alpha) \tau^n(g) = g$  for all  $n$ . If  $\alpha$  were not a unit, this would contradict the finiteness of the number of irreducible factors of  $g$ .  $\square$

**Remark 6.3.** The following observation will be extensively used in the rest of this paper. Let  $\phi$  be a uniformizable Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$  of parameter  $\alpha \in \mathbb{T}_s^\times$ . Then the function  $\exp_\phi$  induces an exact sequence of  $A_s$ -modules:

$$0 \rightarrow \frac{\tilde{\pi}}{\omega_\alpha} A_s \rightarrow \mathbb{T}_s \rightarrow \phi(\mathbb{T}_s) \rightarrow 0,$$

where  $\omega_\alpha$  is defined as in (19). In this case, the module  $\mathbb{T}_s^{\tau_\alpha=1}$  is obviously given by:

$$\mathbb{T}_s^{\tau_\alpha=1} = \frac{1}{\omega_\alpha} k[\underline{t}_s].$$

**Definition 6.4.** Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$ , then  $\mathbb{T}_s$  is an  $A_s$ -module via  $\phi$ , thus if  $f \in A_s \setminus \{0\}$ , we define the  $A_s$ -module of  $f$ -torsion  $\phi[f]$  by:

$$\phi[f] = \{g \in \mathbb{T}_s, \phi_f(g) = 0\}.$$

**Corollary 6.5.** *Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$  of parameter  $\alpha \in \mathbb{T}_s^*$ . The following assertions are equivalent:*

- (1)  $\phi$  is uniformizable,
- (2)  $\mathbb{T}_s^{\tau_\alpha=1}$  is a  $k[\underline{t}_s]$ -module of rank one,
- (3) for any  $f \in A_s \cap \mathbb{T}_s^\times$  we have an isomorphism of  $A_s$ -modules:

$$\phi[f] \simeq \frac{A_s}{fA_s},$$

- (4) there exists  $f \in A_s \cap \mathbb{T}_s^\times$  such that  $\phi[f] \neq (0)$ .

*Proof.* The equivalence of the properties (1) and (2) is already covered by the proof of Proposition 6.2 and by the Remark 6.3.

We show that (1) implies (3). By Remark 6.3 we have that

$$\text{Ker}(\exp_\phi) = \frac{\tilde{\pi}}{\omega_\alpha} A_s.$$

Notice also that  $\exp_\phi$  is surjective so that if  $f \in A_s \cap \mathbb{T}_s^\times$ , we also have that

$$\exp_\phi^{-1}(\phi[f]) = \frac{\tilde{\pi}}{f\omega_\alpha} A_s.$$

It is obvious that (3) implies (4); it remains to show that (4) implies (1). Let  $\alpha$  be the parameter of  $\phi$  and let us assume that for some  $f \in A_s \cap \mathbb{T}_s^\times$ , we have  $\phi[f] \neq (0)$ ; let  $g \in \mathbb{T}_s \setminus \{0\}$  be such that  $\phi_f(g) = 0$ .

We can write, in  $\mathbb{T}_s[\tau]$ :

$$\phi_f = \sum_{i=0}^d c_i \tau_\alpha^i = \sum_{i \geq 0} \alpha \cdots \tau^{i-1}(\alpha) c_i \tau^i,$$

where for  $i = 0, \dots, d$ ,  $c_i \in A_s$ , and  $c_0 = f, c_d \in k^\times$ . We get:

$$\sum_{i=1}^d \alpha \cdots \tau^{i-1}(\alpha) c_i \tau^i(g) = -fg.$$

Since  $f \in \mathbb{T}_s^\times$ , we get  $g = \alpha g_1$ ,  $g_1 \in \mathbb{T}_s \setminus \{0\}$ . Thus (we recall that  $\mathbb{T}_s$  is a unique factorization domain):

$$\sum_{i=1}^d \tau(\alpha) \cdots \tau^i(\alpha) c_i \tau^i(g_1) = -f g_1.$$

Therefore  $g_1 = \tau(\alpha) g_2$ , and  $\alpha \tau(\alpha)$  divides  $g$  in  $\mathbb{T}_s$ . Thus for any  $n \geq 1$ ,  $\alpha \cdots \tau^n(\alpha)$  divides  $g$  in  $\mathbb{T}_s$ . Therefore  $\alpha \in \mathbb{T}_s^\times$  and  $\phi$  is uniformizable by Proposition 6.2.  $\square$

**Remark 6.6.** The definition of uniformizable  $A_s$  module is motivated by Anderson's result [1, Theorem 4]. It is an interesting question to characterize higher rank "uniformizable" Drinfeld modules over  $\mathbb{T}_s$ , that is, Drinfeld modules over  $\mathbb{T}_s$  which have surjective associated exponential function.

**6.1. The elements  $\omega_\alpha$ .** Let  $\alpha \in \mathbb{T}_s(K_\infty)^\times$ . Then, there exists  $\gamma \in \mathbb{T}_s(K_\infty)^\times$  monic (as a power series in  $\theta^{-1}$ ) such that  $\alpha = \rho \gamma$  for some  $\rho \in k^\times$ .

The function  $\omega_\alpha$  defined in (19) is determined up to a factor in  $k^\times$ . Let  $x = \rho \theta^r$  with  $r = -v_\infty(\alpha)$ . Then  $\|\alpha - x\| < \|\alpha\|$ . Therefore:

$$(21) \quad \omega_\alpha = \tilde{\rho} \lambda_\theta^r \prod_{i \geq 0} \left( \frac{\tau^i(\alpha)}{\rho \theta^{r q^i}} \right)^{-1},$$

where  $\tilde{\rho} \in k^{ac}$  is such that  $\tilde{\rho}^{q-1} = (-1)^r \rho$ . From this it is apparent that

$$\omega_\alpha \in \tilde{\rho} \lambda_\theta^r \mathbb{T}_s(K_\infty)^\times$$

and that

$$(22) \quad \|\omega_\alpha\| = q^{\frac{r}{q-1}}.$$

**Remark 6.7.** We have defined  $\tilde{\pi}$  up to a choice of a  $(q-1)$ -th root of  $-\theta$ , which is  $\lambda_\theta$ . When  $\rho = (-1)^r$ , we set  $\tilde{\rho} = 1$ , this defines  $\omega_\alpha$  uniquely by the product (21). From now on, we will always use this normalization.

The proof of the next Lemma is easy and left to the reader.

**Lemma 6.8.** *Let  $\alpha$  be in  $\mathbb{T}_s(K_\infty)^\times$ . The following conditions are equivalent:*

- (1)  $\frac{\tilde{\pi}}{\omega_\alpha} \in \mathbb{T}_s(K_\infty)$ ,
- (2) *If  $r = -v_\infty(\alpha)$ ,  $r \equiv 1 \pmod{q-1}$  and  $-\alpha$  is monic.*

**6.2. Examples.** If  $s = 1$  and  $\alpha = t - \theta$ , we have an important example:

$$(23) \quad \omega_\alpha = \omega = \lambda_\theta \prod_{i \geq 0} \left( 1 - \frac{t}{\theta^{q^i}} \right)^{-1} \in \lambda_\theta K_\infty.$$

This function, introduced in Anderson and Thakur paper [3, Proof of Lemma 2.5.4 p. 177], was also used extensively in [15, 4]. For general  $s$ , it is important to also consider the function  $\omega_\alpha$  associated with the choice of  $\alpha = \beta(t_1 - \theta) \cdots (t_s - \theta)$ ,  $\beta \in k^\times$ . In this case,

$$(24) \quad \omega_\alpha = \tilde{\beta} \omega(t_1) \cdots \omega(t_s),$$

where  $\tilde{\beta}^{q-1} = \beta$ .

## 7. UNIFORMIZABLE DRINFELD MODULES OF RANK ONE DEFINED OVER $A_s$

In this Section, we fix  $\phi$  a uniformizable Drinfeld  $A_s$ -module of rank one *defined over  $A_s$* . Its parameter  $\alpha$  lies in  $A_s \cap \mathbb{T}_s^\times$  and we have a factorization (11) with  $\beta \in k^\times$ . Recall that  $r = -v_\infty(\alpha) \in \mathbb{Z}_{\geq 0}$ .

**7.1. The torsion case.** Here, we assume that  $\alpha$  is as in (11) with  $\beta = 1$ , and  $r \equiv 1 \pmod{q-1}$ . If  $r = 0$  then  $q = 2$  and  $L(1, \phi) = \zeta_C(1)$ ; in this case  $\exp_C(\zeta_C(1)) = 1$  is a torsion point for the Carlitz module.

We begin by the case  $r = 1$ . We then have  $\alpha = x - \theta$  with  $x \in k[\underline{t}_s]$ , so that  $\rho_\alpha(a) = a(x)$  for  $a \in A$ , and in particular,  $\rho_\alpha(\theta) = x$ .

**Lemma 7.1.** *If  $\beta = 1$ ,  $r = 1$ , we have the identity:*

$$L(1, \phi)\omega_\alpha = \frac{\tilde{\pi}}{\theta - x}.$$

*Proof.* Since  $\rho_\alpha(\theta) = x$ , we have:

$$(x - \theta)\omega_\alpha \tilde{\pi}^{-1} = \frac{-\theta\lambda_\theta}{\tilde{\pi}} \prod_{i \geq 1} \left(1 - \frac{x}{\theta q^i}\right)^{-1} \equiv \frac{-\theta\lambda_\theta}{\tilde{\pi}} \equiv -1 \pmod{\mathfrak{m}_{\mathbb{T}_s(K_\infty)}}.$$

Hence, if we write

$$F = (x - \theta)L(1, \phi)\omega_\alpha \tilde{\pi}^{-1},$$

we have

$$F \equiv -1 \pmod{\mathfrak{m}_{\mathbb{T}_s(K_\infty)}}.$$

Since  $\exp_\phi(L(1, \phi)) = 1$  (Remark 5.14), we get  $(x - \theta)L(1, \phi) = (\rho_\alpha(\theta) - \theta)L(1, \phi) \in \text{Ker}(\exp_\phi)$  so that  $F \in A_s$ , and  $F = -1$  (notice that 1 is a point of  $(x - \theta)$ -torsion for  $\phi$  in this case).  $\square$

The above Lemma implies [15, Theorem 1].

**Proposition 7.2.** *The following properties hold.*

- (1) *If  $r \geq q$ ,  $U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A_s$  and  $\exp_\phi(U_\phi) = 0$ .*
- (2) *If  $r = 1, q$ , then  $H_\phi = (0)$  and  $U_\phi^c = U_\phi$ .*
- (3) *If  $r \geq 2q - 1$ , then  $H_\phi$  is a free  $k[\underline{t}_s]$ -module of rank  $u(\alpha)$ . Moreover  $U_\phi/U_\phi^c$  is isomorphic to  $H_\phi$  as a  $k[\underline{t}_s]$ -module.*

*Proof.* (1). By Lemma 6.8 and the identity (22) we see that  $\|\tilde{\pi}/\omega_\alpha\| = q^{u(\alpha)}$  so that

$$(25) \quad \mathbb{T}_s(K_\infty) = \frac{\tilde{\pi}}{\omega_\alpha} A_s \oplus N_\alpha(K_\infty).$$

Let  $f$  be in  $U_\phi$  and let us write  $f = f_1 + f_2$  with  $f_1 \in \frac{\tilde{\pi}}{\omega_\alpha} A_s$  and  $f_2 \in N_\alpha$ . Since  $f_1$  is in the kernel of  $\exp_\phi$ , we have  $\exp_\phi(f) = \exp_\phi(f_2) \in N_\alpha$ . Since  $\exp_\phi$  induces an isometric automorphism of  $N_\alpha$ , the condition  $\exp_\phi(f) \in A_s$  yields  $f_2 = 0$ . This means that  $U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A_s$  as expected.

(2). If  $r = 1, q$ , by Proposition 5.15, we know that  $H_\phi = (0)$  and  $\exp_\phi(U_\phi) = \exp_\phi(U_\phi^c)$ . If  $r = 1$ , thanks to Lemma 7.1, we see that

$$U_\phi^c = \frac{\tilde{\pi}}{(x - \theta)\omega_\alpha} A_s,$$

where  $x$  is the unique root of  $\alpha$ . This module contains the torsion submodule of  $\exp_\phi$  and  $U_\phi = U_\phi^c$  in this case.

If  $r = q$ , we already know that  $U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A[\underline{t}_q]$ . Then,

$$L(1, \phi)A[\underline{t}_q] = U_\phi^c \subset U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A[\underline{t}_q].$$

There exists  $a \in A[\underline{t}_q]$  such that  $L(1, \phi) = a\tilde{\pi}/\omega_\alpha$ . But  $\|L(1, \phi)\| = 1$  and  $\|\tilde{\pi}/\omega_\alpha\| = q^{\frac{q-r}{q-1}} = 1$ . Therefore,  $a \in k[\underline{t}_s]$ . Let us now observe that  $L(1, \phi) \in 1 + \mathfrak{m}_{\mathbb{T}_q(K_\infty)}$  and  $\tilde{\pi}/\omega_\alpha \in -1 + \mathfrak{m}_{\mathbb{T}_q(K_\infty)}$ . This implies that  $a = -1$ .

(3). By (25),  $N_\alpha(K_\infty) = \exp_\phi(\mathbb{T}_s(K_\infty))$  and therefore is a  $A_s$ -module via  $\phi$ . We have:

$$H_\phi = \frac{\phi(\mathbb{T}_s(K_\infty))}{\phi(A_s) \oplus \phi(N_\alpha(K_\infty))}.$$

In particular,  $H_\phi$ , as a  $k[\underline{t}_s]$ -module, is isomorphic to  $A_s\theta^{-u(\alpha)}/A_s$ , and hence is free of rank  $u(\alpha)$ . Finally, in (18), the third arrow maps to zero so that  $U_\phi/U_\phi^c \cong H_\phi$  as a  $k[\underline{t}_s]$ -module.  $\square$

We deduce the next Corollaries.

**Corollary 7.3.** *Let  $\alpha$  be as in (11) with  $\beta = 1$  and  $r = q$ . Let  $\phi$  be a Drinfeld module of rank one with parameter  $\alpha$ . Then, the following formula holds:*

$$L(1, \phi) = -\frac{\tilde{\pi}}{\omega_\alpha}.$$

**Corollary 7.4.** *If  $\beta = 1$ ,  $r \equiv 1 \pmod{q-1}$  and  $r \geq q$ , then*

$$L(1, \phi) \in A_s \frac{\tilde{\pi}}{\omega_\alpha}.$$

**Remark 7.5.** The results of Lemma 7.1 and Corollary 7.4 also justify the terminology *torsion case* because  $\exp_\phi(L(1, \phi))$  is a torsion point for  $\phi$ . And by Lemma 6.8, for  $\phi$  a uniformizable Drinfeld  $A_s$ -module of rank one defined over  $A_s$ ,  $\exp_\phi(L(1, \phi))$  is a torsion point for  $\phi$  if and only if  $r \equiv 1 \pmod{q-1}$  and  $\beta = 1$ .

