Symmetries of Hamiltonian systems on symplectic and Poisson manifolds

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1 Introduction

This text presents some basic notions in symplectic geometry, Poisson geometry, Hamiltonian systems, Lie algebras and Lie groups actions on symplectic or Poisson manifolds, momentum maps and their use for the reduction of Hamiltonian systems. It should be accessible to readers with a general knowledge of basic notions in differential geometry. Full proofs of many results are provided.

1.1 Contents of the paper

Symplectic and Poisson manifolds are defined in Sections 2 and 3, where their basic properties are given, often with detailed proofs. Darboux theorem and the main results about the local structure of Poisson manifolds, however, are given without proof. Actions of a Lie group or of a Lie algebra on a smooth manifold and, when this manifold is endowed with a symplectic or a Poisson structure, symplectic, Poisson and Hamiltonian actions are introduced in Section 4. For Hamiltonian actions of a Lie group on a connected symplectic manifold, the equivariance of the momentum map with respect to an affine action of the group on the dual of its Lie algebra is proven, and the notion of symplectic cocycle is introduced. We prove (4.2.14) that given a Lie algebra symplectic cocycle, there exists on the associated connected and simply connected Lie group a unique corresponding Lie group symplectic cocycle. The Hamiltonian actions of a Lie group on its cotangent bundle obtained by lifting the actions of the group on itself by translations on the left and on the right are fully discussed in Subsection 4.5. We prove that there exists a two-parameter family of deformations of these actions into a pair of mutually symplectically orthogonal Hamiltonian actions whose momentum maps are equivariant with respect to an affine action involving any given Lie group symplectic cocycle (4.5.4). The use of first integrals and, more generally, of momentum maps for the resolution of Hamiltonian dynamical systems, is discussed in Section 5. For a system whose Hamiltonian is invariant under a Hamiltonian Lie algebra action, the Marsden-Weinstein reduction procedure can be used: through the use of Noether's theorem, this procedure leads to a reduced symplectic manifold on which a reduced Hamiltonian system can be solved in a first step. Another way of using the symmetries of the system rests on the use of the Euler-Poincaré equation. This equation can be written for classical Lagrangian mechanical systems when there exists a locally transitive Lie algebra action on their configuration space, or for the corresponding Hamiltonian systems when the Lagrangian is hyper-regular. However, the Euler-Poincaré equation does not always lead to a reduction of the system: such a reduction occurs mainly when the Hamiltonian can be expressed as the momentum map composed with a smooth function defined on the dual of the Lie algebra; the Euler-Poincaré equation is then equivalent to the Hamilton equation written on the dual of the Lie algebra.

Finally in Section 6 three classical examples are considered: the spherical pendulum, the motion of a rigid body around a fixed point and the Kepler problem. For each example the Euler-Poincaré equation is derived (for the Kepler problem a transitive Lie algebra action is obtained by adding the Lie algebra of the group of positive homotheties to the Lie algebra of the group of rotations around the attractive centre), the first integrals linked to symmetries are given. In this Section, the classical concepts of vector calculus on an Euclidean three-dimensional vector space (scalar, vector and mixed products) are used and their interpretation in terms of concepts such as the adjoint or coadjoint action of the group of rotations are explained.

1.2 Further reading

Of course this text is just an introduction. Several important parts of the theory of Hamiltonian systems are not discussed here, for example completely integrable systems (although the three examples presented belong to that class of systems), action-angle coordinates, monodromy, singular reduction, the Kolmogorov-Arnold-Moser theorem, symplectic methods in Hydrodynamics, ... To extend his knowledge of the subject, the reader can consult the books by Ralph Abraham and Jerry Marsden [1], Vladimir Arnold [4], Vladimir Arnold and Boris Khesin [5], Patrick Iglesias-Zemmour [33], Camille Laurent-Gengoux, Anne Pichereau and Pol Vanhaecke [42], Yvette Kosmann-Schwarzbach (editor) [38] on both the scientific and historical aspects of the development of modern Poisson geometry, Izu Vaisman [64].

1.3 Notations

Our notations are those which today are generally used in differential geometry. For example, the tangent and the cotangent bundles to a smooth n-dimensional manifold M are denoted, respectively, by TM and by T^*M, and their canonical projections on M by $\tau_M : TM \to M$ and by $\pi_M : T^*M \to M$. The space of differential forms of degree p, *i.e.* the space of smooth sections of $\bigwedge^p(T^*M)$, the p-th exterior power of the cotangent bundle, is denoted by $\Omega^p(M)$. Similarly the space of multivectors of degree p, *i.e.* the space of smooth sections of $\bigwedge^p(TM)$, the p-th exterior power of the tangent bundle, is denoted by $A^p(M)$. By convention $\Omega^p(M) = A^p(M) = 0$ for $p < 0$ or $p > n$, and $\Omega^0(M) = A^0(M) = C^\infty(M, \mathbb{R})$. The exterior algebras of differential forms and of multivectors are, respectively, $\Omega(M) = \oplus_{p=0}^n \Omega^p(M)$ and $A(M) = \oplus_{p=0}^n A^p(M)$. Their main properties are briefly recalled in Section 3.2.6.

When $f : M \to N$ is a smooth map between two smooth manifolds M and N, the natural lift of f to the tangent bundles is denoted by $Tf : TM \to TN$. The same notation $Tf : \bigwedge^p TM \to \bigwedge^p TN$ is used to denote its natural prolongation to the p-th exterior power of TM. The pull-back by f of a smooth differential form $\alpha \in \Omega(N)$ is denoted by $f^*\alpha$.

When $f : M \to N$ is a smooth diffeomorphism, the push-forward f_*X of a a smooth vector field $X \in A^1(M)$ is the vector field $f_*X \in A^1(N)$ defined by

$$f_*X(y) = Tf\Big(X\big(f^{-1}(y)\big)\Big), \quad y \in N.$$

Similarly, the pull-back of a smooth vector field $Y \in A^1(N)$ is the vector field $f^*Y \in A^1(M)$ defined by

$$f^*Y(x) = Tf^{-1}\Big(Y\big(f(x)\big)\Big), \quad x \in M.$$

The same notation is used for the push-forward of any smooth tensor field on M and the pull-back of any smooth tensor field on N.

2 Symplectic manifolds

2.1 Definition and elementary properties

Definition 2.1.1. A *symplectic form* on a smooth manifold M is a bilinear skew-symmetric differential form ω on that manifold which satisfies the following two properties:

- the form ω is closed; it means that its exterior differential $\mathrm{d}\omega$ vanishes: $\mathrm{d}\omega = 0$;
- the rank of ω is everywhere equal to the dimension of M; it means that for each point $x \in M$ and each vector $v \in T_xM$, $v \neq 0$, there exists another vector $w \in T_xM$ such that $\omega(x)(v,w) \neq 0$.

Equipped with the symplectic form ω, the manifold M is called a *symplectic manifold* and denoted (M,ω). One says also that ω determines a *symplectic structure* on the manifold M.

2.1.2 Elementary properties of symplectic manifolds

Let (M,ω) be a symplectic manifold.

1. For each $x \in M$ and each $v \in T_xM$ we denote by $\mathrm{i}(v)\omega : T_xM \to \mathbb{R}$ the map $w \mapsto \omega(x)(v,w)$; it is a linear form on the vector space T_xM, in other words an element of the cotangent space T_x^*M. Saying that the rank of ω is everywhere equal to the dimension of M amounts to say that the map $v \mapsto \mathrm{i}(v)\omega$ is an isomorphism of the tangent bundle TM onto the cotangent bundle T^*M.

2. Let V be a finite-dimensional vector space, and $\eta : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear form. As above, $v \mapsto \mathrm{i}(v)\eta$ is a linear map defined on V, with values in its dual space V^*. The *rank* of η is the dimension of the image of that map. An easy result in linear algebra is that the rank of a skew-symmetric bilinear form is always an even integer. When (M,ω) is a symplectic manifold, for each $x \in M$ that result can be applied to the bilinear form $\omega(x) : T_xM \times T_xM \to \mathbb{R}$, and we see that the dimension of M must be an even integer $2n$.

3. The *Darboux theorem*, due to the French mathematician Gaston Darboux (1842–1917), states that in a $2n$-dimensional symplectic manifold (M,ω) any point has a neighbourhood on which there exists local coordinates $(x^1,\dots,x^{2n})$ in which the $(2n)\times(2n)$-matrix (ω_{ij}) $(1 \leq i,\ j \leq 2n)$ of components of ω is a constant, skew-symmetric invertible matrix. We recall that

$$\omega_{ij} = \omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right).$$

These local coordinates can even be chosen in such a way that

$$\omega_{ij} = \begin{cases} 1 & \text{if } i-j=n, \\ -1 & \text{if } i-j=-n, \\ 0 & \text{if } |i-j| \neq n, \end{cases} \qquad 1 \leq i,\ j \leq 2n.$$

Local coordinates which satisfy this property are called *Darboux local coordinates*.

