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# ARITHMETIC OF POSITIVE CHARACTERISTIC $L$ -SERIES VALUES IN TATE ALGEBRAS

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WITH AN APPENDIX BY F. DEMESLAY<sup>1</sup>.

ABSTRACT. In this paper we introduce *Drinfeld modules over the Tate algebras*. We study in detail the case of modules of rank one. For some of these modules (defined over a certain polynomial ring) we associate  $L$ -series values following [18] and [5]. The main purpose of this paper is to study these  $L$ -series values at one. By using the results of the appendix written by F. Demeslay (a *class number formula* for  $L$ -series values inspired by the work of Taelman [24]) we show that these  $L$ -series are, in a sense which will be clarified, *circular units*. This leads us to study the *class module* and the *unit module* associated to such Drinfeld modules, again inspired by Taelman's work. As a consequence of our investigations, we will obtain a new proof and a refinement of *Anderson log-algebraic Theorem* for the Carlitz module [3], and a refinement of *Herbrand-Ribet-Taelman Theorem* in [25]. The appendix makes use of the theory of *nuclear operators* as in [23].

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## 1. INTRODUCTION

In the following, we fix a finite field  $k$  with  $q$  elements; we denote by  $p$  its characteristic. We further denote by  $A$  the polynomial ring  $k[\theta]$  in an indeterminate  $\theta$  and we denote by  $K$  its field of fractions. We also consider the field  $K_\infty = k((\theta^{-1}))$ , the completion of  $K$  with respect to the place at infinity; we denote by  $|\cdot|$  the norm of  $K_\infty$  normalized by setting  $|\theta| = q$ . We denote by  $\mathbb{C}_\infty$  the completion of a fixed algebraic closure of  $K_\infty$  and we choose an embedding of an algebraic closure  $K^{ac}$  of  $K$  in  $\mathbb{C}_\infty$ .

Carlitz [9] focused on the so-called *Carlitz zeta values*

$$\zeta_C(n) := \sum_{a \in A_+} a^{-n} \in K_\infty, \quad n > 0$$

as some analogues, up to a certain extent, of the classical zeta values

$$\zeta(n) = \sum_{k > 0} k^{-n} \in \mathbb{R}$$

( $n > 1$ ). In the definition of  $\zeta_C(n)$ ,  $A_+$  denotes the set of monic polynomials in  $A$  and provides a kind of substitute of the set of positive integers. The analogy is however not complete, as, for example, a sum of monic polynomials may not be monic. Nevertheless, the Carlitz zeta values offer interesting analogies with the classical zeta values. Let us look at the archimedean example of the divergent series

$$(1) \quad \zeta(1) = \sum_{k \geq 1} k^{-1} = \prod_p \left(1 - \frac{1}{p}\right)^{-1} = \infty$$

which we have developed as a divergent eulerian product (running over the prime numbers  $p$ ). For a commutative ring  $R$  and a functor  $G$  from  $R$ -algebras to  $R$ -modules, we denote by  $\text{Lie}(G)$  the functor from  $R$ -algebras to  $R$ -modules defined, for  $B$  an  $R$ -algebra, by:

$$\text{Lie}(G)(B) = \text{Ker}(G(B[\epsilon]/(\epsilon^2)) \rightarrow G(B)).$$

The local factor at  $p$  in (1) is

$$\left(1 - \frac{1}{p}\right)^{-1} = \frac{p}{p-1} = \frac{|\text{Lie}(\mathbb{G}_m)(\mathbb{Z}/p\mathbb{Z})|}{|\mathbb{G}_m(\mathbb{Z}/p\mathbb{Z})|},$$

where  $|\cdot|$  denotes here the cardinality of a set. The above cardinalities can also be seen as positive generators of *Fitting ideals* of finite  $\mathbb{Z}$ -modules (see §5.2).

In parallel, let  $C$  be the *Carlitz module* functor from  $A$ -algebras to  $A$ -modules (see §2.1). Then, for  $P$  a *prime* of  $A$  (<sup>1</sup>) the module  $C(A/PA)$  is a finite  $A$ -module and one can easily prove (in different ways; read Goss, [13, Theorem 3.6.3], Taelman, [23, Proposition 1], see also Anderson and Thakur's paper [4, Proposition 1.2.1]) that  $P-1$  is the monic generator of the Fitting ideal of  $M$ . For a finitely generated and torsion  $A$ -module  $M$ ,  $[M]_A$  denotes the monic generator of its Fitting ideal. Then,

$$[C(A/PA)]_A = P-1$$

---

<sup>1</sup>A prime of  $A$  is an irreducible monic polynomial of  $A$ .

and

$$(2) \quad \zeta_C(1) = \prod_P \left(1 - \frac{1}{P}\right)^{-1} = \prod_P \frac{P}{P-1} = \prod_P \frac{[\text{Lie}(C)(A/PA)]_A}{[C(A/PA)]_A}.$$

The tensor powers of the Carlitz module functor introduced by Anderson and Thakur [4] provide a way to interpret the values  $\zeta_C(n)$  as well, and this can be viewed as one of the main sources of analogies between the theory of the Carlitz zeta values and the values of the Riemann zeta function at integers  $n \geq 2$ .

Carlitz proves that for all  $n > 0$  divisible by  $q-1$ ,  $\zeta_C(n)$  is, up to a scalar factor of  $K^\times$ , proportional to  $\tilde{\pi}^n$ , where the quantity  $\tilde{\pi}$  is defined by

$$(3) \quad \tilde{\pi} = \sqrt[q-1]{\theta - \theta^q} \prod_{i \geq 1} \left(1 - \frac{\theta^{q^i} - \theta}{\theta^{q^{i+1}} - \theta}\right) \in \theta \sqrt[q-1]{-\theta} (1 + \theta^{-1}k[[\theta^{-1}]])$$

unique up to multiplication by an element of  $k^\times$ , see Goss, [13, Chapter 3]. It is very easy to show that  $\tilde{\pi}$  is transcendental over  $K$ ; much easier than proving the transcendence of its classical counterpart  $2\pi i \in \mathbb{C}$  (see for example [16]).

If  $q-1$  does not divide  $n$ , the behavior of  $\zeta_C(n)$  is completely different. Only recently, it has been possible to show that  $\tilde{\pi}$  and the values  $\zeta_C(n)$  for  $n > 0$  not divisible by  $q-1$  and by the characteristic  $p$  are algebraically independent over  $K$  (see the work of Chieh-Yu Chang and Jing Yu [10], where they use the Tannaka approach introduced by Papanikolas in [17] and the algebraic independence theory of Anderson, Brownawell, Papanikolas in [1]).

This result is accessible due to a result of Anderson and Thakur [4, Theorem 3.8.3] implying that for all  $n > 0$ ,  $\zeta_C(n)$  is a linear combination of *polylogarithms* of order  $n$  of elements of  $K$  with coefficients in  $K$  <sup>(2)</sup>.

In fact, the transcendence of  $\zeta_C(1)$  was already accessible a long time prior to the work of the above-mentioned authors due to a formula of Carlitz, which is now a special case of Anderson and Thakur result. Let  $\exp_C$  be the *Carlitz exponential* (see §2.1 for the background about this function) Carlitz proved the formula

$$(4) \quad \exp_C(\zeta_C(1)) = 1.$$

Knowing that 1 belongs to the domain of convergence of the Carlitz logarithm  $\log_C$ , reciprocal of  $\exp_C$  (see §2.1.2) and comparing the absolute values of  $\log_C(1)$ ,  $\zeta_C(1)$ ,  $\tilde{\pi}$  we see that the above formula is equivalent to

$$(5) \quad \zeta_C(1) = \log_C(1).$$

Taelman [23] recently exhibited an appropriate setting to interpret the above formula as an instance of the *class number formula*. His approach, involving determinants of Fredholm operators, also relies on the formula (2). He did this in the broader framework of *Drinfeld modules* defined over the ring of integers  $R$  of a finite extension  $L$  of  $K$ . Taelman associated, to such a Drinfeld module  $\phi$ , a finite  $A$ -module called the *class module* (of  $\phi$  over  $L$ ), and a finitely generated  $A$ -module called the *unit module* (of  $\phi$  over  $L$ ). An *L-series value*  $L(\phi/R)$  that he also defines is then equal to the product of the monic generator of the Fitting ideal of the class module times the regulator of the unit module (see Theorem 1 of loc. cit.).

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<sup>2</sup>In our setting, what we call polylogarithms are Hadamard powers of the Carlitz logarithm series, see [4]. We review in §2.1.2 the elementary properties of the Carlitz logarithm.

In the case of  $\phi = C$  the Carlitz module, and  $L = K$ , the  $L$ -series value is equal to  $\zeta_C(1)$ , the class module is trivial, and the regulator of the unit module is  $\log_C(1)$ , the Carlitz logarithm of 1, yielding (4).

**1.1. Our point of view.** The class number formula of Taelman is for us one of the main sources of inspiration. One of the purposes of the present paper is to show how a class of  $L$ -series, recently introduced by the second author [18] and studied by the first and second author in [5], nicely fits in the theory of Taelman. We now describe the novelty of our point of view as well as our results.

The  $L$ -series values of Taelman are typical elements of  $\mathbb{C}_\infty$ . In particular, the Carlitz zeta values  $\zeta_C(n)$  are elements of  $K_\infty$ . It turns out that the theory can be generalized in another direction, introducing  $L$ -series values which are elements of *Tate algebras* of positive dimension. The “classical”  $L$ -series values then are just elements of zero-dimensional Tate algebras so that, with our point of view in mind, the classical theory is the zero-dimensional case.

One way to explain our construction of “generalized Carlitz zeta values” (or  $L$ -series values) is to start examining the special values of Goss abelian  $L$ -functions. We recall from Goss, [13, Chapter 8], that for  $a \in A_+$  squarefree, a *Dirichlet character modulo  $a$*  is a group homomorphism

$$\chi : \left( \frac{A}{aA} \right)^\times \rightarrow (k^{ac})^\times,$$

where  $k^{ac}$  denotes the algebraic closure of  $k$  in  $\mathbb{C}_\infty$ . A Dirichlet character modulo  $a$  is said to be *primitive of conductor  $a$*  if it is not induced by any character modulo a proper divisor of  $a$ . We lift such a primitive character  $\chi$  to a map  $A \rightarrow k^{ac}$  by setting  $\chi(b) = 0$  if  $b$  is not invertible in  $A/aA$  and  $\chi(b) = \chi(b + aA)$  otherwise. Unless otherwise specified, the Dirichlet characters that we use in this paper are all primitive. For an integer  $n > 0$ , the  $L$ -series value at  $n$  associated to a primitive character  $\chi$  of conductor  $a$  is the sum of the convergent series

$$L(n, \chi) = \sum_{b \in A_+} \chi(b) b^{-n} \in \mathbb{C}_\infty.$$

An interesting particular case is when we can write, for  $b \in A$ ,

$$(6) \quad \chi(b) = b(\zeta_1) \cdots b(\zeta_d),$$

where  $\zeta_1, \dots, \zeta_d$  are the roots of  $a$  in  $k^{ac}$ . In this case, the image of  $\chi$  lies in  $k$  and is equal to the map  $b \mapsto \text{Res}_\theta(a, b)$  where  $\text{Res}_\theta(a, b)$  denotes the resultant of  $a, b$  with respect to the indeterminate  $\theta$  as defined in [15, §IV.8] (we have used Proposition 8.3 of loc. cit. and that  $a, b$  are monic); we then have  $L(n, \chi) \in K_\infty$ .

**1.1.1. The definition of  $L$ -series values.** We generalize the notion of character. We recognize that  $k$  is the subfield of  $\mathbb{C}_\infty$  fixed by the Frobenius automorphism  $\tau$  defined by  $\tau(x) = x^q$  for all  $x \in \mathbb{C}_\infty$ . The *Tate algebra*  $\mathbb{T}_s$  of dimension  $s$  is the completion of the polynomial algebra  $\mathbb{C}_\infty[t_1, \dots, t_s]$  for the *Gauss norm* (see §2.2) and we have  $\mathbb{T}_0 = \mathbb{C}_\infty$ . We may legitimately replace, in the above discussion,  $k$  by  $k[t_1, \dots, t_s]$  <sup>(3)</sup>, the fixed subring of the  $k[t_1, \dots, t_s]$ -linear extension of  $\tau$  to  $\mathbb{T}_s$ .

What then generalize the Dirichlet characters (6) in this setting are the maps

$$\rho_\alpha : A \rightarrow k[t_1, \dots, t_s]$$

---

<sup>3</sup>This is only one of the various offered possibilities; see §2.2.1.

defined by

$$\rho_\alpha(b) = \text{Res}_\theta(b, \alpha),$$

where  $\alpha$  is a polynomial of  $A[t_1, \dots, t_s] \setminus \{0\}$  and where  $\text{Res}_\theta(P, Q)$  denotes again the resultant of two polynomials  $P, Q$  in  $\theta$  (the reason for which we choose  $\text{Res}_\theta(b, \alpha)$  instead of  $\text{Res}_\theta(\alpha, b)$  will become clear later; it depends on the way (14) we have customized the factorization of  $\alpha$ ).

The typical  $L$ -series value at  $n > 0$  that we handle in this paper is of the form

$$(7) \quad L(n, \alpha) = \sum_{b \in A_+} \text{Res}_\theta(b, \alpha) b^{-n} = \prod_P \left( 1 - \frac{\rho_\alpha(P)}{P^n} \right)^{-1}$$

(the product is taken over the primes  $P$  of  $A$ ). It is easy to see that this series converges in the Tate algebra  $\mathbb{T}_s$  for the Gauss norm; see §4.

**1.2. The local factors of the  $L$ -series values.** The formula (2) suggests that there should be a structure in the eulerian product (7), at least in the case  $n = 1$ . We realize it by introducing a generalization of Drinfeld modules as follows.

Classically, a Drinfeld module  $\phi$  of rank  $r$  is the datum of an injective  $k$ -algebra homomorphism

$$\phi : A \rightarrow \text{End}_{k\text{-lin}}(\mathbb{C}_\infty),$$

uniquely defined by the image of  $\theta$ , that is, the value  $\phi_\theta$  of  $\phi$  at  $\theta$ , which is of the form

$$(8) \quad \phi_\theta = \theta + \alpha_1 \tau + \dots + \alpha_r \tau^r,$$

where the *parameters*  $\alpha_1, \dots, \alpha_r$  are elements of  $\mathbb{C}_\infty$  with  $\alpha_r \neq 0$ . We use the  $k[t_1, \dots, t_s]$ -linear automorphism  $\tau$  of  $\mathbb{T}_s$  to define *Drinfeld modules of rank  $r$  over  $\mathbb{T}_s$* ; a Drinfeld module  $\phi$  of rank  $r$  over  $\mathbb{T}_s$  is an injective  $k[t_1, \dots, t_s]$ -algebra homomorphism

$$\phi : A[t_1, \dots, t_s] \rightarrow \text{End}_{k[t_1, \dots, t_s]\text{-lin}}(\mathbb{T}_s),$$

with  $\phi_\theta$  as in (8) (this suffices to define  $\phi$  over  $A[t_1, \dots, t_s]$ ) but where the parameters  $\alpha_1, \dots, \alpha_r$  are now allowed to be chosen in  $\mathbb{T}_s$  and, of course,  $\alpha_r \neq 0$ ; see §3.

When the rank of  $\phi$  is one and when the unique parameter  $\alpha_1$  is in  $A[t_1, \dots, t_s]$ , our construction of  $L$ -series values (7) with  $n = 1$  (see §4) is compatible with the construction of Taelman. The reader can see §5.2 and, in particular, Proposition 5.11 where we show that the local factors in (7) are ratios of monic generators of Fitting ideals exactly as in (2).

We will write, all along this paper,  $L(n, \phi) = L(n, \alpha)$  where  $\phi$  is the Drinfeld module of rank one over  $\mathbb{T}_s$  and parameter  $\alpha \in A[t_1, \dots, t_s] \setminus \{0\}$  to stress on the fact that, in the case  $n = 1$  at least, the local factors in (7) are intimately related with the torsion structure of the Drinfeld module  $\phi$ .

With our definition of  $L$ -series values, we will cover already many  $L$ -series values studied by Goss, as well as in [18] and [5]. Of course, Taelman's  $L$ -series values relative to a Drinfeld module of rank one defined over  $A$  can be recovered as well and, in particular, it will be so for the value  $\zeta_C(1)$ .

**Remark 1.1.** There are several notational issues involved with our definition of  $L$ -series values. Depending on the point of view we adopt, these can also be seen as  $\zeta$ -values, or as  $L$ -series, associated to various structures. For example, the first property of Result B (that is,  $\exp_\phi(L(1, \phi)) \in A_s$ ) is a sign that the element  $L(1, \phi)$

of  $\mathbb{T}_s$  behaves as a zeta value. We might then have used the notation  $\zeta(1, \phi)$  to designate it. However, this would not have been compatible with the definition of zeta value at one  $\zeta(R, 1)$  of an  $A$ -module by Taelman in [24]. Additionally, there is a simple link with the global  $L$ -series of a  $\tau$ -sheaf (compare with Böckle and Böckle-Pink [7, 8]). Indeed, it is possible to construct a  $\tau$ -sheaf with base scheme of dimension  $s$  such that if  $P$  is a closed point of the coefficient scheme, then the local  $L$ -factor at  $P$  is the inverse of  $1 - T\rho_\alpha(P) \in \mathbb{F}_q[t_1, \dots, t_s]$ . Since we do not need it in this paper, we will not give the details here. Although not completely satisfactory, our notation look as a good compromise.

**1.3. Results.** The  $L$ -series values that we study being elements of the Tate algebras  $\mathbb{T}_s$ , they have the double status of “numbers” and “functions”. As numbers, the indeterminates  $t_1, \dots, t_s$  are unspecified and the series  $L(n, \phi)$  are handled as elements of  $\mathbb{T}_s$ . As typical results with this point of view in mind, we deduce, from an appendix by F. Demeslay, a *class number formula* for the values  $L(1, \phi)$  (see Theorem 5.13 §5.3). A consequence of this result (Theorem 8.1 in §8) is a refinement of Anderson’s *log-algebraic Theorem* for the Carlitz module as in [3].

As functions, the variables  $t_1, \dots, t_s$  can be specialized and the analytic properties of the functions  $L(n, \phi)$  can be used to obtain arithmetic information e.g. on Carlitz zeta values (this was already noticed in [18] and [5]). Here we study the evaluations of our  $L$ -series in yet another direction; we shall discuss (see Theorem 9.17 §9) a generalization of the *Herbrand-Ribet-Taelman Theorem* (as in Taelman’s paper [25]).

We shall now give a wider overview of our results. For the sake of commodity, we are going to exhibit simplified statements (Result A, Result B, ...) which correspond to more precise and refined statements all along this text.

**1.3.1.  $L$ -series values seen as “numbers”.** Let  $\phi$  be a Drinfeld module of rank one over  $\mathbb{T}_s$  of parameter  $\alpha$  (that is,  $\phi_\theta = \theta + \alpha\tau$ ) in  $A_s := A[t_1, \dots, t_s]$ . We introduce, in §5.1, the *class module*  $H_\phi$  and the *unit module*  $U_\phi$  associated to  $\phi$ . This partially generalizes the constructions of Taelman paper [24] in a new direction. We will study some properties of these  $A_s$ -modules.

The module  $H_\phi$  is of finite rank over  $k[t_1, \dots, t_s]$ . We set  $k_s = k(t_1, \dots, t_s)$  and  $R_s = k_s[\theta] = k_s \otimes_{k[t_1, \dots, t_s]} A_s$ . The  $k_s$ -vector space

$$V_\phi = k_s \otimes_{k[t_1, \dots, t_s]} H_\phi$$

is of finite dimension and endowed with the structure of  $R_s$ -module (Corollary 5.7). Let  $[V_\phi]_{R_s} \in R_s$  be the monic generator of its Fitting ideal. We will see (Proposition 5.4) that

$$k_s \otimes_{k[t_1, \dots, t_s]} U_\phi$$

is a free  $R_s$ -module of rank one, to which we can associate a *regulator*  $[R_s : k_s \otimes_{k[t_1, \dots, t_s]} U_\phi]_{R_s}$ . Then, the class number formula for the  $L$ -series value  $L(1, \phi)$  can be obtained (the notation will be made more precise later in this text):

**Result A. (Theorem 5.13 §5.3)** *The following formula holds:*

$$L(1, \phi) = [V_\phi]_{R_s} [R_s : k_s \otimes_{k[t_1, \dots, t_s]} U_\phi]_{R_s}.$$

This result, keystone of the present paper, is deduced from Theorem 10.4 of the Appendix, by F. Demeslay (<sup>4</sup>). The result of F. Demeslay is obtained by the use of Taelman theory of nuclear operators as in [24].

Now, the properties of the *exponential function*  $\exp_\phi$  (§3.2) strongly influence the properties of  $L(1, \phi)$ ,  $H_\phi$  and  $U_\phi$ . This depends on whether  $\exp_\phi$  is surjective as an endomorphism of  $\mathbb{T}_s$  or not, and on whether, in the affirmative case, its kernel has non-zero intersection with  $\mathbb{T}_s(K_\infty)$ , the completion of  $K_\infty[t_1, \dots, t_s]$  for the Gauss norm. Indeed, we will be able to prove, for  $\phi$  a Drinfeld module of rank one defined over  $A_s$  (with  $A_s = A[t_1, \dots, t_s]$ , so that we have a unique parameter  $\alpha \in A_s$  and  $\phi_\theta = \theta + \alpha\tau$ ):

**Result B. (Extracted from §5, 6, 7)** *The point  $\exp_\phi(L(1, \phi)) \in \mathbb{T}_s$  belongs to  $A_s$ . It is a torsion point for the structure of  $A_s$ -module induced by  $\phi$  if and only if the parameter  $\alpha \in A_s$  is a monic polynomial in  $-\theta$  of degree  $r \equiv 1 \pmod{q-1}$ .*

We will see that the condition of the above result is equivalent to the fact that the function  $\exp_\phi$  is surjective, and its kernel has non-trivial intersection with  $\mathbb{T}_s(K_\infty)$ .

1.3.2. *Examples.* For the sake of completeness, we give some examples. We will not reproduce the computations leading to them in the paper. The computation of the polynomials  $\exp_\phi(L(1, \phi))$  is increasingly difficult with the growth of the degree in  $\theta$  of the parameter  $\alpha$ . In the examples which follow, we work in the Tate algebra which is the completion of the polynomial ring  $\mathbb{C}_\infty[t_0, t_1, \dots, t_s]$  for the Gauss absolute value. If we denote by  $\mathbb{S}_s \in A[t_0, \dots, t_s]$  the polynomial  $\exp_\phi(L(1, \phi))$  with  $\phi$  of parameter  $\alpha = t_0(t_1 - \theta) \cdots (t_s - \theta)$ , then we have the following explicit examples:

$$\begin{aligned} \mathbb{S}_s &= 1 \quad \text{if } 1 \leq s \leq q-1 \\ \mathbb{S}_s &= 1 - t_0 \sum_{i_1 < \dots < i_{s-q}} \prod_{j=1}^{s-q} (t_{i_j} - \theta) \quad \text{if } q \leq s \leq 2q-2 \\ \mathbb{S}_{2q-1} &= (1 - t_0) \left( 1 - t_0 \sum_{1 \leq i_1 < \dots < i_{q-1} \leq 2q-1} \prod_{k=1}^{q-1} (t_{i_k} - \theta) \right). \end{aligned}$$

In fact, when  $\alpha$  is submitted to the equivalent conditions of Result B (we will refer to this case as to the *uniformizable torsion case*), we have a more precise information:

**Result C. (From §7.1.1 and Theorem 7.9)** *If  $\alpha$  is monic as a polynomial in  $-\theta$ , of degree  $r \equiv 1 \pmod{q-1}$  with  $r \geq 2$ , then  $\exp_\phi(L(1, \phi)) = 0$ . In the special case of  $\phi$  of parameter  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$ , we have the formula*

$$L(1, \phi) = \frac{\tilde{\pi} \mathbb{B}_s}{\omega(t_1) \cdots \omega(t_s)},$$

where  $\mathbb{B}_s \in A_s$  is the monic generator of the Fitting ideal of the  $A_s$ -module  $H_\phi$  and  $\omega$  is the Anderson-Thakur function introduced in [4].

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<sup>4</sup> Florent Demeslay informed us that the content of the appendix of the present paper can be generalized in such a way that it implies Taelman results in [24]. This is the subject of a work currently in progress. His result allows to handle Drinfeld modules of rank  $r \geq 1$  over the Tate algebras  $\mathbb{T}_s$  whose parameters  $\alpha_1, \dots, \alpha_r$  belong to  $R[t_1, \dots, t_s]$ , where  $R$  is the ring of integers of a finite extension of  $K$ .



The reader can observe the above phenomenon already with a look at the examples of §1.3.2. Indeed, the polynomials of the right-hand side of the second and the third formulas vanish at  $t_0 = 1$ .

We have few explicit examples of polynomial  $\mathbb{B}_s$ . Here are some:

$$\begin{aligned}\mathbb{B}_q &= -1, \\ \mathbb{B}_{2q-1} &= \theta - \sum_{1 \leq i_1 < \dots < i_{q-1} \leq 2q-1} \prod_{k=1}^q t_{i_k}.\end{aligned}$$

Also, by Lemma 7.1, we can set  $\mathbb{B}_1 = \frac{1}{\theta - t_1}$ .

Thanks to the above result we have a better understanding of the modules  $H_\phi, U_\phi$  in the uniformizable torsion case. Our results in this direction are presented in §7. At once, we also address questions on the structure of these modules; see §9.3.

As a consequence of the class number formula, we shall also mention the *log-algebraic Theorem of Anderson*, in the case of the Carlitz module, see [3, Theorem 3 and Proposition 8 (I)]. In fact, we prove in §8 a result which can be interpreted as an *operator theoretic version*, thus a refinement of Anderson Proposition 8 (I) loc. cit.

The settings here are quite involved so that the reader may need to consult §8 for a precise statement. We introduce a class of formal series in infinitely many indeterminates  $X_i, \tau(X_i), \dots, Z, \tau(Z), \dots$  ( $i = 1, \dots, r$ ) by setting:

$$\mathcal{L}_r(X_1, \dots, X_r; Z) = \sum_{d \geq 0} \left( \sum_{a \in A_{+,d}} C_a(X_1) \cdots C_a(X_r) a^{-1} \right) \tau^d(Z),$$

where  $A_{+,d}$  denotes the set of monic polynomials of degree  $d$  and  $C_a(X_i)$  denotes a certain polynomial in  $X_i, \tau(X_i), \dots$  obtained from the action of the Carlitz module evaluated at  $a$  on the indeterminate  $X_i$ ; for example,  $C_\theta(X_1) = \theta X_1 + \tau(X_1)$ . It turns out that a formal series  $\exp_C$  associated to the Carlitz module can be multiplied on the left of  $\mathcal{L}_r(X_1, \dots, X_r; Z)$  with a composition rule specified in §8. Our result can then be loosely stated as follows:

**Result D. (Theorem 8.1 §8)** *The formal series  $\exp_C(\mathcal{L}_r(X_1, \dots, X_r; Z))$  can be expressed as a polynomial with coefficients in  $A$  in the infinite set of indeterminates  $X_i, \tau(X_i), \dots, Z, \tau(Z), \dots$  ( $i = 1, \dots, r$ ).*

If we substitute, in the above result,  $X_1 = \dots = X_r = X$  and  $\tau^n(X) = X^{q^n}, \tau^n(Z) = Z^{q^n}$  for all  $n \geq 0$ , we recover Anderson's original result asserting that

$$\exp_C \left( \sum_{a \in A_+} Z^{q^{\deg_\theta(a)}} a^{-1} C_a(X)^r \right) \in A[X, Z].$$

Here, we can also replace  $X, Z$  with variables chosen in  $\mathbb{C}_\infty$  provided that  $|Z|$  is small enough but our result allows to handle the more general case in which the variables  $X_i$  and  $Z$  are in  $\mathbb{T}_s$  (with  $Z$  of Gauss norm small enough), case in which Anderson's result cannot be applied.

1.3.3. *Evaluation of L-series values.* The evaluation of  $L$ -series values is the necessary step to deduce from the above results, arithmetic results on the *original* values of Goss abelian  $L$ -series (in the zero-dimensional Tate algebra, that is, in  $\mathbb{C}_\infty$ ). This is what we do in §9. We shall now review our results in this part of the paper.

Let  $\chi$  be a Dirichlet character of *type*  $s$  such that  $s \equiv 1 \pmod{q-1}$  and conductor  $a \in A_+$  (the terminology is described in the text). Let us denote by  $k_a$  the subfield of  $\mathbb{C}_\infty$  obtained adjoining to  $k$  the roots of  $a$ , by  $K_a$  the  $a$ -th *cyclotomic field* and by  $\Delta_a$  the Galois group of  $K_a$  over  $K$ . We denote by  $H_a$  the *Taelman class  $A$ -module* associated to the Carlitz module and relative to the extension  $K_a/K$ . This is a finite  $k[\Delta_a]$ -module and there is a  $k_a[\Delta_a]$ -module structure on the  $A[k_a]$ -module  $k_a \otimes_k H_a$ . Let  $e_\chi$  be the *idempotent element* of  $k_a[\Delta_a]$  associated to  $\chi$ . Then, the  $\chi$ -isotypic component

$$H_\chi = e_\chi(k_a \otimes_k H_a)$$

is a finite  $A[k_a]$ -module endowed with a suitable structure of  $k_a[\Delta_a]$ -module. Our Result C can be used to prove, if  $s \geq 2$ :

**Result E. (Theorem 9.4 §9)** *The Fitting ideal of the  $A[k_a]$ -module  $H_\chi$  is generated by the “evaluation of  $\mathbb{B}_s$  at  $\chi$ ”.*

The above mentioned evaluation is a precise construction described in §9. It is obtained by substituting the variables  $t_i$  (for  $i = 1, \dots, s$ ) by appropriate roots of unity chosen among the roots of the conductor  $a$  in  $k^{ac}$  and depending on the choices of Teichmüller characters at the various primes  $P$  dividing  $a$ . This result serves to demonstrate a generalization of Herbrand-Ribet-Taelman Theorem.

In §9.5 we associate to our character  $\chi$  certain *generalized Bernoulli-Carlitz fractions* denoted by  $\text{BC}_{i, \chi^{-1}}$ . These are elements of the compositum  $K(k_a)$  of  $k_a$  and  $K$  in  $\mathbb{C}_\infty$ . We must choose an appropriate embedding  $\iota_P : K^{ac} \rightarrow \mathbb{C}_P$ , with  $\mathbb{C}_P$  the completion of an algebraic closure of  $\widehat{K}_P$ , the completion of  $K$  at the prime  $P$ . A consequence of Proposition 9.14 is that  $\text{BC}_{q^d - N, \tilde{\chi}^{-1}}$  is  $P$ -integral, that is,  $\iota_P(\text{BC}_{q^d - N, \tilde{\chi}^{-1}}) \in \widehat{A}_P[k_a]$  (this will be made more precise in §9.5).

Let us write

$$\chi = \vartheta_P^N \tilde{\chi},$$

where  $P$  is a prime dividing the conductor  $a$  of  $\chi$  (so that  $a = Pb$  with  $P$  not dividing  $b$ ),  $\tilde{\chi}$  is a Dirichlet character of conductor  $b$ ,  $\vartheta_P$  is the Teichmüller character associated to  $P$ , and  $N$  is an integer between 0 and  $q^d - 2$ , with  $d$  the degree of  $P$ .

If  $\widehat{A}_P$  is the valuation ring of  $\widehat{K}_P$ , we denote by  $\widehat{A}_P[k_a]$  the valuation ring of the field  $\widehat{K}_P(k_a)$  (isomorphic to  $k_a \otimes_{k_P} \widehat{A}_P$ ). We consider the  $\widehat{A}_P[k_a][\Delta_a]$ -module

$$H_a \otimes_A \widehat{A}_P[k_a].$$

Its  $\chi$ -isotypic component has again the “evaluation” of  $\mathbb{B}_s$  at  $\chi$  as a generator of the Fitting  $\widehat{A}_P[k_a]$ -ideal. From this information, we deduce a generalization of Herbrand-Ribet-Taelman Theorem of [24]:

**Result F. (Theorem 9.17 §9.5)** *The  $\widehat{A}_P[k_a][\Delta_a]$ -module  $e_\chi(H_a \otimes_A \widehat{A}_P[k_a])$  is non-trivial if and only if  $\text{BC}_{q^d - N, \tilde{\chi}^{-1}} \equiv 0 \pmod{P}$ .*

The original result of Taelman [24, Theorem 1] corresponds to the case in which  $\tilde{\chi}$  is the trivial character. See also [6, Theorem 8.14].