The terminology is also suggested by the behavior of the higher Carlitz zeta values  $\zeta_C(n) = \sum_{a \in A_+} a^{-n}$ . In [3], Anderson and Thakur constructed a point  $z_n \in \text{Lie}(C^{\otimes n})(K_\infty)$  with last entry  $\Pi(n-1)\zeta_C(n)$  such that  $\text{Exp}_n(z_n) = Z_n$ , where  $\Pi$  denotes the *Carlitz factorial* (see §9.4),  $\text{Exp}_n$  denotes the exponential function of  $C^{\otimes n}$  and  $Z_n$  is a certain  $A$ -valued *special point* of  $C^{\otimes n}$  explicitly constructed in loc. cit. We have that  $Z_n$  is a torsion point for  $C^{\otimes n}$  if and only if  $q-1$  divides  $n$  (see Anderson and Thakur, [3, Corollary 3.8.4] and J. Yu, [24, Corollary 2.6]).

The methods of [4, Theorem 4] can probably be used to show that, more generally,  $\tilde{\pi}^{-n}L(n, \phi)\omega_\alpha$  is rational if and only if  $r \equiv n \pmod{q-1}$  and  $-\alpha$  is monic. It would be nice to see if these are also related to torsion points for the tensor powers of the modules  $\phi$  as in [3] in the case  $s = 0$ .

7.1.1. *The polynomials  $\mathbb{B}_\phi$ .* For  $r \geq q$ , by (1) of Proposition 7.2, we have that

$$\mathbb{B}_\phi := (-1)^{\frac{r-1}{q-1}} L(1, \phi)\omega_\alpha \tilde{\pi}^{-1} \in A_s.$$

We also set for  $r = 1$ :

$$\mathbb{B}_\phi = \frac{1}{\theta - x},$$

where  $x \in k[\underline{t}_s]$  is the unique root of  $\alpha$  in  $k_s$  as a polynomial in  $\theta$ .

The polynomials  $\mathbb{B}_\phi$  have already been studied in [4] in the case of  $\alpha = (t_1 - \theta) \cdots (t_r - \theta)$  with  $r = s$ . If  $r = q$ , we can even deduce the exact value of  $\mathbb{B}_\phi$  (see Corollary 7.3):  $\mathbb{B}_\phi = 1$ . More generally, we have the following:

**Lemma 7.6.** *The polynomial  $\mathbb{B}_\phi \in k[\underline{t}_s][\theta]$  is monic polynomial of degree  $u(\alpha) = \frac{r-q}{q-1}$  in the indeterminate  $\theta$ .*

*Proof.* Let us write:

$$\mathbb{B}_\phi = \sum_{i=0}^m a_i \theta^i,$$

where  $a_i \in k[\underline{t}_s]$ , and  $a_m \neq 0$ . We have that  $v_\infty(\tilde{\pi}^{-1}L(1, \phi)\omega_\alpha) = v_\infty(\mathbb{B}_\phi) = \frac{q-r}{q-1}$ , which implies that

$$m = \frac{r-q}{q-1}$$

and

$$a_m \in k^\times.$$

To compute  $a_m$  it suffices to compute the leading coefficient of the expansion of  $\tilde{\pi}^{-1}L(1, \phi)\omega_\alpha$  as a series of  $k_s((\theta^{-1}))$ . This computation is easy and left to the reader.  $\square$

The importance of the polynomials  $\mathbb{B}_\phi$  is dictated by the next Theorem.

**Theorem 7.7.** *Let  $r \geq q$ , then*

$$\text{Fitt}_{A_s}(H_\phi) = \mathbb{B}_\phi A_s.$$

*Proof.* Since  $H_\phi$  is free of rank  $u(\alpha)$  (part 3 of Proposition 7.2), we have that  $\text{Fitt}_{A_s}(H_\phi) = FA_s$ , where

$$F = \det_{k[\underline{t}_s]}(Z - \phi_\theta|_{H_\phi})|_{Z=\theta},$$

and  $F$  has degree  $u(\alpha)$  as a polynomial in  $\theta$ . Again by (1) of Proposition 7.2, we have  $U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A_s$  and

$$[R_s : k_s U_\phi]_{R_s} = [R_s : \frac{\tilde{\pi}}{\omega_\alpha} R_s]_{R_s} = (-1)^{\frac{r-1}{q-1}} \frac{\tilde{\pi}}{\omega_\alpha},$$

because  $(-1)^{\frac{r-1}{q-1}} \frac{\tilde{\pi}}{\omega_\alpha}$  is monic. It remains to apply Corollary 5.18.  $\square$

We presently do not know much about the irreducible factors of the polynomials  $\mathbb{B}_\phi$ . However, if  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$  (that is, if  $\phi = C_s$ ) with  $s \equiv 1 \pmod{q-1}$ , more can be said.

**Lemma 7.8.** *If  $s \equiv 1 \pmod{q-1}$ ,  $s \geq q$ , then  $\mathbb{B}_{C_s}$  has no non-trivial divisor in  $A$ .*

*Proof.* In this case, we have  $\omega_\alpha = \omega(t_1) \cdots \omega(t_s)$  and  $L(1, C_s) = L(\chi_{t_1} \cdots \chi_{t_s}, 1)$  in the notation of [4]. We can evaluate at  $t_1 = \cdots = t_s = \zeta \in k$  and, by the fact that  $s \equiv 1 \pmod{q-1}$ ,  $L(1, C_s)|_{t_i=\zeta} = \sum_{a \in A_+} \frac{a(\zeta)}{a}$ . By using Lemma 7.1, we obtain

$$\sum_{a \in A_+} \frac{a(\zeta)}{a} = \frac{\tilde{\pi}}{(\theta - \zeta)\omega(\zeta)}.$$

Therefore,

$$\mathbb{B}_{C_s}|_{t_i=\zeta} = \tilde{\pi}^{-1}L(1, C_s)\omega_\alpha|_{t_i=\zeta} = \omega(\zeta)^{r-1}(\theta - \zeta)^{-1} = (\theta - \zeta)^{\frac{r-q}{q-1}} \in A.$$

Now, if  $a \in A^*$  divides  $\mathbb{B}_{C_s}$  in  $A_s$ , then  $a$  divides  $(\theta - \zeta)^{\frac{r-1}{q-1}}$  for all  $\zeta \in k$  so that  $a \in k^\times$ .  $\square$

**Proposition 7.9.** *Let us suppose that  $s \geq 2q - 1$  and  $s \equiv 1 \pmod{q - 1}$ . Then the  $A$ -module  $H_{C_s}$  is torsion-free and not finitely generated.*

*Proof.* We recall that  $H_{C_s}$  is a  $k[\underline{t}_s]$ -module free of rank  $\frac{s-q}{q-1} \geq 1$ . Thus the first assertion is a consequence of Lemma 7.8. According to [14, Chapter XIX, Proposition 2.5], we have the inclusions:

$$(26) \quad \text{Ann}_{A_s}(H_{C_s})^{u(\alpha)} \subset \mathbb{B}_{C_s}A_s \subset \text{Ann}_{A_s}(H_{C_s}).$$

Let us suppose by contradiction that  $H_{C_s}$  is finitely generated as an  $A$ -module. Then, the rank is, say,  $m$ . Let  $g_1, \dots, g_m$  be generators of this  $A$ -module. Then, for all  $i = 1, \dots, m$  the elements  $g_i, t_1g_i, t_1^2g_i, \dots, t_1^m g_i$  are  $A$ -linearly dependent. In particular, there exists a non-zero element  $F \in A[t_1]$  such that

$$(C_s)_F(f) = 0,$$

for all  $f \in H_{C_s}$ , that is  $F \in \text{Ann}_{A_s}(H_{C_s})$ , so that  $\mathbb{B}_{C_s}$  divides  $F^{u(\alpha)}$ .

Since  $s \geq 2q - 1$ ,  $\mathbb{B}_{C_s}$ , which is symmetric in  $t_1, \dots, t_s$ , has positive degree in  $\theta$  but cannot be independent of the variables  $t_1, \dots, t_s$  because this would contradict the first part of the statement of the Proposition (the polynomial is not divisible by any non-constant polynomial of  $A$ ); since  $s > 1$ , the only symmetric polynomials of  $A[t_1]$  are in  $A$  and in particular, we find that  $\mathbb{B}_{C_s} \in A_s \setminus A[t_1]$ . But then, this is again contradictory because a polynomial of  $A[t_1]$  cannot be divisible by a polynomial of  $A_s \setminus A[t_1]$  if  $s \geq 2$  ( $A_s$  is factorial).  $\square$

**7.2. The non-torsion case.** We suppose here that  $s \not\equiv 1 \pmod{q-1}$ . We consider again the case  $\phi = C_s$ . Here we suppose that  $s \geq 2q - 1$ , and we consider the  $A$ -torsion submodule  $M$  of  $H_{C_s}$ . This also is an  $A_s$ -module, and we know that it is a finitely generated  $k[\underline{t}_s]$ -module (Corollary 5.2). Moreover:

**Proposition 7.10.** *The  $A_s$ -submodule  $M$  is a torsion  $k[\underline{t}_s]$ -module.*

*Proof.* We must show that  $M \otimes_{k[\underline{t}_s]} k_s = (0)$ . By the isomorphism (13), it is enough to show that  $[V_{C_s}]_{R_s}$  has no divisors in  $A$ . By part (1) of Proposition 5.4, we know that  $k_s U_{C_s}$  is an  $R_s$ -module free of rank one. The class number formula, Theorem 5.12, yields that

$$[V_{C_s}]_{R_s} k_s U_{C_s} = R_s L(1, C_s).$$

Let  $a \in A^*$  be a divisor of  $[V_{C_s}]_{R_s}$ . Then,  $a^{-1}L(1, C_s) \in k_s U_{C_s}$ . By part (2) of Proposition 5.4, we have that  $\exp_{C_s}(a^{-1}L(1, C_s)) \in R_s$ . Since we also have, at once,  $\exp_{C_s}(a^{-1}L(1, C_s)) \in \mathbb{T}_s(K_\infty)$ , we obtain that

$$\exp_{C_s}(a^{-1}L(1, C_s)) \in A_s.$$

We claim that this is impossible unless  $a \in k^\times$ . To see this, we appeal to Proposition 3.10 which says us that  $\exp_{C_s}(a^{-1}L(1, C_s))$  extends to an entire function in  $s$  variables.



It is here that we use the particular shape of the parameter  $\alpha$ . Indeed,  $\alpha$  vanishes at  $t_s = \theta$ . The evaluation at  $t_s = \theta$  is licit in  $\exp_{C_s}(a^{-1}L(1, C_s))$  and yields an entire function in  $s - 1$  variables  $t_1, \dots, t_s$ . Since

$$(27) \quad \exp_{C_s}(a^{-1}L(1, C_s)) = \sum_{k \geq 0} \sum_{i+j=k} \frac{\alpha \tau(\alpha) \cdots \tau^{i-1}(\alpha)}{a^{q^i} D_i} \sum_{b \in A_{+,j}} \frac{\chi_{t_1}(b) \cdots \chi_{t_s}(b)}{b^{q^j}},$$

evaluating at  $t_s = \theta$  gives,

$$\exp_{C_s}(a^{-1}L(1, C_s))|_{t_s=\theta} = a^{-1}L(0, C_{s-1}) \in a^{-1}k[t_1, \dots, t_{s-1}] \cap R_s$$

where  $C_{s-1}$  is the Drinfeld module of rank one of parameter  $\alpha' = (t_1 - \theta) \cdots (t_{s-1} - \theta)$ . If by contradiction  $a \notin k^\times$ , then  $a^{-1}k[t_1, \dots, t_{s-1}] \cap R_s = (0)$  and

$$\exp_{C_s}(a^{-1}L(1, C_s))|_{t_s=\theta} = L(0, C_{s-1}) = 0.$$

However,  $L(0, C_{s-1}) \neq 0$ . Indeed, by hypothesis,  $s - 1 \not\equiv 0 \pmod{q - 1}$  and it is well known from the theory of the values of the Goss zeta function at negative integers that the value

$$L(0, C_{s-1})|_{t_1=\dots=t_{s-1}=\theta} = \zeta_C(1 - s) \in A$$

is non-zero, which yields a contradiction. Therefore,  $a \in k^\times$ .  $\square$

## 8. ON THE LOG-ALGEBRAICITY THEOREM OF ANDERSON

The class number formula (Theorem 5.12) implies a refined version of Anderson's *log-algebraicity Theorem* in the case of the Carlitz module (cf. [2, Theorem 3]; see also loc. cit. Proposition 8). Let  $r \in \mathbb{N}$ ,  $X_1, \dots, X_r, Z$  be "symbols", let us consider the polynomial ring in infinitely many indeterminates

$$\mathcal{B}_r = \mathbb{C}_\infty[X_1, \dots, X_r, \tau(X_1), \dots, \tau(X_r), \tau^2(X_1), \dots, \tau^2(X_r), \dots].$$

Note that  $\tau$  acts on  $\mathcal{B}_r$  by simply setting  $\tau(\tau^m(X)) = \tau^{m+1}(X)$  with  $X = X_1, \dots, X_r$ , the operator  $\tau$  acting as usual over  $\mathbb{C}_\infty$ . We use the ring  $\mathcal{B}_r$  (and the so far unused indeterminate  $Z$ ) to construct yet another ring, non-commutative, denoted by  $\mathcal{A}_r$ . This is the set of infinite sums

$$\sum_{i \geq 0} c_i \tau^i(Z)$$

with the coefficients  $c_i \in \mathcal{B}_r$ , the sum being the usual one and the product is given by:

$$\text{for } F = \sum_{i \geq 0} f_i \tau^i(Z), G = \sum_{j \geq 0} g_j \tau^j(Z), F \cdot G := \sum_{k \geq 0} \left( \sum_{i+j=k} f_i \tau^i(g_j) \right) \tau^k(Z).$$

Note that the action of  $\tau$  on  $\mathcal{B}_r$  extends to an action of  $\tau$  on  $\mathcal{A}_r$  by setting  $\tau(\tau^i(Z)) = \tau^{i+1}(Z)$ . The series

$$\mathcal{L}_r(X_1, \dots, X_r; Z) = \sum_{d \geq 0} \left( \sum_{a \in A_{+,d}} C_a(X_1) \cdots C_a(X_r) a^{-1} \right) \tau^d(Z)$$

defines an element of  $\mathcal{A}_r$ . Let  $\exp_C = \sum_{i \geq 0} D_i^{-1} \tau^i$  is the operator associated to Carlitz's exponential. Obviously,

$$S_r(X_1, \dots, X_r; Z) := \exp_C(\mathcal{L}_r(X_1, \dots, X_r; Z))$$

is an element of  $\mathcal{A}_r$ . But more is true.