4. On the $2n$-dimensional symplectic manifold (M,ω), the $2n$-form ω^n (the n-th exterior power of ω) is a *volume form* (it means that it is everywhere $\neq 0$). Therefore a symplectic manifold always is *orientable*.

2.2 Examples of symplectic manifolds

2.2.1 Surfaces

A smooth orientable surface embedded in an Euclidean 3-dimensional affine space, endowed with the area form determined by the Euclidean metric, is a symplectic manifold.

More generally, any 2-dimensional orientable manifold, equipped with a nowhere vanishing area form, is a symplectic manifold.

2.2.2 Symplectic vector spaces

A *symplectic vector space* is a finite-dimensional real vector space E equipped with a skew-symmetric bilinear form $\omega : E \times E \to \mathbb{R}$ of rank equal to the dimension of E; therefore $\dim E$ is an even integer $2n$. Considered as a constant differential two-form on E, η is symplectic, which allows us to consider (E, η) as a symplectic manifold.

The canonical example of a symplectic vector space is the following. Let V be a real n-dimensional vector space and let V^* be its dual space. There exists on the direct sum $V \oplus V^*$ a natural skew-symmetric bilinear form

$$\eta\big((x_1,\zeta_1),(x_2,\zeta_2)\big) = \langle \zeta_1, x_2\rangle - \langle \zeta_2, x_1\rangle .$$

The rank of η being $2n$, $(V \oplus V^*, \eta)$ is a symplectic vector space.

Conversely, any $2n$-dimensional symplectic vector space (E, ω) can be identified with the direct sum of any of its n-dimensional vector subspaces V such that the symplectic form ω vanishes identically on $V \times V$, with its dual space V^*. In this identification, the symplectic form ω on E becomes identified with the above-defined symplectic form η on $V \oplus V^*$.

2.2.3 Cotangent bundles

Let N be a smooth n-dimensional manifold. With the notations of 1.3 for the canonical projections of tangent or cotangent bundles onto their base manifold and for prolongation to vectors of a smooth map, we recall that the diagram

$$\begin{array}{ccc} T(T^*N) & \xrightarrow{T\pi_N} & TN \\ \downarrow{\scriptstyle \tau_{T^*N}} & & \downarrow{\scriptstyle \tau_N} \\ T^*N & \xrightarrow{\pi_N} & N \end{array}$$

is commutative. For each $w \in T(T^*N)$, we can therefore write

$$\eta_N(w) = \big\langle \tau_{T^*N}(w), T\pi_N(w)\big\rangle .$$

This formula defines a differential 1-form η_N on the manifold T^*N, called the *Liouville 1-form*. Its exterior differential $d\eta_N$ is a symplectic form, called the *canonical symplectic form* on the cotangent bundle T^*N.

Let $(x^1,\dots,x^n)$ be a system of local coordinates on N, $(x^1,\dots,x^n,p_1,\dots,p_n)$ be the corresponding system of local coordinates on T^*N. The local expressions of the Liouville form η_N and of its exterior differential $d\eta_N$ are

$$\eta_N = \sum_{i=1}^{n} p_i\, dx^i , \quad d\eta_N = \sum_{i=1}^{n} dp_i \wedge dx^i .$$

We see that $(x^1,\dots,x^n,p_1,\dots,p_n)$ is a system of Darboux local coordinates. Therefore any symplectic manifold is locally isomorphic to a cotangent bundle.

2.2.4 The complex plane

The complex plane $\mathbb{C}$ is naturally endowed with a Hermitian form

$$\eta(z_1,z_2) = z_1\overline{z_2}\,, \quad z_1 \text{ and } z_2 \in \mathbb{C}\,,$$

where $\overline{z_2}$ is the conjugate of the complex number z_2. Let us write $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, where x_1, y_1, x_2, y_2 are real, and separate the real and imaginary parts of $\eta(z_1,z_2)$. We get

$$\eta(z_1,z_2) = (x_1x_2 + y_1y_2) + i(y_1x_2 - y_2x_1)\,.$$

The complex plane $\mathbb{C}$ has an underlying structure of real, 2-dimensional vector space, which can be identified with $\mathbb{R}^2$, each complex number $z = x+iy \in \mathbb{C}$ being identified with $(x,y) \in \mathbb{R}^2$. The real and imaginary parts of the Hermitian form η on $\mathbb{C}$ are, respectively, the Euclidean scalar product g and the symplectic form ω on $\mathbb{R}^2$ such that

$$\begin{aligned}\eta(z_1,z_2) &= (x_1x_2 + y_1y_2) + i(y_1x_2 - y_2x_1)\\ &= g\big((x_1,y_1),(x_2,y_2)\big) + i\omega\big((x_1,y_1),(x_2,y_2)\big)\,.\end{aligned}$$

2.2.5 Kähler manifolds

More generally, a n-dimensional *Kähler manifold* (*i.e.* a complex manifold of complex dimension n endowed with a Hermitian form whose imaginary part is a closed two-form), when considered as a real $2n$-dimensional manifold, is automatically endowed with a Riemannian metric and a symplectic form given, respectively, by the real and the imaginary parts of the Hermitian form.

Conversely, it is not always possible to endow a symplectic manifold with a complex structure and a Hermitian form of which the given symplectic form is the imaginary part. However, it is always possible to define, on a symplectic manifold, an *almost complex structure* and an *almost complex 2-form* with which the properties of the symplectic manifold become similar to those of a Kähler manifold (but with change of chart functions which are not holomorphic functions). This possibility was used by *Mikhaïl Gromov* [22] in his theory of *pseudo-holomorphic curves*.

2.3 Remarkable submanifolds of a symplectic manifold

Definitions 2.3.1. Let (V,ω) be a symplectic vector space, and W be a vector subspace of V. The *symplectic orthogonal* of W is the vector subspace

$$\operatorname{orth} W = \{\, v \in V\,;\, \omega(v,w) = 0 \text{ for all } w \in W\,\}\,.$$

The vector subspace W is said to be

- *isotropic* if $W \subset \operatorname{orth} W$,
- *coisotropic* if $W \supset \operatorname{orth} w$,
- *Lagrangian* if $W = \operatorname{orth} W$,
- *symplectic* if $W \oplus \operatorname{orth} W = V$.

2.3.2 Properties of symplectic orthogonality

The properties stated below are easily consequences of the above definitions

1. For any vector subspace W of the symplectic vector space (W,ω), we have $\mathrm{orth}(\mathrm{orth}\,W) = W$.

2. Let $\dim V = 2n$. For any vector subspace W of V, we have $\dim(\mathrm{orth}\,W) = \dim V - \dim W = 2n - \dim W$. Therefore, if W is isotropic, $\dim W \leq n$; if W is coisotropic, $\dim W \geq n$; and if W is Lagrangian, $\dim W = n$.

3. Let W be an isotropic vector subspace of V. The restriction to $W \times W$ of the symplectic form ω vanishes identically. Conversely, if W is a vector subspace such that the restriction of ω to $W \times W$ vanishes identically, W is isotropic.

4. A Lagrangian vector subspace of V is an isotropic subspace whose dimension is the highest possible, equal to half the dimension of V.

5. Let W be a symplectic vector subspace of V. Since $W \cap \mathrm{orth}\,W = \{0\}$, the rank of the restriction to $W \times W$ of the form ω is equal to $\dim W$; therefore $\dim W$ is even and, equipped with the restriction of ω, W is a symplectic vector space. Conversely if, when equipped with the restriction of ω, a vector subspace W of V is a symplectic vector space, we have $W \oplus \mathrm{orth}\,W = V$, and W is a symplectic vector subspace of V in the sense of the above definition.

6. A vector subspace W of V is symplectic if and only if $\mathrm{orth}\,W$ is symplectic.

Definitions 2.3.3. Let (M,ω) be a symplectic manifold. For each $x \in M$, $\big(T_xM, \omega(x)\big)$ is a symplectic vector space. A submanifold N of M is said to be

- *isotropic* if for each $x \in N$, T_xN is an isotropic vector subspace of the symplectic vector space $\big(T_xM, \omega(x)\big)$,
- *coisotropic* if for each $x \in N$, T_xN is a coisotropic vector subspace of $\big(T_xM, \omega(x)\big)$,
- *Lagrangian* if for each $x \in N$, T_xN is a Lagrangian vector subspace of $\big(T_xM, \omega(x)\big)$,
- *symplectic* if for each $x \in N$, T_xN is a symplectic vector subspace of $\big(T_xM, \omega(x)\big)$.

2.4 Hamiltonian vector fields on a symplectic manifold

Let (M,ω) be a symplectic manifold. We have seen that the map which associates to each vector $v \in TM$ the covector $\mathrm{i}(v)\omega$ is an isomorphism from TM onto T^*M. So, for any given differential one-form α, there exists a unique vector field X such that $\mathrm{i}(X)\omega = \alpha$. We are therefore allowed to state the following definitions.

