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Lie, symplectic and Poisson groupoids and their Lie algebroids

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Introduction

Groupoids are mathematical structures able to describe symmetry properties more general than those described by groups. They were introduced (and named) by H. Brandt in 1926. Around 1950, Charles Ehresmann used groupoids with additional structures (topological and differentiable) as essential tools in topology and differential geometry. In recent years, Mickael Karasev, Alan Weinstein and Stanisław Zakrzewski independently discovered that symplectic groupoids can be used for the construction of noncommutative deformations of the algebra of smooth functions on a manifold, with potential applications to quantization. Poisson groupoids were introduced by Alan Weinstein as generalizations of both Poisson Lie groups and symplectic groupoids.

We present here the main definitions and first properties relative to groupoids, Lie groupoids, Lie algebroids, symplectic and Poisson groupoids and their Lie algebroids.

1 Groupoids

1.1 What is a groupoid? Before stating the formal definition of a groupoid, let us explain, in an informal way, why it is a very natural concept. The easiest way to understand that concept is to think of two sets, Γ and Γ_0. The first one, Γ, is called the *set of arrows* or *total space* of the groupoid, and the other one, Γ_0, the *set of objects* or *set of units* of the groupoid. One may think of an element $x \in \Gamma$ as an arrow going from an object (a point in Γ_0) to another object (another point in Γ_0). The word "arrow" is used here in a very general sense: it means a way for going from a point in Γ_0 to another point in Γ_0. One should not think of an arrow as a line drawn in the set Γ_0 joining the starting point of the arrow to its end point: this happens only for some special groupoids. Rather, one should think of an arrow as living outside Γ_0, with only its starting point and its end point in Γ_0, as shown on Figure 1.

The following ingredients enter the definition of a groupoid.

– Two maps $\alpha : \Gamma \to \Gamma_0$ and $\beta : \Gamma \to \Gamma_0$, called the *target map* and the *source map* of the groupoid. If $x \in \Gamma$ is an arrow, $\alpha(x) \in \Gamma_0$ is its end point and $\beta(x) \in \Gamma_0$ its starting point.

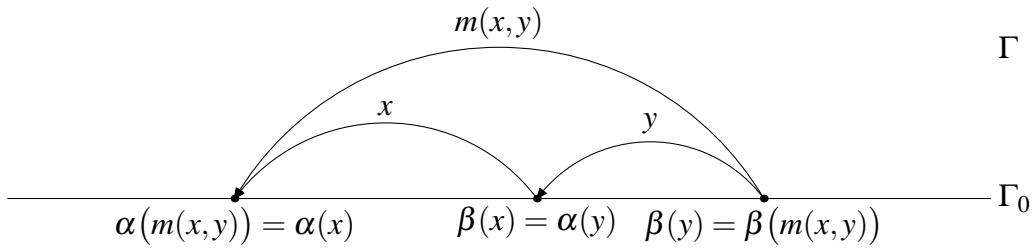

Figure 1: Two arrows x and $y \in \Gamma$, with the target of y, $\alpha(y) \in \Gamma_0$, equal to the source of x, $\beta(x) \in \Gamma_0$, and the composed arrow $m(x,y)$.

– A *composition law* on the set of arrows; we can compose an arrow y with another arrow x, and get an arrow $m(x,y)$, by following first the arrow y, then the arrow x. Of course, $m(x,y)$ is defined if and only if the target of y is equal to the source of x. The source of $m(x,y)$ is equal to the source of y, and its target is equal to the target of x, as illustrated on Figure 1. It is only by convention that we write $m(x,y)$ rather than $m(y,x)$: the arrow which is followed first is on the right, by analogy with the usual notation $f \circ g$ for the composition of two maps g and f. When there will be no risk of confusion we will write $x \circ y$, or $x.y$, or even simply xy for $m(x,y)$.The composition of arrows is associative.

– An *embedding* ε of the set Γ_0 into the set Γ, which associates a unit arrow $\varepsilon(u)$ with each $u \in \Gamma_0$. That unit arrow is such that both its source and its target are u, and it plays the role of a unit when composed with another arrow, either on the right or on the left: for any arrow x, $m\Big(\varepsilon\big(\alpha(x)\big),x\Big) = x$, and $m\Big(x,\varepsilon\big(\beta(x)\big)\Big) = x$.

– Finally, an *inverse map* ι from the set of arrows onto itself. If $x \in \Gamma$ is an arrow, one may think of $\iota(x)$ as the arrow x followed in the reverse sense. We will often write x^{-1} for $\iota(x)$.

Now we are ready to state the formal definition of a groupoid.

1.2 Definition. *A* groupoid *is a pair of sets* (Γ,Γ_0) *equipped with the structure defined by the following data:*

– *an injective map* $\varepsilon : \Gamma_0 \to \Gamma$*, called the* unit section *of the groupoid;*

– *two maps* $\alpha : \Gamma \to \Gamma_0$ *and* $\beta : \Gamma \to \Gamma_0$*, called, respectively, the* target map *and the* source map*; they satisfy*

$$\alpha \circ \varepsilon = \beta \circ \varepsilon = \mathrm{id}_{\Gamma_0}\,; \tag{1}$$

– *a composition law* $m : \Gamma_2 \to \Gamma$*, called the* product*, defined on the subset* Γ_2 *of* $\Gamma \times \Gamma$*, called the* set of composable elements*,*

$$\Gamma_2 = \big\{\,(x,y) \in \Gamma \times \Gamma; \beta(x) = \alpha(y)\,\big\}\,, \tag{2}$$

which is associative, in the sense that whenever one side of the equality

$$m\big(x,m(y,z)\big) = m\big(m(x,y),z\big) \tag{3}$$

is defined, the other side is defined too, and the equality holds; moreover, the composition law m is such that for each $x \in \Gamma$*,*

$$m\Big(\varepsilon\big(\alpha(x)\big),x\Big) = m\Big(x,\varepsilon\big(\beta(x)\big)\Big) = x\,; \tag{4}$$

– *a map* $\iota : \Gamma \to \Gamma$, *called the* inverse, *such that, for every* $x \in \Gamma$, $\big(x, \iota(x)\big) \in \Gamma_2$ *and* $\big(\iota(x), x\big) \in \Gamma_2$, *and*

$$m\big(x, \iota(x)\big) = \varepsilon\big(\alpha(x)\big)\,, \quad m\big(\iota(x), x\big) = \varepsilon\big(\beta(x)\big)\,. \tag{5}$$

The sets Γ *and* Γ_0 *are called, respectively, the* total space *and the* set of units *of the groupoid, which is itself denoted by* $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$.