**Theorem 8.1.** *We have that*

$$S_r(X_1, \dots, X_r; Z) \in A[X_1, \dots, X_r, Z, \tau(X_1), \dots, \tau(X_r), \tau(Z), \dots].$$

*Proof.* We shall use the non-abelian auxiliary Drinfeld module of rank one  $\phi$  whose parameter is

$$\alpha' = t_{r+1}(t_1 - \theta) \cdots (t_r - \theta) \in A[t_1, \dots, t_{r+1}] = A_{r+1}.$$

We also set  $\alpha = (t_1 - \theta) \cdots (t_r - \theta)$ , the parameter of  $C_r$ . We use the  $K$ -linear action of  $K[t_1, \dots, t_{r+1}]$  over  $\mathcal{A}_r$  determined by the relations, for  $m$  an integer,  $t_i \cdot \tau^m(X_j) = \tau^m(X_j)$  if  $i \neq j$ ,  $t_i \cdot \tau^m(Z) = \tau^m(Z)$  ( $i \neq r+1$ ),  $t_i \cdot \tau^m(X_i) = \tau^m(t_i \cdot X_i) = \tau^m(C_\theta(X_i))$  and  $t_{r+1} \cdot \tau^m(X_i) = \tau^m(X_i)$ ,  $t_{r+1} \cdot \tau^m(Z) = \tau^{m+1}(Z)$ . One easily sees that

$$\begin{aligned} \alpha' \cdot \tau^m(X_1 \cdots X_r Z) &= \tau^{m+1}(X_1 \cdots X_r Z), \\ \rho_\alpha(a) \cdot \tau^m(X_1 \cdots X_r) &= \tau^m(C_a(X_1) \cdots C_a(X_r)), \quad a \in A. \end{aligned}$$

One can also prove, by induction, that

$$\alpha' \tau(\alpha') \cdots \tau^{i-1}(\alpha') \cdot Y = \tau^i(Y), \quad Y \in \mathcal{A}_r.$$

If  $F = \sum_{i \geq 0} F_i t_{r+1}^i \in K[t_1, \dots, t_r][[t_{r+1}]]$ , define:

$$F.(X_1 \cdots X_r Z) = \sum_{i \geq 0} F_i t_{r+1}^i \cdot (X_1 \cdots X_r Z) = \sum_{i \geq 0} F_i \cdot (X_1 \cdots X_r \tau^i(Z)) \in \mathcal{A}_r,$$

in particular

$$L(1, \phi).(X_1 \cdots X_r Z) = \mathcal{L}_r(X_1, \dots, X_r; Z).$$

An easy computation shows that if  $F \in K[t_1, \dots, t_r][[t_{r+1}]]$ , then

$$\exp_\phi(F).(X_1 \cdots X_r Z) = \exp_C(F.(X_1 \cdots X_r Z))$$

so that

$$S_r(X_1, \dots, X_r; Z) = \exp_\phi(L(1, \phi).(X_1 \cdots X_r Z))$$

and the result then follows from Corollary 5.13.  $\square$

**Remark 8.2.** Even though it only applies to the Carlitz module, Theorem 8.1 has an advantage if compared to Anderson original result [2, Theorem 3], and this, even if we forget the occurrence of several variables  $X_1, \dots, X_r$ . Indeed, these variables can vary in the Tate algebra  $\mathbb{T}_s$ , while Anderson's result holds if the variable is chosen in  $\mathbb{C}_\infty$ . Let us assume, for the sake of simplicity, that  $X_1 = \cdots = X_r = X$ . In [2, §4.3], Anderson also provides a table of *special polynomials* of small order for small values of  $q$ . For example, if  $q = 3$  and  $r = 4$ , we have the formula (cf. loc. cit. p. 191):

$$(28) \quad \exp_C \left( \sum_{k \geq 0} Z^q{}^k \sum_{a \in A_{+,k}} \frac{C_a(X)^4}{a} \right) = ZX^4 - Z^3 X^6.$$

This formula has to be understood *with the variables  $X, Z$  varying in  $\mathbb{C}_\infty$  so that  $|Z|$  is small enough to ensure convergence*. If the variables are chosen in  $\mathbb{T}_s$ , the formula no longer holds. It can be proved, with an explicit computation, that, again for  $q = 3$ ,

$$\begin{aligned}
S_4(X_1, \dots, X_4; Z) &= ZX_1 \cdots X_4 - \tau(Z)(X_1 X_2 X_3 \tau(X_4) + \\
&\quad X_1 X_2 \tau(X_3) X_4 + X_1 \tau(X_2) X_3 X_4 + \tau(X_1) X_2 X_3 X_4).
\end{aligned}$$

If we choose  $X_1 = \cdots = X_4 = X$ , then we get

$$S_4(X, \dots, X; Z) = ZX^4 - \tau(Z)X^3\tau(X),$$

so that, if  $X, Z \in \mathbb{C}_\infty$ , we recover the original entry of Anderson table (28). Of course, further information about the polynomials  $S(X_1, \dots, X_r; Z)$  can be made explicit in the same spirit of [2, Proposition 8]; we refer the interested reader to a forthcoming work of the authors.

## 9. EVALUATION AT DIRICHLET CHARACTERS

This section is inspired by [6] and [4].

**9.1. Some settings.** Let  $P$  be a prime of  $A$  of degree  $d \geq 1$ . Since we are going to study certain congruences modulo  $P$ , we choose once and for all a  $K$ -embedding

$$\iota_P : K^{ac} \rightarrow \mathbb{C}_P,$$

where  $\mathbb{C}_P$  is the completion of  $\widehat{K_P^{ac}}$ , an algebraic closure of the completion  $\widehat{K_P}$  of  $K$  at the place determined by  $P$ . We normalize the valuation  $v_P$  at the place  $P$  by setting  $v_P(P) = 1$ . The *Teichmüller character* is the unique morphism  $\vartheta_P : \Delta_P \rightarrow (k^{ac})^\times$  such that

$$\forall a \in A \setminus PA, v_P(\iota_P(\vartheta_P(\sigma_a)) - a) \geq 1.$$

Note that there exists a unique root  $\zeta_P$  of  $P$  in  $k^{ac}$  with the property that for  $\sigma_b \in \Delta$  with  $b \in A \setminus PA$ , we have  $\vartheta_P(\sigma_b) = b(\zeta_P)$ .

Then, every character  $\chi \in \widehat{\Delta_P} = \text{Hom}(\Delta_P, (k^{ac})^\times)$  is a power of  $\vartheta_P$ ; we can write

$$(29) \quad \chi = \vartheta_P^i, \quad 0 \leq i \leq q^d - 2.$$

More generally, if  $a \in A_+$  is non-constant and squarefree, we can write  $a = P_1 \cdots P_r$  with  $P_1, \dots, P_r$  distinct primes of respective degrees  $d_1, \dots, d_r$ , and we have that  $\widehat{\Delta_a} \cong \widehat{\Delta_{P_1}} \times \cdots \times \widehat{\Delta_{P_r}}$  so that for every character  $\chi \in \widehat{\Delta_a}$ ,

$$(30) \quad \chi = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$$

where the integers  $N_i$  are such that  $0 \leq N_i \leq q^{d_i} - 2$  for all  $i$ . We call the character like the above  $\chi$  in (30), *Dirichlet character* <sup>(5)</sup> and its conductor is the product  $\prod_{N_i \neq 0} P_i$  (note that the trivial character as conductor 1). We define  $k_a$  as the subfield of  $k^{ac}$  generated by the roots of  $a$  (and we set  $k_1 = k$ ).

Observe that if  $b \in A_+$ , any homomorphism  $\Delta_b \rightarrow (k^{ac})^\times$  comes from a Dirichlet character.

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<sup>5</sup>In fact we should call such characters *tame Dirichlet characters* because the extensions of  $K$  associated to such characters are tamely ramified. However, since this is the only kind of characters to be considered in this section, we call them *Dirichlet characters* for simplicity.

9.1.1. *Gauss-Thakur sums.* We recall that if  $P$  is a prime of  $A$ , the Gauss-Thakur sum associated to a character  $\chi = \vartheta_P^{q^j} \in \widehat{\Delta_P}$  is defined (see Thakur [23, Section 9.8]) by

$$g(\vartheta_P^{q^j}) = - \sum_{\delta \in \Delta_P} \vartheta_P(\delta^{-1})^{q^j} \delta(\lambda_P) \in k_P K_P,$$

where we recall that  $\lambda_P = \exp_C\left(\frac{\tilde{\pi}}{P}\right)$ , and  $k_P K_P$  is the compositum of  $k_P$  and  $K_P = K(\lambda_P)$  in  $K^{ac}$ . Let  $\chi$  be a character of  $\widehat{\Delta_P}$ . We define  $g(\chi)$  by using (29) in the following way. We expand  $i = i_0 + i_1 q + \cdots + i_{d-1} q^{d-1}$  in base  $q$  ( $i_0, \dots, i_r \in \{0, \dots, q-1\}$ ), and along with this expansion, we define,

$$g(\chi) = \prod_{j=0}^{d-1} g(\vartheta_P^{q^j})^{i_j}.$$

We recall that, by [5, Theorem 2.9], we have:

$$g(\vartheta_P^{q^j}) = P'(\zeta_P)^{-q^j} \omega(\zeta_P^{q^j}), \quad j = 0, \dots, d-1,$$

where  $\omega$  is the function of Anderson and Thakur.

The factorization (30) of a Dirichlet character  $\chi$  allows us to associate to it a *Gauss-Thakur sum*. This is done for example in [4, Section 2.3].

Let us expand in base  $q$ :

$$(31) \quad N_i = \sum_{j=0}^{d_i-1} n_{i,j} q^j, \quad i = 1, \dots, r,$$

with  $n_{i,j} \in \{0, \dots, q-1\}$ . For a positive integer  $N$  we denote by  $\ell_q(N)$  the sum of the digits of the expansion in base  $q$  of  $N$  so that  $\ell_q(N_i) = \sum_{j=0}^{d_i-1} n_{i,j}$ . We also set  $s = \sum_i \ell_q(N_i)$ . The integer  $s$  is called the *type of  $\chi$* . We point out that the type  $s$  of  $\chi$  does not depend on the embeddings  $\iota_{P_i}$ . We make the convention that the trivial character is the unique Dirichlet character of type zero. According to loc. cit., the Gauss-Thakur sum associated to  $\chi$  is defined as follows:

$$g(\chi) = \prod_{i=1}^r g(\vartheta_{P_i}^{N_i}) \in k_a K_a,$$

where  $k_a K_a$  denotes the compositum of  $k_a$  and  $K_a$  in  $K^{ac}$ .

9.1.2.  *$k_a[\Delta_a]$ -modules.* For  $v$  a place of  $K_a$ , let us denote by  $\widehat{K_{a,v}}$  the completion of  $K_a$  at  $v$ . If  $v$  divides  $\infty$  and  $a \neq 1$ , then

$$\widehat{K_{a,v}} \cong K_\infty(\tilde{\pi}).$$

We set  $K_{a,\infty} = K_a \otimes_K K_\infty$ . we have  $K_{1,\infty} = K_\infty$  and if  $a$  is non-constant, we have an isomorphism of  $K_\infty$ -algebras:

$$K_{a,\infty} \cong \prod_{v \in S_\infty(K_a)} K_\infty(\tilde{\pi}),$$

where  $S_\infty(K_a)$  is the set of places of  $K_a$  dividing  $\infty$ . Recall that we have an action of  $\Delta_a$  on  $K_{a,\infty}$ ; if  $\sigma \in \Delta_a$  and  $x \otimes y \in K_a \otimes_K K_\infty$ , then:

$$\sigma(x \otimes y) := \sigma(x) \otimes y.$$

The operator  $\tau$  acts on  $K_{a,\infty}$  by exponentiation by  $q$  ( $\tau(x \otimes y) = x^q \otimes y^q$ ) and the actions of  $\Delta_a$  and  $\tau$  commute. We set

$$\Omega_a = K_{a,\infty} \otimes_k k_a.$$

We endow  $\Omega_a$  with a structure of  $k_a[\Delta_a]$ -module by setting (as above), for  $\sigma \in \Delta_a$  and  $x \otimes y \in \Omega_a = K_{a,\infty} \otimes_k k_a$ ,

$$\sigma(x \otimes y) = \sigma(x) \otimes y.$$

This action commutes with the  $k_a$ -linear extension  $\varphi$  of the operator  $\tau$  on  $\Omega_a$ . Explicitly, if  $x \otimes y \in \Omega_a$ ,  $\varphi(x \otimes y) = \tau(x) \otimes y$ . If we identify  $k_a K_\infty$  with  $K_\infty \otimes_k k_a$  which is in a natural way a  $K_\infty$ -subalgebra of  $\Omega_a$ , if  $x = \sum_i x_i \theta^{-i} \in k_a((\theta^{-1})) = k_a K_\infty$ , then

$$\varphi(x) = \sum_i x_i \theta^{-iq}.$$

**9.1.3. Idempotents.** We identify  $k_a K_a$  with  $K_a \otimes_k k_a$  which is a  $K$ -subalgebra of  $\Omega_a$  (note that  $K_a \cap k_a = k$ ). We denote by  $O_a$  the integral closure of  $A$  in  $K_a$ ; then  $O_a[k_a] = \widehat{O_a} \otimes_k k_a$  is the integral closure of  $A[k_a] = A \otimes_k k_a$  in  $k_a K_a$ . To a character  $\chi \in \widehat{\Delta_a}$  as in (30) we associate an idempotent  $e_\chi \in k_a[\Delta_a]$ , defined as follows:

$$e_\chi = |\Delta_a|^{-1} \sum_{\delta \in \Delta_a} \delta^{-1} \chi(\delta).$$

By [4, Lemma 16], we have:

$$e_\chi(k_a K_a) = (k_a K)g(\chi), \quad \text{and} \quad e_\chi(O_a[k_a]) = A[k_a]g(\chi).$$