Definitions 2.4.1. Let (M,ω) be a symplectic manifold and $f : M \to \mathbb{R}$ be a smooth function. The vector field X_f which satisfies

$$\mathrm{i}(X_f)\omega = -\mathrm{d}f$$

is called the *Hamiltonian vector field* associated to f. The function f is called a *Hamiltonian* for the Hamiltonian vector field X_f.

A vector field X on M such that the one-form $\mathrm{i}(X)\omega$ is closed,

$$\mathrm{di}(\Omega) = 0\,,$$

is said to be *locally Hamiltonian.*

Remarks 2.4.2. The function f is not the unique Hamiltonian of the Hamiltonian vector field X_f: any function g such that $\mathrm{i}(X_f)\omega = -dg$ is another Hamiltonian for X_f. Given a Hamiltonian f of X_f, a function g is another Hamiltonian for X_f if and only if $\mathrm{d}(f-g) = 0$, or in other words if and only if $f-g$ keeps a constant value on each connected component of M

Of course, a Hamiltonian vector field is locally Hamiltonian. The converse is not true when the cohomology space $H^1(M,\mathbb{R})$ is not trivial.

Proposition 2.4.3. *On a symplectic manifold (M,ω), a vector field X is locally Hamiltonian if and only if the Lie derivative $\mathcal{L}(X)\omega$ of the symplectic form ω with respect to X vanishes:*

$$\mathcal{L}(X)\omega = 0\,.$$

The bracket $[X,Y]$ of two locally Hamiltonian vector fields X and Y is Hamiltonian, and has as a Hamiltonian the function $\omega(X,Y)$.

Proof. The well known formula which relates the the exterior differential d, the interior product $\mathrm{i}(X)$ and the Lie derivative $\mathcal{L}(X)$ with respect to the vector field X

$$\mathcal{L}(X) = \mathrm{i}(X)\mathrm{d} + \mathrm{d}\mathrm{i}(X)$$

proves that when X is a vector field on a symplectic manifold (M,ω)

$$\mathcal{L}(X)\omega = \mathrm{d}\,\mathrm{i}(X)\omega\,,$$

since $\mathrm{d}\,\omega = 0$. Therefore $\mathrm{i}(X)\omega$ is closed if and only if $\mathcal{L}(X)\omega = 0$.

Let X and Y be two locally Hamiltonian vector fields. We have

$$\begin{aligned}
\mathrm{i}\big([X,Y]\big)\omega &= \mathcal{L}(X)\mathrm{i}(Y)\omega - \mathrm{i}(Y)\mathcal{L}(X)\omega \\
&= \mathcal{L}(X)\mathrm{i}(Y)\omega \\
&= \big(\mathrm{i}(X)\mathrm{d} + \mathrm{d}\,\mathrm{i}(X)\big)\mathrm{i}(Y)\omega \\
&= \mathrm{d}\,\mathrm{i}(X)\mathrm{i}(Y)\omega \\
&= -d\big(\omega(X,Y)\big)\,,
\end{aligned}$$

which proves that $\omega(X,Y)$ is a Hamiltonian for $[X,Y]$. □

2.4.4 Expression in a system of Darboux local coordinates

Let $(x^1,\ldots,x^{2n})$ be a system of Darboux local coordinates. The symplectic form ω can be locally writen as

$$\omega = \sum_{i=1}^{n} \mathrm{d}x^{n+i} \wedge \mathrm{d}x^i\,,$$

so we see that the Hamiltonian vector field X_f associated to a smooth function f can be locally written as

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial x^{n+i}}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^{n+i}}\,.$$

A smooth curve φ drawn in M parametrized by the real variable t is said to be a *solution* of the differential equation determined by X_f, or an *integral curve* of X_f, if it satisfies the equation, called the *Hamilton equation* for the Hamiltonian f,

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = X_f\big(\varphi(t)\big)\,.$$

Its local expression in the considered system of Darboux local coordinates is

$$\begin{cases} \dfrac{\mathrm{d}x^i}{\mathrm{d}t} = \dfrac{\partial f}{\partial x^{n+i}}, \\ \dfrac{\mathrm{d}x^{n+i}}{\mathrm{d}t} = -\dfrac{\partial f}{\partial x^i}, \end{cases} \qquad (1 \leq i \leq n).$$

Definition 2.4.5. Let $\Phi : N \to N$ be a diffeomorphism of a smooth manifold N onto itself. The *canonical lift* of Φ to the cotangent bundle is the transpose of the vector bundles isomorphism $T(\Phi^{-1}) = (T\Phi)^{-1} : TN \to TN$. In other words, denoting by $\widehat{\Phi}$ the canonical lift of Φ to the cotangent bundle, we have for all $x \in N$, $\xi \in T_x^*N$ and $v \in T_{\Phi(x)}N$,

$$\left\langle \widehat{\Phi}(\xi), v \right\rangle = \left\langle \xi, (T\Phi)^{-1}(v) \right\rangle.$$

Remark 2.4.6. With the notations of Definition 2.4.5, we have $\pi_N \circ \widehat{\Phi} = \Phi \circ \pi_N$.

2.4.7 The flow of a vector field

Let X be a smooth vector field on a smooth manifold M. We recall that the *reduced flow* of X is the map Φ, defined on an open subset Ω of $\mathbb{R} \times M$ and taking its values in M, such that for each $x \in M$ the parametrized curve $t \mapsto \varphi(t) = \Phi(t,x)$ is the maximal integral curve of the differential equation

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = X\big(\varphi(t)\big)$$

which satisfies $\varphi(0) = x$. For each $t \in \mathbb{R}$, the set $D_t = \{x \in M; (t,x) \in \Omega\}$ is an open subset of M and when D_t is not empty the map $x \mapsto \Phi_t(x) = \Phi(t,x)$ is a diffeomorphism of D_t onto D_{-t}.

Definitions 2.4.8. Let N be a smooth manifold, TN and T^*N be its tangent and cotangent bundles, $\tau_N : TN \to N$ and $\pi_N : T^*N \to N$ be their canonical projections. Let X be a smooth vector field on N and $\{\Phi_t^X ; t \in \mathbb{R}\}$ be its reduced flow.

1. The *canonical lift* of X to the tangent bundle TN is the unique vector field $\overline{X}$ on TM whose reduced flow $\{\Phi_t^{\overline{X}} ; t \in \mathbb{R}\}$ is the prolongation to vectors of the reduced flow of X. In other words, for each $t \in \mathbb{R}$,

$$\Phi_t^{\overline{X}} = T\Phi_t^X,$$

therefore, for each $v \in TN$,

$$\overline{X}(v) = \frac{\mathrm{d}}{\mathrm{d}t}\big(T\Phi_t^X(v)\big)\,\big|_{t=0}.$$

2. The *canonical lift* of X to the cotangent bundle T^*N is the unique vector field $\widehat{X}$ on T^*M whose reduced flow $\{\Phi_t^{\widehat{X}} ; t \in \mathbb{R}\}$ is the lift to the cotangent bundle of the reduced flow $\{\Phi_t^X ; t \in \mathbb{R}\}$ of X. In other words, for each $t \in \mathbb{R}$,

$$\Phi_t^{\widehat{X}} = \widehat{\Phi_t^X},$$

therefore, for each $\xi \in T^*N$,

$$\widehat{X}(\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\big(\widehat{\Phi_t^X}(\xi)\big)\,\big|_{t=0}.$$

Remark 2.4.9. Let X be a smooth vector field defined on a smooth manifold N. Its canonical lift $\overline{X}$ to the tangent bundle TN (2.4.8) is related to the prolongation to vectors $TX : TN \to T(TN)$ by the formula

$$\overline{X} = \kappa_N \circ TX,$$

where $\kappa_N : T(TN) \to T(TN)$ is the canonical involution of the tangent bundle to TN (see [63]).

