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On Tail Index Estimation based on Multivariate Data

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Abstract: This article is devoted to the study of tail index estimation based on i.i.d. multivariate observations, drawn from a standard heavy-tailed distribution, *i.e.* of which 1-d Pareto-like marginals share the same tail index. A multivariate Central Limit Theorem for a random vector, whose components correspond to (possibly dependent) Hill estimators of the common shape index $1/\alpha$, is established under mild conditions. Motivated by the statistical analysis of extremal spatial data in particular, we introduce the concept of (standard) heavy-tailed random field of tail index α and show how this limit result can be used in order to build an estimator of α with small asymptotic mean squared error, through a proper convex linear combination of the coordinates. Beyond asymptotic results, simulation experiments illustrating the relevance of the approach promoted are also presented.

Primary 60K35, 60K35; secondary 60K35.

Keywords and phrases: Heavy-tailed random field, tail index estimation, Hill estimator, aggregation.

1. Introduction

It is the main purpose of this paper to provide a sound theoretical framework for risk assessment, when dangerous events coincides with the occurrence of extremal values of a (continuous) random field with an intrinsic heavy-tail behavior. Following in the footsteps of Basrak and Segers [2], where the heavy-tail property for multivariate random vectors is extended to time series, we generalize it to random fields in the present article. Originally, the modeling of extreme events relied on (univariate) extreme value theory (EVT) (see Beirlant et al. [6]). The seminal work of de Haan [17] generalized this approach to the functional setup, introducing the concept of max-stable processes, see also Smith [44]. A variety of extensions have been recently proposed, for instance one may refer to Schlatter [43] for spatial phenomena or to Davis et al. [15] for space-time models. Generalized EVT is motivated by a variety of applications, including in particular most empirical studies of extreme climatic events, as in Davis et al. [16]. Recall finally that Buishand et al. [8] introduced the generalized Pareto process, considered as the functional extension of the generalized Pareto distribution, classically used to model the probability of exceedance over a high threshold.

However, these models are essentially relevant to describe the limit distribution of extremal observations. In contrast, the theory of regularly varying functions provides a non asymptotic semi-parametric framework, with the ability to give an appropriate description of heavy-tail phenomena. In risk assessment, this conservative approach avoids underestimating the probability of occurrence of extreme events and is the main mathematical tool to carry out worst-case risk analyses in various fields. It is widely used for risk quantification in Finance (Rachev et al. [39]), Insurance (Mikosch [37]) or for the modeling of natural hazards, see Tawn [45] or Coles and Walshaw [9]. Hult and Lindskog [34] introduces the regularly varying processes of $D([0, 1], \mathbb{R}^d)$, the space of right-continuous functions from $[0, 1]$ in \mathbb{R}^d with left-limit. Yet, a general theory for spatial processes with intrinsic marginal regularly varying behavior has not been developed so far, to the best of our knowledge. Following de Haan [17] and Buishand et al. [8], the present article firstly aims at extending the concept of (multivariate) regular variation to the spatial setup and defining rigorously the notion of standard *heavy-tailed random field* with tail index $\alpha > 0$. The parameter α governing the extremal behavior of 1-d marginals of the random field, we consider the problem of estimating the tail index α of a standard heavy-tailed random field observed on a grid of d vertices. Whereas a variety of statistical techniques for tail index estimation have been proposed in the univariate setup (see Chap. 6 in Embrechts et al. [25] for instance), focus is here on extension of the popular Hill inference method, see Hill [31]. Incidentally, we point out that the analysis carried out in this paper can be extended to alternative estimation procedures.

In the univariate i.i.d. case, several authors investigated the asymptotic normality of the Hill estimator under various assumptions, including Davis and Resnick [14], Beirlant and Teugels [3] or Teugels and H  ausler [46]. In a primary work, Hsing [33] showed a central limit theorem in a weak dependent setting under suitable mixing and stationary conditions. Recently, these conditions have been considerably weakened in Hill [32]. Here, the framework we consider is quite different. The data analyzed correspond to i.i.d observations of a random field on a compact set $\mathcal{S} \subset \mathbb{R}^d$ with $d \geq 1$ and where each margin is stationary with the same tail index. Precisely, the random field is observed on a lattice s_1, \dots, s_l : to each vertex of the lattice corresponds a sequence of $n \geq 1$ i.i.d. random variables with tail index α , the collection of sequences being not independent in general. Denoting by $H_{k_i, n}^{(i)}$ the Hill estimator using the k_i largest observations at location s_i , $1 \leq i \leq d$, the accuracy of the estimator $H_{k_i, n}^{(i)}$ is known to depend dramatically on k_i . There are several ways to choose this parameter, based on the *Hill horror plot* (k_i is picked in a region where the plot looks flat or on resampling procedures for instance, see Danielsson et al. [13]). Eventually, the *optimal* k_i 's are likely to be different, depending highly on the location s_i . Here, we consider the issue of accurately estimating the parameter α based on the collection of estimators $H_{k_1, n}^{(1)}, \dots, H_{k_l, n}^{(l)}$ and investigate the advantage of suitably chosen convex linear combination of the local estimates over a simple uniform average. The study is based on a limit theorem established in this paper, claiming that $\sqrt{k_1} \left(H_{k_1, n}^{(1)} - 1/\alpha, \dots, H_{k_l, n}^{(l)} - 1/\alpha \right)$ is asymptotically

Gaussian under mild assumptions, together with the computation of an estimate of the asymptotic covariance matrix. These results can be used to derive the limit distribution of any linear combination of the local estimators and, as a byproduct, to find an optimal convex linear combination regarding the asymptotic mean squared error (AMSE). For illustration purpose, experimental results are also presented in this article, supporting the use of the approach promoted, for risk assessment in the shipping industry in particular.

The paper is organized as follows. In section 2, the concept of heavy-tailed random fields, extending very naturally the notion of heavy-tailed random vectors, is introduced. In section 3, the main results of this paper are stated, establishing in particular the asymptotic normality of the multivariate statistic H_n whose components coincide with local Hill estimators, and explaining how to derive a tail index estimator with minimum AMSE. The simulation results are provided in section 4. Numerical results are displayed in section 5, while technical proofs are postponed to the Appendix section.

2. Background and Preliminaries

We start off with some background theory on regular variation and the measure of extremal dependence. Next, we rigorously define heavy-tailed random fields and very briefly recall the classical Hill approach to tail index estimation in the context of i.i.d. univariate data drawn from a heavy-tailed distribution. The indicator function of any event \mathcal{E} by $\mathbb{1}(\mathcal{E})$. For all $\mathbf{x} = (x_1, \dots, x_l) \in \mathbb{R}^l$, the cartesian product $\prod_{i=1}^l (x_i, +\infty]$ is denoted by $(\mathbf{x}, +\infty]$. In addition, all operations in what follows are meant component-wise, *e.g.* $1/\mathbf{k} = (1/k_1, \dots, 1/k_l)$ for any $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{N}^{*l}$. In addition, $\text{Cov}[X, Y]$ will denote the variance of two square integrable random variables X and Y and $\text{Cov}[X, X] = \mathbb{V}[X]$.

2.1. Heavy-tailed random fields

By definition, heavy-tail phenomena are those which are ruled by very large values, occurring with a far from negligible probability and with significant impact on the system under study. When the phenomenon of interest is described by the distribution of a univariate r.v., the theory of regularly varying functions provides the appropriate mathematical framework for heavy-tail analysis. For clarity's sake and in order to introduce some notations which shall be widely used in the sequel, we recall some related theoretical background before extending notions pertaining to standard EVT to the spatial context. One may refer to Resnick [41] for an excellent account of the theory of regularly varying functions and its application to heavy-tail analysis.

Let $\alpha > 0$, we set

$$\mathcal{RV}_{-\alpha} = \{U : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ borelian} \mid \lim_{t \rightarrow \infty} \frac{U(tx)}{U(t)} = x^{-\alpha}\}$$

the space of regularly varying functions (at infinity) with (tail-)index α . Let X be a random variable with cumulative distribution function (*cdf* in short) F and survival function $\bar{F} = 1 - F$. It is said to belong to the set \mathcal{H}_α of random variables with a heavy (right) tail of index α if $\bar{F} \in \mathcal{RV}_{-\alpha}$. In addition, the heavy-tail property can be classically formulated in terms of vague convergence to a homogeneous Radon measure. Indeed, the random variable X belongs to $\mathcal{RV}_{-\alpha}$ if and only if:

$$n\mathbb{P}\left(X/F^{-1}(1 - 1/n) \in \cdot\right) \xrightarrow[n \rightarrow \infty]{w} \nu_\alpha(\cdot) \text{ in } M_+(0, \infty],$$

where $\xrightarrow[n \rightarrow \infty]{w}$ denotes weak convergence (the reader may refer to Resnick [42, chap 3.], Hult and Lindskog [34] and Hult and Lindskog [35]) for further details), $F^{-1}(u) = \inf\{t : F(t) \geq u\}$ denotes F 's generalized inverse, $\nu_\alpha(x, \infty] = x^{-\alpha}$, $M_+(0, \infty]$ the set of nonnegative Radon measures on $(0, \infty]$.

Based on this characterization, the heavy-tail model can be straightforwardly extended to the multivariate setup. Now, consider a d -dimensional random vector $\mathbf{X} = (X_1, \dots, X_d)$ taking its values in \mathbb{R}_+^d and where each margin has the same tail index α . It is said to be a *standard* heavy tailed random vector with tail index $\alpha > 0$, we then shall write $\mathbf{X} \in \mathcal{H}_\alpha$ for simplicity, if there exists a non null positive Radon measure ν on $(0, \infty]^d$ such that:

$$x\mathbb{P}(\mathbf{X}/F_1^{-1}(1 - 1/x) \in \cdot) \xrightarrow[x \rightarrow \infty]{w} \nu(\cdot), \quad (2.1)$$

where F_1 is the cdf of the first component. In such a case, ν fulfills the homogeneity property $\nu(tC) = t^{-\alpha}\nu(C)$ for all $t > 0$ and any Borel set C of $(0, \infty]^d$, and all components are tail equivalent: $X_i \in \mathcal{RV}_{-\alpha}$ for $i = 1, \dots, d$.

Regularly varying random fields. We now give the definition of a regularly varying random field. It is characterized by its finite dimensional marginal distributions.

Definition 2.1. Let $(X_s)_{s \in \mathbb{R}^d}$, be a random field taking its value in \mathbb{R}_+^* and for any $s \in \mathbb{R}^d$, $X_s \in \mathcal{RV}_{-\alpha}$. We say that the field X is regularly varying if: $\forall k \in \mathbb{N}$, $\forall \mathbf{t} = (t_1, \dots, t_k) \in \mathbb{R}^{dk}$, $Y_{\mathbf{t}} := (X_{t_1}, \dots, X_{t_k})$ is regularly varying in the sens of multivariate regular variations.

By virtue of the Kolmogorov existence theorem (see Billingsley [7] chap 7. theorem 36.2), this defines a valid continuous random process on \mathbb{R}^d . This definition of a regularly varying random field is the natural continuous extension of the notion of multivariate regular variations.

2.2. The Hill method for tail index estimation

A variety of estimators of α have been proposed in the statistical literature in the context of univariate i.i.d. observations drawn from a heavy-tailed distribution, see Hill [31], Pickands [38], de Haan and Resnick [19], Mason [36], Davis and

Resnick [14], Cs  rg   et al. [11], Dekkers et al. [22]. In this paper, focus is on the popular Hill estimator but ideas to extend this work to other estimators are introduced in section 8. The Hill estimator is defined as follows. Let X_1, \dots, X_n be observations drawn from a heavy-tailed probability distribution with tail index α and denote $X(1) \geq \dots \geq X(n)$ the corresponding order statistics. Let k such that $k \rightarrow \infty$ and $k/n \rightarrow 0$; the Hill estimator of the shape parameter $1/\alpha$ based on the k -largest observations is given by

$$H_{k,n} = \frac{1}{k} \sum_{i=1}^k \log \left(\frac{X(i)}{X(k+1)} \right) = \int_1^\infty \frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{X_i}{X(k+1)} > x \right) \frac{dx}{x}.$$

The asymptotic behavior of this estimator has been extensively investigated. Weak consistency is shown by Mason [36] for i.i.d. sequences, by Hsing [33] for weakly dependent sequences, and by Resnick and Starica [40] for linear processes. Strong consistency is proved in Deheuvels et al. [21] in the i.i.d. setup under the additional assumption that $k/\log \log n \rightarrow \infty$. The asymptotic normality with a deterministic centering by $1/\alpha$ requires additional assumptions on the distribution F of X . Under the von Mises Condition (recalled later in the present paper) the asymptotic normality of the Hill estimator has been established in Teugels and H  ausler [46], de Haan and Resnick [20], Geluk et al. [28], Hill [32]. In this case,

$$\sqrt{k} (H_{k,n} - 1/\alpha^2) \Rightarrow \mathcal{N}(0, 1/\alpha^2),$$

where \Rightarrow means convergence in distribution, and k is a function of β . However, depending on the choice of k and on the property of F regarding second order regular variation, the Hill estimator can be significantly biased. This is studied for instance in de Haan and Peng [18].