1.3 Identification and notations In what follows, by means of the injective map ε, we will identify the set of units Γ_0 with se subset $\varepsilon(\Gamma_0)$ of Γ. Therefore ε will be the canonical injection in Γ of its subset Γ_0.

For x and $y \in \Gamma$, we will sometimes write $x.y$, or even simply xy for $m(x,y)$, and x^{-1} for $\iota(x)$. Also we will write "the groupoid Γ" for "the groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$.

1.4 Properties and comments The above definitions have the following consequences.

1.4.1 Involutivity of the inverse map The inverse map ι is involutive:

$$\iota \circ \iota = \mathrm{id}_\Gamma\,. \tag{6}$$

We have indeed, for any $x \in \Gamma$,

$$\begin{aligned}\iota \circ \iota(x) &= m\big(\iota \circ \iota(x), \beta\big(\iota \circ \iota(x)\big)\big) = m\big(\iota \circ \iota(x), \beta(x)\big) = m\big(\iota \circ \iota(x), m\big(\iota(x), x\big)\big)\\ &= m\big(m\big(\iota \circ \iota(x), \iota(x)\big), x\big) = m\big(\alpha(x), x\big) = x\,.\end{aligned}$$

1.4.2 Unicity of the inverse Let x and $y \in \Gamma$ be such that

$$m(x,y) = \alpha(x) \quad \text{and} \quad m(y,x) = \beta(x)\,.$$

Then we have

$$\begin{aligned}y &= m\big(y, \beta(y)\big) = m\big(y, \alpha(x)\big) = m\big(y, m\big(x, \iota(x)\big)\big) = m\big(m(y,x), \iota(x)\big)\\ &= m\big(\beta(x), \iota(x)\big) = m\big(\alpha\big(\iota(x)\big), \iota(x)\big) = \iota(x)\,.\end{aligned}$$

Therefore for any $x \in \Gamma$, the unique $y \in \Gamma$ such that $m(y,x) = \beta(x)$ and $m(x,y) = \alpha(x)$ is $\iota(x)$.

1.4.3 The fibers of α and β and the isotropy groups The target map α (resp. the source map β) of a groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ determines an equivalence relation on Γ: two elements x and $y \in \Gamma$ are said to be α-equivalent (resp. β-equivalent) if $\alpha(x) = \alpha(y)$ (resp. if $\beta(x) = \beta(y)$). The corresponding equivalence classes are called the α-*fibers* (resp. the β-*fibers*) of the groupoid. They are of the form $\alpha^{-1}(u)$ (resp. $\beta^{-1}(u)$), with $u \in \Gamma_0$.

For each unit $u \in \Gamma_0$, the subset

$$\Gamma_u = \alpha^{-1}(u) \cap \beta^{-1}(u) = \big\{\, x \in \Gamma; \alpha(x) = \beta(x) = u \,\big\} \tag{7}$$

is called the *isotropy group* of u. It is indeed a group, with the restrictions of m and ι as composition law and inverse map.

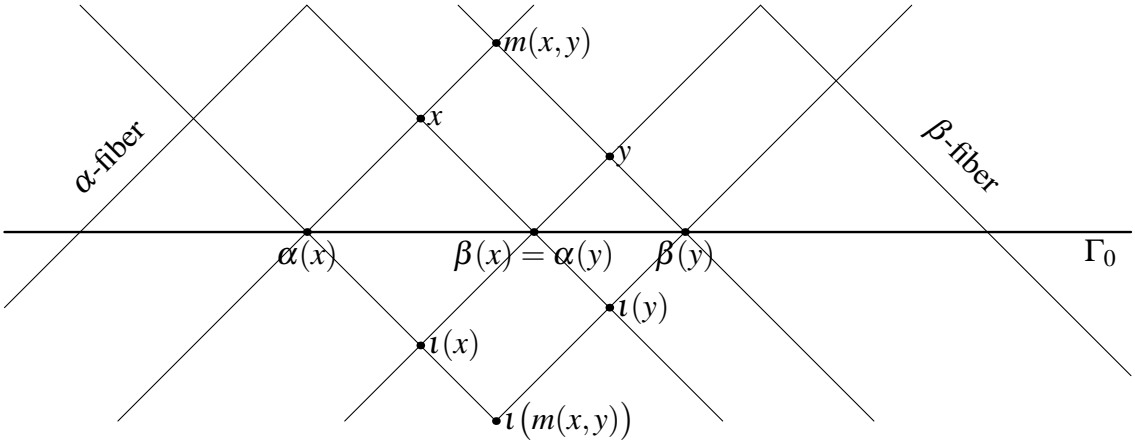

Figure 2: A way to visualize groupoids.

1.4.4 A way to visualize groupoids We have seen (Figure 1) a way in which groupoids may be visualized, by using arrows for elements in Γ and points for elements in Γ_0. There is another, very useful way to visualize groupoids, shown on Figure 2.

The total space Γ of the groupoid is represented as a plane, and the set Γ_0 of units as a straight line in that plane. The α-fibers (resp. the β-fibers) are represented as parallel straight lines, transverse to Γ_0.

1.5 Examples of groupoids

1.5.1 The groupoid of pairs Let E be a set. The *groupoid of pairs* of elements in E has, as its total space, the product space $E \times E$. The diagonal $\Delta_E = \big\{ (x,x); x \in E \big\}$ is its set of units, and the target and source maps are

$$\alpha : (x,y) \mapsto (x,x)\,, \quad \beta : (x,y) \mapsto (y,y)\,.$$

Its composition law m and inverse map ι are

$$m\big((x,y),(y,z)\big) = (x,z)\,, \quad \iota\big((x,y)\big) = (x,y)^{-1} = (y,x)\,.$$

1.5.2 Groups A group G is a groupoid with set of units $\{e\}$, with only one element e, the unit element of the group. The target and source maps are both equal to the constant map $x \mapsto e$.