Proposition 2.4.10. *Let $\Phi : N \to N$ be a diffeomorphism of a smooth manifold N onto itself and $\widehat{\Phi} : T^*N \to T^*N$ the canonical lift of Φ to the cotangent bundle. Let η_N be the Liouville form on T^*N. We have*

$$\widehat{\Phi}^* \eta_N = \eta_N .$$

Let X be a smooth vector field on N, and $\widehat{X}$ be the canonical lift of X to the cotangent bundle. We have

$$\mathcal{L}(\widehat{X})(\eta_N) = 0 .$$

Proof. Let $\xi \in T^*N$ and $v \in T_\xi(T^*N)$. We have

$$\widehat{\Phi}^* \eta_N(v) = \eta_N\big(T\widehat{\Phi}(v)\big) = \big\langle \tau_{T^*N} \circ T\widehat{\Phi}(v), T\pi_N \circ T\widehat{\Phi}(v) \big\rangle .$$

But $\tau_{T^*N} \circ T\widehat{\Phi} = \widehat{\Phi} \circ \tau_{T^*N}$ and $T\pi_N \circ T\widehat{\Phi} = T(\pi_N \circ \widehat{\Phi}) = T(\Phi \circ \pi_N)$. Therefore

$$\widehat{\Phi}^* \eta_N(v) = \big\langle \widehat{\Phi} \circ \tau_{T^*N}(v), T(\Phi \circ \pi_N)(v) \big\rangle = \big\langle \tau_{T^*N}(v), T\pi_N(v) \big\rangle = \eta_N(v)$$

since $\widehat{\Phi} = (T\Phi^{-1})^T$. Now let X be a smooth vector field on N, $\{\Phi_t^X ; t \in \mathbb{R}\}$ be its reduced flow, and $\widehat{X}$ be the canonical lift of X to the cotangent bundle. We know that the reduced flow of $\widehat{X}$ is $\{\widehat{\Phi_t^X} ; t \in \mathbb{R}\}$, so we can write

$$\mathcal{L}(\widehat{X})\eta_N = \frac{\mathrm{d}}{\mathrm{d}t}\big(\widehat{\Phi_t^X}^* \eta_N\big) \,\big|_{t=0} .$$

Since $\widehat{\Phi_t^X}^* \eta_N = \eta_N$ does not depend on t, $\mathcal{L}(\widehat{X})\eta_N = 0$. □

The following Proposition, which presents an important example of Hamiltonian vector field on a cotangent bundle, will be used when we will consider Hamiltonian actions of a Lie group on its cotangent bundle.

Proposition 2.4.11. *Let N be a smooth manifold, T^*N be its cotangent bundle, η_N be the Liouville form and $\mathrm{d}\eta_N$ be the canonical symplectic form on T^*N. Let X be a smooth vector field on N and $f_X : T^*N \to \mathbb{R}$ be the smooth function defined by*

$$f_X(\xi) = \Big\langle \xi, X\big(\pi_N(\xi)\big) \Big\rangle , \quad \xi \in T^*N .$$

On the symplectic manifold $(T^*N, \mathrm{d}\eta_N)$, the vector field $\widehat{X}$, canonical lift to T^*N of the vector field X on N in the sense defined above (2.4.8), is a Hamiltonian field which has the function f_X as a Hamiltonian. In other words

$$\mathrm{i}(\widehat{X})\mathrm{d}\eta_N = -\mathrm{d}f_X .$$

Moreover,

$$f_X = \mathrm{i}(\widehat{X})\eta_N .$$

Proof. We have seen (Proposition 2.4.10) that $\mathcal{L}(\widehat{X})\eta_N = 0$. Therefore

$$\mathrm{i}(\widehat{X})\mathrm{d}\eta_N = \mathcal{L}(\widehat{X})\eta_N - \mathrm{d}\mathrm{i}(\widehat{X})\eta_N = -\mathrm{d}\mathrm{i}(\widehat{X})\eta_N ,$$

which proves that $\widehat{X}$ is Hamiltonian and admits $\mathrm{i}(\widehat{X})\eta_N$ as Hamiltonian. For each $\xi \in T^*N$

$$\mathrm{i}(\widehat{X})\eta_N(\xi) = \eta_N(\widehat{X})(\xi) = \Big\langle \xi, T\pi_N\big(\widehat{X}(\xi)\big) \Big\rangle = \Big\langle \xi, X\big(\pi_N(\xi)\big) \Big\rangle = f_X(\xi) . \quad \square$$

2.5 The Poisson bracket

Definition 2.5.1. The *Poisson bracket* of an ordered pair (f,g) of smooth functions defined on the symplectic manifold (M,ω) is the smooth function $\{f,g\}$ defined by the equivalent formulae

$$\{f,g\} = \mathrm{i}(X_f)\,dg = -\mathrm{i}(X_g)\,df = \omega(X_f,X_g)\,,$$

where X_f and X_g are the Hamiltonian vector fields on M with, respectively, the functions f and g as Hamiltonian.

Lemma 2.5.2. *Let (M,ω) be a symplectic manifold, let f and g be two smooth functions on M and let X_f and X_g be the associated Hamiltonian vector fields. The bracket $[X_f,X_g]$ is a Hamiltonian vector field which admits $\{f,g\}$ as Hamiltonian.*

Proof. This result is an immediate consequence of Proposition 2.4.3. □

Proposition 2.5.3. *Let (M,ω) be a symplectic manifold. The Poisson bracket is a bilinear composition law on the space $C^\infty(M,\mathbb{R})$ of smooth functions on M, which satisfies the following properties.*
1. *It is skew-symmetric: $\{g,f\} = -\{f,g\}$.*
2. *It satisfies the Leibniz identity with respect to the ordinary product of functions:*

$$\{f,gh\} = \{f,g\}h + g\{f,h\}\,.$$

3. *It satisfies the Jacobi identity, which is a kind of Leibniz identity with respect to the Poisson bracket itself:*

$$\big\{f,\{g,h\}\big\} = \big\{\{f,g\},h\big\} + \big\{g,\{f,h\}\big\}\,,$$

which can also be written, when the skew-symmetry of the Poisson bracket is taken into account,

$$\big\{\{f,g\},h\big\} + \big\{\{g,h\},f\big\} + \big\{\{h,f\},g\big\} = 0\,.$$

Proof. The proofs of Properties (i) and (ii) are very easy and left to the reader. Let us proove Property (iii).

We have

$$\big\{\{f,g\},h\big\} = \omega(X_{\{f,g\}},X_h) = -\mathrm{i}(X_{\{f,g\}})\mathrm{i}(X_h)\omega = \mathrm{i}(X_{\{f,g\}})\mathrm{d}h\,.$$

By Lemma 2.5.2, $X_{\{f,g\}} = [X_f,X_g]$ so we have

$$\big\{\{f,g\},h\big\} = \mathrm{i}\big([X_f,X_g]\big)\mathrm{d}h = \mathcal{L}\big([X_f,X_g]\big)h\,.$$

We also have

$$\big\{\{g,h\},f\big\} = -\mathcal{L}(X_f)\circ\mathcal{L}(X_g)h\,,\quad \big\{\{h,f\},g\big\} = \mathcal{L}(X_g)\circ\mathcal{L}(X_f)h\,.$$

Taking the sum of these three terms, and taking into account the identity

$$\mathcal{L}\big([X_f,X_g]\big) = \mathcal{L}(X_f)\circ\mathcal{L}(X_g) - \mathcal{L}(X_g)\circ\mathcal{L}(X_f)\,,$$

we see that the Jacobi identity is satisfied. □

Remarks 2.5.4.

1. In a system of Darboux local coordinates $(x^1,\dots,x^{2n})$, the Poisson bracket can be written

$$\{f,g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^{n+i}} \frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^{n+i}} \right) .$$

2. Let H be a smooth function on the symplectic manifold (M,ω), and X_H be the associaled Hamiltonian vector field. By using the Poisson bracket, one can write in a very concise way the Hamilton equation for X_H. Let $t \mapsto \varphi(t)$ be any integral curve of X_H. Then for any smooth function $f : M \to \mathbb{R}$

$$\frac{\mathrm{d}f\big(\varphi(t)\big)}{\mathrm{d}t} = \{H,f\}\big(\varphi(t)\big) .$$

By succesively taking for f the coordinate functions $x^1,\dots,x^{2n}$ of a system of Darboux local coordinates, we recover the equations

$$\begin{cases} \dfrac{\mathrm{d}x^i}{\mathrm{d}t} = \dfrac{\partial H}{\partial x^{n+i}} , \\ \dfrac{\mathrm{d}x^{n+i}}{\mathrm{d}t} = -\dfrac{\partial H}{\partial x^i} , \end{cases} \qquad (1 \leq i \leq n) .$$

3 Poisson manifolds

3.1 The inception of Poisson manifolds

Around the middle of the XX-th century, several scientists felt the need of a frame in which Hamiltonian differential equations could be considered, more general than that of symplectic manifolds. Paul Dirac for example proposed such a frame in his famous 1950 paper *Generalized Hamiltonian dynamics* [17, 18].

In many applications in which, starting from a symplectic manifold, another manifold is built by a combination of processes (products, quotients, restriction to a submanifold, ...), there exists on that manifold a structure, more general than a symplectic structure, with which a vector field can be associated to each smooth function, and the bracket of two smooth functions can be defined. It was also known that on a (odd-dimensional) contact manifold one can define the bracket of two smooth functions.

Several generalizatons of symplectic manifolds were defined and investigated by André Lichnerowicz during the years 1975–1980. He gave several names to these generalizations: *canonical, Poisson, Jacobi* and *locally conformally symplectic* manifolds [45, 46].

In 1976 Alexander Kirillov published a paper entitled *Local Lie algebras* [35] in which he determined all the possible structures on a manifold allowing the definition of a bracket with which the space of smooth functions becomes a local Lie algebra. *Local* means that the value taken by the bracket of two smooth functions at each point only depends of the values taken by these functions on an arbitrarily small neighbourhood of that point. The only such structures are those called by Lichnerowicz *Poisson structures*, *Jacobi structures* and *locally conformally symplectic structures*.