Hence, the practical issue of choosing k is particularly important and has been addressed in various papers. They mostly rely on the second order regular variations and seek to achieve the best trade-off between bias and variance. Drees and Kaufmann [23] derive a sequential estimator of the optimal k that does not require prior knowledge of the second order parameters. In Danielsson et al. [13] a subsample bootstrap procedure is proposed, where the sample fraction that minimizes the asymptotic mean-squared error is adaptively determined. Graphical procedures are also available. In Drees et al. [24] the popular Hill horror plot is compared to the AltHill Plot that is proved to be more accurate if F is not strictly Pareto.

3. Tail index estimation for a heavy-tailed field

Consider a heavy-tailed random field $X = (X_s)_{s \in \mathcal{S}}$, where $\mathcal{S} \in \mathbb{R}^d$, $d \geq 1$. Although the field is not assumed to be stationary in the subsequent analysis, we suppose that all the 1-d marginal distributions of X have the same (unknown) tail index α . Suppose also that the field is observed on a lattice $\mathbf{s} = (s_1, \dots, s_l) \in$

\mathcal{S}^l , $l \geq 2$, and that, at each vertex $s_k \in \mathcal{S}$, $n \geq 1$ observations $(X_{s_k,1}, \dots, X_{s_k,n})$ of X_{s_k} are available.

In order to state the main results of the paper, we introduce some additional notations. For simplicity, we abusively set $X_{s_k} := X_k$ and $X_{s_k,i} := X_{k,i}$ for $1 \leq k \leq l$ and $1 \leq i \leq n$. Denote respectively by F and \bar{F} the cdf and the survival function of the r.v. $\mathbf{X} = (X_1, \dots, X_l)$ and by F_k and \bar{F}_k those of X_k , $k = 1, \dots, l$. Here and throughout, $X_i(1) > \dots > X_i(n)$ are the order statistics related to the sample $(X_{i,1}, \dots, X_{i,n})$ and $H_{k,n}^{(i)}$ is the Hill estimator based on the k -largest values observed at location i . The quantile of order $(n - k + 1)/n$ of F_i is denoted by $a^{(i)}(n/k)$ and we set $\mathbf{a}(n/k) = (a^{(1)}(n/k), \dots, a^{(l)}(n/k))$. Finally, recall that there exists a Radon measure $\nu_{\mathbf{s}}$ such that the following convergence holds true (see Chapter 6 in Resnick [42] for instance):

$$\frac{n}{k} \mathbb{P} \left(\frac{X_1}{a^{(1)}(n/k)} > x_1, \dots, \frac{X_l}{a^{(l)}(n/k)} > x_l \right) \xrightarrow{w} \nu_{\mathbf{s}}(\mathbf{x}), \quad (3.1)$$

where $(x_1, \dots, x_l) \in \mathbb{R}_+^l$. We also set $\nu_{i,j}(x_i, x_j)$ as the limit of $\nu_{\mathbf{s}}(x_1, \dots, x_l)$ when all the components but the i -th and the j -th tend to 0.

We point out that all the results of this section can be extended to alternative estimators of the tail index, such as those studied in Danielsson et al. [13], Dekkers et al. [22] or Danielsson et al. [12]. Technical details are deferred to Appendix section 8.

It is the goal pursued in this section to show how to combine, in a linear and convex fashion, the local Hill estimators in order to refine the estimation of α in the AMSE sense.

3.1. A multivariate functional central limit theorem

As a first go, we start with establishing a theorem stating the convergence of the tail empirical process toward a Gaussian process. This result is next used to prove a CLT with a random centering for the random vector whose components correspond to the local Hill estimators, the latter being all viewed as functionals of the tail empirical process. Under some additional assumptions, the random centering is removed and replaced by α . The case where the number of observations involved in the local Hill estimator components depends on the location considered is also dealt with. The main application of this latter result is that Hill estimator can be replaced by an alternative estimator when the mean squared error of this latter is smaller.

Theorem 3.1. (A FUNCTIONAL CENTRAL LIMIT THEOREM) *Equipped with the notations previously introduced, the following weak convergence (in the space of continuous functions from \mathbb{R}_+^d to \mathbb{R}) holds : as $n, k \rightarrow +\infty$ such that $k = o(n)$, we have*

$$\sqrt{k} \left(\frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{\mathbf{X}_i}{\mathbf{a}(n/k)} > \mathbf{x} \right) - \frac{n}{k} \bar{F}(\mathbf{a}(n/k)\mathbf{x}) \right) \xrightarrow[n \rightarrow \infty]{w} W(\mathbf{x}), \quad (3.2)$$

where $\mathbf{x} = (x_1, \dots, x_l)$, $W(\mathbf{x})$ is a centered Gaussian random field with covariance given by:

$$\forall(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^l \times \mathbb{R}^l, \quad \mathbb{E}[W(\mathbf{x})W(\mathbf{y})] = \nu(\max(\mathbf{x}, \mathbf{y})).$$

In order to generalize the result stated above to the situation where the sample fraction k/n of observations involved in the local Hill estimator possibly varies with the location, we exploit a property of the inverse of a regularly varying function: if $1 - F$ is regularly varying with index $-\alpha$, then $F^{-1}(1 - 1/.)$ is regularly varying with index $1/\alpha$, see Beirlant et al. [6].

Corollary 3.2. (A FUNCTIONAL CENTRAL LIMIT THEOREM (BIS)) *Let $\mathbf{k} = (k_1, \dots, k_l) \in \mathbb{N}^{*l}$ with $k_i = k_i(n) \rightarrow \infty$ and $k_i/n \rightarrow 0$ as $n \rightarrow +\infty$. Suppose that, for all $i \in \{1, \dots, l\}$, $c_i = \lim_{n \rightarrow \infty} k_1/k_i$ is well-defined and belongs to $]0, +\infty[$. Set $\mathbf{a}(n/\mathbf{k}) = (a^{(1)}(n/k_1), \dots, a^{(l)}(n/k_l))$ and $\mathbf{x}' = (x_1, c_2^{1/\alpha} x_2, \dots, c_l^{1/\alpha} x_l)$. We have*

$$\sqrt{k_1} \left(\frac{1}{k_1} \sum_{i=1}^n \mathbb{1} \left(\frac{\mathbf{Z}_i}{\mathbf{a}(n/\mathbf{k})} > \mathbf{x} \right) - \frac{n}{k_1} \bar{F} \left(\mathbf{a} \left(\frac{n}{\mathbf{k}} \right) \mathbf{x} \right) \right) \xrightarrow[n \rightarrow \infty]{w} W(\mathbf{x}), \quad (3.3)$$

where $W(\mathbf{x})$ is a centered Gaussian field with same covariance operator as that involved in Theorem 3.1.

Refer to the Appendix section for the technical proof. Since the local Hill estimators are functionals of the tail empirical process, a Central Limit Theorem for the random vector formed by concatenating the latter can be immediately derived from Theorem 3.1.

Theorem 3.3. *For any $l \geq 1$ and $\mathbf{s} = (s_1, \dots, s_l) \in \mathcal{S}^l$, we have, as $n, k \rightarrow +\infty$ so that $k = o(n)$:*

$$\sqrt{k} \left(H_{k,n}^{(1)} - \int_{X_1(k)}^{\infty} \frac{n}{k} \bar{F}_1(x) \frac{dx}{x}, \dots, H_{k,n}^{(l)} - \int_{X_l(k)}^{\infty} \frac{n}{k} \bar{F}_l(x) \frac{dx}{x} \right) \Rightarrow \mathcal{N}(0, \Sigma), \quad (3.4)$$

$$\text{where } \Sigma_{i,j} = \int_1^{\infty} \int_1^{\infty} \nu_{i,j}(x, y) x^{-1} y^{-1} dx dy.$$

The following corollary relaxes the assumption that all local Hill estimators involve the same number of observations.

Corollary 3.4. *Equipped with the assumptions and notations of Corollary 3.2, for any $l \geq 1$ and any $\mathbf{s} = (s_1, \dots, s_l) \in \mathcal{S}^l$, we have*

$$\sqrt{k_1} \left(H_{k_1,n}^{(1)} - \int_{X_1(k_1)}^{\infty} \frac{n}{k_1} \bar{F}_1(x) \frac{dx}{x}, \dots, H_{k_l,n}^{(l)} - \int_{X_l(k_l)}^{\infty} \frac{n}{k_l} \bar{F}_l(x) \frac{dx}{x} \right) \Rightarrow \mathcal{N}(0, \Sigma'), \quad (3.5)$$

$$\text{with } \Sigma'_{i,j} = \int_{c_j^{1/\alpha}}^{\infty} \int_{c_i^{1/\alpha}}^{\infty} \nu_{i,j}(x, y) x^{-1} y^{-1} dx dy, \text{ for } 1 \leq i, j \leq l.$$

We now address the issue of removing the random centering. From a practical perspective indeed, in order to recover a pivotal statistic and build (asymptotic) confidence intervals the random centering should be replaced by $1/\alpha$. The key point is that $\int_{X_i(k_i)}^{\infty} \frac{n}{k_i} \bar{F}_i(x) \frac{dx}{x}$ can be substituted for $\int_{a^{(i)}}^{\infty} \frac{n}{k_i} \bar{F}_i(x) \frac{dx}{x}$, along with (the second order) Condition (3.6) for the latter term. This condition is used when trying to establish a Central Limit Theorem in the univariate setup (see Resnick [42]):

$$\forall i \in \{1, \dots, l\}, \lim_{n \rightarrow \infty} \sqrt{k} \int_1^{\infty} \frac{n}{k} \bar{F}_i \left(a^{(i)} \left(\frac{n}{k} \right) x \right) - x^{-\alpha} \frac{dx}{x} = 0. \quad (3.6)$$

This immediately implies that the random vector

$$\begin{aligned} \sqrt{k} \left(H_{k,n}^{(1)} - \frac{1}{\alpha}, \dots, H_{k,n}^{(l)} - \frac{1}{\alpha} \right) + \\ \sqrt{k} \left(\int_{a^{(1)}(n/k)}^{X_1(k)} \frac{n}{k} \bar{F}_1(x) \frac{dx}{x}, \dots, \int_{a^{(l)}(n/k)}^{X_l(k)} \frac{n}{k} \bar{F}_l(x) \frac{dx}{x} \right) \end{aligned} \quad (3.7)$$

converges in distribution to $\mathcal{N}(0, \Sigma')$. As shown by expression (3.7), recentering by $(1/\alpha, \dots, 1/\alpha)$ requires to incorporate a term due to the possible correlation between the random centering and the local Hill estimators into the asymptotic covariance matrix. Indeed, from Eq. (3.7), we straightforwardly get that

$$\sqrt{k} \left(H_{k,n}^{(1)} - \frac{1}{\alpha}, \dots, H_{k,n}^{(l)} - \frac{1}{\alpha} \right) \Rightarrow \mathcal{N}(0, \Omega), \quad (3.8)$$

as n and $k = o(n)$ both tend to infinity, provided that, for $1 \leq i \neq j \leq l$, the expectation of the quantity

$$\begin{aligned} k \int_{a^{(i)}(n/k)}^{X_i(k)} \frac{n}{k} \bar{F}_i(x) \frac{dx}{x} \int_{a^{(j)}(n/k)}^{X_j(k)} \frac{n}{k} \bar{F}_j(x) \frac{dx}{x} \\ + k \left(H_{k,n}^{(j)} - \frac{1}{\alpha} \right) \int_{a^{(i)}(n/k)}^{X_i(k)} \frac{n}{k} \bar{F}_i(x) \frac{dx}{x} + k \left(H_{k,n}^{(i)} - \frac{1}{\alpha} \right) \int_{a^{(j)}(n/k)}^{X_j(k)} \frac{n}{k} \bar{F}_j(x) \frac{dx}{x}. \end{aligned}$$

converges, the limit being then equal to $\int_1^{\infty} \int_1^{\infty} \nu_{i,j}(x, y)/(xy) dx dy - \Omega_{i,j}$, while $\Omega_{i,i} = 1/\alpha^2$ for all $i \in \{1, \dots, l\}$.