Therefore:

$$(32) \quad \Omega_a = \bigoplus_{\chi \in \widehat{\Delta_a}} e_\chi(\Omega_a), \quad \text{and} \quad e_\chi(\Omega_a) = k_a K_\infty g(\chi), \quad \chi \in \widehat{\Delta_a}.$$

**9.1.4. Evaluation map.** Let  $\chi$  be a Dirichlet character of conductor  $a = \prod_{i=1}^r P_i$ ,  $\chi = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$ ,  $1 \leq N_i \leq q^{d_i} - 2$ , where  $d_i$  is the degree of  $P_i$ . We recall that the type of  $\chi$  is  $s = \sum_i \ell_q(N_i)$ . Consider an  $s$ -tuple of variables

$$\underline{t}_s = \left( \underbrace{t_{1,0,1}, \dots, t_{1,0,n_{1,0}}}_{n_{1,0}}, \dots, \underbrace{t_{1,d_0-1,1}, \dots, t_{1,d_0-1,n_{1,d_0-1}}}_{n_{1,d_0-1}}, \dots, \dots, \underbrace{t_{r,d_r-1,n_{r,d_r-1}}}_{n_{r,d_r-1}} \right).$$

$\underbrace{\hspace{15em}}_{\ell_q(N_1)} \qquad \underbrace{\hspace{15em}}_{\ell_q(N_r)}$

We define the  $\mathbb{C}_\infty$ -algebra homomorphism ‘‘evaluation’’ at  $\chi$ :

$$\text{ev}_\chi : \mathbb{T}_s \rightarrow \mathbb{C}_\infty$$

by setting  $\text{ev}_\chi(t_{i,j,k}) = \zeta_{P_i}^{q^j}$  for all  $i, j, k$ . If we restrict  $\text{ev}_\chi$  to the subring of  $\mathbb{T}_s$  whose elements are symmetric in  $t_1, \dots, t_s$ , then  $\text{ev}_\chi$  only depends on  $\chi$ . Observe

that:

$$\begin{aligned}
g(\chi) &= g(\vartheta_{P_1}^{N_1}) \cdots g(\vartheta_{P_r}^{N_r}) \\
&= P_1'(\zeta_1)^{-N_1} \cdots P_r'(\zeta_r)^{-N_r} \prod_{i=1}^r \prod_{j=0}^{d_i-1} \omega(\zeta_i^{q^j})^{n_{i,j}} \\
(33) \quad &= \text{ev}_\chi \left( \prod_{i=1}^r \prod_{j=0}^{d_i-1} \prod_{k=1}^{n_{i,j}} P_i'(t_{i,j,k})^{-1} \omega(t_{i,j,k}) \right).
\end{aligned}$$

9.1.5. *An equivariant isomorphism.* We recall that we have introduced, in §9.1.2, a  $k_a$ -linear endomorphism  $\varphi$  of  $\Omega_a$ . We choose a Dirichlet character

$$\chi = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$$

as in (30) and we expand the integers  $N_i$  in base  $q$  as in (31);

$$N_i = \sum_{j \geq 0} n_{i,j} q^j.$$

By (33) and by the functional equation  $\tau(\omega) = (t - \theta)\omega$  of the function  $\omega$  of Anderson and Thakur, we see that

$$\varphi(g(\chi)) = \left( \prod_{i=1}^r \prod_{j=0}^{d_i-1} (\vartheta_{P_i}(\theta)^{q^j} - \theta)^{n_{i,j}} \right) g(\chi) = \left( \prod_{i=1}^r \prod_{j=0}^{d_i-1} (\zeta_i^{q^j} - \theta)^{n_{i,j}} \right) g(\chi).$$

We now come back to the identity (32) and we consider the isomorphism

$$\nu_\chi : e_\chi(\Omega_a) \rightarrow k_a K_\infty$$

defined by  $\nu_\chi(y) = yg(\chi)^{-1}$ . Then,

$$\nu_\chi(\varphi(x)) = \tilde{\varphi}(\nu_\chi(x)),$$

where

$$\tilde{\varphi}(x) := \left( \prod_{i=1}^r \prod_{j=0}^{d_i-1} (\zeta_i^{q^j} - \theta)^{n_{i,j}} \right) \varphi(x).$$

The Taelman class module associated to the Carlitz module and relative to the extension  $K_a/K$  (see [19] and [6]) is defined by

$$(34) \quad H_a = \frac{C(K_a \otimes_K K_\infty)}{\exp_C(K_a \otimes_K K_\infty) + C(O_a)}.$$

Let  $C^\varphi : k_a \otimes_k A \rightarrow \Omega_a[\varphi]$  be the homomorphism of  $k_a$ -algebras defined by

$$C_\theta^\varphi = \theta + \varphi.$$

Let us additionally set

$$\exp_C^\varphi = \sum_{i \geq 0} D_i^{-1} \varphi^i,$$

which gives rise to a  $k_a$ -linear continuous function  $\Omega_a \rightarrow \Omega_a$ .

We then have an isomorphism of  $A[k_a]$ -modules:

$$H_a \otimes_k k_a \cong \frac{C^\varphi(\Omega_a)}{\exp_C^\varphi(\Omega_a) + C^\varphi(O_a[k_a])}.$$

Now, we consider, with an analogue meaning of the symbols, the  $k_a$ -linear endomorphisms  $C^{\tilde{\varphi}}$  and  $\exp_C^{\tilde{\varphi}}$  of  $k_a K_\infty$ , and the  $A[k_a]$ -module:

$$H_\chi = \frac{C^{\tilde{\varphi}}(k_a K_\infty)}{\exp_C^{\tilde{\varphi}}(k_a K_\infty) + C^{\tilde{\varphi}}(A[k_a])}.$$

The previous discussions imply the next Lemma.

**Lemma 9.1.** *The map  $\nu_\chi$  induces an isomorphism of  $A[k_a]$ -modules*

$$e_\chi(H_a \otimes_k k_a) \cong H_\chi.$$

9.1.6. *Link between  $H_{C_s}$  and  $H_\chi$ .* Let  $\chi$  be a Dirichlet character of conductor  $a = \prod_{i=1}^r P_i$ ,  $\chi = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$ ,  $1 \leq N_i \leq q^{d_i} - 2$ , where  $d_i$  is the degree of  $P_i$ . Write  $N_i$  in base  $q$ :  $N_i = \sum_{j=0}^{d_i-1} n_{i,j} q^j$ , with  $n_{i,j} \in \{0, \dots, q-1\}$ .

With  $s = \sum_{i=1}^r \ell_q(N_i)$  the type of  $\chi$  we consider the uniformizable Drinfeld module of rank one  $C_s$ , with parameter

$$(35) \quad \alpha = \prod_{i=1}^r \prod_{j=0}^{d_i-1} (t_{i,j,k} - \theta)^{n_{i,j}}.$$

Note that  $\text{ev}_\chi(k[\underline{t}_s]) = k_a$  and that we have a field isomorphism  $k[\underline{t}_s]/I_\chi \cong k_a$ , where

$$I_\chi = \text{Ker}(\text{ev}_\chi) \cap k[\underline{t}_s],$$

which yields an isomorphism

$$(36) \quad \frac{A_s}{I_\chi A_s} \cong A[k_a] = k_a[\theta].$$

**Proposition 9.2.** *The evaluation map  $\text{ev}_\chi$  induces an isomorphism of  $A[k_a]$ -modules*

$$\psi_\chi : \frac{H_{C_s}}{I_\chi H_{C_s}} \rightarrow H_\chi.$$

*Proof.* The evaluation map  $\text{ev}_\chi : \mathbb{T}_s(K_\infty) \rightarrow k_a K_\infty$  satisfies:

$$\text{ev}_\chi(\tau_\alpha(f)) = \tilde{\varphi}(\text{ev}_\chi(f)), \quad f \in \mathbb{T}_s(K_\infty).$$

In particular, for all  $b \in A[k_a]$ ,  $\tilde{b} \in A_s$  such that  $b = \text{ev}_\chi(\tilde{b})$  and  $f$  in  $\mathbb{T}_s(K_\infty)$ , we have (with  $\phi = C_s$ ):

$$(37) \quad C_b^{\tilde{\varphi}}(\text{ev}_\chi(f)) = \text{ev}_\chi(\phi_{\tilde{b}}(f))$$

and

$$(38) \quad \exp_C^{\tilde{\varphi}}(\text{ev}_\chi(f)) = \text{ev}_\chi(\exp_\phi(f)).$$

We consider the composition  $w = \text{pr} \circ \text{ev}_\chi$  of two surjective  $k$ -linear maps:

$$\mathbb{T}_s(K_\infty) \rightarrow k_a K_\infty \rightarrow \frac{k_a K_\infty}{\exp_C^{\tilde{\varphi}}(k_a K_\infty) + A[k_a]},$$

where the first map is  $\text{ev}_\chi$  and the second map  $\text{pr}$  is the projection. We deduce from (37) that, with  $b$  and  $\tilde{b}$  such that  $b = \text{ev}_\chi(\tilde{b})$ ,

$$w(\phi_{\tilde{b}}(f)) = C_b^{\tilde{\varphi}}(w(f))$$

for all  $f \in \mathbb{T}_s(K_\infty)$  and  $b \in A_s$ .

Let  $f$  be an element of  $\mathbb{T}_s(K_\infty)$ . We have  $w(f) = 0$  if and only if

$$\text{ev}_\chi(f) \in \exp_{C_s}^{\tilde{\varphi}}(k_a K_\infty) + A[k_a].$$

By (38), we have

$$\exp_{C_s}^{\tilde{\varphi}}(k_a K_\infty) + A[k_a] = \text{ev}_\chi(\exp_{C_s}(\mathbb{T}_s(K_\infty)) + A_s)$$

which means that  $w(f) = 0$  if and only if

$$f \in I_\chi \mathbb{T}_s(K_\infty) + \exp_{C_s}(\mathbb{T}_s(K_\infty)) + A_s,$$

where

$$I_\chi \mathbb{T}_s(K_\infty) = \left\{ \sum_{i \geq m} x_i \theta^{-i}; x_i \in I_\chi, m \in \mathbb{Z} \right\}.$$

Now, we notice the isomorphisms of  $k$ -vector spaces

$$\frac{H_{C_s}}{I_\chi H_{C_s}} = \frac{\mathbb{T}_s(K_\infty)}{I_\chi \mathbb{T}_s(K_\infty) + \exp_{C_s}(\mathbb{T}_s(K_\infty)) + A_s}$$

which shows, with (36), that the map of the Proposition is an isomorphism of  $A[k_a]$ -modules.  $\square$

**Corollary 9.3.** *Let  $\chi$  be a Dirichlet character of type  $s \geq q$ , with  $s \equiv 1 \pmod{q-1}$  and with conductor  $a$ . Then:*

$$\text{Fitt}_{A[k_a]}(H_\chi) = \text{ev}_\chi(\mathbb{B}_{C_s})A[k_a].$$

*Proof.* By Theorem 7.7, we get:

$$\text{Fitt}_{A[k_a]} \left( \frac{H_{C_s}}{I_\chi H_{C_s}} \right) = \text{ev}_\chi(\mathbb{B}_{C_s})A[k_a].$$

We conclude by applying Proposition 9.2.  $\square$

Florent Demesley informed us that the above result can also be deduced from an equivariant class number formula similar to Theorem A of [6].

## 9.2. On the structure of $H_{C_s}$ and the isotypic components $H_\chi$ .

**Definition 9.4.** Let  $\mathcal{P}$  be a property on the set of all the Dirichlet characters of type  $s$ . We say that  $\mathcal{P}$  holds for *almost all characters of type  $s$*  if there exists a non-zero polynomial  $F \in k[\underline{t}_s]$  such that if  $\chi$  is one of such characters with  $\text{ev}_\chi(F) \neq 0$ , then  $\mathcal{P}$  is true for  $\chi$ .

Let  $R$  be a commutative ring and  $M$  an  $R$ -module. Let  $N$  be an element of  $R$ . We denote the  $N$ -torsion submodule of  $M$  by  $M[N] = \{m \in M, N.m = 0\}$ .

**Proposition 9.5.** *Let us suppose that  $s \geq 1$  and let  $N$  be in  $A_+$ . Then, for almost all Dirichlet characters of type  $s$ , we have  $H_\chi[N] = (0)$ .*

*Proof.* Thanks to Proposition 5.15, we can suppose that  $s \geq 2q-1$ . By Proposition 7.9, if  $s \equiv 1 \pmod{q-1}$ ,  $H_{C_s}[N] = (0)$ . If  $s \not\equiv 1 \pmod{q-1}$ , then, by Proposition 7.10,  $H_{C_s}[N] \otimes_{k[\underline{t}_s]} k_s = (0)$ . Hence, in all cases,

$$H_{C_s}[N] \otimes_{k[\underline{t}_s]} k_s = (0).$$



We now consider the exact sequence of  $k[\underline{t}_s]$ -modules of finite type (the middle map is the multiplication by  $N$ ):

$$0 \rightarrow H_{C_s}[N] \rightarrow H_{C_s} \rightarrow H_{C_s} \rightarrow \frac{H_{C_s}}{NH_{C_s}} \rightarrow 0.$$

Taking the tensor product of the above exact sequence with  $k_s$ , we get:

$$\frac{H_{C_s}}{NH_{C_s}} \otimes_{k[\underline{t}_s]} k_s = (0).$$

In particular, the  $k[\underline{t}_s]$ -module of finite type  $\frac{H_{C_s}}{NH_{C_s}}$  is torsion and there exists  $F_N \in k[\underline{t}_s] \setminus \{0\}$  such that  $F_N H_{C_s} \subset NH_{C_s}$ . It remains to apply Proposition 9.2.  $\square$

We now suppose that  $s \geq 1$  is such that  $s \not\equiv 1 \pmod{q-1}$ . Recall that, by Remark 5.19, the element  $m_s = [V_{C_s}]_{R_s}$  is a monic polynomial in  $A_s$ .

**Theorem 9.6.** *For almost all characters  $\chi$  of type  $s$ , we have that*

$$\text{Fitt}_{A[k_a]}(H_\chi) = \text{ev}_\chi(m_s)A[k_a],$$

where  $a$  is the conductor of  $\chi$ .