## 2. NOTATION AND BACKGROUND

The basic list of notation of this paper is the following:

- $k$ : a fixed finite field with  $q$  elements.
- $p$ : the characteristic of  $k$ .

- $\theta$ : an indeterminate over  $k$ .
- $A$ : the polynomial ring  $k[\theta]$ .
- $K = k(\theta)$ : the fraction field of  $A$ .
- $K_\infty = k((\theta^{-1}))$ : the completion of  $K$  at the infinite prime.
- $\mathbb{C}_\infty$ : the completion of an algebraic closure of  $K_\infty$ .
- $v_\infty$ : the valuation on  $\mathbb{C}_\infty$  normalized so that  $v_\infty(\theta) = -1$  (we make the convention that  $v_\infty(0) = +\infty$ ).
- $|\cdot|$ : the absolute value of  $\mathbb{C}_\infty$  defined by  $|\alpha| = q^{-v_\infty(\alpha)}$  for  $\alpha \in \mathbb{C}_\infty$  and  $|0| = 0$ .

To this basic list, we also add the following list of very common notation; this list is not exhaustive.

- $k_s$ : the field  $k(t_1, \dots, t_s)$ .
- $A_s$ : the ring  $A[t_1, \dots, t_s]$ .
- $\mathbb{T}_s$ : the completion of  $\mathbb{C}_\infty[t_1, \dots, t_s]$  for the Gauss norm  $\|\cdot\|$  ( $s$ -th dimensional Tate algebra).
- $R_s$ : the ring  $k_s[\theta]$ .
- $K_{s,\infty}$ : the field  $k_s((\theta^{-1}))$ .

**2.1. The Carlitz exponential.** The Carlitz exponential is the function

$$\exp_C : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$$

defined by

$$\exp_C(X) = \sum_{i \geq 0} \frac{X^{q^i}}{D_i}, \quad X \in \mathbb{C}_\infty,$$

where  $(D_i)_{i \geq 0}$  is the sequence of  $A$  defined by  $D_0 = 1$  and, for  $i \geq 1$ ,

$$D_i = (\theta^{q^i} - \theta)D_{i-1}^q.$$

This function,  $k$ -linear, is entire because  $|D_i| = q^{iq^i}$  so that  $|{}^q\sqrt{D_i}| = q^i$  and

$$\lim_{i \rightarrow \infty} |{}^q\sqrt{D_i}| = \infty,$$

therefore  $\exp_C$  is surjective.

The kernel of  $\exp_C$  is the  $A$ -module  $\tilde{\pi}A$ , where  $\tilde{\pi}$  is the *period* defined in (3). We have

$$|\tilde{\pi}| = q^{\frac{q}{q-1}}.$$

We can expand the function  $\exp_C$  in a convergent infinite product:

$$\exp_C(X) = X \prod_{a \in A \setminus \{0\}} \left(1 - \frac{X}{\tilde{\pi}a}\right), \quad X \in \mathbb{C}_\infty.$$

From this product expansion one deduces that  $\exp_C$  induces an isometric automorphism of the disk

$$D_{\mathbb{C}_\infty}(0, q^{\frac{q}{q-1}}) = \{z \in \mathbb{C}_\infty; |z| < q^{\frac{q}{q-1}}\}.$$

2.1.1. *The Carlitz module.* The  $\mathbb{C}_\infty$ -algebra of the  $k$ -linear algebraic endomorphisms of  $\mathbb{G}_a(\mathbb{C}_\infty)$

$$\text{End}_{k\text{-lin.}}(\mathbb{G}_a(\mathbb{C}_\infty))$$

can be identified with the skew polynomial ring  $\mathbb{C}_\infty[\tau]$  whose elements are the finite sums  $\sum_{i \geq 0} c_i \tau^i$  with the  $c_i$ 's in  $\mathbb{C}_\infty$ , subject to the product rule defined by  $\tau x = x^q \tau$  for all  $x \in \mathbb{C}_\infty$ . If  $X \in \mathbb{C}_\infty$  and  $P = \sum_{i=0}^d P_i \tau^i$  is an element of  $\mathbb{C}_\infty[\tau]$ , the evaluation of  $P$  at  $X$  is defined by setting

$$P(X) = \sum_{i=0}^d P_i X^{q^i}.$$

For example, the evaluation of  $\tau$  at  $X$  is  $\tau(X) = X^q$ .

The *Carlitz module* is the unique  $k$ -algebra homomorphism

$$C : A \rightarrow \text{End}_k(\mathbb{G}_a(\mathbb{C}_\infty))$$

determined by

$$C_\theta = \theta + \tau,$$

with  $\tau$  the endomorphism of  $\mathbb{C}_\infty$  such that  $\tau(c) = c^q$  for all  $c \in \mathbb{C}_\infty$ . The Carlitz module can also be viewed, more generally, as a functor from the category of  $A$ -algebras to the category of  $A$ -modules.

If  $a \in A_{+,d}$ , we denote by  $C_a$  the evaluation of  $C$  at  $a$ . We have  $C_a = a_0 \tau^0 + a_1 \tau^1 + \cdots + a_{d-1} \tau^{d-1} + \tau^d$  with  $a_0 = a$ , and if  $X \in \mathbb{C}_\infty$ , we will write, in particular,

$$C_a(X) = a_0 X + a_1 X^q + \cdots + a_{d-1} X^{q^{d-1}} + X^{q^d}.$$

This endows  $\mathbb{C}_\infty$  with a structure of  $A$ -module that will be denoted by  $C(\mathbb{C}_\infty)$  and we have

$$C_a(\exp_C(X)) = \exp_C(aX)$$

for all  $a \in A$  and  $X \in \mathbb{C}_\infty$ . The Carlitz module  $C$  allows to make the exact sequence of  $k$ -vector spaces  $0 \rightarrow \tilde{\pi}A \rightarrow \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty \rightarrow 0$  induced by  $\exp_C$  into an exact sequence of  $A$ -modules

$$0 \rightarrow \tilde{\pi}A \rightarrow \mathbb{C}_\infty \rightarrow C(\mathbb{C}_\infty) \rightarrow 0.$$

2.1.2. *The Carlitz logarithm.* The Carlitz logarithm is the rigid analytic function defined, for  $X \in \mathbb{C}_\infty$  such that  $|X| < q^{\frac{q}{q-1}}$ , by the convergent series

$$\log_C(X) = \sum_{i \geq 0} \frac{X^{q^i}}{l_i} = \sum_{i \geq 0} (-1)^i \frac{X^{q^i}}{L_i},$$

where  $(L_i)_{i \geq 0}$  is the sequence defined by  $L_0 = 1$  and, for  $i \geq 1$ ,

$$L_i = (\theta^{q^i} - \theta)L_{i-1},$$

and where  $l_i := (-1)^i L_i$  for all  $i \geq 0$ .

The convergence property is due to the fact that  $|L_i| = |l_i| = q^{q \frac{q^i - 1}{q-1}}$ . We then have, for all  $X \in \mathbb{C}_\infty$  such that  $|X| < q^{\frac{q}{q-1}}$ ,

$$(9) \quad |X| = |\exp_C(X)| = |\log_C(X)|$$

and

$$(10) \quad \log_C(\exp_C(X)) = \exp_C(\log_C(X)) = X, \quad |X| < q^{\frac{q}{q-1}}.$$

Let  $a$  be in  $A$  with degree  $d > 0$  and let  $X$  in  $\mathbb{C}_\infty$  be such that  $|X| < q^{\frac{q}{q-1}-d}$ . We have that  $C_a(X) = \exp_C(a \log_C(X))$  so that  $|C_a(X)| < q^{\frac{q}{q-1}}$  and  $\log_C(C_a(X))$  is well defined over  $D(0, q^{\frac{q}{q-1}-d})$ . Since the functions  $\log_C(C_a(x))$  and  $a \log_C(x)$  agree on a neighborhood of 0, we conclude that if  $|X| < q^{\frac{q}{q-1}-d}$ , then

$$\log_C(C_a(X)) = a \log_C(X).$$

2.1.3. *The Carlitz torsion.* A monic irreducible element in  $A_+$  will be called a prime. For  $a \in A_+$ , we set

$$\lambda_a = \exp_C\left(\frac{\tilde{\pi}}{a}\right) \in \mathbb{C}_\infty.$$

The subfield of  $\mathbb{C}_\infty$

$$K_a = K(\lambda_a),$$

a finite extension of  $K$ , will be called the  $a$ -th *cyclotomic function field*. A reference for the basic theory of these fields is [22, Chapter 12]. Here, we recall that  $K_a/K$  is a finite abelian extension unramified outside  $a$  and  $\infty$ . Its Galois group

$$\Delta_a = \text{Gal}(K_a/K)$$

is isomorphic to the unit group

$$\left(\frac{A}{aA}\right)^\times.$$

If  $b \in A$  is prime to  $a$ , there exists  $\sigma_b \in \Delta_a$  such that:

$$\sigma_b(\lambda_a) = C_b(\lambda_a).$$

2.2. **Tate algebras.** Let  $s \geq 1$  be an integer and let  $t_1, \dots, t_s$  be indeterminates over  $\mathbb{C}_\infty$ . If  $s = 1$ , we will also write  $t = t_1$ . Let  $L$  be a complete extension of  $K_\infty$  contained in  $\mathbb{C}_\infty$ , that is, a subfield  $L$  of  $\mathbb{C}_\infty$  which is an extension  $L/K_\infty$ , together with a valuation  $L \rightarrow \mathbb{R}$  such that its restriction to  $K$  is  $v_\infty$  (and again denoted by  $v_\infty$ ). Let us consider a polynomial  $f \in L[t_1, \dots, t_s]$ , expanded as a finite sum

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} x_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}, \quad x_{i_1, \dots, i_s} \in L.$$

We set

$$v_\infty(f) = \inf\{v_\infty(x_{i_1, \dots, i_s}), i_1, \dots, i_s \in \mathbb{N}\}.$$

We then have, for  $f, g \in L[t_1, \dots, t_s]$ :

$$v_\infty(f + g) \geq \inf(v_\infty(f), v_\infty(g)).$$

Furthermore, we have

$$v_\infty(fg) = v_\infty(f) + v_\infty(g)$$

so that  $v_\infty$  is a valuation, called the *Gauss valuation*.

Let us set, for  $f \in L[t_1, \dots, t_s]$ ,  $\|f\| = q^{-v_\infty(f)}$  if  $f \neq 0$  and  $\|0\| = 0$ . We have  $\|f + g\| \leq \max\{\|f\|, \|g\|\}$ ,  $\|fg\| = \|f\|\|g\|$  and  $\|f\| = 0$  if and only if  $f = 0$ ; the function  $\|\cdot\|$  is an  $L$ -algebra norm on  $L[t_1, \dots, t_s]$  and an absolute value, called the *Gauss absolute value*.

We denote by  $\mathbb{T}_s(L)$  the completion of  $L[t_1, \dots, t_s]$  with respect to  $\|\cdot\|$ . When  $s = 1$ , we also write  $\mathbb{T}(L)$  for  $\mathbb{T}_1(L)$  and we adopt the convention that  $\mathbb{T}_0(L) = L$ .

Equipped with the Gauss norm,  $\mathbb{T}_s(L)$  is an  $L$ -Banach algebra and also an ultrametric ring, that can be identified with the set of formal series of  $f \in L[[t_1, \dots, t_s]]$  such that, writing

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} x_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}, \quad x_{i_1, \dots, i_s} \in L,$$

we have

$$\lim_{i_1 + \dots + i_s \rightarrow +\infty} x_{i_1, \dots, i_s} = 0.$$

The Gauss norm of  $f$  as above is then given by

$$\|f\| = \sup\{|x_{i_1, \dots, i_s}|, (i_1, \dots, i_s) \in \mathbb{N}^s\}.$$

When  $L = \mathbb{C}_\infty$ , we shall write  $\mathbb{T}_s, \mathbb{T}$  instead of  $\mathbb{T}_s(\mathbb{C}_\infty), \mathbb{T}_1(\mathbb{C}_\infty)$ . We refer the reader to [12, Chapter 3] for the basic properties of Tate algebras.

We denote by  $\mathfrak{o}_L$  the valuation ring of  $L$  (whose elements  $x$  are characterized by the fact that  $|x| \leq 1$ ). We denote by  $\mathfrak{m}_L$  the maximal ideal of  $\mathfrak{o}_L$  whose elements  $x$  are such that  $|x| < 1$ , and by  $\overline{L} = \frac{\mathfrak{o}_L}{\mathfrak{m}_L}$  the residual field of  $L$ . We further denote by  $\mathfrak{o}_{\mathbb{T}_s(L)}$  the subring of elements  $f \in \mathbb{T}_s(L)$  such that  $\|f\| \leq 1$  and by  $\mathfrak{m}_{\mathbb{T}_s(L)}$  the prime ideal of  $\mathfrak{o}_{\mathbb{T}_s(L)}$  whose elements are the  $f$  such  $\|f\| < 1$ . Then, we have that

$$\overline{\mathbb{T}_s(L)} := \frac{\mathfrak{o}_{\mathbb{T}_s(L)}}{\mathfrak{m}_{\mathbb{T}_s(L)}} = \overline{L}[t_1, \dots, t_s].$$

If  $L/K_\infty$  is a finite complete extension, let  $\pi_L$  be a uniformizer of  $L$ . Then, we have that  $L = \overline{L}((\pi_L))$ ,  $\mathfrak{o}_L = \overline{L}[[\pi_L]]$ . In particular:

$$\mathbb{T}_s(L) = \overline{L}[t_1, \dots, t_s]((\pi_L)).$$

### 2.2.1. Some difference rings.

**Definition 2.1.** A *difference ring*  $(R, \tau)$  is the datum of a ring  $R$  together with a ring endomorphism  $\tau$ . A *difference field* is a difference ring which is a field. Given a difference ring  $(R, \tau)$ , the *constant subring*  $R^\tau$  is the subset of  $R$  whose elements are the  $r \in R$  such that  $\tau(r) = r$ ; it inherits from  $R$  the structure of a ring. If  $R$  is a field,  $R^\tau$  is a subfield of  $R$ . If the choice of  $\tau$  in a difference ring  $(R, \tau)$  is understood, we will denote the latter with  $R$ . An inclusion of difference rings  $R \subset R'$  is the datum of two difference rings  $(R, \tau), (R', \tau')$  such that  $R \subset R'$  and such that  $\tau'|_R = \tau$ .

The dimension  $s$  of  $\mathbb{T}_s$  is an important parameter in our investigations. We look at the chain of inclusions of difference rings

$$k \subset A \subset K_\infty$$

(with the  $k$ -linear Frobenius endomorphism  $\tau$ ) as occurring in the case when the parameter  $s$  is set to 0. There are many ways to extend it for higher values of  $s$ .

We replace  $k$  with a finitely generated commutative  $k$ -algebra  $\mathfrak{k}$  which is a domain, of dimension  $s$ . This yields the chain of inclusions:

$$(11) \quad \mathfrak{k} \subset \mathfrak{k} \otimes_k A \subset \mathfrak{k} \otimes_k K_\infty$$

and the endomorphism  $\tau$  acts  $\mathfrak{k}$ -linearly on each of these algebras by sending  $x \otimes y$  to  $x \otimes y^q$ , with  $x \in \mathfrak{k}$  and  $y \in A$  or  $y \in K_\infty$ . Obviously, we have  $(\mathfrak{k} \otimes_k A)^\tau = (\mathfrak{k} \otimes_k K_\infty)^\tau = \mathfrak{k}$ . In this paper, we will focus on certain choices of  $\mathfrak{k}$ . With  $t_1, \dots, t_s$  indeterminates over  $K$ , we will essentially discuss two cases determined by the

choices  $\mathfrak{k} = k[t_1, \dots, t_s]$  and  $\mathfrak{k} = k(t_1, \dots, t_s)$  respectively. We will use very often the following notation:

$$k_s = k(t_1, \dots, t_s), \quad A_s = A[t_1, \dots, t_s], \quad R_s = k_s[\theta], \quad K_{s,\infty} = k_s((\theta^{-1})).$$

Then, the chain of inclusions (11) becomes:

- (1)  $k[t_1, \dots, t_s] \subset A_s \subset \mathbb{T}_s(K_\infty)$ ,
- (2)  $k_s \subset R_s \subset K_{s,\infty}$ .

In §9 we will also examine a third, zero-dimensional case; indeed, we will use, at the place of  $k$ , a field  $k_a$  obtained by adding to  $k$  the roots of a polynomial  $a \in A$  (the conductor of a Dirichlet character of the first kind). This will also lead us to examine the chain

$$k_a \subset A[k_a] \subset k_a((\theta^{-1}))$$

with the  $k_a$ -linear endomorphisms  $\tau$  defined by  $\tau(\theta) = \theta^q$ .

2.2.2.  $k[t_1, \dots, t_s]$ -linear endomorphisms of  $\mathbb{T}_s$ . Let

$$\tau \in \text{End}_{k[t_1, \dots, t_s]\text{-lin.}}(\mathbb{T}_s)$$

be the  $k[t_1, \dots, t_s]$ -linear extension of the previous operator  $\tau$  defined as follows: for  $f \in \mathbb{T}_s$  with

$$f = \sum_{i_1, \dots, i_s \in \mathbb{N}} x_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}, \quad x_{i_1, \dots, i_s} \in \mathbb{C}_\infty,$$

we set

$$\tau(f) = \sum_{i_1, \dots, i_s \in \mathbb{N}} x_{i_1, \dots, i_s}^q t_1^{i_1} \cdots t_s^{i_s}.$$

This is a  $k[t_1, \dots, t_s]$ -linear automorphism of  $\mathbb{T}_s$ . In fact,  $\tau$  is also an automorphism for the structure of  $k[t_1, \dots, t_s]$ -algebra of  $\mathbb{T}_s$ . Obviously,  $\mathbb{T}_s^\tau = k[t_1, \dots, t_s]$  and

$$\|\tau^n(f)\| = \|f\|^{q^n}, \quad n \geq 0, \quad f \in \mathbb{T}_s.$$

With the action of  $\tau$  on  $\mathbb{T}_s$ , we have the non-commutative skew polynomial rings  $\mathbb{T}_s[\tau]$  and  $\mathbb{T}_s[[\tau]]$ . The latter is, as a set, constituted of the formal series  $\sum_{i \geq 0} f_i \tau^i$  with  $f_i \in \mathbb{T}_s$  for all  $i$ , and the elements of the former are the formal series whose sequences of coefficients are eventually zero. The commutation rule defining the product is given by

$$\tau f = \tau(f)\tau,$$

for  $f \in \mathbb{T}_s$ . Moreover, the ring  $\mathbb{T}_s[\tau]$  acts on  $\mathbb{T}_s$ : if  $P = \sum_{i=0}^d P_i \tau^i \in \mathbb{T}_s[\tau]$  and  $f \in \mathbb{T}_s$ , then we set

$$P(f) = \sum_{i=0}^d P_i \tau^i(f) \in \mathbb{T}_s.$$

For  $f \in \mathbb{T}_s$ , we will sometimes write:

$$f^{(n)} = \tau^n(f), \quad n \geq 0.$$

3. DRINFELD  $A_s$ -MODULES OVER  $\mathbb{T}_s$ .

From now on, we will write  $\underline{t}_s$  for the set of variables  $\{t_1, \dots, t_s\}$ . If  $s = 0$ , this set is empty. If the value of  $s$  is clear from the context, we will more simply write  $\underline{t}$ . For example, we will write  $k[\underline{t}_s]$  or  $k[\underline{t}]$  instead of  $k[t_1, \dots, t_s]$  and  $k[\underline{t}_0] = k$ .

**Definition 3.1.** Let  $r \geq 1$  be an integer. A *Drinfeld  $A_s$ -module of rank  $r$  over  $\mathbb{T}_s$*  is a homomorphism of  $k[\underline{t}_s]$ -algebras

$$\phi : A_s \rightarrow \mathbb{T}_s[\tau]$$

defined by

$$\phi_\theta = \theta + \alpha_1 \tau + \dots + \alpha_r \tau^r,$$

for elements  $\alpha_1, \dots, \alpha_r \in \mathbb{T}_s$  with  $\alpha_r$  non-zero. The vector

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{T}_s^r$$

will be called the *parameter* of  $\phi$ . If  $r = 1$ , we identify the parameter with its unique entry  $\alpha_1$ .

Given a Drinfeld  $A_s$ -module  $\phi$  of rank  $r$  over  $\mathbb{T}_s$ , if  $M$  is a sub- $k[\underline{t}_s]$ -module of  $\mathbb{T}_s$  such that  $\phi_\theta(M) \subset M$ , we denote by  $\phi(M)$  the  $k[\underline{t}_s]$ -module  $M$  equipped with the  $A_s$ -module structure induced by  $\phi$ . In particular, we will often work in the module  $\phi(\mathbb{T}_s)$ .

If  $s \leq s'$  then we have the embedding  $\mathbb{T}_s \subset \mathbb{T}_{s'}$  induced by the inclusion  $k[\underline{t}_s] \subset k[t_1, \dots, t_s, t_{s+1}, \dots, t_{s'}]$ . Every Drinfeld  $A_s$ -module over  $\mathbb{T}_s$  can be extended in a natural way to a Drinfeld  $A_{s'}$ -module over  $\mathbb{T}_{s'}$  of the same rank, which will be denoted again by  $\phi$  for the sake of simplicity.

**Definition 3.2.** Let  $\phi, \phi'$  be two Drinfeld  $A_s$ -modules over  $\mathbb{T}_s$ . We will say that  $\phi$  is *isomorphic* to  $\phi'$  if there exists  $u \in \mathbb{T}_s^\times$  ( $\mathbb{T}_s^\times$  denotes the multiplicative group of the units of  $\mathbb{T}_s$ ) such that, in  $\mathbb{T}_s[\tau]$ :

$$\phi_\theta u = u \phi'_\theta.$$

If  $\phi$  and  $\phi'$  are isomorphic Drinfeld modules, they must have the same rank and we shall also write  $\phi \cong \phi'$ .

**Remark 3.3.** When two Drinfeld  $A_s$ -modules are isomorphic, it is understood that they are isomorphic *over*  $\mathbb{T}_s$ .

Let  $\phi, \phi'$  be Drinfeld modules of rank  $r > 0$  over  $\mathbb{T}_s$  of respective parameters

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_r), \quad \underline{\alpha}' = (\alpha'_1, \dots, \alpha'_r) \in \mathbb{T}_s^r.$$

Then, the condition  $\phi \cong \phi'$  amounts to the existence of  $u \in \mathbb{T}_s^\times$  such that

$$\alpha_i \tau^i(u) = \alpha'_i u, \quad i = 1, \dots, r.$$

**Remark 3.4.** If  $s = 0$ , all the Drinfeld  $A$ -modules of rank one are isomorphic to the Carlitz module  $C$ . This is no longer true for Drinfeld  $A_s$ -modules of rank one if  $s \geq 1$ ; for example, the Drinfeld modules of rank 1 of parameters  $\alpha = 1$  (Carlitz module) and  $\alpha = t$  are not isomorphic.

If  $f \in \mathbb{T} \setminus \{0\}$  (case  $s = 1$ ) then we can write  $f$  as a product of a unit of  $\mathbb{T}^\times$  and a non-zero polynomial of  $\mathbb{C}_\infty[t]$  having all its zeroes  $x$  such that  $|x| \leq 1$ . Hence, the set of classes of isomorphism of Drinfeld  $A_1$ -modules of rank one over  $\mathbb{T}$  is in bijection with the set of finite formal combinations  $\sum_i n_i [x_i]$  with  $|x_i| \leq 1$  and  $n_i \geq 0$  for all  $i$ .



From now on, we will be focused on the Drinfeld modules of rank 1, leaving the analysis of the higher rank case for subsequent works; we notice that one of such works is currently in progress by Florent Demeslay.

A remarkable sequence of Drinfeld modules of rank one emerge. For this, we shall adopt a particular notation.

**Definition 3.5.** We will denote by  $C_s$  the Drinfeld module of rank one over  $\mathbb{T}_s$  with parameter

$$\alpha = (t_1 - \theta) \cdots (t_s - \theta).$$

Of course, if  $s = 0$ , we get  $C_0 = C$ , the Carlitz module.

**3.1.  $\mathfrak{A}$ -modules.** In parallel, we will also be interested in studying the analogue structures associated to other choices of the algebra  $\mathfrak{k}$  in (11). Let us write  $\mathfrak{A} = \mathfrak{k} \otimes_k A$  and  $\mathfrak{K}_\infty = \mathfrak{k} \otimes_k A$ . The definition of Drinfeld  $\mathfrak{A}$ -module is essentially the same as Definition 3.2. Identifying  $\mathfrak{K}_\infty[\tau]$  with a sub-algebra of the algebra of  $\mathfrak{k}$ -linear endomorphisms of  $\mathfrak{K}_\infty$ , we have the following definition.

**Definition 3.6.** A *Drinfeld  $\mathfrak{A}$ -module of rank  $r$  over  $\mathfrak{K}_\infty$*  is a homomorphism of  $\mathfrak{k}$ -algebras

$$\phi : \mathfrak{A} \rightarrow \mathfrak{K}_\infty[\tau]$$

defined by

$$\phi_\theta = \theta + \alpha_1 \tau + \cdots + \alpha_r \tau^r,$$

for elements  $\alpha_1, \dots, \alpha_r \in \mathfrak{K}_\infty$  with  $\alpha_r$  non-zero.

In this paper, we will also study Drinfeld  $R_s$ -modules.

**3.2. Exponential and logarithm.** Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one defined over  $\mathbb{T}_s$  with parameter  $\alpha \in \mathbb{T}_s$ . We also set  $\tau_\alpha = \alpha\tau$ . This yields a  $k[\underline{t}_s]$ -linear endomorphism of  $\mathbb{T}_s$ ; notice however that, in general, this is not an endomorphism for the structure of  $k[\underline{t}_s]$ -algebra unless  $\alpha = 1$ . Explicitly, for any  $n \geq 0$ , we have

$$\tau_\alpha^n = \alpha\tau(\alpha) \cdots \tau^{n-1}(\alpha)\tau.$$

We will be particularly interested in the formal series of  $\mathbb{T}_s[[\tau]]$

$$\begin{aligned} \exp_\phi &= \sum_{n \geq 0} \frac{1}{D_n} \tau_\alpha^n, \\ \log_\phi &= \sum_{n \geq 0} \frac{1}{l_n} \tau_\alpha^n, \end{aligned}$$

respectively called the *exponential series* and the *logarithm series* associated to  $\phi$ .

It is easy to show that, in  $\mathbb{T}_s[[\tau]]$ , we have:

$$\exp_\phi \log_\phi = \log_\phi \exp_\phi = 1, \quad \exp_\phi \theta = \phi_\theta \exp_\phi.$$

A routine computation also shows the identities in  $\mathbb{T}_s[[\tau]]$ :

$$\phi_a \exp_\phi = \exp_\phi a, \quad \log_\phi \phi_a = a \log_\phi, \quad \text{for all } a \in A_s.$$

We observe that

$$\|D_n^{-1} \alpha\tau(\alpha) \cdots \tau^{n-1}(\alpha)\| = \|\alpha\| \frac{q^n - 1}{q - 1} q^{-nq^n}$$

so that for all  $f \in \mathbb{T}_s$ , the series

$$\exp_\phi(f) := \sum_{n \geq 0} \frac{\tau_\alpha^n(f)}{D_n} = \sum_{n \geq 0} \frac{\alpha \tau(\alpha) \cdots \tau^{n-1}(\alpha)}{D_n} \tau^n(f)$$

converges in  $\mathbb{T}_s$ . We call the element  $\exp_\phi(f) \in \mathbb{T}_s$  defined in this way the *evaluation* of  $\exp_\phi$  at  $f$  <sup>(5)</sup>. The  $k[\underline{t}_s]$ -linear map

$$\exp_\phi : \mathbb{T}_s \rightarrow \mathbb{T}_s$$

defined by  $f \mapsto \exp_\phi(f)$  is called the *exponential function* of  $\phi$ . Notice that, except for the case  $s = 0$ , this is not an entire function. In particular, there are bounded subsets of  $\mathbb{T}_s$  containing infinitely many of its zeros. In particular, it does not allow a Weierstrass product expansion, property shared by entire functions  $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ . At least, it is open and continuous, as the reader can easily check, and its kernel, a  $k[\underline{t}_s]$ -module, is topologically discrete. Also, if  $\alpha \in \mathbb{T}_s(K_\infty)$ , then  $\exp_\phi$  induces a  $k[\underline{t}_s]$ -linear map  $\mathbb{T}_s(K_\infty) \rightarrow \mathbb{T}_s(K_\infty)$ .

If  $B$  is a normed ring with ultrametric norm  $\|\cdot\|$ , and if  $r \geq 0$ , we shall denote by  $D_B(0, r)$  (resp.  $\overline{D}_B(0, r)$ ) the set  $\{z \in B; \|z\| < r\}$  (resp.  $\{z \in B; \|z\| \leq r\}$ ). We notice that, for all  $r \geq 0$ , the sets  $D_{\mathbb{T}_s}(0, r)$  and  $\overline{D}_{\mathbb{T}_s}(0, r)$  are  $k[t_1, \dots, t_s]$ -submodules of  $\mathbb{T}_s$ . We observe that

$$\|l_n^{-1} \alpha \tau(\alpha) \cdots \tau^{n-1}(\alpha)\| = \|\alpha\| \frac{q^n - 1}{q - 1} q^{-q \frac{q^n - 1}{q - 1}}.$$

Let us set  $r = -v_\infty(\alpha)$ . For all  $f \in \mathbb{T}_s$  such that  $v_\infty(f) > \frac{r - q}{q - 1}$  (that is,  $f \in D_{\mathbb{T}_s}(0, q^{\frac{q-r}{q-1}})$ ), the series

$$\log_\phi(f) := \sum_{n \geq 0} \frac{\tau_\alpha^n(f)}{l_n}$$

also converges in  $\mathbb{T}_s$ . We call this element in  $\mathbb{T}_s$  the *evaluation* of  $\log_\phi$  at  $f$ . The  $k[\underline{t}_s]$ -linear map

$$\log_\phi : D_{\mathbb{T}_s}(0, q^{\frac{q-r}{q-1}}) \rightarrow \mathbb{T}_s$$

defined by  $f \mapsto \log_\phi(f)$  is called the *logarithm function* of  $\phi$ . With  $\alpha \in \mathbb{T}_s \setminus \{0\}$ ,  $r = -v_\infty(\alpha)$ , we set

$$N_\alpha = D_{\mathbb{T}_s}(0, q^{\frac{q-r}{q-1}}).$$

Let  $\phi$  be the Drinfeld  $A_s$ -module of rank one defined by  $\phi_\theta = \theta + \tau_\alpha$ . As a consequence of the above discussion we have the next Lemma.

**Lemma 3.7.** *The functions  $\exp_\phi, \log_\phi$  induce isometric automorphisms of  $N_\alpha$  inverse of each other.*

---

<sup>5</sup>The reader is warned that we are using the same symbols to denote completely different entities. Indeed, at once,  $\exp_\phi$  denotes a formal series of  $\mathbb{T}_s[[\tau]]$  and a continuous endomorphism of  $\mathbb{T}_s$ . The same remark can be made for  $\log_\phi$ . This should not lead to confusion and contributes to easily manageable notation, but it is important to be aware of these facts.

