Microstructural effects in elastic composites

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MICROSTRUCTURAL EFFECTS IN ELASTIC COMPOSITES

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Abstract—In this paper, the static microstructural effects of periodic elastic composites were studied by the homogenization method. This approach is based on the analysis of the momentum balance equations which appear at higher orders. The physical meaning of the corrective terms due to the existence of heterogeneities is discussed in detail. We show that the higher terms introduce the successive gradients of macroscopic strain and tensors characteristic of the microstructure, which result in non-local effects. The boundary conditions for solving problems up to third-order are also given. This analysis is used to define a kinematic criterion for the occurrence of these microstructural effects, and a procedure to assess them. The obtained macroscopic description constitutes a generalization of the second gradient theory, but it is not in agreement with the mechanics of Cosserat media. Moreover, these results can be used to approach the *emergence* of localization phenomenon. Finally, an application of the method is given using periodically stratified composites as an example.

1. INTRODUCTION

The materials used in civil engineering (concrete, rocks, steel, etc.) or in industry (composites, metals, etc.) often present a structure in which different components are assembled, therefore leading to a heterogeneous material. However the design rules of the buildings made with these materials are essentially issued from the mechanics of continuous media applied to homogeneous media. The good mechanical behavior of many constructions built according to these principles proves that this method has stood the test of time. This simple remark led us to think that, under certain conditions, these heterogeneous materials can be regarded as homogeneous continuous media.

The homogenization method of periodic media developed by Sanchez-Palencia (1980), proves that a heterogeneous medium can behave as a homogeneous material in compliance with the equations of continuous media, provided that macroscopic size, L, is infinitely large in comparison with size l of its heterogeneities ($\varepsilon = l/L \to 0$). However, in reality, this limit is never reached because the microstructure size does not equal zero ($l \neq 0$) and the macroscopic size characteristic of the load applied to the material does not give an infinite value ($L \neq \infty$). For these reasons, it has become interesting to investigate how the description of the medium deviates from the homogenized behavior which complies with standard continuum mechanics.

This study deals with macroscopic phenomena whose characteristic size L *is large, but not very large*, with respect to size l. Under static loading, these phenomena can be observed in samples constituted of few heterogeneities, in cases of stress concentration, for example at the vicinity of a load or near an angular boundary, and also during the formation of a shear band (Fig. 1). In dynamics, this situation corresponds to wavelengths which are about 10–100 times greater than heterogeneities.

In this article, we will consider elastic periodic composites submitted to static loadings. Although the elastic behavior is very simple, this example is instructive since it allows one to clearly understand the influence of the microstructure on overall behavior in determining under which conditions this influence is negligible, and, on the other hand, when there is an influence, to show how it modifies the response of the material.

The homogenization method which, according to its principles, takes into account the existence of a microstructure, is particularly well adapted for this type of analysis. This

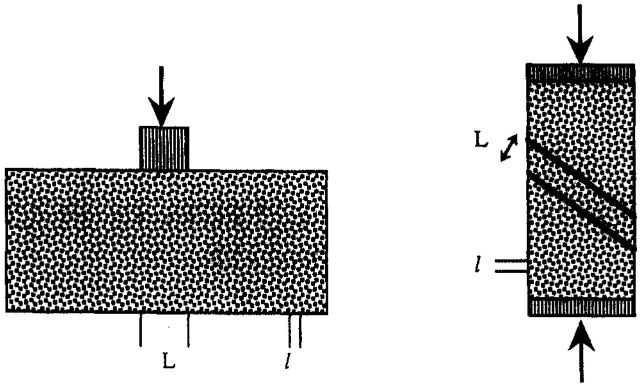

Fig. 1. Cases where the macroscopic size L is large but not very large in comparison with l.

technique of asymptotic expansion allows description improvement by exploiting the terms of higher orders and considering their role in macroscopic behavior. Such an approach, which, contrary to the common process, is not limited to the first significant terms, has already been proposed by Gambin and Kröner (1989) and Boutin (1993) in statics, and also developed in dynamics by Boutin (1991) and Boutin and Auriault (1993).

In the second part of this article we recall the basic principles of the homogenization method. Then, the equations governing the macroscopic static behavior of an elastic composite are presented, up to third-order. In the third chapter, the results obtained are discussed, and their interest in the study of microstructural effects is presented. Then, a comparison of the homogenization method with two other microstructural approaches—second gradient and Cosserat media theories—is given, and the application of the method to the study of the emergence of localization is discussed. Finally, in the fourth section this approach is applied to periodic stratified composites.

2. HOMOGENIZATION OF ELASTIC COMPOSITES (TO THE THIRD ORDER)

Before going further into the details of the method, let us first present some general considerations about homogenization.

The aim of any homogenization method is to give a description of a phenomenon in a heterogeneous material, where the local fluctuations due to the heterogeneities do not appear explicitly. Obviously the possibility of such a description depends on the studied case. In which condition is it, then, relevant to employ a homogenized description?

First of all, there is no way to obtain such a description if the studied phenomenon essentially varies at the level of the heterogeneities. For this reason, a basic reason to have a homogenized description is that the phenomenon actually presents a characteristic size of evolution, L, much larger than that of the heterogeneities. This means that one variable (at least) related to the phenomenon varies according to a macroscopic length L. Thus, the research of a homogenized behavior makes sense only if the following two conditions of *scale separation* are fulfilled. The first concerns the material, which must be sufficiently regular so that one can define a characteristic microscopic size l. The second is related to the phenomenon for which the characteristic size of its variations must be large in comparison with l. From these conditions it results that "some invariance property" has to be fulfilled at the local scale by *both* material and phenomenon.

For the material, the simultaneous existence of a characteristic size and a local invariance, means that it is possible to define a *representative elementary volume*. Practically, it is difficult to propose a rigorous definition of such a volume, likewise the definition of the particle in continuum mechanics is not strict. In fact there is some similarity between these two notions since both have to be:

- not too small
  - to obey the continuity assumption (continuum mechanics)
  - to be representative (homogenization);

- not too large
  - to be considered as infinitesimal (continuum mechanics)
  - to respect the scale separation (homogenization).

However there is a kind of material, namely periodic material, for which this difficulty disappears since the period defines the representative elementary volume without any ambiguity. In this case the "invariance property" is expressed by the periodicity of variables.

In the same way, there is no strict definition of macroscopic length. Often this size is estimated from the typical size of the overall volume studied, or by the size of the tested sample. However, with respect to the problem in consideration, L can also be determined by the boundary conditions or by a size related to the physics of the phenomenon (like wavelength, thickness of diffusion layer,...). As a matter of fact, a first assessment of L can be deduced from the dimensional analysis.

The conditions of "homogenizability" necessarily induce some specificities for homogenizable situations.

A priori, the behavior of a heterogeneous material depends on the whole of the heterogeneities. However the existence of an elementary representative volume allows, by definition, one to extrapolate the properties of this volume to those of the material. Moreover, since a characteristic variable of the phenomenon varies at the macroscopic scale, the gradients of this quantity Q are constants close to zero at the level of the representative volume. So, under homogenizable situations, the solicitation applied to the representative volume is not arbitrary. Consequently, it is not necessary to know all the properties of the representative volume, but only those related to forced homogeneous solicitations in gradients of Q.

In other words, some of the properties of the representative volume do not intervene, and therefore cannot be put into evidence, under homogenizable situations. This remark allows us to understand more clearly why the response of a material can be significantly different in homogenizable and non-homogenizable situations.

Finally we note that a leading hypothesis is the *local invariance*. In fact the periodicity is only a simple and efficient way to express this assumption. For this reason, when considering the same physics at microscopic level, the macroscopic description will be of the *same kind* whether the microstructure is periodic or not. As already underlined by Auriault (1991) the *structure* of the equations will be the same, but will have different coefficients. This point, which has been experimentally confirmed for numerous phenomena (Darcy's law, suspension's viscosities, thermal and elastic coefficients, etc.) allows, in some extent, the use of the periodicity assumption in order to analyze real materials.

It is important to note that these considerations are only valid for homogenizable situations. Obviously, when there is no scale separation, the phenomenon is essentially determined by the local organization, so that descriptions for periodic or disordered media are structurally different. Experiments such as high frequency wave diffraction are also in agreement with this intuitive idea.

2.1. *Principles of the homogenization method*

In this chapter we will describe the main principles of this method which is now commonly used to solve multiple scale problems. Basically, it is an asymptotic method with double variables, initially developed by Benssoussan *et al.* (1978) and Sanchez–Palencia (1980).

The microscopic scale, described by variable y, is determined by the microstructure, having l as a characteristic length. Variable x is associated with the macroscopic scale, the characteristic length, L, of which being determined by the loading or the boundary conditions applied to the material. The small parameter ε is defined by the ratio of the two scales:

$$\varepsilon = l/L \quad y = \varepsilon^{-1}x.$$

The use of the two variables x and y leads us to transform the common spatial derivatives into $\partial_x + \varepsilon^{-1}\partial_y$ and to put the variables in the form of asymptotic expansions in power of ε. For example, for the displacements:

$$u(x, y) = \sum \varepsilon^i u^i(x, y) \quad \text{with: } O(u^i) = u^0.$$

Moreover, the periodicity of the microstructure induces the same periodicity of the functions u^i according to variable y. The homogenization process consists of, expressing in the form of expansions the equations governing the physics at local scale, identifying the terms with the same power of ε, and finally, solving the problems obtained in series.

In principle, this method of asymptotic expansions is all the more reliable as ε is small in comparison with l, that is, when the separation of scales is clear. In this case, the description obtained at the first significant order corresponds to the macroscopic behavior of the material, *to an accuracy of* ε. Note that for a given medium, length l is fixed, and consequently, the real behavior is all the better described by the homogenized behavior the larger L is in comparison with l.

2.2. *Homogenization applied to elastic composites*

A number of studies, such as Sanchez–Palencia (1980) and Auriault and Bonnet (1985) address homogenization of elastic composites under static or dynamic loadings, according to period geometry, the contrasts of the mechanical properties of the period components etc. In these works, the macroscopic description is given by the first significant order obtained in the homogenization process. The determination of higher orders was initiated for the elastostatics of composites by Gambin and Kröner (1989) who gave their expressions without analysis of the physical insight. Following this work the developments at higher orders were derived and their physical meanings analyzed, for dynamic cases Boutin (1991) and Boutin and Auriault (1993) for static cases, Boutin (1993), and for thermal conduction, Boutin (1995).

2.2.1. *The set of problems to be solved.* We consider a finely heterogeneous material for which the elastic tensor varies locally according to a period Ω (Fig. 2). Under static conditions, a field of volume forces of the type $\rho(y)f^0(x)$ [where $\rho(y)$ is the density of the cell components] is applied to the material. The problem is governed by the following equations:

$$\nabla . \sigma + \rho f^0 = 0 \quad \boldsymbol{\sigma} = \mathbf{c}\,..\,\mathbf{e}(u),$$

where u is the displacement field, $\boldsymbol{\sigma}$ the stress tensor, $\mathbf{e}(u)$ the small strain tensor and $\mathbf{c}$ the elasticity tensor. The dot means contraction (.. double contraction, etc.) and ∇ is the Nabla

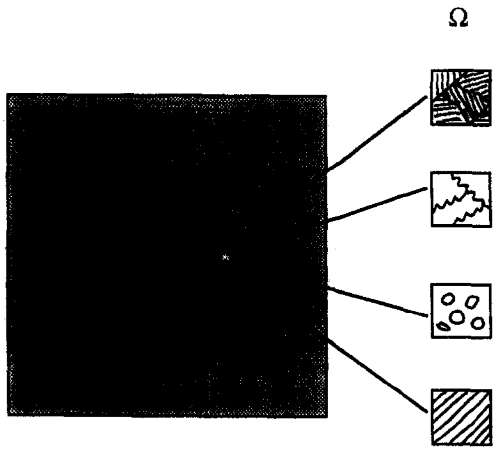

Fig. 2. Microheterogeneous media with different kind of local structure (crystalline, fissured, porous, composite).

operator. If the variations of $\mathbf{c}$ are not continuous, these equations have to be taken in the sense of the distributions, and u and $\boldsymbol{\sigma}.n$ must be continuous through the discontinuity surfaces (having n as normal).

The existence of two well distinct scales is expressed by using the system of double variables x and y. Then the stress tensor takes the following form:

$$\boldsymbol{\sigma} = \mathbf{c}..[\mathbf{e}_x(u)+\varepsilon^{-1}\mathbf{e}_y(u)],$$

where $\mathbf{e}_y$ and $\mathbf{e}_x$ are the strain tensors calculated according to the variables y and x, respectively. The momentum balance is thus written:

$$\varepsilon^{-1}\nabla_y.\boldsymbol{\sigma}+\nabla_x.\sigma+\rho f^0 = 0.$$

Using the displacement fields only, the initial problem is thus transformed as follows:

$$\varepsilon^{-2}L^{-2}(u)+\varepsilon^{-1}L^{-1}(u)+L^0(u)+\rho f^0 = 0,$$

where

$$\begin{aligned} L^{-2}(u) &= \nabla_y.[\mathbf{c}..\mathbf{e}_y(u)] \\ L^{-1}(u) &= \nabla_y.[\mathbf{c}..\mathbf{e}_x(u)]+\nabla_x.[\mathbf{c}..\mathbf{e}_y(u)] \\ L^0(u) &= \nabla_x.[\mathbf{c}..\mathbf{e}_x(u)]. \end{aligned}$$

When introducing into these differential equations asymptotic expansions, the following problems are to be solved in series:

$$\begin{aligned}
&L^{-2}(u^0) = 0 && \text{i.e.: } \nabla_y.[\sigma^{-1}] = 0 && \text{(1a)}\\
&L^{-2}(u^1) = -L^{-1}(u^0) && \nabla_y.[\sigma^0] = -\nabla_x.[\sigma^{-1}] && \text{(1b)}\\
&L^{-2}(u^2) = -L^{-1}(u^1)-L^0(u^0)-\rho f^0 && \nabla_y.[\sigma^1] = -\nabla_x.[\sigma^0]-\rho f^0 && \text{(1c)}\\
&L^{-2}(u^3) = -L^{-1}(u^2)-L^0(u^1) && \nabla_y.[\sigma^2] = -\nabla_x.[\sigma^1] && \text{(1d)}\\
&L^{-2}(u^4) = -L^{-1}(u^3)-L^0(u^2) && \nabla_y.[\sigma^3] = -\nabla_x.[\sigma^2]. && \text{(1e)}
\end{aligned}$$

2.2.2. *Solution.* Only general ideas are given below. The detailed solutions of the problems for successive orders, close to that proposed by Gambin and Kröner (1989) and Boutin and Auriault (1993), *but with different volume forces*, are given in Appendix A. All the problems to be solved successively address the search for Ω-periodic displacements fields v, such as:

$$\nabla_y.[\sigma^{i+1}] = -\nabla_x.[\sigma^i]-F^i \quad \text{with} \quad \boldsymbol{\sigma}^{i+1} = \mathbf{c}..[\mathbf{e}_y(v)+\mathbf{E}^i] \tag{2}$$

where the volume force distributions are known ($F^0 = \rho f^0$ and $F^i = 0$ for $i > 0$) and tensors $\boldsymbol{\sigma}^i$ and $\mathbf{E}^i$ have already been determined by the previous problems.