*Proof.* Let us consider the torsion sub- $k[\underline{t}_s]$ -module  $M$  of  $H_{C_s}$ . Set:

$$\widetilde{H}_{C_s} := \frac{H_{C_s}}{M}$$

Note that the  $R_s$ -module  $\widetilde{H}_{C_s} \otimes_{k[\underline{t}_s]} k_s$  is isomorphic to  $V_{C_s}$  and therefore its Fitting ideal over  $R_s$  is generated by  $m_s$ . Since  $M$  is a finitely generated and torsion  $k[\underline{t}_s]$ -module, we notice that for almost all Dirichlet characters  $\chi$  of type  $s$ , we have that  $M \subset I_\chi H_{C_s}$ . Thus, for almost all the Dirichlet characters  $\chi$  of type  $s$ , the  $A_s/I_\chi A_s$ -module

$$\frac{\widetilde{H}_{C_s}}{I_\chi \widetilde{H}_{C_s}}$$

is well defined. For almost all Dirichlet characters  $\chi$  of type  $s$ , we have that

$$\text{Fitt}_{\frac{A_s}{I_\chi A_s}} \left( \frac{\widetilde{H}_{C_s}}{I_\chi \widetilde{H}_{C_s}} \right) = \frac{\text{Fitt}_{A_s}(\widetilde{H}_{C_s}) + I_\chi A_s}{I_\chi A_s}.$$

But by Proposition 9.2, for almost all characters  $\chi$  of type  $s$ , we also have an isomorphism of  $A[k_a]$ -modules  $\frac{H_{C_s}}{I_\chi H_{C_s}} \cong H_\chi$ . The Theorem follows easily.  $\square$

### 9.3. Pseudo-null and pseudo-cyclic modules.

**Definition 9.7.** Let  $M$  be a finitely generated  $A_s$ -module which also is finitely generated as a  $k[\underline{t}_s]$ -module. We say that  $M$  is *pseudo-null* if  $M \otimes_{k[\underline{t}_s]} k_s = \{0\}$ . We say that  $M$  is *pseudo-cyclic* if there exists  $m \in M$  such that  $\frac{M}{mA_s}$  is pseudo-null.

In this Section, we investigate the property of pseudo-cyclicity and pseudo-nullity for the modules  $H_{C_s}$  (that is, in the case of the Drinfeld module  $C_s$  of parameter  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$ ). We are concerned by the following Questions which we leave open.

**Question 9.8.** *Is  $H_{C_s}$  pseudo-cyclic?*

**Question 9.9.** *Assuming that  $s \not\equiv 1 \pmod{q-1}$ , is  $H_{C_s}$  pseudo-null?*

9.3.1. *The torsion case.* We are here in the case  $s \equiv 1 \pmod{q-1}$ . Recall that  $H_{C_s} = (0)$  for  $s = 1, q$ . We can suppose without loss of generality, that  $s \geq 2q - 1$ . We have the following Proposition.

**Proposition 9.10.** *The following conditions are equivalent.*

- (1)  $H_{C_s}$  is pseudo-cyclic,
- (2) For almost all Dirichlet characters  $\chi$  of type  $s$ ,  $H_\chi$  is a cyclic  $k_a[\theta]$ -module, where  $a$  is the conductor of  $\chi$ ,
- (3) There exists a Dirichlet characters  $\chi$  of type  $s$  such that  $H_\chi$  is a cyclic  $k_a[\theta]$ -module, where  $a$  is the conductor of  $\chi$ .

*Proof.* The first condition implies the second by means of the equivariant isomorphism of the Proposition 9.2. The second condition obviously implies the third. It remains to show that the third condition implies the first. We have to show that  $\text{Fitt}_{R_s}(V_{C_s}) = \text{Ann}_{R_s}(V_{C_s})$ , where we recall that  $V_{C_s} = H_{C_s} \otimes_{k[\underline{L}_s]} k_s$ . By Theorem 7.7, the polynomial  $\mathbb{B}_{C_s}$  is the monic generator of  $\text{Fitt}_{A_s}(H_{C_s})$  and  $\text{Fitt}_{R_s}(V_{C_s}) = \mathbb{B}_{C_s} R_s$ . Let  $m$  be the monic generator of  $\text{Ann}_{R_s}(V_{C_s})$ , observe that  $m \in A_s$  and  $m$  divides  $\mathbb{B}_{C_s}$ . By Corollary 9.3,

$$\text{Ann}_{A[k_a]}(H_\chi) = \text{ev}_\chi(\mathbb{B}_{C_s})A[k_a].$$

Thus,  $\text{ev}_\chi(\mathbb{B}_{C_s})$  divides  $\text{ev}_\chi(m)$  and therefore  $\mathbb{B}_{C_s} = m$ .  $\square$

9.3.2. *The non-torsion case.* Here we suppose that  $s \not\equiv 1 \pmod{q-1}$  and  $s \geq 1$ .

**Proposition 9.11.** *The following assertions are equivalent.*

- (1)  $H_{C_s}$  is pseudo-cyclic,
- (2)  $\text{Ann}_{R_s} \left( \frac{U_{C_s}}{U_{C_s}^c} \otimes_{k[\underline{L}_s]} k_s \right) = \text{Ann}_{R_s}(V_{C_s})$ ,
- (3) For almost all Dirichlet characters  $\chi$  of type  $s$ , the  $A[k_a]$ -module  $H_\chi$  is a cyclic module, where  $a$  is the conductor of  $\chi$ .

*Proof.* We know that  $U_{C_s}/U_{C_s}^c$  is a cyclic  $A_s$ -module and we already know that it is of finite rank over  $k[\underline{L}_s]$  and free (see Remark 5.19). By Corollary 5.18,

$$\text{Ann}_{R_s} \left( \frac{U_{C_s}}{U_{C_s}^c} \otimes_{k[\underline{L}_s]} k_s \right) = \text{Fitt}_{R_s} \left( \frac{U_{C_s}}{U_{C_s}^c} \otimes_{k[\underline{L}_s]} k_s \right) = \text{Fitt}_{R_s}(V_{C_s}).$$

Then,  $H_{C_s}$  is pseudo-cyclic if and only if  $\text{Fitt}_{R_s}(V_{C_s}) = \text{Ann}_{R_s}(V_{C_s})$ . This implies the equivalence of the first condition and the second condition. That these conditions are also equivalent to the third condition follows in a way which is very similar to that used in the proof of Proposition 9.10.  $\square$

**Remark 9.12.** We notice that  $H_{C_s}$  is pseudo-null if and only if

$$\text{Fitt}_{R_s}(V_{C_s}) = R_s.$$

Thus  $H_{C_s}$  is pseudo-null if and only if  $U_{C_s} = U_{C_s}^c$ . Moreover,  $H_{C_s}$  is pseudo-null if and only if  $m_s = [V_{C_s}]_{R_s} = 1$  which is equivalent, by Theorem 9.6, to the fact that, for almost all Dirichlet character  $\chi$  of type  $s$  we have  $H_\chi = (0)$ .

**9.4. Evaluation of the polynomials  $\mathbb{B}_{C_s}$ .** For  $s \geq 1$  with  $s \equiv 1 \pmod{q-1}$ , to simplify our notation, we write  $\mathbb{B}_s$  instead of  $\mathbb{B}_{C_s}$ . Note that  $\mathbb{B}_1 = \frac{1}{\theta-t}$ .

Let  $\chi$  be a character of conductor  $a$  and of type  $s$ , write  $a = P_1 \cdots P_r$  for distinct primes  $P_1, \dots, P_r \in A$ , so that  $\chi = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$  with  $N_i \leq q^{d_i} - 2$ ,  $d_i$  being the degree in  $\theta$  of  $P_i$  for all  $i$ .

We recall that the special value at  $n \geq 1$  of Goss Dirichlet  $L$ -series (see [12], chapter 8) associated to  $\chi$  is defined by

$$\forall n \in \mathbb{Z}, L(n, \chi) = \sum_{m \geq 0} \sum'_{b \in A_{+,m}} \chi(\sigma_b) b^{-n} \in \mathbb{C}_\infty,$$

where the sum runs over the elements  $b$  which are relatively prime to  $a$ .

In [4] and [15] it is shown that these  $L$ -series values can be obtained from the evaluation of  $L$ -series values  $L(n, C_s)$ . More precisely, for all  $b$  relatively prime to  $a$ ,

$$\chi(\sigma_b) = \text{ev}_\chi(\rho_\alpha(b))$$

( $\alpha$  being the parameter of  $C_s$ ), and therefore we get

$$L(n, \chi) = \text{ev}_\chi(L(n, C_s)).$$

Let  $N \in \mathbb{N}$  that we expand in base  $q$  as  $N = \sum_{j=0}^k n_j q^j$  ( $n_0, \dots, n_k \in \{0, \dots, q-1\}$ ) and we set  $s' = s + \ell_q(N)$ . We then have the evaluation map (we recall that  $\mathbb{E}_s$  is the sub-algebra of  $\mathbb{T}_s$  of entire functions §3.3).

$$\text{ev}_N : \mathbb{E}_{s'} \rightarrow \mathbb{E}_s$$

defined by replacing the family of variables  $(t_1, \dots, t_s, t_{s+1}, \dots, t_{s+\ell_q(N)})$  by

$$(t_1, \dots, t_s, \underbrace{\theta, \dots, \theta}_{n_0}, \underbrace{\theta^q, \dots, \theta^q}_{n_1}, \dots, \underbrace{\theta^{q^k}, \dots, \theta^{q^k}}_{n_k}).$$

If  $N = 0$ , this map is obviously the identity map of  $\mathbb{E}_s$ . We shall work, in this subsection, with the evaluation map  $\text{ev}_{\chi, N} : \mathbb{E}_{s'} \rightarrow \mathbb{C}_\infty$  defined by

$$\text{ev}_{N, \chi} = \text{ev}_\chi \circ \text{ev}_N.$$

In particular:

$$\text{ev}_{N, \chi}(A[t_1, \dots, t_{s+\ell_q(N)}]) = A[k_a].$$

If  $C_{s'}$  is the Drinfeld module of rank one of parameter  $\alpha = (t_1 - \theta) \cdots (t_{s'} - \theta)$ , then this evaluation map allows to obtain the special values of the Dirichlet  $L$ -series of Goss from  $L(1, C_{s'})$ . Indeed, for all  $j \in \mathbb{N}$ ,

$$\text{ev}_{N, \chi}(\tau^j(L(1, C_{s'}))) = L(q^j - N, \chi).$$

To  $N$  as above, with its expansion  $N = \sum_i n_i q^i$  in base  $q$ , we associate the *Carlitz factorial*  $\Pi(N)$ , defined by

$$\Pi(N) = \prod_{i \geq 0} D_i^{n_i}.$$

We apply the evaluations  $\text{ev}_{N, \chi}$  in two different ways.

9.4.1. *First way to apply  $\text{ev}_{N,\chi}$ .* Recall that  $s' = s + \ell_q(N)$  and we assume here that  $s' \equiv 1 \pmod{q-1}$ ,  $s' \geq 1$ . Furthermore if  $s = 0$  and  $\ell_q(N) = 1$ , we assume that  $N \geq 2$ . For a polynomial  $a \in A$ , we denote by  $a'$  its derivative in the indeterminate  $\theta$ . Observe that  $\text{ev}_N(\mathbb{B}_{s'})$  is well defined. By [4, Corollary 8], the function

$$L(1, C_{s'}) = \sum_{d \geq 0} \sum_{b \in A_{+,d}} \chi_{t_1}(b) \cdots \chi_{t_{s'}}(b) b^{-1}$$

is in  $\mathbb{E}_{s'}$ , that is, entire in the set of variables  $\underline{t}_{s'}$  (it is denoted by  $L(\chi_{t_1} \cdots \chi_{t_{s'}}, 1)$  in [4]). By [4, Lemma 5], it vanishes at any point of the form

$$(t_1, \dots, t_s, t_{s+1}, \dots, t_{s+j-1}, \theta^{q^l}, t_{s+j+1}, \dots, t_{s'}), \quad l \geq 0, \quad t_i \in \mathbb{C}_\infty.$$

We will use in the sequel that the function  $\omega(t_k)$  has a simple pole at  $\theta^{q^l}$  of residue  $-\tilde{\pi}^{q^l} D_l^{-1}$  (for all  $l \geq 0$ ). We rename the variables  $t_{s+1}, \dots, t_{s'}$  in a way which is compatible with the expansion of  $N$  in base  $q$  by writing:

$$(t_{s+1}, \dots, t_{s'}) = (t_{0,0}, \dots, t_{0,n_0}, \dots, t_{k,0}, \dots, t_{k,n_k}).$$

We have, with  $\Delta$  the differential operator

$$\Delta = \frac{\partial}{\partial t_{0,0}} \cdots \frac{\partial}{\partial t_{0,n_0}} \cdots \frac{\partial}{\partial t_{k,0}} \cdots \frac{\partial}{\partial t_{k,n_k}},$$

$$\begin{aligned} \text{ev}_N \left( \omega(t_1) \cdots \omega(t_{s'}) \sum_{d \geq 0} \sum_{b \in A_{+,d}} b^{-1} b(t_1) \cdots b(t_{s'}) \right) &= \\ &= \omega(t_1) \cdots \omega(t_s) (-\tilde{\pi})^{n_0} \cdots (-\tilde{\pi})^{n_k q^k} D_0^{-n_0} \cdots D_k^{-n_k} \times \\ &\quad \times \left[ \Delta \left( \sum_{d \geq 0} \sum_{b \in A_{+,d}} b^{-1} b(t_1) \cdots b(t_s) b(t_{0,0}) \cdots b(t_{k,n_k}) \right) \right]_{t_{i,j} = \theta^{q^i}} \\ &= (-1)^N \omega(t_1) \cdots \omega(t_s) \tilde{\pi}^N \Pi(N)^{-1} \times \\ &\quad \times \sum_{d \geq 0} \sum_{b \in A_{+,d}} b^{-1} b(t_1) \cdots b(t_s) [b'(t_{0,0}) \cdots b'(t_{k,n_k})]_{t_{i,j} = \theta^{q^i}} \\ &= (-1)^N \omega(t_1) \cdots \omega(t_s) \tilde{\pi}^N \Pi(N)^{-1} \sum_{b \in A_{+,d}} b^{-1} b(t_1) \cdots b(t_s) (b')^N, \end{aligned}$$

which yields the formula

$$\text{ev}_N(\mathbb{B}_{s'}) = (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \tilde{\pi}^{N-1} \Pi(N)^{-1} \omega(t_1) \cdots \omega(t_s) \sum_{d \geq 0} \sum_{b \in A_{+,d}} \rho_\alpha(b) \frac{(b')^N}{b},$$

where we notice that the series in the right-hand side, with  $\alpha$  the parameter of  $C_s$ , is convergent for the Gauss norm of  $\mathbb{T}_s$ .