3.2.1. *Example.* If  $s \geq 0$  and  $\alpha \in \mathbb{C}_\infty^\times$ , we have

$$\phi = u^{-1}Cu$$

where  $u \in \mathbb{C}_\infty$  is a root of  $X^{q-1} = \alpha$  in  $\mathbb{C}_\infty$ . Hence  $\phi$  is isomorphic to  $C$  seen as a  $k[\underline{t}_s]$ -algebra homomorphism. Then, the kernel of the function  $\exp_\phi$  is

$$\text{Ker}(\exp_\phi) = u^{-1}\tilde{\pi}A_s$$

and  $\exp_\phi$  induces a short exact sequence of  $A_s$ -modules:

$$0 \rightarrow \tilde{\pi}u^{-1}A_s \rightarrow \mathbb{T}_s \rightarrow \phi(\mathbb{T}_s) \rightarrow 0.$$

The exponential function  $\exp_\phi$  and the logarithm function  $\log_\phi$  provide isometric automorphisms of  $D(0, |u|^{-1}q^{\frac{q}{q-1}})$  which are inverse of each other.

3.2.2. *More about the modules  $N_\alpha$ .* Because this will be needed in the computations of §5, we give here some properties of the modules

$$N_\alpha(K_\infty) = D_{\mathbb{T}_s(K_\infty)}(0, q^{\frac{q-r}{q-1}}) = \{f \in \mathbb{T}_s(K_\infty), \|f\| < q^{\frac{q-r}{q-1}}\}.$$

Observe that we have a direct sum of  $k[\underline{t}_s]$ -modules:

$$\mathbb{T}_s(K_\infty) = A_s \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}.$$

Let us assume that  $\alpha \in \mathbb{T}_s(K_\infty)$ . Since for  $f \in \mathbb{T}_s(K_\infty)$  we have  $v_\infty(f) \in \mathbb{Z}$ , we can also write that  $N_\alpha(K_\infty) = \{f \in \mathbb{T}_s(K_\infty) : v_\infty(f) \geq -\nu(\alpha)\}$ , where  $\nu(\alpha) = \frac{q-r}{q-1} - 1$  if  $r \equiv 1 \pmod{q-1}$  and  $\nu(\alpha) = \lfloor \frac{q-r}{q-1} \rfloor$  otherwise ( $\lfloor x \rfloor$  denotes the biggest integer  $\leq x$ ). Then,  $-\nu(\alpha) = \lfloor \frac{r-q}{q-1} \rfloor + 1$ .

We now investigate the mutual positions of  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)}$ ,  $N_\alpha(K_\infty)$  in function of the value of  $r$ . We denote by  $u(\alpha)$  the maximum of the lower integer part of  $\frac{r-q}{q-1}$  and zero:

$$u(\alpha) = \max \left\{ 0, \left\lfloor \frac{r-q}{q-1} \right\rfloor \right\} = \max\{0, -\nu(\alpha) - 1\}.$$

Here is a table of small values of  $\nu, u$ :

$r$	0	1	...	$q-1$	$q$	...	$2q-2$	$2q-1$	...	$3q-2$	...
$\nu(\alpha)$	1	0	...	0	-1	...	-1	-2	...	-2	...
$u(\alpha)$	0	0	...	0	0	...	0	1	...	1	...

We notice that  $u(\alpha) > 0$  if and only if  $r \geq 2q-1$ . There are some critical values of  $r$  which discriminate part of our results; they are listed below.

*Case  $r \leq q-1$ .* In this case  $\mathfrak{o}_{\mathbb{T}_s(K_\infty)} \subset N_\alpha(K_\infty)$ . In fact, there is equality except in the case  $r=0$ , when the inclusion is strict.

*Case  $q \leq r < 2q-1$ .* In this case  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)} = N_\alpha(K_\infty)$ .

*Case  $r \geq 2q-1$ .* We have  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)} \supset N_\alpha(K_\infty)$  and  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)}^{u(\alpha)+1} = \mathfrak{m}_{\mathbb{T}_s(K_\infty)}^{-\nu(\alpha)} = N_\alpha(K_\infty)$  <sup>(6)</sup>.

The proof of the next Lemma is easy and left to the reader.

<sup>6</sup>We will no longer use the function  $\nu$  but the identities above may be useful for the reader to keep track of certain computations.

**Lemma 3.8.** *Let us assume that  $r \geq 2q - 1$ . Then, we have a direct sum of  $k[\underline{t}_s]$ -modules:*

$$\mathfrak{m}_{\mathbb{T}_s(K_\infty)} = N_\alpha(K_\infty) \oplus \theta^{-u(\alpha)} \langle \theta, \dots, \theta^{u(\alpha)-1} \rangle_{k[\underline{t}_s]},$$

where  $\langle \dots \rangle_{k[\underline{t}_s]}$  denotes the  $k[\underline{t}_s]$ -span of a set of elements of  $\mathbb{T}_s$ .

We denote by  $M_\alpha$  the module

$$(12) \quad M_\alpha = \theta^{-u(\alpha)} \langle 1, \dots, \theta^{u(\alpha)-1} \rangle_{k[\underline{t}_s]},$$

(by convention, for  $r < 2q - 1$ , we have  $M_\alpha = (0)$ ). Then, if  $r \geq 2q - 1$ ,

$$(13) \quad \mathbb{T}_s(K_\infty) = A_s \oplus M_\alpha \oplus N_\alpha(K_\infty).$$

3.2.3. *An example of  $\exp_\phi$  injective and not surjective.* We shall consider here the case of  $\alpha = t \in \mathbb{T}$  and describe in detail some properties of the associated exponential function  $\exp_\phi$ , given by

$$\exp_\phi = \sum_{i \geq 0} \frac{t^i \tau^i}{D_i}.$$

This map is obviously injective. We also have  $\exp_\phi(\mathbb{C}_\infty) \cap \mathbb{C}_\infty = (0)$  and  $\mathbb{T}^{\tau\alpha} = (0)$  in this case; properties which are very easy to check. However, the map is not surjective. To see this, let us extend  $\exp_\phi$  to  $\mathbb{C}_\infty[[t]]$  by  $k[t]$ -linearity. We then have the next Lemma.

**Lemma 3.9.** *Let  $y$  be an element of  $\mathbb{C}_\infty$ . There exists a unique formal series  $x = \sum_{i \geq 0} x_n t^n \in \mathbb{C}_\infty[[t]]$  such that  $\exp_\phi(x) = y$ . Furthermore, let  $\epsilon \in \mathbb{R}$  be such that*

$$|y| = q^{\frac{q-\epsilon}{q-1}}.$$

Then, for all  $n \geq 0$ ,

$$|x_n| = q^{\frac{q-q^n\epsilon}{q-1}}.$$

In particular,  $x \in \mathbb{T}$  if and only if  $|y| < q^{\frac{q}{q-1}}$ .

*Proof.* The existence of  $x = \sum_{i \geq 0} x_i t^i \in \mathbb{C}_\infty[[t]]$  such that  $\exp_\phi(x) = y$  amounts to solving the equations

$$x_0 = y, \quad \sum_{j=0}^i x_j^{q^{i-j}} D_{i-j}^{-1} = 0, \quad i > 0.$$

This system certainly has a unique solution provided inductively by

$$x_n = -(x_0^{q^n} D_n^{-1} + x_1^{q^{n-1}} D_{n-1}^{-1} + \dots + x_{n-1}^q D_1^{-1}), \quad n \geq 0.$$

We shall prove the identity for  $|x_n|$  by induction on  $n$ . For  $n = 0$  this is clear. Let us assume that  $n > 0$  is an integer such that for all  $k = 0, \dots, n-1$ ,  $|x_k| = q^{\frac{q-q^k\epsilon}{q-1}}$  and let us first compute, for  $k = 0, \dots, n-1$ :

$$\begin{aligned} |x_k^{q^{n-k}} D_{n-k}^{-1}| &= q^{\frac{q-q^k\epsilon}{q-1} q^{n-k} - (n-k)q^{n-k}} \\ &= q^{\frac{1}{q-1}(q^{n-k}(n-k-q(n-k-1)) - q^n\epsilon)}. \end{aligned}$$

We now observe that if  $k$  is chosen in the interval  $\{1, \dots, n-1\}$  then the term  $n-k-q(n-k-1)$  is either  $\leq 0$  (this happens if  $k \leq n-2$ ) either equal to one and the latter equality holds in the case of  $k = n-1$  only. Hence,

$$|x_n| = \max_{k=0, \dots, n-1} \{|x_k^{q^{n-k}} D_{n-k}^{-1}|\} = |x_{n-1}^q D_1^{-1}| = q^{\frac{q-q^n\epsilon}{q-1}}.$$

This yields the conclusion of the proof of the Lemma.  $\square$

In particular, if  $|y| < q^{\frac{a}{q-1}}$ , then  $x = \exp^{-1}(y) \in \mathbb{T}^\times$  and  $\|x\| = |y|$ . More generally, if  $y = \sum_{i \geq 0} y_i t^i \in D_{\mathbb{T}}(0, q^{\frac{a}{q-1}})$ , then, setting  $x_i = \exp_{\phi}^{-1}(y_i) \in \mathbb{T}^\times$  with  $\|x_i\| = |y_i|$  we see, writing  $x = \sum_{i \geq 0} x_i t^i$ , that  $\exp_{\phi}(x) = y$  so that  $\exp_{\phi}$  induces an isometric automorphism of  $D_{\mathbb{T}}(0, q^{\frac{a}{q-1}})$ . The series  $\log_{\phi} = \sum_{i \geq 0} \frac{t^i \tau^i}{\ell_i}$  converges in the same disk and provides a reciprocal of  $\exp_{\phi}$  there (Lemma 3.7).

It may also be worthy of noticing that the composition of  $\exp_{\phi}$  with the evaluation at  $t = 0$  yields the identity map, while the composition with the evaluation at  $t = 1$  yields the Carlitz exponential  $\exp_C$ .

3.2.4. *Entire operators.* Let

$$f = \sum_{i_1, \dots, i_s} f_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s}$$

be an element of  $\mathbb{T}_s$  (in particular, the coefficients  $f_{i_1, \dots, i_s}$  lie in  $\mathbb{C}_{\infty}$ ). We say that  $f$  is an *entire function* if

$$\frac{v_{\infty}(f_{i_1, \dots, i_s})}{i_1 + \cdots + i_s} \rightarrow \infty$$

for the Fréchet filter (that is, the cofinite filter). The subset  $\mathbb{E}_s$  of entire functions of  $\mathbb{T}_s$  is a subring containing the subring of polynomials  $\mathbb{C}_{\infty}[\underline{t}_s]$ . The operator  $\tau$  induces an automorphism of it.

Let us consider a sequence of entire functions  $(F_n)_{n \geq 0}$  and an operator

$$F = \sum_{n \geq 0} F_n \tau^n \in \mathbb{E}_s[[\tau]].$$

We say that  $F$  is an *entire operator* if  $\lim_{n \rightarrow \infty} v_{\infty}(F_n) q^{-n} = +\infty$ . In particular, we have a well defined evaluation of the operator  $F$  at an element  $f \in \mathbb{T}_s$  defined by the element

$$F(f) = \sum_{n \geq 0} F_n \tau^n(f) \in \mathbb{T}_s.$$

We have the following Lemma.

**Lemma 3.10.** *Let  $F = \sum_{n \geq 0} F_n \tau^n$  be an entire operator. Then,  $F(\mathbb{E}_s) \subset \mathbb{E}_s$ .*

*Proof.* We use here a multi-index notation. With  $\underline{i}$  we shall denote a multi-index  $(i_1, \dots, i_s)$  whose entries are non-negative integers. We denote by  $|\underline{i}|$  the integer  $i_1 + \cdots + i_s$ , and if  $\underline{i}, \underline{j}$  are such multi-indices, then  $\underline{i} + \underline{j}$  denotes their component-wise sum. We also write  $t^{\underline{i}}$  for the monomial  $t_1^{i_1} \cdots t_s^{i_s}$ . Hence, we have  $f = \sum_{\underline{j}} f_{\underline{j}} t^{\underline{j}}$ . We expand each entire function  $F_n$  in series

$$F_n = \sum_{\underline{i}} F_{n, \underline{i}} t^{\underline{i}},$$

where by hypothesis,  $\frac{v_{\infty}(F_{n, \underline{i}})}{|\underline{i}|} \rightarrow \infty$  for the Fréchet filter.

Now, we verify easily that  $F(f) = \sum_{\underline{k}} c_{\underline{k}} t^{\underline{k}} \in \mathbb{T}_s$  where

$$c_{\underline{k}} = \sum_{\underline{i} + \underline{j} = \underline{k}} \sum_{n \geq 0} F_{n, \underline{i}} f_{\underline{j}}^q.$$

Since

$$v_\infty(c_k) \geq \inf_{i+j=k, n \geq 0} (v_\infty(F_{n,i}) + q^n v_\infty(f_j)),$$

and since  $\lim_{n \rightarrow \infty} v_\infty(F_n)q^{-n} = \lim_{\underline{i}} v_\infty(f_{\underline{i}})|\underline{i}|^{-1} = \lim_{\underline{j}} v_\infty(F_{n_0, \underline{j}})|\underline{j}|^{-1} = \infty$  (the limits are taken along the Fréchet filter) for all  $n_0$ , we get  $\lim_{\underline{k}} v_\infty(g_{\underline{k}})|\underline{k}|^{-1} = \infty$  and  $F(f) \in \mathbb{E}_s$ .  $\square$

Let  $\alpha$  be an element of  $\mathbb{E}_s$ . Then,  $\alpha\tau(\alpha) \cdots \tau^{i-1}(\alpha)$  is also entire for all  $i$  and

$$\lim_{i \rightarrow \infty} v_\infty(\alpha\tau(\alpha) \cdots \tau^{i-1}(\alpha)D_i^{-1})q^{-i} \rightarrow +\infty.$$

Therefore we deduce the next Proposition which will be of some help later on in this paper.

**Proposition 3.11.** *Let  $\phi$  be a Drinfeld module of rank one over  $\mathbb{T}_s$ , let us assume that the parameter  $\alpha$  is an entire function of  $\mathbb{E}_s$  and let  $\exp_\phi(f)$  be the exponential function of  $\phi$ . Then,  $\exp_\phi$  is an entire operator so that, for every element  $f \in \mathbb{E}_s$ ,  $\exp_\phi(f) \in \mathbb{E}_s$ .*

#### 4. L-SERIES VALUES

In this section we consider a Drinfeld  $A_s$ -module  $\phi$  of rank one over  $\mathbb{T}_s$  with parameter

$$\alpha \in A_s.$$

We are going to associate to such a parameter  $\alpha \in A_s^*$  (<sup>7</sup>) an *L-series value*. Although the hypotheses can be relaxed (in various directions; see many remarks along this text), this will provide for us already quite a large spectrum of properties to investigate. The forthcoming paper by Florent Demeslay [11] will deepen some of the issues presented in this paper.

**4.1. Definition of L-series values.** Let us expand  $\alpha = \alpha_r\theta^r + \alpha_{r-1}\theta^{r-1} + \cdots + \alpha_0 \in A_s^*$ , with  $\alpha_r, \dots, \alpha_0 \in k[\underline{t}_s]$  and  $\alpha_r \neq 0$ . Then, in a fixed algebraic closure  $k_s^{ac}$  of  $k_s$  we can find elements  $x_1, \dots, x_r$  and  $\beta \in k[\underline{t}_s]^*$  so that, in  $k_s^{ac}[\theta]$ ,

$$(14) \quad \alpha = \beta(x_1 - \theta) \cdots (x_r - \theta).$$

We point out that there exists a unique extension of the norm  $\|\cdot\|$  to  $k_s^{ac}((\theta^{-1}))$ ; this will be again denoted by  $\|\cdot\|$ . If  $x \in k_s^{ac}$ , then  $\|x\| = 1$  or  $\|x\| = 0$  depending on whether  $x \neq 0$  or  $x = 0$  and this is sufficient to define the norm  $\|\cdot\|$ .

**4.1.1. Characters  $A^* \rightarrow k_s^\times$ .** Our characters can be considered, in some way, as some kind of higher dimensional generalization of Dirichlet characters. With  $\alpha \in A_s^*$  the parameter of  $\phi$  as in (14) we study

$$\rho_\alpha : A \rightarrow k_s^{ac}$$

the map defined by  $\rho_\alpha(0) = 0$  and

$$\rho_\alpha(a) = \beta^{\deg_\theta(a)} a(x_1) \cdots a(x_r), \quad a \in A^*.$$

An alternative way to write it is by means of the resultant. We have

$$\rho_\alpha(a) = \text{Res}_\theta(a, \alpha) \in k[\underline{t}_s],$$

---

<sup>7</sup>For a ring  $R$ ,  $R^*$  denotes the multiplicative semigroup whose elements are the non-zero elements of  $R$ .

where  $\text{Res}_\theta(P, Q)$  denotes the resultant of two polynomials in the indeterminate  $\theta$  <sup>(8)</sup>. In particular, with  $P$  a prime of  $A$ ,  $\rho_\alpha(P) = 0$  if and only if  $P$  divides  $\alpha$  in  $A_s$ .

If  $a, b \in A$ , then  $\rho_\alpha(ab) = \rho_\alpha(a)\rho_\alpha(b)$  and if  $\alpha_1, \alpha_2$  are polynomials of  $A_s$ , then

$$\rho_{\alpha_1\alpha_2}(a) = \rho_{\alpha_1}(a)\rho_{\alpha_2}(a), \quad a \in A.$$

This map will be called the *character associated to  $\phi$* . When  $s = 0$ , we get Dirichlet characters  $A^* \rightarrow k^\times$ . The character associated to the Carlitz module is easily seen to be the trivial one, sending the whole  $A^*$  to one.

**Definition 4.1.** Let  $\phi$  be the Drinfeld  $A_s$ -module of rank one of parameter  $\alpha$ , let  $n \geq 1$  be an integer. The *L-series value of order  $n$  associated to  $\phi$*  is the unit of norm one of  $\mathbb{T}_s(K_\infty)$  defined by

$$L(n, \phi) = \sum_{a \in A_+} \frac{\rho_\alpha(a)}{a^n} = \prod_{\mathfrak{p} \text{ prime of } A} \left(1 - \frac{\rho_\alpha(\mathfrak{p})}{\mathfrak{p}^n}\right)^{-1}.$$

The above *L-series value* (in short *L-value*) will be one of the main objects of interest of the present paper.

**Remark 4.2.** We can also associate *L-series values*  $L(n, \phi)$  to Drinfeld  $R_s$ -modules of rank one defined over  $R_s$  (we recall that  $R_s = k_s[\theta]$ ), see Definition 3.6. In the sequel, we will also work with such modules and *L-series values*, but the most interesting examples discussed here will arise from the case of  $A_s$ -modules defined over  $A_s$ .

We will be mainly interested in the case  $n = 1$  in this paper.

4.1.2. *First examples with  $s = 0$ .* If  $s = 0$  and  $\alpha = 1$ , we have  $\phi = C$  and

$$L(n, \phi) = L(n, C) = \zeta_C(n),$$

where  $\zeta_C(n)$  is, for  $n > 0$ , the *Carlitz zeta value*

$$\zeta_C(n) = \sum_{a \in A^+} a^{-n} \in 1 + \theta^{-1}k[[\theta^{-1}]].$$

If  $\alpha \in A^*$ , then we can write  $\alpha = \beta \mathfrak{p}_1^{\nu_1} \cdots \mathfrak{p}_m^{\nu_m}$  with  $\beta \in k^\times$ , for primes  $\mathfrak{p}_1, \dots, \mathfrak{p}_m$  of respective degrees  $d_1, \dots, d_m$  so that  $\sum_i d_i \nu_i = r = \deg_\theta(\alpha)$ , we have, for  $a \in A^+$ ,

$$\rho_\alpha(a) = \beta^{\deg_\theta(a)} \prod_{i=1}^m \prod_{j=1}^{d_i} a(\zeta_{i,j})^{\nu_i},$$

where  $\zeta_{i,1}, \dots, \zeta_{i,d_i}$  are the zeros of  $\mathfrak{p}_i$  in  $k^{ac}$  for all  $i$ . This implies, in the case  $\beta = 1$  (that is,  $\alpha \in A^+$ ), that the series  $L(n, \phi)$  is the special value of a Dirichlet *L-series*:

$$L(n, \phi) = \sum_{a \in A_+} a^{-n} \prod_{i=1}^m \prod_{j=1}^{d_i} a(\xi_{i,j})^{\nu_i} \in K_\infty.$$

---

<sup>8</sup>We recall that if  $P = P_0\theta^d + P_1\theta^{d-1} + \cdots + P_d$  and  $Q = Q_0\theta^r + Q_1\theta^{r-1} + \cdots + Q_r$  are polynomials with roots respectively  $\zeta_i$  and  $x_j$ , then, for the resultant  $\text{Res}_\theta(P, Q)$ , we have the identity  $\text{Res}_\theta(P, Q) = P_0^r \prod_i Q(\zeta_i) = (-1)^{dr} Q_0^d \prod_j P(x_j)$ .

4.1.3. *Case of  $\alpha = t$ .* It is understood here that  $s = 1$  so that we are in  $\mathbb{T} = \mathbb{T}_1$ . This case directly refers to the example of Drinfeld module  $\phi$  treated in 3.2.3. We have then

$$L(n, \phi) = \sum_{d \geq 0} t^d \sum_{a \in A_{+,d}} a^{-n} \in \mathbb{T} \cap K[[t]]$$

if  $n > 0$ . It is easy to see that

$$L(1, \phi) = \sum_{i \geq 0} t^i \ell_i^{-1} = \log_{\phi}(1) \in \mathbb{T}.$$

In his Ph. D. Dissertation [20], R. Perkins proves a recursive formula for certain *special polynomials* associated to the values at negative integers of the series occurring in [18] and [5]. With his result it is possible to show,  $\phi$  being a Drinfeld  $A_s$ -module defined over  $A_s$ , that

$$L(\phi, -n) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \rho_{\alpha}(a) a^n \in A_s, \quad n \geq 0.$$

For example, it is well known that  $\sum_{a \in A_{+,d}} a^j = 0$  for all  $d$  big enough if  $j \geq 0$ . Thus, we have (with  $\phi$  as in 3.2.3), for  $j \geq 0$ , that the series

$$L(-j, \phi) := \sum_{d \geq 0} t^d \sum_{a \in A_{+,d}} a^j$$

defines an element of  $A[t]$  and

$$L(-j, \phi) = z(t^{-1}, -j)$$

where the function  $z$  is defined as in Goss' book [13, Remark 8.12.1]. In loc. cit. Goss computes recursively the polynomial  $z(t^{-1}, -j) \in A[t]$  for all  $j \geq 0$ .

4.1.4. *Case in which  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$ .* In this case we have  $\phi = C_s$  with  $C_s$  the Drinfeld modules of Definition 3.5 and we recover the functions

$$L(\chi_{t_1} \cdots \chi_{t_s}, n) = \sum_{a \in A^+} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{a^n} \in \mathbb{T}_s(K_{\infty})^{\times}$$

studied in [5]. The case  $s = 1$  and  $\alpha = t - \theta$  yields the functions  $L(\chi_t, n)$  of [18].

4.1.5. *A further example.* We shall also trace a connection with the Goss zeta functions, especially the functions considered by Goss in [14], see also [5, Section 3.1]. This also explains why we distinguished, at least at the beginning of this Section,  $L$ -series from  $L$ -series values. We recall, from Section 3.1 of loc. cit. the definition of the  $L$ -series  $L(\chi_{t_1} \cdots \chi_{t_s}, x, y)$ , with  $(x, y)$  in the topological group  $\mathbb{C}_{\infty}^{\times} \times \mathbb{Z}_p$  denoted by  $\mathbb{S}_{\infty}$  there:

$$L(\chi_{t_1} \cdots \chi_{t_s}, x, y) = \sum_{k \geq 0} x^{-k} \sum_{a \in A_{k,+}} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \langle a \rangle^{-y},$$

where  $\langle a \rangle$  is the 1-unit  $a/\theta^{\deg_{\theta}(a)}$ , of which the  $p$ -adic exponentiation by  $-y$  is well defined. For fixed  $(x, y) \in \mathbb{S}_{\infty}$ , the above series is a well defined unit element of  $\mathbb{T}_s$ . Thanks to [5, Proposition 32], we know that for any choice of  $n \geq 1$ , the above series in  $\mathbb{T}_s$  also defines an entire function  $\mathbb{C}_{\infty}^s \rightarrow \mathbb{C}_{\infty}$ .



We have, for  $\beta \in k^\times$ :

$$\begin{aligned} L(\chi_{t_1} \cdots \chi_{t_s}, \beta^{-1}\theta^n, n) &= \sum_{k \geq 0} \beta^k \theta^{-kn} \sum_{a \in A_{k,+}} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \theta^{kn} a^{-n} \\ &= \sum_{k \geq 0} \beta^k \sum_{a \in A_{k,+}} \chi_{t_1}(a) \cdots \chi_{t_s}(a) a^{-n}. \end{aligned}$$

This equals  $L(n, C_s)$  if  $\beta = 1$ .

## 5. THE CLASS NUMBER FORMULA

We introduce the *class module* and the *unit module* associated to a given Drinfeld module of rank one of parameter  $\alpha \in A_s^*$ .

**5.1. Class and unit modules.** Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one with parameter  $\alpha \in A_s^*$ . We now introduce the *unit module* and the *class module* of  $\phi$ . The definitions are inspired by Taelman's work [24, 25].

**5.1.1. The class module.** Again because  $\alpha \in A_s^*$ , we have the  $A_s$ -submodule

$$\exp_\phi(\mathbb{T}_s(K_\infty)) + \phi(A_s)$$

of  $\phi(\mathbb{T}_s(K_\infty))$ . We define the class module  $H_\phi$  as the quotient  $A_s$ -module:

$$H_\phi := \frac{\phi(\mathbb{T}_s(K_\infty))}{\exp_\phi(\mathbb{T}_s(K_\infty)) + \phi(A_s)}.$$

**5.1.2. The unit module.** As a set, it is defined by:

$$U_\phi := \{f \in \mathbb{T}_s(K_\infty); \exp_\phi(f) \in A_s\}.$$

Since  $\phi_\theta(A_s) \subset A_s$  (because  $\alpha \in A_s$ ),  $U_\phi$  has a structure of  $A_s$ -submodule of  $\mathbb{T}_s(K_\infty)$ ; the action of  $a \in A_s$  on an element  $f \in U_\phi$  is just the product  $af$ .

**5.1.3. Computation of  $H_\phi$  when  $r < 2q - 1$ .** We use here the computations of §3.2.2. We have

$$\mathbb{T}_s(K_\infty) = A_s \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}.$$

If  $r < 2q - 1$ , then  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)} \subset N_\alpha$  and  $U_\phi \cap \mathfrak{m}_{\mathbb{T}_s(K_\infty)} = (0)$ . Hence,  $\exp_\phi(U_\phi \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}) \subset A_s \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}$ . Then, the exponential function  $\exp_\phi$  induces a morphism

$$\frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}} \rightarrow \frac{\mathbb{T}_s(K_\infty)}{A_s \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}}$$

the cokernel of which is  $H_\phi$ . But the module on the right is zero; we deduce that  $H_\phi = (0)$  and that  $U_\phi$  is non-trivial.

**Lemma 5.1.** *Let us suppose that  $r \geq 2q - 1$ , so that  $u(\alpha) > 0$ . The exponential function  $\exp_\phi : \mathbb{T}_s(K_\infty) \rightarrow \mathbb{T}_s(K_\infty)$  induces an injective homomorphism of  $k[\underline{t}_s]$ -modules*

$$\frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus N_\alpha(K_\infty)} \rightarrow \frac{\mathbb{T}_s(K_\infty)}{A_s \oplus N_\alpha(K_\infty)}$$

whose cokernel is  $H_\phi$ .

*Proof.* This is plain by the fact that  $\exp_\phi$  restricted to  $N_\alpha$  is an isometric isomorphism and the fact that  $\exp_\phi^{-1}(A_s) = U_\phi$ .  $\square$

**Corollary 5.2.** *For all Drinfeld modules  $\phi$  as above,  $H_\phi$  is a finitely generated  $k[\underline{t}_s]$ -module of rank  $\leq u(\alpha)$ .*

**Remark 5.3.** When  $r \geq 2q - 1$  we have constructed a short exact sequence of  $k[\underline{t}_s]$ -modules

$$(15) \quad 0 \rightarrow \frac{\mathbb{T}_s(K_\infty)}{U_\phi + N_\alpha(K_\infty)} \rightarrow \frac{\mathbb{T}_s(K_\infty)}{A_s \oplus N_\alpha(K_\infty)} \rightarrow H_\phi \rightarrow 0.$$

Note that in general, this may not be an exact sequence of  $A_s$ -modules. On the other hand, there is an isomorphism of  $k[\underline{t}_s]$ -modules between  $M_\alpha$  (the module defined in (12)) and

$$\frac{\mathbb{T}_s(K_\infty)}{A_s \oplus N_\alpha(K_\infty)}.$$

Therefore the modules

$$\frac{\mathbb{T}_s(K_\infty)}{U_\phi + N_\alpha(K_\infty)}, \quad H_\phi$$

have finite ranks and their ranks add up to  $u(\alpha)$ , which is the rank of  $M_\alpha$ . This tells us in particular that  $U_\phi$  is non-zero.

For  $\phi$  a Drinfeld  $A_s$ -module of rank one defined over  $A_s$ , we denote by  $k_s U_\phi$  the  $R_s$ -module  $U_\phi \otimes_{k[\underline{t}_s]} k_s$ .

**Proposition 5.4.** *The following properties hold.*

- (1) *The  $R_s$ -module  $k_s U_\phi$  is free of rank one.*
- (2) *We have:*

$$k_s U_\phi = \{f \in K_{s,\infty}, \exp_\phi(f) \in R_s\}.$$

*Proof.* (1). Let  $r$  be  $\geq 2q - 1$ . Since  $\exp_\phi$  is injective on  $N_\alpha$  and  $U_\phi$  is non-trivial, there exists an element  $f \in k_s U_\phi$  with  $\|f\| > 0$  minimal. Indeed, by the fact that  $\exp_\phi$  induces an isometric isomorphism of  $N_\alpha$ ,  $k_s U_\phi$  is discrete, that is,  $k_s U_\phi \cap \mathfrak{m}_{K_{s,\infty}}^n = (0)$  for  $n$  big enough.

Let  $g$  be another element of  $k_s U_\phi$ . Then, by the euclidean algorithm of the division in  $R_s$ , there exists a polynomial  $h$  of  $R_s$  such that  $g = hf + r$  where  $r \in K_{s,\infty}$  is such that  $\|r\| < \|f\|$ . Since  $k_s U_\phi$  is an  $R_s$  module, we get  $r \in k_s U_\phi$  so that  $r = 0$ . This means that  $k_s U_\phi$  is free of rank one. The proof for  $r < 2q - 1$  is similar and left to the reader.