Because of the periodicity of $\boldsymbol{\sigma}^{i+1}$, a first condition (said "of compatibility") can be obtained directly by integrating eqn (2) over the period:

$$\int_\Omega \nabla_y.[\sigma^{i+1}]\,\mathrm{d}v = \int_{\partial\Omega}[\sigma^{i+1}].n\,\mathrm{d}s = 0 = -\int_\Omega(\nabla_x.[\sigma^i]+F^i)\,\mathrm{d}v$$

i.e.:

$$\nabla_x . [\langle \sigma^i \rangle] + F^i = 0 \quad \text{with} \quad \langle . \rangle = |\Omega|^{-1} \int_\Omega . \, dv.$$

These are fundamental equations, since they involve only the macroscopic variable x, and express the balance of the forces of order ε^i acting on the cell. This compatibility condition being established, the determination of Ω-periodic field $v(x, y)$ is done by using the variational formulation of the problem. To avoid indetermination due to fields of rigid translation $V(x)$, we then look for the solution in the vectorial space W defined by:

$$W = \{w/w \ \Omega\text{-periodic}, \langle w \rangle = 0\}.$$

Taking the scalar product of eqn (2) by any test field w and integrating over Ω, we get:

$$\begin{aligned} \int_\Omega \nabla_y . [\sigma^{i+1}] . w \, dv &= -\int_\Omega \sigma^{i+1} .. \mathbf{e}_y(w) \, dv + \int_{\partial\Omega} [\sigma^{i+1}] . n . w \, ds \\ &= -\int_\Omega (\nabla_x . [\sigma^i] + F^i) . w \, dv. \end{aligned}$$

The boundary term being zero by periodicity, we obtain after introducing the compatibility equation:

$$\forall w \in W \quad \int_\Omega \mathbf{c} .. \mathbf{e}_y(v) .. \mathbf{e}_y(w) \, dv = \int_\Omega \nabla_x . [\langle \sigma^i \rangle - \sigma^i] . w \, dv + \int_\Omega \mathbf{c} .. \mathbf{E}^i .. \mathbf{e}(w) \, dv,$$

$\mathbf{c}$ satisfies an ellipticity condition since it is an elastic tensor. Therefore the Lax–Milgram lemma ensures the existence and uniqueness of a field v_w of W, solution which linearly depends on the forcing terms. The general solution is obtained by adding to v_w any rigid translation $V(x)$.

2.2.3. *Results*. The main results of Appendix A are the following.
Displacements. The displacement fields can be expressed in the form:

$$\begin{aligned} u(x,y) = {} & U^0(x) \\ & + \varepsilon[U^1(x) + \mathbf{X}(y) .. \mathbf{e}_x(U^0)] \\ & + \varepsilon^2[U^2(x) + \mathbf{X}(y) .. \mathbf{e}_x(U^1) + \mathbf{Y}(y) ... \nabla_x \mathbf{e}_x(U^0)] \\ & + \varepsilon^3[U^3(x) + \mathbf{X}(y) .. \mathbf{e}_x(U^2) + \mathbf{Y}(y) ... \nabla_x \mathbf{e}_x(U^1) + \mathbf{Z}(y) .... \nabla_x \nabla_x \mathbf{e}_x(U^0)]. \end{aligned}$$

Tensors $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ of rank 3, 4 and 5, respectively, are obtained from particular solutions v_w. Those latter having a zero average, the mean displacement $U(x)$ is given by:

$$\langle u(x,y) \rangle = U(x) = U^0(x) + \varepsilon U^1(x) + \varepsilon^2 U^2(x) + \varepsilon^3 U^3(x) + \cdots$$

Strains. We denote by ${}^s\nabla_x$ and ${}^s\nabla_y$ the symmetrized gradients calculated according to x and y. The field of strain is in the form:

$$\begin{aligned} \mathbf{e}_{(x,y)} = {} & \mathbf{e}_x(U^0) + {}^s\nabla_y[\mathbf{X}(y) .. \mathbf{e}_x(U^0)] \\ & + \varepsilon\{\mathbf{e}_x(U^1) + {}^s\nabla_y[\mathbf{X}(y) .. \mathbf{e}_x(U^1)] + {}^s\nabla_x[\mathbf{X}(y) .. \mathbf{e}_x(U^0)] + {}^s\nabla_y[\mathbf{Y}(y) ... \nabla_x \mathbf{e}_x(U^0)]\} \\ & + \varepsilon^2\{\mathbf{e}_x(U^2) + {}^s\nabla_y[\mathbf{X}(y) .. \mathbf{e}_x(U^2)] + {}^s\nabla_x[\mathbf{X}(y) .. \mathbf{e}_x(U^1)] + {}^s\nabla_y[\mathbf{Y}(y) ... \nabla_x \mathbf{e}_x(U^1)] \\ & + {}^s\nabla_x[\mathbf{Y}(y) ... \nabla_x \mathbf{e}_x(U^0)] + {}^s\nabla_y[\mathbf{Z}(y) .... \nabla_x \nabla_x \mathbf{e}_x(U^0)]\} + \cdots \end{aligned}$$

As tensors **X**, **Y**, **Z** are periodic with a zero mean value, we obtain by averaging:

$$\langle \mathbf{e}_{(x,y)} \rangle = \mathbf{e}_x(U^0) + \mathbf{e}_x(\varepsilon U^1) + \mathbf{e}_x(\varepsilon^2 U^2) + \mathbf{e}_x(\varepsilon^3 U^3) + \cdots = \mathbf{e}_x(U).$$

This means that the *mean strain is the strain of the mean displacement.*

Stresses. The tensor of local stresses is directly calculated from $\mathbf{c}(y)$ and $\mathbf{e}(x, y)$:

$$\boldsymbol{\sigma}(x, y) = \mathbf{c}(y) \mathbin{..} \mathbf{e}(x, y).$$

The averaged values of these stresses allow us to define tensors $\mathbf{C}^0$, $\varepsilon\mathbf{C}^1$, $\varepsilon^2\mathbf{C}^2$, which characterize the macroscopic behavior:

$$\begin{aligned}\langle \boldsymbol{\sigma}(x, y) \rangle = {} & \mathbf{C}^0 \mathbin{..} \mathbf{e}_x(U^0) \\ & + \mathbf{C}^0 \mathbin{..} \mathbf{e}_x(\varepsilon U^1) + \varepsilon \mathbf{C}^1 \mathbin{...} \nabla_x \mathbf{e}_x(U^0) \\ & + \mathbf{C}^0 \mathbin{..} \mathbf{e}_x(\varepsilon^2 U^2) + \varepsilon \mathbf{C}^1 \mathbin{...} \nabla_x \mathbf{e}_x(\varepsilon U^1) + \varepsilon^2 \mathbf{C}^2 \mathbin{....} \nabla_x \nabla_x \mathbf{e}_x(U^0),\end{aligned}$$

where, using formal writing,

$$\mathbf{C}^0 = |\Omega|^{-1} \int_\Omega (\mathbf{c} + \mathbf{c} \mathbin{..} {}^s\nabla_y(\mathbf{X}))\, dv: \quad \text{4th rank tensor} \tag{3a}$$

$$\mathbf{C}^1 = |\Omega|^{-1} \int_\Omega (\mathbf{c} \cdot \mathbf{X} + \mathbf{c} \mathbin{..} {}^s\nabla_y(\mathbf{Y}))\, dv: \quad \text{5th rank tensor} \tag{3b}$$

$$\mathbf{C}^2 = |\Omega|^{-1} \int_\Omega (\mathbf{c} \cdot \mathbf{Y} + \mathbf{c} \mathbin{..} {}^s\nabla_y(\mathbf{Z}))\, dv: \quad \text{6th rank tensor.} \tag{3c}$$

The expressions of $\mathbf{C}^1$ and $\mathbf{C}^2$ given by eqns (3b, 3c) show that:

$$\mathbf{C}^1 = O(\mathbf{c}\mathrm{l}_m) \quad \text{and} \quad \mathbf{C}^2 = O(\mathbf{c}\mathrm{l}_m^2),$$

where l_m is the dimension of the period expressed according to the *system of dilated variables* y. Consequently:

$$\varepsilon\mathbf{C}^1 = O(\mathbf{c}l) \quad \text{and} \quad \varepsilon^2\mathbf{C}^2 = O(\mathbf{c}l^2),$$

where l is the dimension of the cell expressed in the *system of reference variables* x. Therefore, in macroscopic equations, we necessarily have to use the effective tensors $\mathbf{C}' = \varepsilon\mathbf{C}^1$ and $\mathbf{C}'' = \varepsilon^2\mathbf{C}^2$, which can be directly calculated from known geometry and mechanical characteristics of period components, *independently of* ε.

It should be noted that the closer the period comes being homogeneous, the more the values of $\mathbf{C}'$ and $\mathbf{C}''$ are small. Finally, note that tensor $\mathbf{C}'$ is of odd rank and therefore anisotropic. It results that if the material is macroscopically isotropic (up to the second order), $\mathbf{C}' = 0$.

Momentum balance. At the three first significant orders, the macroscopic momentum balances are given by:

$$\nabla_x \cdot [\langle \sigma^0 \rangle] + \langle \rho \rangle f^0 = 0; \quad \langle \sigma^0 \rangle = \mathbf{C}^0 \mathbin{..} \mathbf{e}_x(U^0) \tag{4a}$$

$$\nabla_x \cdot [\varepsilon \langle \sigma^1 \rangle] = 0 \quad \langle \varepsilon\sigma^1 \rangle = \mathbf{C}^0 \mathbin{..} \mathbf{e}_x(\varepsilon U^1) + \varepsilon \mathbf{C}^1 \mathbin{...} \nabla_x \mathbf{e}_x(U^0) \tag{4b}$$

$$\nabla_x \cdot [\varepsilon^2\langle\sigma^2\rangle] = 0 \quad \varepsilon^2\langle\sigma^2\rangle = \mathbf{C}^0 \,.\,.\, \mathbf{e}_x(\varepsilon^2 U^2) + \varepsilon\mathbf{C}^1 \ldots \nabla_x \mathbf{e}_x(\varepsilon U^1) + \varepsilon^2\mathbf{C}^2 \ldots . \nabla_x \nabla_x \mathbf{e}_x(U^0). \tag{4c}$$

The variables used in the macroscopic description are the volume averages of variables defined at the microscopic scale. Their physical meaning should thus be specified.

2.2.4. *Physical meaning of macroscopic displacements and strains.* The macroscopic displacements of any order $U^i(x)$ are volume averages of local fields $u^i(x, y)$. These displacements $U^i(x)$ can therefore be interpreted as the translation of geometric center G of the cell ($OG = |\Omega|^{-1}\int_\Omega OM \,.\, \mathrm{d}v$). It should be noted that the whole period is translated without local rotation.

There is no difficulty in interpreting macroscopic strains of order i, since they correspond to strains associated with macroscopic displacements of the same order i.

2.2.5. *Physical meaning of averaged stresses.* The homogenization introduces *volume averages* of local stresses at order i. Do the quantities thus defined correspond to the usual physical definition of stresses which are fluxes of forces obtained through *surface averaging* of elementary forces?

In order to examine this point, we transform volume integrals into surface integrals, using the identity:

$$\partial(\boldsymbol{\sigma}_{ij} \,.\, y_k)/\partial y_j = y_k \,.\, \partial(\boldsymbol{\sigma}_{ij})/\partial y_j + \boldsymbol{\sigma}_{ik},$$

i.e. by integrating over a volume V and using the divergence theorem,

$$\int_{\partial V} \boldsymbol{\sigma}_{ij} \,.\, y_k \,.\, n_j \,\mathrm{d}s = -\int_V (\nabla_y \,.\, [\boldsymbol{\sigma}])_i \,.\, y_k \,\mathrm{d}v + \int_V \boldsymbol{\sigma}_{ik} \,\mathrm{d}v. \tag{5}$$

Stress of zero order. Stress $\boldsymbol{\sigma}^0$ is of zero divergence (according to y), and therefore when applying eqn (5) to the cell Ω (Fig. 3), we get:

$$\langle\boldsymbol{\sigma}^0_{ik}\rangle = |\Omega|^{-1} \int_{\partial\Omega} \boldsymbol{\sigma}^0_{ij} \,.\, y_k \,.\, n_j \,\mathrm{d}s \tag{6}$$

Considering the periodicity of $\boldsymbol{\sigma}^0$, only the integrals on the boundaries S_{+k} and S_{-k} (where normal $n = \pm\mathbf{e}_k$) are not equal to zero, the other ones equating to zero two by two:

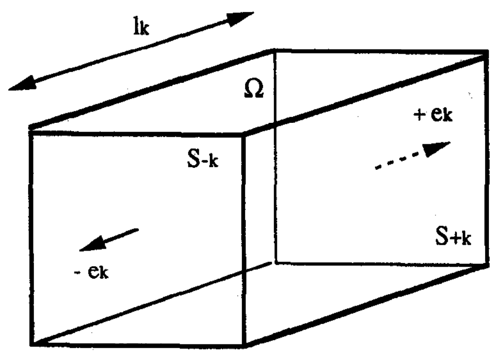

Fig. 3. Period Ω and surfaces $S_{\pm k}$.

$$\langle\sigma_{ik}^0\rangle = |\Omega|^{-1}\int_{S_{+k}} \sigma_{ij}^0 \,.\, y_k \,.\, \delta_{jk}\, ds - |\Omega|^{-1}\int_{S_{-k}} \sigma_{ij}^0 \,.\, y_k \,.\, \delta_{jk}\, ds$$

l_k being the period length in $\mathbf{e}_k$ direction, we have $|\Omega| = l_k \cdot S_{+k}$. Using the periodicity of $\boldsymbol{\sigma}^0$ we finally get:

$$\langle\boldsymbol{\sigma}_{ik}^0\rangle = l_k \,.\, |\Omega|^{-1}\int_{S_{+k}} \boldsymbol{\sigma}_{ik}^0\, ds = |S_{+k}|^{-1}\int_{S_{+k}} \boldsymbol{\sigma}_{ik}^0\, ds.$$

Thus, the volume average $\langle\boldsymbol{\sigma}_{ik}^0\rangle$ is actually the surface average of elementary forces directed according to $\mathbf{e}_i$ acting on the face having $\mathbf{e}_k$ as normal.

