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Abstract

The identification of the most dangerous combinations of excitations that a non-linear mechanical system can be confronted to is not an easy task. Indeed, in such cases, the link between the maximal values of the inputs and of the outputs is not direct, as the system can be more sensitive to a problematic succession of excitations of low amplitudes than to high amplitudes for each kind of excitations. This work presents therefore an innovative method to identify the combined shapes of excitations that are the most correlated to problematic responses of the studied mechanical system.

I. Introduction

For years, mechanical engineers have been working on the identification of the most dangerous excitations that a mechanical system can be confronted to during its lifecycle. In the case of a linear system excited by a scalar stationary process, these researches have been mostly based on correlation analysis between the spectral density function of the excitation and the eigenfrequencies of the studied system. When confronted to non-linear mechanical systems excited by vector-valued stochastic processes that are *a priori* non-stationary, such comparisons cannot be achieved. In this prospect, Section II proposes an original sensitivity method based on a scaled Karhunen-Loève expansion. This method is then illustrated on an application based on experimental data in Section III.

II. Karhunen-Loève based sensitivity analysis

Theoretical frame

For all $P \geq 1$, let $\mathcal{P}^{(P)}(\Omega)$ be the space of all the second-order $\mathbb{R}^P$-valued stochastic processes, indexed by the compact interval $\Omega = [0, T]$, where $T < +\infty$. Let $\mathbb{H}_P = L^2(\Omega, \mathbb{R}^P)$ be the space

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of square integrable functions on Ω, with values in $\mathbb{R}^P$, equipped with the inner product $\langle \cdot, \cdot \rangle_P$, such that for all $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{H}_P$,

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_P = \int_\Omega \boldsymbol{u}^T(t)\boldsymbol{v}(t)dt. \tag{1}$$

For P and Q in $\mathbb{N}^*$, let $\boldsymbol{X} \in \mathcal{P}^{(P)}(\Omega)$ and $\boldsymbol{Y} \in \mathcal{P}^{(Q)}(\Omega)$ be two vector-valued stochastic processes. It is then supposed that $\boldsymbol{X}$ gathers the time evolution of the excitations, whereas $\boldsymbol{Y}$ refers to the time evolution of the corresponding quantities of interest of a dynamical mechanical system $\mathcal{M}$ ($\mathcal{M}$ is a differential operator that is linear or not, whose coefficients depend on t or not), such that:

$$\boldsymbol{Y}(t) = \mathcal{M}\left[t, \{\boldsymbol{X}(s),\ 0 \leq s \leq t\}\right],\ t \in \Omega. \tag{2}$$

Let $\boldsymbol{d}$ be a particular element of $\mathbb{H}_Q$, which is supposed to correspond to the time evolution of a problematic response for system $\mathcal{M}$. For instance, function $\boldsymbol{d}$ can refer to a dangerous combination of oscillations for the components of $\boldsymbol{Y}$. In a sensitivity analysis prospect, we are thus interested in identifying the evolution for input $\boldsymbol{X}$, which would be the most correlated to such a response $\boldsymbol{d}$ for $\boldsymbol{Y}$. In other words, we therefore search the optimal function $\boldsymbol{f}$ in $\mathbb{H}_P$, such that:

$$\boldsymbol{f} = \arg\max_{\boldsymbol{u}\in\mathbb{H}_P} E\left\{\langle \boldsymbol{X}, \boldsymbol{u} \rangle_P \langle \boldsymbol{Y}, \boldsymbol{d} \rangle_Q\right\}. \tag{3}$$