(2). We recall that  $\mathbb{T}_s(K_\infty) = A_s \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}$ . This implies that

$$K_{s,\infty} = k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty) + \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}.$$

It is clear that  $k_s U_\phi \subset \{f \in K_{s,\infty}, \exp_\phi(f) \in R_s\}$ . Now, let  $f \in K_{s,\infty}$  such that  $\exp_\phi(f) \in R_s$ . We can write  $f$  as a sum  $g + h$ , where  $g \in k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty)$  and  $h \in \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}$ . We get:

$$\exp_\phi(h) = \exp_\phi(f) - \exp_\phi(g) \in k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty).$$

This implies that:

$$\exp_\phi(h) \in k_s \mathfrak{m}_{\mathbb{T}_s(K_\infty)}^{u(\alpha)+1} = \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1} \cap k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty).$$

Therefore  $h \in k_s \otimes_{k[\underline{t}_s]} \mathfrak{m}_{\mathbb{T}_s(K_\infty)}^{u(\alpha)+1}$  and thus  $f \in k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty)$ . We conclude that  $f \in k_s U_\phi$ .  $\square$

**Corollary 5.5.** *The  $A_s$ -module  $U_\phi$  is free of rank one.*

*Proof.* By Proposition 5.4, we see that  $k_s U_\phi = f R_s$ , with  $f \in \mathbb{T}_s(K_\infty)$ . Without loss of generality, we can also suppose that if  $h$  divides  $f$  in  $\mathbb{T}_s(K_\infty)$ , with  $h \in k[\underline{t}_s]$ , then  $h \in k^\times$ . Clearly,  $U_\phi \supset f A_s$ . Let us consider now an element  $g$  of  $U_\phi$ . We have that  $g \in f R_s$  because  $U_\phi \subset k_s U_\phi$ , and  $k_s U_\phi$  is free of rank one by Proposition 5.4. Hence, we can write  $g = af/\delta$  where  $a \in A_s$  and  $\delta \in k[\underline{t}_s]^*$ . This means that  $\delta$  divides  $af$  in  $\mathbb{T}_s(K_\infty)$  so that  $g \in A_s f$ .  $\square$

**Remark 5.6.** The Drinfeld module  $\phi$  acts on  $K_{s,\infty}$  and on  $R_s$  as well. It gives rise to a Drinfeld  $R_s$ -module of rank one over  $K_{s,\infty}$ . We define the  $R_s$ -module:

$$V_\phi = \frac{\phi(K_{s,\infty})}{\phi(R_s) + \exp_\phi(K_{s,\infty})}.$$

Just as in the proof of the second part of Proposition 5.4, we see in fact that for all  $n \geq 1$ ,

$$K_{s,\infty} = k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty) + \mathfrak{m}_{K_{s,\infty}}^n.$$

For all  $n$  big enough,  $\exp_\phi$  induces an isometric automorphism of  $\mathfrak{m}_{K_{s,\infty}}^n$  (for instance, it suffices to take  $n \geq u(\alpha) + 1$ ). Therefore, for such a choice of  $n$ , we have the isomorphisms of  $k_s$ -vector spaces

$$(16) \quad V_\phi = \frac{k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty) + \mathfrak{m}_{K_{s,\infty}}^n}{R_s + \exp_\phi(k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty)) + \mathfrak{m}_{K_{s,\infty}}^n} \cong \frac{k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty)}{R_s + \exp_\phi(k_s \otimes_{k[\underline{t}_s]} \mathbb{T}_s(K_\infty))}.$$

The last  $k_s$ -vector space is

$$H_\phi \otimes_{k[\underline{t}_s]} k_s,$$

of finite dimension  $\leq u(\alpha)$  by Corollary 5.2.

This yields:

**Corollary 5.7.** *The  $R_s$ -module  $V_\phi$  is a finite dimensional  $k_s$ -vector space of dimension at most  $u(\alpha)$ .*

**5.2. Local factors of the  $L$ -series values.** Let  $R$  be a unitary commutative ring. Let  $M$  be a finitely generated  $R$ -module. As *Fitting ideal* of  $M$  we mean the *initial Fitting ideal* as defined in [15, Chapter XIX]. By Chapter XIX, Corollary 2.9 of loc. cit., if  $M$  is a finite direct sum of cyclic modules,

$$M = \bigoplus_{i=1}^n \frac{R}{\mathfrak{a}_i}, \quad \mathfrak{a}_i \text{ ideal of } R,$$

then

$$\text{Fitt}(M) = \mathfrak{a}_1 \cdots \mathfrak{a}_n.$$

Let  $\theta$  be an indeterminate over a field  $F$ , write  $R = F[\theta]$ , and let us consider an  $R$ -module  $M$  which also is an  $F$ -vector space of finite dimension. Let  $e_\theta$  be the endomorphism of  $M$  induced by the multiplication by  $\theta$ . Then, we write

$$[M]_R = \det_R(Z - e_\theta|_M)|_{Z=\theta} \in R$$

for the characteristic polynomial of  $e_\theta$ , where the indeterminate  $Z$  is replaced with  $\theta$ . This is a monic polynomial in  $\theta$  of  $R = F[\theta]$  and its important property is that it is the monic generator of  $\text{Fitt}_R(M)$ .

We now focus on the case  $R = R_s = k_s[\theta]$ . In this Section we will establish a result (Proposition 5.11) leading us to apply Theorem 10.4 of the Appendix §10 of F. Demeslay. This step will be crucial as it furnishes access to most of the results of this paper.

Let  $\alpha$  be an element of  $R_s^*$  and let us consider the Drinfeld  $R_s$ -module of rank one and parameter  $\alpha$ , that is, the injective homomorphism of  $k_s$ -algebras

$$\phi : R_s \rightarrow \text{End}_{k_s\text{-lin.}}(K_{s,\infty})$$

given by  $\phi_\theta = \theta + \alpha\tau$ . For all  $a \in A$ , the resultant  $\rho_\alpha(a) = \text{Res}_\theta(a, \alpha)$  is a well defined element of  $k_s$  making the series (and the corresponding eulerian product)

$$L(n, \phi) = \sum_{a \in A_+} \rho_\alpha(a) a^{-n} = \prod_P \left( 1 - \frac{\rho_\alpha(P)}{P^n} \right)^{-1}, \quad n > 0$$

well defined and convergent in  $K_{s,\infty}$  (for the Gauss absolute value).

**Lemma 5.8.** *Let  $P$  be a prime of  $A$  of degree  $d$ . Then, the following congruence holds:*

$$\phi_P \equiv \rho_\alpha(P) \tau^d \pmod{PR_s[\tau]}.$$

*Proof.* Let  $a \in A_+$  of degree  $d$ . We expand in  $R_s[\tau]$ :

$$\phi_a = \sum_{i=0}^d (a)_i \tau^i,$$

where it is easy to see that  $(a)_0 = a$ ,  $(a)_i \in R_s$  for  $i = 0, \dots, d$ , and

$$(a)_d = \alpha\tau(\alpha) \cdots \tau^{d-1}(\alpha).$$

From the relation  $\phi_a \phi_\theta = \phi_\theta \phi_a$  we get, by induction on  $i = 1, \dots, d-1$ ,

$$(a)_i = \frac{\alpha\tau((a)_{i-1}) - \tau^{i-1}(\alpha)(a)_{i-1}}{\theta^{q^i} - \theta}.$$

Since  $P$  does not divide  $\theta^{q^i} - \theta$  if  $i < d$ , we have, again by induction,  $P|(a)_i$  for all  $i < d-1$  if  $P|a$ . Thus, if  $P$  is a prime of degree  $d$ , we get for  $i = 0, \dots, d-1$ ,  $(P)_i \equiv 0 \pmod{PR_s}$ . This implies that

$$\phi_P \equiv \alpha\tau(\alpha) \cdots \tau^{d-1}(\alpha) \tau^d \pmod{PR_s[\tau]}.$$

Now, we observe that, if  $\zeta_1, \dots, \zeta_d$  are the roots of  $P$  in  $k^{ac}$  and if  $\zeta$  is one of these roots,

$$\begin{aligned} \rho_\alpha(P) &= \text{Res}_\theta(P, \alpha) \\ &= \prod_{j=1}^d \alpha|_{\theta=\zeta_j} \\ &= \alpha\tau(\alpha) \cdots \tau^{d-1}(\alpha)|_{\theta=\zeta} \\ &\equiv \alpha\tau(\alpha) \cdots \tau^{d-1}(\alpha) \pmod{PR_s}. \end{aligned}$$

□

**Lemma 5.9.** *Let  $(L, \sigma)$  be a difference field. Let  $r \geq 0$  be an integer strictly less than the order of  $\sigma$ , let  $a_0, \dots, a_{r-1}$  be  $r$  elements in  $L$ . We consider the set*

$$V = \left\{ x \in L; \sigma^r(x) + \sum_{i=0}^{r-1} a_i \sigma^i(x) = 0 \right\}.$$

*Then  $V$  is an  $L^\sigma$ -vector space of dimension not exceeding  $r$ .*

*Proof.* A sketch of proof will be enough as this is essentially well known, see [21, §1.2]. Let  $n \geq 1$  be an integer and let  $\mathcal{A}$  be a matrix with entries in  $L$ . Let  $v_1, \dots, v_r$  be vectors of  $L^n$  such that  $\sigma(v_i) = \mathcal{A}v_i, i = 1, \dots, r$ . Then by the proof of [21, Lemma 1.7], if the vectors  $v_1, \dots, v_r$  are linearly dependent over  $L$ , they are also linearly dependent over  $L^\sigma$ . This implies that the  $L^\sigma$ -vector space  $W = \{v \in L^n, \sigma(v) = \mathcal{A}v\}$  satisfies:

$$\dim_{L^\sigma}(W) \leq n.$$

Let  $\mathcal{A}$  be the *companion matrix* of the equation

$$\sigma^r(x) + \sum_{i=0}^{r-1} a_i \sigma^i(x) = 0$$

(see [21, p. 8]). Let  $W = \{v \in L^r, \sigma(v) = \mathcal{A}v\}$ . Then the map  $V \mapsto W$ ,

$$x \mapsto {}^t(x, \sigma(x), \dots, \sigma^{r-1}(x))$$

(the sign  ${}^t$  means transposition) is an isomorphism of  $L^\sigma$ -vector spaces.  $\square$

In the next Lemma,  $\phi$  denotes a Drinfeld  $R_s$ -module of rank one over  $K_{s,\infty}$  with parameter  $\alpha \in R_s^*$ .

**Lemma 5.10.** *Let  $P$  be a prime of  $A$  of degree  $d$ . Then, we have an isomorphism of  $R_s$ -modules:*

$$\phi \left( \frac{R_s}{PR_s} \right) \cong \frac{R_s}{(P - \rho_\alpha(P))R_s}.$$

*Proof.* By Lemma 5.8, we have:

$$(P - \rho_\alpha(P)) \cdot \phi \left( \frac{R_s}{PR_s} \right) = (0),$$

where the dot indicates the multiplication by the polynomial  $P - \rho_\alpha(P) \in R_s$  for the structure of  $R_s$ -module induced by  $\phi$ .

We set  $L = R_s/PR_s$ ; then  $\tau$  induces an automorphism of  $L$  and  $L^\tau = k_s$ , so that  $\tau$  is  $k_s$ -linear and its order is  $d$ . Also,  $\phi$  induces a morphism of  $k_s$ -algebras  $\phi : R_s \rightarrow L[\tau]$ . For  $a$  in  $R_s$  we set  $\text{Ker}(\phi_a) = \{x \in L; \phi_a(x) = 0\}$ . We write  $b = P - \rho_\alpha(P)$ . We notice that  $d = \dim_{k_s}(L) = \deg_\theta(b)$  and  $L = \text{Ker}(\phi_b)$ . We have, by Lemma 5.9,  $\dim_{k_s}(\text{Ker}(\phi_b)) \leq d$ , but  $L \subset \text{Ker}(\phi_b)$  so that  $L = \text{Ker}(\phi_b)$ . This implies (see for example [13, proof of Theorem 6.3.2]) that we have an isomorphism of  $R_s$ -modules  $\text{Ker}(\phi_b) \cong R_s/bR_s$ .  $\square$

Let  $P$  be a prime of  $A$ . By Lemma 5.10, we have  $[\phi(\frac{R_s}{PR_s})]_{R_s} = P - \rho_\alpha(P)$ . We now introduce an infinite product running over the primes  $P$  of  $A$ :

$$(17) \quad \mathcal{L}(\phi/R_s) = \prod_P \frac{[\frac{R_s}{PR_s}]_{R_s}}{[\phi(\frac{R_s}{PR_s})]_{R_s}}.$$

**Proposition 5.11.** *Let  $\phi$  be a Drinfeld  $R_s$ -module of rank one of parameter  $\alpha \in R_s$ . The product  $\mathcal{L}(\phi/R_s)$  in (17) converges in  $K_{s,\infty}$  and we have*

$$\mathcal{L}(\phi/R_s) = L(1, \phi).$$

*Proof.* We have, by Lemma 5.10,

$$\frac{[\frac{R_s}{PR_s}]_{R_s}}{[\phi(\frac{R_s}{PR_s})]_{R_s}} = \frac{P}{P - \rho_\alpha(P)} = \left(1 - \frac{\rho_\alpha(P)}{P}\right)^{-1}.$$

The factors of the infinite product defining  $\mathcal{L}(\phi/R_s)$  agree with the eulerian factors of  $L(1, \phi)$ . Since the product  $L(1, \phi)$  converges in  $K_{s,\infty}$ , this implies that the product  $\mathcal{L}(\phi/R_s)$  converges to  $L(1, \phi)$  in  $K_{s,\infty}$ .  $\square$

**Remark 5.12.** Proposition 5.11 agrees with the local factor expression of (2).

**5.3. The class number formula.** An element

$$f = \sum_{i \geq i_0} f_i \theta^{-i} \in K_{s,\infty}, \quad f_i \in k_s$$

is *monic* if the leading coefficient  $f_{i_0}$  is equal to one. We shall write

$$[R_s : k_s U_\phi]_{R_s}$$

for the unique monic element  $f$  in  $K_{s,\infty}$  such that  $k_s U_\phi = f R_s$ , the existence of which is guaranteed by the first part of Proposition 5.4.

We recall from Corollary 5.7 that the  $R_s$ -module  $V_\phi$  is a  $k_s$ -vector space of finite dimension  $\leq u(\alpha)$ . The next Theorem directly follows from Theorem 10.4, proved in the appendix by Florent Demeslay, by means of Proposition 5.11.

**Theorem 5.13** (The class number formula). *Let  $\phi$  be a Drinfeld  $R_s$ -module of rank one of parameter  $\alpha \in R_s$ . The following equality holds in  $K_{s,\infty}$ :*

$$L(1, \phi) = [V_\phi]_{R_s} [R_s : k_s U_\phi]_{R_s}.$$

Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$  with parameter  $\alpha \in A_s^*$ . The following Corollary to the class number formula will be crucial:

**Corollary 5.14.** *We have:*

$$\exp_\phi(L(1, \phi)) \in A_s.$$

*Proof.* By definition,  $[R_s : k_s U_\phi]_{R_s} \in k_s U_\phi$  and obviously,  $[V_\phi]_{R_s} \in R_s$ . Since  $k_s U_\phi$  is an  $R_s$ -module (with the structure of module issued from the multiplication in  $K_{s,\infty}$ ) we have

$$L(1, \phi) \in k_s U_\phi.$$

By Part (2) of Proposition 5.4,  $\exp_\phi(L(1, \phi)) \in R_s$ . At once, by construction,  $L(1, \phi) \in \mathbb{T}_s(K_\infty)$  so that  $\exp_\phi(L(1, \phi)) \in \mathbb{T}_s(K_\infty)$ . But then,

$$\exp_\phi(L(1, \phi)) \in R_s \cap \mathbb{T}_s = A_s. \quad \square$$

**Remark 5.15.** If  $\alpha$  is as in (14) and  $0 \leq r \leq q-1$  we have that  $L(1, \phi) \in N_\alpha$  and  $L(1, \phi) - 1 \in \mathfrak{m}_{\mathbb{T}_s(K_\infty)}$  (see §3.2.2). Since  $\exp_\phi$  is an isometric automorphism of  $N_\alpha$  we also get  $\exp_\phi(L(1, \phi)) - 1 \in \mathfrak{m}_{\mathbb{T}_s(K_\infty)}$  but  $R_s \cap \mathfrak{m}_{\mathbb{T}_s(K_\infty)} = (0)$ . We have obtained the identity

$$(18) \quad \exp_\phi(L(1, \phi)) = 1.$$

This can be rewritten as

$$(19) \quad L(1, \phi) = \log_\phi(1)$$

because  $1 \in N_\alpha$  thanks again to the hypothesis on  $r$ .

5.3.1. *The circular unit module.* This is the sub- $A_s$ -module

$$U_\phi^c = L(1, \phi)A_s \subset U_\phi.$$

Let us now focus on the modules  $U_\phi$  and  $U_\phi^c$ . Since  $L(1, \phi)$  is a unit of norm one in  $\mathbb{T}_s(K_\infty)$ , we have the decomposition

$$\mathbb{T}_s(K_\infty) = U_\phi^c \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}.$$

Let  $f$  be in  $U_\phi$ . We can then write, according to the above decomposition,

$$(20) \quad f = aL(1, \phi) \oplus \mu, \quad a \in A_s, \quad \mu \in \mathfrak{m}_{\mathbb{T}_s(K_\infty)}.$$

If  $1 \leq r < 2q - 1$ , Since  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)} \subset N_\alpha$ , we have the decomposition:

$$\exp_\phi(\mathbb{T}_s(K_\infty)) = \exp_\phi(U_\phi^c \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}) = \exp_\phi(U_\phi^c) \oplus \mathfrak{m}_{\mathbb{T}_s(K_\infty)}$$

which implies that  $\mu = 0$  in (20) (by the fact that  $\exp_\phi$  induces an isometric isomorphism of  $N_\alpha$ , hence on  $\mathfrak{m}_{\mathbb{T}_s(K_\infty)}$ ). This yields that  $\exp_\phi(U_\phi) = \exp_\phi(U_\phi^c)$ . We have obtained:

**Proposition 5.16.** *If  $r < 2q - 1$  then  $H_\phi = (0)$  and  $\exp_\phi(U_\phi) = \exp_\phi(U_\phi^c)$ .*

**Remark 5.17.** If the function  $\exp_\phi$  is injective we get, with the hypotheses of the above Proposition,  $U_\phi = U_\phi^c$ . With the study of *uniformizable* Drinfeld modules of rank one of §6, we will be able to say when this occurs.

**Proposition 5.18.** *Let us suppose that  $r \geq 2q - 1$  and that the restriction of  $\exp_\phi$  is injective over  $\mathbb{T}_s(K_\infty)$ . Then,  $\frac{U_\phi}{U_\phi^c}$  and  $H_\phi$  are finitely generated  $k[\underline{t}_s]$ -modules of equal rank at most  $u(\alpha)$ .*

*Proof.* We have that  $u(\alpha) \geq 1$ . By Lemma 3.8, we have a direct sum of  $k[\underline{t}_s]$ -modules:

$$\mathbb{T}_s(K_\infty) = \theta^{-u(\alpha)}U_\phi^c \oplus N_\alpha(K_\infty).$$

Then, we have an exact sequence of  $k[\underline{t}_s]$ -modules:

$$(21) \quad 0 \rightarrow \frac{U_\phi}{U_\phi^c} \rightarrow \frac{\theta^{-u(\alpha)}U_\phi^c \oplus N_\alpha(K_\infty)}{U_\phi^c \oplus N_\alpha(K_\infty)} \rightarrow \frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus N_\alpha(K_\infty)} \rightarrow 0$$

with all arrows induced by the identity map over  $\mathbb{T}_s(K_\infty)$ . Observe that the  $k[\underline{t}_s]$ -module in the middle is free of rank  $u(\alpha)$ . Thus  $\frac{U_\phi}{U_\phi^c}$  is a finitely generated torsion-free  $k[\underline{t}_s]$ -module of rank not bigger than  $u(\alpha)$ .

Recall from Remark 5.3 that if  $r \geq 2q - 1$ , then  $\exp_\phi$  induces an exact sequence of finitely generated  $k[\underline{t}_s]$ -modules:

$$0 \rightarrow \frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus N_\alpha(K_\infty)} \rightarrow \frac{\mathbb{T}_s(K_\infty)}{A_s \oplus N_\alpha(K_\infty)} \rightarrow H_\phi \rightarrow 0.$$

Since there is an isomorphism of  $k[\underline{t}_s]$ -modules

$$\frac{\theta^{-u(\alpha)}U_\phi^c \oplus N_\alpha(K_\infty)}{U_\phi^c \oplus N_\alpha(K_\infty)} \cong \frac{\mathbb{T}_s(K_\infty)}{A_s \oplus N_\alpha(K_\infty)}$$

the modules  $\frac{U_\phi}{U_\phi^c}$  and  $H_\phi$  have the same rank over  $k[\underline{t}_s]$ .  $\square$

We denote by  $k_s U_\phi^c$  the  $R_s$ -module  $k_s \otimes_{k[\underline{t}_s]} U_\phi^c$ . We deduce, from Theorem 5.13:

**Corollary 5.19.** *We have*

$$\text{Fitt}_{R_s}(H_\phi \otimes_{k[\underline{t}_s]} k_s) = [k_s U_\phi : k_s U_\phi^c]_{R_s} R_s.$$

*Proof.* We have:

$$[R_s : k_s U_\phi]_{R_s} = [R_s : R_s L(1, \phi)]_{R_s} \left[ \frac{k_s U_\phi}{R_s L(1, \phi)} \right]_{R_s}^{-1}.$$

Then, by Theorem 5.13, we obtain:

$$\left[ \frac{k_s U_\phi}{R_s L(1, \phi)} \right]_{R_s} = [V_\phi]_{R_s}.$$

□

**Remark 5.20.** The Proposition 5.18 uses the injectivity of  $\exp_\phi$  over  $\mathbb{T}_s(K_\infty)$ . For example, if  $s = 1$  and  $\phi_\theta = \theta + t\tau$  we know from §3.2.3 that  $\exp_\phi$  is injective (even over  $\mathbb{T}_s$ ). In this case, it is easy to show that  $L(1, \phi) = \log_\phi(1)$  generates the module  $U_\phi$ .

To proceed further, we need a precise characterization of the  $A_s$ -modules whose exponential function is injective. This is in fact closely related to the non-surjectivity of  $\exp_\phi$  and will be investigated in the next Section.

## 6. UNIFORMIZABLE DRINFELD MODULES OF RANK ONE.

In this Section we consider general Drinfeld modules of rank one defined over  $\mathbb{T}_s$ . If  $F \in \mathbb{T}_s[\tau]$ , we set

$$\mathbb{T}_s^F = \{g \in \mathbb{T}_s, F(g) = g\}.$$

This is a  $k[\underline{t}_s]$ -submodule of  $\mathbb{T}_s$ . Observe that if  $\alpha = 1$  (case in which  $\phi = C$ , the Carlitz module), then  $\exp_C : \mathbb{T}_s \rightarrow \mathbb{T}_s$  is a surjective homomorphism of  $k[\underline{t}_s]$ -modules. Furthermore, we have  $\tau_\alpha = \tau$ ,

$$\text{Ker}(\exp_C) = \tilde{\pi} A_s,$$

and

$$\mathbb{T}_s^{\tau_\alpha} = k[\underline{t}_s].$$

We are going to study a class of Drinfeld  $A_s$ -modules of rank one defined over  $\mathbb{T}_s$  which have similar properties. We will focus on the next Definition.

**Definition 6.1.** Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$ . We say that  $\phi$  is *uniformizable* if  $\exp_\phi$  is surjective on  $\mathbb{T}_s$ .

**Proposition 6.2.** *Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$  and let  $\alpha \in \mathbb{T}_s \setminus \{0\}$  be its parameter. The following conditions are equivalent.*

- (1)  $\phi$  is uniformizable,
- (2)  $\mathbb{T}_s^{\tau_\alpha} \neq (0)$ ,
- (3)  $\alpha \in \mathbb{T}_s^\times$ ,
- (4)  $\phi$  is isomorphic to the Carlitz module.



*Proof.* We begin by proving that (3) implies (4). Since  $\alpha \in \mathbb{T}_s^\times$ , there exists  $x \in \mathbb{C}_\infty^\times$  such that  $v_\infty(\alpha - x) > v_\infty(\alpha)$ . Observe that:

$$v_\infty \left( \frac{\alpha^{(i)}}{x^{(i)}} - 1 \right) \geq q^i (v_\infty(\alpha - x) - v_\infty(\alpha)).$$

Thus the product  $\prod_{i \geq 0} \left( \frac{x^{(i)}}{\alpha^{(i)}} \right)$  converges in  $\mathbb{T}_s^\times$ . Now let us choose an element  $\gamma \in \mathbb{C}_\infty^\times$  such that:

$$\gamma^{q-1} = x.$$

We set:

$$(22) \quad \omega_\alpha = \gamma \prod_{i \geq 0} \left( \frac{x^{(i)}}{\alpha^{(i)}} \right) \in \mathbb{T}_s^\times.$$

At first sight, these functions depend on the choice of  $x$  but it is easy to show that they are defined up to a scalar factor of  $k^\times$ . We also notice that, for  $\alpha_1, \alpha_2 \in \mathbb{T}_s^\times$ ,

$$\omega_{\alpha_1 \alpha_2} \in k^\times \omega_{\alpha_1} \omega_{\alpha_2}, \quad \alpha_1, \alpha_2 \in \mathbb{T}_s^\times.$$

Then:

$$(23) \quad \omega_\alpha^{(1)} = \alpha \omega_\alpha.$$

This implies that, in  $\mathbb{T}_s[\tau]$ , we have:

$$C_\theta \omega_\alpha = \omega_\alpha \phi_\theta,$$

that is,  $\phi$  and  $C$  are isomorphic.

In fact, it is also easy to show that (4) implies (3). Indeed, assuming that the Drinfeld module of rank one  $\phi$  is isomorphic to  $C$ , we see directly that the parameter  $\alpha$  of  $\phi$  must satisfy  $\tau(u)/u = \alpha$  for a unit  $u$  of  $\mathbb{T}_s$ , but this implies that  $\alpha$  is a unit as well.

Next, we prove that (4) implies (1). By hypothesis, there exists  $\vartheta \in \mathbb{T}_s^\times$  such that, in  $\mathbb{T}_s[[\tau]]$ :

$$\vartheta \tau_\alpha = \tau \vartheta$$

We get in  $\mathbb{T}_s[[\tau]]$ :

$$\exp_\phi = \vartheta^{-1} \exp_C \vartheta.$$

Since  $\exp_C$  is surjective on  $\mathbb{T}_s$ , we obtain that  $\exp_\phi$  is also surjective.

We prove that (1) implies (2). Let us then suppose that  $\exp_\phi$  is surjective. The map  $\mathbb{T}_s \rightarrow \mathbb{T}_s$  defined by  $f \mapsto \phi_\theta(f)$  is surjective. Explicitly, for all  $f \in \mathbb{T}_s$  there exists  $g \in \mathbb{T}_s$  such that:

$$\alpha \tau(g) + \theta g = f.$$

Therefore:

$$\alpha \tau \left( \frac{g}{\lambda_\theta} \right) - \frac{g}{\lambda_\theta} = \frac{f}{\lambda_\theta^q} = -\frac{f}{\lambda_\theta \theta}$$

(we recall that we have set  $\lambda_\theta = \exp_C(\tilde{\pi}/\theta)$ ). This implies that the map  $\tau_\alpha - 1$  is surjective on  $\mathbb{T}_s$ .

Hence, we can assume, without loss of generality, that  $\|\alpha\| = 1$ . Let us suppose, by contradiction, that  $\mathbb{T}_s^{\tau_\alpha} = (0)$ . Then, the map  $f \mapsto \tau_\alpha(f) - f$  is an isomorphism of  $k[\underline{t}_s]$ -modules which satisfies  $\|\alpha \tau(f) - f\| < 1$  if and only if  $\|f\| < 1$  and  $\|\alpha \tau(f) - f\| > 1$  if and only if  $\|f\| > 1$ . In particular, this map induces an automorphism of the  $k[\underline{t}_s]$ -module  $\{f \in \mathbb{T}_s : \|f\| = 1\}$ .

Reducing modulo  $\mathfrak{m}_{\mathbb{T}_s}$ , the above map induces the  $k[\underline{t}_s]$ -linear endomorphism of  $k^{ac}[\underline{t}_s]$  given by  $\bar{f} \mapsto \bar{\alpha}\tau(\bar{f}) - \bar{f}$  where  $\bar{\alpha} \neq 0$  is the image of  $\alpha$  in  $k^{ac}[\underline{t}_s]$  and  $\bar{f} \in k^{ac}[\underline{t}_s]$ . However, this endomorphism cannot be an automorphism.

For instance, if  $\bar{\alpha}$  is a non-constant polynomial in the indeterminates  $t_1, \dots, t_s$ , then, it is not surjective (no polynomial  $\bar{g} \in k[t_1, \dots, t_s]$  of total degree strictly less than the total degree of  $\bar{\alpha}$  can lie in the image). On the other hand, if  $\bar{\alpha} \in k^{ac}$ , the endomorphism is not injective. Indeed, the kernel of the map  $\bar{f} \mapsto \bar{\alpha}\tau(\bar{f}) - \bar{f}$  is a  $k[\underline{t}_s]$ -module of dimension one. This constitutes a contradiction with the assumption that  $\mathbb{T}_s^{\tau\alpha} = (0)$ .

We finally prove that (2) implies (3). Let  $g$  be a non-zero element of  $\mathbb{T}_s^{\tau\alpha}$ . By  $\alpha\tau(g) = g$  we deduce that  $\alpha\tau(\alpha) \cdots \tau^{n-1}(\alpha)\tau^n(g) = g$  for all  $n$ . If  $\alpha$  were not a unit, this would contradict the finiteness of the number of irreducible factors of  $g$ .  $\square$

**Remark 6.3.** The following observation will be extensively used in the rest of this paper. Let  $\phi$  be a uniformizable Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$  of parameter  $\alpha \in \mathbb{T}_s^\times$ . Then its exponential function  $\exp_\phi$  induces an exact sequence of  $A_s$ -modules:

$$0 \rightarrow \frac{\tilde{\pi}}{\omega_\alpha} A_s \rightarrow \mathbb{T}_s \rightarrow \phi(\mathbb{T}_s) \rightarrow 0,$$

where  $\omega_\alpha$  is defined as in (22). In this case, the module  $\mathbb{T}_s^{\tau\alpha}$  is obviously given by:

$$\mathbb{T}_s^{\tau\alpha} = \frac{1}{\omega_\alpha} k[\underline{t}_s].$$

We recall that we have set  $N_\alpha = D_{\mathbb{T}_s}(0, q^{\frac{\alpha-r}{q-1}})$ . By our initial remarks on  $\exp_C$  and  $\log_C$ , the series  $\exp_\phi$  and  $\log_\phi$  induce isometric automorphisms of  $N_\alpha$  each one reciprocal of the other; we have recovered the conclusion of Lemma 3.7 in a different way.

**Definition 6.4.** Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$ , then  $\mathbb{T}_s$  is an  $A_s$ -module via  $\phi$ , thus if  $f \in A_s \setminus \{0\}$ , we define the  $A_s$ -module of  $f$ -torsion  $\phi[f]$  by:

$$\phi[f] = \{g \in \mathbb{T}_s, \phi_f(g) = 0\}.$$

Proposition 6.2 can be used to prove the following.

**Corollary 6.5.** *Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$  of parameter  $\alpha \in \mathbb{T}_s^*$ . The following assertions are equivalent:*

- (1)  $\phi$  is uniformizable.
- (2)  $\mathbb{T}_s^{\tau\alpha}$  is a  $k[\underline{t}_s]$ -module of rank one.
- (3) For any  $f \in A_s \cap \mathbb{T}_s^\times$  we have an isomorphism of  $A_s$ -modules:

$$\phi[f] \simeq \frac{A_s}{fA_s}.$$

- (4) There exists  $f \in A_s \cap \mathbb{T}_s^\times$  such that  $\phi[f] \neq (0)$ .

*Proof.* The equivalence of the properties (1) and (2) is already covered by the proof of Proposition 6.2 and by the Remark 6.3.

We show that (1) implies (3). By Remark 6.3 we have that

$$\text{Ker}(\phi) = \frac{\tilde{\pi}}{\omega_\alpha} A_s.$$

Notice also that  $\exp_\phi$  is surjective so that if  $f \in \mathbb{T}_s^\times$ , we also have that

$$\exp_\phi^{-1}(\phi[f]) = \frac{\tilde{\pi}}{f\omega_\alpha} A_s.$$

It is obvious that (3) implies (4); it remains to show that (4) implies (1). Let  $\alpha$  be the parameter of  $\phi$  and let us assume that for some  $f \in A_s \cap \mathbb{T}_s^\times$ , we have  $\phi[f] \neq (0)$ ; let  $g \in \mathbb{T}_s \setminus \{0\}$  be such that  $\phi_f(g) = 0$ .