This result can be generalized to stresses applied to any oriented surface $|S| \cdot \mathbf{m} = \mathbf{A} \wedge \mathbf{B}$, where vectors $\mathbf{A}$ and $\mathbf{B}$ are linear combinations—with integer coefficients—of the vectors defining the elementary cell. Let us prove that the vector $\mathbf{T}$ obtained by surface averaging of elementary forces acting on S:

$$\mathbf{T}(\mathbf{m}) = |S|^{-1}\int_S \boldsymbol{\sigma}^0 \,.\, \mathbf{m}\, dS \quad \text{is equal to} \quad \langle\boldsymbol{\sigma}^0\rangle \,.\, \mathbf{m}.$$

In this aim, we consider the cell Ω' (Fig. 4), defined by $(\mathbf{A}, \mathbf{B}, \mathbf{m}l')$, the distance l' between the two the faces of normal $\pm\mathbf{m}$ being such that: $l' = |\Omega'|/S = l_j(\mathbf{e}_j \cdot \mathbf{m})$ (no summation over j). When applying eqn (5) to Ω' and taking the scalar product by m, one obtains:

$$\int_{\Omega'} \boldsymbol{\sigma}^0 \,.\, \mathbf{m}\, dv = \int_{\partial\Omega'} (\boldsymbol{\sigma}^0 \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{m})\, ds.$$

From the definition of Ω', $\boldsymbol{\sigma}^0$ is also Ω' periodic (see Fig. 4), and we have:

$$|\Omega'|^{-1}\int_{\Omega'} \boldsymbol{\sigma}^0\, ds = |\Omega|^{-1}\int_{\Omega} \boldsymbol{\sigma}^0\, ds = \langle\boldsymbol{\sigma}^0\rangle$$

hence

$$\langle\boldsymbol{\sigma}^0\rangle \,.\, \mathbf{m} = |\Omega'|^{-1}\int_{\partial\Omega'} (\boldsymbol{\sigma}^0 \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{m})\, ds.$$

As previously stated, due to the periodicity, only the terms associated with the surfaces

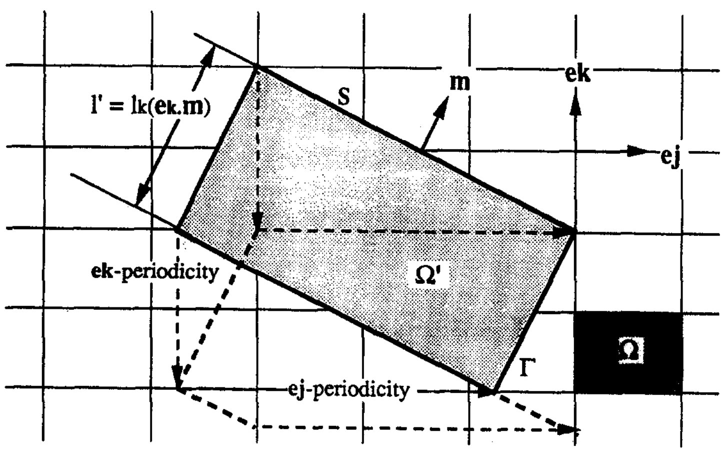

Fig. 4. Volume Ω' and surface S (representation in the two-dimensional case).

having $\pm m$ as normal, stay present in the boundary integral. Since the distance between these faces is l' we get:

$$\langle \boldsymbol{\sigma}^0 \rangle . \mathbf{m} = (1'/1' . |S|) \int_S \boldsymbol{\sigma}^0 . \mathbf{m} \, \mathrm{d}s = \mathbf{T}(\mathbf{m}).$$

This equality is valid for any discrete orientations $\mathbf{m}$ (such that $\mathbf{m} = \mathbf{A} \wedge \mathbf{B}$). As at the macroscopic level an elementary surface includes numerous cells, one can always assimilate the macroscopic orientation to one of the discrete orientations. Therefore $\langle \boldsymbol{\sigma}^0 \rangle = \mathbf{C}^0 .. \mathbf{e}_x(U^0)$ actually defines a *macroscopic Cauchy stress tensor*. In the same way all the terms $\mathbf{C}^0 .. \mathbf{e}_x(U^i)$ are also Cauchy stress tensors. We note that local stresses are symmetric and therefore these macroscopic tensors are also symmetric.

Stresses at higher orders. On the contrary to $\boldsymbol{\sigma}^0$, the stresses at order p ($p > 0$) satisfy an equation of the form:

$$\nabla_y . [\boldsymbol{\sigma}^p] = s^p,$$

where s^p is a periodic term different from zero, having a zero mean value in y. Following the same reasoning as above, we have

$$\langle \boldsymbol{\sigma}^p \rangle . \mathbf{m} = |S|^{-1} \int_S \boldsymbol{\sigma}^p . \mathbf{m} \, \mathrm{d}s + |\Omega'|^{-1} \int_{\Omega'} s^p(\mathbf{y} . \mathbf{m}) \, \mathrm{d}v. \tag{7}$$

Equation (7) leads us to make two remarks concerning the surface average of $\boldsymbol{\sigma}^p$:

(a) Force vector $\mathbf{T}^p$ ($\mathbf{m}$) by unit surface acting on face S,

$$\mathbf{T}^p(\mathbf{m}) = |S|^{-1} \int_S \boldsymbol{\sigma}^p . \mathbf{m} \, \mathrm{d}s$$

is not a macroscopic quantity. As a matter of fact, we have

$$\mathbf{T}^p(\mathbf{m}) = \langle \boldsymbol{\sigma}^p \rangle \mathbf{m} - |\Omega'|^{-1} \int_{\Omega'} s^p(\mathbf{y} . \mathbf{m}) \, \mathrm{d}v$$

and as $s^p.(\mathbf{y} . \mathbf{m})$ is *not a periodic* term, the value of $\mathbf{T}^p(\mathbf{m})$ depends on the choice of the period used for the calculation (i.e. on the variable y). That means that the surface average of $\boldsymbol{\sigma}^p$ varies at the microscopic scale.

(b) Moreover the *surface average* of elementary forces due to $\boldsymbol{\sigma}^p$ *do not define a tensor*: in the frame associated with the normals of the faces S_k, we consider the components Σ_{ik} (no symmetry) defined from the surface averaging of the ith component of the force due to $\boldsymbol{\sigma}^p$ acting on face S_k:

$$\Sigma_{ik} = \mathbf{T}_i^p(\mathbf{e}_k) = |S_k|^{-1} \int_{Sk} \sigma_{ik}^p \, \mathrm{d}s.$$

These components define a tensor $\boldsymbol{\Sigma}$. Now, from eqn (7) applied to Ω_m and Ω, it is easy to see that the components of $\mathbf{T}^p(\mathbf{m})$ are given by:

$$\mathbf{T}_i^{p'} = \Sigma_{ik} m_k + |\Omega|^{-1} \left\{ \int_\Omega s_i^p(\mathbf{y} . \mathbf{m}) \, \mathrm{d}v - \int_{\Omega'} s_i^p(\mathbf{y} . \mathbf{m}) \, \mathrm{d}v \right\}$$

hence

$$\mathbf{T}_i^p \neq \Sigma_{ik} m_k = \mathbf{T}_i^p(\mathbf{e}_k) \,.\, m_k.$$

This inequality proves that the operator connecting $\mathbf{m}$ to $\mathbf{T}^p(\mathbf{m})$ is not linear, and consequently, the *surface average of stresses* σ^p *does not define a tensor.* On the contrary, volume averages do define a tensor. But these can not be interpreted in terms of common Cauchy stress. We will see in the next section what is the meaning of the macroscopic equations which involve $\langle \boldsymbol{\sigma}^p \rangle$ tensors.

3. ANALYSIS OF THE DESCRIPTION INCLUDING HIGHER ORDER TERMS

Gambin and Kröner (1989) considered the influence of the higher order terms in terms of macroscopic "behavior" law. It was shown that the relation which links the averaged stresses and strains include weak non-local effects—associated with the successive gradients of the strain tensor. However, *stricto sensu*, this relation is not a behavior law, since averaged stresses $\langle \boldsymbol{\sigma}^p \rangle$ $(p > 0)$ are not Cauchy stresses.

Here, taking into account this difficulty of interpretation, we study the role of the higher order terms from macroscopic balance equations.

3.1. *Interpretation of balance equations*

3.1.1. *Macroscopic elastostatics.* The balance equation (4a) at zero order, corresponds exactly to the common elastostatic formulation. Field U^0 which results from the distribution $\langle \rho \rangle f^0$, is what would appear in a continuum elastic medium characterized by elastic tensor $\mathbf{C}^0$. However, this description is only valid to an accuracy of ε.

3.1.2. *Force distributions due to microstructure.* Let us now examine eqns. (4b, c) of order 1 and 2. As tensor $\langle \boldsymbol{\sigma}^p \rangle$ $(p > 0)$ is not a stress tensor, these equations cannot be interpreted directly in this form. Considering that the terms $\mathbf{C}^0 \,..\, \mathbf{e}_x(U^p)$ in $\langle \boldsymbol{\sigma}^p \rangle$ are actually stress tensors, we can give a physical meaning if we rewrite eqn (4) in the following forms:

$$\nabla_x \,.\, [\mathbf{C}^0 \,..\, \mathbf{e}_x(U^0)] = -\langle \rho \rangle f^0 \tag{8a}$$

$$\nabla_x \,.\, [\mathbf{C}^0 \,..\, \mathbf{e}_x(\varepsilon U^1)] = -\nabla_x \,.\, [\mathbf{C}' \ldots \nabla_x \mathbf{e}_x(U^0)] \tag{8b}$$

$$\nabla_x \,.\, [\mathbf{C}^0 \,..\, \mathbf{e}_x(\varepsilon^2 U^2)] = -\nabla_x \,.\, [\mathbf{C}' \ldots \nabla_x \mathbf{e}_x(\varepsilon U^1)] - \nabla_x \,.\, [\mathbf{C}'' \,....\, \nabla_x \nabla_x \mathbf{e}_x(U^0)]. \tag{8c}$$

Equations (8b, c) are elastostatic equations applied to fields εU^1 and $\varepsilon^2 U^2$, respectively, where source terms resulting from displacements fields at lower orders appear. These terms are similar to diffracted sources in the dynamic case (Boutin and Auriault, (1993)). These force distributions arise from the fact that the equations which express the momentum balance of the cell at order ε^i do not take into account contributions of order ε^{i+1}. So, these latter become forcing terms in the balance equation at order ε^{i+1}. Consequently, in order to balance these sources, a displacement field U^{i+1} occurs, satisfying the elastostatic equations of the *homogeneous equivalent medium.*

Unlike what would happen in a perfect homogeneous material (for which $\mathbf{C}^i = 0$), the presence of a microstructure results in a distribution of forces which generates a series of displacement fields of a lower and lower amplitude: fields of order strictly inferior to i generate sources at order i which themselves generates a field of order i, and so on.

The source terms are connected with the successive gradients of the strain tensor, and therefore, they introduce a non-local effect in the material behavior. It is clear that the more the deformation comes to being homogeneous, the weaker the sources will be. Conversely, when strain gradients are significant, the corrective terms become significant, so they amplify the non-local effects and modify the solution U^0 of the equivalent homogeneous medium.

The sharper the heterogeneities are at the microscopic scale then the larger are the values of $\mathbf{C}'$ and $\mathbf{C}''$ and the more the non-local terms become significant (for the same inhomogeneity level of the macroscopic deformation). Finally, note that if the material presents an isotropic macroscopic behavior (up to the second order), then $\mathbf{C}' = 0$ and the first significant source term appears at the order ε^2. Consequently the non-local effects would generally be more important in anisotropic materials than in isotropic materials.

3.2. *Solving boundary condition problems*

For a material of known microstructure, tensors $\mathbf{C}^0$, $\mathbf{C}'$, $\mathbf{C}''$ can be calculated. But limit conditions to be applied to the $U^i(x)$ fields should be known in order to determine the macroscopic solution, up to third-order, for given boundary condition problem.

3.2.1. *Macroscopic solution* (*without edge effects*). On the macroscopic scale, let us consider a body $\mathbf{B}$—the boundary of which being $\partial\mathbf{B}$—submitted to the macroscopic force density $\langle\rho\rangle f(x)$ and to the following conditions on the border:

- on the portion $\partial\mathbf{B}_S$ of $\partial\mathbf{B}$, imposed stress vector $\mathbf{S}(x)$;
- on the portion $\partial\mathbf{B}_D = \partial\mathbf{B} - \partial\mathbf{B}_S$, imposed displacement vector $\mathbf{D}(x)$.

At zero order, the field $U^0(x)$ is determined by the elastostatic eqn (8a) and the following boundary conditions:

$$\begin{aligned} &\mathbf{C}^0 \,..\, \mathbf{e}_x(U^0(x)) \,.\, N = \mathbf{S}(x) && \text{on } \partial\mathbf{B}_S \text{ (having } N \text{ as normal)} \\ &U^0(x) = \mathbf{D}(x) && \text{on } \partial\mathbf{B}_D. \end{aligned} \tag{9a}$$

The calculation of the field $\varepsilon U^1(x)$ is performed by solving another elastostatic problem governed by eqn (8b) where the microstructural forces are directly deduced from the knowledge of $U^0(x)$. The boundary conditions applied to $\varepsilon U^1(x)$ must be such that the *global* field $U^0(x) + \varepsilon U^1(x)$ meets macroscopic conditions imposed on $\partial\mathbf{B}$, i.e.

$$\begin{aligned} &\mathbf{C}^0 \,..\, \mathbf{e}_x(U^0(x) + \varepsilon U^1(x)) \,.\, N = \mathbf{S}(x) && \text{on } \partial\mathbf{B}_S \\ &U^0(x) + \varepsilon U^1(x) = \mathbf{D}(x) && \text{on } \partial\mathbf{B}_D. \end{aligned}$$

Taking into account equation (9a), $\varepsilon U^1(x)$ follows a Neumann condition on $\partial\mathbf{B}_S$, and a Dirichlet condition on $\partial\mathbf{B}_D$:

$$\begin{aligned} &\mathbf{C}^0 \,..\, \mathbf{e}_x(\varepsilon U^1(x)) \,.\, N = 0 && \text{on } \partial\mathbf{B}_S \\ &\varepsilon U^1(x) = 0 && \text{on } \partial\mathbf{B}_D. \end{aligned} \tag{9b}$$

In the same way, the macroscopic field $\varepsilon^2 U^2(x)$ is totally determined by the elastostatic equation (8c), where the microstructural forces are calculated from the fields $U^0(x)$ and $\varepsilon U^1(x)$ already obtained and boundary conditions of zero stresses and displacements, respectively, on $\partial\mathbf{B}_S$ and $\partial\mathbf{B}_D$ [i.e. conditions (9b) where $\varepsilon^2 U^2(x)$ substitutes $\varepsilon U^1(x)$].