In what follows we will mainly consider Poisson manifolds.

3.2 Definition and structure of Poisson manifolds

Definition 3.2.1. A *Poisson structure* on a smooth manifold M is the structure determined by a bilinear, skew-symmetric composition law on the space of smooth functions, called the *Poisson bracket* and denoted by $(f,g) \mapsto \{f,g\}$, satisfying the Leibniz identity

$$\{f,gh\} = \{f,g\}h + g\{f,h\}$$

and the Jacobi identity

$$\big\{\{f,g\},h\big\} + \big\{\{g,h\},f\big\} + \big\{\{h,f\},g\big\} = 0\,.$$

A manifold endowed with a Poisson structure is called a *Poisson manifold*.

Proposition 3.2.2. *On a Poisson manifold M, there exists a unique smooth bivector field Λ, called the* Poisson bivector field *of M, such that for any pair (f,g) of smooth functions defined on M, the Poisson bracket $\{f,g\}$ is given by the formula*

$$\{f,g\} = \Lambda(\mathrm{d}f,\mathrm{d}g)\,.$$

Proof. The existence, uniqueness and skew-symmetry of Λ are easy consequences of the the Leibniz identity and of the skew-symmetry of the Poisson bracket. It does not depend on the Jacobi identity. □

Remark 3.2.3. The Poisson bivector field Λ determines the Poisson structure of M, since it allows the calculation of the Poisson bracket of any pair of smooth functions. For this reason a Poisson manifold M is often denoted by (M,Λ).

Definition 3.2.4. Let (M,Λ) be a Poisson manifold. We denote by $\Lambda^\sharp : T^*M \to TM$ the vector bundle homomorphism such that, for each $x \in M$ and each $\alpha \in T_x^*M$, $\Lambda^\sharp(\alpha)$ is the unique element in T_xM such that, for any $\beta \in T_x^*M$,

$$\big\langle \beta, \Lambda^\sharp(\alpha)\big\rangle = \Lambda(\alpha,\beta)\,.$$

The subset $C = \Lambda^\sharp(T^*M)$ of the tangent bundle TM is called the *characteristic field* of the Poisson manifold (M,Λ)

The following theorem, due to Alan Weinstein [66], proves that, loosely speaking, a Poisson manifold is the disjoint union of symplectic manifolds, arranged in such a way that the union is endowed with a differentiable structure.

Theorem 3.2.5. *Let (M,Λ) be a Poisson manifold. Its characteristic field C is a completely integrable generalized distribution on M. It means that M is the disjoint union of immersed connected submanifolds, called the* symplectic leaves *of (M,Λ), with the following properties: a leaf S is such that, for each $x \in S$, $T_xS = T_xM \cap C$; moreover, S is maximal in the sense that any immersed connected submanifold S' containing S and such that for each $x \in S'$, $T_xS' = T_xM \cap C$, is equal to S.*

Moreover, the Poisson structure of M determines, on each leaf S, a symplectic form ω_S, such that the restriction to S of the Poisson bracket of two smooth functions defined on M only depends on the restrictions of these functions to S, and can be calculated as the Poisson bracket of these restrictions, using the symplectic form ω_S.

The reader may look at [66] or at [44] for a proof of this theorem.

3.2.6 The Schouten-Nijenhuis bracket

Let M be a smooth n-dimensional manifold. We recall that the exterior algebra $\Omega(M)$ of differential forms on M is endowed with an associative composition law, the *exterior product*, which associates to a pair (η,ζ), with $\eta\in\Omega^p(M)$ and $\zeta\in\Omega^q(M)$ the form $\eta\wedge\zeta\in\Omega^{p+q}(M)$, with the following properties.

1. When $p=0$, $\eta\in\Omega^0(M)\equiv C^\infty(M,\mathbb{R})$; the exterior product $\eta\wedge\zeta$ is the usual product $\eta\zeta$ of the differential form ζ of degree q by the function η.

2. The exterior product satisfies

$$\zeta\wedge\eta=(-1)^{pq}\eta\wedge\zeta\,.$$

3. When $p\geq 1$ and $q\geq 1$, $\eta\wedge\zeta$ evaluated on the $p+q$ vector fields $v_i\in A^1(M)$ $(1\leq i\leq p+q)$ is expressed as

$$\eta\wedge\zeta(v_1,\ldots,v_{p+q})=\sum_{\sigma\in\mathcal{S}_{(p,q)}}\varepsilon(\sigma)\eta(v_{\sigma(1)},\ldots,v_{\sigma(p)})\zeta(v_{\sigma(p+1)},\ldots,v_{\sigma(p+q)})\,.$$

We have denoted by $\mathcal{S}_{(p,q)}$ the set of permutations σ of $\{1,\ldots,p+q\}$ which satisfy

$$\sigma(1)<\sigma(2)<\cdots<\sigma(p)\quad\text{and}\quad\sigma(p+1)<\sigma(p+2)<\cdots<\sigma(p+q)\,,$$

and set

$$\varepsilon(\sigma)=\begin{cases}1 & \text{if } \sigma \text{ is even,}\\ -1 & \text{if } \sigma \text{ is odd.}\end{cases}$$

The exterior algebra $\Omega(M)$ is endowed with a linear map $\mathrm{d}:\Omega(M)\to\Omega(M)$ called the *exterior differential*, with the following properties.

1. The exterior differential d is a graded map of degree 1, which means that $\mathrm{d}\big(\Omega^p(M)\big)\subset\Omega^{p+1}(M)$.

2. It is a *derivation* of the exterior algebra $\Omega(M)$, which means that when $\eta\in\Omega^p(M)$ and $\zeta\in\Omega^q(M)$,

$$\mathrm{d}(\eta\wedge\zeta)=(\mathrm{d}\eta)\wedge\zeta+(-1)^p\eta\wedge\mathrm{d}\zeta\,.$$

3. It satisfies

$$\mathrm{d}\circ\mathrm{d}=0\,.$$

Similarly, the exterior algebra $A(M)$ of smooth multivector fields on M is endowed with an associative composition law, the *exterior product*, which associates to a pair (P,Q), with $P\in A^p(M)$ and $Q\in A^q(M)$, the multivector field $P\wedge Q\in A^{p+q}(M)$. It is defined by the formulae given above for the exterior product of differential forms, the only change being the exchange of the roles of $\Omega^p(M)$ and $A^p(M)$. Its properties are essentially the same as those of the exterior product of differential forms.

There is a natural pairing of elements of same degree in $A(M)$ and in $\Omega(M)$. It is first defined for decomposable elements: let $\eta=\eta_1\wedge\cdots\wedge\eta_p\in\Omega^p(M)$ and $P=X_1\wedge\cdots\wedge X_p\in A^p(M)$. We set

$$\langle\eta,P\rangle=\det\big(\langle\eta_i,X_j\rangle\big)\,.$$

Then this pairing can be uniquely extended to $\Omega^p(M)\times A^p(M)$ by bilinearity.

With any $P\in A^p(M)$ we can associate a graded endomorphism $\mathrm{i}(P)$ of the exterior algebra of differential forms $\Omega(M)$, of degree $-p$, which means that when $\eta\in\Omega^q(M)$, $\mathrm{i}(P)\eta\in\Omega^{q-p}(M)$.

This endomorphism, which extends to multivector fields the interior product of forms with a vector field, is determined by the formula, in which $P \in A^p(M)$, $\eta \in \Omega^q(M)$ and $R \in A^{q-p}(M)$,

$$\big\langle \mathrm{i}(P)\eta, R\big\rangle = (-1)^{(p-1)p/2}\langle \eta, P \wedge Q\rangle\,.$$

Besides the exterior product, there exists on the graded vector space $A(M)$ of multivector fields another bilinear composition law, which naturally extends to multivector fields the Lie bracket of vector fields. It associates to $P \in A^p(M)$ and $Q \in A^q(M)$ an element denoted $[P,Q] \in A^{p+q-1}(M)$, called the *Schouten-Nijenhuis bracket* of P and Q. The Schouten-Nijenhuis bracket $[P,Q]$ is defined by the following formula, which gives the expression of the corresponding graded endomorphism of $\Omega(M)$,

$$\mathrm{i}\big([P,Q]\big) = \Big[\big[\mathrm{i}(P),\mathrm{d}\big],\mathrm{i}(Q)\Big]\,.$$

The brackets in the right hand side of this formula are the *graded commutators* of graded endomorphisms of $\Omega(M)$. Let us recall that if E_1 and E_2 are graded endomorphisms of $\Omega(M)$ of degrees e_1 and e_2 respectively, their graded commutator is

$$[E_1,E_2] = E_1 \circ E_2 - (-1)^{e_1 e_2} E_2 \circ E_1\,.$$

The following properties of the Schouten-Nijenhuis bracket can be deduced from the above formulae.