We can write, in  $\mathbb{T}_s[\tau]$ :

$$\phi_f = \sum_{i=0}^d c_i \tau_\alpha^i = \sum_{i \geq 0} \alpha \cdots \alpha^{(i-1)} c_i \tau^i,$$

where for  $i = 0, \dots, d$ ,  $c_i \in A_s$ , and  $c_0 = f, c_d \in k^\times$ . We get:

$$\sum_{i=1}^d \alpha \cdots \alpha^{(i-1)} c_i g^{(i)} = -fg.$$

Since  $f \in \mathbb{T}_s^\times$ , we get  $g = \alpha g_1$ ,  $g_1 \in \mathbb{T}_s \setminus \{0\}$ . Thus (we recall that  $\mathbb{T}_s$  is a unique factorization domain):

$$\sum_{i=1}^d \alpha^{(1)} \cdots \alpha^{(i)} c_i g_1^{(i)} = -fg_1.$$

Therefore  $g_1 = \alpha^{(1)} g_2$ , and  $\alpha \alpha^{(1)}$  divides  $g$  in  $\mathbb{T}_s$ . Thus for any  $n \geq 1$ ,  $\alpha \cdots \alpha^{(n)}$  divides  $g$  in  $\mathbb{T}_s$ . Therefore  $\alpha \in \mathbb{T}_s^\times$  and  $\phi$  is uniformizable by Proposition 6.2.  $\square$

**Remark 6.6.** The definition of uniformizable  $A_s$  module is motivated by Anderson's result [2, Theorem 4]. It is an interesting question to characterize higher rank "uniformizable" Drinfeld modules over  $\mathbb{T}_s$ , that is, Drinfeld modules over  $\mathbb{T}_s$  which have surjective associated exponential function. We plan to come to this question in another work.

**6.1. The elements  $\omega_\alpha$ .** We focus on the case  $\alpha \in \mathbb{T}_s(K_\infty)^\times$ . Let  $\alpha \in \mathbb{T}_s(K_\infty)^\times$ . Then, there exists  $\gamma \in \mathbb{T}_s(K_\infty)^\times$  monic (as a power series in  $\theta^{-1}$ ) such that  $\alpha = \beta\gamma$  for some  $\beta \in k^\times$ .

The function  $\omega_\alpha$  defined in (22) is determined up to a factor in  $k^\times$ . In the present case, the element  $x \in \mathbb{C}_\infty^\times$  such that  $\|\alpha - x\| < \|\alpha\|$  is  $x = \beta\theta^r$  with  $r = -v_\infty(\alpha)$  so that we can rewrite  $\omega_\alpha$  in the following form:

$$(24) \quad \omega_\alpha = \tilde{\beta} \lambda_\theta^r \prod_{i \geq 0} \left( \frac{\alpha^{(i)}}{\beta \theta^{r q^i}} \right)^{-1},$$

where  $\tilde{\beta} \in k^{ac}$  is such that  $\tilde{\beta}^{q-1} = \beta$ . From this it is apparent that

$$\omega_\alpha \in \tilde{\beta} \lambda_\theta^r \mathbb{T}_s(K_\infty)^\times$$

and that

$$(25) \quad \|\omega_\alpha\| = q^{\frac{r}{q-1}}.$$

We also recall the equation (23).

**Remark 6.7.** In (3), we have defined  $\tilde{\pi}$  up to a choice of a  $(q-1)$ -th root of  $-\theta$ , which is  $\lambda_\theta$ . This defines  $\omega_\alpha$  uniquely by the product (24). From now on, we will always use this normalization.

The proof of the next Lemma is easy and left to the reader.

**Lemma 6.8.** *Let  $\alpha$  be in  $\mathbb{T}_s(K_\infty)^\times$ . The following conditions are equivalent:*

- (1)  $\frac{\tilde{\pi}}{\omega_\alpha} \in \mathbb{T}_s(K_\infty)$ .
- (2) *If  $r = -v_\infty(\alpha)$ ,  $r \equiv 1 \pmod{q-1}$  and  $(-1)^r \alpha$  is monic as a polynomial in  $\theta$ .*

**Remark 6.9.** Let  $\phi$  be a Drinfeld  $A_s$ -module of rank one with parameter in  $\mathbb{T}_s(K_\infty)$ . The exponential function  $\exp_\phi : \mathbb{T}_s \rightarrow \mathbb{T}_s$  is either surjective, either injective. As for the restriction to  $\mathbb{T}_s(K_\infty)$ , we have three cases.

*The non-uniformizable case.* The exponential function  $\exp_\phi : \mathbb{T}_s \rightarrow \mathbb{T}_s$  is injective and not surjective.

*The uniformizable non-torsion case.* Here,  $\phi$  is uniformizable but  $\tilde{\pi}/\omega_\alpha \notin \mathbb{T}_s(K_\infty)$ . We then have  $\exp_\phi : \mathbb{T}_s \rightarrow \mathbb{T}_s$  surjective and not injective, but the restriction  $\exp_\phi : \mathbb{T}_s(K_\infty) \rightarrow \mathbb{T}_s(K_\infty)$  is injective (and not surjective). The terminology is justified by the fact that  $\exp_\phi(L(1, \phi))$  is not, in this case, a point of torsion for  $\phi$  (see Remark 7.6).

*The uniformizable torsion case.* In this case,  $\tilde{\pi}/\omega_\alpha \in \mathbb{T}_s(K_\infty)$ . We have that  $\exp_\phi : \mathbb{T}_s \rightarrow \mathbb{T}_s$  is surjective, and the restriction  $\exp_\phi : \mathbb{T}_s(K_\infty) \rightarrow \mathbb{T}_s(K_\infty)$  is not injective, with kernel generated by  $\tilde{\pi}/\omega_\alpha$ . This can be used to apply Proposition 5.18 effectively once the factorization (14) of  $\alpha$  over  $k_s^{ac}$  is known. In this case  $\exp_\phi(L(1, \phi))$  is a point of torsion for  $\phi$  (see Remark 7.6).

6.1.1. *Examples in the uniformizable case.* If  $s = 0$  and  $\alpha = \beta\theta^r$ , we get, as a particular case,  $\omega_\alpha = \tilde{\beta}\lambda_\theta^r$ . More generally, if  $\alpha \in A^*$ , we have

$$(26) \quad \omega_\alpha^{q-1} = \alpha.$$

If  $s = 1$  and  $\alpha = t - \theta$ , we have an important example (uniformizable torsion case):

$$(27) \quad \omega_\alpha = \omega = \lambda_\theta \prod_{i \geq 0} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1} \in \lambda_\theta K_\infty.$$

This function, introduced in Anderson and Thakur paper [4, Proof of Lemma 2.5.4 p. 177], was also used extensively in [18, 5]. For general  $s$ , it is important to also consider the function  $\omega_\alpha$  associated with the choice of  $\alpha = \beta(t_1 - \theta) \cdots (t_s - \theta)$ ,  $\beta \in k^\times$ . In this case,

$$(28) \quad \omega_\alpha = \tilde{\beta} \omega(t_1) \cdots \omega(t_s).$$

The infinite product

$$\prod_{i \geq 0} \left(1 - \frac{x}{\theta^{q^i}}\right)^{-1}$$

converges in  $k_s^{ac}((\theta^{-1}))$  for all  $x \in k_s^{ac}$  and allows to furnish a well defined notion of evaluation  $\omega(x)$  of the function  $\omega$  at a given element  $x \in k_s^{ac}$ . In particular, for  $\alpha$  as in (14) with  $\beta \in k^\times$ , we have in  $K_{s,\infty}$ :

$$\omega_\alpha = \tilde{\beta} \omega(x_1) \cdots \omega(x_s).$$

**Remark 6.10.** Another way to rewrite the identity noticed in Remark 5.15 is, for  $\alpha$  as in (14) with  $\beta \in k^\times$  and  $1 \leq r \leq q-1$ :

$$L(1, \phi)\omega_\alpha = \log_C(\omega_\alpha),$$

which yields, as a particular case, Theorem 1.2 in Perkins paper [19].

## 7. UNIFORMIZABLE DRINFELD MODULES OF RANK ONE DEFINED OVER $A_s$

In this Section, we fix a uniformizable Drinfeld  $A_s$ -module of rank one over  $\mathbb{T}_s$  defined over  $A_s$ , that is, such that its parameter  $\alpha$  is in  $A_s$ . Then,  $\alpha$  lies in  $A_s \cap \mathbb{T}_s^\times$  and we have a factorization (14) with  $\beta \in k^\times$ .

**7.1. The modules  $H_\phi$ ,  $U_\phi$  in the uniformizable torsion case.** Here, we assume that  $\alpha$  is as in (14) with  $\beta = 1$ , and  $r \equiv 1 \pmod{q-1}$ . We begin by inspecting the case  $r = 1$ . Here we have  $\alpha = x - \theta$  with  $x \in k[\underline{t}_s]$ , so that  $\rho_\alpha(a) = a(x)$  for  $a \in A$ , and in particular,  $\rho_\alpha(\theta) = x$ .

**Lemma 7.1.** *If  $\beta = 1$ ,  $r = 1$ , we have the identity:*

$$L(1, \phi)\omega_\alpha = \frac{\tilde{\pi}}{\theta - x}.$$

*Proof.* Since  $\rho_\alpha(\theta) = x$ , we have:

$$(x - \theta)\omega_\alpha \tilde{\pi}^{-1} = \frac{-\theta\lambda_\theta}{\tilde{\pi}} \prod_{i \geq 1} \left(1 - \frac{x}{\theta q^i}\right)^{-1} \equiv \frac{-\theta\lambda_\theta}{\tilde{\pi}} \equiv -1 \pmod{\mathfrak{m}_{\mathbb{T}_s(K_\infty)}}.$$

Hence, if we write

$$F = (x - \theta)L(1, \phi)\omega_\alpha \tilde{\pi}^{-1},$$

we have

$$F \equiv -1 \pmod{\mathfrak{m}_{\mathbb{T}_s(K_\infty)}}.$$

Since  $\exp_\phi(L(1, \phi)) = 1$  (Remark 5.15), we get  $(x - \theta)L(1, \phi) = (\rho_\alpha(\theta) - \theta)L(1, \phi) \in \text{Ker}(\exp_\phi)$  so that  $F \in A_s$ , and  $F = -1$  (notice that 1 is a point of  $(t - \theta)$ -torsion for  $\phi$  in this case).  $\square$

The above Lemma implies [18, Theorem 1]. We further have the following Proposition.

**Proposition 7.2.** *In the above hypotheses (uniformizable torsion case with  $r = -v_\infty(\alpha)$  and  $r \equiv 1 \pmod{q-1}$ ) the following properties hold.*

- (1) *If  $r \geq q$ ,  $U_\phi = \frac{\tilde{\pi}}{\omega_\alpha}A_s$  and  $\exp_\phi(U_\phi) = 0$ .*
- (2) *If  $r = 1, q$ , then  $H_\phi = (0)$  and  $U_\phi^c = U_\phi$ .*
- (3) *If  $r \geq 2q - 1$ , then  $H_\phi$  is a free  $k[\underline{t}_s]$ -module of rank  $u(\alpha)$  spanned by the image of  $M_\alpha$  by the next to last map in (15). Moreover,  $U_\phi$  is equal to the rank one  $A_s$ -module generated by  $\frac{\tilde{\pi}}{\omega_\alpha}$  and  $U_\phi/U_\phi^c$  is isomorphic to  $H_\phi$  as a  $k[\underline{t}_s]$ -module.*

*Proof.* (1). By Lemma 6.8 and the identity (25) we see that  $\|\tilde{\pi}/\omega_\alpha\| = q^{u(\alpha)}$  so that

$$(29) \quad \mathbb{T}_s(K_\infty) = \frac{\tilde{\pi}}{\omega_\alpha}A_s \oplus N_\alpha(K_\infty).$$

Let  $f$  be in  $U_\phi$  and let us write  $f = f_1 \oplus f_2$  with  $f_1 \in \frac{\tilde{\pi}}{\omega_\alpha}A_s$  and  $f_2 \in N_\alpha$ . Since  $f_1$  is in the kernel of  $\exp_\phi$ , we have  $\exp_\phi(f) = \exp_\phi(f_2) \in N_\alpha$ . Since  $\exp_\phi$  induces

an isometric automorphism of  $N_\alpha$ , the condition  $\exp_\phi(f) \in A_s$  yields  $f_2 = 0$ . This means that  $U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A_s$  as expected.

(2). If  $r = 1, q$ , we already know that  $H_\phi = (0)$  (Corollary 5.2). In fact, we know it for all  $r < 2q - 1$ . By Proposition 5.16, we also know that the identity  $\exp_\phi(U_\phi) = \exp_\phi(U_\phi^c)$  holds. If  $r = 1$ , thanks to Lemma 7.1, we see that

$$U_\phi^c = \frac{\tilde{\pi}}{(x - \theta)\omega_\alpha} A_s,$$

where  $x$  is the unique root of  $\alpha$ . This module contains the torsion submodule of  $\exp_\phi$  and  $U_\phi = U_\phi^c$  in this case.

If  $r = q$ , we already know that  $U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A[\underline{t}_q]$ . Then,

$$L(1, \phi)A[\underline{t}_q] = U_\phi^c \subset U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A[\underline{t}_q].$$

There exists  $a \in A[\underline{t}_q]$  such that  $L(1, \phi) = a\tilde{\pi}/\omega_\alpha$ . But  $\|L(1, \phi)\| = 1$  and  $\|\tilde{\pi}/\omega_\alpha\| = q^{\frac{q-r}{q-1}} = 1$ . Therefore,  $a \in k[\underline{t}_s]$ . Let us now observe that  $L(1, \phi) \in 1 + \mathfrak{m}_{\mathbb{T}_q(K_\infty)}$  and  $\tilde{\pi}/\omega_\alpha \in -1 + \mathfrak{m}_{\mathbb{T}_q(K_\infty)}$ . This implies that  $a = -1$ .

(3). Following the decomposition (29), the  $k[\underline{t}_s]$ -module  $N_\alpha$  acquires in the uniformizable torsion case a structure of  $A_s$ -module from  $\phi$ . By Lemma 5.1 and Remark 5.3, the exact sequence (15) becomes an exact sequence of  $A_s$ -modules:

$$0 \rightarrow \frac{\mathbb{T}_s(K_\infty)}{U_\phi \oplus N_\alpha(K_\infty)} \rightarrow \frac{\phi(\mathbb{T}_s(K_\infty))}{\phi(A_s) \oplus \phi(N_\alpha(K_\infty))} \rightarrow H_\phi \rightarrow 0.$$

But in our uniformizable torsion case,

$$H_\phi = \frac{\phi(\mathbb{T}_s(K_\infty))}{\exp_\phi(\mathbb{T}_s(K_\infty)) + \phi(A_s)} = \frac{\phi(\mathbb{T}_s(K_\infty))}{\phi(A_s) \oplus \phi(N_\alpha(K_\infty))}$$

and the third arrow of the above exact sequence is an isomorphism (and the second arrow, induced by the exponential function  $\exp_\phi$ , is trivial), so that  $H_\phi$ , as a  $k[\underline{t}_s]$ -module, is isomorphic to  $A_s\theta^{-u(\alpha)}/A_s \cong M_\alpha$ , which is free of rank  $u(\alpha)$ , by Lemma 5.1. Finally, in (21), the third arrow maps to zero so that  $U_\phi/U_\phi^c \cong H_\phi$  as a  $k[\underline{t}_s]$ -module. In particular,  $U_\phi/U_\phi^c$  is free over  $k[\underline{t}_s]$  of rank  $u(\alpha)$ , which can also be proved directly, by the fact that  $U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A_s$  and  $U_\phi^c = L(1, \phi)A_s$ .  $\square$

We deduce the next Corollaries.

**Corollary 7.3.** *Let  $\alpha$  be as in (14) with  $\beta = 1$  and  $r = q$ . Let  $\phi$  be a Drinfeld module of rank one with parameter  $\alpha$ . Then, the following formula holds:*

$$L(1, \phi) = -\frac{\tilde{\pi}}{\omega_\alpha}.$$

**Corollary 7.4.** *If  $\beta = 1$ ,  $r \equiv 1 \pmod{q-1}$  and  $r \geq q$  then*

$$L(1, \phi) \in A_s \frac{\tilde{\pi}}{\omega_\alpha}.$$

**Remark 7.5.** If  $\alpha$  is as in (14) with  $\beta = 1$  and  $r \equiv 1 \pmod{q-1}$ , the isomorphism of (16) is an isomorphism of  $R_s$ -modules. Indeed, in this case,  $N_\alpha$  is an  $A_s$ -module and  $k_s \otimes_{k[\underline{t}_s]} N_\alpha$  is an  $R_s$ -module. These modules are respectively denoted by  $\phi(N_\alpha)$  and  $\phi(k_s N_\alpha)$ .

**Remark 7.6.** The results Proposition 7.2 and Corollary 7.4 also justify the terminology *uniformizable torsion case* and *uniformizable non-torsion case* of Remark 6.9. We can see,  $\phi$  being a Drinfeld  $A_s$ -module of rank one of parameter  $\alpha$  defined over  $A_s$ , that  $\tilde{\pi}^{-1}L(1, \phi)\omega_\alpha$  is rational if and only if  $\phi$  is in the uniformizable torsion case described in the above mentioned Remark. This is equivalent to the fact that  $L(1, \phi)$  is a torsion point for  $\phi$  and is also equivalent to the fact that  $r = \deg_\theta(\alpha)$  is congruent to one modulo  $q - 1$  and  $(-1)^r\alpha$  is monic (as a polynomial in  $\theta$ ).

The terminology is also suggested by the behavior of the higher Carlitz zeta values  $\zeta_C(n) = \sum_{a \in A_+} a^{-n}$ . In [4], Anderson and Thakur constructed a point  $z_n \in \text{Lie}(C^{\otimes n})(K_\infty)$  with last entry  $\Pi(n-1)\zeta_C(n)$  such that  $\text{Exp}_n(z_n) = Z_n$ , where  $\Pi$  denotes the *Carlitz factorial* (see §9.4),  $\text{Exp}_n$  denotes the exponential function of  $C^{\otimes n}$  and  $Z_n$  is a certain  $A$ -valued *special point* of  $C^{\otimes n}$  explicitly constructed in loc. cit. We have that  $Z_n$  is a torsion point for  $C^{\otimes n}$  if and only if  $q - 1$  divides  $n$  (see Anderson and Thakur, [4, Corollary 3.8.4] and J. Yu, [27, Corollary 2.6]).

The methods of [5, Theorem 4] can probably be used to show that, more generally,  $\tilde{\pi}^{-n}L(n, \phi)\omega_\alpha$  is rational if and only if  $r \equiv n \pmod{q - 1}$  and  $(-1)^r\alpha$  is monic. It would be nice to see if these are also related to torsion points for the tensor powers of the modules  $\phi$  as in [4] in the case  $s = 0$ .

7.1.1. *The polynomials  $\mathbb{B}_\phi$ .* Let us suppose again that  $\alpha$  is as in (14) with  $\beta = 1$  and that  $r \equiv 1 \pmod{q - 1}$ ,  $r \geq q$ . By (1) of Proposition 7.2, we have that

$$\mathbb{B}_\phi := (-1)^{\frac{r-1}{q-1}}L(1, \phi)\omega_\alpha\tilde{\pi}^{-1} \in A_s.$$

**Remark 7.7.** We additionally set, when  $\alpha$  is as in (14) with  $\beta = 1$  and that  $r = 1$ ,

$$\mathbb{B}_\phi = \frac{1}{\theta - x}$$

where  $x \in k[\underline{t}_s]$  is the unique root of  $\alpha$  in  $k_s$  as a polynomial in  $\theta$ .

The polynomials  $\mathbb{B}_\phi$  have already been studied in [5] in the case of  $\alpha = (t_1 - \theta) \cdots (t_r - \theta)$  with  $r = s$ . If  $r = q$ , we can even deduce the exact value of  $\mathbb{B}_\phi$  (see Corollary 7.3):  $\mathbb{B}_\phi = -1$ . More generally, we have the following:

**Lemma 7.8.** *In the above hypotheses, the polynomial  $(-1)^{\frac{r-1}{q-1}}\mathbb{B}_\phi \in k[\underline{t}_s][\theta]$  is monic polynomial of degree  $u(\alpha) = \frac{r-q}{q-1}$  in the indeterminate  $\theta$ .*

*Proof.* Let us write:

$$\mathbb{B}_\phi = \sum_{i=0}^m a_i \theta^i,$$

where  $a_i \in k[\underline{t}_s]$ , and  $a_m \neq 0$ . We have that  $v_\infty(\tilde{\pi}^{-1}L(1, \phi)\omega_\alpha) = v_\infty(\mathbb{B}_\phi) = \frac{q-r}{q-1}$ , which implies that

$$m = \frac{r-q}{q-1}$$

and

$$a_m \in k^\times.$$

To compute  $a_m$  it suffices to compute the leading coefficient of the expansion of  $\tilde{\pi}^{-1}L(1, \phi)\omega_\alpha$  as a series of  $k[\underline{t}_s] \otimes_k k((\theta^{-1}))$ . This computation is easy and left to the reader.  $\square$

The importance of the polynomials  $\mathbb{B}_\phi$  is dictated by the next Theorem.

**Theorem 7.9.** *If  $\alpha$  is as in (14) with  $\beta = 1$  and  $r \geq q, r \equiv 1 \pmod{q-1}$ , then*

$$\text{Fitt}_{A_s}(H_\phi) = \mathbb{B}_\phi A_s.$$

*Proof.* Since  $H_\phi$  is free of rank  $u(\alpha)$  (part 3 of Proposition 7.2), we have that  $\text{Fitt}_{A_s}(H_\phi) = F A_s$ , where

$$F = \det_{k[\underline{t}_s]}(Z - \phi_\theta|_{H_\phi})|_{Z=\theta},$$

and  $F$  has degree  $u(\alpha)$  as a polynomial in  $\theta$ . Moreover, by Remark 5.6, there is an isomorphism of  $R_s$ -modules

$$H_\phi \otimes_{k[\underline{t}_s]} k_s \rightarrow V_\phi,$$

which implies

$$F = [V_\phi]_{R_s}.$$

Again by (1) of Proposition 7.2, we have  $U_\phi = \frac{\tilde{\pi}}{\omega_\alpha} A_s$  and

$$[R_s : k_s U_\phi]_{R_s} = [R_s : \frac{\tilde{\pi}}{\omega_\alpha} R_s]_{R_s} = (-1)^{\frac{r-1}{q-1}} \frac{\tilde{\pi}}{\omega_\alpha},$$

because  $(-1)^{\frac{r-1}{q-1}} \frac{\tilde{\pi}}{\omega_\alpha}$  is monic. Therefore by the class number formula (Theorem 5.13):

$$F = (-1)^{\frac{r-1}{q-1}} \mathbb{B}_\phi$$

hence proving our result.  $\square$

We presently do not know much about the irreducible factors of the polynomials  $\mathbb{B}_\phi$ . However, if  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$  (that is, if  $\phi = C_s$ ) with  $s \equiv 1 \pmod{q-1}$ , more can be said.

**Lemma 7.10.** *If  $s \equiv 1 \pmod{q-1}$ , then  $\mathbb{B}_{C_s}$  has no non-trivial divisor in  $A$ .*

*Proof.* In this case, we have  $\omega_\alpha = \omega(t_1) \cdots \omega(t_s)$  and  $L(1, C_s) = L(\chi_{t_1} \cdots \chi_{t_s}, 1)$  in the notations of [5]. We can evaluate at  $t_1 = \cdots = t_s = \zeta \in k$  and, by the fact that  $s \equiv 1 \pmod{q-1}$ ,  $L(1, C_s)|_{t_i=\zeta} = \sum_{a \in A_+} \frac{a(\zeta)}{a}$ . By using Lemma 7.1, we obtain

$$\sum_{a \in A_+} \frac{a(\zeta)}{a} = \frac{\tilde{\pi}}{(\theta - \zeta)\omega(\zeta)}.$$

Therefore,

$$\mathbb{B}_{C_s}|_{t_i=\zeta} = \tilde{\pi}^{-1} L(1, C_s) \omega_\alpha|_{t_i=\zeta} = \omega(\zeta)^{r-1} (\theta - \zeta)^{-1} = (\theta - \zeta)^{\frac{r-q}{q-1}} \in A.$$

Now, if  $a \in A^*$  divides  $\mathbb{B}_{C_s}$  in  $A_s$ , then  $a$  divides  $(\theta - \zeta)^{\frac{r-1}{q-1}}$  for all  $\zeta \in k$  so that  $a \in k^\times$ .  $\square$

**7.1.2. Further properties of the class module in the uniformizable torsion case.** The polynomials  $\mathbb{B}_\phi$  of §7.1.1 are likely to encode deep information about the modules  $H_\phi$  in the uniformizable torsion case. We show here how to deduce, from few known properties of the  $\mathbb{B}_\phi$ , pieces of information of the modules  $H_\phi$  in the case  $\phi = C_s$ .

Lemma 7.10 yields supplementary information in the case of  $\phi = C_s$  (see Definition 3.5):

**Proposition 7.11.** *Let us suppose that  $s \geq 2q - 1$  and  $s \equiv 1 \pmod{q-1}$ . Then, via the structure of  $A$ -module inherited by  $C_s$ ,  $H_{C_s}$  is torsion-free and not finitely generated.*



*Proof.* We already know that  $[V_{C_s}]_{R_s} = \mathbb{B}_{C_s}$ . By Lemma 7.10, for all  $a \in A^*$ ,  $a$  and  $\mathbb{B}_{C_s}$  are relatively prime in  $R_s$ . This means that the  $R_s$ -module  $V_{C_s}$  does not have non-trivial  $a$ -torsion points for  $a \in A^*$ . But there is an isomorphism of  $R_s$ -modules  $H_{C_s} \otimes_{k[\underline{t}_s]} k_s \cong V_{C_s}$  which are finite dimensional  $k_s$ -vector spaces and  $H_{C_s}$  is a free  $k[\underline{t}_s]$ -module. Thus,  $H_{C_s}$  has no non-trivial  $a$ -torsion points for  $a \in A^*$ .

For the second part of the proof of our statement, we recall that by Corollary 5.2,  $H_{C_s}$  is a  $k[\underline{t}_s]$ -module free of rank  $u(\alpha) \geq 1$ . According to [15, Chapter XIX, Proposition 2.5], we have then the inclusions involving the annihilator of  $H_{C_s}$ :

$$(30) \quad \text{Ann}_{A_s}(H_{C_s})^{u(\alpha)} \subset \mathbb{B}_{C_s} A_s \subset \text{Ann}_{A_s}(H_{C_s}).$$

Let us suppose by contradiction that  $H_{C_s}$  is finitely generated as an  $A$ -module. Then, the rank is, say,  $m$ . Let  $g_1, \dots, g_m$  be generators of this  $A$ -module. Then, for all  $i = 1, \dots, m$  the elements  $g_i, t_1 g_i, t_1^2 g_i, \dots, t_1^m g_i$  are  $A$ -linearly dependent. In particular, there exists a non-zero element  $F \in A[t_1]$  such that

$$(C_s)_F(f) = 0,$$

for all  $f \in H_{C_s}$ , that is  $F \in \text{Ann}_{A_s}(H_{C_s})$ , so that  $\mathbb{B}_{C_s}$  divides  $F^{u(\alpha)}$ .

Since  $s \geq 2q - 1$ ,  $\mathbb{B}_{C_s}$ , which is symmetric in  $t_1, \dots, t_s$ , has positive degree in  $\theta$  but cannot be independent of the variables  $t_1, \dots, t_s$  because this would contradict the first part of the statement of the Proposition (the polynomial is not divisible by any non-constant polynomial of  $A$ ); since  $s > 1$ , the only symmetric polynomials of  $A[t_1]$  are in  $A$  and in particular, we find that  $\mathbb{B}_{C_s} \in A_s \setminus A[t_1]$ . But then, this is again contradictory because a polynomial of  $A[t_1]$  cannot be divisible by a polynomial of  $A_s \setminus A[t_1]$  if  $s \geq 2$  ( $A_s$  is factorial).  $\square$

**7.1.3. Further properties in the uniformizable non-torsion case.** We suppose here that  $s \not\equiv 1 \pmod{q-1}$ . We consider again the case  $\phi = C_s$ . Here we suppose that  $s \geq 2q - 1$ , and we consider the  $A$ -torsion submodule  $M$  of  $H_{C_s}$ . This also is an  $A_s$ -module, and we know that it is a finitely generated  $k[\underline{t}_s]$ -module (Corollary 5.2). Moreover:

**Proposition 7.12.** *The  $A$ -torsion  $A_s$ -submodule  $M$  is a torsion  $k[\underline{t}_s]$ -module.*

*Proof.* We must show that  $M \otimes_{k[\underline{t}_s]} k_s = (0)$ . This follows if we show that  $[V_{C_s}]_{R_s}$  has no divisors in  $A$ , where we recall that  $V_{C_s}$  is the vector space  $H_{C_s} \otimes_{k[\underline{t}_s]} k_s$ .

By part (1) of Proposition 5.4, we know that  $k_s U_{C_s}$  is an  $R_s$ -module free of rank one. The class number formula, Theorem 5.13, yields that

$$[V_{C_s}]_{R_s} k_s U_{C_s} = R_s L(1, C_s).$$

Let  $a \in A^*$  be a divisor of  $[V_{C_s}]_{R_s}$ . Then,  $a^{-1}L(1, C_s) \in k_s U_{C_s}$ . By part (2) of Proposition 5.4, we have that  $\exp_{C_s}(a^{-1}L(1, C_s)) \in R_s$ . Since we also have, at once,  $\exp_{C_s}(a^{-1}L(1, C_s)) \in \mathbb{T}_s(K_\infty)$ , we obtain that

$$\exp_{C_s}(a^{-1}L(1, C_s)) \in A_s.$$

We claim that this is impossible unless  $a \in k^\times$ . To see this, we appeal to Proposition 3.11 which says us that  $\exp_{C_s}(a^{-1}L(1, C_s))$  extends to an entire function in  $s$  variables.

It is here that we use the particular shape of the parameter  $\alpha$ . Indeed,  $\alpha$  vanishes at  $t_s = \theta$ . The evaluation at  $t_s = \theta$  is licit in  $\exp_{C_s}(a^{-1}L(1, C_s))$  and yields an

entire function in  $s - 1$  variables  $t_1, \dots, t_s$ . Since

$$(31) \quad \exp_{C_s}(a^{-1}L(1, C_s)) = \sum_{k \geq 0} \sum_{i+j=k} \frac{\alpha \tau(\alpha) \cdots \tau^{i-1}(\alpha)}{a^{q^i} D_i} \sum_{b \in A_{+,j}} \frac{\chi_{t_1}(b) \cdots \chi_{t_s}(b)}{b^{q^j}},$$

evaluating at  $t_s = \theta$  gives,

$$\exp_{C_s}(a^{-1}L(1, C_s))|_{t_s=\theta} = a^{-1}L(0, C_{s-1}) \in a^{-1}k[t_1, \dots, t_{s-1}] \cap R_s$$

where  $C_{s-1}$  is the Drinfeld module of rank one of parameter  $\alpha' = (t_1 - \theta) \cdots (t_{s-1} - \theta)$ . If by contradiction  $a \notin k^\times$ , then  $a^{-1}k[t_1, \dots, t_{s-1}] \cap R_s = (0)$  and

$$\exp_{C_s}(a^{-1}L(1, C_s))|_{t_s=\theta} = L(0, C_{s-1}) = 0.$$

However,  $L(0, C_{s-1}) \neq 0$ . Indeed, by hypothesis,  $s - 1 \not\equiv 0 \pmod{q - 1}$  and it is well known from the theory of the values of the Goss zeta function at negative integers that the value

$$L(0, C_{s-1})|_{t_1=\dots=t_{s-1}=\theta} = \zeta_C(1 - s) \in A$$

is non-zero, which yields a contradiction. Therefore,  $a \in k^\times$ .  $\square$

7.1.4. *Some special values of  $L(1, \phi)$ .* We suppose that  $0 \leq r \leq q - 1$ ,  $s = r$ , and that  $\alpha = \beta(t_1 - \theta) \cdots (t_r - \theta)$  with  $\beta \in k^\times$ . We recall that  $\tilde{\beta} \in k^{ac}$  is such that  $\tilde{\beta}^{q-1} = \beta$ . We thus have the formula (19):

$$L(1, \phi) = \frac{\log_C(\omega_\alpha)}{\omega_\alpha},$$

which holds in  $\mathbb{T}_r$ . We notice that  $\beta^{\frac{q^i-1}{q-1}} = \beta^i$  for all  $i \geq 0$ . We then have

$$(32) \quad L(1, \phi) = \sum_{i \geq 0} \beta^i \ell_i^{-1} b_i(t_1) \cdots b_i(t_r)$$

where the sequence of polynomials  $b_i(t)$  of  $A[t]$  is defined by  $b_0 = 1$  and

$$b_i = \prod_{j=0}^{i-1} (t - \theta^{q^j}).$$

The right-hand side of (32) is an entire function of  $\mathbb{E}_r$  hence providing an analytic extension of  $L(1, \phi)$  to  $\mathbb{C}_\infty^r$ ; this also follows from §4.1.5. We now identify  $L(1, \phi)$  with an entire function and we proceed in deducing some properties for its special values in the same spirit as in [18, Section 1] (we have more freedom in choosing our specializations). For simplicity, we assume from now on that  $s = 1$  and that  $\alpha = \beta(t - \theta)$  with  $\beta \in k^\times$ . We denote by  $\phi_\beta$  the Drinfeld  $A_s$ -module defined by

$$\phi_{\beta, \theta} = \theta + \beta(t - \theta)\tau.$$

Until the end of this sub-section, we will focus on these modules as well as on their  $L$ -series.