1.6 Definitions. *A* topological groupoid *is a groupoid* $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ *for which* Γ *is a (maybe non Hausdorff) topological space,* Γ_0 *a Hausdorff topological subspace of* Γ*,* α *and* β *surjective continuous maps,* $m : \Gamma_2 \to \Gamma$ *a continuous map and* $\iota : \Gamma \to \Gamma$ *an homeomorphism.*

A Lie groupoid *is a groupoid* $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ *for which* Γ *is a smooth (maybe non Hausdorff) manifold,* Γ_0 *a smooth Hausdorff submanifold of* Γ*,* α *and* β *smooth surjective submersions (which implies that* Γ_2 *is a smooth submanifold of* $\Gamma \times \Gamma$*),* $m : \Gamma_2 \to \Gamma$ *a smooth map and* $\iota : \Gamma \to \Gamma$ *a smooth diffeomorphism.*

1.7 Properties of Lie groupoids

1.7.1 Dimensions Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid. Since α and β are submersions, for any $x \in \Gamma$, the α-fiber $\alpha^{-1}\big(\alpha(x)\big)$ and the β-fiber $\beta^{-1}\big(\beta(x)\big)$ are submanifolds of Γ, both of dimension $\dim\Gamma - \dim\Gamma_0$. The inverse map ι, restricted to the α-fiber through x (resp. the β-fiber through x) is a diffeomorphism of that fiber onto the β-fiber through $\iota(x)$ (resp. the α-fiber through $\iota(x)$). The dimension of the submanifold Γ_2 of composable pairs in $\Gamma \times \Gamma$ is $2\dim\Gamma - \dim\Gamma_0$.

1.7.2 The tangent bundle of a Lie groupoid Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid. Its tangent bundle $T\Gamma$ is a Lie groupoid, with $T\Gamma_0$ as set of units, $T\alpha : T\Gamma \to T\Gamma_0$ and $T\beta : T\Gamma \to T\Gamma_0$ as target and source maps. Let us denote by Γ_2 the set of composable pairs in $\Gamma \times \Gamma$, by $m : \Gamma_2 \to \Gamma$ the composition law and by $\iota : \Gamma \to \Gamma$ the inverse. Then the set of composable pairs in $T\Gamma \times T\Gamma$ is simply $T\Gamma_2$, the composition law on $T\Gamma$ is $Tm : T\Gamma_2 \to T\Gamma$ and the inverse is $T\iota : T\Gamma \to T\Gamma$.

When the groupoid Γ is a Lie group G, the Lie groupoid TG is a Lie group too.

We will see below that the cotangent bundle of a Lie groupoid is a Lie groupoid, and more precisely a symplectic groupoid.

1.7.3 Isotropy groups For each unit $u \in \Gamma_0$ of a Lie groupoid, the isotropy group Γ_u (defined in 1.4.3) is a Lie group.

1.8 Examples of topological and Lie groupoids

1.8.1 Topological groups and Lie groups A topological group (resp. a Lie group) is a topological groupoid (resp. a Lie groupoid) whose set of units has only one element e.

1.8.2 Vector bundles A smooth vector bundle $\pi : E \to M$ on a smooth manifold M is a Lie groupoid, with the base M as set of units (identified with the image of the zero section); the source and target maps both coincide with the projection π, the product and the inverse maps are the addition $(x,y) \mapsto x+y$ and the opposite map $x \mapsto -x$ in the fibers.

1.8.3 The fundamental groupoid of a topological space Let M be a topological space. A *path* in M is a continuous map $\gamma : [0,1] \to M$. We denote by $[\gamma]$ the homotopy class of a path γ and by $\Pi(M)$ the set of homotopy classes of paths in M (with fixed endpoints). For $[\gamma] \in \Pi(M)$, we set $\alpha\big([\gamma]\big) = \gamma(1)$, $\beta\big([\gamma]\big) = \gamma(0)$, where γ is any representative of the class $[\gamma]$. The concatenation of paths determines a well defined composition law on $\Pi(M)$, for which $\Pi(M) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$ is a topological groupoid, called the *fundamental groupoid* of M. The inverse map is $[\gamma] \mapsto [\gamma^{-1}]$, where γ is any representative of $[\gamma]$ and γ^{-1} is the path $t \mapsto \gamma(1-t)$. The set of units is M, if we identify a point in M with the homotoppy class of the constant path equal to that point.

When M is a smooth manifold, the same construction can be made with piecewise smooth paths, and the fundamental groupoid $\Pi(M) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} M$ is a Lie groupoid.

2 Symplectic and Poisson groupoids

2.1 Symplectic and Poisson geometry Let us recall some definitions and results in symplectic and Poisson geometry, used in the next sections.

2.1.1 Symplectic manifolds A *symplectic form* on a smooth manifold M is a differential 2-form ω, which is closed, *i.e.* which satisfies

$$d\omega = 0\,, \tag{8}$$

and nondegenerate, *i.e.* such that for each point $x \in M$ an each nonzero vector $v \in T_xM$, there exists a vector $w \in T_xM$ such that $\omega(v,w) \neq 0$. Equipped with the symplectic form ω, a smooth manifold M is called a *symplectic manifold* and denoted by (M,ω).

The dimension of a symplectic manifold is always even.

2.1.2 The Liouville form on a cotangent bundle Let N be a smooth manifold, and T^*N be its cotangent bundle. The *Liouville form* on T^*N is the 1-form θ such that, for any $\eta \in T^*N$ and $v \in T_\eta(T^*N)$,

$$\theta(v) = \big\langle \eta, T\pi_N(v) \big\rangle\,, \tag{9}$$

where $\pi_N : T^*N \to N$ is the canonical projection.

The 2-form $\omega = d\theta$ is symplectic, and is called the *canonical symplectic form* on the cotangent bundle T^*N.

2.1.3 Poisson manifolds A *Poisson manifold* is a smooth manifold P equipped with a bivector field (i.e. a smooth section of $\bigwedge^2 TP$) Π which satisfies

$$[\Pi,\Pi] = 0\,, \tag{10}$$

the bracket on the left hand side being the Schouten bracket. The bivector field Π will be called the *Poisson structure* on P. It allows us to define a composition law on the space $C^\infty(P,\mathbb{R})$ of smooth functions on P, called the *Poisson bracket* and denoted by $(f,g) \mapsto \{f,g\}$, by setting, for all f and $g \in C^\infty(P,\mathbb{R})$ and $x \in P$,

$$\{f,g\}(x) = \Pi\big(df(x), dg(x)\big)\,. \tag{11}$$

That composition law is skew-symmetric and satisfies the Jacobi identity, therefore turns $C^\infty(P,\mathbb{R})$ into a Lie algebra.