By this reasoning at the macroscopic scale, we obtain the macroscopic displacement, which appears within body $\mathbf{B}$, up to third-order. However, this solution does not take into account the edge effects existing on a thin layer in the proximity of $\partial\mathbf{B}$.

3.2.2. *Remarks on edge effects.* It is known that real boundary conditions are only satisfied on average (on a period) by the homogenized description (see for example Sanchez-Palencia, 1985). For this reason, boundary layers have to be introduced in order to match real and homogenized conditions. The analysis of these layers, up to third-order, requires specific developments and will not be addressed in this paper. However, the following remarks can be made:

(a) Matching boundary conditions up to third-order requires the introduction of boundary layers at each order.

(b) For homogenized problems at zero order, the boundary layer concerns the stresses on $\partial\mathbf{B}_S$. It generates corrective displacement of the first order which is defined up to a macroscopic term. This latter will be determined by a continuity condition between the displacements given, on one side by the solution on $\partial\mathbf{B}_S$ of the macroscopic problem at first order, and, on the other side, by the inner limit of the corrective term.

(c) For homogenized problems at higher orders, other boundary layers will expand near $\partial\mathbf{B}_S$ for matching stresses conditions. Moreover, as the displacement fields vary on the local scale (contrary to u^0), another type of boundary layer has to be developed on $\partial\mathbf{B}_D$ to match the homogenized and the real imposed displacements (it is clear that for these new boundary layers there is no displacement indetermination).

As a conclusion, let us note that the boundary layers describing edge effects modify the fields at the very vicinity of the boundaries, since the limit layers have a "thickness" of about one period, Dumontet (1986). Therefore the macroscopic solution obtained by the procedure given in Section 3.1.3 will be valid for the body **B**, except for these limit zones.

3.3. *The measurement of ε to quantify non local effects*

Until now, we spoke about the macroscopic size L and $\varepsilon = l/L$ without giving an accurate definition. But this is a key point in assessing the role of the microstructure in the overall material response. We propose here two approximated methods aimed at assessing L and ε.

3.3.1. *Rough evaluation of ε*. The more simple approach, Boutin and Auriault (1991), consists of observing that the homogenization process leads to displacement fields of the type:

$$U^0(x)+\varepsilon u^1(x,y)+\cdots \quad \text{with } u^1 = O(U^0).$$

If we consider the increase of U_1^0 in the direction x_1 over a given period, we necessarily must have:

$$|U_1^0(x_1+l)-U_1^0(x_1)| \leqslant O(\varepsilon\,.\,|U_1^0(x_1)|).$$

At a macroscopic scale, l is very small and we can write:

$$l\,.\,|\partial(U_1^0)/\partial x_1|/|U_1^0| \leqslant O(\varepsilon),$$

which gives a lower boundary of ε. But, as ε is the measurement of the macroscopic description accuracy, the optimum value corresponds to the smallest allowable value, that is:

$$O(\varepsilon) = |\partial(U_1^0)/\partial x_1|\,.\,(l/|U_1^0|) \text{ or also: } L = O(|U_1^0|\,.\,|\partial(U_1^0)/\partial x_1|^{-1}).$$

This latter expression of L is what would be obtained by the dimensional analysis of the phenomena at the macroscopic scale. In the general case for a three-dimensional field, we get:

$$L = O(\min\{|U_i^0|\,.\,|\partial(U_i^0)/\partial x_j|^{-1}\}) \tag{10}$$

and:

$$\varepsilon = O(\max\{(l/|U_i^0|)\,.\,|\partial(U_i^0)/\partial x_j|\}). \tag{11}$$

An order of magnitude of ε is thus locally given by the relative variation of the displacement on the cell.

For a given boundary condition problem, we can solve the elastostatic equations, and calculate the field U^0. Then, using eqns (10) and (11), we, respectively, obtain the order of magnitude of the macroscopic size L and of ε in any point of the material.

3.3.2. *Accurate assessment of* ε. The second method used to assess ε is more precise but needs two integrations. From field U^0 previously calculated and the knowledge of $\mathbf{C}'$ (or $\mathbf{C}''$ if $\mathbf{C}' = 0$), we can determine by a second integration the field $\varepsilon U^1(x)$ (or $\varepsilon^2 U^2(x)$). By comparison with U^0 we deduce a theoretical local assessment of ε:

$$\varepsilon = O(|\varepsilon U^1|/|U^0|) \quad \text{or: } \varepsilon = O(\sqrt{|\varepsilon^2 U^2|/|U^0|}). \tag{12}$$

3.3.3. *A kinematic criterion for non local effects*. Assessed values (11), (12) of ε could be used as a *criterion of occurrence* of microstructural effects (except on borders). For a given boundary problem, the drawing of isovalues could allow defining different zones for the response of the material:

(a) at very low values of ε ($\leqslant 10^{-2}$), the macroscopic behavior corresponds to that of the equivalent homogeneous material;

(b) the areas with large gradients, where $\varepsilon \approx 0.1$, corresponds to the domains where the role of the microstructure is significant and for which it is necessary to introduce corrective fields εU^1 (or $\varepsilon^2 U^2$);

(c) if regions with $\varepsilon > 1$ are observed, the homogenization approach is no longer valid, and in this domain, the physics of the phenomenon have to be treated at the microscopic scale.

3.4. *Nonlocal terms for different loadings*

These results have been obtained assuming that the medium was submitted to volume forces of type $\rho(y)f^0(x)$, i.e. forces which vary at local scale. The developed formalism can also be applied to problems where volume forces are different—for example zero—on the cell. However, the equilibrium of the period is modified by this new force distribution, and consequently, the expressions of tensors $\mathbf{C}'$ and $\mathbf{C}''$ are *different*, even though $\mathbf{C}^0$ remains the same (see Appendix A).

Thus, it appears that if the zero order description can be considered very clearly as one of a continuous homogeneous medium, it is not the case for the macroscopic descriptions which include the higher order terms. As a matter of fact, the effective tensors are not strictly material tensors since they depend on the volume forces applied to the medium. Moreover, the *structure* of the equations depends on the kind of load, as it can be seen when comparing static and dynamic descriptions (see Boutin and Auriault, 1993).

However, it should be noted that, for a *given* distribution of volume forces, the tensors remain the same whatever the boundary conditions, since these latter do not change the cell equilibrium. From this point of view, the effective tensors can be considered as quasi-material tensors.

3.5. *Comparison with other microstructural approaches*

In this section, an attempt was made to compare the homogenization approach with two other theories, which deviate from the usual frame of continuum mechanics, by introducing a characteristic length of the microstructure. In order to study the localization phenomena, the double gradient theory was used, for example, by Lasry and Belytschko (1988), De Borst and Mülhaus (1991), and the Cosserat mechanics theory by Besdo (1985) and Mülhaus and Vardoulakis (1986).

3.5.1. *Comparison with the second gradient approach*. Through the addition of the balance equations at the three first-orders, we can obtain a single equation, valid up to an accuracy of ε^3, which includes the global macroscopic displacement field.

$$\nabla_x . [\mathbf{C}^0 \,..\, \mathbf{e}_x(U)] + \langle\rho\rangle f^0 = -\nabla_x . [\mathbf{C}' \,...\, \nabla_x \mathbf{e}_x(U)] - \nabla_x . [\mathbf{C}'' \,....\, \nabla_x \nabla_x \mathbf{e}_x(U)].$$

In the more general case, where $\mathbf{C}' \neq 0$, we get the following equation to ε^2, when considering only the first corrective term:

$$\nabla_x . [\mathbf{C}^0 \,..\, \mathbf{e}_x(U)] + \langle\rho\rangle f^0 = -\nabla_x . [\mathbf{C}' \,...\, \nabla_x \mathbf{e}_x(U)]. \tag{13}$$

Note that in eqn (13), two operators are applied to U, one of them elliptic of the second order, the other one hyperbolic of the third order. This point seems to pose a problem of solution stability. Actually this difficulty is artificially introduced when regrouping into one equation the two equations at zero and first orders. If we respect the non-coupling of the orders given by homogenization, it is clear that each field U^i is determined by the elliptic operator of elasticity without any problem of stability.

For the cases where $\mathbf{C}' = 0$, which correspond for example to a macroscopic isotropic material, we get the following equation to ε^3:

$$\nabla_x . [\mathbf{C}^0 \,..\, \mathbf{e}_x(U)] + \langle\rho\rangle f^0 = -\nabla_x . [\mathbf{C}'' \,....\, \nabla_x \nabla_x \mathbf{e}_x(U)]. \tag{14}$$

This constitutive law eqn (14) is similar to that proposed in the second gradient approach, and leads us to make the following comments:

(a) On the basis of the homogenization approach assumptions, the existence of non-local terms linked to the successive gradients of the strain tensor, without any postulate, has been rigorously established. In contrast to the usual one-dimensional analysis, we present a tensorial description dealing with three-dimensional problems. An advantage of this method is that it shows that the first significant non-local term is generally not associated with the second gradient, but to the *single gradient*, which introduces an *anisotropic* effect [eqn (13)]. This fact is not highlighted by one-dimensional analysis. The non-local terms only need the second gradient of the strain tensor [eqn (14)], in particular cases, as for example macroscopic isotropy (up to second order) or stratified composites (see Section 4). They may be isotropic or anisotropic according to the nature of tensor $\mathbf{C}''$.

(b) Moreover, these results enable a good understanding of the physics of the corrective terms originating from the microstructure. It is interesting to note that non-local effects do not result from a correction of macroscopic strain tensor by its second gradient, as proposed by Lasry and Belytschko (1988). In the same way, non-local phenomena cause a distribution of volume forces, and not a modification of the stress–strain relation (since for $p > 0$, tensors $\langle\boldsymbol{\sigma}^p\rangle$ are not stress tensors and the surface average of $\boldsymbol{\sigma}^p$ does not define a tensor).

(c) Another advantage of homogenization is the clear definition of its validity domain: the obtained description—like that obtained from the second gradient method—is correct only if there remains a clear scale separation. Thus, any phenomenon for which the physics at the local scale is determinant, cannot be described by this kind of approach. From a numerical point of view, the results of the homogenization process present the interest of uncoupled equations at the different orders of magnitude. Thus, when $\mathbf{C}' = 0$, the determination of the global displacement field requires an integration of two differential second order problems, while the second gradient approach necessitates the resolution of a much more complex differential fourth order problem.

3.5.2. *Comparison with Cosserat media.* Here the question becomes: do the higher terms change the elastic macroscopic behavior, into a behavior of a Cosserat medium? Let us recall that for those media, the stress state is described by a non-symmetric *Cauchy tensor* and the particulate kinetics include both translations and rotations.

Higher terms and Cosserat media. The homogenization development shows that higher order expressions introduce corrective translation terms into the kinematic of the cell, without any rotation. Moreover, the physical macroscopic stresses, i.e. in the form

$\mathbf{C}^0 \,..\, \mathbf{e}_x(U)$, are necessarily symmetric. Finally, the source terms are force densities, which do not introduce micropolar couples. Consequently, even when taking into account higher order terms, *under homogenizable conditions*, the macroscopic behavior of the elastic periodic composites can not be identified with the behavior of a Cosserat medium.

It is clear that this conclusion comes from the essential assumption of local invariance of the variables, which is expressed in our case by periodicity. As a matter of fact, this hypothesis does not allow rigid rotation of the cell, and implies the symmetry of the macroscopic tensor stress (see Section 2.2.5).

Surface average of non-invariant stresses. At this point another question arises: could we find a Cosserat media by dropping the hypothesis of periodicity of the variables, i.e. in non-homogenizable situations?

Using the same notations, eqn (6) can be written in the more compact form:

$$\int_V \boldsymbol{\sigma} \,.\, \mathbf{m} \, \mathrm{d}v = \int_{\partial V} (\boldsymbol{\sigma} \cdot \mathbf{n})(\mathbf{y} \,.\, \mathbf{m}) \, \mathrm{d}s. \tag{15}$$

Let's apply this relation to the cell Ω, where $\mathbf{m}$ is equal to $\mathbf{e}_i$ and $\mathbf{e}_j$, respectively:

$$\int_\Omega \boldsymbol{\sigma} \,.\, \mathbf{e}_i \, \mathrm{d}v = \int_{\partial\Omega} (\boldsymbol{\sigma} \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{e}_i) \, \mathrm{d}s = l_i \int_{S_i} (\boldsymbol{\sigma} \,.\, \mathbf{e}_i) \, \mathrm{d}s + \int_{S\pm k \quad k \neq i} (\boldsymbol{\sigma} \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{e}_i) \, \mathrm{d}s$$

$$\int_\Omega \boldsymbol{\sigma} \,.\, \mathbf{e}_j \, \mathrm{d}v = \int_{\partial\Omega} (\boldsymbol{\sigma} \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{e}_j) \, \mathrm{d}s = l_j \int_{Sj} (\boldsymbol{\sigma} \,.\, \mathbf{e}_j) \, \mathrm{d}s + \int_{S\pm k \quad k \neq j} (\boldsymbol{\sigma} \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{e}_j) \, \mathrm{d}s.$$

By dividing by $|\Omega|$, taking the scalar product by $\mathbf{e}_j$ and $\mathbf{e}_i$, respectively, subtracting the two equations, and finally using the symmetry of $\boldsymbol{\sigma}$, we obtain:

$$[|S_i|^{-1} \int_{Si} (\boldsymbol{\sigma} \,.\, \mathbf{e}_i) \, \mathrm{d}s] \,.\, e_j - [|S_j|^{-1} \int_{Sj} (\boldsymbol{\sigma} \,.\, \mathbf{e}_j) \, \mathrm{d}s] \,.\, e_i$$

$$= \int_{S\pm k \quad k \neq j} (\boldsymbol{\sigma}_{ik})(\mathbf{y} \,.\, \mathbf{e}_j) \, \mathrm{d}s - \int_{S\pm k \quad k \neq i} (\boldsymbol{\sigma}_{jk})(\mathbf{y} \,.\, \mathbf{e}_i) \, \mathrm{d}s.$$

We notice that when the periodicity is not assumed the surface integrals do not vanish, and therefore the surface averaged stresses are not symmetric.