A tractable expression for Ω can be derived from the Bahadur-Kiefer representation of high order quantiles, see Cs  rg   and R  v  sz [10], under the additional hypothesis (3.9) which can be viewed as a multivariate counterpart of Condition (3.6):

$$\begin{aligned} \text{For } 1 \leq i \neq j \leq l, \sup_{x, y > 0} \left| \frac{n}{k} \bar{F}_{i,j} \left(a^{(i)} \left(\frac{n}{k} \right) x, a^{(j)} \left(\frac{n}{k} \right) y \right) - \nu_{i,j}(x, y) \right| \\ = o \left(\frac{1}{\log k} \right) \text{ as } n, k \rightarrow +\infty. \end{aligned} \quad (3.9)$$

This condition permits to establish the next theorem, which provides the form of the asymptotic covariance of the r.v. obtained by concatenating the local Hill estimators, when all recentered by $1/\alpha$. Corollary 3.6 offers a generalization to the situation where the number of extremal observations involved in the local tail index estimation possibly depends on the location.

Theorem 3.5. *Suppose that Condition (3.6) and Condition (3.9) hold true, together with the von Mises conditions: $\forall i \in \{1, \dots, l\}$*

$$\lim_{s \rightarrow \infty} \alpha(s) := \frac{sF'_i(s)}{1 - F_i(s)} = \alpha. \quad (3.10)$$

Then we have the convergence in distribution

$$\sqrt{k} \left(H_{k,n}^{(1)} - \frac{1}{\alpha}, \dots, H_{k,n}^{(l)} - \frac{1}{\alpha} \right) \Rightarrow \mathcal{N}(0, \Omega), \quad (3.11)$$

$$\text{where } \Omega_{i,j} = \begin{cases} \frac{\nu_{i,j}(1,1)}{\alpha^2} & \text{if } 1 \leq i \neq j \leq l. \\ \frac{1}{\alpha^2} & \text{otherwise.} \end{cases}$$

Corollary 3.6. *Suppose that the assumptions of Corollary 3.2 are fulfilled and that, for any integer $l \geq 1$ and any $\mathbf{s} = (s_1, \dots, s_l) \in \mathcal{S}^l$, conditions (3.6), (3.9) and (3.10) hold. Then, we have*

$$\sqrt{k_1} \left(H_{k_1,n}^{(1)} - \frac{1}{\alpha}, \dots, H_{k_1,n}^{(l)} - \frac{1}{\alpha} \right) \Rightarrow \mathcal{N}(0, \Gamma), \quad (3.12)$$

where, for $1 \leq i \neq j \leq l$,

$$\Gamma_{i,j} = \frac{\nu_{i,j}(c_i^{1/\alpha}, c_j^{1/\alpha})}{\alpha^2}.$$

Before showing how the results above apply to the aggregation of the local tail index estimators, we exhibit a distribution fulfilling the conditions involved in the previous analysis.

Example 3.7. *The l -dimensional Gumbel copula C_ν with dependence coefficient $\nu \geq 1$ is given by*

$$C_\nu(u_1, \dots, u_l) = \exp \left(- \left((-\log u_1)^\nu + \dots + (-\log u_l)^\nu \right)^{1/\nu} \right).$$

Let $X^{(1)}, \dots, X^{(l)}$ be heavy-tailed r.v.'s defined on the same probability space with tail index α and survival functions $\bar{F}_i = 1 - F_i$, $i = 1 \dots l$ and with joint distribution $F = C_\nu(F_1, \dots, F_l)$. In this case, we have:

$$\forall 1 \leq i \neq j \leq l, \quad \nu_{i,j}(x, y) = x^{-\alpha} + y^{-\alpha} - (x^{-\nu\alpha} + y^{-\nu\alpha})^{1/\nu}.$$

In addition, Condition (3.9) is satisfied if, as $n, k \rightarrow \infty$,

$$\text{for } 1 \leq i \leq l, \quad \sup_{x>0} \left| \bar{F}_i \left(a^{(i)} \left(\frac{n}{k} \right) x \right) - x^{-\alpha} \right| = o \left(\frac{1}{\log k} \right). \quad (3.13)$$

For instance, if F_i is the GPD, Condition (3.13) is satisfied, since in this case $\sup_{x>0} |\bar{F}_i(a_k x) - x^{-\alpha}| = O((k/n)^{1/\alpha})$. The proof is given in the Appendix 7 therein.

3.2. Application to AMSE minimization.

Based on the asymptotic results of the previous section, we now consider the problem of building an estimator of the form of a convex sum of the local Hill estimators $H_{k_1,n}^{(1)}, \dots, H_{k_l,n}^{(l)}$, namely $H_{\mathbf{k},n}(\lambda) = \sum_{i=1}^l \lambda_i H_{k_i,n}^{(i)}$ where the λ_i 's are non negative real numbers such that $\sum_{i=1}^l \lambda_i = 1$, with minimum asymptotic variance. Precisely, the asymptotic mean square error (AMSE in abbreviated form) is defined as $AMSE(\lambda) = k \mathbb{E} \left[\left(H_{\mathbf{k},n}(\lambda) - \frac{1}{\alpha} \right)^2 \right]$, for $k \in \{k_1, \dots, k_l\}$. Hence, the goal is to find a solution $\lambda^* = (\lambda_1^*, \dots, \lambda_l^*)$ of the minimization problem

$$\min_{\lambda=(\lambda_i)_{1 \leq i \leq l} \in [0,1]^l} AMSE(\lambda) \text{ subject to } \sum_{i=1}^l \lambda_i = 1. \quad (3.14)$$

Observe that we have

$$\begin{aligned} AMSE(\lambda) &= k \mathbb{E} \left[\left(\sum_{i=1}^l \lambda_i H_{k_i,n}^{(i)} - \frac{1}{\alpha} \right)^2 \right] = k \mathbb{E} \left[\left(\sum_{i=1}^l \lambda_i \left(H_{k_i,n}^{(i)} - \frac{1}{\alpha} \right) \right)^2 \right] \\ &= k \sum_{i=1}^l \sum_{j=1}^l \lambda_i \lambda_j \mathbb{E} \left[\left(H_{k_i,n}^{(i)} - \frac{1}{\alpha} \right) \left(H_{k_j,n}^{(j)} - \frac{1}{\alpha} \right) \right] \\ &= \sum_{i=1}^l \sum_{j=1}^l \lambda_i \lambda_j \Gamma_{i,j} = {}^t \lambda \Gamma \lambda \end{aligned}$$

The minimization problem (3.14) thus boils down to solving the quadratic problem:

$$\underset{C\lambda \leq d}{\operatorname{argmin}} {}^t \lambda \Gamma \lambda,$$

where the constraint matrix C and the vector d are given by:

$$C = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{pmatrix} \text{ and } d = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

A variety of procedures can be readily used to solve this quadratic problem, including Uzawa’s algorithm for instance, see Glowinski [29].

Now, going back to statistical estimation, suppose that a consistent estimator $\hat{\alpha}$ of α is at our disposal. As the matrix Γ is unknown in practice, we define

$$\hat{\nu}_{i,j}(x, y) = \frac{1}{k} \sum_{m=1}^n \mathbb{1} \left(\frac{X_m^{(i)}}{X_i(k)} > x, \frac{X_m^{(j)}}{X_j(k)} > y \right) \text{ and } \hat{\Gamma}_{i,j} = \frac{\hat{\nu}_{i,j}(c_i^{1/\hat{\alpha}}, c_j^{1/\hat{\alpha}})}{\hat{\alpha}^2}.$$

Then, we compute

$$\hat{\lambda}^{opt} = \underset{C\lambda \leq d}{\operatorname{argmin}}^t \lambda \hat{\Gamma} \lambda.$$

The quantity $H_{\mathbf{k},n}(\hat{\lambda}^{opt})$ is then referred to as Best Empirical AggRegation (BEAR) estimator. The performance of the BEAR estimator is investigated from an empirical perspective by means of simulation experiments in the next section.

4. Simulations

For illustrative purposes, we computed the BEAR estimator for simulated random fields observed on regular grids of dimension 2×1 , 2×2 , 3×3 . The distributions of the margins were chosen among the Student- t distribution with degree of freedom α , the Generalized Pareto Distribution (GPD) with shape parameter $1/\alpha$ and the Fr  chet distribution with shape parameter α , these three distributions sharing the same tail index α . The main barrier in practice is the choice of the optimal sample fraction k used to compute the marginal Hill estimators. This choice had to be automated. We implemented the procedures introduced in Beirlant et al. [4, 5], Danielsson et al. [13]. Unfortunately, they lead to inaccurate choices for small sample sizes (except for the t -distribution), significantly overestimating the optimal k for the GPD and the Fr  chet distributions. The corresponding results are not documented. For the Student- t , the GPD and the Fr  chet distribution, it is possible to determine an optimal value k_{opt} in terms of AMSE. Hence, for each simulated sample we decided to choose at random the optimal k in the interval $[\max(30, 0.5k_{opt}), \min(n/3, 1.5k_{opt})]$ where n is the sample size. The largest admissible value is bounded by $n/3$ because we considered that including a larger fraction of the sample would lead to a highly biased estimate. Similarly, we bounded the smallest admissible value in order not to obtain estimates with a too large variance. In addition the interval $[0.5k_{opt}, 1.5k_{opt}]$ seemed to be a reasonable choice to account for the error in the selection of the optimal sample fraction.

We tried two different choices of copulas to describe the asymptotic dependence structure of the field, namely the Gumbel copula (Gudendorf and Segers [30]) and the t -copula (Frahm et al. [26]). Both yielded very similar results and only the results for the Gumbel copula with dependence parameter $\nu = 3$ ($\nu = 1$ for exact independence and $\nu \rightarrow \infty$ for exact dependence) are displayed.

The results for different values are presented in Table 1 where n is the sample size. The mean squared error (MSE) of the BEAR estimator is compared to

the MSE of two other estimators. Firstly, the Average estimator (Ave.) is the weighted average over all the marginal Hill estimators with weights equal to the marginal optimal k . Secondly, the MSE of the so-called Marginal estimator (Marg.) is the weighted average (weights marginal k) of the marginal MSE's. The figure in brackets indicates ratio of MSE of the estimator and of the BEAR.

The results indicate a very good behavior of the BEAR estimator, even for relatively small sample sizes n except when α or n becomes too large (bold values). We point out that when n is too small the exponent measure is estimated with less accuracy and the error contaminates the covariance matrix $\hat{\Gamma}$ (italic values). However, for large n the gain in MSE is significant.

The bad behavior of BEAR for some values of α can be understood through second order approximations of the marginal and multivariate distributions and is the subject of current research.