We also have the formula

$$\text{ev}_\chi(\omega(t_1) \cdots \omega(t_s)) = \vartheta_{P_1}(\sigma_{P_1'})^{N_1} \cdots \vartheta_{P_r}(\sigma_{P_r'})^{N_r} g(\chi) = P_1'(\zeta_1)^{N_1} \cdots P_r'(\zeta_r)^{N_r} g(\chi)$$

where  $\zeta_i = \zeta_{P_i}$ . Therefore, composing with  $\text{ev}_\chi$  now gives the identity

$$\begin{aligned} \text{ev}_{N,\chi}(\mathbb{B}_{s'}) &= \\ &= (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \tilde{\pi}^{N-1} \Pi(N)^{-1} \vartheta_{P_1}(\sigma_{P_1})^{N_1} \cdots \vartheta_{P_r}(\sigma_{P_r})^{N_r} g(\chi) \times \\ &\quad \sum_{d \geq 0} \sum_{b \in A_{+,d}} \chi(b) \frac{(b')^N}{b}. \end{aligned}$$

9.4.2. *Second way to apply  $\text{ev}_{N,\chi}$ .* Here again  $s' \equiv 1 \pmod{q-1}$ ,  $s' \geq 1$ . Let us consider an integer  $d \geq 1$  such that  $q^d > N$ . The functions  $\tau^d(L(1, C_{s'})) = L(q^d, C_{s'})$  are also entire and we have

$$\tau^d(\mathbb{B}_{s'}) = (-1)^{\frac{s'-1}{q-1}} \tilde{\pi}^{-q^d} L(q^d, C_{s'}) b_d(t_1) \cdots b_d(t_{s'}) \omega(t_1) \cdots \omega(t_{s'}),$$

where  $b_i = (t - \theta)(t - \theta^q) \cdots (t - \theta^{q^{i-1}})$  for  $i > 0$  and  $b_0 = 1$ . We observe, as in [4, §3.2], that

$$\text{ev}_N(b_d(t_{s+1}) \cdots b_d(t_{s'}) \omega(t_{s+1}) \cdots \omega(t_{s'})) = \prod_{i=0}^r L_{d-i-1}^{n_i q^i} \tilde{\pi}^{-N}.$$

Again by (33) we have

$$\begin{aligned} \text{ev}_\chi(L(q^d, C_{s'}) b_d(t_1) \omega(t_1) \cdots b_d(t_s) \omega(t_s)) &= \\ &= L(q^d - N, \chi) \vartheta_{P_1}(\sigma_{P_1})^{N_1} \cdots \vartheta_{P_r}(\sigma_{P_r})^{N_r} g(\chi). \end{aligned}$$

Hence, we obtain the formula

$$\begin{aligned} \text{ev}_{N,\chi}(\tau^d(\mathbb{B}_{s'})) &= \\ &= (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \tilde{\pi}^{N-q^d} \vartheta_{P_1}(\sigma_{P_1})^{N_1} \cdots \vartheta_{P_r}(\sigma_{P_r})^{N_r} g(\chi) \times \\ &\quad L(q^d - N, \chi) \prod_{i=0}^r L_{d-i-1}^{n_i q^i} \text{ev}_\chi(b_d(t_1) \cdots b_d(t_s)). \end{aligned}$$

We set

$$\rho_{N,\chi,d} := \frac{\text{ev}_{N,\chi}(\tau^d(\mathbb{B}_{s'}))}{\vartheta_{P_1}^{N_1}(\sigma_{P_1}) \cdots \vartheta_{P_r}^{N_r}(\sigma_{P_r})} \in k_a K.$$

**Proposition 9.13.** *Let  $s' = s + \ell_q(N) \equiv 1 \pmod{q-1}$ ,  $s' \geq 1$ . The following properties hold.*

(1) *If  $s = 0$  and  $\ell_q(N) = 1$ , we assume that  $N \geq 2$ . We have:*

$$\rho_{N,\chi,0} = (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \tilde{\pi}^{N-1} g(\chi) \Pi(N)^{-1} \sum'_{d \geq 0} \sum_{b \in A_{+,d}} \chi(b) b'^N b^{-1}.$$

(2) *Let  $d \geq 1$  be an integer such that  $q^d > N$ . Then,*

$$\rho_{N,\chi,d} = (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} g(\chi) L(q^d - N, \chi) \tilde{\pi}^{N-q^d} \text{ev}_\chi(b_d(t_1) \cdots b_d(t_s)) \prod_{i=0}^r L_{d-1-i}^{n_i q^i}.$$

**9.5. A refinement of Herbrand-Ribet-Taelman Theorem.** As in the previous Section, let  $\chi$  be a Dirichlet character of type  $s \geq 0$  and conductor  $a$ . Following [4, §2.4], we introduce the *generalized Bernoulli-Carlitz numbers*  $\text{BC}_{n,\chi^{-1}}$  by means of the following generating series:

$$\frac{g(\chi)}{a} \sum_{b \in (A/aA)^\times} \frac{\chi(b)X}{\exp_C(\frac{X}{a}) - \sigma_b(\lambda_a)} = \sum_{i \geq 0} \frac{\text{BC}_{i,\chi^{-1}}}{\Pi(i)} X^i.$$

If  $s = 0$ , when  $a = 1$ , we set  $\lambda_a = 0$  in the above formula so that we get in this case

$$\text{BC}_{i,\chi^{-1}} = \text{BC}_i$$

for  $i \geq 0$ . From [4, Proposition 17], we deduce easily the following:

**Lemma 9.14.** *The following properties hold:*

- (1) For all  $i \geq 0$ , we have  $\text{BC}_{i,\chi^{-1}} \in k_a K$ .
- (2) If  $i \not\equiv s \pmod{q-1}$ , then  $\text{BC}_{i,\chi^{-1}} = 0$ .
- (3) We have  $\text{BC}_{0,\chi^{-1}} = 0$  if  $s \geq 1$ .
- (4) If  $i \geq 1$  is such that  $i \equiv s \pmod{q-1}$ , then

$$L(i, \chi)g(\chi) = \tilde{\pi}^i \text{BC}_{i,\chi^{-1}} \Pi(i)^{-1}.$$

We now consider integers  $s, s', N$  with  $s + \ell_q(N) = s'$ ,  $s \geq 1$ ,  $s' \equiv 1 \pmod{q-1}$  and a character  $\chi$  of type  $s'$  and conductor  $a = Pb$  such that

$$\chi = \vartheta_P^N \tilde{\chi},$$

with  $P$  a prime not dividing the conductor  $b$  of  $\tilde{\chi}$ , and such that  $N \leq q^d - 2$ ,  $d$  being the degree of  $P$ . The valuation ring of the compositum  $k_a \widehat{K_P}$  of  $k_a$  and  $\widehat{K_P}$  in  $\mathbb{C}_P$  is the ring  $\widehat{A_P}[k_a]$  where  $\widehat{A_P}$  the valuation ring of  $\widehat{K_P}$ .

We highlight that the congruences for the above generalized Bernoulli-Carlitz numbers that will be used in the proof of Theorem 9.15 are well defined thanks to the choice of the embedding of  $K^{ac}$  in  $\mathbb{C}_P$  that we made at the beginning of §9.1.

**9.5.1. An example.** We consider the simplest non-trivial case of a character  $\chi$  of type  $s' = 1$ . Here, the factor  $\tilde{\chi}$  is the trivial character and therefore  $s = 0$  and  $N = q^i$  so that  $\chi = \vartheta_P^{q^i}$  with  $d-1 \geq i \geq 0$ . The case  $i = 0$  is somewhat exceptional so that we assume that  $i > 0$ . By §5.1.3, we have  $H_{C_1} = (0)$  which implies, by Proposition 9.2, the triviality of  $H_{\chi^{q^j}}$ ,  $j \in \mathbb{N}$ . By Lemma 9.1,  $e_{\chi^{q^j}}(H_a \otimes_k k_a) = 0$  for all  $j \in \mathbb{N}$ , and therefore  $e_\chi(H_a \otimes_A \widehat{A_P}[k_a]) = (0)$ . Now we observe, by the second part of the Proposition 9.13, that

$$\frac{\text{BC}_{q^d - q^i} L_{d-1-i}^{q^i}}{\Pi(q^d - q^i)} = -\frac{1}{\theta^{q^d} - \theta^{q^i}},$$

where for  $n \in \mathbb{N}$ ,  $\text{BC}_n$  denotes the  $n$ th Bernoulli-Carlitz number (see [12, Chapter 9, paragraph 9.2]). In particular, the Bernoulli-Carlitz number  $\text{BC}_{q^d - q^i}$  is in this case  $P$ -integral and reduces to a unit modulo  $P$ .

**Theorem 9.15** (Refinement of Herbrand-Ribet-Taelman Theorem [21]). *Let  $\chi$  be a Dirichlet character with conductor  $a$  and type  $s' \equiv 1 \pmod{q-1}$ ,  $s' \geq 1$ . Let  $P$  be a prime dividing  $a$ , of degree  $d$ , and let us write  $\chi = \vartheta_P^N \tilde{\chi}$  with  $1 \leq N \leq q^d - 2$  and with  $\tilde{\chi}$  a Dirichlet character of conductor prime to  $P$ . We further suppose that if*

$s' = 1$ , then  $N$  is at least 2. The generalized Bernoulli-Carlitz number  $\text{BC}_{q^d-N, \tilde{\chi}^{-1}}$  is  $P$ -integral. Furthermore,

$$e_{\chi}(H_a \otimes_A \widehat{A}_P[k_a]) \neq (0)$$

if and only if

$$\text{BC}_{q^d-N, \tilde{\chi}^{-1}} \equiv 0 \pmod{P}.$$

*Proof.* We have already considered the case  $s' = 1$  in §9.5.1 so we may now suppose that  $s' \geq 2$ . We work in  $k_a \widehat{K}_P$ . Note that the Dirichlet character  $\tilde{\chi}$  is of type  $s \geq 0$  and that we have the congruence  $\tau^d(\mathbb{B}_s) \equiv \mathbb{B}_s \pmod{P}$ . Since obviously,  $\text{ev}_{N, \tilde{\chi}}(\mathbb{B}_{s'}) \equiv \text{ev}_{\chi}(\mathbb{B}_{s'}) \pmod{P}$ , we have that

$$e_{N, \tilde{\chi}}(\tau^d(\mathbb{B}_s)) \equiv \text{ev}_{\chi}(\mathbb{B}_s) \pmod{P}.$$

Let us write now  $\tilde{\chi} = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$ , where  $b = P_1 \cdots P_r$  is the conductor of  $\tilde{\chi}$  (we recall that  $N = \sum_{i=0}^{d-1} n_i q^i$ ,  $n_i \in \{0, \dots, q-1\}$ ). By the second part of the Proposition 9.13, we have:

$$\rho_{N, \tilde{\chi}, d} = (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \Pi(q^d - N)^{-1} \text{BC}_{q^d-N, \tilde{\chi}^{-1}} \text{ev}_{\tilde{\chi}}(b_d(t_1) \cdots b_d(t_s)) \prod_{i=0}^{d-1} L_{d-1-i}^{n_i q^i}.$$

This implies that  $\text{BC}_{q^d-N, \tilde{\chi}^{-1}}$  is  $P$ -integral. Moreover,  $\text{BC}_{q^d-N, \tilde{\chi}^{-1}} \equiv 0 \pmod{P}$  if and only if  $\text{ev}_{\chi}(\mathbb{B}_s) \equiv 0 \pmod{P}$ . We now set

$$[\chi] = \{\chi^{q^i}, i \geq 0\}$$

and we consider the element in  $A[k_a][\Delta_a]$ :

$$F = \sum_{\psi \in [\chi]} \text{ev}_{\psi}(\mathbb{B}_s) e_{\psi}.$$

In fact, we have that  $F \in A[\Delta_a]$ . We also set:

$$e_{[\chi]} = \sum_{\psi \in [\chi]} e_{\psi} \in k[\Delta_a].$$

We deduce from Corollary 9.3 that

$$\text{Fitt}_{e_{[\chi]} A[\Delta_a]} e_{[\chi]}(H_a) = F e_{[\chi]} A[\Delta_a].$$

This implies that

$$\text{Fitt}_{e_{[\chi]} \widehat{A}_P[k_a][\Delta_a]} e_{[\chi]}(H_a \otimes_A \widehat{A}_P[k_a]) = F e_{[\chi]} \widehat{A}_P[k_a][\Delta_a].$$

Therefore,

$$\text{Fitt}_{\widehat{A}_P[k_a]} e_{\chi}(H_a \otimes_A \widehat{A}_P[k_a]) = \text{ev}_{\chi}(\mathbb{B}_s) \widehat{A}_P[k_a].$$

□

## 10. LINK WITH OTHER TYPES OF $L$ -SERIES

In this Section we shortly explain the link between our  $L$ -series values and the global  $L$ -functions of Goss, Taguchi-Wan and Böckle-Pink (see [7, 9, 22]). The notation of this Section slightly differs from the notation of the rest of this paper.

The Carlitz module is usually seen as a functor from  $A$ -algebras to  $A$ -modules. In fact, it can also be viewed more naturally as a functor defined over the larger category of  $\tau$ -modules. We present this construction.

**10.1. The Carlitz functor over  $\tau$ -modules.** We consider a finitely generated  $k$ -algebra  $\mathbf{A}$  and the ring  $R = A \otimes_k \mathbf{A}$ , endowed with the  $\mathbf{A}$ -linear endomorphism  $\tau$  defined by  $\tau(a \otimes b) = a^q \otimes b$ .

**Definition 10.1.** A  $\tau$ -module  $M$  is an  $R$ -module  $M$  together with an  $\mathbf{A}$ -linear endomorphism  $\tau_M$  such that, for  $a \in R$  and  $m \in M$ ,  $\tau_M(am) = \tau(a)\tau_M(m)$  (we say that  $\tau_M$  is *semi-linear*). A *morphism*  $f : M_1 \rightarrow M_2$  of  $\tau$ -modules  $M_1, M_2$  is a morphism of  $R$ -modules which commutes with the  $\tau$ -module structures.

We define the *Carlitz functor*  $C$  from the category of  $\tau$ -modules to the category of  $R$ -modules in the following way. Let  $M$  be a  $\tau$ -module, with semi-linear endomorphism  $\tau_M$ . Then,  $C(M)$  is the  $R$ -module having  $M$  as underlying  $\mathbf{A}$ -module, and where the multiplication  $C_\theta$  by  $\theta$  is given by  $C_\theta = \theta + \tau_M$ . It is easy to show that this gives rise to a functor. This functor is faithful, but it is not fully faithful.