(1). We begin by considering the substitution  $t = \theta$ . In this case,  $b_i(\theta) = 0$  for all  $i \geq 1$  so that

$$L(1, \phi_\beta)|_{t=\theta} = 1.$$

(2). More generally, we can replace  $t = \theta^{q^k}$  when  $k \geq 0$ . Since  $b_i(\theta^{q^j}) = 0$  for all  $j < i$ , we get

$$L(1, \phi_\beta)|_{t=\theta^{q^k}} = \sum_{i=0}^k \beta^i S_i(q^k - 1).$$

In particular:

**Corollary 7.13.** *Let us consider the uniformizable Drinfeld  $A[t]$ -module  $\phi_\beta$  with  $\beta \in k^\times$ . For all  $k \geq 0$ ,*

$$L(1, \phi_\beta)|_{t=\theta^{q^k}} \in A.$$

This can be obtained in a different way by using [13, Remark 8.12.1], indeed,

$$L(1, \phi_\beta)|_{t=\theta^{q^k}} = L(1 - q^k, \psi_\beta) = z(\beta^{-1}, 1 - q^k) \in A,$$

where  $\psi_\beta$  is defined by  $\psi_{\beta, \theta} = \theta + \beta\tau$  and  $z$  is the function discussed by Goss in loc. cit.

If  $\beta \neq 1$ , then  $L(1, \phi_\beta)$  cannot only vanish at all the points  $\theta^{q^k}$  ( $k > 0$ ) with multiplicity one. Indeed, otherwise, it would be proportional to  $L(1, \phi_1)$  with proportionality factor equal to one. From the definition, this would imply

$$\exp_C \left( \frac{\tilde{\pi}\tilde{\beta}}{\theta - t} \right) = \tilde{\beta} \exp_C \left( \frac{\tilde{\pi}}{\theta - t} \right),$$

that is,  $\tilde{\beta} \in k$ , which would happen only if  $\beta = 1$ .

(3). Keeping again the same hypotheses, that  $\alpha = \beta(t - \theta)$  in  $A[t]$ , we now replace  $t = \theta$  in  $\tau^k(L(1, \phi_\beta))$ , with  $k > 0$ . Since for all  $i \geq 0$ ,

$$\tau^k(b_i) = \tau^i(b_k)b_k^{-1}b_i,$$

we get, by using that  $b_i = \tau(b_{i-1})(t - \theta)$  and  $b_k = \tau(b_{k-1})(t - \theta)$  (for  $i, k > 0$ ),

$$\begin{aligned} \tau^k(L(1, \phi_\beta)) &= \sum_{i \geq 0} \beta^i \ell_i^{-q^k} \tau^i(b_k)b_k^{-1}b_i \\ &= 1 + \tau(b_{k-1})^{-1} \sum_{i > 0} \beta^i \ell_i^{-q^k} \tau^i(b_k)\tau(b_{i-1}). \end{aligned}$$

If we write  $b_k = \sum_{j=0}^k c_j t^j \in A[t]$  with  $c_j \in A$  for all  $j$  and we write

$$\text{Li}_{\beta, N} = \sum_{i \geq 0} \beta^i \ell_i^{-N} \tau^i = \tilde{\beta}^{-1} \text{Li}_{1, N}(\tilde{\beta}),$$

with  $\text{Li}_{1, N}$  the  $N$ -th order polylogarithm, then,

$$\begin{aligned} \tau^k(L(1, \phi_\beta))|_{t=\theta} &= 1 + \ell_k^{-1} \sum_{j=0}^k \theta^j \sum_{i > 0} \beta^i \ell_i^{1-q^k} c_j^{q^i} \\ &= \ell_k^{-1} \sum_{j=0}^k \theta^j \text{Li}_{\beta, q^k-1}(c_j). \end{aligned}$$

We deduce the formulas

$$L(q^k, \phi_\beta)|_{t=\theta} = \sum_{a \in A^+} \beta^{\deg_\theta(a)} a^{1-q^k} = \ell_k^{-1} \sum_{j=0}^k \theta^j \text{Li}_{\beta, q^k-1}(c_j).$$

## 8. ON THE LOG-ALGEBRAIC THEOREM OF ANDERSON

The class number formula (Theorem 5.13) implies a refined version of Anderson *Log-algebraic Theorem* in the case of the Carlitz module (cf. [3, Theorem 3]; see also loc. cit. Proposition 8). We describe these issues in this Section. We shall use an  $s$ -dimensional “coefficient Tate algebra”  $\mathbb{T}'_s$  whose variables are denoted by  $t'_1, \dots, t'_s$ , not to be confused with the algebra  $\mathbb{T}_{r+1}$ , with variables  $t_1, \dots, t_r, t_{r+1}$  that will be introduced later.

Let  $X_1, \dots, X_r, Z$  be “symbols”, let us consider the polynomial ring in infinitely many indeterminates

$$\mathcal{B}_r = \mathbb{T}'_s[X_1, \dots, X_r, \tau(X_1), \dots, \tau(X_r), \tau^2(X_1), \dots, \tau^2(X_r), \dots],$$

that we endow with the structure of a difference ring  $(\mathcal{B}_r, \tau)$  in the obvious way, by simply setting  $\tau(\tau^m(X)) = \tau^{m+1}(X)$  with  $X = X_1, \dots, X_r$ , the operator  $\tau$  acting as usual over  $\mathbb{C}_\infty$ . We use the ring  $\mathcal{B}_r$  (and the so far unused indeterminate  $Z$ ) to construct yet another ring, non-commutative, denoted by  $\mathcal{A}_r$ . This is the set of finite sums

$$\sum_{i=0}^n c_i \tau^i(Z)$$

with the coefficients  $c_i \in \mathcal{B}_r$ , usual sum and product given by composition: if  $F = \sum_{i \geq 0} f_i \tau^i(Z)$  and  $G = \sum_{j \geq 0} g_j \tau^j(Z)$  are two elements of  $\mathcal{A}_r$ , their product is defined by

$$F \cdot G := \sum_{k \geq 0} \left( \sum_{i+j=k} f_i \tau^i(g_j) \right) \tau^k(Z).$$

The ring  $\mathcal{A}_r$  is itself given with a structure of a  $\tau$ -difference ring  $(\mathcal{A}_r, \tau)$  defined by setting  $\tau(\tau^i(Z)) = \tau^{i+1}(Z)$ . We embed  $(\mathcal{A}_r, \tau)$  in the difference ring  $(\widehat{\mathcal{A}}_r, \tau)$  whose elements are the formal series  $\sum_{i \geq 0} c_i \tau^i(Z)$  with  $c_i \in \mathcal{B}_r$  for all  $i$ ; the product and the difference structure to specialize those of  $\mathcal{A}_r$  and are uniquely determined by this condition.

The left composition with the series

$$\mathbf{exp}_C = \sum_{i \geq 0} D_i^{-1} \tau^i(Z)$$

determines a  $k[t'_1, \dots, t'_s]$ -linear endomorphism of  $\widehat{\mathcal{A}}_r$ . Also, the series

$$\mathcal{L}_r(X_1, \dots, X_r; Z) = \sum_{d \geq 0} \left( \sum_{a \in A_{+,d}} C_a(X_1) \cdots C_a(X_r) a^{-1} \right) \tau^d(Z)$$

defines an element of  $\widehat{\mathcal{A}}_r$ . It is easy to show that, if we now choose  $X_1, \dots, X_r \in \mathbb{T}'_s$  then, with  $Z \in \mathbb{T}'_s$  with Gauss norm small enough depending on the Gauss norms of the  $X_i$ 's, the series  $\mathcal{L}_r(X_1, \dots, X_r; Z)$  converges to a well defined element of  $\mathbb{T}'_s$ . If we choose  $X_i = \omega(t'_i)$  for all  $i$ , then the choice  $Z = 1$  suffices to ensure convergence. Obviously,

$$S_r(X_1, \dots, X_r; Z) := \mathbf{exp}_C(\mathcal{L}_r(X_1, \dots, X_r; Z))$$

is an element of  $\widehat{\mathcal{A}}_r$ . But more is true.

**Theorem 8.1.** *We have that*

$$S_r(X_1, \dots, X_r; Z) \in A[X_1, \dots, X_r, Z, \tau(X_1), \dots, \tau(X_r), \tau(Z), \dots] \cap \mathcal{A}_r.$$

*Proof.* We shall use the non-uniformizable auxiliary Drinfeld module of rank one  $\phi$  whose parameter is

$$\alpha' = t_{r+1}(t_1 - \theta) \cdots (t_r - \theta) \in A[t_1, \dots, t_{r+1}] =: A_{r+1}.$$

We also set  $\alpha = (t_1 - \theta) \cdots (t_r - \theta)$ , the parameter of  $C_r$ . We use the  $A$ -linear action of  $A_{r+1}$  over  $\widehat{\mathcal{A}}_r$  determined by the conditions that, for all  $f \in B$ , the map  $\widehat{\mathcal{A}}_r \rightarrow \widehat{\mathcal{A}}_r$  defined by  $Y \mapsto f.Y$  is a ring homomorphism and, for  $m$  an integer, by the relations  $t_i.\tau^m(X_j) = \tau^m(X_j)$  if  $i \neq j$ ,  $t_i.\tau^m(Z) = \tau^m(Z)$  ( $i \neq r+1$ ),  $t_i.\tau^m(X_i) = \tau^m(t_i.X_i) = \tau^m(C_\theta(X_i))$  and  $t_{r+1}.\tau^m(X_i) = \tau^m(X_i)$ ,  $t_{r+1}.\tau^m(Z) = \tau^{m+1}(Z)$ . One easily sees that

$$\begin{aligned} \alpha' . \tau^m(X_1 \cdots X_r Z) &= \tau^{m+1}(X_1 \cdots X_r Z), \\ \rho_\alpha(a) . \tau^m(X_1 \cdots X_r) &= \tau^m(C_a(X_1) \cdots C_a(X_r)), \quad a \in A. \end{aligned}$$

One can also prove, by induction, that

$$\alpha' \tau(\alpha') \cdots \tau^{i-1}(\alpha') . Y = \tau^i(Y), \quad Y \in \widehat{\mathcal{A}}_r.$$

We now focus on the Drinfeld module  $\phi$ . By Theorem 5.13 and especially by its Corollary 5.14, we have that

$$\mathbf{exp}_\phi(L(1, \phi)) \in A_{r+1}.$$

But, in  $\mathbb{T}_{r+1}$ ,

$$\begin{aligned} \mathbf{exp}_\phi(L(1, \phi)) &= \sum_{i \geq 0} t_{r+1}^i D_i^{-1} \alpha \tau(\alpha) \cdots \tau^{i-1}(\alpha) \sum_{j \geq 0} t_{r+1}^j \sum_{a \in A_{+,j}} \rho_\alpha(a) a^{-q^i} \\ &= \sum_{k \geq 0} t_{r+1}^k \sum_{i+j=k} D_i^{-1} \alpha \tau(\alpha) \cdots \tau^{i-1}(\alpha) \sum_{a \in A_{+,j}} \rho_\alpha(a) a^{-q^i}. \end{aligned}$$

We deduce that for all  $k \geq 0$ ,

$$U_k := \sum_{i+j=k} D_i^{-1} \alpha \tau(\alpha) \cdots \tau^{i-1}(\alpha) \sum_{a \in A_{+,j}} \rho_\alpha(a) a^{-q^i} \in A_r = A[\underline{t}_r]$$

(where  $\underline{t}_r = (t_1, \dots, t_r)$ ) and for all  $k$  big enough,  $U_k = 0$ . To end the proof of the Theorem it suffices now to compute the action of the polynomial  $\mathbf{exp}_\phi(L(1, \phi))$  over the monomial  $X_1 \cdots X_r Z$  in  $\widehat{\mathcal{A}}_r$ . Since by definition

$$t_{r+1}^{\deg(a)} \rho_\alpha(a) . (X_1 \cdots X_r Z) = C_a(X_1) \cdots C_a(X_r) \tau^{\deg(a)}(Z)$$

we get:

$$\begin{aligned}
& \mathbf{exp}_\phi(L(1, \phi)).(X_1 \cdots X_r Z) = \\
&= \sum_{k \geq 0} t_{r+1}^k U_k.(X_1 \cdots X_r Z) \\
&= \sum_{k \geq 0} \tau^k(Z) \sum_{i+j=k} D_i^{-1} \sum_{a \in A_{+,j}} \tau^i(C_a(X_1) \cdots C_a(X_r)) a^{-q^i} \\
&= \mathbf{exp}_C \left( \sum_{k \geq 0} \tau^k(Z) \sum_{a \in A_{+,k}} C_a(X_1) \cdots C_a(X_r) a^{-1} \right) \\
&= S_r(X_1, \dots, X_r; Z),
\end{aligned}$$

( $\mathbf{exp}_C = \sum_{i \geq 0} D_i^{-1} \tau^i(Z)$  is the operator associated to Carlitz's exponential) which makes it clear that  $S_r(X_1, \dots, X_r; Z)$  is in  $A[X_1, \dots, X_r, Z, \dots]$   $\square$

**Remark 8.2.** Even though it only applies to the Carlitz module, Theorem 8.1 has an advantage if compared to Anderson original result [3, Theorem 3], and this, even if we forget the occurrence of several variables  $X_1, \dots, X_r$ . Indeed, these variables can vary in the Tate algebra  $\mathbb{T}'_s$ , while Anderson's result holds if the variable is chosen in  $\mathbb{C}_\infty$  <sup>(9)</sup>. Let us assume, for the sake of simplicity, that  $X_1 = \dots = X_r = X$ . In [3, §4.3], Anderson also provides a table of *special polynomials* of small order for small values of  $q$ . For example, if  $q = 3$  and  $r = 4$ , we have the formula (cf. loc. cit. p. 191):

$$(33) \quad \mathbf{exp}_C \left( \sum_{k \geq 0} Z^{q^k} \sum_{a \in A_{+,k}} \frac{C_a(X)^4}{a} \right) = ZX^4 - Z^3 X^6.$$

This formula has to be understood *with the variables  $X, Z$  varying in  $\mathbb{C}_\infty$  so that  $|Z|$  is small enough to ensure convergence*. If the variables are chosen in  $\mathbb{T}'_s$ , the formula no longer holds. If  $s = 1$ , let us make the choice of  $X = \omega(t')$  and  $Z \in \mathbb{C}_\infty$  (so we are in  $\mathbb{T}' = \mathbb{T}'_1$ ). Then,

$$\sum_{k \geq 0} Z^{q^k} \sum_{a \in A_{+,k}} \frac{C_a(\omega(t'))^4}{a} = \omega(t')^4 \sum_{k \geq 0} Z^{q^k} \sum_{a \in A_{+,k}} \frac{a(t')^4}{a}$$

and the left-hand side of (33) has a meromorphic extension to  $\mathbb{C}_\infty$  with a pole of order four at  $t' = \theta$ , while the right-hand side has a pole of order six so that no identity can hold in this case. It can be proved, with an explicit computation, that, again for  $q = 3$ ,

$$\begin{aligned}
& S_4(X_1, \dots, X_4; Z) = \\
&= ZX_1 \cdots X_4 - \tau(Z)(X_1 X_2 X_3 \tau(X_4) + \\
&\quad X_1 X_2 \tau(X_3) X_4 + X_1 \tau(X_2) X_3 X_4 + \tau(X_1) X_2 X_3 X_4).
\end{aligned}$$

If we choose  $X_1 = \dots = X_4 = X$ , then we get

$$S_4(X, \dots, X; Z) = ZX^4 - \tau(Z)X^3\tau(X),$$

---

<sup>9</sup>Similar properties may hold for the so-called Drinfeld modular forms. We hope to come back to this question in the near future.

so that, if  $X, Z \in \mathbb{C}_\infty$ , we recover the original entry of Anderson table (33). Replacing  $X_i$  by  $\omega(t'_i)$  (in  $\mathbb{T}'_4$ ) and setting  $Z = 1$  (licit), we find the formula

$$\exp_{C_4}(L(1, C_4)) = 1 + \theta - t'_1 - t'_2 - t'_3 - t'_4.$$

Of course, further information about the polynomials  $S(X_1, \dots, X_r; Z)$  can be made explicit in the same spirit of [3, Proposition 8]; we refrain from doing this here.

## 9. EVALUATION AT DIRICHLET CHARACTERS

We shall give here further application of our previous investigations. We shall deal here with the computation of the isotypic components of the class modules introduced by Taelman in [24] when they are associated to the Carlitz module. This section is inspired by the ideas of [6] and [5]; we shall analyze the case of  $\phi = C_s$ , that is, the Drinfeld module of rank one of parameter  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$  (Definition 3.5). We will end up with a refinement of Taelman's Herbrand-Ribet Theorem [25] (our Theorem 9.17). We now present our settings.

**9.1. Some settings.** Let  $P$  be a prime of  $A$  of degree  $d \geq 1$ . We are going to study certain congruences modulo  $P$  for which we need to introduce additional settings. For this, we choose once and for all a  $K$ -embedding

$$\iota_P : K^{ac} \rightarrow \mathbb{C}_P,$$

where  $\mathbb{C}_P$  is the completion of  $\widehat{K_P^{ac}}$ , an algebraic closure of the completion  $\widehat{K_P}$  of  $K$  at the place determined by  $P$ . We further normalize the valuation  $v_P$  at the place  $P$  by setting  $v_P(P) = 1$ . The *Teichmüller character*  $\vartheta_P : \Delta_P \rightarrow (A/PA)^\times$  is determined by the unique Dirichlet character of conductor  $P$  such that

$$v_p(\iota_P(\vartheta_P(\sigma_\theta)) - \theta) = 1.$$

In particular, this means that there exists a unique root  $\zeta$  of  $P$  in the finite field  $A/PA$  with  $q^d$  elements with the property that for all  $\sigma \in \Delta$  there is  $b \in A$  such that we have  $\vartheta_P(\sigma) = b(\zeta)$ . We then write, in this case,  $\sigma = \sigma_b$ . At once, we have a well defined  $A$ -algebra homomorphism

$$A \rightarrow A/PA$$

sending  $b$  to  $b(\vartheta_P(\theta))$ . Then, every character  $\chi \in \widehat{\Delta_P} = \text{Hom}(\Delta_P, (k^{ac})^\times)$  is a power of  $\vartheta_P$ ; we can write

$$(34) \quad \chi = \vartheta_P^i, \quad 0 \leq i \leq q^d - 2.$$

More generally, if  $a \in A_+$  is non-constant and squarefree, we can write  $a = P_1 \cdots P_r$  with  $P_1, \dots, P_r$  distinct primes of respective degrees  $d_1, \dots, d_r$ , and we have that  $\widehat{\Delta_a} \cong \widehat{\Delta_{P_1}} \times \cdots \times \widehat{\Delta_{P_r}}$  so that for every character  $\chi \in \widehat{\Delta_a}$ ,

$$(35) \quad \chi = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$$

where the integers  $N_i$  are such that  $0 \leq N_i \leq q^{d_i} - 2$  for all  $i$ . We call the characters like the above  $\chi$  in (35), *Dirichlet characters of the first kind of conductor  $a$* . We define  $k_a$  as the subfield of  $k^{ac}$  generated by the roots of  $a$ . We notice that  $k_a$  is also the field obtained adjoining to  $k$  the images of the various characters of  $\widehat{\Delta_a}$ , because

$$k_a = k(\zeta_1, \dots, \zeta_r)$$

with  $\zeta_i$  root of  $P_i$ . We will handle the trivial character as a Dirichlet character. By convention, its conductor is 1 and we shall set  $k_a = k$  in this case.

9.1.1. *Gauss-Thakur sums.* We recall that if  $P$  is a prime of  $A$ , the Gauss-Thakur sum associated to a character  $\chi = \vartheta_P^{q^j} \in \widehat{\Delta_P}$  is defined (see Thakur [26, Section 9.8]) by

$$g(\vartheta_P^{q^j}) = \sum_{\delta \in \Delta_P} \vartheta_P(\delta^{-1})^{q^j} \delta(\lambda_P) \in k_P K_P,$$

where we recall that  $\lambda_P = \exp_C\left(\frac{\tilde{\pi}}{P}\right)$ , where  $k_P K_P$  is the compositum of  $k_P$  and  $K_P = K(\lambda_P)$  in  $K^{ac}$  that we identify with  $k_P \otimes_k K_P$  (notice that  $k_P$  and  $K_P$  are linearly disjoint over  $k$ ). In fact, we have more than that;  $g(\vartheta_P^{q^j}) \in k_P[\lambda_P]$ . Let  $\chi$  be a character of  $\widehat{\Delta_P}$  with  $P$  a prime of  $A$ . We define  $g(\chi)$  by using (34) in the following way. We expand  $i = i_0 + i_1 q + \cdots + i_{d-1} q^{d-1}$  in base  $q$  ( $i_0, \dots, i_r \in \{0, \dots, q-1\}$ ), and along with this expansion, we define,

$$g(\chi) = \prod_{j=0}^{d-1} g(\vartheta_P^{q^j})^{i_j}.$$

Furthermore, the factorization (35) of a Dirichlet character of the first kind allows us to associate to it a *Gauss-Thakur sum*. This was done in [5, Section 2.1.1]. Let us set  $\zeta_i = \vartheta_{P_i}(\sigma_\theta)$  ( $i = 1, \dots, r$ ).

Let us expand in base  $q$ :

$$(36) \quad N_i = \sum_{j=0}^{d_i-1} n_{i,j} q^j, \quad i = 1, \dots, r,$$

with  $n_{i,j} \in \{0, \dots, q-1\}$ . For a positive integer  $N$  we denote by  $\ell_q(N)$  the sum of the digits of the expansion in base  $q$  of  $N$  so that  $\ell_q(N_i) = \sum_{j=0}^{d_i-1} n_{i,j}$ . We also set  $s = \sum_i \ell_q(N_i)$ . The integer  $s$  is called the *type of  $\chi$* . We make the convention that the trivial character is the unique Dirichlet character of type zero. According to loc. cit., the Gauss-Thakur sum associated to  $\chi$  is defined as follows:

$$g(\chi) = \prod_{i=1}^r g(\vartheta_{P_i}^{N_i}) \in k_a K_a,$$

where  $k_a K_a$  denotes the compositum  $k_a \otimes_k K_a$  of  $k_a$  and  $K_a$  in  $K^{ac}$ .

9.1.2.  *$k_a[\Delta_a]$ -modules.* We identify  $k_a K$  with  $k_a \otimes_k K = k_a(\theta)$ . Then, the field extension  $k_a K_a / k_a K$  is uniformizable of Galois group  $\Delta_a$ . There also is, on  $k_a K_a = k_a \otimes_k K_a$ , a structure of  $k_a[\Delta_a]$ -module defined by setting, for  $\sigma \in \Delta_a$  and  $x \otimes y \in k_a \otimes_k K_a$ ,

$$\sigma(x \otimes y) = x \otimes \sigma(y).$$

This action is well defined because  $k_a \cap K_a = k$ . Although the notation  $k_a \otimes_k K_a$  is an appropriate one, we will often prefer the more compact notation  $k_a K_a$ ; the reader is warned that  $k_a K_a$  is also enriched with the above  $k_a[\Delta_a]$ -module structure.

We also consider the field  $k_a K_\infty = k_a \otimes_k K_\infty = k_a((\theta^{-1}))$ , the compositum of  $k_a$  and  $K_\infty$  in  $\mathbb{C}_\infty$ . For  $v$  a place of  $K_a$ , let us denote by  $\widehat{K_{a,v}}$  the completion of  $K_a$  at  $v$ . If  $v$  divides  $\infty$  and  $a \neq 1$ , then

$$\widehat{K_{a,v}} \cong K_\infty(\tilde{\pi}).$$



We set  $K_{a,\infty} = K_a \otimes_K K_\infty$ , a  $K_\infty$ -algebra either equal to  $K_\infty$  (case of  $a = 1$ ) either satisfying

$$K_{a,\infty} \cong \prod_{v \in S_\infty(K_a)} K_\infty(\tilde{\pi}),$$

where  $S_\infty(K_a)$  is the set of places of  $K_a$  dividing  $\infty$ . We have an action of  $\Delta_a$  on  $K_{a,\infty}$  as well; if  $\sigma \in \Delta_a$  and  $x \otimes y \in K_a \otimes_K K_\infty$ , then, by definition,

$$\sigma(x \otimes y) := \sigma(x) \otimes y.$$

The operator  $\tau$  acts on  $K_{a,\infty}$  by exponentiation by  $q$  ( $\tau(x \otimes y) = x^q \otimes y^q$ ) and the actions of  $\Delta_a$  and  $\tau$  commute.

With the  $K_\infty$ -algebra  $K_{a,\infty}$ , we now construct certain modules  $\Omega_a$  that will be used in §9.1.3. We set

$$\Omega_a = K_{a,\infty} \otimes_k k_a = (K_a \otimes_K K_\infty) \otimes_k k_a.$$

We endow  $\Omega_a$  with a structure of  $k_a[\Delta_a]$ -module by setting (as above), for  $\sigma \in \Delta_a$  and  $x \otimes y \in \Omega_a = K_{a,\infty} \otimes_k k_a$ ,

$$\sigma(x \otimes y) = \sigma(x) \otimes y.$$

This action commutes with the  $k_a$ -linear extension  $\varphi$  of the operator  $\tau$  on  $\Omega_a$ . Explicitly, if  $x \otimes y \in \Omega_a$ ,  $\varphi(x \otimes y) = \tau(x) \otimes y$ . In particular, if  $x = \sum_i x_i \theta^{-i} \in k_a((\theta^{-1})) = k_a K_\infty$ , then

$$\varphi(x) = \sum_i x_i \theta^{-iq}.$$

**9.1.3. Idempotents.** We denote by  $O_a$  the integral closure of  $A$  in  $K_a$ ; then  $O_a[k_a] = k_a \otimes_k O_a$  is the integral closure of  $A[k_a] = k_a \otimes_k A = k_a[\theta]$  in  $k_a K_a$ . To a character  $\chi \in \widehat{\Delta_a}$  as in (35) we associate an idempotent  $e_\chi \in k_a[\Delta_a]$ , defined as follows:

$$e_\chi = (-1)^r \sum_{\delta \in \Delta_a} \delta^{-1} \chi(\delta) = |\Delta_a|^{-1} \sum_{\delta \in \Delta_a} \delta^{-1} \chi(\delta),$$

where  $r$  is the number of prime factors of  $a$ . The properties of relevance for us are contained in [5, Lemma 11]. In particular, we have that

$$e_\chi(k_a K_a) = (k_a K)g(\chi), \quad \text{and} \quad e_\chi(O_a[k_a]) = A[k_a]g(\chi).$$

This allows to compute the action of  $e_\chi$  on the modules  $\Omega_a$ . We have:

$$(37) \quad \Omega_a = \bigoplus_{\chi \in \widehat{\Delta_a}} e_\chi(\Omega_a), \quad \text{and} \quad e_\chi(\Omega_a) = k_a K_\infty g(\chi), \quad \chi \in \widehat{\Delta_a}.$$

In [5, Theorem 3] it is proved that for  $P$  a prime of  $A$  and  $\zeta$  the root of  $P$  such that  $\vartheta_P(\theta) = \zeta$ ,

$$g(\vartheta_P^{q^j}) = P'(\zeta)^{-q^j} \omega(\zeta^{q^j}), \quad j = 0, \dots, d-1,$$

where  $\omega$  is the function of Anderson and Thakur.

9.1.4. *Evaluation map.* We recall that we have set  $s = \sum_i \ell_q(N_i)$ . Let us associate, to the datum of the integers  $N_1, \dots, N_r$  and their expansions in base  $q$  (36), an  $s$ -tuple of variables

$$\underline{t}_s = \underbrace{(t_{1,0,1}, \dots, t_{1,0,n_{0,0}})}_{n_{0,0}} \underbrace{, \dots, t_{1,d_0-1,1}, \dots, t_{1,d_0-1,n_{0,d_0-1}}}_{n_{0,d_0-1}} \dots \dots \dots \underbrace{, \dots, t_{r,d_r-1,n_{r,d_r-1}}}_{n_{r,d_r-1}}.$$

$\ell_q(N_1)$   $\ell_q(N_r)$

We also define the  $\mathbb{C}_\infty$ -algebra homomorphism “evaluation” at  $\chi$ :

$$\text{ev}_\chi : \mathbb{T}_s \rightarrow \mathbb{C}_\infty$$

by setting  $\text{ev}_\chi(t_{i,j,k}) = \zeta_i^{q^j}$  for all  $i, j, k$ . We point out that the type  $s$  of  $\chi$  does not depend on the embeddings  $\iota_{P_i}$  and is invariant under permutation of the  $P_i$ . Furthermore, if we restrict  $\text{ev}_\chi$  to the subring of  $\mathbb{T}_s$  whose elements are symmetric in  $t_1, \dots, t_s$ , then, in spite of the definition,  $\text{ev}_\chi$  really only depends on  $\chi$ .

With the above parameters we can compute  $g(\chi)$  for a Dirichlet character of the first kind  $\chi$  as in (35). We have:

$$\begin{aligned} g(\chi) &= g(\vartheta_{P_1}^{N_1}) \cdots g(\vartheta_{P_r}^{N_r}) \\ &= P'_1(\zeta_1)^{-N_1} \cdots P'_r(\zeta_r)^{-N_r} \prod_{i=1}^r \prod_{j=0}^{d_i-1} \omega(\zeta_i^{q^j})^{n_{i,j}} \\ (38) \quad &= \text{ev}_\chi \left( \prod_{i=1}^r \prod_{j=0}^{d_i-1} \prod_{k=1}^{n_{i,j}} P'_i(t_{i,j,k})^{-1} \omega(t_{i,j,k}) \right). \end{aligned}$$

9.1.5. *An equivariant isomorphism.* We recall that we have introduced, in §9.1.2, a  $k_a$ -linear endomorphism  $\varphi$  of  $\Omega_a$ . We choose a Dirichlet character

$$\chi = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$$

as in (35) and we expand the integers  $N_i$  in base  $q$  as in (36);

$$N_i = \sum_{j \geq 0} n_{i,j} q^j.$$

By (38) and by the functional equation  $\tau(\omega) = (t - \theta)\omega$  of the function  $\omega$  of Anderson and Thakur, we see that

$$\varphi(g(\chi)) = \left( \prod_{i=1}^r \prod_{j=0}^{d_i-1} (\vartheta_{P_i}(\theta)^{q^j} - \theta)^{n_{i,j}} \right) g(\chi) = \left( \prod_{i=1}^r \prod_{j=0}^{d_i-1} (\zeta_i^{q^j} - \theta)^{n_{i,j}} \right) g(\chi).$$

We now come back to the identity (37) and we consider the isomorphism

$$\nu_\chi : e_\chi(\Omega_a) \rightarrow k_a K_\infty$$

defined by  $\nu_\chi(y) = yg(\chi)^{-1}$ . Then,

$$\nu_\chi(\varphi(x)) = \tilde{\varphi}(\nu_\chi(x)),$$

where

$$\tilde{\varphi}(x) := \left( \prod_{i=1}^r \prod_{j=0}^{d_i-1} (\zeta_i^{q^j} - \theta)^{n_{i,j}} \right) \varphi(x).$$

The Taelman class module associated to the Carlitz module and relative to the extension  $K_a/K$  (see [23] and [6]) is defined by

$$(39) \quad H_a = \frac{C(K_a \otimes_K K_\infty)}{\exp_C(K_a \otimes_K K_\infty) + C(O_a)}.$$

Let  $C^\varphi : k_a \otimes_k A \rightarrow \Omega_a[\varphi]$  be the homomorphism of  $k_a$ -algebras defined by

$$C_\theta^\varphi = \theta + \varphi.$$

Let us additionally set

$$\exp_C^\varphi = \sum_{i \geq 0} D_i^{-1} \varphi^i,$$

which gives rise to a  $k_a$ -linear continuous function  $\Omega_a \rightarrow \Omega_a$  (in fact, it separately induces  $k_a$ -linear entire functions on each component of  $\Omega_a$ ). We then have

$$k_a \otimes_k H_a = \frac{C^\varphi(\Omega_a)}{\exp_C^\varphi(\Omega_a) + C^\varphi(O_a[k_a])},$$

the  $k_a \otimes_k A$ -module structure being induced by the action of  $C^\varphi$ .