5. Example : sloshing data tail index inference

In the liquefied natural gas (LNG) shipping industry, *sloshing* refers to an hydraulic phenomenon which arises when the cargo is set in motion, [27]. Following incidents experienced by the ships Larbi Ben M'Hidi and more recently by Catalunya Spirit, these being two LNG carriers faced with severe sloshing phenomena, rigorous risk assessments have become a strong requirement for designers, certification organizations (seaworthiness) and ship owners. In addition, sloshing has also been a topic of interest in other industries (for instance, see [1] for a contribution in the field of aerospace engineering). Gaztransport & Technigaz (GTT) is a French company which designs the most widely used cargo containment system (CCS) for conveying LNG, namely the membrane containment system. The technology developed by GTT uses the hull structure of the vessel itself: the tanks are effectively part of the ship. The gas in the cargo is liquefied and kept at a very low temperature (-163°C) and atmospheric pressure, thanks to a thermal insulation system which prevents the LNG from evaporating. Although this technology is highly reliable, it can be susceptible to sloshing: waves of LNG apply very high pressures (over 20 bar) on the tank walls on impact and may possibly damage the CCS. Due to its high complexity, the sloshing phenomenon is modeled as a random process. The phenomenon is being studied by GTT experimentally on instrumented small-scale replica tanks (1/40 scale) as the one in Fig. 1. The tanks are shaken by a jack system to reproduce the motion of the ship and induce the occurrence of sloshing, with the associated high pressures being recorded by the sensors.

GTT studies the sloshing phenomenon during experiments on small scaled tanks replica (1/40), like the one of Fig. 1. The tank is instrumented with a collection of pressure sensors, divided in arrays. A pressure measurement is considered as a sloshing impact if the maximal pressure recorded is above 0.05 bar. As soon as a sensor records a pressure above 0.05 bar (we call it an event), the pressures measured by all the other sensors of the array are also recorded at a frequency of 200Hz. The recording stops when the pressures measured by

α	dim.	n	BEAR	Ave. (Improv.)	Marg. (Improv.)
$\alpha = 1$	2×2	1000	7.0×10^{-3}	7.3×10^{-3} (4.8%)	1.0×10^{-2} (44.8%)
		2500	3.5×10^{-3}	3.7×10^{-3} (6.8%)	5.3×10^{-3} (52.4%)
		10000	9.4×10^{-4}	1.1×10^{-3} (19.9%)	1.7×10^{-3} (76.0%)
	3×3	1000	7.6×10^{-3}	7.1×10^{-3} (-6.2%)	1.0×10^{-2} (35.9%)
		2500	3.5×10^{-3}	3.6×10^{-3} (2.4%)	5.1×10^{-3} (45.8%)
		10000	7.9×10^{-4}	1.0×10^{-3} (27.9%)	1.7×10^{-3} (110.0%)
	4×4	1000	7.4×10^{-3}	5.4×10^{-3} (-27.8%)	8.7×10^{-3} (17.4%)
		2500	4.4×10^{-3}	3.1×10^{-3} (-18.5%)	6.5×10^{-3} (27.7%)
		10000	9.2×10^{-4}	1.0×10^{-3} (9.6%)	1.7×10^{-3} (82.0%)
$\alpha = 2$	2×2	1000	2.1×10^{-3}	2.2×10^{-3} (3.1%)	2.7×10^{-3} (28.9%)
		2500	1.2×10^{-3}	1.2×10^{-3} (5.0%)	1.9×10^{-3} (59.7%)
		10000	4.6×10^{-4}	5.1×10^{-4} (11.5%)	8.0×10^{-4} (72.3%)
	3×3	1000	2.2×10^{-3}	2.6×10^{-3} (18.2%)	4.2×10^{-3} (92.4%)
		2500	1.1×10^{-3}	1.3×10^{-3} (12.3%)	2.1×10^{-3} (87.9%)
		10000	4.3×10^{-4}	4.8×10^{-4} (12.1%)	8.3×10^{-4} (94.8%)
	4×4	1000	2.2×10^{-3}	2.5×10^{-3} (14.9%)	4.2×10^{-3} (95.8%)
		2500	1.2×10^{-3}	1.3×10^{-3} (4.0%)	2.2×10^{-3} (83.8%)
		10000	5.0×10^{-4}	5.2×10^{-4} (3.4%)	9.1×10^{-4} (80.5%)
$\alpha = 5$	2×2	1000	1.1×10^{-3}	6.0×10^{-4} (-46.5%)	2.0×10^{-3} (75.2%)
		2500	2.4×10^{-4}	2.6×10^{-4} (6.2%)	6.8×10^{-4} (178.8%)
		10000	6.8×10^{-5}	8.4×10^{-5} (22.8%)	2.2×10^{-4} (224.1%)
	3×3	1000	1.4×10^{-3}	6.4×10^{-4} (-52.8%)	2.3×10^{-3} (73.3%)
		2500	2.5×10^{-4}	2.8×10^{-4} (14.3%)	8.7×10^{-4} (253.5%)
		10000	6.0×10^{-5}	8.6×10^{-5} (42.8%)	2.7×10^{-4} (347.8%)
	4×4	1000	1.8×10^{-3}	7.4×10^{-4} (-58.7%)	2.7×10^{-3} (49.6%)
		2500	3.0×10^{-4}	3.3×10^{-4} (8.9%)	9.9×10^{-4} (227.2%)
		10000	5.3×10^{-5}	9.1×10^{-5} (70.2%)	3.3×10^{-4} (522.4%)

TABLE 1
Simulation results. MSEs of the three estimators

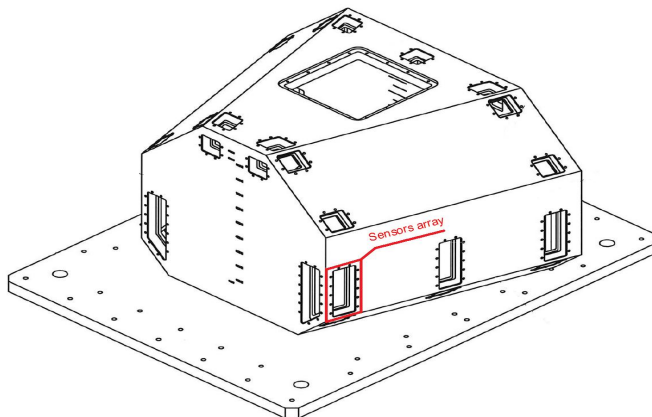


FIG 1. Scheme of the small scaled tank (1:40). The empty compartment are where the sensors are nested.

all the sensors are back to zero. For each event and for each sensor, we have a collection of measures. For each sensor and each event, we only keep the highest pressure.

GTT provided us with a low filling configuration data set: the tanks are nearly empty (the level of LNG in the tank is 10% of the height of the tank so that only the lower parts of the tank are instrumented with sensors). We consider the observations of the sensors array emphasized in Fig. 1 and represented in Fig. 2. This is a 7×3 sensors array. 31197 events were recorded by the 21 sensors.

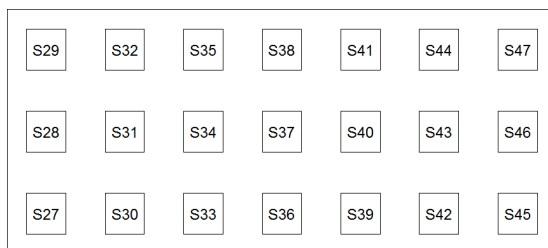


FIG 2. sensors array (after a 90° rotation)

It is the assumption of GTT that the tail index is the same for the observations of all the sensors, even though the field is not supposed to be stationary. This totally fits in the framework of this paper and we use our methodology to estimate the tail index α .

First step : Marginal estimation of α . At each location s_i , $i = 1 \dots 21$, we randomly chose half the observations and draw a Hill plot based on these observations. We used it to determine graphically k_i the optimal number of extremes

3.7 (3.2 – 4.2)	3.5 (2.8 – 4.1)	4.1 (3.7 – 4.6)	4.1 (3.6 – 4.5)	3.4 (3.1 – 3.6)	3.6 (3.3 – 4.0)	3.3 (2.8 – 3.7)
3.9 (3.5 – 4.4)	3.4 (3.0 – 3.9)	3.3 (2.8 – 3.8)	4.2 (3.6 – 4.8)	4.0 (3.3 – 4.7)	3.2 (2.4 – 4.0)	3.7 (3.2 – 4.2)
3.6 (3.2 – 4.0)	3.7 (3.4 – 4.0)	3.4 (3.2 – 3.7)	3.9 (3.6 – 4.3)	3.6 (3.1 – 4.1)	3.4 (3.0 – 3.8)	3.6 (3.3 – 4.0)

TABLE 2

Marginal tail index estimation with 95% confidence intervals.

to be used. We computed the Hill estimator $H_{k_i,n}^{(i)}$ with the other half of the observations. The Hill plots for four sensors are presented in Fig. 3. The solid line corresponds to the Hill plot used to determine k . We compared it (for validation purpose) to the Hill plot drawn using the second half of the data (dashed line). The estimation $\hat{\alpha}$ of α used for the aggregation procedure is the average over all these estimations. We found $\hat{\alpha} = 3.6$, with an estimated standard deviation of 1.1×10^{-2} .

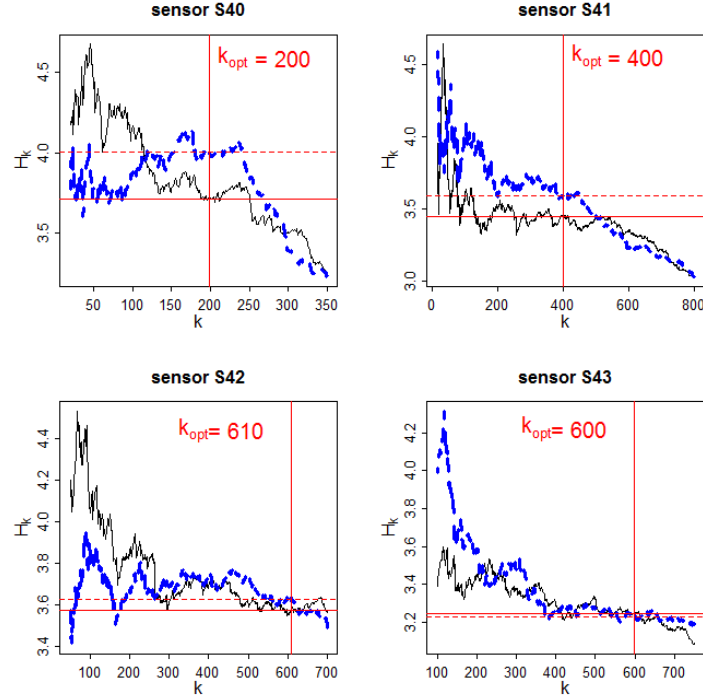


FIG 3. Hill plots for the sensors S35, S36, S37 and S38.

Second step : Aggregation. We used the methodology described in section 3.2 to compute the BEAR estimator. We found $\hat{\alpha}^{opt} = 3.4$ with an estimated standard deviation of 5.2×10^{-3} .

6. Conclusion

This paper introduces the concept of (standard) heavy-tail random field and proposes the so-termed BEAR estimator to estimate its tail index. Incidentally, the BEAR estimator can also be used in the context of regularly varying random processes (citepHult05) or in any heavy-tailed multivariate framework, as long as all the margins share the same tail index. Beyond the asymptotic analysis, it was shown to be highly accurate even for small sample sizes. When α increases, some approximations needed to derive asymptotic result are bad, especially for the GPD distribution. This can be understood with second order conditions of the margins. It is the subject of further research to estimate the bias of the BEAR estimator. This study could help deciding which marginal estimator to choose (Hill, Moment and J) in order to minimize the asymptotic mean squared error.

Appendix - Technical Proofs

Convention for the remaining of the paper: Without loss of generality and for ease of notation, the proofs are given in dimension ($l = 2$). To lighten, we set $X := X_{s_1}$, $Y := X_{s_2}$, $\nu := \nu_{s_1, s_2}$, $a(n/k) = a^{(1)}(n/k)$ and $b(n/k) = a^{(2)}(n/k)$. The survival functions of X and Y are denoted by \bar{F}_X and \bar{F}_Y respectively, and the survival function of (X, Y) is denoted by \bar{F} . We observe an n -sample $\left((X_1, Y_1), \dots, (X_n, Y_n)\right)$ of (X, Y) and for any $i = 1 \dots k$, we set $U_i = F_X(X_i)$ and $V_i = F_Y(Y_i)$.