2.1.4 Hamiltonian vector fields Let (P,Π) be a Poisson manifold. We denote by $\Pi^\sharp : T^*P \to TP$ the vector bundle map defined by

$$\big\langle \eta, \Pi^\sharp(\zeta) \big\rangle = \Pi(\zeta,\eta)\,, \tag{12}$$

where ζ and η are two elements in the same fiber of T^*P. Let $f : P \to \mathbb{R}$ be a smooth function on P. The vector field $X_f = \Pi^\sharp(df)$ is called the *Hamiltonian vector field* associated to f. If $g : P \to \mathbb{R}$ is another smooth function on P, the Poisson bracket $\{f,g\}$ can be written

$$\{f,g\} = \big\langle dg, \Pi^\sharp(df) \big\rangle = -\big\langle df, \Pi^\sharp(dg) \big\rangle\,. \tag{13}$$

2.1.5 The canonical Poisson structure on a symplectic manifold Every symplectic manifold (M,ω) has a Poisson structure, associated to its symplectic structure, for which the vector bundle map $\Pi^\sharp : T^*M \to M$ is the inverse of the vector bundle isomorphism $v \mapsto -i(v)\omega$. We will always consider that a symplectic manifold is equipped with that Poisson structure, unless otherwise specified.

2.1.6 The KKS Poisson structure Let $\mathcal{G}$ be a finite-dimensional Lie algebra. Its dual space $\mathcal{G}^*$ has a natural Poisson structure, for which the bracket of two smooth functions f and g is

$$\{f,g\}(\xi) = \big\langle \xi, \big[df(\xi), dg(\xi)\big]\big\rangle\,, \tag{14}$$

with $\xi \in \mathcal{G}^*$, the differentials $df(\xi)$ and $dg(\xi)$ being considered as elements in $\mathcal{G}$, identified with its bidual $\mathcal{G}^{**}$. It is called the KKS Poisson structure on $\mathcal{G}^*$ (for Kirillov, Kostant and Souriau).

2.1.7 Poisson maps Let (P_1,Π_1) and (P_2,Π_2) be two Poisson manifolds. A smooth map $\varphi : P_1 \to P_2$ is called a *Poisson map* if for every pair (f,g) of smooth functions on P_2,

$$\{\varphi^* f, \varphi^* g\}_1 = \varphi^*\{f,g\}_2\,. \tag{15}$$

2.1.8 Product Poisson structures The product $P_1 \times P_2$ of two Poisson manifolds (P_1,Π_1) and (P_2,Π_2) has a natural Poisson structure: it is the unique Poisson structure for which the bracket of functions of the form $(x_1,x_2) \mapsto f_1(x_1)f_2(x_2)$ and $(x_1,x_2) \mapsto g_1(x_1)g_2(x_2)$, where f_1 and $g_1 \in C^\infty(P_1,\mathbb{R})$, f_2 and $g_2 \in C^\infty(P_2,\mathbb{R})$, is

$$(x_1,x_2) \mapsto \{f_1,g_1\}_1(x_1)\{f_2,g_2\}_2(x_2)\,.$$

The same property holds for the product of any finite number of Poisson manifolds.

2.1.9 Symplectic orthogonality Let (V,ω) be a symplectic vector space, that means a real, finite-dimensional vector space V with a skew-symmetric nondegenrate bilinear form ω. Let W be a vector subspace of V. The *symplectic orthogonal* of W is

$$\operatorname{orth} W = \big\{\, v \in V; \omega(v,w) = 0 \text{ for all } w \in W \,\big\}\,. \tag{16}$$

It is a vector subspace of V, which satisfies

$$\dim W + \dim(\operatorname{orth} W) = \dim V\,, \quad \operatorname{orth}(\operatorname{orth} W) = W\,.$$

The vector subspace W is said to be *isotropic* if $W \subset \operatorname{orth} W$, *coisotropic* if $\operatorname{orth} W \subset W$ and *Lagrangian* if $W = \operatorname{orth} W$. In any symplectic vector space, there are many Lagrangian subspaces, therefore the dimension of a symplectic vector space is always even; if $\dim V = 2n$, the dimension of an isotropic (resp. coisotropic, resp. Lagrangian) vector subspace is $\leq n$ (resp. $\geq n$, resp. $= n$).

2.1.10 Coisotropic and Lagrangian submanifolds A submanifold N of a Poisson manifold (P,Π) is said to be *coisotropic* if the bracket of two smooth functions, defined on an open subset of P and which vanish on N, vanishes on N too. A submanifold N of a symplectic manifold (M,ω) is coisotropic if and only if for each point $x \in N$, the vector subspace T_xN of the symplectic vector space $\big(T_xM, \omega(x)\big)$ is coisotropic. Therefore, the dimension of a coisotropic submanifold in a $2n$-dimensional symplectic manifold is $\geq n$; when it is equal to n, the submanifold N is said to be *Lagrangian*.

2.1.11 Poisson quotients Let $\varphi : M \to P$ be a surjective submersion of a symplectic manifold (M,ω) onto a manifold P. The manifold P has a Poisson structure Π for which φ is a Poisson map if and only if $\mathrm{orth}(\ker T\varphi)$ is integrable. When that condition is satisfied, that Poisson structure on P is unique.

2.1.12 Poisson Lie groups A *Poisson Lie group* is a Lie group G with a Poisson structure Π, such that the product $(x,y) \mapsto xy$ is a Poisson map from $G \times G$, endowed with the product Poisson structure, into (G,Π). The Poisson structure of a Poisson Lie group (G,Π) always vanishes at the unit element e of G. Therefore the Poisson structure of a Poisson Lie group never comes from a symplectic structure on that group.

2.2 Definitions. *A* symplectic groupoid *(resp. a* Poisson groupoid*) is a Lie groupoid* $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ *with a symplectic form* ω *on* Γ *(resp. with a Poisson structure* Π *on* Γ*) such that the graph of the composition law* m

$$\left\{ (x,y,z) \in \Gamma \times \Gamma \times \Gamma ; (x,y) \in \Gamma_2 \text{ and } z = m(x,y) \right\}$$

is a Lagrangian submanifold (resp. a coisotropic submanifold) of $\Gamma \times \Gamma \times \overline{\Gamma}$ *with the product symplectic form (resp. the product Poisson structure), the first two factors* Γ *being endowed with the symplectic form* ω *(resp. with the Poisson structure* Π*), and the third factor* $\overline{\Gamma}$ *being* Γ *with the symplectic form* $-\omega$ *(resp. with the Poisson structure* $-\Pi$*).*

The next theorem states important properties of symplectic and Poisson groupoids.