Optimality of the Karhunen-Loève expansion

Mathematically, the Karhunen-Loève (KL) expansion associated with any stochastic process $\boldsymbol{Z}$ in $\mathcal{P}^{(N)}(\Omega)$, $N \in \mathbb{N}$, corresponds to the orthogonal projection theorem in separable Hilbert spaces, for which the Hilbertian basis, $\mathcal{K} = \left\{\boldsymbol{k}^{(m)},\ m \geq 1\right\}$, is constructed as the eigenfunctions of the covariance operator of $\boldsymbol{Z}$, which is assumed, for instance, to be square integrable on $\Omega \times \Omega$. This Hilbertian basis is thus optimal in the sense that, for all $M \geq 1$, it can be extracted from $\mathcal{K}$ the M-dimensional family that minimizes the total mean square error associated with $\boldsymbol{Z}$ among all the M-dimensional families of $\mathbb{H}_N$. In particular, it can be shown that the element of $\mathcal{K}$ of highest eigenvalue, that is denoted by $\boldsymbol{k}^*$, verifies the following equality:

$$\boldsymbol{k}^* = \arg\max_{\boldsymbol{v}\in\mathbb{H}_N} E\left\{\langle \boldsymbol{Z}, \boldsymbol{v} \rangle_N^2\right\}. \tag{4}$$

Sensitivity analysis

For any κ in $]0, 1[$, let $\boldsymbol{Z}(\kappa) \in \mathcal{P}^{(P+1)}(\Omega)$ be the *scaled* stochastic process such that:

$$\boldsymbol{Z}(\kappa) = \begin{pmatrix} \kappa\boldsymbol{X} \\ (1-\kappa)\langle \boldsymbol{Y}, \boldsymbol{d} \rangle_Q I \end{pmatrix},\quad I = \{I(t) = 1,\ t \in \Omega\} \in \mathbb{H}_1, \tag{5}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are the input and output stochastic processes previously defined. Based on the developments achieved in [Perrin et al., 2013], it can be shown that the KL expansion associated with such a *scaled* stochastic process, $\boldsymbol{Z}(\kappa)$, allows defining projection basis that can favor or put at a disadvantage *on purpose* the characterization of the last component of $\boldsymbol{Z}(\kappa)$. In particular, let $\boldsymbol{k}^*(\kappa)$ be the eigenfunction of highest eigenvalue associated with the covariance operator of $\boldsymbol{Z}(\kappa)$, which can be decomposed as:

$$\boldsymbol{k}^*(\kappa) = \begin{pmatrix} \boldsymbol{h}(\kappa) \\ g(\kappa) \end{pmatrix},\quad \boldsymbol{h}(\kappa) \in \mathbb{H}_P,\quad g(\kappa) \in \mathbb{H}_1. \tag{6}$$

In a sensitivity analysis prospect, such a method can therefore be used to identify, at a very low computational cost, very relevant approximations of optimal function $\boldsymbol{f}$, defined by Eq. (3), as it can be deduced from Eqs. (4) and (6) that:

$$\lim_{\kappa \to 0} \boldsymbol{h}(\kappa) = \boldsymbol{f}. \tag{7}$$

III. Application based on experimental data

A railway simulation can be seen as the dynamic response of a complex and strongly non-linear system (the train) excited by a vector-valued stochastic process (the track irregularities). For the last decades, the identification of the most dangerous and most uncomfortable track irregularities for the train dynamics has been based on correlation analysis between dynamical quantities of the train, such as vertical and transverse accelerations of any mass bodies and loads between the rail and the wheel, and the amplitudes, the wavelengths or the maximal values of the corresponding measured track irregularities. However, due to the high non-linearities of the train suspensions and of the wheel-rail contact, even if the maximal amplitudes or the variances of the track irregularities seem to be representative quantities for the track quality, such analysis have shown that they are not at all sufficient. Indeed, it can easily be extracted track conditions with high track irregularities that would less excite the train than track conditions with low track irregularities. On the contrary, based on available measurements of the track geometry and of the corresponding train dynamical responses, it can be shown that the method developed in Section II is very relevant to identify the successions of track irregularities that are the most correlated to problematic responses of the train [Perrin, 2013].

IV. Conclusions

An innovative sensitivity analysis method has been presented in this work. Based on an original scaled KL expansion, it has therefore been shown to what extent this method can open new ways to identify problematic successions of excitations for the analysis of non-linear mechanical systems. At last, the possibilities of such a method, which can be applied to any scientific field, are illustrated in the railway fields.

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