However, a global description corresponding to Cosserat media will be obtained when the condition that, the state of "global stress" is actually a *tensorial* state of stress is obeyed. Consequently we have to examine if the surface average of local non-periodic stresses define a Cauchy tensor. In this aim, we follow the same reasoning as in Section 2.2.5. Applying relation eqn (15) to the volumes Ω and Ω', respectively, we get:

$$\int_\Omega \boldsymbol{\sigma} \,.\, \mathbf{m} \, \mathrm{d}v = \int_{\partial\Omega} (\boldsymbol{\sigma} \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{m}) \, \mathrm{d}s = l_k \int_{Sk} (\boldsymbol{\sigma} \,.\, \mathbf{m}) \, \mathrm{d}s + \int_{S\pm j \quad j \neq k} (\boldsymbol{\sigma} \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{m}) \, \mathrm{d}s$$

$$\int_{\Omega'} \boldsymbol{\sigma} \,.\, \mathbf{m} \, \mathrm{d}v = \int_{\partial\Omega'} (\boldsymbol{\sigma} \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{m}) \, \mathrm{d}s = l' \int_{S} (\boldsymbol{\sigma} \,.\, \mathbf{m}) \, \mathrm{d}s + \int_{\Gamma\pm} (\boldsymbol{\sigma} \,.\, \mathbf{n})(\mathbf{y} \,.\, \mathbf{m}) \, \mathrm{d}s.$$

Then, the surface average of stresses acting on the face S, $\mathbf{T}(\mathbf{m}) = |S|^{-1} \int_S (\boldsymbol{\sigma} \,.\, \mathbf{m}) \, \mathrm{d}s$, is related to the surface average of stresses acting on a faces S_k by:

$$\mathbf{T}(\mathbf{m}) - m_k |S_k|^{-1} \int_{Sk} (\boldsymbol{\sigma} \cdot \mathbf{e}_k) \, \mathrm{d}s = |\Omega'|^{-1} \int_{\Omega'} \boldsymbol{\sigma} \cdot \mathbf{m} \, \mathrm{d}v - |\Omega|^{-1} \int_{\Omega} \boldsymbol{\sigma} \cdot \mathbf{m} \, \mathrm{d}v$$

$$- |\Omega'|^{-1} \int_{\Gamma\pm} (\boldsymbol{\sigma} \cdot \mathbf{n})(\mathbf{y} \cdot \mathbf{m}) \, \mathrm{d}s + |\Omega|^{-1} \int_{S\pm j \quad j \neq k} (\boldsymbol{\sigma} \cdot \mathbf{n})(\mathbf{y} \cdot \mathbf{m}) \, \mathrm{d}s.$$

In case of periodicity, the two volume integrals were equal and each surface integral equals zero. Now, when relaxing the assumption of periodicity, these equalities are no longer true (when $\mathbf{m}$ is different from the vectors $\mathbf{e}_k$). Consequently:

$$\mathbf{T} - m_k |S_k|^{-1} \int_{Sk} (\boldsymbol{\sigma} \cdot \mathbf{e}_k) \, \mathrm{d}s \neq 0$$

It results that the surface average of stresses in a direction $\mathbf{m}$ cannot be deduced from the knowledge of the surface average of stresses acting on the faces defined by the reference frame (as it would be for Cauchy stresses). Thus, when stresses are not periodic—more generally, do not present a local invariance—it is not possible to define a tensorial state of global stress, and therefore the overall behavior does not correspond to the one of a Cosserat medium.

In conclusion, when studying elastic composites, one is confronted with the following alternatives:

(a) Either, the situation is homogenizable and then a macroscopic *tensorial symmetric* state of stresses exist. The microstructural effects are described by a microstructural forces which implies non-local effects.

(b) Alternatively, the situation is not homogenizable. Then the surface average of stresses are *not symmetric*, but these surface averages of stresses *do not define a Cauchy stress state*. As this point is in contradiction with a basic assumption of Cosserat mechanics, it results that the medium can not be described by Cosserat mechanics. Moreover, the use of "non-Cauchy global stresses" has to be done very carefully for the following reasons:

- Since "global stresses" are not tensorial, there is no tensorial "global stress"–strain relationship. For example if a "global stress"–strain relationship is established in a given frame, the usual tensorial rules are not applicable to deduce the relationship for any other frame. More generally this result poses the question of the definition, and even the existence, of a constitutive law (linking strains with . . . ?).
- The use of non-tensorial "global stresses" do not allow to express boundary conditions on a surface of various orientations. In this case (i.e. when the boundaries of the medium are not plane) the only possibility is to express the conditions at the microscopic level.

Discussion. Since these results are not in agreement with the approaches developed for stratified elastic half plane by Mülhaus and Vardoulakis (1986) or Papamichos *et al.* (1990) let us examine more precisely their work.

In these articles the "stress-state" in a stratified medium is described by surface averaged stresses. The equations governing these non symmetric "global stresses" are deduced and correspond to the moment balance of a Cosserat continuum, provided that adequate calibrated coefficients are introduced. For this reason, the authors formally identify stratified media to Cosserat media.

However, this is not sufficient to justify the application of this theory: it must also be verified that the others assumptions are fulfilled. This is not the case here since, when "global stresses" are not symmetric, they do not define a tensor. To our knowledge, this essential problem was never addressed.

Nevertheless, the consequence of non-tensoriality of "stresses" only appears when considering inclined surfaces in the reference frame. Thus, as the treated problems only involve plane boundary conditions, this difficulty disappears and the solution obtained by this approach, *for this kind of problem*, is correct, and therefore the Mülhaus approach becomes relevant (independently of the inappropriate use of the term "Cosserat medium" for the stratified material). Thus, the case of stratified half plane is a very interesting since *non-homogenizable situations can be modelized* by using "non-Cauchy global stresses". This feature, which is due to the "one-dimensional aspect" of both medium and boundaries, is specific to this problem and the obtained results cannot easily be generalized neither for other medium, nor for other boundary conditions.

3.5.3. *Application to the study of the occurrence of the localization.* At present, numerous experimental studies—for example Scarpelli and Wood (1982), Desrues (1984) and Boulon (1988)—address the emergence of shear bands in heterogeneous materials such as soils, rocks, and concrete. It is now well established and admitted that this loss of homogeneity in the macroscopic deformation of the sample—or the studied structure—is linked with the presence of a microstructure in the material. In other terms, when the localization happens, the microstructure is involved in the response of the material. Thus, the mechanics of continuum media is not the most adapt theory to describe this kind of phenomena. Moreover, while a shear band occurs, experimental observations show that the deformation is localized in a domain having a thickness of about 10–20 grains or heterogeneities (i.e. L roughly equals $20 \times l$ so $\varepsilon \approx 5 \times 10^{-2}$), and more generally the localization expands into the high gradient zones.

Finally, the use of microphotogrammetry for rocks and concretes has clearly shown that the strain inhomogeneity can happen even when the behavior of the material can still be considered as elastic (at least macroscopically).

From a theoretical point of view, localization phenomena, are usually considered as the expression of a mechanic instability, linked with non-linear behavior, and resulting in bifurcations in the solution (Hill, 1962; Mandel, 1966; and Rice, 1977). These theories, developed in continuum mechanics, lead to a shear band of zero thickness, which is not in agreement with observations made. In order to have a better description of the experimental reality, the use of constitutive laws of Cosserat, non-local or double gradient types, which introduce microstructural effects, have been proposed since the beginning of the 1980s, particularly in the papers mentioned in the two preceding sections. The results published show that the instability analysis and the microstructural approach complement one another.

In order to illustrate the interest of our results, let us consider a rock which can be considered as an elastic composite. A sample of this rock is submitted to a test which is deliberately non-homogeneous (for example, the Brazilian test) or it becomes non-homogeneous because of uncontrolled effects (binding, eccentricity, surface unevenness, etc.). In areas where the deformation remains quasi-homogeneous, the material will respond as the equivalent homogeneous continuum. Therefore, the effects of the microstructure, needed for the emergence of localization, will not appear. But, in the domains where the deformation is clearly non-homogeneous, these microstructural effects will develop. This reasoning leads us to believe that the localization can only appear in these areas, which can be identified using the kinematic criterion obtained from the assessment of ε.

However, this approach is limited to the characterization of the *potential* regions for the emergence of localization within the body. As a matter of fact, inside a well developed shear band, there is no more scale separation and the *homogenization approach breaks down*. These conditions where $L \approx l$ allow new local kinematics, different from those corresponding to $L \gg l$. Particularly, rotations, slips, etc. of the grains can be observed. The physics governing these phenomena is often different from the one determined by homogeneous tests: the role of grain angularity, granulometry, shape, etc. then becomes essential. Consequently, we observe non-homogenizable phenomena resulting from microscopic effects.

Let us remark that the second gradient approach is also limited by the same reasons, but they are not explicitly expressed. Finally, the fact that the differential Cosserat equations present a positive definite principal part do not insure the relevance of the model for describing the physical phenomena.

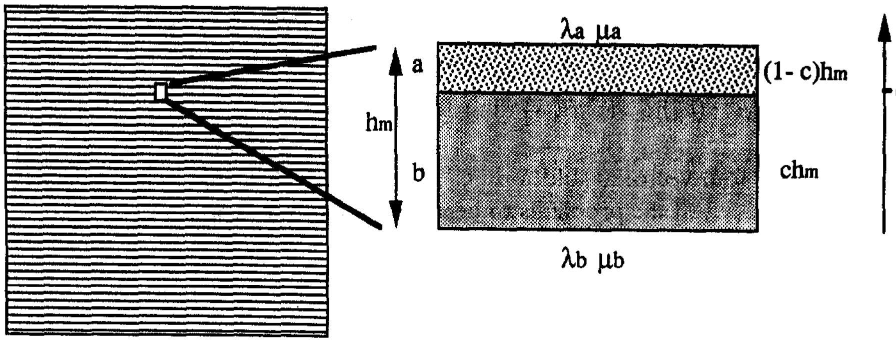

Fig. 5. Description of the studied stratified composite.

4. THE EXAMPLE OF STRATIFIED COMPOSITES

The preceding results have been applied to the periodic stratified composites (Fig. 5) which have already been studied in a number of papers, for example Biot (1965), Auriault and Bonnet (1985) and Papamichos *et al.* (1990).

This kind of microstructure presents the advantage of simplicity, which allows an analytical expression of tensors the $\mathbf{C}^0$ and $\mathbf{C}''$. But, we will see that its one-dimensional structure exhibits some specific properties.

The determination of tensor $\mathbf{C}''$ in periodic stratified composites is interesting for two reasons. First, it gives an exact expression which can be used in second gradient theories, second, the knowledge of $\mathbf{C}''$ allows one to solve problems with any kind of homogenizable boundary conditions which cannot be treated by "non-Cauchy stresses". As an example, the determination of the microstructural effects in the presence of any cavity—provided that its size is large in comparison with the stratification thickness—can be addressed. This kind of analysis is important and relevant in composite engineering (fragilization of structures by holes), civil engineering (tunnels in stratified rocks), etc.

We assume that the period (Fig. 5) is constituted of two isotropic elastic layers a and b, with respective thicknesses $(l-\tau).h$ and $\tau.h$. As we use microscopic variables to solve elementary problems, in this system of variable, the period size is $h_m = \varepsilon^{-1}h$. Let λ_a, μ_a, λ_b, μ_b, equal the Lamé constants of the layers a and b. We will note λ $(\mu, \ldots)$ the function having the value λ_a $(\mu_a, \ldots)$ in layer a and λ_b $(\mu_b \ldots)$ in layer b.

Owing to the one-dimensional geometry, the fields depend locally only on variable y_1 (which we will note simply as y). Thus, the differential operators L^{-2}, L^{-1}, L^0 take the following expressions:

$$L^{-2}(u) = \left| \begin{array}{l} \partial[(\lambda+2\mu).\partial(u_1)/\partial_y]/\partial_y \\ \partial[\mu.\partial(u_2)/\partial_y]/\partial_y \\ \partial[\mu.\partial(u_3)/\partial_y]/\partial_y \end{array} \right.$$

$$L^{-1}(u) = \left| \begin{array}{l} \partial[\lambda.\{\partial(u_1)/\partial x_1+\partial(u_2)/\partial x_2+\partial(u_3)/\partial x_3\}+2\mu.\partial(u_1)/\partial x_1]/\partial_y \\ \qquad +\partial[(\lambda+2\mu).\partial(u_1)/\partial_y]/\partial x_1+\partial[\mu.\partial(u_2)/\partial_y]/\partial x_2+\partial[\mu.\partial(u_2)/\partial_y]/\partial x_3 \\ \partial[\mu\{\partial(u_1)/\partial x_2+\partial(u_2)/\partial x_1\}]/\partial_y+\partial[\mu\partial/(u_2)\,\partial_y]/\partial x_1+\partial[\mu\partial(u_1)/\partial_y]/\partial x_2 \\ \partial[\mu\{\partial(u_1)/\partial x_3+\partial(u_3)/\partial x_1\}]/\partial_y+\partial[\mu\partial/(u_3)\,\partial_y]/\partial x_1+\partial[\mu\partial(u_1)/\partial_y]/\partial x_3 \end{array} \right.$$

L^0 is the common isotropic elastostatic operator expressed according to the x variables.