**Remark 10.2.** We can also define the Carlitz functor on the even larger category of  $\tau$ -sheaves.

Essentially, the basic case of this paper is  $\mathbf{A} = k[t_1, \dots, t_s]$ . We also considered  $\mathbf{A} = k_s = k(t_1, \dots, t_s)$  and  $\mathbf{A}$  an algebraic extension of  $k$ . We have studied the case of  $M$  free of rank one, that is,  $M = R = A \otimes_k \mathbf{A}$ , with  $\tau_M = \alpha\tau$ , with  $\alpha \in R^* = R \setminus \{0\}$ . Indeed, if  $M = R = A_s$  and  $\tau_M = \alpha\tau$  with  $\alpha \in R^*$ , then  $C(M)$  is the structure of  $A_s$ -module induced on  $A_s$  by the Drinfeld module of rank one of parameter  $\alpha$ .

Note that this is in apparent conflict with Definition 3.1 where we have defined Drinfeld modules over Tate algebras (and the parameter varies in Tate algebras). In fact, in the settings of the present Section, Definition 3.1 corresponds, with  $M$  as above, to an analytic realization (at the place infinity) of  $C(M)$ , just as the Carlitz module structure on  $\mathbb{C}_\infty$  is usually considered as an analytic realization of the Carlitz module structure  $C(A)$  over  $A$ .

**10.2. Exponential functions revisited.** Let  $M$  be a  $\tau$ -module. We suppose that it is endowed with a norm  $\|\cdot\|_M$  such that  $\|am\|_M = \|a\|\|m\|_M$  (we set  $\|x \otimes y\| = |x|$  in  $R$  if  $y \neq 0$ ), where  $a \in R$  and  $m \in M$  and that it is complete for this norm. Then, the *exponential function*  $\exp_M$  is the well defined, continuous, open map

$$\exp_M : \widehat{M} \rightarrow C(\widehat{M})$$

with  $\widehat{M} = M \widehat{\otimes} K_\infty$ , defined by  $\exp_M(m) = \sum_{i \geq 0} D_i^{-1} \tau_M^i(m)$ . It satisfies

$$\exp_M(am) = a \cdot \exp_M(m),$$

where the action of  $a \in R$  in the right is that given by the module structure of  $C(\widehat{M})$ . This notably happens when  $R = A[t_1, \dots, t_s]$  and  $M = \mathbb{T}_s \supset R$  with  $\tau_M = \alpha\tau$ ,  $\alpha \in R \setminus \{0\}$  so we recover the exponential functions of §3.1.

**10.3. L-series values revisited.** We consider here a new variable  $T$ . Let  $M$  be a  $\tau$ -module which is free of finite rank over  $R$  and  $P$  a prime of  $A$ . Then, the module  $M/PM$  is free of finite rank over  $\mathbf{A}$ . the *L-series value at one* of  $M$  is:

$$L(M, 1) = \prod_P \det_{\mathbf{A}[T]} (1 - T\tau_M | M/PM)^{-1} |_{T^{\deg_\theta(P)} \mapsto P^{-1}} \in 1 + \theta^{-1} \mathbf{A}[[\theta^{-1}]].$$

By [8, Lemma 8.2] we note that for all  $P$ ,  $\det_{\mathbf{A}[T]} (1 - T\tau_M | M/PM)^{-1} \in 1 + T^d \mathbf{A}[[T^d]]$ , where  $d$  is the degree of  $P$ . Hence, we can replace  $T^d$  by  $P^{-1}$  in the above



formal series and we get  $\det_{\mathbf{A}[T]}(1 - T\tau_M|M/PM)^{-1}|_{T^d=P-1} \in 1 + P^{-1}\mathbf{A}[[P^{-1}]]$ . Since for all  $d$  there are finitely many primes  $P$  with degree  $d$ , the product defining  $L(M, 1)$  converges in  $K_\infty \widehat{\otimes} \mathbf{A}$  to an element which belongs to  $1 + \theta^{-1}\mathbf{A}[[\theta^{-1}]]$ . This is a variant of the value of the *global L-function of M at one*, following Goss, Taguchi-Wan, and Böckle-Pink (see [9] for the definition of the global L-function associated to a  $\tau$ -sheaf).

With such a module  $M$ , we have that for all  $P$ ,  $C(M/PM)$  is also free of finite rank over  $\mathbf{A}$  (over  $\mathbf{A}$ , there is an isomorphism between  $M/PM$  and  $C(M/PM)$ ). The  $\mathcal{L}$ -value at one of  $C(M)$  is

$$\mathcal{L}(C(M), 1) = \prod_P \frac{[M/PM]_R}{[C(M/PM)]_R} \in 1 + \theta^{-1}\mathbf{A}[[\theta^{-1}]].$$

Here,  $[N]_R$  denotes the monic generator (in  $\theta$ ) of the Fitting ideal of an  $R$ -module  $N$  which is free and finitely generated over  $\mathbf{A}$ . If  $\mu_\theta$  denotes the product by  $\theta$  for this module structure, then  $[N]_R = \det_{\mathbf{A}[T]}(T - \mu_\theta|N)$ . This is essentially the  $L$ -series value of Taelman in [20]; the product converges as a consequence of the next Lemma:

**Lemma 10.3.** *Let  $M$  be a  $\tau$ -module which is free of finite rank over  $R$ . Then,  $L(M, 1) = \mathcal{L}(C(M), 1)$ .*

We omit the proof as this follows essentially the same lines of various other proofs in our paper. In particular, if  $\phi$  is a Drinfeld  $A_s$ -module of rank one of parameter  $\alpha \in A_s^*$ ,  $L(1, \phi) = \mathcal{L}(C(A_s), 1)$  can also be constructed starting from the Euler factors of the  $L$ -series  $L(M, 1)$  of the  $\tau$ -module  $M = A_s$  with  $\tau_M = \alpha\tau$ . In this respect, the  $L$ -series values that we introduce in this paper can be viewed as first examples of an alternative way of defining global  $L$ -functions, which moreover are rigid analytic, in contrast with the Goss'  $L$ -functions.

**10.4. The log-algebraic Theorem revisited.** As a final remark, we point out that also Theorem 8.1 can be viewed as a statement of integrality of the value of an exponential function associated to a certain  $\tau$ -module.

We consider again the algebra  $\mathcal{A}_r$  of §8; with the  $k$ -endomorphism  $\tau_{\mathbb{M}} := \tau$  and with the structure of  $R$ -module defined there (with  $R = A_{r+1}$ ), it becomes a  $\tau$ -module that we denote by  $\mathbb{M}$ . But this is not sufficient to interpret Theorem 8.1.

Now, as we have seen, the structure of  $A_{r+1}$ -module extends to a structure of  $\mathbb{T}_{r+1}$ -module. We then have the exponential function  $\exp_{\mathbb{M}} : \mathbb{M} \rightarrow C(\mathbb{M})$  and we have proved that

$$\exp_{\mathbb{M}}(L(1, \phi).[ZX_1 \cdots X_r]) = \exp_{\phi}(L(1, \phi)).[ZX_1 \cdots X_r],$$

where  $\phi$  is the Drinfeld module of rank one of parameter  $\alpha = t_0(t_1 - \theta) \cdots (t_s - \theta)$ ,  $\exp_{\phi}$  its exponential function, and  $L(1, \phi)$  its  $L$ -series value at one. Of course, this is just a way to rephrase Theorem 8.1.

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## 11. APPENDIX BY FLORENT DEMESLAY.

We shall work with Drinfeld  $R_s$ -modules rather than with  $A_s$ -modules<sup>(6)</sup>. As we have already seen, the benefit of this assumption comes from the fact that  $R_s$  is a principal ideal domain. We keep using the same notation adopted in the previous sections.

Let us choose  $\alpha \in R_s^*$  and let us consider the Drinfeld  $R_s$ -module of rank one and parameter  $\alpha$ , that is, the injective homomorphism of  $k_s$ -algebras

$$\phi : R_s \rightarrow \text{End}_{k_s\text{-lin.}}(K_{s,\infty})$$

given by  $\phi_\theta = \theta + \alpha\tau$ , where we recall that  $\tau : K_{s,\infty} \rightarrow K_{s,\infty}$  is the continuous morphism of  $k_s$ -algebras given by  $\tau(\theta) = \theta^q$ . Let  $M$  be an  $R_s$ -algebra together with a  $k_s$ -endomorphism  $\tau_M : M \rightarrow M$  such that

$$\tau_M(fm) = \tau(f)\tau_M(m), \quad f \in R_s, \quad m \in M.$$

We denote by  $\phi(M)$  the  $k_s$ -vector space  $M$  equipped with the structure of  $R_s$ -module induced by  $\phi$ , e.g.  $\forall m \in \phi(M)$  :

$$\theta.m = \alpha\tau_M(m) + \theta m.$$

We recall that we have the exponential function associated to  $\phi$ , which is a  $k_s$ -linear endomorphism of  $K_{s,\infty}$  defined by

$$\exp_\phi = \sum_{i \geq 0} \frac{1}{D_i} \tau_\alpha^i,$$

where  $\tau_\alpha = \alpha\tau$ . It also is a morphism of  $R_s$ -modules. We recall that if  $u(\alpha)$  is the maximum of the integer part of  $\frac{r-q}{q-1}$  and 0, with  $r = -v_\infty(\alpha)$ , then  $\exp_\phi$  induces an isometric  $k_s$ -linear automorphism  $\mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1} \rightarrow \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}$ . Notice that  $\mathfrak{m}_{K_{s,\infty}} = \theta^{-1}k_s[[\theta^{-1}]]$ .

**Definition 11.1.** A sub- $k_s$ -vector space  $M$  of  $K_{s,\infty}$  is said to be *discrete* if there exists an integer  $n \geq 1$  such that  $M \cap \mathfrak{m}_{K_{s,\infty}}^n = \{0\}$  (note that this is equivalent to the fact that the intersection  $M \cap \mathfrak{m}_{K_{s,\infty}}$  is a finite dimensional  $k_s$ -vector space).

**Lemma 11.2.** *Let  $M \neq (0)$  be a sub- $R_s$ -module of  $K_{s,\infty}$ . The following assertions are equivalent:*

- (1)  $M$  is discrete,
- (2)  $M$  is a free  $R_s$ -module of rank one.

*Proof.* The fact that the property (2) implies the property (1) is clear. Let us prove that the property (1) implies the property (2). Let  $f$  be a non-zero element of  $M$ . Then,  $R_s \subset f^{-1}M$  and  $f^{-1}M$  is discrete in  $K_{s,\infty}$ . Thus, we can assume that  $R_s \subset M$ . We now observe that we have a direct sum of  $k_s$ -vector spaces:

$$K_{s,\infty} = R_s \oplus \mathfrak{m}_{K_{s,\infty}} = R_s \oplus \theta^{-1}k_s[[\theta^{-1}]].$$

Since  $M$  is discrete we deduce from the above decomposition that  $M/R_s$  is a finite  $k_s$ -vector space. But  $M/R_s$  is also a  $R_s$ -module, hence a torsion  $R_s$ -module. Therefore there exists  $r \in R_s \setminus (0)$  such that  $rM \subset R_s$ . Since  $R_s$  is a noetherian

<sup>6</sup>The results of this appendix can be generalized to  $R_s$ -Drinfeld modules of arbitrary rank defined over  $L(t_1, \dots, t_s)$  where  $L/K$  is a finite extension. We refer the interested reader to a forthcoming work of the author of this appendix.

ring, we obtain that  $M$  is a finitely generated  $R_s$ -module of rank 1. Since  $R_s$  is a principal ideal domain we deduce that  $M$ , as an  $R_s$ -module, is free of rank one.  $\square$

**Remark 11.3.** Let us consider the following  $R_s$ -module:

$$V_\phi = \frac{\phi(K_{s,\infty})}{\phi(R_s) + \exp_\phi(K_{s,\infty})}.$$

One can show as in Corollary 5.7 that  $V_\phi$  is a  $k_s$ -vector space of dimension  $\leq u(\alpha)$ . We notice that  $R_s + \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1} \subset R_s + \exp_\phi(K_{s,\infty})$ . We observe that  $R_s$  and  $\text{Ker}(\exp_\phi)$  are discrete sub- $R_s$ -modules of  $K_{s,\infty}$  which implies that  $\exp_\phi^{-1}(R_s)$  is a discrete sub- $R_s$ -module of  $K_{s,\infty}$ . The exponential  $\exp_\phi$  then produces an exact sequence of  $k_s$ -vector spaces

$$0 \rightarrow \frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s) + \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}} \rightarrow \frac{K_{s,\infty}}{R_s + \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}} \rightarrow V_\phi \rightarrow 0.$$

In particular,  $\exp_\phi^{-1}(R_s) \neq (0)$  and we obtain that  $\exp_\phi^{-1}(R_s)$  is free of rank one by using Lemma 11.2.

11.1. *L-series.* Let  $P$  be a prime of  $A$ . Then the  $R_s$ -module  $\phi(\frac{R_s}{PR_s})$  is finitely generated and torsion. One can show that the product over the primes of  $A$

$$\mathcal{L}(\phi/R_s) = \prod_P \frac{[\frac{R_s}{PR_s}]_{R_s}}{[\phi(\frac{R_s}{PR_s})]_{R_s}}$$

converges in  $K_{s,\infty}$ . We will only give a sketch of proof of the next Theorem, as the proof is very close to ideas developed by Taelman in [20].

**Theorem 11.4** (class number formula for  $\mathcal{L}(\phi/R_s)$ ). *The following identity holds in  $K_{s,\infty}$ :*

$$\mathcal{L}(\phi/R_s) = [V_\phi]_{R_s} [R_s : \exp_\phi^{-1}(R_s)]_{R_s}.$$

11.2. **Nuclear operators and determinants.** This Section is inspired by [20, Section 2]. Let  $(V, \|\cdot\|)$  be a  $k_s$ -vector space equipped with a non-archimedean absolute value which is trivial on  $k_s$  and such that every open  $k_s$ -subspace  $U \subset V$  is of finite  $k_s$ -co-dimension. Let's give a typical example of such objects: let  $M$  be a non-trivial, discrete  $R_s$ -submodule of  $K_{s,\infty}$ , then  $V = \frac{K_{s,\infty}}{M}$  satisfies the above hypothesis.