Proof of Theorem 3.1. We set

$$\alpha_n(x, y) = \sqrt{k} \left(\frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{X_i}{a(n/k)} > x, \frac{Y_i}{b(n/k)} > y \right) - \frac{n}{k} \bar{F} \left(a \left(\frac{n}{k} \right) x, b \left(\frac{n}{k} \right) y \right) \right)$$

Finite dimensional convergence: We first show than any finite dimensional projection of $\alpha_n(x, y)$ converges in distribution toward a Gaussian random vector. We use a result of citep[p.41]Araujo80, providing a multivariate version of the Lindeberg-Feller theorem. Let $(x_1, y_1), \dots, (x_l, y_l)$ a finite subset of \mathbb{R}_+^2 . Denote, for $1 \leq i \leq n$ and $1 \leq j \leq l$

$$Z_{i,n}(x_j, y_j) := \frac{1}{\sqrt{k}} \mathbb{1} \left(\frac{X_i}{a(n/k)} > x_j, \frac{Y_i}{b(n/k)} > y_j \right) - \frac{1}{\sqrt{k}} \bar{F} \left(a \left(\frac{n}{k} \right) x_j, b \left(\frac{n}{k} \right) y_j \right).$$

Firstly, notice that $\mathbb{E}[Z_{i,n}(x_j, y_j)] = 0$, $1 \leq i \leq n$ and $1 \leq j \leq l$.

Lindeberg condition: With $Z_{i,n} = (Z_{i,n}(x_1, y_1), \dots, Z_{i,n}(x_l, y_l))$ and the euclidean norm $\|\cdot\|$, we have $\|Z_{i,n}\|^2 = \sum_{j=1}^t Z_{i,n}(x_j, y_j)^2 \leq t/k$ which gives

$$\forall \varepsilon > 0, \quad \sum_{j=1}^n \int_{\|Z_{i,n}\| > \varepsilon} \|Z_{i,n}\|^2 d\mathbb{P} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now, let $r, s < l$. We have

$$\begin{aligned} & \sum_{j=1}^n \mathbb{E} [Z_{i,n}(x_r, y_r) Z_{i,n}(x_s, y_s)] \\ &= \frac{n}{k} \mathbb{P} \left(\frac{X_1}{a(n/k)} > \max(x_r, x_s), \frac{Y_1}{b(n/k)} > \max(y_r, y_s) \right) \\ & \quad - F \left(a \left(\frac{n}{k} \right) x_r, b \left(\frac{n}{k} \right) y_r \right) F \left(a \left(\frac{n}{k} \right) x_s, b \left(\frac{n}{k} \right) y_s \right) \\ & \quad \rightarrow \nu(\max(x_r, x_s), \max(y_r, y_s)). \end{aligned}$$

Hence $(\alpha_n(x_1, y_1), \dots, \alpha_n(x_n, y_n)) \Rightarrow \mathcal{N}(0, V)$, with $V_{i,j} = \nu(\max(x_i, x_j), \max(y_i, y_j))$.

Tightness: Let $M > 0$. We prove the tightness of α_n on $[0, M]$ from which we deduce that the process is asymptotically tight on \mathbb{R}^+ (for technical details, refer to [Th. 4]Schmidt06). To do so we show the asymptotic uniform equicontinuity in probability of α_n : $\forall \xi > 0, \forall \eta > 0, \exists \delta \in (0, 1), m_0 \in \mathbb{N}$, s.t. $\forall m > m_0$:

$$\mathbb{P} \left(\sup_{\substack{|x_1 - x_2|^2 + |y_1 - y_2|^2 < \delta \\ x_j, y_j \in [0, M], i=1,2}} |\alpha_n(x_1, y_1) - \alpha_n(x_2, y_2)| > \xi \right) < \eta.$$

In the sequel we introduce the following additional notations:
 $\forall 0 \leq x_1, x_2, y_1, y_2 \leq \infty$, we set

- $S_k^n(x_1, x_2, y_1, y_2) := \frac{1}{n} \sum_{i=1}^n \mathbb{1} \left(x_1 \leq \frac{X_i}{a(n/k)} \leq x_2, \frac{Y_i}{b(n/k)} < y_1 \right)$
- $F_k^n(x_1, x_2, y_1, y_2) := F \left(\left(a \left(\frac{n}{k} \right) x_1, a \left(\frac{n}{k} \right) x_2 \right], \left(b \left(\frac{n}{k} \right) y_1, b \left(\frac{n}{k} \right) y_2 \right] \right).$

We partition the square $[0, M]^2$ into cubes with partition points $(Ml_1/L, Ml_2/L), l_i \in \{0, \dots, L\}, L \in \mathbb{N}, i = 1, 2$. Now, choose $\xi > 0$ and $\delta \in (0, 1)$ such that $1/L \geq \delta$. We first have

$$\begin{aligned} & \mathbb{P} \left(\sup_{\substack{|x_1-x_2|^2+|y_1-y_2|^2 < \delta \\ x_j, y_j \in [0, M], i=1,2}} |\alpha_n(x_1, y_1) - \alpha_n(x_2, y_2)| > \xi \right) \\ & \leq \mathbb{P} \left(3 \max_{\substack{1 \leq l_i \leq L \\ i=1,2}} \sup_{\substack{(x_1, y_1), \\ (x_2, y_2) \in I_{l_i, L}}} |\alpha_n(x_1, y_1) - \alpha_n(x_2, y_2)| > \xi \right) := I_1. \end{aligned}$$

Without loss of generality, assume $x_1 < x_2$, $y_1 < y_2$. Then, denoting by $\xi' := \xi/3$, we have that I_1 is smaller than

$$\begin{aligned} & \sum_{\substack{1 \leq l_i \leq L \\ i=1,2}} \mathbb{P} \left(\sup_{\substack{(y_1), (y_2) \in I_{l_i, L}}} \frac{n}{\sqrt{k}} \left| S_k^n(x_1, x_2, y_1, \infty) - F_k^n(x_1, x_2, y_1, \infty) \right| > \xi' \right) \\ & + \mathbb{P} \left(\sup_{\substack{(y_1), (y_2) \in I_{l_i, L}}} \frac{n}{\sqrt{k}} \left| S(x_1, \infty, y_1, y_2) - F(x_1, \infty, y_1, y_2) \right| > \xi' \right) \\ & := I_2 + I_3 \end{aligned}$$

We show that I_2 tends to 0 as $n, k \rightarrow \infty$. The quantity I_3 can be dealt with exactly the same way. We have

$$\begin{aligned} & \sum_{\substack{1 \leq l_i \leq L \\ i=1,2}} \mathbb{P} \left(\sup_{\substack{(y_1), (y_2) \in I_{l_i, L}}} \frac{n}{\sqrt{k}} \left| S_k^n(x_1, x_2, y_1, \infty) - F_k^n(x_1, x_2, y_1, \infty) \right| > \xi' \right) \\ & = \sum_{\substack{1 \leq l_i \leq L \\ i=1,2}} \left\{ \mathbb{P} \left(\sup_{\substack{(y_1), (y_2) \in I_{l_i, L}}} \frac{n}{\sqrt{k}} \left| S_k^n(x_1, x_2, 0, \infty) - F_k^n(x_1, x_2, 0, \infty) \right. \right. \right. \\ & \quad \left. \left. - \left(S_k^n(x_1, x_2, 0, y_1) - F_k^n(x_1, x_2, 0, y_1) \right) \right| > \xi' \right) \right\} \\ & \leq \sum_{\substack{1 \leq l_i \leq L \\ i=1,2}} \left\{ \mathbb{P} \left(\sup_{\substack{(y_1), (y_2) \in I_{l_i, L}}} \left| S_k^n(x_1, x_2, 0, \infty) - F_k^n(x_1, x_2, 0, \infty) \right| > \frac{\xi' \sqrt{k}}{n} \right) \right. \\ & \quad \left. + \mathbb{P} \left(\sup_{\substack{(y_1), (y_2) \in I_{l_i, L}}} \left| S_k^n(x_1, x_2, 0, y_1) - F_k^n(x_1, x_2, 0, y_1) \right| > \frac{\xi' \sqrt{k}}{n} \right) \right\} \end{aligned}$$

$$\begin{aligned} \leq & \exp \left(-\frac{k\xi'^2}{32n^2 F(C_{1,n}^{l_i,L})} \cdot \psi \left(\frac{\sqrt{k}\xi'}{4n^{3/2} F(C_{1,n}^{l_i,L})} \right) \right) \\ & + \exp \left(-\frac{k\xi'^2}{32n^2 F(C_{2,n}^{l_i,L})} \cdot \psi \left(\frac{\sqrt{k}\xi'}{4n^{3/2} F(C_{2,n}^{l_i,L})} \right) \right) := E_1 + E_2. \end{aligned}$$

This inequality can be found in citepRuymgaart84 with ψ satisfying the following properties: (i) $x \mapsto \psi(x)$ is decreasing, (ii) $x \mapsto x\psi(x)$ is increasing, and:

$$\begin{aligned} C_{1,n}^{l_i,L} &:= \left(M \frac{a(n/k)(l_1-1)}{L}, M \frac{b(n/k)l_1}{L} \right] \times (0, \infty] \\ C_{2,n}^{l_i,L} &:= \left(M \frac{a(n/k)(l_1-1)}{L}, M \frac{b(n/k)l_1}{L} \right] \times \left(0, M \frac{a(n/k)l_2}{L} \right] \end{aligned}$$

Notice that $F(C_{2,n}^{l_i,L}) < \frac{Mb(n/k)}{Ln}$. We distinguish two cases. First, for E_1 , we have $\frac{\sqrt{k}\xi'}{n^{3/2} F(C_{1,n}^{l_i,L})} \leq 1$. In this case

$$\exp \left(-\frac{k\xi'^2}{32n^2 F(C_{1,n}^{l_i,L})} \cdot \psi \left(\frac{\sqrt{k}\xi'}{4n^{3/2} F(C_{1,n}^{l_i,L})} \right) \right) \leq \exp \left(-\frac{kL}{8na(n/k)} \cdot \psi(1) \right).$$

Second, for E_2 , we have $\frac{\sqrt{k}\xi'}{n^{3/2} F(C_{2,n}^{l_i,L})} > 1$. In this case the bound is

$$\exp \left(-\frac{k\xi'^2}{32n^2 F(C_{2,n}^{l_i,L})} \cdot \psi \left(\frac{\sqrt{k}\xi'}{4n^{3/2} F(C_{2,n}^{l_i,L})} \right) \right) \leq \exp \left(-\frac{\sqrt{k}\xi'}{8\sqrt{n}} \cdot \psi(1) \right).$$

We hence have

$$I_2 \leq 2 \cdot L^2 \left(\exp \left(-\frac{kL}{8nb(n/k)} \cdot \psi(1) \right) + \exp \left(-\frac{\sqrt{k}\xi'}{8\sqrt{n}} \cdot \psi(1) \right) \right) \xrightarrow{m \rightarrow 0} 0.$$

As the result holds for I_3 , we obtain the tightness on $[0, M]^2$ for every M . \square

Proof of Corollary 3.2. The regular variation property of $a^{(i)}(n/k_i)$ gives immediately $a^{(i)}(n/k_i)/a^{(i)}(n/k_1) \xrightarrow{n \rightarrow \infty} c_i^{1/\alpha}$.

Now, to get the result, we compose the convergence and plug this limit in equation (3.2), as we do hereinafter in equation (6.1). \square

Proof of Theorem 3.3. In order to obtain Eq. (3.4), we apply a transform on the tail empirical process of Eq.(3.2). The tail empirical process is evaluated at $\mathbf{x}_1 = (x, 0)$ and $\mathbf{x}_2 = (0, y)$. The left-hand term of Eq. (3.4) is then obtained by replacing $a(n/k)$ and $b(n/k)$ by their empirical counterpart. The result is integrated over $(1, \infty]$ to obtain the desired convergence. In addition, in the sequel $k := k(n)$ is a sequence such that $k \rightarrow \infty$, $k = o(n)$.