2.3 Theorem. *Let* $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ *be a symplectic groupoid with symplectic* 2*-form* ω *(resp. a Poisson groupoid with Poisson structure* Π*). We have the following properties.*

1. *For a symplectic groupoid, given any point* $c \in \Gamma$*, each one of the two vector subspaces of the symplectic vector space* $\big(T_c\Gamma, \omega(c)\big)$*,* $T_c\big(\beta^{-1}\big(\beta(c)\big)\big)$ *and* $T_c\big(\alpha^{-1}\big(\alpha(c)\big)\big)$*, is the symplectic orthogonal of the other one. For a symplectic or Poisson groupoid, if* f *is a smooth function whose restriction to each* α*-fiber is constant, and* g *a smooth function whose restriction to each* β*-fiber is constant, then the Poisson bracket* $\{f,g\}$ *vanishes identically.*

2. *The submanifold of units* Γ_0 *is a Lagrangian submanifold of the symplectic manifold* (Γ,ω) *(resp. a coisotropic submanifold of the Poisson manifold* (Γ,Π)*).*

3. *The inverse map* $\iota : \Gamma \to \Gamma$ *is an antisymplectomorphism of* (Γ,ω)*, i.e. it satisfies* $\iota^*\omega = -\omega$ *(resp an anti-Poisson diffeomerphism of* (Γ,Π)*, i.e. it satisfies* $\iota_*\Pi = -\Pi$*).*

2.4 Corollary. *Let* $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ *be a symplectic groupoid with symplectic* 2*-form* ω *(resp. a Poisson groupoid with Poisson structure* Π*). There exists on* Γ_0 *a unique Poisson structure* Π_0 *for which* $\alpha : \Gamma \to \Gamma_0$ *is a Poisson map, and* $\beta : \Gamma \to \Gamma_0$ *an anti-Poisson map (*i.e. β *is a Poisson map when* Γ_0 *is equipped with the Poisson structure* $-\Pi_0$*).*

2.5 Examples of symplectic and Poisson groupoids

2.5.1 The cotangent bundle of a Lie groupoid Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid.

We have seen above that its tangent bundle $T\Gamma$ has a Lie groupoid structure, determined by that of Γ. Similarly (but much less obviously) the cotangent bundle $T^*\Gamma$ has a Lie groupoid structure determined by that of Γ. The set of units is the conormal bundle to the submanifold Γ_0 of Γ, denoted by $\mathcal{N}^*\Gamma_0$. We recall that $\mathcal{N}^*\Gamma_0$ is the vector sub-bundle of $T^*_{\Gamma_0}\Gamma$ (the restriction to Γ_0 of the cotangent bundle $T^*\Gamma$) whose fiber $\mathcal{N}^*_p\Gamma_0$ at a point $p \in \Gamma_0$ is

$$\mathcal{N}^*_p\Gamma_0 = \left\{ \eta \in T^*_p\Gamma; \langle \eta, v \rangle = 0 \text{ for all } v \in T_p\Gamma_0 \right\}.$$

To define the target and source maps of the Lie algebroid $T^*\Gamma$, we introduce the notion of *bisection* through a point $x \in \Gamma$. A bisection through x is a submanifold A of Γ, with $x \in A$, transverse both to the α-fibers and to the β-fibers, such that the maps α and β, when restricted to A, are diffeomorphisms of A onto open subsets $\alpha(A)$ and $\beta(A)$ of Γ_0, respectively. For any point $x \in M$, there exist bisections through x. A bisection A allows us to define two smooth diffeomorphisms between open subsets of Γ, denoted by L_A and R_A and called the *left* and *right translations by* A, respectively. They are defined by

$$L_A : \alpha^{-1}\big(\beta(A)\big) \to \alpha^{-1}\big(\alpha(A)\big), \quad L_A(y) = m\big(\beta|_A^{-1} \circ \alpha(y), y\big),$$

and

$$R_A : \beta^{-1}\big(\alpha(A)\big) \to \beta^{-1}\big(\beta(A)\big), \quad R_A(y) = m\big(y, \alpha|_A^{-1} \circ \beta(y)\big).$$

The definitions of the target and source maps for $T^*\Gamma$ rest on the following properties. Let x be a point in Γ and A be a bisection through x. The two vector subspaces, $T_{\alpha(x)}\Gamma_0$ and $\ker T_{\alpha(x)}\beta$, are complementary in $T_{\alpha(x)}\Gamma$. For any $v \in T_{\alpha(x)}\Gamma$, $v - T\beta(v)$ is in $\ker T_{\alpha(x)}\beta$. Moreover, R_A maps the fiber $\beta^{-1}\big(\alpha(x)\big)$ onto the fiber $\beta^{-1}\big(\beta(x)\big)$, and its restriction to that fiber does not depend on the choice of A; its depends only on x. Therefore $TR_A\big(v - T\beta(v)\big)$ is in $\ker T_x\beta$ and does not depend on the choice of A. We can define the map $\widehat{\alpha}$ by setting, for any $\xi \in T^*_x\Gamma$ and any $v \in T_{\alpha(x)}\Gamma$,

$$\big\langle \widehat{\alpha}(\xi), v \big\rangle = \big\langle \xi, TR_A\big(v - T\beta(v)\big) \big\rangle.$$

Similarly, we define $\widehat{\beta}$ by setting, for any $\xi \in T^*_x\Gamma$ and any $w \in T_{\beta(x)}\Gamma$,

$$\big\langle \widehat{\beta}(\xi), w \big\rangle = \big\langle \xi, TL_A\big(w - T\alpha(w)\big) \big\rangle.$$

We see that $\widehat{\alpha}$ and $\widehat{\beta}$ are unambiguously defined, smooth and take their values in the submanifold $\mathcal{N}^*\Gamma_0$ of $T^*\Gamma$. They satisfy

$$\pi_\Gamma \circ \widehat{\alpha} = \alpha \circ \pi_\Gamma, \quad \pi_\Gamma \circ \widehat{\beta} = \beta \circ \pi_\Gamma,$$

where $\pi_\Gamma : T^*\Gamma \to \Gamma$ is the cotangent bundle projection.