Let  $f$  be a continuous endomorphism of  $V$ , we say that  $f$  is *locally contracting* if there exists an open subspace  $U \subset V$  and a real number  $0 < c < 1$  such that, for all  $v \in U$ ,

$$\|f(v)\| \leq c\|v\|.$$

Any such open subspace  $U \subset V$  which moreover satisfies  $f(U) \subset U$  is called a *nucleus* for  $f$ . Observe that any locally contracting continuous endomorphism of  $V$  has a nucleus. Let's give an example: the map

$$\tau_\alpha : \frac{K_{s,\infty}}{R_s} \rightarrow \frac{K_{s,\infty}}{R_s}$$

is locally contracting and the image of  $\mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+3}$  in  $\frac{K_{s,\infty}}{R_s}$  is a nucleus.

Observe that any finite collection of locally contracting endomorphisms of  $V$  has a common nucleus (see for example [20], Proposition 6). Furthermore if  $f$  and  $g$  are locally contracting, then so are the sum  $f + g$  and the composition  $fg$ .

For any integer  $N \geq 0$ , we set:

$$\frac{V[[Z]]}{Z^N} = V \otimes_{k_s} \frac{k_s[[Z]]}{Z^N},$$

and we denote by  $V[[Z]]$  the inverse limit of the  $V[[Z]]/Z^N$  equipped with the limit topology. Observe that any continuous  $k_s[[Z]]$ -endomorphism

$$F : V[[Z]] \rightarrow V[[Z]]$$

is of the form:

$$F = \sum_{n \geq 0} f_n Z^n,$$

where  $f_n$  is a continuous  $k_s$ -endomorphism of  $V$ . Similarly, any continuous  $\frac{k_s[[Z]]}{Z^N}$ -linear endomorphism of  $\frac{V[[Z]]}{Z^N}$  is of the form  $\sum_{n=0}^{N-1} f_n Z^n$ . We say that a continuous  $k_s[[Z]]$ -linear endomorphism  $F$  of  $V[[Z]]$  (resp. of  $\frac{V[[Z]]}{Z^N}$ ) is *nuclear* if for all  $n$  (resp. for all  $n < N$ ), the  $k_s$ -endomorphism  $f_n$  of  $V$  is locally contracting.

Let  $F$  be a nuclear endomorphism of  $\frac{V[[Z]]}{Z^N}$ . Let  $U_1$  and  $U_2$  be common nuclei for the  $f_n$ ,  $n < N$ . Since Proposition 8 in [20] is valid in our context,

$$\det_{\frac{k_s[[Z]]}{Z^N}} \left( 1 + F|_{\frac{V}{U_i}[[Z]]/Z^N} \right) \in \frac{k_s[[Z]]}{Z^N}$$

is independent of  $i \in \{1, 2\}$ . We denote this determinant by

$$\det_{\frac{k_s[[Z]]}{Z^N}} (1 + F|_V).$$

If  $F$  is a nuclear endomorphism of  $V[[Z]]$ , then we denote by  $\det_{k_s[[Z]]} (1 + F|_V)$  the unique power series that reduces to  $\det_{\frac{k_s[[Z]]}{Z^N}} (1 + F|_V)$  modulo  $Z^N$  for any  $N$ .

Note that Proposition 9, Proposition 10, Theorem 2 and Corollary 1 of [20] are also valid in our context.

**11.3. Taelman's trace formula.** Observe that any element in  $R_s[\tau]\tau$  induces a  $k_s$ -linear continuous endomorphism of  $\frac{K_{s,\infty}}{R_s}$  which is locally contracting. Denote by  $R_s[\tau][[Z]]$  the ring of formal power series in  $Z$  with coefficients in  $R_s[\tau]$ , the variable  $Z$  being central (i.e.  $\tau Z = Z\tau$ ).

Let  $P_1, \dots, P_n$  be  $n$  distinct primes of  $A$ . Let us set

$$R = R_s \left[ \frac{1}{P_1}, \dots, \frac{1}{P_r} \right].$$

Let  $P$  be a monic prime of  $A$ . Let  $K_{s,P}$  be the  $P$ -adic completion of  $k_s(\theta)$  (with respect to the  $P$ -adic valuation on  $k_s(\theta)$  which is trivial on  $k_s$  and the usual one on  $K$ ). Observe that every element of  $K_{s,P}$  can be written in a unique way:

$$\sum_{i \geq m} x_i P^i,$$

where  $m \in \mathbb{Z}$ ,  $x_i \in R_s$  of degree in  $\theta$  strictly less than  $\deg_\theta P$ .

We also define:

$$K_{s,S} = K_{s,\infty} \times K_{s,P_1} \times \dots \times K_{s,P_r}.$$

Observe that  $R$  is discrete in  $K_{s,S}$ . Let  $P$  be a prime of  $A$ ,  $P \neq P_1, \dots, P_r$ . Let  $R_{s,P}$  be the valuation ring of  $K_{s,P}$ . Then:

$$K_{s,P} = R_{s,P} + R[1/P].$$

Furthermore, the inclusion  $R_s \subset R$  induces an isomorphism:

$$\frac{R_s}{PR_s} \simeq \frac{R}{PR}.$$

Let  $F \in R[\tau][[Z]]\tau Z$ . Then  $F$  defines  $k_s$ -endomorphisms of  $\frac{K_{s,\infty}}{R}[[Z]]$ ,  $\frac{R}{PR}[[Z]]$ ,  $\frac{K_{s,\infty} \times K_{s,P}}{R[1/P]}[[Z]]$ . Now Taelman's localization Lemma ([20] Lemma 1) remains valid in our case:

**Lemma 11.5.** *Let us choose  $F \in R[\tau][[Z]]\tau Z$ . Then:*

$$\det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{R}{PR}} \right) = \frac{\det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{K_{s,S} \times K_{s,P}}{R[1/P]}} \right)}{\det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{K_{s,S}}{R}} \right)}.$$

We also have in our case:

**Theorem 11.6.** *Let  $F \in R_s[\tau][[Z]]\tau Z$ . Then we have an equality in  $k_s[[Z]]$  :*

$$\prod_{P \text{ monic prime of } A} \det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{R_s}{PR_s}} \right) = \det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{K_{s,\infty}}{R_s}} \right)^{-1}.$$

*Proof.* This is a consequence of Lemma 11.5 and the proof of Theorem 3 in [20]. Note that in our case in [20] page 383 line  $-5$  we replace the original assumption of Taelman by the assumption that  $R$  has no maximal ideal of the form  $PR$ ,  $P$  monic prime of  $A$ , such that  $\dim_{k_s} \left( \frac{R}{PR} \right) < D$ .  $\square$

**11.4. Fitting ideals.** Let  $f : K_{s,\infty} \rightarrow K_{s,\infty}$  be a continuous  $k_s$ -linear map. Let  $M_1$  and  $M_2$  be two free  $R_s$ -modules of rank one in  $K_{s,\infty}$  such that  $f(M_1) \subset M_2$ . Then  $f$  induces a  $k_s$ -continuous linear map

$$f : \frac{K_{s,\infty}}{M_1} \rightarrow \frac{K_{s,\infty}}{M_2}.$$

We say that  $f$  is *infinitely tangent to the identity* on  $K_{s,\infty}$  if for any  $N \geq 0$  there exists an open  $k_s$ -subspace  $U_N \subset K_{s,\infty}$  such that the following properties hold:

- (1)  $U_N \cap M_1 = U_N \cap M_2 = \{0\}$ ,
- (2)  $f$  restricts to an isometry between the images of  $U_N$ ,
- (3)  $\forall u \in U_N$ ,  $v_\infty(f(u) - u) \geq N + v_\infty(u)$ .

If  $f \in K_{s,\infty}[[\tau]]$  is such that this power series is convergent on  $K_{s,\infty}$  and satisfies that  $f(M_1) \subset M_2$ , for some free  $R_s$ -modules of rank one  $M_1$  and  $M_2$ , then, by the proof of Proposition 12 in [20],  $f$  is infinitely tangent to the identity on  $K_{s,\infty}$ . A typical example is given by:  $M_1 = \exp_\phi^{-1}(R_s)$ ,  $M_2 = R_s$  and  $f = \exp_\phi$ .

Now let  $M_1, M_2$  be two free  $R_s$ -modules of rank one in  $K_{s,\infty}$ . Let  $H_1, H_2$  be two finite dimensional  $k_s$ -vector spaces that are also  $R_s$ -modules. Set:

$$L_i = \frac{K_{s,\infty}}{M_i} \times H_i.$$

Let  $f : L_1 \rightarrow L_2$  be a  $k_s$ -linear map which is bijective and continuous. We shall write:

$$\Delta_f = \frac{1 - f^{-1}\theta f Z}{1 - \theta Z} - 1 = \sum_{n \geq 1} (\theta - f^{-1}\theta f) \theta^{n-1} Z^n.$$

We observe that  $\Delta_f$  defines a  $k_s$ -endomorphism of  $L_1[[Z]]$ . Let's assume that  $f$  induces (see [20] page 385 line -6) a continuous  $k_s$ -linear map

$$\frac{K_{s,\infty}}{M_1} \rightarrow \frac{K_{s,\infty}}{M_2}$$

which is infinitely tangent to the identity on  $K_{s,\infty}$ . Then, by the proof of Theorem 4 in [20], we get that  $\Delta_f$  is nuclear, and

$$\det_{k_s[[Z]]}(1 + \Delta_f|_{L_1})_{Z=\theta^{-1}} = [M_1 : M_2]_{R_s} \frac{[H_2]_{R_s}}{[H_1]_{R_s}}.$$

**11.5. Proof of Theorem 11.4.** We set, as in [20] paragraph 5:

$$F = \frac{1 - \phi_\theta Z}{1 - \theta Z} - 1 = \sum_{n \geq 1} (\theta - \phi_\theta) \theta^{n-1} Z^n \in R_s[\tau][[Z]]\tau Z.$$

We have:

$$\mathcal{L}(\phi/R_s) = \prod_{P \text{ monic prime in } A} \left( \det_{k_s[[Z]]}(1 + F|_{\frac{R_s}{PR_s}}) \right)_{Z=\theta^{-1}}^{-1}.$$

By Theorem 11.6, we get:

$$\mathcal{L}(\phi/R_s) = \det_{k_s[[Z]]} \left( 1 + F|_{\frac{K_{s,\infty}}{R_s}} \right) |_{Z=\theta^{-1}}.$$

We consider the short exact sequence of  $R_s$ -modules induced by  $\exp_\phi$ :

$$0 \rightarrow \frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s)} \rightarrow \frac{\phi(K_{s,\infty})}{\phi(R_s)} \rightarrow V_\phi \rightarrow 0.$$

Since  $R_s$  is a principal ideal domain and the  $R_s$ -module  $\frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s)}$  is divisible, this sequence splits. The choice of a section gives an  $R_s$ -isomorphism:

$$\frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s)} \times V_\phi \simeq \frac{\phi(K_{s,\infty})}{\phi(R_s)}.$$

This isomorphism gives rise to an isomorphism of  $k_s$ -vector space:

$$\frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s)} \times V_\phi \simeq \frac{K_{s,\infty}}{R_s}.$$

We denote this map by  $f$ . Then, by the proof of Lemma 6 in [20],  $f$  is infinitely tangent to the identity on  $K_{s,\infty}$ . But observe that on  $\frac{K_{s,\infty}}{R_s}[[Z]]$ :

$$1 + F = \frac{1 - f\theta f^{-1}Z}{1 - \theta Z}.$$

Thus:

$$\det_{k_s[[Z]]} \left( 1 + F|_{\frac{K_{s,\infty}}{R_s}} \right) |_{Z=\theta^{-1}} = [V_\phi]_{R_s} [R_s : \exp_\phi^{-1}(R_s)]_{R_s}.$$

The proof of our Theorem follows.

## REFERENCES

- [1] G. W. Anderson. *t-motives*. Duke Math. J. Volume 53 (1986), 457-502.
- [2] G. W. Anderson. *Log-Algebraicity of Twisted A -Harmonic Series and Special Values of L-Series in Characteristic p*. J. of Number Theory 60 (1996) ,165-209.
- [3] G. Anderson, D. Thakur, *Tensor powers of the Carlitz module and zeta values*. Annals of Math. 132 (1990), 159-191.
- [4] B. Anglès, F. Pellarin, *Functional identities for L-series values in positive characteristic*. To appear in J. of Number Theory, hal-00955673.
- [5] B. Anglès, F. Pellarin, *Universal Gauss-Thakur sums and L-series*. Preprint (2014) see also arXiv: 1301.3608.
- [6] B. Anglès, L. Taelman, *Arithmetic of characteristic p special L-values*. Preprint (2012), arXiv: 1205.2794.
- [7] G. Böckle. *Global L-functions over function fields*. Math. Ann. **323** (2002), 737-795.
- [8] G. Böckle. *Cohomological theory of crystals over function fields and applications*. CRM, Bellaterra, 2010.
- [9] G. Böckle & R. Pink. *Cohomological Theory of Crystals over Function Fields*. Tracts in Mathematics 9, European Mathematical Society (2010).
- [10] L. Carlitz. *On certain functions connected with polynomials in a Galois field*. Duke Math. J. 1 (1935), 137-168.
- [11] J. Fresnel, M. van der Put, *Rigid Analytic Geometry and Its Applications*. Birkhäuser, 2004.
- [12] D. Goss, *Basic Structures of Function Field Arithmetic*. Springer, Berlin, 1996.
- [13] D. Goss, *On the L-series of Pellarin*. J. of Number Theory 133 (2013), 955-962.
- [14] S. Lang, *Algebra*. Revised third edition, Springer, 2002.
- [15] F. Pellarin, *Values of certain L-series in positive characteristic*. Ann. of Math. 176 (2012), 2055-2093.
- [16] R. Perkins, *On Pellarin's L-series*. To appear in Proc. of the A.M.S. arXiv: 1201.0030.
- [17] M. van der Put & M. F. Singer, *Galois theory of linear differential equations*. Springer, 2003.
- [18] M. Rosen, *Number Theory in Function Fields*. Springer, 2002.
- [19] L. Taelman, *A Dirichlet unit theorem for Drinfeld modules*. Math. Ann. 348 (2010), 899-907.
- [20] L. Taelman, *Special L-values of Drinfeld modules*. Annals of Math. 75 (2012), 369-391.
- [21] L. Taelman, *A Herbrand-Ribet theorem for function fields*. Invent. Math. 188 (2012), 253-275.
- [22] Y. Taguchi & D. Wan. *L-functions of  $\varphi$ -sheaves and Drinfeld modules*. J. of the Amer. Math. Soc. **9**, (1996).
- [23] D. Thakur, *Gauss sums for  $\mathbb{F}_q[t]$* . Invent. Math. 94 (1988), 105-112.
- [24] J. Yu. *Transcendence and Special Zeta Values in Characteristic p*. Ann. of Math. 134 (1991), 1-23.

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