4.1. *Elastic tensor $\mathbf{C}^0$*

Denoting by E_{ij} the components of the macroscopic strain tensor $\mathbf{e}_x(U^0)$, the problem $L^{-2}(u_w^1)+L^{-1}(U^0) = 0$ is written in the form:

$$\left|\begin{array}{l}\partial[(\lambda+2\mu)\,.\,\partial(u_1)/\partial y+(\lambda+2\mu)E_{11}+\lambda E_{22}+\lambda E_{33}]/\partial y=0\\ \partial[\mu\,.\,\partial(u_2)/\partial y+2\mu E_{12}]/\partial y=0\\ \partial[\mu\,.\,\partial(u_3)/\partial y+2\mu E_{13}]/\partial y=0.\end{array}\right.$$

We now define four vectorial solutions $\mathbf{X}^{11}$, $\mathbf{X}^{22}$, $\mathbf{X}^{33}$, $\mathbf{X}^{12}$, $\mathbf{X}^{13}$, ($\mathbf{X}^{23}=0$) associated with E_{11}, E_{22}, E_{33}, E_{12}, E_{13}. All the problems to be solved are of the type:

$$\partial[\alpha\,.\,\partial(X)/\partial y+\beta]/\partial y=0$$

with $\mathbf{X}$; $(\alpha\cdot\partial(X)/\partial y+\beta)$ continuous and h_m-periodic, and α, β constant in each layer a and b. The solution is as follows:

$$\mathbf{X}(y)=f(y)\cdot\underline{\alpha}\mathbf{D}(1/\alpha).\mathbf{D}(\beta)/\mathbf{D}(\alpha)$$

where, by convention, for each function Ψ taking the a constant value Ψ_a in layer a and Ψ_b in layer b, we introduce the notations:

$$\underline{\Psi}=[(1-\tau)/\Psi_a+\tau/\Psi_b]^{-1}\quad \mathbf{D}(\Psi)=\tau(1-\tau)h_m(\Psi_a-\Psi_b)$$

and $f(y)$ being the function:

$$f(y)=\left|\begin{array}{ll}[y/h_m-(1-\tau)/2]/(1-\tau) & \text{in layer } a\\ -[y/h_m+\tau/2]/\tau & \text{in layer } b.\end{array}\right.$$

Thus, we obtain for the only components different from zero:

$$X_1^{11}=k.f\qquad\qquad k=(\underline{\lambda+2\mu})\,.\,D(1/(\lambda+2\mu))$$

$$X_1^{22}=X_1^{33}=k\,.\,[\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu)]\,.f\quad\text{with}$$

$$X_2^{12}=X_3^{13}=2m\,.f\qquad\qquad m=(\underline{\mu})\,.\,D(1/\mu).$$

Consequently, the stress fields $c_{ij}^{0\,kl}$ associated with E_{kl} are:

$$c_{ij}^{0\,11}=(\underline{\lambda+2\mu})(1,\eta,\eta,0,0,0)\qquad \eta=\lambda/(\lambda+2\mu)$$

$$c_{ij}^{0\,22}=(\underline{\lambda+2\mu})(\langle\eta\rangle,\gamma,\phi,0,0,0)$$

$$c_{ij}^{0\,33}=(\underline{\lambda+2\mu})(\langle\eta\rangle,\phi,\gamma,0,0,0)\quad \gamma=\langle\eta\rangle\eta+\lambda(1-\eta)/(\underline{\lambda+2\mu})$$

$$c_{ij}^{0\,12}=2\underline{\mu}(0,0,0,0,0,1)$$

$$c_{ij}^{0\,13}=2\underline{\mu}(0,0,0,0,1,0)\qquad \phi=\gamma+2\mu/(\underline{\lambda+2\mu}).$$

$$c_{ij}^{0\,23}=(0,0,0,2\mu,0,0)$$

We get the macroscopic elastic tensor by averaging these elementary stress fields. The components which do not equal zero are the following:

$$C_{11}^{0\ \ 11}=(\underline{\lambda+2\mu})$$

$$C_{22}^{0\ \ 22}=C_{33}^{0\ \ 33}=(\underline{\lambda+2\mu})\langle\gamma\rangle+\langle 2\mu\rangle$$

$$C_{22}^{0\ \ 11}=C_{33}^{0\ \ 11}=C_{11}^{0\ \ 22}=C_{11}^{0\ \ 33}=(\underline{\lambda+2\mu})\langle\eta\rangle$$

$$C^0_{33}{}^{22} = C^0_{22}{}^{33} = (\underline{\lambda+2\mu})\langle\gamma\rangle$$

$$C^0_{23}{}^{23} = \langle 2\mu\rangle \quad C^0_{12}{}^{12} = C^0_{13}{}^{13} = 2\underline{\mu}. \tag{16}$$

Macroscopically, we obtain a cylindrical orthotropic elastic material. It is easy to verify that the tensor given by the expression eqn (16) is valid for any one-dimensional distribution of the elastic coefficients λ and μ which generalize the results of Auriault and Bonnet (1985).

4.2. *Tensor C′*

The tensor $\mathbf{C}^1$ is determined from the elementary solutions of the following problem:

$$L^2(u^2_w)+L^{-1}(u^1_w) = \nabla_x . \{[\mathbf{C}^0-c^0]\,..\,\mathbf{e}_x(U^0)\}.$$

The solutions u^2_w depend on the gradient of strain tensor $\nabla_x\mathbf{e}_x(U^0)$ whose components are noted $E_{ij,k}$. Taking into account the cylindrical symmetry around axis 1, axes 2 and 3 are equivalent, so that we can restrict the number of problems to be solved (other solutions being obtained by index permutation). Thus we then treat:

$$\left|\begin{array}{l}\partial[(\lambda+2\mu)\partial(u_1)/\partial_y+(\lambda+2\mu)(X^{11}_1 . E_{11,1}+X^{22}_1 . E_{22,1})+\lambda X^{12}_2\cdot E_{12,2}]/\partial_y = 0\\ \partial[\mu\partial(u_2)/\partial_y+\mu(X^{11}_1 . E_{11,2}+X^{22}_1 . E_{22,2})+\lambda X^{12}_2 . E_{12,1}]/\partial y\\ \quad = (\lambda+2\mu)\{(\langle\eta\rangle-\eta)E_{11,2}+(\langle\phi\rangle-\phi)E_{22,2}+(\langle\gamma\rangle-\gamma)E_{33,2}\}+2(\langle\mu\rangle-\mu)E_{23,3}.\end{array}\right.$$

All the problems are of the following type:

$$\partial[\alpha . \partial(\mathbf{Y})/\partial_y+\beta\mathbf{X}]/\partial_y = \langle\chi\rangle-\chi$$

with $\mathbf{Y}$; $(\alpha\cdot\partial(\mathbf{Y})/\partial_y+\beta\mathbf{X})$ continuous and h_m-periodic and α; β; χ constant in each layer. The solution is of the form:

$$\mathbf{Y} = \mathbf{D}(\chi)[\langle F/\alpha\rangle-F/\alpha]+\langle\textstyle\int \mathbf{X}\beta/\alpha\rangle-\int\mathbf{X}\beta/\alpha$$

where;

$$F(y) = \int f(u)\,\mathrm{d}u = \left|\begin{array}{ll} y[y/h_m-(1-\tau)]/2(1-\tau) & \text{in layer } a\\ -y[y/h_m+\tau]/2\tau & \text{in layer } b.\end{array}\right.$$

Thus, we get the components which do not equal zero of Y^{jkl}_i, whose expressions are given in Appendix B:

$$\begin{array}{ccc} Y^{111}_1 & Y^{221}_1 = Y^{331}_1 & Y^{122}_1 = Y^{133}_1\\ Y^{112}_2 = Y^{113}_3 & Y^{222}_2 = Y^{333}_3 & Y^{332}_2 = Y^{223}_3\\ Y^{121}_2 = Y^{131}_3 & Y^{233}_2 = Y^{232}_3. & \end{array}$$

The stress fields c^1 resulting from the gradient of the macroscopic strain is deduced (Appendix B). The only fields which do not equal zero are those associated with:

$$E_{12,2}, E_{13,3};\quad E_{11,2}, E_{11,3};\quad E_{22,2}, E_{33,3};\quad E_{22,3}, E_{33,2};\quad E_{23,2}, E_{23,3}.$$

However, the calculations show that even though these stresses of order ε do not equal zero, they have a zero average on the cell. Thus, this one-dimensional microscopic geometry presents the particularity of defining an anisotropic material, for which:

$$\mathbf{C}^1 = 0 \quad \text{thus}\, \mathbf{C}' = 0.$$

This is due to the fact that the strains of first order are constant in each constituent. In the more general case where the microstructure has a two-dimensional or three-dimensional geometry, this point is no longer true and $\mathbf{C}' \neq 0$.

4.3. *Tensor* $\mathbf{C}''$

As $\mathbf{C}^1 = 0$, tensor $\mathbf{C}''$ is determined from the elementary solutions of:

$$L^2(u_w^3) + L^{-1}(u_w^2) = -\nabla_x . \{c^1 \ldots \mathbf{V}_x \mathbf{e}_x(U^0)\}.$$

Taking into account the expressions of u_w^2, c^1, and the cylindrical symmetry, we now solve the elementary problems;

$$\left|\begin{array}{l}\partial[(\lambda+2\mu)\partial(u_1)/\partial_y + (\lambda+2\mu)(Y_1^{111} . E_{11,11} + Y_1^{221}E_{22,11} + Y_1^{122}E_{12,21}) + \lambda(Y_2^{112} . E_{11,22} \\ \qquad + Y_2^{222}E_{22,22} + Y_2^{332}E_{33,22} + Y_2^{121}E_{12,12} + Y_2^{233}E_{23,32})]/\partial_y = 0 \\ \partial[\mu\partial(u_2)/\partial_y + \mu(X_1^{11} . E_{11,2} + X_1^{22} . E_{22,2}) + \lambda X_2^{12} . E_{12,1}]/\partial_y \\ \qquad = (\lambda+2\mu)\{(\langle\eta\rangle-\eta)E_{11,2} + ((\langle\phi\rangle-\phi)E_{22,2}) + (\langle\gamma\rangle-\gamma)E_{33,2}\} + 2(\langle\mu\rangle-\mu)E_{23,3}.\end{array}\right.$$

These problems can be expressed in the form:

$$\partial[\alpha . \partial(\mathbf{Z})/\partial_y + \beta\mathbf{Y}]/\partial_y = \chi . f$$

with $\mathbf{Z}$; $(\alpha \cdot \partial(\mathbf{Z})/\partial_y + \beta\mathbf{Y})$ continuous and h_m-periodic and α; β; χ constant in each layer. In order to determine $\mathbf{C}^2$, it suffices to state the expression of $\partial(\mathbf{Z})/\partial_y$ as:

$$\partial(\mathbf{Z})/\partial_y = \chi F/\alpha - \langle\chi F/\alpha\rangle\alpha/\alpha + (\alpha/\alpha)\langle\mathbf{Y}\beta/\alpha\rangle - \mathbf{Y}\beta/\alpha.$$

We can thus determine the stress fields c^2 associated with the second gradient of the strain tensor. After averaging, we obtain the components (which do not equal zero) of tensor $\mathbf{C}''$ which are given in Appendix B. We note that this tensor is not isotropic. Moreover it clearly appears that $\mathbf{C}''$ is homogeneous to $Pa.m^2$ and involves the square of the size h of the microstructure and the mechanical properties of the constituents of the cell. Finally, it should be noted that for a given concentration τ, the more the contrast of Lamé coefficient is obvious, the more the values of $\mathbf{C}''$ are large, and, for a given mechanical contrast, $\mathbf{C}''$ is even larger when the heterogeneity concentration is high ($\tau \approx 0.5$).

5. CONCLUSION

The homogenization method developed at higher orders is useful when analyzing microstructural effects. Indeed this approach highlights under which conditions an heterogeneous material behaves as a continuous or discontinuous media, from a qualitative and quantitative point of view. In brief, we revert back to continuum mechanics only if the period is an "infinitesimal" volume, relative to the studied phenomena. Conversely, the less negligible the period size is—in comparison with the phenomena (i.e. more inhomogeneous the macroscopic deformation is)—the more important the microstructural effects are.

From the detailed analysis of the macroscopic variables at higher orders, we prove that, even in the simple case of elastic composites, the microstructural effects do not correspond to a modification of the strain tensor or the usual strain–stress relation, but can be simulated by a chain of microstructural volume forces and displacement fields of increasingly lower amplitude.

The source terms introduce non-local effects since they involve the successive gradients of deformation tensors. This point shows that the more inhomogeneous the macroscopic deformation is, the more important the effects of the microstructure become. Moreover, new effective tensors $\mathbf{C}'$, $\mathbf{C}''$, etc. which depend on the microstructure appear in the source terms. Thus, we clearly show that the role of the heterogeneities is not only of topological nature, but includes also local mechanical properties, and we demonstrate that the incidence of the microstructure is stronger for higher contrasts of elastic coefficients.

These effective tensors depend also on the macroscopic volume forces applied to the material, so they are not material tensor, *stricto sensu*. However they can be considered as "quasi-material" since for a *given* distribution of macroscopic volume forces—which is usually the case—these tensors are independent of boundary conditions. Finally, we have to keep in mind that the macroscopic descriptions including higher terms do not strictly define behaviors.

The boundary conditions for solving the involved equations up to the third-order are given. Thus, for a given problem in a given material, we can get a local assessment of ε which can be used as a kinematic criterion for the occurrence of microstructural effects (without considering edge effects).

The comparison between these results and other microstructural theories show that the tensorial description obtained by homogenization is a generalization of the more usual second gradient approach. Indeed, the first significant microstructural effects usually involve anisotropic terms linked with the simple gradient of strain, and only for specific cases such as macroscopic isotropy (or stratified composites) is the correction of a higher order, linked with the second gradient of strain.

Conversely, according to this approach, it appears that the microstructural terms can not be simulated by Cosserat mechanics.

Numerical simulations and comparisons between those three theories and the more general non-local approach developed by Eringen (1972) will be performed later.

A concrete application of these theoretical results to the periodic stratified elastic composites is presented, for which we show that $\mathbf{C}' = 0$ and we also give the expression of $\mathbf{C}''$.

These theoretical results can be used for the experimental study of the *occurrence* of the localization. We saw that this phenomenon is linked with the characteristic size of the macroscopic strain, this size being associated with the inhomogeneity of the deformation.

However, the experiments are usually designed in order to have samples with the most homogeneous macroscopic deformation possible. For this reason the formation of the localization is due to parasite effects which are not easily controllable.

This difficulty could be reduced if—after behavior determination under homogeneous conditions—tests with prescribed inhomogeneous macroscopic strain were performed. One alternative consists of using the propagation of waves having a length close to that of the grain size. However, this measurement can only be qualitative, since the inertial effects modify the description established in the static case. In statics, this imposed inhomogeneity could be governed by boundary conditions (loads or displacements) applied to the sample.

Finally, let us recall that these results have been obtained with small strains, for elastic periodic microstructures and can only be used if there remains two distinct scales. The case of random materials is still an open issue. However it has been shown by Auriault (1991) that when a phenomenon is homogenizable, the *structure* of the equations remains the same in both cases. This point is not established for higher order terms, but the agreement of the theoretical and experimental results for the Rayleigh diffraction seems to confirm this assumption.

When there is no more scale distinction, as is the case in a well developed shear band, the approach proposed here is no longer applicable. Moreover, these conditions where $L \approx l$, allow local kinematics different from those which appear when $L \doteq l$ and the physics governing these phenomena is different from that obtained from homogeneous tests. Consequently the observed response at the "macroscopic" scale will result from this non-homogenizable physics at local scale.

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APPENDIX A

ELASTOSTATICS OF PERIODIC MICROSTRUCTURES

In this appendix we give the theoretical developments which lead us to the macroscopic balance equations up to third order.

A1.1 *Resolution of the problems at different orders*

In order to simplify the expressions we will write $\mathbf{c}$ and ρ instead of $\mathbf{c}(y)$ and $\rho(y)$.