1. For f and $g \in A^0(M) = C^\infty(M,\mathbb{R})$, $[f,g] = 0$.

2. For a vector field $V \in A^1(M)$, $q \in \mathbb{Z}$ and $Q \in A^q(M)$, the Schouten-Nijenhuis bracket $[V,Q]$ is the Lie derivative $\mathcal{L}(V)(Q)$.

3. For two vector fields V and $W \in A^1(M,E)$, the Schouten-Nijenhuis bracket $[V,W]$ is the usual Lie bracket of these vector fields.

4. For all p and $q \in \mathbb{Z}$, $P \in A^p(M)$, $Q \in A^q(M)$,

$$[P,Q] = -(-1)^{(p-1)(q-1)}[Q,P]\,.$$

5. Let $p \in \mathbb{Z}$, $P \in A^p(M)$. The map $Q \mapsto [P,Q]$ is a derivation of degree $p-1$ of the graded exterior algebra $A(M)$. In other words, for q_1 and $q_2 \in \mathbb{Z}$, $Q_1 \in A^{q_1}(M)$ and $Q_2 \in A^{q_2}(M)$,

$$[P, Q_1 \wedge Q_2] = [P,Q_1] \wedge Q_2 + (-1)^{(p-1)q_1} Q_1 \wedge [P,Q_2]\,.$$

6. Let p, q and $r \in \mathbb{Z}$, $P \in A^p(M)$, $Q \in A^q(M)$ and $R \in A^r(M)$. The Schouten-Nijenhuis bracket satisfies the *graded Jacobi identity*

$$\begin{aligned}(-1)^{(p-1)(r-1)}\big[[P,Q],R\big] + (-1)^{(q-1)(p-1)}\big[[Q,R],P\big]& \\ + (-1)^{(r-1)(q-1)}\big[[R,P],Q\big]& \\ = 0\,.&\end{aligned}$$

For more information about the Schouten-Nijenhuis bracket, the reader may look at [40] or [49].

Proposition 3.2.7. *Let Λ be a smooth bivector field on a smooth manifold M. Then Λ is a Poisson bivector field (and (M,Λ) is a Poisson manifold) if and only if $[\Lambda,\Lambda] = 0$.*

Proof. We define the vector bundle homomorphism $\Lambda^\sharp : T^*M \to TM$ by setting, for all $x \in M$, α and $\beta \in T^*_x M$,

$$\big\langle \beta, \Lambda^\sharp(\alpha)\big\rangle = \Lambda(\alpha,\beta)\,.$$

For any pair (f,g) of smooth functions we set

$$X_f = \Lambda^\sharp(df)\,, \quad \{f,g\} = \mathrm{i}(X_f)(\mathrm{d}g) = \Lambda(\mathrm{d}f,\mathrm{d}g)\,.$$

This bracket is a bilinear skew-symmetric composition law on $C^\infty(M,\mathbb{R})$ which satisfies the Leibniz identity. Therefore Λ is a Poisson bivector field if and only if the above defined bracket of functions satisfies the Jacobi identity.

Let f, g and h be three smooth functions on M. We easily see that X_f and $\{f,g\}$ can be expressed in terms of the Schouten-Nijenhuis bracket. Indeed we have

$$X_f = -[\Lambda,f] = -[f,\Lambda]\,, \quad \{f,g\} = \big[[\Lambda,f],g\big]\,.$$

Therefore

$$\big\{\{f,g\},h\big\} = \Big[\Big[\Lambda,\big[[\Lambda,f],g\big]\Big],h\Big]\,.$$

By using the graded Jacobi identity satisfied by Schouten-Nijenhuis bracket, we see that

$$\Big[\Lambda,\big[[\Lambda,f],g\big]\Big] = -\big[[g,\Lambda],[f,\Lambda]\big] + 2\Big[\big[[\Lambda,\Lambda],f\big],g\Big]\,.$$

Using the equalities $X_f = -[\Lambda,f] = -[f,\Lambda]$ and $X_g = -[\Lambda,g] = -[g,\Lambda]$ we obtain

$$\begin{aligned}\big\{\{f,g\},h\big\} &= \big[[X_f,X_g],h\big] + 2\Big[\Big[\big[[\Lambda,\Lambda],f\big],g\Big],h\Big] \\ &= \mathcal{L}\big([X_f,X_g]\big)h + 2\Big[\Big[\big[[\Lambda,\Lambda],f\big],g\Big],h\Big]\,.\end{aligned}$$

On the other hand, we have

$$\big\{\{g,h\},f\big\} = -\mathcal{L}(X_f)\circ\mathcal{L}(X_g)h\,, \quad \big\{\{h,f\},g\big\} = \mathcal{L}(X_g)\circ\mathcal{L}(X_f)h\,.$$

Taking into account the equality

$$\mathcal{L}\big([X_f,X_g]\big) = \mathcal{L}(X_f)\circ\mathcal{L}(X_g) - \mathcal{L}(X_g)\circ\mathcal{L}(X_f)$$

we obtain

$$\big\{\{f,g\},h\big\} + \big\{\{g,h\},f\big\} + \big\{\{h,f\},g\big\} = 2\Big[\Big[\big[[\Lambda,\Lambda],f\big],g\Big],h\Big]\,.$$

By using the formula which defines the Schouten-Nijenhuis bracket, we check that for any $P \in A^3(M)$

$$\Big[\big[[P,f],g\big],h\Big] = P(df,dg,dh)\,.$$

Therefore

$$\big\{\{f,g\},h\big\} + \big\{\{g,h\},f\big\} + \big\{\{h,f\},g\big\} = 2[\Lambda,\Lambda](df,dg,dh)\,,$$

so Λ is a Poisson bivector field if and only if $[\Lambda,\Lambda] = 0$. □

3.3 Some properties of Poisson manifolds

Definitions 3.3.1. Let (M,Λ) be a Poisson manifold.

1. The *Hamiltonian vector field* associated to a smooth function $f \in C^\infty(M,\mathbb{R})$ is the vector field X_f on M defined by

$$X_f = \Lambda^\sharp(\mathrm{d}f)\,.$$

The function f is called a *Hamiltonian* for the Hamiltonian vector field X_f.

2. A *Poisson vector field* is a vector field X which satisfies

$$\mathcal{L}(X)\Lambda = 0\,.$$

Example 3.3.2. On a symplectic manifold (M,ω) we have defined the Poisson bracket of smooth functions. That bracket endows M with a Poisson structure, said to be *associated* to its symplectic structure. The Poisson bivector field Λ is related to the symplectic form ω by

$$\Lambda(df,dg) = \omega(X_f,X_g)\,, \quad f \text{ and } g \in C^\infty(M,\mathbb{R})\,.$$

The map $\Lambda^\sharp : T^*M \to TM$ such that, for any $x \in M$, α and $\beta \in T_x^*M$,

$$\big\langle \beta, \Lambda^\sharp(\alpha) = \Lambda(\alpha,\beta)$$

is therefore the inverse of the map $\omega^\flat : TM \to T^*M$ such that, for any $x \in M$, v and $w \in T_xM$,

$$\big\langle \omega^\flat(v), w\big\rangle = -\big\langle i(v)\omega, w\big\rangle = \omega(w,v)\,.$$

Hamiltonian vector fields for the symplectic structure of M coincide with Hamiltonian vector fields for its Poisson structure. The Poisson vector fields on the symplectic manifold (M,ω) are the locally Hamiltonian vector fields. However, on a general Poisson manifold, Poisson vector fields are more general than locally Hamiltonian vector fields: even restricted to an arbitrary small neighbourhood of a point, a Poisson vector field may not be Hamiltonian.

Remarks 3.3.3.

1. Another way in which the Hamiltonian vector field X_f associated to a smooth function f can be defined is by saying that, for any other smooth function g on the Poisson manifold (M,Λ),

$$\mathrm{i}(X_f)(\mathrm{d}g) = \{f,g\}\,.$$

2. A smooth function g defined on the Poisson manifold (M,Λ) is said to be a *Casimir* if for any other smooth function h, we have $\{g,h\} = 0$. In other words, a Casimir is a smooth function g whose associated Hamiltonian vector field is $X_g = 0$. On a general Poisson manifold, there may exist Casimirs other than the locally constant functions.