Let us now define the composition law $\widehat{m}$ on $T^*\Gamma$. Let $\xi \in T^*_x\Gamma$ and $\eta \in T^*_y\Gamma$ be such that $\widehat{\beta}(\xi) = \widehat{\alpha}(\eta)$. That implies $\beta(x) = \alpha(y)$. Let A be a bisection through x and B a bisection through y. There exist a unique $\xi_{h\alpha} \in T^*_{\alpha(x)}\Gamma_0$ and a unique $\eta_{h\beta} \in T^*_{\beta(y)}\Gamma_0$ such that

$$\xi = (L_A^{-1})^*\big(\widehat{\beta}(\xi)\big) + \alpha_x^*\xi_{h\alpha}, \quad \eta = (R_B^{-1})^*\big(\widehat{\alpha}(\xi)\big) + \beta_y^*\eta_{h\beta}.$$

Then $\widehat{m}(\xi, \eta)$ is given by

$$\widehat{m}(\xi, \eta) = \alpha_{xy}^*\xi_{h\alpha} + \beta_{xy}^*\eta_{h\beta} + (R_B^{-1})^*(L_A^{-1})^*\big(\widehat{\beta}(x)\big).$$

We observe that in the last term of the above expression we can replace $\widehat{\beta}(\xi)$ by $\widehat{\alpha}(\eta)$, since these two expressions are equal, and that $(R_B^{-1})^*(L_A^{-1})^* = (L_A^{-1})^*(R_B^{-1})^*$, since R_B and L_A commute.

Finally, the inverse $\widehat{\iota}$ in $T^*\Gamma$ is ι^*.

With its canonical symplectic form, $T^*\Gamma \underset{\widehat{\beta}}{\overset{\widehat{\alpha}}{\rightrightarrows}} \mathcal{N}^*\Gamma_0$ is a symplectic groupoid.

When the Lie groupoid Γ is a Lie group G, the Lie groupoid T^*G is not a Lie group, contrary to what happens for TG. This shows that the introduction of Lie groupoids is not at all artificial: when dealing with Lie groups, Lie groupoids are already with us! The set of units of the Lie groupoid T^*G can be identified with $\mathcal{G}^*$ (the dual of the Lie algebra $\mathcal{G}$ of G), identified itself with T_e^*G (the cotangent space to G at the unit element e). The target map $\widehat{\alpha} : T^*G \to T_e^*G$ (resp. the source map $\widehat{\beta} : T^*G \to T_e^*G$) associates to each $g \in G$ and $\xi \in T_g^*G$, the value at the unit element e of the right-invariant 1-form (resp., the left-invariant 1-form) whose value at x is ξ.

2.5.2 Poisson Lie groups as Poisson groupoids Poisson groupoids were introduced by Alan Weinstein as a generalization of both symplectic groupoids and Poisson Lie groups. Indeed, a Poisson Lie group is a Poisson groupoid with a set of units reduced to a single element.

3 Lie algebroids

The notion of a Lie algebroid, due to Jean Pradines, is related to that of a Lie groupoid in the same way as the notion of a Lie algebra is related to that of a Lie group.

3.1 Definition. *A* Lie algebroid *over a smooth manifold M is a smooth vector bundle $\pi : A \to M$ with base M, equipped with*

- *a composition law $(s_1, s_2) \mapsto \{s_1, s_2\}$ on the space $\Gamma^\infty(\pi)$ of smooth sections of π, called the* bracket*, for which that space is a Lie algebra,*

- *a vector bundle map $\rho : A \to TM$, over the identity map of M, called the* anchor map*, such that, for all s_1 and $s_2 \in \Gamma^\infty(\pi)$ and all $f \in C^\infty(M, \mathbb{R})$,*

$$\{s_1, fs_2\} = f\{s_1, s_2\} + \big((\rho \circ s_1).f\big)s_2 \,. \tag{17}$$

3.2 Examples

3.2.1 Lie algebras A finite-dimensional Lie algebra is a Lie algebroid (with a base reduced to a point and the zero map as anchor map).

3.2.2 Tangent bundles and their integrable sub-bundless A tangent bundle $\tau_M : TM \to M$ to a smooth manifold M is a Lie algebroid, with the usual bracket of vector fields on M as composition law, and the identity map as anchor map. More generally, any integrable vector sub-bundle F of a tangent bundle $\tau_M : TM \to M$ is a Lie algebroid, still with the bracket of vector fields on M with values in F as composition law and the canonical injection of F into TM as anchor map.

3.2.3 The cotangent bundle of a Poisson manifold Let (P,Π) be a Poisson manifold. Its cotangent bundle $\pi_P : T^*P \to P$ has a Lie algebroid structure, with $\Pi^\sharp : T^*P \to TP$ as anchor map. The composition law is the *bracket of* 1*-forms*. It will be denoted by $(\eta,\zeta) \mapsto [\eta,\zeta]$ (in order to avoid any confusion with the Poisson bracket of functions). It is given by the formula, in which η and ζ are 1-forms and X a vector field on P,

$$\big\langle [\eta,\zeta],X\big\rangle = \Pi\big(\eta, d\langle\zeta,X\rangle\big) + \Pi\big(d\langle\eta,X\rangle,\zeta\big) + \big(\mathcal{L}(X)\Pi\big)(\eta,\zeta)\,. \tag{18}$$

We have denoted by $\mathcal{L}(X)\Pi$ the Lie derivative of the Poisson structure Π with respect to the vector field X. Another equivalent formula for that composition law is

$$[\zeta,\eta] = \mathcal{L}(\Pi^\sharp\zeta)\eta - \mathcal{L}(\Pi^\sharp\eta)\zeta - d\big(\Pi(\zeta,\eta)\big)\,. \tag{19}$$

The bracket of 1-forms is related to the Poisson bracket of functions by

$$[df,dg] = d\{f,g\} \quad \text{for all } f \text{ and } g \in C^\infty(P,\mathbb{R})\,. \tag{20}$$

3.3 Properties of Lie algebroids Let $\pi : A \to M$ be a Lie algebroid with anchor map $\rho : A \to TM$.

3.3.1 A Lie algebras homomorphism For any pair (s_1,s_2) of smooth sections of π,

$$\rho\circ\{s_1,s_2\} = [\rho\circ s_1, \rho\circ s_2]\,,$$

which means that the map $s \mapsto \rho\circ s$ is a Lie algebra homomorphism from the Lie algebra of smooth sections of π into the Lie algebra of smooth vector fields on M.

3.3.2 The generalized Schouten bracket The composition law $(s_1,s_2) \mapsto \{s_1,s_2\}$ on the space of sections of π extends into a composition law on the space of sections of exterior powers of (A,π,M), which will be called the *generalized Schouten bracket*. Its properties are the same as those of the usual Schouten bracket. When the Lie algebroid is a tangent bundle $\tau_M : TM \to M$, that composition law reduces to the usual Schouten bracket . When the Lie algebroid is the cotangent bundle $\pi_P : T^*P \to P$ to a Poisson manifold (P,Π), the generalized Schouten bracket is the *bracket of forms* of all degrees on the Poisson manifold P, introduced by J.-L. Koszul, which extends the bracket of 1-forms used in 3.2.3.