First, we make a change of variable so that we can re-use some results given in [resnick2007heavy](#) chap.9. Theorem [3.1](#) gives

$$\begin{aligned} & \sqrt{k} \left(\frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{X_i}{a(n/k)} > x, \frac{Y_i}{b(n/k)} > 0 \right) - \frac{n}{k} \bar{F}_X(a(n/k)x), \right. \\ & \left. \frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{X_i}{a(n/k)} > 0, \frac{Y_i}{b(n/k)} > y \right) - \frac{n}{k} \bar{F}_Y(b(n/k)y) \right) \Rightarrow \left(W(\mathbf{x}_1^{-\alpha}), W(\mathbf{x}_2^{-\alpha}) \right), \end{aligned}$$

where W is the process defined in Theorem [3.1](#), $(W(\mathbf{x}_1^{-\alpha}), W(\mathbf{x}_2^{-\alpha})) \sim \mathcal{N}(0, \Sigma)$ and

$$\Sigma = \begin{pmatrix} 1 & \nu(x^{-\frac{1}{\alpha}}, y^{-\frac{1}{\alpha}}) \\ \nu(x^{-\frac{1}{\alpha}}, y^{-\frac{1}{\alpha}}) & 1 \end{pmatrix}.$$

One also have ([Eq. \(4.17\)](#) [resnick2007heavy](#))

$$\left(\frac{X(k)}{a(n/k)}, \frac{Y(k)}{b(n/k)} \right) \xrightarrow{\mathbb{P}} (1, 1).$$

Hence from Proposition [3.1](#) in [resnick2007heavy](#) we have

$$\begin{aligned} & \left(\sqrt{k} \left(\frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{X_i}{a(n/k)} > x, \frac{Y_i}{b(n/k)} > 0 \right) - \frac{n}{k} \bar{F}_X(a(n/k)x), \right. \right. \\ & \quad \left. \frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{X_i}{a(n/k)} > 0, \frac{Y_i}{b(n/k)} > y \right) - \frac{n}{k} \bar{F}_Y(b(n/k)y) \right), \\ & \quad \left. \left(\frac{X(k)}{a(n/k)}, \frac{Y(k)}{b(n/k)} \right) \right) \Rightarrow \left((W(\mathbf{x}_1^{-\alpha}), W(\mathbf{x}_2^{-\alpha})), (1, 1) \right). \end{aligned}$$

Now, we apply the composition map $(x(t), p) \mapsto x(tp)$ which gives

$$\begin{aligned} & \sqrt{k} \left(\frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{X_i}{X(k)} > x, \frac{Y_i}{Y(k)} > 0 \right) - \frac{n}{k} \bar{F}_X(X(k)x), \right. \\ & \quad \left. \frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{X_i}{X(k)} > 0, \frac{Y_i}{Y(k)} > y \right) - \frac{n}{k} \bar{F}_Y(Y(k)y) \right) \Rightarrow (W(\mathbf{x}_1^{-\alpha}), W(\mathbf{x}_2^{-\alpha})), \end{aligned} \tag{6.1}$$

Equation [\(6.1\)](#) yields, again by

citep[[p.297-298]resnick2007heavy, as $n, k \rightarrow \infty$:

$$\begin{aligned} \sqrt{k} \left(\int_1^\infty \frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{X_i}{X(k)} > x \right) \frac{dx}{x} - \int_1^\infty \frac{n}{k} \bar{F}_X(X(k)x) \frac{dx}{x}, \right. \\ \left. \int_1^\infty \frac{1}{k} \sum_{i=1}^n \mathbb{1} \left(\frac{Y_i}{Y(k)} > y \right) \frac{dy}{y} - \int_1^\infty \frac{n}{k} \bar{F}_Y(Y(k)y) \frac{dy}{y} \right) \\ \Rightarrow \left(\int_1^\infty W(\mathbf{x}_1^{-\alpha}) \frac{dx}{x}, \int_1^\infty W(\mathbf{x}_2^{-\alpha}) \frac{dy}{y} \right) \quad (6.2) \end{aligned}$$

with

$$\begin{aligned} \mathbb{V} \left[\int_1^\infty W(\mathbf{x}_1^{-\alpha}) \frac{dx}{x} \right] &= \frac{2}{\alpha^2}. \\ \mathbb{Cov} \left[\int_1^\infty W(\mathbf{x}_1^{-\alpha}) \frac{dx}{x}, \int_1^\infty W(\mathbf{x}_2^{-\alpha}) \frac{dy}{y} \right] &= \int_1^\infty \int_1^\infty \nu(x, y) \frac{dx dy}{xy} := \sigma^2. \end{aligned}$$

Equation (6.2) is equivalent to

$$\sqrt{k} \left(H_{k,n}^X - \int_1^\infty \frac{n}{k} \bar{F}_X(X(k)x) \frac{dx}{x}, H_{k,n}^Y - \int_1^\infty \frac{n}{k} \bar{F}_Y(Y(k)y) \frac{dy}{y} \right) \Rightarrow K \quad (6.3)$$

$$\text{where } K \sim \mathcal{N} \left(0, \begin{pmatrix} \frac{2}{\alpha^2} & \sigma^2 \\ \sigma^2 & \frac{2}{\alpha^2} \end{pmatrix} \right).$$

□

Proof of Corollary 3.4. The argument is the same as for Corollary 3.2. □

Proof of Theorem 3.5. In order to prove Theorem 3.5, we need to give an analytic expression to the covariance matrix in Eq. (3.11). This is the object of the Lemmas (6.1) and (6.2).

Lemma 6.1. *Under Condition (3.10), we have*

$$\lim_{k \rightarrow \infty} \mathbb{E} \left[k \int_{a(n/k)}^{X(k)} \frac{n}{k} \bar{F}_X(x) \frac{dx}{x} \int_{b(n/k)}^{Y(k)} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} \right] = \frac{\nu(1, 1)}{\alpha^2}.$$

Lemma 6.2. *Under Condition (3.9), we have*

$$\lim_{k \rightarrow \infty} \mathbb{E} \left[k \left(H_{k,n}^X - \frac{1}{\alpha} \right) \int_{b(n/k)}^{Y(k)} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} \right] = \frac{1}{\alpha} \int_1^\infty \frac{\nu(x, 1)}{x} dx - \frac{\nu(1, 1)}{\alpha^2}.$$

To show the Lemmas 6.1 and 6.2, we linearize functional of the order statistics $X(1), \dots, X(k), Y(1), \dots, Y(k)$ of X and Y as series of the original observations $X_1, \dots, X_n, Y_1, \dots, Y_n$. This is done, using Taylor series and the Bahadur-Kiefer representation of the order statistics (citepBahadur66), in Lemmas 6.3 and 6.4. The Bahadur-Kiefer representation

involves a remainder term (see citepKiefer67) that needs to be controlled. This is the object of Lemmas 6.5, 6.6 and 6.7. The Lemmas 6.1 and 6.2 put together all the results of the aforementioned Lemmas. Lemma 6.8 simplifies the expressions given in Lemmas 6.1 and 6.2. □

Lemma 6.3 (BAHADUR-KIEFER REPRESENTATIONS). *We set $p_i = \frac{n-i+1}{n}$ and $\bar{p}_i = 1 - p_i$ $i = 1 \dots k$, we have the almost sure equalities*

$$X(i) = a(n/i) - \frac{1}{n} \sum_{j=1}^n \frac{\mathbb{1}(U_j \leq p_i) - p_i}{f_X(a(n/i))} + T_n(p_i) \quad (6.4)$$

$$\int_{a(n/k)}^{X(k)} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} = -\frac{1}{n} \sum_{j=1}^n \frac{\mathbb{1}(U_j \leq p_k) - p_k}{a(n/k) f_X(a(n/k))} + \frac{T_n(p_k)}{a(n/k)}. \quad (6.5)$$

$$\log X(i) = \log a(n/i) - \frac{1}{n} \sum_{j=1}^n \frac{\mathbb{1}(U_j \leq p_i) - p_i}{a(n/i) f_X(a(n/i))} + O\left(\frac{T_n(p_i)}{a(n/i)}\right). \quad (6.6)$$

where T_n is a remainder terms.

Proof of Lemma 6.3. Eq. (6.4) is just the Bahadur Kiefer representation of $X(i)$. For Eq. (6.5) we have almost surely

$$\begin{aligned} \int_{a(n/k)}^{X(k)} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} &= \int_1^{\frac{X(k)}{a(n/k)}} \frac{n}{k} \bar{F}_X\left(a\left(\frac{n}{k}\right)x\right) \frac{dx}{x} \\ &= \frac{1}{\alpha} \left(1 - \left(\frac{X(k)}{a(n/k)}\right)^{-\alpha}\right) + O\left(\frac{1}{\sqrt{k}}\right) \\ &= \frac{X(k) - a(n/k)}{a(n/k)} + O\left(\left(\frac{X(k) - a(n/k)}{a(n/k)}\right)^2\right). \end{aligned}$$

Eq. (6.6) follows from a Taylor expansion of the logarithm function. □

Lemma 6.4 (CONTROL OF THE REMAINDER TERMS). *Assuming Conditions (3.2) and (3.4) in citepCsorgo1978, we have almost surely:*

$$\sup_{0 \leq y \leq 1} |T_n(y)| = O\left(n^{-3/4} (\log \log n)^{-1/4} (\log n)^{-1/2}\right). \quad (6.7)$$

$$\int_{b(n/k)}^{Y(k)} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} = O\left(\frac{1}{a(n/k)} n^{-3/4} (\log \log n)^{-1/4} (\log n)^{-1/2}\right). \quad (6.8)$$

Proof of Lemma 6.4. Eq.(6.7) follows directly from Th.E in citepCsorgo1978.

In order to prove Eq. (6.8), we start recalling some fact about uniform approximation of the generalized quantile process. We set

$$\rho_n^Y(p_i) = \sqrt{n} \left(Y(i) - a(n/i) \right) f_Y \left(a(n/i) \right) \quad (6.9)$$

$$u_n^Y(p_i) = \sqrt{n} (U(i) - p_i) \quad (6.10)$$

It is known from citepCsorgo1978 that under specific conditions satisfied by regularly varying survival functions, we have

$$\sup_{0 \leq y \leq 1} |\rho_n^Y(y) - u_n^Y(y)| = O \left(n^{-1/2} \log \log n \right) a.s. \quad (6.11)$$

$$\sup_{0 \leq y \leq 1} |u_n^Y(y)| = O \left(n^{-1/4} (\log \log n)^{-1/4} (\log n)^{-1/2} \right) a.s., \quad (6.12)$$

cf
citep[see][cond. (i) to (iv) p.18]Csorgo2004. We deduce for Eq. (6.11) and (6.12) that

$$\sup_{0 \leq y \leq 1} |\rho_n^Y(y)| = O \left(n^{-1/4} (\log \log n)^{-1/4} (\log n)^{-1/2} \right) a.s. \quad (6.13)$$

Now, for Eq. (6.8), notice that

$$\int_{b(n/k)}^{Y(k)} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} = \frac{1}{\alpha} \frac{\rho_n(p_k)}{\sqrt{n} b(n/k)} + O \left(\left(\frac{\rho_n(p_k)}{\sqrt{n} b(n/k)} \right)^2 \right) a.s.$$

and conclude by means of Eq. (6.13) □

Lemma 6.5 (COVARIANCE COMPUTATION (I)).

$$\mathbb{E} \left[\log X(i) \int_{b(n/k)}^{Y(k)} \frac{n}{k} \bar{F}_X(x) \frac{dx}{x} \right] = M_n(i) + R_{n,1}(k) + R_{n,2}(k),$$

where

$$M_n(i) = \frac{\mathbb{P} \left(X > a(n/i), Y > b(n/k) \right) - \bar{p}_i \bar{p}_k}{na(n/i) f_X \left(a(n/i) \right) b(n/k) f_Y \left(b(n/k) \right)},$$

$$R_{n,1}(k) = O \left(\frac{n^{-3/2} (\log \log n)^{-1/2} (\log n)^{-1}}{a(n/k) b(n/k)} \right),$$

$$R_{n,2}(k) = O \left(c \cdot b(n/k)^{-1} n^{-3/4} (\log \log n)^{-1/4} (\log n)^{-1/2} \right).$$