3. A smooth vector field X on the Poisson manifold (M,Λ) is a Poisson vector field if and only if, for any pair (f,g) of smooth functions,

$$\mathcal{L}(X)\big(\{f,g\}\big) = \big\{\mathcal{L}(X)f,g\big\} + \big\{f,\mathcal{L}(X)g\big\}\,.$$

Indeed we have

$$\begin{aligned}\mathcal{L}(X)\big(\{f,g\}\big) &= \mathcal{L}(X)\big(\Lambda(\mathrm{d}f,\mathrm{d}g)\big)\\ &= \big(\mathcal{L}(X(\Lambda)\big)(\mathrm{d}f,\mathrm{d}g) + \Lambda\big(\mathcal{L}(X)(\mathrm{d}f),\mathrm{d}g\big) + \Lambda\big(\mathrm{d}f,\mathcal{L}(X)(\mathrm{d}g)\big)\\ &= \big(\mathcal{L}(X)(\Lambda)\big)(\mathrm{d}f,\mathrm{d}g) + \big\{\mathcal{L}(X)f,g\big\} + \big\{f,\mathcal{L}(X)g\big\}\,.\end{aligned}$$

3. Any Hamiltonian vector field X_f is a Poisson vector field. Indeed, if f is a Hamiltonian for X_f, g and h two other smooth functions,we have according to the Jacobi identity

$$\begin{aligned}\mathcal{L}(X_f)\big(\{g,h\}\big) &= \big\{f,\{g,h\}\big\} = \big\{\{f,g\},h\big\} + \big\{g,\{f,h\}\big\} \\ &= \big\{\mathcal{L}(X_f)g,h\big\} + \big\{g,\mathcal{L}(X_f)h\big\}\,.\end{aligned}$$

4. Since the characteristic field of the Poisson manifold (M,Λ) is generated by the Hamiltonian vector fields, any Hamiltonian vector field is everywhere tangent to the symplectic foliation. A Poisson vector field may not be tangent to that foliation.

Proposition 3.3.4. *Let (M,Λ) be a Poisson manifold, $H \in C^\infty(M,\mathbb{R})$ be a smooth function and $X_H = \Lambda^\sharp(\mathrm{d}H) \in A^1(M)$ be the associated Hamiltonian vector field. A smooth function $g \in C^\infty(M,\mathbb{R})$ keeps a constant value on each integral curve of X_H if and only if $\{H,g\} = 0$ identically. Such a function g is said to be a* first integral *of X_H.*

A specially important first integral of X_H, called the energy first integral*, is the Hamiltonian H.*

Proof. Let $\varphi : I \to M$ be an integral curve of X_H, defined on an open interval I of $\mathbb{R}$. For each $t \in I$

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = X_H\big(\varphi(t)\big)\,.$$

The function $g \circ \varphi$ is smooth and satisfies

$$\frac{\mathrm{d}(g\circ\varphi)(t)}{\mathrm{d}t} = \mathrm{i}(X_H)(\mathrm{d}g)\big(\varphi(t)\big) = \{H,g\}\big(\varphi(t)\big)\,.$$

Since I is connected, $g \circ \varphi$ keeps a constant value if and only if, for each $t \in I$, $\dfrac{\mathrm{d}(g\circ\varphi)(t)}{\mathrm{d}t} = 0$, and the above equality proves that such is the case if and only if $\{H,g\}\big(\varphi(t)\big) = 0$. The indicated result follows from the fact that for any point $x \in M$, there exists an integral curve $\varphi : I \to M$ of X_H and an element t in I such that $\varphi(t) = x$.

The skew-symmetry of the Poisson bracket implies $\{H,H\} = 0$, therefore the Hamiltonian H is a first integral of X_H. □

Remark 3.3.5. Some Hamiltonian mechanical systems encountered in Mechanics, defined on a Poisson manifold (M,Λ), have as Hamiltonian a smooth function H defined on $\mathbb{R}\times M$ rather than on the manifold M. Such a function H is said to be a *time-dependent Hamiltonian*. The associated Hamiltonian vector field X_H is no more an ordinary vector field on M, *i.e.* a smooth map $M \to TM$ wich associates to each $x \in M$ an element in T_xM, but rather a *time-dependent vector field*, *i.e.* a smooth map $X_H : \mathbb{R}\times M \to TM$ such that, for each $t \in \mathbb{R}$ and each $x \in M$ $X_H(t,x) \in T_xM$. For each fixed value of $t \in \mathbb{R}$, the map $x \mapsto X_H(t,x)$ is the Hamiltonian vector field on M whose Hamiltonian is the function $H_t : M \to \mathbb{R}$, defined by

$$H_t(x) = H(t,x)\,, \quad x \in M\,.$$

Therefore

$$X_H(t,x) = \Lambda^\sharp\big(\mathrm{d}H_t\big)(x)\,, \quad x \in M\,,\ t \in \mathbb{R}\,.$$

A smooth parametrized curve $\varphi : I \to M$, defined on an open interval I of $\mathbb{R}$, is an integral curve of the time-dependent vector field X_H if for each $t \in I$ it satisfies the non-autonomous differential equation

$$\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = X_H\big(t,\varphi(t)\big)\,.$$

The time-dependent Hamiltonian $H : \mathbb{R} \times M \to \mathbb{R}$ is no more a first integral of X_H since, for each integral curve $\varphi : I \to M$ of X_H and each $t \in I$,

$$\frac{\mathrm{d}(H \circ \varphi)(t)}{\mathrm{d}t} = \frac{\partial H\big(t, \varphi(t)\big)}{\partial t}.$$

Proposition 3.3.6. *Let (M_1, Λ_1) and (M_2, Λ_2) be two Poisson manifolds and let $\varphi : M_1 \to M_2$ be a smooth map. The following properties are equivalent.*

1. *For any pair (f, g) of smooth functions defined on M_2*

$$\{\varphi^* f, \varphi^* g\}_{M_1} = \varphi^* \{f, g\}_{M_2}.$$

2. *For any smooth function $f \in C^\infty(M_2, \mathbb{R})$ the Hamiltonian vector fields $\Lambda_2^\sharp(df)$ on M_2 and $\Lambda_1^\sharp\big(d(f \circ \varphi)\big)$ on M_1 are φ-compatible, which means that for each $x \in M_1$*

$$T_x\varphi\Big(\Lambda_1^\sharp\big(d(f \circ \varphi)(x)\big)\Big) = \Lambda_2^\sharp\Big(df\big(\varphi(x)\big)\Big).$$

3. *The bivector fields Λ_1 on M_1 and Λ_2 on M_2 are φ-compatible, which means that for each $x \in M_1$*

$$T_x\varphi\big(\Lambda_1(x)\big) = \Lambda_2\big(\varphi(x)\big).$$

A map $\varphi : M_1 \to M_2$ which satisfies these equivalent properties is called a Poisson map.

Proof. Let f and g be two smooth functions defined on M_2. For each $x \in M_1$, we have

$$\begin{aligned}\{\varphi^* f, \varphi^* g\}_{M_1}(x) &= \{f \circ \varphi, g \circ \varphi\}(x) = \Lambda_1(x)\big(\mathrm{d}(f \circ \varphi)(x), \mathrm{d}(g \circ \varphi)(x)\big) \\ &= \Big\langle \mathrm{d}(g \circ \varphi)(x), \Lambda_1^\sharp\big(\mathrm{d}(f \circ \varphi(x))\big)\Big\rangle.\end{aligned}$$

We have also

$$\begin{aligned}\varphi^*\{f, g\}_{M_2}(x) &= \{f, g\}_{M_2}\big(\varphi(x)\big) \\ &= \Big\langle \mathrm{d}g\big(\varphi(x)\big), \Lambda_2^*\big(\mathrm{d}f\big(\varphi(x)\big)\big)\Big\rangle.\end{aligned}$$

These formulae show that Properties 1 and 2 are equivalent.

We recall that $T_x\varphi\big(\Lambda_1(x)\big)$ is, by its very definition, the bivector at $\varphi(x) \in M_2$ such that, for any pair (f, g) of smooth functions on M_2

$$T_x\varphi\big(\Lambda_1(x)\big)\Big(\mathrm{d}f\big(\varphi(x)\big), \mathrm{d}g\big(\varphi(x)\big)\Big) = \Lambda_1\big(\mathrm{d}(f \circ \varphi)(x), \mathrm{d}(g \circ \varphi)(x)\big).$$

The above equalities therefore prove that Properties 2 and 3 are equivalent. □

Poisson manifolds often appear as quotients of symplectic manifolds, as indicated by the following Proposition, due to Paulette Libermann [43].

Proposition 3.3.7. *Let (M, ω) be a symplectic manifold and let $\varphi : M \to P$ be a surjective submersion of M onto a smooth manifold P whose fibres are connected (it means that for each $y \in P$, $\varphi^{-1}(y)$ is connected). The following properties are equivalent.*

1. *On the manifold M, the distribution* $\mathrm{orth}(\ker T\varphi)$ *is integrable.*

2. *For any pair (f,g) of smooth functions defined on P, the Poisson bracket $\{f\circ\varphi, g\circ\varphi\}$ is constant on each fibre $\varphi^{-1}(y)$ of the submersion φ (with $y\in P$).*

When these two equivalent properties are satisfied, there exists on P a unique Poisson structure for which $\varphi : M\to P$ is a Poisson map (the manifold M being endowed with the Poisson structure associated to its symplectic structure).