Order ε^{-2}. The first problem one must tackle is the following eqn (1a):

$$L^{-2}(u^0) = \nabla_y \,.\, [c \,..\, \mathbf{e}_y(u^0)] = 0$$

whose evident solutions are constant fields on the period: $u^0 = U^0(x)$.

Order ε^{-1}. At this order we have the system eqn (1b):

$$L^{-2}(u^1) = -L^{-1}(U^0)$$

which can be written:

$$\nabla_y \cdot [c \mathbin{..} \mathbf{e}_x(u^1)] + \nabla_x \cdot [c \mathbin{..} \mathbf{e}_y(U^0)] = 0.$$

As a consequence of the linearity of the problem, the general solution becomes:

$$u^1(x, y) = U^1(x) + \mathbf{X}(y) \mathbin{..} \mathbf{e}_x(U^0).$$

The third-rank tensor $\mathbf{X}$ is constructed from the particular solutions $\mathbf{X}^{kl}$ such as:

$$c^0(y) = c + c \mathbin{..} {}^S\nabla_y[X] \quad \nabla_y \cdot [c^0(y)] = 0 \quad \langle \mathbf{X} \rangle = 0$$

(${}^S\nabla_y$ is the symmetrized gradient tensor). The variational formulation is the following:

$$\int_\Omega c^0(y) \mathbin{..} \mathbf{e}_y(w)\, \mathrm{d}v = \int_\Omega [\mathbf{e}_y(w) \mathbin{..} c \mathbin{..} \mathbf{e}_y(X) + c \mathbin{..} \mathbf{e}_y(w)]\, \mathrm{d}v = 0.$$

Using index notations, we have more explicitly:

$$u_i^1 = U^1 + \mathbf{X}_i^{kl} \mathbf{e}_{xkl}(U^0).$$

The vectors $\mathbf{X}^{kl}$ being the solutions to the systems:

$$c_{ij}^{0\,kl} = c_{ij}^{kl} + c_{ij}^{rt} \mathbf{e}_{yrt}[X^{kl}] \quad (c_{ij}^{0\,kl})_{,j} = 0 \quad \langle \mathbf{X}_i^{kl} \rangle = 0.$$

Order ε^0. With respect to this order the macroscopic balance equations are no longer obvious. We obtain them by integrating over the cell the considered system (1c):

$$L^{-2}(u^2) = -L^{-1}(u^1) - L^0(U^0)$$

which is more convenient to write in its equivalent form:

$$\nabla_y \cdot \boldsymbol{\sigma}^1 + \nabla_x \cdot \boldsymbol{\sigma}^0 + \rho(y) f^0(x) = 0.$$

Taking into account the stress periodicity, we have:

$$\nabla_x \cdot [\langle \boldsymbol{\sigma}^0 \rangle] + \langle \rho \rangle f^0 = 0 \quad \langle \boldsymbol{\sigma}^0 \rangle = \langle \mathbf{c} \mathbin{..} [\mathbf{e}_y(u^1) + \mathbf{e}_x(U^0)] \rangle.$$

Then, by putting the expression of u^1 into $\boldsymbol{\sigma}^0$, we can deduce the macroscopic momentum equation at zero order eqn (8a):

$$\nabla_x \cdot [\mathbf{C}^0 \mathbin{..} \mathbf{e}_x(U^0)] = -\langle \rho \rangle f^0 \quad \mathbf{C}^0 = \langle c^0 \rangle.$$

The set of equations allowing the determination of u^2 is:

$$\nabla_y \cdot [c \mathbin{..} (\mathbf{e}_y(u^2) + \mathbf{e}_x(u^1))] = -[\nabla_x \cdot [c \mathbin{..} (\mathbf{e}_y(u^1) + \mathbf{e}_x(U^0))] + \rho(y) f^0].$$

In this equation we substitute u^1 by its expression and $\rho(y) f^0$ by the macroscopic dynamics eqn (8a) obtained above. We then get:

$$\nabla_y \cdot [c \mathbin{..} (\mathbf{e}_y(u^2) + \mathbf{e}_x(\mathbf{X} \mathbin{..} \mathbf{e}_x(U^0)))] + \nabla_y \cdot [c \mathbin{..} \mathbf{e}_x(U^1)] = -\nabla_x \cdot [c^0 \mathbin{..} \mathbf{e}_x(U^0) - \beta \mathbf{C}^0 \mathbin{..} \mathbf{e}_x(U^0)].$$

In order to simplify the notation, we introduced $\beta(y)$, the ratio of the density to the average density:

$$\beta(y) = \rho(y) / \langle \rho \rangle.$$

We observe that the solution u^2 depends on two forcing terms:

- The first one is associated with $\mathbf{e}_x(U^1)$,
- The second one is associated with $\nabla_x \mathbf{e}_x(U^0)$ i.e. the *gradient* of the strain tensor of the displacement U^0 at order zero.

As a consequence of the linearity of the system, the field solution is a linear combination of particular solutions associated with each of these forcing terms. It is important to notice that the problems linked to the displacement at first order are identical to those already treated at zero order. Consequently we have:

$$u^2(x, y) = U^2(x) + \mathbf{X}(y) \mathbin{..} \mathbf{e}_x(U^1) + \mathbf{Y}(y) \mathbin{...} \nabla_x \mathbf{e}_x(U^0).$$

The fourth-rank tensor $\mathbf{Y}$ is constructed from the particular solutions $\mathbf{Y}^{klm}$ and verifies:

$$c^1(y) = c \cdot \mathbf{X} + c \mathbin{..} {}^S\nabla_y[Y] \quad \nabla_y \cdot [c^1(y)] = -[c^0(y) - \beta(y) \mathbf{C}^0] \quad \langle \mathbf{Y} \rangle = 0,$$

which corresponds to the variational formulation

$$\int_{\Omega} c^1(y)\,..\,\mathbf{e}_y(w)\,\mathrm{d}v = \int_{\Omega} [\mathbf{e}_y(w)\,..\,c\,..\,\mathbf{e}_y(Y) + c\,.\,\mathbf{X}\,..\,\mathbf{e}_y(w)]\,\mathrm{d}v = \int_{\Omega} [c^0(y) - \beta(y)\mathbf{C}^0]\,.\,w\,\mathrm{d}v.$$

Or when using the indicial notation:

$$u_i^2 = U^2 + \mathbf{X}_i^{pq}\mathbf{e}_{xpq}(U^1) + \mathbf{Y}_i^{klm}\nabla_x\mathbf{e}_x(U^0)_{klm}.$$

The vectors $\mathbf{Y}^{klm}$ being the solutions to the systems:

$$c_{ij}^{1\,klm} = c_{ij}^{mr}\mathbf{X}_r^{kl} + c_{ij}^{st}\mathbf{e}_{yst}[\mathbf{Y}^{klm}]$$

$$(c_{ij}^{1\,klm}),j = \beta\mathbf{C}_{im}^{0\;\;kl} - \mathbf{c}_{im}^{0\;\;kl} \quad \langle \mathbf{Y}_i^{klm} \rangle = 0.$$

Order ε^1. As above, we first establish the balance equation at this order. We obtain:

$$\nabla_x\,.\,[\langle \boldsymbol{\sigma}^1 \rangle] = 0 \quad \langle \boldsymbol{\sigma}^1 \rangle = \langle c\,..\,[\mathbf{e}_y(u^2) + e_x(u^1)] \rangle.$$

In order to have an equation where only average displacements appear, we introduce the expressions of the fields that have already been determined. Thus, we get:

$$\boldsymbol{\sigma}^1 = c\,..\,[\mathbf{e}_y(\mathbf{Y}\,...\,\nabla_x\mathbf{e}_x(U^0) + \mathbf{X}\,..\,\mathbf{e}_x(U^1)) + \mathbf{e}_x(\mathbf{X}\,..\,\mathbf{e}_x(U^0) + U^1)] = c^1\,...\,\nabla_x\mathbf{e}_x(U^0) + c^0\,..\,\mathbf{e}_x(U^1).$$

Consequently, the momentum balance at the first-order is eqn (8):

$$\nabla_x\,.\,[\mathbf{C}^0\,..\,\mathbf{e}_x(U^1)] = -\nabla_x\,.\,[\mathbf{C}^1\,..\,\nabla_x\mathbf{e}_x(U^0)] \quad \mathbf{C}^1 = \langle \mathbf{c}^1 \rangle.$$

The determination of the field u^3, is achieved by solving:

$$\nabla_y\,.\,[c\,..\,\mathbf{e}_y(u^3) + \mathbf{e}_x(u^2)] = -\nabla_x\,.\,[c\,..\,\mathbf{e}_y(u^2) + \mathbf{e}_x(u^1)].$$

That is, when expressing the different fields:

$$\begin{aligned}&\nabla_y\,.\,c\,..\,[\mathbf{e}_y(u^3) + \mathbf{e}_x(\mathbf{Y}\,...\,\nabla_x\mathbf{e}_x(U^0))] + \nabla_y\,.\,e_x(\mathbf{X}\,..\,\mathbf{e}_x(U^1)) + \nabla_y\,.\,e_x(U^2)\\ &\qquad = -\nabla_x\,.\,[c\,..(\mathbf{e}_y(\mathbf{Y}\,...\,\nabla_x\mathbf{e}_x(U^0))) + \mathbf{e}_x(\mathbf{X}\,..\,\mathbf{e}_x(U^0))] - \nabla_x\,.\,[c\,..(\mathbf{e}_y(\mathbf{X}\,..\,\nabla_x\mathbf{e}_x(U^1))) + \mathbf{e}_x(U^1)]\end{aligned}$$

so, introducing the tensor $\mathbf{C}^0$ and $\mathbf{c}^1$:

$$\nabla_y\,.\,c\,..\,[\mathbf{e}_y(u^3) + \mathbf{e}_x(\mathbf{Y}\,...\,\nabla_x\mathbf{e}_x(U^0))] + \nabla_y\,.\,e_x(\mathbf{X}\,..\,\mathbf{e}_x(U^1)) + \nabla_y\,.\,e_x(U^2) = -\nabla_x\,.\,[c^1\,...\,\nabla_x\mathbf{e}_x(U^0) + c^0\,...\,\mathbf{e}_x(U^1)].$$

Let us now substract from this equation the momentum balance at the first order [eqn (8b)] multiplied by β:

$$\begin{aligned}&\nabla_y\,.\,c\,..\,[\mathbf{e}_y(u^3) + \mathbf{e}_x(\mathbf{Y}\,...\,\nabla_x\mathbf{e}_x(U^0))] + \nabla_y\,.\,e_x(\mathbf{X}\,..\,\mathbf{e}_x(U^1)) + \nabla_y\,.\,e_x(U^2)\\ &\qquad = -\nabla_x\,.\,[c^1 - \beta\mathbf{C}^1]\,...\,\nabla_x\mathbf{e}_x(U^0) - \nabla_x\,.\,[c^0 - \beta\mathbf{C}^0]\,..\,\mathbf{e}_x(U^1).\end{aligned}$$

We notice that the solution u^3 depends on three forcing terms associated with:

(a) the strain tensor of the average displacement at second-order:
(b) the gradient of strain tensor of the average displacement at first-order;
(c) the double gradient of strain tensor of the displacement at zero-order.

The first two terms lead to the preceding solved problems and only the latter, introduces a new problem to be solved. Because of the linearity, the solution is as follows:

$$u(x,y) = U^3(x) + \mathbf{X}(y)\,..\,\mathbf{e}_x(U^2) + \mathbf{Y}(y)\,...\,\nabla_x\,.\,e_x(U^1) + \mathbf{Z}(y)\,....\,\nabla_x\,.\,\nabla_x\mathbf{e}_x(U^0).$$

$\mathbf{Z}$ is the fifth rank tensor constructed from the particular solutions $\mathbf{Z}^{klmn}$. It verifies the equations:

$$c^2(y) = c\,.\,Y + c\,..\,{}^s\nabla_y[Z] \quad \langle \mathbf{Z} \rangle = 0.$$

$$\nabla_y\,.\,[c^2(y)] = -[c^1(y) - \beta(y)\mathbf{C}^1].$$

Or, with indicial notation:

$$u_i^3 = U_i^3 + \mathbf{X}_i^{st}\mathbf{e}_{xst}(U^2) + \mathbf{Y}_i^{pqr}\nabla_x\mathbf{e}_x(U^1)_{pqr} + \mathbf{Z}_i^{klmn}\nabla_x\nabla_x\mathbf{e}_x(U^0)_{klmn},$$

where the vectors $\mathbf{Z}^{klmn}$ are the solutions to the systems;

$$c_{ij}^{2\,klmn} = c_{ij}^{nr}\mathbf{Y}_r^{klm} + c_{ij}^{st}\mathbf{e}_{yst}[\mathbf{Z}^{klmn}] \quad \langle \mathbf{Z}_i^{klmn}\rangle = 0$$

$$(c_{ij}^{2\,klmn})_{,j} = \beta\mathbf{C}_{in}^{1\,klm} - c_{in}^{1\,klm}.$$

Order ε^2. At this order we are only interested in the momentum balance, which is obtained as above:

$$\nabla_x\,.\,[\langle\boldsymbol{\sigma}^2\rangle] = 0 \quad \langle\boldsymbol{\sigma}^2\rangle = \langle c\,.\,.\,[\mathbf{e}_y(u^3)+\mathbf{e}_x(u^2)]\rangle.$$

After replacing u^2 and u^3 by their expression, we get eqn (8c):

$$\nabla_x\,.\,[\mathbf{C}^0\,.\,.\,\mathbf{e}_x(U^2)] = -\nabla_x\,.\,[\mathbf{C}^1\,.\,.\,.\,\nabla_x\mathbf{e}_x(U^1)] - \nabla_x\,.\,[\mathbf{C}^2\,.\,.\,.\,.\,\nabla_x\nabla_x\mathbf{e}_x(U^0)] \quad \mathbf{C}^2 = \langle c^2\rangle.$$

Note. These results have been obtained in the case of a volume force $\rho(y)f^0(x)$. The case of constant volume force—and therefore no volume force—can easily be deduced by putting in all the developments $\beta = 1$. Let us mention that the formalism remains the same but the expressions of tensors $\mathbf{C}^1$ and $\mathbf{C}^2$ would be different.