Proof of Lemma 6.5. One may write

$$\begin{aligned} & \mathbb{E} \left[\log X(i) \int_{b(n/k)}^{Y(k)} \frac{n}{k} \overline{F}_X(x) \frac{dx}{x} \right] = \\ & \mathbb{E} \left[\left(\log a(n/i) - \frac{1}{n} \sum_{j=1}^n \frac{\mathbb{1}(U_j \leq p_i) - p_i}{a(n/i) f_X(a(n/i))} \right) \left(-\frac{1}{n} \sum_{j=1}^n \frac{\mathbb{1}(V_j \leq p_k) - p_k}{b(n/k) f_Y(b(n/k))} \right) \right] \\ & \quad + \mathbb{E} \left[O \left(\frac{T_n(p_i)}{a(n/i)} \int_{b(n/k)}^{Y(k)} \frac{n}{k} \overline{F}_Y(x) \frac{dx}{x} \right) \right] + \mathbb{E} \left[\log X(i) \frac{T_n(p_k)}{b(n/k)} \right] \\ & = M_n(i) + R_{n,1}(k) + R_{n,2}(k). \end{aligned}$$

We have

$$\begin{aligned} M_n(i) &= \mathbb{E} \left[\left(\frac{1}{n^2} \sum_{j=1}^n \frac{(\mathbb{1}(U_j \leq p_i) - p_i)(\mathbb{1}(V_j \leq p_k) - p_k)}{na(n/i)f_X(a(n/i))b(n/k)f_Y(b(n/k))} \right) \right] \\ &= \frac{\mathbb{P}(X > a(n/i), Y > b(n/k)) - \bar{p}_i \bar{p}_k}{na(n/i)f_X(a(n/i))b(n/k)f_Y(b(n/k))}. \end{aligned}$$

By Lemma 6.4, we have

$$R_{n,1}(k) = O \left(\frac{n^{-3/2} (\log \log n)^{-1/2} (\log n)^{-1}}{a(n/k) b(n/k)} \right).$$

In addition, by virtue of Cauchy-Schwarz inequality, we have

$$\begin{aligned} R_{n,2}(i) &= \mathbb{E} \left[\log X(i) \frac{T_n(p_k)}{b(n/k)} \right] \\ &\leq b(n/k)^{-1} \sqrt{\mathbb{E} [\log^2 X(i)] \mathbb{E} [T_n(p_k)^2]}. \end{aligned}$$

For any positive value of α , $\mathbb{E} [\log^2 X(i)] < +\infty$. Hence, we have

$$R_{n,2}(i) = O \left(b(n/k)^{-1} n^{-3/4} (\log \log n)^{-1/4} (\log n)^{-1/2} \right).$$

□

Lemma 6.6 (COVARIANCE COMPUTATION (II)). *The following asymptotic relationships hold*

$$\forall i = 1 \dots k, \quad M_n(i) \sim \frac{1}{\alpha^2} \frac{n}{ik} \mathbb{P}(X > a(n/i), Y > b(n/k)).$$

In the particular case $i = k$, we have

$$M_n(k) \sim \frac{\nu(1,1)}{k\alpha^2}.$$

Proof of Lemma 6.6. Von Mises's Conditions (3.10) give

$$a(n/i)f_X\left(a(n/i)\right)b(n/k)f_Y\left(b(n/k)\right) \sim \alpha^2 \frac{ik}{n^2}. \quad (6.14)$$

With Eq. (6.14), this yields

$$\frac{\mathbb{P}\left(X > a(n/i), Y > b(n/k)\right) - \bar{p}_k \bar{p}_k}{na(n/i)f_X\left(a(n/i)\right)b(n/k)f_X\left(b(n/k)\right)} \sim \frac{1}{\alpha^2} \frac{n}{ik} \mathbb{P}\left(X > a(n/i), Y > b(n/k)\right).$$

□

Lemma 6.7 (COVARIANCE COMPUTATION (III)). *The following convergence holds*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^k \frac{n}{ki} \mathbb{P}\left(X > a(n/i), Y > b(n/k)\right) = \int_1^\infty \frac{\nu(x,1)}{x} dx$$

Proof. Write, for any $i = 1 \dots k$

$$\begin{aligned} \frac{n}{ki} \mathbb{P}\left(X > a(n/i), Y > b(n/k)\right) &= \frac{n}{ki} \mathbb{P}\left(X > a\left(\frac{n}{k}\right) \frac{a\left(\frac{n}{i}\right)}{a\left(\frac{n}{k}\right)}, Y > b\left(\frac{n}{k}\right)\right) \\ &= \frac{n}{ki} \bar{F}\left(a\left(\frac{n}{k}\right) \frac{a\left(\frac{n}{i}\right)}{a\left(\frac{n}{k}\right)}, b\left(\frac{n}{k}\right)\right). \end{aligned}$$

We have

$$\begin{aligned} \sup_{x>0} \left| \sum_{i=1}^k \frac{\frac{n}{k} \bar{F}\left(a\left(\frac{n}{k}\right)x, b\left(\frac{n}{k}\right)\right)}{i} - \frac{\nu(x,1)}{i} \right| \\ \leq \sup_{x>0} \sum_{i=1}^k \frac{1}{i} \left| \frac{n}{k} \bar{F}\left(a\left(\frac{n}{k}\right)x, b\left(\frac{n}{k}\right)\right) - \nu(x,1) \right| \\ \leq \sum_{i=1}^k \frac{1}{i} \sup_{x>0} \left| \frac{n}{k} \bar{F}\left(a\left(\frac{n}{k}\right)x, b\left(\frac{n}{k}\right)\right) - \nu(x,1) \right| \\ \sim \log k \cdot \sup_{x>0} \left| \frac{n}{k} \bar{F}\left(a\left(\frac{n}{k}\right)x, b\left(\frac{n}{k}\right)\right) - \nu(x,1) \right| \\ \sim o(1) \quad \text{by Condition (3.9).} \end{aligned}$$

Then,

$$\begin{aligned} & \left| \sum_{i=1}^k \frac{1}{i} \left(\frac{n}{k} \overline{F} \left(a \left(\frac{n}{k} \right) \frac{a \left(\frac{n}{i} \right)}{a \left(\frac{n}{k} \right)}, b \left(\frac{n}{k} \right) \right) - \nu \left(\frac{a \left(\frac{n}{i} \right)}{a \left(\frac{n}{k} \right)}, 1 \right) \right) \right| \\ & \leq \sup_{x>0} \sum_{i=1}^k \frac{1}{i} \left| \frac{n}{k} \overline{F} \left(a \left(\frac{n}{k} \right) x, b \left(\frac{n}{k} \right) \right) - \nu(x, 1) \right| \end{aligned}$$

Hence

$$\left| \sum_{i=1}^k \frac{1}{i} \frac{n}{k} \overline{F} \left(a \left(\frac{n}{k} \right) \frac{a \left(\frac{n}{i} \right)}{a \left(\frac{n}{k} \right)}, b \left(\frac{n}{k} \right) \right) - \sum_{i=1}^k \frac{1}{i} \nu \left(\frac{a \left(\frac{n}{i} \right)}{a \left(\frac{n}{k} \right)}, 1 \right) \right| \xrightarrow{n \rightarrow \infty} 0$$

In addition, using Potter's Bound, for any $\varepsilon > 0$ if n is large enough we have

$$\begin{aligned} (1 - \varepsilon) \left(\frac{k}{i} \right)^{1/\alpha - \varepsilon} & \leq \frac{a \left(\frac{n}{i} \right)}{a \left(\frac{n}{k} \right)} \leq (1 + \varepsilon) \left(\frac{k}{i} \right)^{1/\alpha + \varepsilon} \\ \nu \left((1 - \varepsilon) \left(\frac{k}{i} \right)^{1/\alpha - \varepsilon}, 1 \right) & \leq \nu \left(\frac{a \left(\frac{n}{i} \right)}{a \left(\frac{n}{k} \right)}, 1 \right) \leq \nu \left((1 + \varepsilon) \left(\frac{k}{i} \right)^{1/\alpha + \varepsilon}, 1 \right) \\ \frac{1}{k} \sum_{i=1}^k \frac{k}{i} \nu \left((1 - \varepsilon) \left(\frac{k}{i} \right)^{1/\alpha - \varepsilon}, 1 \right) & \leq \frac{1}{k} \sum_{i=1}^k \frac{k}{i} \nu \left(\frac{a \left(\frac{n}{i} \right)}{a \left(\frac{n}{k} \right)}, 1 \right) \\ & \leq \frac{1}{k} \sum_{i=1}^k \frac{k}{i} \nu \left((1 + \varepsilon) \left(\frac{k}{i} \right)^{1/\alpha + \varepsilon}, 1 \right). \quad (6.15) \end{aligned}$$

As $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$, the bounds of Eq. (6.15) converges towards $\int_0^1 \nu(x^{-1/\alpha}, 1) x^{-1} dx$. We deduce from the above that

$$\frac{1}{k} \sum_{i=1}^k \frac{k}{i} \nu \left(\frac{a \left(\frac{n}{i} \right)}{a \left(\frac{n}{k} \right)}, 1 \right) \xrightarrow{n \rightarrow \infty} \int_0^1 \frac{\nu(x^{-1/\alpha}, 1)}{x} dx = \alpha \int_1^\infty \frac{\nu(x, 1)}{x} dx$$

Finally, we obtain the desired convergence

$$\sum_{i=1}^k \frac{1}{i} \frac{n}{k} \overline{F} \left(a \left(\frac{n}{k} \right) \frac{a \left(\frac{n}{i} \right)}{a \left(\frac{n}{k} \right)}, b \left(\frac{n}{k} \right) \right) \xrightarrow{n \rightarrow \infty} \alpha \int_1^\infty \frac{\nu(x, 1)}{x} dx$$

□

Proof of Lemma 6.1. We have

$$\begin{aligned}
 & k\mathbb{E} \left[\int_{a(n/k)}^{X^{(k)}} \frac{n}{k} \bar{F}_X(x) \frac{dx}{x} \int_{b(n/k)}^{Y^{(k)}} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} \right] = \\
 & k\mathbb{E} \left[\left(-\frac{1}{n} \sum_{j=1}^n \frac{\mathbb{1}(U_j \leq p_k) - p_k}{a(n/k) f_X(a(n/k))} + \frac{T_n(p_k)}{a(n/k)} \right) \right. \\
 & \quad \left. \times \left(-\frac{1}{n} \sum_{j=1}^n \frac{\mathbb{1}(V_j \leq p_k) - p_k}{b(n/k) f_Y(b(n/k))} + \frac{T_n(p_k)}{b(n/k)} \right) \right] \\
 & = k\mathbb{E} \left[\frac{1}{n^2} \sum_{j=1}^n \frac{(\mathbb{1}(U_j \leq p_k) - p_k)(\mathbb{1}(V_j \leq p_k) - p_k)}{a(n/k) f_X(a(n/k)) b(n/k) f_Y(b(n/k))} \right] \\
 & \quad + \mathbb{E} \left[\frac{T_n(p_k)}{a(n/k)} \int_{b(n/k)}^{Y^{(k)}} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} \right] + \mathbb{E} \left[\frac{T_n(p_k)}{b(n/k)} \int_{a(n/k)}^{X^{(k)}} \frac{n}{k} \bar{F}_X(x) \frac{dx}{x} \right] \\
 & = k \frac{\mathbb{P}(X > a(n/k), Y > b(n/k)) - \bar{p}_k \bar{p}_k}{nb(n/k) f_X(b(n/k)) a(n/k) f_X(a(n/k))} \\
 & \quad + O \left(k \frac{n^{-3/2} (\log \log n)^{-1/2} (\log n)^{-1}}{a(n/k) b(n/k)} \right).
 \end{aligned}$$