Now, we consider, with an analogue meaning of the symbols, the  $k_a$ -linear endomorphisms  $C^{\tilde{\varphi}}$  and  $\exp_C^{\tilde{\varphi}}$  of  $k_a \otimes_k K_\infty$ , and the  $C^{\tilde{\varphi}}$ -module:

$$H_\chi = \frac{C^{\tilde{\varphi}}(k_a K_\infty)}{\exp_C^{\tilde{\varphi}}(k_a K_\infty) + C^{\tilde{\varphi}}(A[k_a])}.$$

The previous discussions imply the next Lemma.

**Lemma 9.1.** *The map  $\nu_\chi$  induces an isomorphism of  $A[k_a]$ -modules*

$$e_\chi(k_a \otimes_k H_a) \cong H_\chi$$

which is equivariant for the two structures of  $A[k_a]$ -modules induced by  $\varphi$  and  $\tilde{\varphi}$ .

9.1.6. *Link between  $H_{C_s}$  and  $H_\chi$ .* Let  $\chi$  be a Dirichlet character of the first kind as in (35). With  $s$  the type of  $\chi$  as above we consider the uniformizable Drinfeld module of rank one  $C_s$ , with parameter

$$(40) \quad \alpha = \prod_{i=1}^r \prod_{j=0}^{d_i-1} (t_{i,j,k} - \theta)^{n_{i,j}}.$$

We now relate the class module  $H_{C_s}$  to the module  $H_\chi$ . Firstly, we notice that  $\text{ev}_\chi(k[\underline{t}_s]) = k_a$  and that we have a field isomorphism  $k[\underline{t}_s]/I_\chi \cong k_a$ , where

$$I_\chi = \text{Ker}(\text{ev}_\chi) \cap k[\underline{t}_s],$$

which yields an isomorphism

$$(41) \quad \frac{A_s}{I_\chi A_s} \cong A[k_a] = k_a[\theta].$$

We shall show the next Proposition.

**Proposition 9.2.** *The evaluation map  $\text{ev}_\chi$  induces an isomorphism of  $A[k_a]$ -modules*

$$\psi_\chi : \frac{H_{C_s}}{I_\chi H_{C_s}} \rightarrow H_\chi.$$

*Proof.* The evaluation map  $\text{ev}_\chi : \mathbb{T}_s(K_\infty) \rightarrow k_a \otimes_k K_\infty$  satisfies:

$$\text{ev}_\chi(\tau_\alpha(f)) = \tilde{\varphi}(\text{ev}_\chi(f)), \quad f \in \mathbb{T}_s(K_\infty).$$

In particular, for all  $b \in A[k_a]$ ,  $\tilde{b} \in A_s$  such that  $b = \text{ev}_\chi(\tilde{b})$  and  $f$  in  $\mathbb{T}_s(K_\infty)$ , we have (with  $\phi = C_s$ ):

$$(42) \quad C_b^{\tilde{\varphi}}(\text{ev}_\chi(f)) = \text{ev}_\chi(\phi_{\tilde{b}}(f))$$

and

$$(43) \quad \exp_C^{\tilde{\varphi}}(\text{ev}_\chi(f)) = \text{ev}_\chi(\exp_\phi(f)).$$

We consider the composition  $w = \text{pr} \circ \text{ev}_\chi$  of two surjective  $k$ -linear maps:

$$\mathbb{T}_s(K_\infty) \rightarrow k_a \otimes_k K_\infty \rightarrow \frac{k_a \otimes_k K_\infty}{\exp_C^{\tilde{\varphi}}(k_a \otimes_k K_\infty) + A[k_a]},$$

where the first map is  $\text{ev}_\chi$  and the second map  $\text{pr}$  is the projection. We deduce from (42) that, with  $b$  and  $\tilde{b}$  such that  $b = \text{ev}_\chi(\tilde{b})$ ,

$$w(\phi_{\tilde{b}}(f)) = C_b^{\tilde{\varphi}}(w(f))$$

for all  $f \in \mathbb{T}_s(K_\infty)$  and  $b \in A_s$ .

Let  $f$  be an element of  $\mathbb{T}_s(K_\infty)$ . We have  $w(f) = 0$  if and only if

$$\text{ev}_\chi(f) \in \exp_C^{\tilde{\varphi}}(k_a K_\infty) + A[k_a].$$

By (43), we have

$$\exp_C^{\tilde{\varphi}}(k_a K_\infty) = \text{ev}_\chi(\exp_{C_s}(\mathbb{T}_s(K_\infty)) + A_s)$$

which means that  $w(f) = 0$  if and only if

$$f \in I_\chi \mathbb{T}_s(K_\infty) + \exp_{C_s}(\mathbb{T}_s(K_\infty)) + A_s,$$

where

$$I_\chi \mathbb{T}_s(K_\infty) = \left\{ \sum_{i \geq m} x_i \theta^{-i}; x_i \in I_\chi, m \in \mathbb{Z} \right\}.$$

Now, we notice the isomorphisms of  $k$ -vector spaces

$$\frac{H_{C_s}}{I_\chi H_{C_s}} = \frac{\mathbb{T}_s(K_\infty)}{I_\chi \mathbb{T}_s(K_\infty) + \exp_{C_s}(\mathbb{T}_s(K_\infty)) + A_s}$$

which shows, with (41), that the map of the Proposition is an isomorphism of  $A[k_a]$ -modules.  $\square$

**Remark 9.3.** We can deduce from Proposition 9.2 the Theorem B of [6] for ‘‘tamely ramified characters’’. This follows from the next Theorem, where  $C_s$  is the Drinfeld module of rank one with parameter  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$ :

**Theorem 9.4.** *Let  $\chi$  be a Dirichlet character of type  $s \geq q$ , with  $s \equiv 1 \pmod{q-1}$  and with conductor  $a$ . Then:*

$$\text{Fitt}_{A[k_a]}(H_\chi) = \text{ev}_\chi(\mathbb{B}_{C_s})A[k_a].$$

*Proof.* By Theorem 7.9, we get:

$$\text{Fitt}_{A[k_a]} \left( \frac{H_{C_s}}{I_\chi H_{C_s}} \right) = \text{ev}_\chi(\mathbb{B}_{C_s})A[k_a].$$

We conclude by applying Proposition 9.2.  $\square$

We inform the reader that Florent Demeslay recently obtained the above result with a different proof. He deduced it from an equivariant class number formula similar to Theorem A of [6]. This will appear in the forthcoming work [11].

## 9.2. On the structure of $H_{C_s}$ and the isotypic components $H_\chi$ .

**Definition 9.5.** Let  $\mathcal{P}$  be a property on the set of all the Dirichlet characters of type  $s$ . We say that  $\mathcal{P}$  holds for *almost all characters of type  $s$*  if there exists a non-zero polynomial  $F \in k[\underline{t}_s]$  such that if  $\chi$  is one of such characters with  $\text{ev}_\chi(F) \neq 0$ , then  $\mathcal{P}$  is true for  $\chi$ .

Let  $R$  be a ring and  $M$  an  $R$ -module. Let  $N$  be an element of  $R$ . We denote by  $M[N]$  the  $N$ -torsion submodule of  $M$ , that is, the  $R$ -module of elements  $m \in M$  such that  $N.m = 0$  (where  $N.m$  denotes the action of  $N$  over an element  $m$  of  $M$ ). In particular, we have the  $A[k_a]$ -modules  $H_\chi$  and, for  $N \in A[k_a]$ , the  $N$ -torsion submodule  $H_\chi[N]$ ; we recall that the structure of  $A[k_a]$ -module is as before that induced by  $C^\varphi$  as in the proof of Proposition 9.2. We have the next proposition.

**Proposition 9.6.** *Let us suppose that  $s \geq 1$  and let  $N$  be in  $A_+$ . Then, for almost all Dirichlet characters of type  $s$ , we have  $H_\chi[N] = (0)$ .*

*Proof.* Thanks to Proposition 5.16, we can suppose that  $s \geq 2q - 1$ . By Proposition 7.11, if  $s \equiv 1 \pmod{q-1}$ ,  $H_{C_s}[N] = (0)$ . If  $s \not\equiv 1 \pmod{q-1}$ , then, by Proposition 7.12,  $H_{C_s}[N] \otimes_{k[\underline{t}_s]} k_s = (0)$ . Hence, in all cases,

$$H_{C_s}[N] \otimes_{k[\underline{t}_s]} k_s = (0).$$

We now consider the exact sequence of  $k[\underline{t}_s]$ -modules of finite type (the middle map is the multiplication by  $N$ ):

$$0 \rightarrow H_{C_s}[N] \rightarrow H_{C_s} \rightarrow H_{C_s} \rightarrow \frac{H_{C_s}}{NH_{C_s}} \rightarrow 0.$$

Taking the tensor product  $(\cdot) \otimes_{k[\underline{t}_s]} k_s$  of the above exact sequence with  $k_s$  we obtain that the multiplication by  $N$  is injective in  $H_{C_s}$  which means that

$$\frac{H_{C_s}}{NH_{C_s}} \otimes_{k[\underline{t}_s]} k_s = (0).$$

In particular, the  $k[\underline{t}_s]$ -module of finite type  $\frac{H_{C_s}}{NH_{C_s}}$  is torsion and there exists  $F_N \in k[\underline{t}_s] \setminus \{0\}$  such that  $F_N H_{C_s} \subset NH_{C_s}$ . This yields the desired conclusion.  $\square$

We now suppose that  $s \geq 1$  is such that  $s \not\equiv 1 \pmod{q-1}$ . The element  $m_s = [V_{C_s}]_{R_s}$  is a monic polynomial in  $R_s = k_s[\theta]$  and  $\text{ev}_\chi(m_s)$  is well defined for almost all Dirichlet characters of type  $s$ .

**Theorem 9.7.** *For almost all characters  $\chi$  of type  $s$ , we have that*

$$\text{Fitt}_{A[k_a]}(H_\chi) = \text{ev}_\chi(m_s)A[k_a],$$

where  $a$  is the conductor of  $\chi$ .

*Proof.* Let us consider the torsion sub- $k[\underline{t}_s]$ -module  $M$  of  $H_{C_s}$ . Then  $M$  is an  $A_s$ -module which is finitely generated as a  $k[\underline{t}_s]$ -module. We notice that for almost all Dirichlet characters  $\chi$  of type  $s$ , we have that  $M \subset I_\chi H_{C_s}$ . The  $k[\underline{t}_s]$ -module

$$\widetilde{H_{C_s}} := \frac{H_{C_s}}{M}$$

is an  $A_s$ -module which is finitely generated as a  $k[\underline{t}_s]$ -module and, for almost all the Dirichlet characters  $\chi$  of type  $s$ , the  $A_s/I_\chi A_s$ -module

$$\frac{\widetilde{H_{C_s}}}{I_\chi \widetilde{H_{C_s}}}$$

is well defined. For almost all Dirichlet characters  $\chi$  of type  $s$ , we have that

$$\text{Fitt}_{\frac{A_s}{I_\chi A_s}} \left( \frac{\widetilde{H_{C_s}}}{I_\chi \widetilde{H_{C_s}}} \right) = \frac{\text{Fitt}_{A_s}(\widetilde{H_{C_s}}) + I_\chi A_s}{I_\chi A_s}.$$

The  $k_s$ -vector space  $\widetilde{H_{C_s}} \otimes_{k[\underline{t}_s]} k_s$  is isomorphic to  $V_{C_s}$  and is also an  $R_s$ -module; its Fitting ideal over  $R_s$  is then generated by  $m_s$ . This implies that

$$k_s \otimes_{k[\underline{t}_s]} \text{Fitt}_{\frac{A_s}{I_\chi A_s}} \left( \frac{\widetilde{H_{C_s}}}{I_\chi \widetilde{H_{C_s}}} \right) = R_s m_s.$$

But by Proposition 9.2, for almost all characters  $\chi$  of type  $s$ , we also have an isomorphism of  $A[k_a]$ -modules  $\frac{\widetilde{H_{C_s}}}{I_\chi \widetilde{H_{C_s}}} \cong H_\chi$ .  $\square$

### 9.3. Pseudo-null and pseudo-cyclic modules.

**Definition 9.8.** Let  $M$  be a finitely generated  $A_s$ -module which also is finitely generated as a  $k[\underline{t}_s]$ -module. We will say that  $M$  is *pseudo-null* if  $M$  is a torsion  $k[\underline{t}_s]$ -module, that is, if its rank is zero as a module over  $k[\underline{t}_s]$ . We will also say that  $M$  is *pseudo-cyclic* if there exists  $m \in M$  such that  $M/A_s m$  is pseudo-null.

In particular, a module as above which is pseudo-null is also pseudo-cyclic. If  $M$  is a finitely generated  $A_s$ -module which also is finitely generated as a  $k[\underline{t}_s]$ -module, then  $M$  is pseudo-null if and only if  $k_s \otimes_{k[\underline{t}_s]} M = (0)$ . On the other hand,  $M$  is pseudo-cyclic if and only if  $k_s \otimes_{k[\underline{t}_s]} M$  is cyclic as an  $R_s$ -module, that is, if there exists  $m \in M$  such that, for all  $f \in F$ , we have that  $f \in R_s m$ . A pseudo-cyclic module  $M$  need not to be cyclic.

Let  $M$  be an  $A_s$ -module as above. A useful way to detect if there is pseudo-cyclicity for  $M$  is the following. One computes the ideals  $\text{Fitt}_{R_s}(k_s \otimes_{k[\underline{t}_s]} M)$  and  $\text{Ann}_{R_s}(k_s \otimes_{k[\underline{t}_s]} M)$ . These ideals of  $R_s$  are one contained in the other (see (30)) and coincide if and only if  $M$  is pseudo-cyclic. Obviously,  $M$  is pseudo-null if and only if  $\text{Fitt}_{R_s}(k_s \otimes_{k[\underline{t}_s]} M) = R_s$ .

In this Section, we investigate the property of pseudo-cyclicity and pseudo-nullity for the modules  $H_{C_s}$  (that is, in the case of the Drinfeld module  $C_s$  of parameter  $\alpha = (t_1 - \theta) \cdots (t_s - \theta)$ ). We are concerned by the following Questions which we leave open.

**Question 9.9.** *Is  $H_{C_s}$  pseudo-cyclic?*

**Question 9.10.** *Assuming that  $s \not\equiv 1 \pmod{q-1}$ , Is  $H_{C_s}$  pseudo-null?*

We give here some conditions equivalent to pseudo-cyclicity and pseudo-nullity for  $H_{C_s}$ . They seem to support the hypothesis that the Questions above have affirmative answer. We begin with the uniformizable torsion case.

9.3.1. *Pseudo-cyclicity in the uniformizable torsion case.* The computation of  $H_{C_s}$  has been completed in the cases  $s = 1, q$ ; we have found that  $H_{C_s} = (0)$ . We can suppose without loss of generality, that  $s \geq 2q - 1$ . We have the following Proposition.

**Proposition 9.11.** *Let  $s \geq 2q - 1$  be an integer such that  $s \equiv 1 \pmod{q - 1}$ . The following conditions are equivalent.*

- (1)  $H_{C_s}$  is pseudo-cyclic.
- (2) For almost all Dirichlet characters  $\chi$  of type  $s$ ,  $H_\chi$  is a cyclic  $k_a[\theta]$ -module, where  $a$  is the conductor of  $\chi$ .
- (3) There exists a Dirichlet characters  $\chi$  of type  $s$  such that  $H_\chi$  is a cyclic  $k_a[\theta]$ -module, where  $a$  is the conductor of  $\chi$ .

*Proof.* The first condition implies the second by means of the equivariant isomorphism of the Proposition 9.2, and Theorem 9.4. The second condition obviously implies the third. It remains to show that the third condition implies the first. We have to show that  $\text{Fitt}_{R_s}(V_{C_s}) = \text{Ann}_{R_s}(V_{C_s})$ , where we recall that  $V_{C_s} = H_{C_s} \otimes_{k[\underline{t}_s]} k_s$ . By Theorem 7.9, the polynomial  $\mathbb{B}_{C_s}$  is the monic generator of  $\text{Fitt}_{A_s}(H_{C_s})$  and  $\text{Fitt}_{R_s}(V_{C_s}) = \mathbb{B}_{C_s} R_s$ . Let  $m$  be the monic generator of  $\text{Ann}_{R_s}(V_{C_s})$ . By (30),  $m$  divides  $\mathbb{B}_{C_s}$ . By Theorem 9.4,

$$\text{Ann}_{A[k_a]}(H_\chi) = \text{ev}_\chi(\mathbb{B}_{C_s})A[k_a].$$

Thus,  $\text{ev}_\chi(\mathbb{B}_{C_s})$  divides  $\text{ev}_\chi(m)$  and therefore  $\mathbb{B}_{C_s} = m$ .  $\square$

9.3.2. *Pseudo-cyclicity in the uniformizable non-torsion case.* In the uniformizable non-torsion case we have the next Proposition. We keep studying here the case of  $C_s$ , the parameter of which is  $(t_1 - \theta) \cdots (t_s - \theta)$ , and we suppose that  $s \not\equiv 1 \pmod{q - 1}$  and  $s \geq 1$ .

**Proposition 9.12.** *Under the above assumptions ( $s \not\equiv 1 \pmod{q - 1}$ ), the following assertions are equivalent.*

- (1)  $H_{C_s}$  is pseudo-cyclic.
- (2)  $\text{Ann}_{R_s} \left( \frac{U_{C_s}}{U_{C_s}^c} \otimes_{k[\underline{t}_s]} k_s \right) = \text{Ann}_{R_s}(V_{C_s})$ .
- (3) For almost all Dirichlet characters  $\chi$  of type  $s$ , the  $A[k_a]$ -module  $H_\chi$  is a cyclic module, where  $a$  is the conductor of  $\chi$ .

*Proof.* By Proposition 5.4,  $k_s U_{C_s}$  is a free  $R_s$ -module of rank one. This implies that  $U_{C_s}/U_{C_s}^c$  is pseudo-cyclic and we already know that it is of finite rank over  $k[\underline{t}_s]$  and torsion-free (Proposition 5.18). By Corollary 5.19,

$$\text{Ann}_{R_s} \left( \frac{U_{C_s}}{U_{C_s}^c} \otimes_{k[\underline{t}_s]} k_s \right) = \text{Fitt}_{R_s} \left( \frac{U_{C_s}}{U_{C_s}^c} \otimes_{k[\underline{t}_s]} k_s \right) = \text{Fitt}_{R_s}(V_{C_s}).$$

Then,  $H_{C_s}$  is pseudo-cyclic if and only if  $\text{Fitt}_{R_s}(V_{C_s}) = \text{Ann}_{R_s}(V_{C_s})$ . This implies the equivalence of the first condition and the second condition. That these conditions are also equivalent to the third condition follows in a way which is very similar to that used in the proof of Proposition 9.11, left to the reader.  $\square$

**Remark 9.13.** We notice that  $H_{C_s}$  is pseudo-null if and only if

$$\text{Fitt}_{R_s}(V_{C_s}) = R_s.$$

But, as we have already seen (proof of Proposition 5.18),  $\frac{U_{C_s}}{U_{C_s}^c}$  is a torsion-free  $k[\underline{t}_s]$ -module. Thus  $H_{C_s}$  is pseudo-null if and only if  $U_{C_s} = U_{C_s}^c$ . Moreover,  $H_{C_s}$  is pseudo-null if and only if  $m_s = 1$  which is equivalent, by Theorem 9.7, to the fact that, for almost all Dirichlet character  $\chi$  of type  $s$  we have  $H_\chi = (0)$ .

**9.4. Evaluation of the polynomials  $\mathbb{B}_{C_s}$  and  $\mathbb{V}_{C_s}$ .** Here we study again the polynomials  $\mathbb{B}_{C_s}$  for  $s \geq q$  with  $s \equiv 1 \pmod{q-1}$ . To simplify our notation, we write  $\mathbb{B}_s$  instead of  $\mathbb{B}_{C_s}$ . We will also consider the polynomials  $\mathbb{V}_{C_s} = \exp_{C_s}(L(1, C_s)) \in A_s$ , which will be denoted by  $\mathbb{V}_s$ , again for the sake of simplicity. We additionally set  $\mathbb{B}_1 = (\theta - t)^{-1} \in \mathbb{T} = \mathbb{T}_1$ .

We refer again to the settings in §9.1. Let  $\chi$  be a character of conductor  $a$  and type  $s = \sum_{j=1}^r \ell_q(N_j)$  as in (35), with associated evaluation map  $\text{ev}_\chi$  defined as in §9.1.4. We have that  $a = P_1 \cdots P_r$  for distinct primes  $P_1, \dots, P_r \in A$ , so that  $\chi = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$  with  $N_i \leq q^{d_i} - 2$ ,  $d_i$  being the degree in  $\theta$  of  $P_i$  for all  $i$ .

We recall that the special value at  $n \geq 1$  of Goss Dirichlet  $L$ -series (see [13], chapter 8) associated to  $\chi$  is defined by

$$L(n, \chi) = \sum'_{b \in A_+} \chi(\sigma_b) b^{-n} \in \mathbb{C}_\infty,$$

where the sum runs over the elements  $b \in A$  which are relatively prime to  $a$ .

In [5] and [18] it has been used that these  $L$ -series values can be obtained from the evaluation of  $L$ -series values  $L(n, C_s)$ . More precisely, if  $s$  is the type of  $\chi$ , for all  $b$  relatively prime with the conductor  $a$ ,

$$\chi(\sigma_b) = \text{ev}_\chi(\rho_\alpha(b))$$

( $\alpha$  being the parameter of  $C_s$ ). In particular, we get  $L(n, \chi) = \text{ev}_\chi(L(n, C_s))$ .

We additionally consider an integer  $N \geq 0$  that we expand in base  $q$  as  $N = \sum_{j=0}^k n_j q^j$  ( $n_0, \dots, n_k \in \{0, \dots, q-1\}$ ) and we set  $s' = s + \ell_q(N)$ . We then have the evaluation map (we recall that  $\mathbb{E}_s$  is the sub-algebra of  $\mathbb{T}_s$  of entire functions §3.2.4).

$$\text{ev}_N : \mathbb{E}_{s'} \rightarrow \mathbb{E}_s$$

defined replacing the family of variables  $(t_1, \dots, t_s, t_{s+1}, \dots, t_{s+\ell_q(N)})$  by

$$(t_1, \dots, t_s, \underbrace{\theta, \dots, \theta}_{n_0}, \underbrace{\theta^q, \dots, \theta^q}_{n_1}, \dots, \underbrace{\theta^{q^k}, \dots, \theta^{q^k}}_{n_k}).$$

If  $N = 0$ , this map is obviously the identity map of  $\mathbb{E}_s$ . We shall work, in this subsection, with the evaluation map  $\text{ev}_{\chi, N} : \mathbb{E}_{s'} \rightarrow \mathbb{C}_\infty$  defined by

$$\text{ev}_{N, \chi} = \text{ev}_\chi \circ \text{ev}_N.$$

In particular:

$$\text{ev}_{N, \chi}(A[t_1, \dots, t_{s+\ell_q(N)}]) = A[k_a].$$

If  $C_{s'}$  is the Drinfeld module of rank one of parameter  $\alpha = (t_1 - \theta) \cdots (t_{s'} - \theta)$ , then this evaluation map allows to obtain the special values of the Dirichlet  $L$ -series of Goss from  $L(1, C_{s'})$ . Indeed, for all  $j > N$ ,

$$\text{ev}_{N, \chi}(L(1, \chi)) = L(q^j - N, \chi).$$



To  $N$  as above, with its expansion  $N = \sum_i N_i q^i$  in base  $q$ , we associate the *Carlitz factorial*  $\Pi(N)$ , defined by

$$\Pi(N) = \prod_{i \geq 0} D_i^{N_i}.$$

We apply the evaluations  $\text{ev}_{N, \chi}$  in two different ways.

9.4.1. *First way to apply*  $\text{ev}_{N, \chi}$ . For a polynomial  $a \in A$ , we denote by  $a'$  its derivative in the indeterminate  $\theta$ . The function

$$(44) \quad \mathbb{B}_{s'} = (-1)^{\frac{s'-1}{q-1}} \tilde{\pi}^{-1} L(\chi_{t_1} \cdots \chi_{t_{s'}}, 1) \omega(t_1) \cdots \omega(t_{s'})$$

is a polynomial of  $A_{s'}$  and  $\text{ev}_N(\mathbb{B}_{s'})$  is a well defined polynomial of  $A_s$ . By [5, Corollary 34], the function

$$L(1, C_{s'}) = \sum_{d \geq 0} \sum_{b \in A_{+, d}} \chi_{t_1}(b) \cdots \chi_{t_{s'}}(b) b^{-1}$$

is in  $\mathbb{E}_{s'}$ , that is, entire in the set of variables  $\underline{t}_{s'}$  (it is denoted by  $L(\chi_{t_1} \cdots \chi_{t_{s'}}, 1)$  in [5]). By [5, Lemma 31], it vanishes at any point of the form

$$(t_1, \dots, t_s, t_{s+1}, \dots, t_{s+j-1}, \theta^l, t_{s+j+1}, \dots, t_{s'}), \quad l \geq 0, \quad t_i \in \mathbb{C}_\infty.$$

We follow now the computations in [5, §4.1.2] and we use that the function  $\omega(t_k)$  has a simple pole at  $\theta^l$  of residue  $-\tilde{\pi}^l D_l^{-1}$  (for all  $l \geq 0$ ). We introduce some additional notation. We rename the variables  $t_{s+1}, \dots, t_{s'}$  in a way which is compatible with the expansion of  $N$  in base  $q$  by writing:

$$(t_{s+1}, \dots, t_{s'}) = (t_{0,0}, \dots, t_{0,n_0}, \dots, t_{k,0}, \dots, t_{k,n_k})$$

and we have to evaluate the right-hand side of (44) at  $t_{i,j} = \theta^{q^i}$  for  $i = 0, \dots, k$ . We have, with  $\Delta$  the differential operator

$$\Delta = \frac{\partial}{\partial t_{0,0}} \cdots \frac{\partial}{\partial t_{0,n_0}} \cdots \frac{\partial}{\partial t_{k,0}} \cdots \frac{\partial}{\partial t_{k,n_k}},$$

$$\begin{aligned} \text{ev}_N \left( \omega(t_1) \cdots \omega(t_{s'}) \sum_{d \geq 0} \sum_{b \in A_{+, d}} b^{-1} b(t_1) \cdots b(t_{s'}) \right) &= \\ &= \omega(t_1) \cdots \omega(t_s) (-\tilde{\pi})^{n_0} \cdots (-\tilde{\pi})^{n_k q^k} D_0^{-n_0} \cdots D_k^{-n_k} \times \\ &\quad \times \left[ \Delta \left( \sum_{d \geq 0} \sum_{b \in A_{+, d}} b^{-1} b(t_1) \cdots b(t_s) b(t_{0,0}) \cdots b(t_{k,n_k}) \right) \right]_{t_{i,j} = \theta^{q^i}} \\ &= (-1)^N \omega(t_1) \cdots \omega(t_s) \tilde{\pi}^N \Pi(N)^{-1} \times \\ &\quad \times \sum_{d \geq 0} \sum_{b \in A_{+, d}} b^{-1} b(t_1) \cdots b(t_s) [b'(t_{0,0}) \cdots b'(t_{k,n_k})]_{t_{i,j} = \theta^{q^i}} \\ &= (-1)^N \omega(t_1) \cdots \omega(t_s) \tilde{\pi}^N \Pi(N)^{-1} \sum_{b \in A_{+, d}} b^{-1} b(t_1) \cdots b(t_s) (b')^N, \end{aligned}$$

which yields the formula

$$\text{ev}_N(\mathbb{B}_{s'}) = (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \tilde{\pi}^{N-1} \Pi(N)^{-1} \omega(t_1) \cdots \omega(t_s) \sum_{d \geq 0} \sum_{b \in A_{+, d}} \rho_\alpha(b) \frac{(b')^N}{b},$$

where we notice that the series in the right-hand side, with  $\alpha$  the parameter of  $C_s$ , is convergent for the Gauss norm of  $\mathbb{T}_s$ .

We also have the formula

$\text{ev}_\chi(\omega(t_1) \cdots \omega(t_s)) = \vartheta_{P_1}(\sigma_{P'_1})^{N_1} \cdots \vartheta_{P_r}(\sigma_{P'_r})^{N_r} g(\chi) = P'_1(\zeta_1)^{N_1} \cdots P'_r(\zeta_r)^{N_r} g(\chi)$   
for a suitable choice of  $\zeta_i$  root of  $P_i$  ( $1 \leq i \leq r$ , see (38) or [5, Theorem 3]). Therefore, composing with  $\text{ev}_\chi$  now gives the identity

$$\begin{aligned} \text{ev}_{N,\chi}(\mathbb{B}_{s'}) &= \\ &= (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \tilde{\pi}^{N-1} \Pi(N)^{-1} \vartheta_{P_1}(\sigma_{P'_1})^{N_1} \cdots \vartheta_{P_r}(\sigma_{P'_r})^{N_r} g(\chi) \times \\ &\quad \sum_{d \geq 0} \sum_{b \in A_{+,d}} \chi(b) \frac{(b')^N}{b}. \end{aligned}$$

9.4.2. *Second way to apply  $\text{ev}_{N,\chi}$ .* Let us consider an integer  $d \geq 1$  such that  $q^d > N$ . The functions  $\tau^d(L(1, C_{s'})) = L(q^d, C_{s'})$  are also entire and we have

$$\tau^d(\mathbb{B}_{s'}) = (-1)^{\frac{s'-1}{q-1}} \tilde{\pi}^{-q^d} L(q^d, C_{s'}) b_d(t_1) \cdots b_d(t_{s'}) \omega(t_1) \cdots \omega(t_{s'}),$$

where we recall that  $b_i = (t - \theta)(t - \theta^q) \cdots (t - \theta^{q^{i-1}})$  for  $i > 0$  and  $b_0 = 1$ . We observe, as in [5, §4.2], that

$$\text{ev}_N(b_d(t_{s+1}) \cdots b_d(t_{s'}) \omega(t_{s+1}) \cdots \omega(t_{s'})) = \Pi(q^d - N)^{-1} \prod_{i=0}^r L_{d-i-1}^{N_i q^i} \tilde{\pi}^{-N}.$$

Again by (38) we have

$$\begin{aligned} \text{ev}_\chi(L(q^d, C_{s'}) b_d(t_1) \omega(t_1) \cdots b_d(t_s) \omega(t_s)) &= \\ &= L(q^d - N, \chi) \vartheta_{P_1}(\sigma_{P'_1})^{N_1} \cdots \vartheta_{P_r}(\sigma_{P'_r})^{N_r} g(\chi). \end{aligned}$$

Hence, we obtain the formula

$$\begin{aligned} \text{ev}_{N,\chi}(\tau^d(\mathbb{B}_{s'})) &= \\ &= (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \tilde{\pi}^{N-q^d} \Pi(q^d - N)^{-1} \vartheta_{P_1}(\sigma_{P'_1})^{N_1} \cdots \vartheta_{P_r}(\sigma_{P'_r})^{N_r} g(\chi) \times \\ &\quad L(q^d - N, \chi) \prod_{i=0}^r L_{d-i-1}^{N_i q^i} \text{ev}_\chi(b_d(t_1) \cdots b_d(t_s)). \end{aligned}$$

We collect the above formulas in the next Proposition, where all the already mentioned settings are used. To simplify our notation we additionally set

$$\rho_{N,\chi,d} := \frac{\text{ev}_{N,\chi}(\tau^d(\mathbb{B}_{s'}))}{\vartheta_{P'_1}^{N_1}(\sigma_{P'_1}) \cdots \vartheta_{P'_r}^{N_r}(\sigma_{P'_r})} \in k_a K.$$

**Proposition 9.14.** *With the above assumptions and assuming in particular that  $s \geq 1$  and that  $s' = s + \ell_q(N) \equiv 1 \pmod{q-1}$ , the following properties hold.*

(1)

$$\rho_{N,\chi,0} = (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \tilde{\pi}^{N-1} g(\chi) \Pi(N)^{-1} \sum'_{d \geq 0} \sum_{b \in A_{+,d}} \chi(b) b'^N b^{-1}.$$

(2) *Let  $d \geq 1$  be an integer such that  $q^d > N$ . Then,*

$$\rho_{N,\chi,d} = (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} g(\chi) L(q^d - N, \chi) \tilde{\pi}^{N-q^d} \text{ev}_\chi(b_d(t_1) \cdots b_d(t_s)) \prod_{i=0}^r L_{d-1-i}^{N_i q^i}.$$

9.4.3. *Examples.* If  $N = 0$  and  $s = s' \geq 1$ ,  $s \equiv 1 \pmod{q-1}$ , that is, we just apply the operator  $\text{ev}_\chi$  to  $\mathbb{B}_{C_s}$ , then we get from the first part of the Proposition 9.14,

$$\rho_{N,\chi,0} = \rho_{0,\chi,0} = (-1)^{\frac{s-1}{q-1}} L(1, \chi) g(\chi) \tilde{\pi}^{-1}.$$

The case  $s = 0$  can be handled as well if  $N \geq 1$ ; in this case  $\chi$  is the trivial character (of type 0 and conductor 1), then we get from the second part of Proposition 9.14 with the conditions  $q^d > N$ ,  $N \geq 1$ ,  $N \equiv 1 \pmod{q-1}$ ,

$$\rho_{N,1,d} = (-1)^{\ell_q(N) + \frac{\ell_q(N)-1}{q-1}} \text{BC}_{q^d-N} \prod_{i=0}^r L_{d-1-i}^{N_i q^i} \Pi(q^d - N)^{-1},$$

where  $\text{BC}_n$  denotes the  $n$ -th Bernoulli-Carlitz fraction (see [13, Chapter 9]).