3.3.3 The dual bundle of a Lie algebroid Let $\varpi : A^* \to M$ be the dual bundle of the Lie algebroid $\pi : A \to M$. There exists on the space of sections of its exterior powers a graded endomorphism d_ρ, of degree 1 (that means that if η is a section of $\bigwedge^k A^*$, $d_\rho(\eta)$ is a section of $\bigwedge^{k+1} A^*$). That endomorphism satisfies

$$d_\rho \circ d_\rho = 0\,,$$

and its properties are essentially the same as those of the exterior derivative of differential forms. When the Lie algebroid is a tangent bundle $\tau_M : TM \to M$, d_ρ is the usual exterior derivative of differential forms.

We can develop on the spaces of sections of the exterior powers of a Lie algebroid and of its dual bundle a differential calculus very similar to the usual differential calculus of vector and multivector fields and differential forms on a manifold. Operators such as the

interior product, the exterior derivative and the Lie derivative can still be defined and have properties similar to those of the corresponding operators for vector and multivector fields and differential forms on a manifold.

The total space A^* of the dual bundle of a Lie algebroid $\pi : A \to M$ has a natural Poisson structure: a smooth section s of π can be considered as a smooth real-valued function on A^* whose restrictiion to each fiber $\varpi^{-1}(x)$ ($x \in M$) is linear; that property allows us to extend the bracket of sections of π (defined by the Lie algebroid structure) to obtain a Poisson bracket of functions on A^*. When the Lie algebroid A is a finite-dimensional Lie algebra $\mathcal{G}$, the Poisson structure on its dual space $\mathcal{G}^*$ is the KKS Poisson structure discussed in 2.1.6.

3.4 The Lie algebroid of a Lie groupoid Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ be a Lie groupoid. Let $A(\Gamma)$ be the intersection of $\ker T\alpha$ and $T_{\Gamma_0}\Gamma$ (the tangent bundle $T\Gamma$ restricted to the submanifold Γ_0). We see that $A(\Gamma)$ is the total space of a vector bundle $\pi : A(\Gamma) \to \Gamma_0$, with base Γ_0, the canonical projection π being the map which associates a point $u \in \Gamma_0$ to every vector in $\ker T_u\alpha$. We will define a composition law on the set of smooth sections of that bundle, and a vector bundle map $\rho : A(\Gamma) \to T\Gamma_0$, for which $\pi : A(\Gamma) \to \Gamma_0$ is a Lie algebroid, called the *Lie algebroid* of the Lie groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$.

We observe first that for any point $u \in \Gamma_0$ and any point $x \in \beta^{-1}(u)$, the map $L_x : y \mapsto L_x y = m(x,y)$ is defined on the α-fiber $\alpha^{-1}(u)$, and maps that fiber onto the α-fiber $\alpha^{-1}\big(\alpha(x)\big)$. Therefore T_uL_x maps the vector space $A_u = \ker T_u\alpha$ onto the vector space $\ker T_x\alpha$, tangent at x to the α-fiber $\alpha^{-1}\big(\alpha(x)\big)$. Any vector $w \in A_u$ can therefore be extended into the vector field along $\beta^{-1}(u)$, $x \mapsto \widehat{w}(x) = T_uL_x(w)$. More generally, let $w : U \to A(\Gamma)$ be a smooth section of the vector bundle $\pi : A(\Gamma) \to \Gamma_0$, defined on an open subset U of Γ_0. By using the above described construction for every point $u \in U$, we can extend the section w into a smooth vector field $\widehat{w}$, defined on the open subset $\beta^{-1}(U)$ of Γ, by setting, for all $u \in U$ and $x \in \beta^{-1}(u)$,

$$\widehat{w}(x) = T_uL_x\big(w(u)\big)\,.$$

We have defined an injective map $w \mapsto \widehat{w}$ from the space of smooth local sections of $\pi : A(\Gamma) \to \Gamma_0$, onto a subspace of the space of smooth vector fields defined on open subsets of Γ. The image of that map is the space of smooth vector fields $\widehat{w}$, defined on open subsets $\widehat{U}$ of Γ of the form $\widehat{U} = \beta^{-1}(U)$, where U is an open subset of Γ_0, which satisfy the two properties:

(i) $T\alpha \circ \widehat{w} = 0$,

(ii) for every x and $y \in \widehat{U}$ such that $\beta(x) = \alpha(y)$, $T_yL_x\big(\widehat{w}(y)\big) = \widehat{w}(xy)$.

These vector fields are called *left invariant vector fields* on Γ.

The space of left invariant vector fields on Γ is closed under the bracket operation. We can therefore define a composition law $(w_1, w_2) \mapsto \{w_1, w_2\}$ on the space of smooth sections of the bundle $\pi : A(\Gamma) \to \Gamma_0$ by defining $\{w_1, w_2\}$ as the unique section such that

$$\widehat{\{w_1, w_2\}} = [\widehat{w}_1, \widehat{w}_2]\,.$$

Finally, we define the anchor map ρ as the map $T\beta$ restricted to $A(\Gamma)$. With that composition law and that anchor map, the vector bundle $\pi : A(\Gamma) \to \Gamma_0$ is a Lie algebroid, called the *Lie algebroid of* the Lie groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$.

We could exchange the roles of α and β and use *right invariant* vector fields instead of left invariant vector fields. The Lie algebroid obtained remains the same, up to an isomorphism.

When the Lie groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}}$ is a Lie group, its Lie algebroid is simply its Lie algebra.

3.5 The Lie algebroid of a symplectic groupoid Let $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ be a symplectic groupoid, with symplectic form ω. As we have seen above, its Lie algebroid $\pi : A \to \Gamma_0$ is the vector bundle whose fiber, over each point $u \in \Gamma_0$, is $\ker T_u\alpha$. We define a linear map $\omega_u^\flat : \ker T_u\alpha \to T_u^*\Gamma_0$ by setting, for each $w \in \ker T_u\alpha$ and $v \in T_u\Gamma_0$,

$$\left\langle \omega_u^\flat(w), v \right\rangle = \omega_u(v, w)\,.$$

Since $T_u\Gamma_0$ is Lagrangian and $\ker T_u\alpha$ complementary to $T_u\Gamma_0$ in the symplectic vector space $\big(T_u\Gamma, \omega(u)\big)$, the map $\omega_u^\flat$ is an isomorphism from $\ker T_u\alpha$ onto $T_u^*\Gamma_0$. By using that isomorphism for each $u \in \Gamma_0$, we obtain a vector bundle isomorphism of the Lie algebroid $\pi : A \to \Gamma_0$ onto the cotangent bundle $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$.