The result follows from Lemma 6.6. □

Proof of Lemma 6.2. Using the notations above, we can write

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \mathbb{E} \left[k \left(H_{k,n}^X - \frac{1}{\alpha} \right) \int_{b(n/k)}^{Y(k)} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} \right] \\
&= \lim_{k \rightarrow \infty} \frac{k}{\alpha} \mathbb{E} \left[\left(\frac{1}{k} \sum_{i=1}^k \log \frac{X(i)}{X(k)} - \frac{1}{\alpha} \right) \int_{b(n/k)}^{Y(k)} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} \right] \\
&= \lim_{k \rightarrow \infty} \frac{k}{\alpha} \mathbb{E} \left[\left(\frac{1}{k} \sum_{i=1}^k \log X(i) - \log X(k) - \frac{1}{\alpha} \right) \int_{b(n/k)}^{Y(k)} \frac{n}{k} \bar{F}_Y(x) \frac{dx}{x} \right] \\
&= \lim_{k \rightarrow \infty} \sum_{i=1}^k M_n(i) - M_n(k) + \lim_{k \rightarrow \infty} \sum_{i=1}^k R_{n,1}(i) + R_{n,2}(i) - R_{n,1}(k) - R_{n,2}(k) \\
&= \lim_{k \rightarrow \infty} \sum_{i=1}^k M_n(i) - M_n(k) \quad \text{by Lemma 6.4.} \\
&= \lim_{n \rightarrow \infty} \frac{1}{\alpha^2} \sum_{i=1}^k \frac{n}{ki} \mathbb{P}(X > a(n/i), Y > b(n/k)) - \frac{\nu(1,1)}{\alpha^2} \quad \text{by Lemma 6.6.} \\
&= \frac{1}{\alpha} \int_1^\infty \frac{\nu(x,1)}{x} dx - \frac{\nu(1,1)}{\alpha^2} \quad \text{by Lemma 6.7.}
\end{aligned}$$

□

Lemma 6.8.

$$\int_1^\infty \int_1^\infty \frac{\nu(x,y)}{xy} dx dy - \frac{1}{\alpha} \int_1^\infty \frac{\nu(x,1)}{x} dx - \frac{1}{\alpha} \int_1^\infty \frac{\nu(1,y)}{y} dy.$$

Proof of Lemma 6.8. For $1 \leq i \neq j \leq l$, we have

$$\begin{aligned}
& \frac{1}{\alpha} \int_1^\infty \frac{\nu_{i,j}(x,1)}{x} dx = \int_1^\infty \int_y^\infty \frac{\nu_{i,j}(x,y)}{xy} dx dy \\
& \text{and } \frac{1}{\alpha} \int_1^\infty \frac{\nu_{i,j}(1,y)}{y} dy = \int_1^\infty \int_x^\infty \frac{\nu_{i,j}(x,y)}{xy} dx dy.
\end{aligned}$$

Now, just notice that

$$\int_1^\infty \int_1^\infty \frac{\nu_{i,j}(x,y)}{xy} dx dy = \int_1^\infty \int_y^\infty \frac{\nu_{i,j}(x,y)}{xy} dx dy + \int_1^\infty \int_x^\infty \frac{\nu_{i,j}(x,y)}{xy} dx dy.$$

□

7. Example - Technical details

We first treat the case $l = 2$. We set $F_1 = G$, $F_2 = H$, $a^{(1)}(n/k) = a_k$ and $a^{(2)}(n/k) = b_k$

We have $\overline{F}(a_k x, b_k y) = \overline{G}(a_k x) + \overline{H}(b_k y) - 1 + C_\nu \left(G(a_k x), H(b_k y) \right)$, and

$$\begin{aligned} & \frac{n}{k} C_\nu \left(G(a_k x), H(b_k y) \right) - \frac{n}{k} \\ &= \left(\left(\frac{n}{k} \overline{G}(a_k x) \right)^\nu + \left(\frac{n}{k} \overline{H}(b_k y) \right)^\nu \right)^{1/\nu} + O \left(\left(\frac{n}{k} \overline{G}(a_k x) \right)^\nu + \left(\frac{n}{k} \overline{H}(b_k y) \right)^\nu \right)^{1/\nu} \\ &= \left(\left(\frac{n}{k} \overline{G}(a_k x) - x^{-\alpha} + x^{-\alpha} \right)^\nu + \left(\frac{n}{k} \overline{H}(b_k y) + y^{-\alpha} - y^{-\alpha} \right)^\nu \right)^{1/\nu} \\ &\quad + O \left(\left(\frac{n}{k} \overline{G}(a_k x) \right)^\nu + \left(\frac{n}{k} \overline{H}(b_k y) \right)^\nu \right)^{1/\nu} \\ &= \left(x^{-\nu\alpha} \left(1 + \frac{\frac{n}{k} \overline{G}(a_k x) - x^{-\alpha}}{x^{-\alpha}} \right)^\nu + \frac{1}{y^{\alpha\nu}} \left(1 + \frac{\frac{n}{k} \overline{H}(b_k y) - y^{-\alpha}}{y^{-\alpha}} \right)^\nu \right)^{1/\nu} \\ &= (x^{-\nu\alpha} + y^{-\nu\alpha})^{1/\nu} + O \left(\frac{\frac{n}{k} \overline{G}(a_k x) - x^{-\alpha}}{x^{-\alpha}} + \frac{\frac{n}{k} \overline{H}(b_k y) - y^{-\alpha}}{y^{-\alpha}} \right) \end{aligned}$$

This gives

$$\sup_{x, y > 0} \left| \frac{n}{k} \overline{F}(a_k x, b_k y) - \nu(x, y) \right| = O \left(\sup_{x > 0} \left| \overline{G}(a_k x) - \frac{1}{x^\alpha} \right| + \sup_{y > 0} \left| \overline{H}(b_k y) - \frac{1}{y^\alpha} \right| \right).$$

Now, in the general case, it can easily be shown that for $x_i > 0$, $i = 1, \dots, d$, we have

$$\nu(x_1, \dots, x_l) = \sum_{k=1}^l \sum_{1 \leq i_1 \neq \dots \neq i_k \leq l} (-1)^{k+1} (x_{i_1}^{-\alpha\nu} + \dots + x_{i_k}^{-\alpha\nu})^\nu.$$

It follows that for any $1 \leq i \neq j \leq l$, we have:

$$\nu_{i,j}(x, y) = (x^{-\nu\alpha} + y^{-\nu\alpha})^{1/\nu}.$$

8. Extensions to alternative tail index estimation methods

We now give an insight into the way the BEAR procedure can be generalized to alternative local estimators of the tail index.

Given a sample X_1, \dots, X_n with distribution in $\mathcal{RV}_{-\alpha}$, for some $\alpha > 0$, and the related order statistics $X(1) > \dots > X(n)$, citepDekkers89 introduced the moment estimator $M_{k,n}$ defined as

$$M_{k,n} = H_{k,n} + 1 - \frac{1}{2} \left(1 - \frac{H_{k,n}^2}{L_{k,n}} \right)^{-1}, \quad (8.1)$$

where

$$L_{k,n} = \frac{1}{k} \sum_{i=1}^k \log^2 \left(\frac{X(i)}{X(k+1)} \right). \quad (8.2)$$

Note that $L_{n,k}$ is an estimator of $2/\alpha^2$.
citepVries1996 also introduced the estimator $J_{k,n}$ defined as

$$J_{k,n} = \frac{L_{k,n}}{2H_{k,n}} \quad (8.3)$$

This estimator was used in
citepDanielsson2001 to derive the optimal number k of upper order statistics through a bootstrap method.

A Central Limit Theorem for $L_{k,n}$ and $J_{k,n}$. We use the same notations as those introduced at the beginning of section 3 and consider a l -dimensional vector \mathbf{X} of regularly varying margins with index $-\alpha$. Following steps by step the proofs of the main results of section 3, one may adapt them to obtain a multivariate CLT for $(L_{k,n}^{(1)}, \dots, L_{k,n}^{(l)})$ and for $(J_{k,n}^{(1)}, \dots, J_{k,n}^{(l)})$. First, replacing integrals $\int_1^\infty \cdot \frac{dx}{x}$ in Eq.(6.2) by $2 \int_1^\infty \cdot \log x \frac{dx}{x}$ yields:

$$\begin{aligned} \sqrt{k} \left(L_{k,n}^{(1)} - 2 \int_1^\infty \frac{n}{k} \bar{F}_1(X_1(k)x) \log x \frac{dx}{x}, \dots, \right. \\ \left. L_{k,n}^{(l)} - 2 \int_1^\infty \frac{n}{k} \bar{F}_Y(Y(k)y) \log y \frac{dy}{y} \right) \Rightarrow K_2. \end{aligned} \quad (8.4)$$

Then, under Condition (3.6), the random centering can be removed. The variant of Theorem 3.11 follows:

$$\sqrt{k} \left(L_{k,n}^{(1)} - \frac{2}{\alpha^2}, \dots, L_{k,n}^{(l)} - \frac{1}{\alpha} \right) \Rightarrow \mathcal{N}(0, \Omega_2). \quad (8.5)$$

More generally

$$\sqrt{k} \left(I_{k,n}^{(1)} - \alpha^{(1)}, \dots, I_{k,n}^{(l)} - \alpha^{(l)} \right) \Rightarrow \mathcal{N}(0, \Omega_I), \quad (8.6)$$

where, for any $1 \leq i \leq l$, $I_{k,n}^{(i)}$ can be either $L_{k,n}^{(i)}$ with $\alpha_i = 2/\alpha^2$ or $H_{k,n}^{(i)}$ with $\alpha_i = 1/\alpha$. The generalization of Theorem 3.11 for $J_{k,n}$ can be established by means of the Delta method. Indeed,

$$\begin{aligned} J_{k,n} &= \frac{L_{k,n}}{2H_{k,n}} = \frac{L_{k,n} - \frac{2}{\alpha^2} + \frac{2}{\alpha^2}}{2H_{k,n} - \frac{2}{\alpha} + \frac{2}{\alpha}} = \frac{1}{\alpha} \frac{1 + \left(\frac{\alpha^2}{2} L_{k,n} - 1 \right)}{1 + \left(\alpha H_{k,n} - 1 \right)} \\ &= \frac{1}{\alpha} \left(1 + \left(\frac{\alpha^2}{2} L_{k,n} - 1 \right) \right) \left(1 - \left(\alpha H_{k,n} - 1 \right) + o \left(\left(\alpha H_{k,n} - 1 \right) \right) \right) \quad a.s.. \end{aligned}$$

Hence,

$$J_{k,n} = \frac{1}{\alpha} + \frac{2}{\alpha} \left(L_{k,n} - \frac{2}{\alpha^2} \right) - \left(H_{k,n} - \frac{1}{\alpha} \right) + o \left(\left(\alpha H_{k,n} - 1 \right) \right) \quad a.s. \quad (8.7)$$

For any $(i, j) \in \{1, \dots, l\}^2$, set:

- $C_{LH}(i, j) = \lim_{k \rightarrow \infty} \text{Cov} \left(\sqrt{k} \left(L_{k,n}^{(i)} - \frac{2}{\alpha^2} \right), \sqrt{k} \left(H_{k,n}^{(j)} - \frac{1}{\alpha} \right) \right),$
- $C_{LL}(i, j) = \lim_{k \rightarrow \infty} \text{Cov} \left(\sqrt{k} \left(L_{k,n}^{(i)} - \frac{2}{\alpha^2} \right), \sqrt{k} \left(L_{k,n}^{(j)} - \frac{2}{\alpha^2} \right) \right),$
- $C_{HH}(i, j) = \lim_{k \rightarrow \infty} \text{Cov} \left(\sqrt{k} \left(H_{k,n}^{(i)} - \frac{1}{\alpha} \right), \sqrt{k} \left(H_{k,n}^{(j)} - \frac{1}{\alpha} \right) \right).$

We then have:

$$\lim_{k \rightarrow \infty} \text{Cov} \left(\sqrt{k} \left(J_{k,n}^{(i)} - \frac{1}{\alpha} \right), \sqrt{k} \left(J_{k,n}^{(j)} - \frac{1}{\alpha} \right) \right) = \frac{\alpha^2}{4} C_{LL}(i, j) - \frac{\alpha}{2} C_{LH}(i, j) - \frac{\alpha}{2} C_{LH}(j, i) + C_{HH}(i, j). \quad (8.8)$$

The same kind of relation holds for the asymptotic covariance of the Moment estimator $\lim_{k \rightarrow \infty} \text{Cov} \left(\sqrt{k} \left(M_{k,n}^{(i)} - \frac{1}{\alpha} \right), \sqrt{k} \left(M_{k,n}^{(j)} - \frac{1}{\alpha} \right) \right)$. Hence, the BEAR method can be straightforwardly generalized to a combination of local Moment estimators (respectively J -estimators).

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