Relation between tensors $\mathbf{c}^1$ *and* $\mathbf{c}^0$. From the variational formulations we can link $\mathbf{c}^1$ and $\mathbf{c}^0$.
Let us transform the term including $\mathbf{Y}$, in the expression of the average value of:

$$c^1(y) = c\,.\,\mathbf{X} + c\,.\,.^s\nabla_y[Y] \quad c_{ij}^{1\,klm} = c_{ij}^{mr}\mathbf{X}_r^{kl} + c_{ij}^{st}\mathbf{e}_{yst}[\mathbf{Y}^{klm}].$$

In order to achieve this, we use the variational formulations associated with the fields $\mathbf{X}^{kl}$. They express that any continuous periodic field w verifies:

$$\int_\Omega c_{ij}^{st}\,.\,.\,\mathbf{e}_{yst}(w)\,\mathrm{d}v = -\int_\Omega \mathbf{e}_y(w)\,.\,.\,c\,.\,.\,\mathbf{e}_y(\mathbf{X}^{ij})\,\mathrm{d}v.$$

When we chose $w = \mathbf{Y}^{klm}$, taking into account the symmetry of c, we obtain:

$$\int_\Omega [c_{ij}^{1\,klm} - c_{ij}^{mu}\mathbf{X}_u^{kl}]\,\mathrm{d}v = \int_\Omega [c_{ij}^{st}\mathbf{e}_{yst}(\mathbf{Y}^{klm})]\,\mathrm{d}v = -\int_\Omega \mathbf{e}_y(\mathbf{Y}^{klm})\,.\,.\,c\,.\,.\,\mathbf{e}_y(\mathbf{X}^{ij})\,\mathrm{d}v.$$

Now, using $\mathbf{X}^{ij}$ as test field in the variational formulation associated with $\mathbf{Y}^{klm}$, we get:

$$-\int_\Omega \mathbf{e}_y(\mathbf{Y}^{klm})\,.\,.\,c\,.\,.\,\mathbf{e}_y(X^{ij})\,\mathrm{d}v = \int_\Omega [(\nabla_y\,.\,c^{lklm})X^{ij} + c_{st}^{mu}\mathbf{X}_u^{kl}\mathbf{e}_{yst}(X^{ij})]\,\mathrm{d}v.$$

So:

$$\int_\Omega c_{ij}^{1\,klm}\,\mathrm{d}v = \int_\Omega (\nabla_y\,.\,c^{lklm})X^{ij}\,\mathrm{d}v + \int_\Omega [c_{st}^{mu}\mathbf{X}_u^{kl}\mathbf{e}_{yst}(\mathbf{X}^{ij}) + c_{ij}^{mu}\mathbf{X}_u^{kl}]\,\mathrm{d}v.$$

That is, when introducing c^0;

$$\int_\Omega c_{ij}^{1\,klm}\,\mathrm{d}v = \int_\Omega (\nabla_y\,.\,c^{lklm})\mathbf{X}^{ij}\,\mathrm{d}v + \int_\Omega c_{mu}^{0\;\;ij}\mathbf{X}_u^{kl}\,\mathrm{d}v.$$

And, replacing the divergence by its expression, we establish the identity:

$$\mathbf{C}_{ij}^{1\,klm} = \mathbf{C}_{pm}^{0\;\;kl}\langle\rho\mathbf{X}_p^{ij}\rangle/\langle\rho\rangle + \langle\mathbf{c}_{mu}^{0\;\;ij}\mathbf{X}_u^{kl} - \mathbf{c}_{mu}^{0\;\;kl}\mathbf{X}_u^{ij}\rangle$$

Note. In the case of constant volume forces this relation becomes:

$$\mathbf{C}_{ij}^{1\,klm} = \langle\mathbf{c}_{mu}^{0\;\;ij}\mathbf{X}_u^{kl} - \mathbf{c}_{mu}^{0\;\;kl}\mathbf{X}_u^{ij}\rangle$$

which proves the antisymmetry of tensor $\mathbf{C}^1$ in regards with the (i,j) and (k,l) index.

APPENDIX B

CALCULATION OF TENSORS C′ AND C″ IN STRATIFIED COMPOSITES

Notations
For any function ψ taking the constant value ψ_a in the layer a and ψ_b in the layer b:

$$\psi^{-1} = (1-\mathbf{c})/\psi_a + \mathbf{c}/\psi_b \qquad \langle\psi\rangle = (1-\tau)\psi_a + \tau\psi_b \qquad \mathbf{D}(\psi) = \tau(1-\tau)h_m(\psi_a - \psi_b)$$

$$k = (\underline{\lambda+2\mu})\mathbf{D}(1/(\lambda+2\mu)) \qquad m = \underline{\mu}\mathbf{D}(1/\mu)$$

$$\eta = \lambda/(\lambda+2\mu) \qquad \gamma = \langle\eta\rangle\eta + \lambda(1-\eta)/(\underline{\lambda+2\mu}) \qquad \phi = \gamma + 2\mu/(\underline{\lambda+2\mu})$$

$f(y)$ is the function defined by: $f(y) = \begin{cases} [y/h_m - (1-\tau)/2]/(1-\tau) & \text{in layer } a \\ -[y/h_m + \tau/2]/\tau & \text{in layer } b \end{cases}$

$F(y)$ is the function: $F(y) = \int f(u)\,\mathrm{d}u = \begin{cases} y[y/h_m - (1-\tau)]/2(1-\tau) & \text{in layer } a \\ -y[y/h_m + \tau]/2\tau & \text{in layer } b \end{cases}$

Determination of $\mathbf{C}'$.

The components of $\mathbf{Y}$ which do not equal zero are the following:

$$Y_1^{111} = k.(\langle F\rangle - F)$$

$$Y_1^{221} = Y_1^{331} = k.(\langle F\rangle - F)\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu)$$

$$Y_1^{122} = Y_1^{133} = 2m.(\langle\eta F\rangle - \eta F)$$

$$Y_2^{112} = Y_3^{113} = k.(\langle F\rangle - F) + (\langle F/\mu\rangle - F/\mu)(\underline{\lambda+2\mu})\mathbf{D}(\eta)$$

$$Y_2^{222} = Y_3^{333} = k.(\langle F\rangle - F)\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu) + (\langle F/\mu\rangle - F/\mu)(\underline{\lambda+2\mu})\mathbf{D}(\phi)$$

$$Y_2^{332} = Y_3^{223} = k.(\langle F\rangle - F)\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu) + (\langle F/\mu\rangle - F/\mu)(\underline{\lambda+2\mu})\mathbf{D}(\gamma)$$

$$Y_2^{121} = Y_3^{131} = 2m.(\langle F\rangle - F)$$

$$Y_2^{233} = Y_3^{232} = 2(\langle F/\mu\rangle - F/\mu)\mathbf{D}(\mu).$$

We deduce the expressions of the stress fields (different from zero) which result from the gradient of the macroscopic strain tensor:

$$c^1_{22}{}^{122} = c^1_{33}{}^{133} = (\underline{\lambda+2\mu})(\phi - \eta\mathbf{D}(\eta))2m.f$$

$$c^1_{33}{}^{122} = c^1_{33}{}^{122} = (\underline{\lambda+2\mu})(\gamma - \eta\mathbf{D}(\eta))2m.f$$

$$c^1_{12}{}^{112} = c^1_{13}{}^{113} = -(\underline{\lambda+2\mu})\mathbf{D}(\eta).f$$

$$c^1_{12}{}^{222} = c^1_{13}{}^{333} = -(\underline{\lambda+2\mu})\mathbf{D}(\phi).f$$

$$c^1_{12}{}^{332} = c^1_{13}{}^{223} = -(\underline{\lambda+2\mu})\mathbf{D}(\gamma).f$$

$$c^1_{13}{}^{232} = c^1_{12}{}^{233} = -\mathbf{D}(\mu).f$$

Note that the components of the stresses $\mathbf{c}^1$ are of the type $\psi \cdot f$ and consequently:

$$\mathbf{C}^1 = \langle\mathbf{c}^1\rangle = 0 \quad \text{so} \quad \mathbf{C}' = 0.$$

Determination of $\mathbf{C}''$.

The expressions below give all the components of $\mathbf{C}''$ which do not equal zero. The non-explicitly written terms are obtained either by substituting index 2 by 3—if index 2 appears alone (for example $\mathbf{C}''_{22}{}^{1212} = \mathbf{C}''_{33}{}^{1313}$) —or by permutation of index 2 and 3 (for example $\mathbf{C}''_{12}{}^{2331} = \mathbf{C}''_{13}{}^{3221}$). In order to simplify the writing we note:

$$\mathbf{D}'(1/(\lambda+2\mu);\eta) = \langle\eta\rangle\mathbf{D}(1/(\lambda+2\mu)) - \mathbf{D}(\eta)/(\underline{\lambda+2\mu})$$

$$\mathbf{C}''_{11}{}^{1212} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\eta).m/6$$

$$\mathbf{C}''_{22}{}^{1212} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\phi).m/6$$

$$\mathbf{C}''_{33}{}^{1212} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\gamma).m/6$$

$$\mathbf{C}''_{11}{}^{2332} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\mu)\{\mathbf{D}(\eta)/\underline{\mu} + \mathbf{D}(1/(\lambda+2\mu))/2\}/6$$

$$\mathbf{C}''_{22}{}^{2332} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\mu)\{\mathbf{D}(\phi)/\underline{\mu} + \mathbf{D}'(1/(\lambda+2\mu);\eta)/2\}/6$$

$$\mathbf{C}''_{33}{}^{2332} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\mu)\{\mathbf{D}(\gamma)/\underline{\mu} + \mathbf{D}'(1/(\lambda+2\mu);\eta)/2\}/6$$

$$\mathbf{C}''_{11}{}^{1122} = h^2(\underline{\lambda+2\mu})\{\mathbf{D}(\eta)[k + \mathbf{D}(\eta)(\underline{\lambda+2\mu})/\underline{\mu}] + (\underline{\lambda+2\mu})\mathbf{D}(1/(\lambda+2\mu))\mathbf{D}(\eta)\}/12$$

$$\mathbf{C}''_{22}{}^{1122} = h^2(\underline{\lambda+2\mu})\{\mathbf{D}(\phi)[k+\mathbf{D}(\eta)(\underline{\lambda+2\mu})/\underline{\mu}] + (\underline{\lambda+2\mu})\mathbf{D}'(1/(\lambda+2\mu)\,;\eta)\mathbf{D}(\eta)\}/12$$

$$\mathbf{C}''_{33}{}^{1122} = h^2(\underline{\lambda+2\mu})\{\mathbf{D}(\gamma)[k+\mathbf{D}(\eta)(\underline{\lambda+2\mu})/\underline{\mu}] + (\underline{\lambda+2\mu})\mathbf{D}'(1/(\lambda+2\mu)\,;\eta)\mathbf{D}(\eta)\}/12$$

$$\mathbf{C}''_{11}{}^{2222} = h^2(\underline{\lambda+2\mu})\{\mathbf{D}(\eta)[k\,.\,\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu)+\mathbf{D}(\phi)(\underline{\lambda+2\mu})/\underline{\mu}] + (\underline{\lambda+2\mu})\mathbf{D}(1/(\lambda+2\mu))\,.\,\mathbf{D}(\phi)\}/12$$

$$\mathbf{C}''_{22}{}^{2222} = h^2(\underline{\lambda+2\mu})\{\mathbf{D}(\phi)[k\,.\,\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu)+\mathbf{D}(\phi)(\underline{\lambda+2\mu})/\underline{\mu}] + (\underline{\lambda+2\mu})\mathbf{D}'(1/(\lambda+2\mu)\,;\eta)\mathbf{D}(\phi)\}/12$$

$$\mathbf{C}''_{33}{}^{2222} = h^2(\underline{\lambda+2\mu})\{\mathbf{D}(\gamma)[k\,.\,\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu)+\mathbf{D}(\phi)(\underline{\lambda+2\mu})/\underline{\mu}] + (\underline{\lambda+2\mu})\mathbf{D}'(1/(\lambda+2\mu)\,;\eta)\mathbf{D}(\phi)\}/12$$

$$\mathbf{C}''_{11}{}^{3322} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\mu)\{\mathbf{D}(\eta)[k\,.\,\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu)+\mathbf{D}(\gamma)(\underline{\lambda+2\mu})/\underline{\mu}] + (\underline{\lambda+2\mu})\mathbf{D}(1/(\lambda+2\mu))\mathbf{D}(\gamma)\}/12$$

$$\mathbf{C}''_{22}{}^{3322} = h^2(\underline{\lambda+2\mu})\{\mathbf{D}(\phi)[k\,.\,\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu)+\mathbf{D}(\gamma)(\underline{\lambda+2\mu})/\underline{\mu}] + (\underline{\lambda+2\mu})\mathbf{D}'(1/(\lambda+2\mu)\,;\eta)\mathbf{D}(\gamma)\}/12$$

$$\mathbf{C}''_{33}{}^{3322} = h^2(\underline{\lambda+2\mu})\{\mathbf{D}(\gamma)[k\,.\,\mathbf{D}(\gamma)/\mathbf{D}(\lambda+2\mu)+\mathbf{D}(\gamma)(\underline{\lambda+2\mu})/\underline{\mu}] + (\underline{\lambda+2\mu})\mathbf{D}'(1/(\lambda+2\mu)\,;\eta)\mathbf{D}(\gamma)\}/12$$

$$\mathbf{C}''_{23}{}^{1123} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\mu)\{\mathbf{D}(1/(\lambda+2\mu))+\mathbf{D}(\eta)/\underline{\mu}\}/12$$

$$\mathbf{C}''_{23}{}^{2223} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\mu)\{\mathbf{D}(1/(\lambda+2\mu))\,.\,\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu)+\mathbf{D}(\phi)/\underline{\mu}\}/12$$

$$\mathbf{C}''_{23}{}^{3333} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\mu)\{\mathbf{D}(1/(\lambda+2\mu))\,.\,\mathbf{D}(\lambda)/\mathbf{D}(\lambda+2\mu)+\mathbf{D}(\gamma)/\underline{\mu}\}/12$$

$$\mathbf{C}''_{23}{}^{1213} = h^2 m\mathbf{D}(\mu)/6$$

$$\mathbf{C}''_{23}{}^{2333} = h^2\mathbf{D}(\mu/\underline{\mu})\mathbf{D}(\mu)/6$$

$$\mathbf{C}''_{12}{}^{1332} = h^2(\underline{\lambda+2\mu})m^2\langle\eta\mathbf{D}(\eta)-\phi\rangle/6$$

$$\mathbf{C}''_{12}{}^{1222} = h^2(\underline{\lambda+2\mu})m^2\langle\eta\mathbf{D}(\eta)-\gamma\rangle/6$$

$$\mathbf{C}''_{12}{}^{1121} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\eta)\,.\,m/12$$

$$\mathbf{C}''_{12}{}^{2221} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\phi)\,.\,m/12$$

$$\mathbf{C}''_{12}{}^{3321} = h^2(\underline{\lambda+2\mu})\mathbf{D}(\gamma)\,.\,m/12$$

$$\mathbf{C}''_{12}{}^{3321} = h^2\mathbf{D}(\mu)\,.\,m\,.\,/12.$$