As seen in Corollary 2.4, the submanifold of units Γ_0 has a unique Poisson structure Π for which $\alpha : \Gamma \to \Gamma_0$ is a Poisson map. Therefore, as seen in 3.2.3, the cotangent bundle $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$ to the Poisson manifold (Γ_0, Π) has a Lie algebroid structure, with the bracket of 1-forms as composition law. That structure is the same as the structure obtained as a direct image of the Lie algebroid structure of $\pi : A(\Gamma) \to \Gamma_0$, by the above defined vector bundle isomorphism of $\pi : A \to \Gamma_0$ onto the cotangent bundle $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$. The Lie algebroid of the symplectic groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ can therefore be identified with the Lie algebroid $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$, with its Lie algebroid structure of cotangent bundle to the Poisson manifold (Γ_0, Π).

3.6 The Lie algebroid of a Poisson groupoid The Lie algebroid $\pi : A(\Gamma) \to \Gamma_0$ of a Poisson groupoid has an additional structure: its dual bundle $\varpi : A(\Gamma)^* \to \Gamma_0$ also has a Lie algebroid structure, compatible in a certain sense (indicated below) with that of $\pi : A(\Gamma) \to \Gamma_0$ (K. Mackenzie and P. Xu, Y. Kosmann-Schwarzbach, Z.-J. Liu and P. Xu).

The compatibility condition between the two Lie algebroid structures on the two vector bundles in duality $\pi : A \to M$ and $\varpi : A^* \to M$ can be written as follows:

$$d_*[X, Y] = \mathcal{L}(X)d_*Y - \mathcal{L}(Y)d_*X\,, \tag{21}$$

where X and Y are two sections of π, or, using the generalized Schouten bracket (3.3.2) of sections of exterior powers of the Lie algebroid $\pi : A \to M$,

$$d_*[X, Y] = [d_*X, Y] + [X, d_*Y]\,. \tag{22}$$

In these formulae d_* is the generalized exterior derivative, which acts on the space of sections of exterior powers of the bundle $\pi : A \to M$, considered as the dual bundle of the Lie algebroid $\varpi : A^* \to M$, defined in 3.3.3.

These conditions are equivalent to the similar conditions obtained by exchange of the roles of A and A^*.

When the Poisson groupoid $\Gamma \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \Gamma_0$ is a symplectic groupoid, we have seen (3.5) that its Lie algebroid is the cotangent bundle $\pi_{\Gamma_0} : T^*\Gamma_0 \to \Gamma_0$ to the Poisson manifold Γ_0 (equipped with the Poisson structure for which α is a Poisson map). The dual bundle is the tangent bundle $\tau_{\Gamma_0} : T\Gamma_0 \to \Gamma_0$, with its natural Lie algebroid structure defined in 3.2.2.

When the Poisson groupoid is a Poisson Lie group (G, Π), its Lie algebroid is its Lie algebra $\mathcal{G}$. Its dual space $\mathcal{G}$ has a Lie algebra structure, compatible with that of $\mathcal{G}$ in the above defined sense, and the pair $(\mathcal{G}, \mathcal{G}^*)$ is called a *Lie bialgebra*.

Conversely, if the Lie algebroid of a Lie groupoid is a Lie bialgebroid (that means, if there exists on the dual vector bundle of that Lie algebroid a compatible structure of Lie algebroid, in the above defined sense), that Lie groupoid has a Poisson structure for which it is a Poisson groupoid (K. Mackenzie and P. Xu).

3.7 Integration of Lie algebroids According to Lie's third theorem, for any given finite-dimensional Lie algebra, there exists a Lie group whose Lie algebra is isomorphic to that Lie algebra. The same property is not true for Lie algebroids and Lie groupoids. The problem of finding necessary and sufficient conditions under which a given Lie algebroid is isomorphic to the Lie algebroid of a Lie groupoid remained open for more than 30 years, although partial results were obtained. A complete solution of that problem was recently obtained by M. Crainic and R.L. Fernandes. Let us briefly sketch their results.

Let $\pi : A \to M$ be a Lie algebroid and $\rho : A \to TM$ its anchor map. A smooth path $a : I = [0,1] \to A$ is said to be *admissible* if, for all $t \in I$, $\rho \circ a(t) = \frac{d}{dt}(\pi \circ a)(t)$. When the Lie algebroid A is the Lie algebroid of a Lie groupoid Γ, it can be shown that each admissible path in A is, in a natural way, associated to a smooth path in Γ starting from a unit and contained in an α-fiber. When we do not know whether A is the Lie algebroid of a Lie goupoid or not, the space of admissible paths in A still can be used to define a topological groupoid $\mathcal{G}(A)$ with connected and simply connected α-fibers, called the *Weinstein groupoid* of A. When $\mathcal{G}(A)$ is a Lie groupoid, its Lie algebroid is isomorphic to A, and when A is the Lie algebroid of a Lie groupoid Γ, $\mathcal{G}(A)$ is a Lie groupoid and is the unique (up to an isomorphism) Lie groupoid with connected and simply connected α-fibers with A as Lie algebroid; moreover, $\mathcal{G}(A)$ is a covering groupoid of an open subgroupoid of Γ. Crainic and Fernandes have obtained computable necessary and sufficient conditions under which the topological groupoid $\mathcal{G}(A)$ is a Lie groupoid, *i.e.* necessary and sufficient conditions under which A is the Lie algebroid of a Lie groupoid.

Key words

Groupoids, Lie groupoids, Lie algebroids, symplectic groupoids, Poisson groupoids, Poisson Lie groups, bisections.

Further reading

The reader will find more about groupoids in the very nice survey paper [10], and in the books [1], [6]. More information about symplectic and Poisson geometry can be found in [5], [9], [8]. The Schouten bracket is discussed in [9]. The paper [11] presents many

properties of Poisson groupoids. Integration of Lie algebroids is fully discussed in [2]. Papers [4] and [12] introduced Lie groupoids for applications to quantization. The books [3] and [7] are symposia proceedings which contain several papers about Lie, symplectic and Poisson groupoids, and many references.

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