9.4.4. *Third way to apply  $\text{ev}_{N,\chi}$ .* The content of the present subsection will not be used elsewhere in the paper but we report it here for the sake of completeness. So far, we did not consider the case  $s' \not\equiv 1 \pmod{q-1}$ . In this case, we do not have a polynomial  $\mathbb{B}_s$  to evaluate ( $L(1, C_{s'})$  is a point of infinite order of  $C_{s'}$ ), but we know that  $\mathbb{V}_{s'} := \exp_{C_{s'}}(L(1, C_{s'})) \in A_s$ , so that we can apply  $\text{ev}_{N,\chi}$  to  $\mathbb{V}_{s'}$ . We assume here that  $s, N$  are integers  $\geq 0$  such that  $s + \ell_q(N) \not\equiv 1 \pmod{q-1}$ , and we assume that  $N \geq 1$ . We consider as above a Dirichlet character  $\chi$  of type  $s$  and we consider  $q^j$  the greatest power of  $q$  which divides  $N$ . We recall from [13] that the value of Dirichlet  $L$ -series at a non-positive integer  $-n$

$$L(-n, \chi) = \sum_{d \geq 0} \sum_{b \in A_{+,d}} \chi(\sigma_b) b^n$$

defines an element of  $k_a \otimes_k A$ . We have the next Proposition.

**Proposition 9.15.** *With the above settings and assumptions, we have*

$$\text{ev}_{N,\chi}(\mathbb{V}_{s'}) = \sum_{i=0}^j D_i^{-1} \text{ev}_{N,\chi}(b_i(t_1) \cdots b_i(t_{s'})) L(q^i - N, \chi).$$

*Proof.* This is a simple application of the identity (31).  $\square$

9.4.5. *Example.* In particular, if  $q$  does not divide  $N$ , then

$$\text{ev}_{N,\chi}(\mathbb{V}_{s'}) = L(1 - N, \chi).$$

**9.5. A refinement of Taelman Herbrand-Ribet Theorem.** An in the previous Section, let  $\chi$  be a Dirichlet character of type  $s \geq 0$  and conductor  $a$ . Following [5, §3.3], we introduce the *generalized Bernoulli-Carlitz fractions*  $\text{BC}_{n,\chi^{-1}}$  by means of the following generating series:

$$\frac{g(\chi)}{a} \sum_{b \in (A/aA)^\times} \frac{\chi(b)X}{\exp_C(\frac{X}{a}) - \sigma_b(\lambda_a)} = \sum_{i \geq 0} \frac{\text{BC}_{i,\chi^{-1}}}{\Pi(i)} X^i.$$

If  $s = 0$ , when  $a = 1$ , we set  $\lambda_a = 0$  in the above formula so that we get in this case

$$\text{BC}_{i,\chi^{-1}} = \text{BC}_i$$

for  $i \geq 0$ . From [5, Proposition 38], we deduce easily the following:

**Lemma 9.16.** *The following properties hold:*

- (1) For all  $i \geq 0$ , we have  $\text{BC}_{i,\chi^{-1}} \in k_a K$ .
- (2) If  $i \not\equiv s \pmod{q-1}$ , then  $\text{BC}_{i,\chi^{-1}} = 0$ .

- (3) We have  $\text{BC}_{0, \chi^{-1}} = 0$  if  $s \geq 1$ .  
(4) If  $i \geq 1$  is such that  $i \equiv s \pmod{q-1}$ , then

$$L(i, \chi)g(\chi) = \tilde{\pi}^i \text{BC}_{i, \chi^{-1}} \Pi(i)^{-1}.$$

We now consider integers  $s, s', N$  with  $s + \ell_q(N) = s'$ ,  $s \geq 1$ ,  $s' \equiv 1 \pmod{q-1}$  and a character  $\chi$  of type  $s'$  and conductor  $a = Pb$  such that

$$\chi = \vartheta_P^N \tilde{\chi},$$

with  $P$  a prime not dividing the conductor  $b$  of  $\tilde{\chi}$ , and such that  $N \leq q^d - 2$ ,  $d$  being the degree of  $P$ . We also consider the completion  $\widehat{K}_P$  of  $K$  at the prime  $P$ , with valuation  $v_P$  normalized by  $v_P(P) = 1$ . The valuation ring of the complete field  $k_a \otimes_{k_P} \widehat{K}_P$  (embedded in  $\mathbb{C}_P$  as explained in §9.1) is the ring  $\widehat{A}_P[k_a] = k_a \otimes_{k_P} \widehat{A}_P$ , with  $\widehat{A}_P$  the valuation ring of  $\widehat{K}_P$ .

In the next result, which can be considered as a refinement of an analogue of the Herbrand-Ribet Theorem by Taelman [25], we show that the fractions  $\text{BC}_{q^d - N, \tilde{\chi}^{-1}}$  are  $P$ -integral and that the triviality of the component  $e_\chi(H_a \otimes_{A[k_a]} \widehat{A}_P[k_a])$  precisely amounts to the condition that  $\text{BC}_{q^d - N, \tilde{\chi}^{-1}}$  is a unit in  $A[k_a]$ . We highlight that the congruences for the above generalized Bernoulli-Carlitz numbers are well defined thanks to the choice of the embedding of  $K^{ac}$  in  $\mathbb{C}_P$  that we made at the beginning of §9.1.

9.5.1. *An example.* To illustrate the above principle, we consider the simplest non-trivial case of a character  $\chi$  of type  $s' = 1$ . Here, the factor  $\tilde{\chi}$  can be assumed to be the trivial character ( $s = 0$  and  $N = q^i$ ) so that  $\chi = \vartheta_P^{q^i}$  with  $i \geq 0$ . The case  $i = 0$  is somewhat exceptional so that we assume that  $i > 0$ , and we have of course that  $i < d$ . By §5.1.3, we have  $H_{C_1} = (0)$  which implies, by Proposition 9.2, the triviality of  $H_\chi$ . By Lemma 9.1,  $e_\chi(k_a \otimes_k H_a) = 0$  and at once,  $e_\chi(H_a \otimes_{A[k_a]} \widehat{A}_P[k_a]) = (0)$ . Now we observe, by the second part of the Proposition 9.14 or §9.4.3, that

$$\frac{\text{BC}_{q^d - q^i} L_{d-1-i}^{q^i}}{\Pi(q^d - q^i)} = -\frac{1}{\theta^{q^d} - \theta^{q^i}}.$$

In particular, the Bernoulli-Carlitz fraction  $\text{BC}_{q^d - q^i}$  is in this case  $P$ -integral and reduces to a unit modulo  $P$ , in agreement with the above principles. In the next Theorem, we show that this phenomenon holds in wider generality.

**Theorem 9.17** (Refinement of Herbrand-Ribet-Taelman Theorem). *Let  $\chi$  be a Dirichlet character with conductor  $a$  and type  $s'$ . Let  $P$  be a prime dividing  $a$ , of degree  $d$ , and let us write  $\chi = \vartheta_P^N \tilde{\chi}$  with  $1 \leq N \leq q^d - 2$  and with  $\tilde{\chi}$  a Dirichlet character of conductor prime to  $P$ . We further suppose that if  $s' = 1$ , then  $N$  is at least 2. The generalized Bernoulli-Carlitz fraction  $\text{BC}_{q^d - N, \tilde{\chi}^{-1}}$  is  $P$ -integral. Furthermore,*

$$e_\chi(H_a \otimes_A \widehat{A}_P[k_a]) \neq (0)$$

*if and only if*

$$\text{BC}_{q^d - N, \tilde{\chi}^{-1}} \equiv 0 \pmod{P}.$$

*Proof.* We have already considered the case  $s' = 1$  in §9.5.1 so we may now suppose that  $s' \geq 2$ . We work in  $k_a \otimes_{k_P} \widehat{K}_P$ . We have the Dirichlet character  $\tilde{\chi}$  which is of

type  $s \geq 0$ . Just as in [5, §4.2], we observe the congruence  $\tau^d(\mathbb{B}_s) \equiv \mathbb{B}_s \pmod{P}$ . Since obviously,  $\text{ev}_{N, \tilde{\chi}}(\mathbb{B}_{s'}) \equiv \text{ev}_{\chi}(\mathbb{B}_{s'}) \pmod{P}$ , we have that

$$e_{N, \tilde{\chi}}(\tau^d(\mathbb{B}_s)) \equiv \text{ev}_{\chi}(\mathbb{B}_s) \pmod{P}.$$

Let us write now  $\tilde{\chi} = \vartheta_{P_1}^{N_1} \cdots \vartheta_{P_r}^{N_r}$ , where  $b = P_1 \cdots P_r$  is the conductor of  $\tilde{\chi}$  (we recall that  $N = \sum_{i=0}^{d-1} N_i q^i$ ,  $N_i \in \{0, \dots, q-1\}$ ). By the second part of the Proposition 9.14, we have:

$$\rho_{N, \tilde{\chi}, d} = (-1)^{\ell_q(N) + \frac{s'-1}{q-1}} \Pi(q^d - N)^{-1} \text{BC}_{q^d - N, \tilde{\chi}^{-1}} \text{ev}_{\tilde{\chi}}(b_d(t_1) \cdots b_d(t_s)) \prod_{i=0}^{d-1} L_{d-1-i}^{N_i q^i}.$$

This implies that  $\text{BC}_{q^d - N, \tilde{\chi}^{-1}}$  is  $P$ -integral. Moreover,  $\text{BC}_{q^d - N, \tilde{\chi}^{-1}} \equiv 0 \pmod{P}$  if and only if  $\text{ev}_{\chi}(\mathbb{B}_s) \equiv 0 \pmod{P}$ . We now set

$$[\chi] = \{\chi^{q^i}, i \geq 0\}$$

and we consider the element in  $A[k_a][\Delta_a]$ :

$$F = \sum_{\psi \in [\chi]} \text{ev}_{\psi}(\mathbb{B}_s) e_{\psi}.$$

In fact, by construction, we have that  $F \in A[\Delta_a]$ . where we recall that  $e_{\psi} = \frac{1}{|\Delta_a|} \sum_{\sigma \in \Delta_a} \psi^{-1}(\sigma) \sigma$  lies in  $k_a[\Delta_a]$ . We also set:

$$e_{[\chi]} = \sum_{\psi \in [\chi]} e_{\psi} \in k_a[\Delta_a],$$

an element of  $k[\Delta_a]$ . We deduce from Theorem 9.4 that

$$\text{Fitt}_{e_{[\chi]} A[\Delta_a]} e_{[\chi]}(H_a) = F e_{[\chi]} A[\Delta_a]$$

<sup>(10)</sup>. This implies that

$$\text{Fitt}_{e_{[\chi]} \widehat{A_P}[k_a][\Delta_a]} e_{[\chi]}(H_a \otimes_A \widehat{A_P}[k_a]) = F e_{[\chi]} \widehat{A_P}[k_a][\Delta_a].$$

Therefore,

$$\text{Fitt}_{\widehat{A_P}[k_a]} e_{\chi}(H_a \otimes_A \widehat{A_P}[k_a]) = \text{ev}_{\chi}(\mathbb{B}_s) \widehat{A_P}[k_a].$$

The Theorem follows at once.  $\square$

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<sup>10</sup>We notice that for  $\psi \in [\chi]$ ,  $e_{\psi}(H_a)$  is not necessarily well defined because  $H_a$  is a  $k[\Delta_a]$ -module while  $e_{\psi} \in k_a[\Delta_a]$ .

## 10. APPENDIX BY FLORENT DEMESLAY

We shall work with Drinfeld  $R_s$ -modules rather than with  $A_s$ -modules. The benefit of this assumption comes from the fact that  $R_s$  is euclidean. We keep using the same notation adopted in the previous sections of the present paper.

Let us choose  $\alpha \in R_s^*$  and let us consider the Drinfeld  $R_s$ -module of rank one and parameter  $\alpha$ , that is, the injective homomorphism of  $k_s$ -algebras

$$\phi : R_s \rightarrow \text{End}_{k_s\text{-lin.}}(K_{s,\infty})$$

given by  $\phi_\theta = \theta + \tau_\alpha$ . Let  $M$  be an  $R_s$ -algebra together with a  $k_s$ -endomorphism  $\tau_M : M \rightarrow M$  such that

$$\tau_M(fm) = \tau(f)\tau_M(m), \quad f \in R_s, \quad m \in M.$$

We denote by  $\phi(M)$  the  $k_s$ -vector space  $M$  equipped with the structure of  $R_s$ -module induced by  $\phi$ . If we denote by  $\phi_\theta(m)$  the action of  $\theta \in R_s$  over  $m \in M$  (the notation  $\theta.m$  will sometimes be used)

$$\phi_\theta(m) = \alpha\tau_M(m) + \theta m$$

and this completely determines the structure of  $R_s$ -module  $\phi(M)$ . We recall that we have the exponential function associated to  $\phi$ , which is a  $k_s$ -linear endomorphism of  $K_{s,\infty}$  defined by

$$\exp_\phi = \sum_{i \geq 0} \frac{\tau_\alpha^i}{D_i}.$$

It also is a morphism of  $R_s$ -modules. We recall that if  $u(\alpha)$  is the maximum of the integer part of  $\frac{r-q}{q-1}$  and 0, with  $r = -v_\infty(\alpha)$ , then  $\exp_\phi$  induces an isometric  $k_s$ -linear automorphism  $\mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1} \rightarrow \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}$ . Notice that  $\mathfrak{m}_{K_{s,\infty}} = \theta^{-1}k_s[[\theta^{-1}]]$ .

**Definition 10.1.** A sub- $k_s$ -vector space  $M$  of  $K_{s,\infty}$  is said to be *discrete* if the intersection  $M \cap \mathfrak{m}_{K_{s,\infty}}$  is a finite dimensional  $k_s$ -vector space.

**Lemma 10.2.** *Let  $M \neq (0)$  be a sub- $R_s$ -module of  $K_{s,\infty}$ . The following assertions are equivalent:*

- (1)  $M$  is discrete,
- (2)  $M$  is a free  $R_s$ -module of rank one.

*Proof.* The fact that the property (2) implies the property (1) is clear. Let us prove that the property (1) implies the property (2). Let  $f$  be a non-zero element of  $M$ . Then,  $R_s \subset f^{-1}M$  and  $f^{-1}M$  is discrete in  $K_{s,\infty}$ . Thus, we can assume that  $R_s \subset M$ . We now observe that we have a direct sum of  $k_s$ -vector spaces:

$$K_{s,\infty} = R_s \oplus \mathfrak{m}_{K_{s,\infty}} = R_s \oplus \theta^{-1}k_s[[\theta^{-1}]].$$

Since  $M$  is discrete we deduce from the above decomposition that  $M/R_s$  is a finite  $k_s$ -vector space. But  $M/R_s$  is also a  $R_s$ -module, hence a torsion  $R_s$ -module. Therefore there exists  $r \in R_s \setminus (0)$  such that  $rM \subset R_s$ . Since  $R_s$  is a noetherian ring, we obtain that  $M$  is a finitely generated  $R_s$ -module of rank 1. Since  $R_s$  is a principal ideal domain we deduce that  $M$ , as an  $R_s$ -module, is free of rank one.  $\square$

**Remark 10.3.** We recall that we have also considered the  $R_s$ -module:

$$V_\phi = \frac{\phi(K_{s,\infty})}{\phi(R_s) + \exp_\phi(K_{s,\infty})}.$$

By Corollary 5.7,  $V_\phi$  is a  $k_s$ -vector space of dimension  $\leq u(\alpha)$ . We notice that  $R_s + \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1} \subset R_s + \exp_\phi(K_{s,\infty})$ . We observe that  $R_s$  and  $\text{Ker}(\exp_\phi)$  are discrete sub- $R_s$ -modules of  $K_{s,\infty}$  which implies that  $\exp_\phi^{-1}(R_s)$  is a discrete sub- $R_s$ -module of  $K_{s,\infty}$ . The exponential  $\exp_\phi$  then produces an exact sequence of  $k_s$ -vector spaces

$$(0) \rightarrow \frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s) + \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}} \rightarrow \frac{K_{s,\infty}}{R_s + \mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+1}} \rightarrow V_\phi \rightarrow (0).$$

In particular,  $\exp_\phi^{-1}(R_s) \neq (0)$  and we obtain that  $\exp_\phi^{-1}(R_s)$  is free of rank one by using Lemma 10.2.

**10.1.  $L$ -series values at one.** We recall from §5.2 the definition of the product  $\mathcal{L}(\phi/R_s)$  for a given Drinfeld  $R_s$ -module of rank one defined over  $R_s$ . Let  $P$  be a prime of  $A$ . If  $P$  does not divide  $\alpha$  or  $P \neq \theta$ , then the  $R_s$ -module  $\phi(\frac{R_s}{PR_s})$  is finitely generated and torsion. If  $P = \theta$  divides  $\alpha$ , we shall set

$$\left[ \phi \left( \frac{R_s}{PR_s} \right) \right]_{R_s} = \theta.$$

Then, the product over the primes of  $A$

$$\mathcal{L}(\phi/R_s) = \prod_P \frac{[\frac{R_s}{PR_s}]_{R_s}}{[\phi(\frac{R_s}{PR_s})]_{R_s}}$$

converges in  $K_{s,\infty}$  to the  $L$ -series value  $L(1, \phi)$  (Proposition 5.11). We will only give a sketch of proof of the next Theorem, as the proof is very close to ideas developed by Taelman in [24].

**Theorem 10.4** (class number formula for  $\mathcal{L}(\phi/R_s)$ ). *The following identity holds in  $K_{s,\infty}$ :*

$$\mathcal{L}(\phi/R_s) = [V_\phi]_{R_s} [R_s : \exp_\phi^{-1}(R_s)]_{R_s}.$$

**10.2. Nuclear operators and determinants.** This Section is inspired by [24, Section 2]. Let  $(V, \|\cdot\|)$  be a  $k_s$ -vector space equipped with a non-archimedean absolute value which is trivial on  $k_s$  and such that every open  $k_s$ -subspace  $U \subset V$  is of finite  $k_s$ -co-dimension. Let's give a typical example of such objects: let  $M$  be a non-trivial, discrete  $R_s$ -submodule of  $K_{s,\infty}$ , then  $V = \frac{K_{s,\infty}}{M}$  satisfies the above hypothesis.

Let  $f$  be a continuous endomorphism of  $V$ , we say that  $f$  is *locally contracting* if there exists an open subspace  $U \subset V$  and a real number  $0 < c < 1$  such that, for all  $v \in U$ ,

$$\|f(v)\| \leq c\|v\|.$$

Any such open subspace  $U \subset V$  which moreover satisfies  $f(U) \subset U$  is called a *nucleus* for  $f$ . Observe that any locally contracting continuous endomorphism of  $V$  has a nucleus. Let's give an example: the map

$$\tau_\alpha : \frac{K_{s,\infty}}{R_s} \rightarrow \frac{K_{s,\infty}}{R_s}$$

is locally contracting and the image of  $\mathfrak{m}_{K_{s,\infty}}^{u(\alpha)+3}$  in  $\frac{K_{s,\infty}}{R_s}$  is a nucleus.

Observe that any finite collection of locally contracting endomorphisms of  $V$  has a common nucleus (see for example [24], Proposition 6). Furthermore if  $f$  and  $g$  are locally contracting, then so are the sum  $f + g$  and the composition  $fg$ .

For any integer  $N \geq 0$ , we set:

$$\frac{V[[Z]]}{Z^N} = V \otimes_{k_s} \frac{k_s[[Z]]}{Z^N},$$

and we denote by  $V[[Z]]$  the inverse limit of the  $V[[Z]]/Z^N$  equipped with the limit topology. Observe that any continuous  $k_s[[Z]]$ -endomorphism

$$F : V[[Z]] \rightarrow V[[Z]]$$

is of the form:

$$F = \sum_{n \geq 0} f_n Z^n,$$

where  $f_n$  is a continuous  $k_s$ -endomorphism of  $V$ . Similarly, any continuous  $\frac{k_s[[Z]]}{Z^N}$ -linear endomorphism of  $\frac{V[[Z]]}{Z^N}$  is of the form  $\sum_{n=0}^{N-1} f_n Z^n$ . We say that a continuous  $k_s[[Z]]$ -linear endomorphism  $F$  of  $V[[Z]]$  (resp. of  $\frac{V[[Z]]}{Z^N}$ ) is *nuclear* if for all  $n$  (resp. for all  $n < N$ ), the  $k_s$ -endomorphism  $f_n$  of  $V$  is locally contracting.

Let  $F$  be a nuclear endomorphism of  $\frac{V[[Z]]}{Z^N}$ . Let  $U_1$  and  $U_2$  be common nuclei for the  $f_n$ ,  $n < N$ . Since Proposition 8 in [24] is valid in our context,

$$\det_{\frac{k_s[[Z]]}{Z^N}} \left( 1 + F|_{\frac{V}{U_i}[[Z]]/Z^N} \right) \in \frac{k_s[[Z]]}{Z^N}$$

is independent of  $i \in \{1, 2\}$ . We denote this determinant by

$$\det_{\frac{k_s[[Z]]}{Z^N}} (1 + F|_V).$$

If  $F$  is a nuclear endomorphism of  $V[[Z]]$ , then we denote by  $\det_{k_s[[Z]]} (1 + F|_V)$  the unique power series that reduces to  $\det_{\frac{k_s[[Z]]}{Z^N}} (1 + F|_V)$  modulo  $Z^N$  for any  $N$ .

Note that Proposition 9, Proposition 10, Theorem 2 and Corollary 1 of [24] are also valid in our context.

**10.3. Taelman's trace formula.** Observe that any element in  $R_s[\tau]\tau$  induces a  $k_s$ -continuous endomorphism of  $\frac{K_{s,\infty}}{R_s}$  which is locally contracting. Denote by  $R_s[\tau][[Z]]$  the ring of formal power series in  $Z$  with coefficients in  $R_s[\tau]$ , the variable  $Z$  being central (i.e.  $\tau Z = Z\tau$ ).

Let  $P_1, \dots, P_n$  be  $n$  distinct primes of  $A$ . Let us set

$$R = R_s \left[ \frac{1}{P_1}, \dots, \frac{1}{P_r} \right].$$

Let  $P$  be a monic prime of  $A$ . Let  $K_{s,P}$  be the  $P$ -adic completion of  $k_s(\theta)$  (with respect to the  $P$ -adic valuation on  $k_s(\theta)$  which is trivial on  $k_s$  and the usual one on  $K$ ). Observe that every element of  $K_{s,P}$  can be written in a unique way:

$$\sum_{i \geq m} x_i P^i,$$

where  $m \in \mathbb{Z}$ ,  $x_i \in R_s$  of degree in  $\theta$  strictly less than  $\deg_\theta P$ .

We also define:

$$K_{s,S} = K_{s,\infty} \times K_{s,P_1} \times \dots \times K_{s,P_r}.$$

Observe that  $R$  is discrete in  $K_{s,S}$ . Let  $P$  be a prime of  $A$ ,  $P \neq P_1, \dots, P_r$ . Let  $R_{s,P}$  be the valuation ring of  $K_{s,P}$ . Then:

$$K_{s,P} = R_{s,P} \oplus R[1/P].$$



Furthermore, the inclusion  $R_s \subset R$  induces an isomorphism:

$$\frac{R_s}{PR_s} \simeq \frac{R}{PR}.$$

Let  $F \in R[\tau][[Z]]\tau Z$ . Then  $F$  defines  $k_s$ -endomorphisms of  $\frac{K_{s,\infty}}{R}[[Z]]$ ,  $\frac{R}{PR}[[Z]]$ ,  $\frac{K_{s,\infty} \times K_{s,P}}{R[1/P]}[[Z]]$ . Now Taelman's localization Lemma ([24] Lemma 1) remains valid in our case:

**Lemma 10.5.** *Let us choose  $F \in R[\tau][[Z]]\tau Z$ . Then:*

$$\det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{R}{PR}} \right) = \frac{\det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{K_{s,S} \times K_{s,P}}{R[1/P]}} \right)}{\det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{K_{s,S}}{R}} \right)}.$$

We also have in our case:

**Theorem 10.6.** *Let  $F \in R_s[\tau][[Z]]\tau Z$ . Then we have an equality in  $k_s[[Z]]$  :*

$$\prod_{P \text{ monic prime of } A} \det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{R_s}{PR_s}} \right) = \det_{k_s[[Z]]} \left( 1 + F \Big|_{\frac{K_{s,\infty}}{R_s}} \right)^{-1}.$$

*Proof.* This is a consequence of Lemma 10.5 and the proof of Theorem 3 in [24]. Note that in our case in [24] page 383 line  $-5$  we replace the original assumption of Taelman by the assumption that  $R$  has no maximal ideal of the form  $PR$ ,  $P$  monic prime of  $A$ , such that  $\dim_{k_s} \left( \frac{R}{PR} \right) < D$ .  $\square$

**10.4. Fitting ideals.** Let  $f : K_{s,\infty} \rightarrow K_{s,\infty}$  be a continuous  $k_s$ -linear map. Let  $M_1$  and  $M_2$  be two free  $R_s$ -modules of rank one in  $K_{s,\infty}$  such that  $f(M_1) \subset M_2$ . Then  $f$  induces a  $k_s$ -continuous linear map

$$f : \frac{K_{s,\infty}}{M_1} \rightarrow \frac{K_{s,\infty}}{M_2}.$$

We say that  $f$  is *infinitely tangent to the identity* on  $K_{s,\infty}$  if for any  $N \geq 0$  there exists an open  $k_s$ -subspace  $U_N \subset K_{s,\infty}$  such that the following properties hold:

- (1)  $U_N \cap M_1 = U_N \cap M_2 = \{0\}$ ,
- (2)  $f$  restricts to an isometry between the images of  $U_N$ ,
- (3)  $\forall u \in U_N$ ,  $v_\infty(f(u) - u) \geq N + v_\infty(u)$ .

If  $f \in K_{s,\infty}[[\tau]]$  is such that this power series is convergent on  $K_{s,\infty}$  and satisfies that  $f(M_1) \subset M_2$ , for some free  $R_s$ -modules of rank one  $M_1$  and  $M_2$ , then, by the proof of Proposition 12 in [24],  $f$  is infinitely tangent to the identity on  $K_{s,\infty}$ . A typical example is given by:  $M_1 = \exp_\phi^{-1}(R_s)$ ,  $M_2 = R_s$  and  $f = \exp_\phi$ .

Now let  $M_1, M_2$  be two free  $R_s$ -modules of rank one in  $K_{s,\infty}$ . Let  $H_1, H_2$  be two finite dimensional  $k_s$ -vector spaces that are also  $R_s$ -modules. Set:

$$L_i = \frac{K_{s,\infty}}{M_i} \times H_i.$$

Let  $f : L_1 \rightarrow L_2$  be a  $k_s$ -linear map which is bijective and continuous. We shall write:

$$\Delta_f = \frac{1 - f^{-1}\theta f Z}{1 - \theta Z} - 1 = \sum_{n \geq 1} (\theta - f^{-1}\theta f) \theta^{n-1} Z^n.$$

We observe that  $\Delta_f$  defines a  $k_s$ -endomorphism of  $L_1[[Z]]$ . Let's assume that  $f$  induces (see [24] page 385 line -6) a continuous  $k_s$ -linear map

$$\frac{K_{s,\infty}}{M_1} \rightarrow \frac{K_{s,\infty}}{M_2}$$

which is infinitely tangent to the identity on  $K_{s,\infty}$ . Then, by the proof of Theorem 4 in [24], we get that  $\Delta_f$  is nuclear, and

$$\det_{k_s[[Z]]}(1 + \Delta_f|_{L_1})_{Z=\theta^{-1}} = [M_1 : M_2]_{R_s} \frac{[H_2]_{R_s}}{[H_1]_{R_s}}.$$

**10.5. Proof of Theorem 10.4.** We set, as in [24] paragraph 5:

$$F = \frac{1 - \phi_\theta Z}{1 - \theta Z} - 1 = \sum_{n \geq 1} (\theta - \phi_\theta) \theta^{n-1} Z^n \in R_s[\tau][[Z]]\tau Z.$$

By Lemma 5.10, we get:

$$\mathcal{L}(\phi/R_s) = \prod_{P \text{ monic prime in } A} \left( \det_{k_s[[Z]]}(1 + F|_{\frac{R_s}{PR_s}}) \right)_{Z=\theta^{-1}}^{-1}.$$

Now by Theorem 10.6, we have:

$$\mathcal{L}(\phi/R_s) = \det_{k_s[[Z]]} \left( 1 + F|_{\frac{K_{s,\infty}}{R_s}} \right) |_{Z=\theta^{-1}}.$$

We consider the short exact sequence of  $R_s$ -modules induced by  $\exp_\phi$ :

$$0 \rightarrow \frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s)} \rightarrow \frac{\phi(K_{s,\infty})}{\phi(R_s)} \rightarrow V_\phi \rightarrow 0.$$

Since  $R_s$  is a principal ideal domain and the  $R_s$ -module  $\frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s)}$  is divisible, this sequence splits. The choice of a section gives an  $R_s$ -isomorphism:

$$\frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s)} \times V_\phi \simeq \frac{\phi(K_{s,\infty})}{\phi(R_s)}.$$

This isomorphism gives rise to an isomorphism of  $k_s$ -vector space:

$$\frac{K_{s,\infty}}{\exp_\phi^{-1}(R_s)} \times V_\phi \simeq \frac{K_{s,\infty}}{R_s}.$$

We denote this map by  $f$ . Then, by the proof of Lemma 6 in [24],  $f$  is infinitely tangent to the identity on  $K_{s,\infty}$ . But observe that on  $\frac{K_{s,\infty}}{R_s}[[Z]]$ :

$$1 + F = \frac{1 - f\theta f^{-1}Z}{1 - \theta Z}.$$

Thus:

$$\det_{k_s[[Z]]} \left( 1 + F|_{\frac{K_{s,\infty}}{R_s}} \right) |_{Z=\theta^{-1}} = [V_\phi]_{R_s} [R : \exp_\phi^{-1}(R_s)]_{R_s}.$$

The proof of our Theorem follows.

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