Proof. On the manifold M, $\ker T\varphi$ is a an integrable distribution of rank $\dim M - \dim P$ whose integral submanifolds are the fibres of the submersion φ. Its symplectic orthog(onal orth$(\ker T\varphi)$ is therefore a distribution of rank $\dim P$. Let f and g be two smooth functions defined on M_2. On M_1, the Hamiltonian vector fields $X_{f\circ\varphi}$ and $X_{g\circ\varphi}$ take their values in orth$(\ker T\varphi)$. We have

$$[X_{f\circ\varphi}, X_{g\circ\varphi}] = X_{\{f\circ\varphi, g\circ\varphi\}}\,.$$

Therefore $[X_{f\circ\varphi}, X_{g\circ\varphi}]$ takes its values in orth$(\ker T\varphi)$ if and only if the function $\{f\circ\varphi, g\circ\varphi\}$ is constant on each fibre $\varphi^{-1}(y)$ of the submersion φ. The equivalence of Properties 1 and 2 easily follows.

Let us now assume that the equivalent properties 1 and 2 are satisfied. Since the map $\varphi : M\to P$ is a submersion with connected fibres, the map which associates to each function $f\in C^\infty(M_2,\mathbb{R})$ the function $f\circ\varphi$ is an isomorphism of $C^\infty(M_2,\mathbb{R})$ onto the subspace of $C^\infty(M_1,\mathbb{R})$ made by smooth functions which are constant on each fibre of φ. The existence and unicity of a Poisson structure on M_2 for which φ is a Poisson map follows. □

Remark 3.3.8. Poisson manifolds obtained as quotients of symplectic manifolds often come by pairs. Let us assume indeed that (M,ω) is a symplectic manifold and that the above Proposition can be applied to a smooth surjective submersion with connected fibres $\varphi : M\to P$, and defines a Poisson structure on P for which φ is a Poisson map. Since orth$(\ker T\varphi)$ is integrable, it defines a foliation of M, which is said to be *simple* when the set of leaves Q of that foliation has a smooth manifold structure such that the map $\psi : M\to Q$, which associates to each point in M the leaf through this point, is a submersion. Then the maps $\varphi : M\to P$ and $\psi : M\to Q$ play similar parts, so there exists on Q a unique Poisson structure for which ψ is a Poisson map. Alan Weinstein [66] has determined the links which exist between the local structures of the two Poisson manifolds P and Q at corresponding points (that means, at points which are the images of the same point in M by the maps φ and ψ).

Several kinds of remarkable submanifolds of a Poisson manifold can be defined [66]. The most important are the *coisotropic* submanifolds, defined below.

Definition 3.3.9. A submanifold N of a Poisson manifold (M,Λ) is said to be *coisotropic* if for any point $x\in N$ and any pair (f,g) of smooth functions defined on a neighbourhood U of x in M whose restrictions to $U\cap N$ are constants, the Poisson bracket $\{f,g\}$ vanishes on $U\cap N$.

3.4 Examples of Poisson manifolds

3.4.1 Symplectic manifolds

We have seen above that any symplectic manifold is a Poisson manifold.

3.4.2 Dual spaces of finite-dimensional Lie algebras

Let $\mathcal{G}$ be a finite-dimensional Lie algebra, and $\mathcal{G}^*$ its dual space. The Lie algebra $\mathcal{G}$ can be considered as the dual of $\mathcal{G}^*$, that means as the space of linear functions on $\mathcal{G}^*$, and the bracket of the Lie algebra $\mathcal{G}$ is a composition law on this space of linear functions. This composition law can be extended to the space $C^\infty(\mathcal{G}^*,\mathbb{R})$ by setting

$$\{f,g\}(x) = \Big\langle x, \big[\mathrm{d}f(x), \mathrm{d}g(x)\big]\Big\rangle, \quad f \text{ and } g \in C^\infty(\mathcal{G}^*,\mathbb{R}), \quad x \in \mathcal{G}^*.$$

This bracket on $C^\infty(\mathcal{G}^*,\mathbb{R})$ defines a Poisson structure on $\mathcal{G}^*$, called its *canonical Poisson structure*. It implicitly appears in the works of Sophus Lie, and was rediscovered by Alexander Kirillov [34], Bertram Kostant [39] and Jean-Marie Souriau [60]. Its existence can be seen as an application of Proposition 3.3.7. Let indeed G be the connected and simply connected Lie group whose Lie algebra is $\mathcal{G}$. We know that the cotangent bundle T^*G has a canonical symplectic structure. One can check easily that for this symplectic structure, the Poisson bracket of two smooth functions defined on T^*G and invariant with respect to the lift to T^*G of the action of G on itself by left translations, is too invariant with respect to that action. Application of Proposition 3.3.7, the submersion $\varphi : T^*G \to \mathcal{G}^*$ being the left translation which, for each $g \in G$, maps T^*_gG onto $T^*_eG \equiv \mathcal{G}^*$, yields the above defined Poisson structure on $\mathcal{G}^*$. If instead of translations on the left, we use translation on the right, we obtain on $\mathcal{G}^*$ the opposite Poisson structure. This illustrates Remark 3.3.8, since, as we will see later, each one of the tangent spaces at a point $\xi \in T^*G$ to the orbits of that point by the lifts to T^*G of the actions of G on itself by translations on the left and on the right, is the symplectic orthogonal of the other.

The symplectic leaves of $\mathcal{G}^*$ equipped with the above defined Poisson structure are the coadjoint orbits.

3.4.3 Symplectic cocycles

A *symplectic cocycle* of the Lie algebra $\mathcal{G}$ is a skew-symmetric bilinear map $\widetilde{\Theta} : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ which satisfies

$$\widetilde{\Theta}\big([X,Y],Z\big) + \widetilde{\Theta}\big([Y,Z],X\big) + \widetilde{\Theta}\big([Z,X],Y\big) = 0.$$

The above defined canonical Poisson structure on $\mathcal{G}^*$ can be modified by means of a symplectic cocycle $\widetilde{\Theta}$ by defining the new bracket (see for example [44])

$$\{f,g\}_{\widetilde{\Theta}}(x) = \Big\langle x, \big[\mathrm{d}f(x), \mathrm{d}g(x)\big]\Big\rangle - \widetilde{\Theta}\big(\mathrm{d}f(x), \mathrm{d}g(x)\big),$$

where f and $g \in C^\infty(\mathcal{G}^*,\mathbb{R})$, $x \in \mathcal{G}^*$. This Poisson structure is called the *modified canonical Poisson structure* by means of the symplectic cocycle $\widetilde{\Theta}$. We will see in Section 4.5 that the symplectic leaves of $\mathcal{G}^*$ equipped with this Poisson structure are the orbits of an affine action whose linear part is the coadjoint action, with an additional term determined by $\widetilde{\Theta}$.

4 Symplectic, Poisson and Hamiltonian actions

4.1 Actions on a smooth manifold

Let us first recall some definitions and facts about actions of a Lie algebra or of a Lie group on a smooth manifold.

Definition 4.1.1. An *action on the left* (resp. an *action on the right*) of a Lie group G on a smooth manifold M is a smooth map $\Phi : G \times M \to M$ (respectively, $\Psi : M \times G \to M$) such that, for any $x \in M$, g_1 and $g_2 \in G$, $e \in G$ being the neutral element,

- for an action on the left

$$\Phi\big(g_1, \Phi(g_2, x)\big) = \Phi(g_1 g_2, x)\,, \quad \Phi(e, x) = x\,,$$

- for an action on the right

$$\Psi\big(\Psi(x, g_1), g_2\big) = \Psi(x, g_1 g_2)\,, \quad \Psi(x, e) = x\,.$$

4.1.2 Consequences

Let $\Phi : G \times M \to M$ be an action on the left of the Lie group G on the smooth manifold M. For each $g \in G$, we denote by $\Phi_g : M \to M$ the map

$$\Phi_g(x) = \Phi(g, x)\,.$$

The map $g \mapsto \Phi_g$ is a groups homomorphism of G into the group of smooth diffeomorphisms of M. In other words, for each $g \in G$, Φ_g is a diffeomorphism of M, and we have

$$\Phi_g \circ \Phi_h = \Phi_{gh}\,, \quad (\Phi_g)^{-1} = \Phi_{g^{-1}}\,, \quad g \text{ and } h \in G\,.$$

Similarly, let $\Psi : M \times G \to M$ be an action on the right of the Lie group G on the smooth manifold M. For each $g \in G$, we denote by $\Psi_g : M \to M$ the map

$$\Psi_g(x) = \Psi(x, g)\,.$$

The map $g \mapsto \Psi_g$ is a groups anti-homomorphism of G into the group of smooth diffeomorphisms of M. In other words, for each $g \in G$, Ψ_g is a diffeomorphism of M, and we have

$$\Psi_g \circ \Psi_h = \Psi_{hg}\,, \quad (\Psi_g)^{-1} = \Psi_{g^{-1}}\,, \quad g \text{ and } h \in G\,.$$

Definition 4.1.3. Let $\Phi : G \times M \to M$ be an action on the left (resp. let $\Psi : M \times G \to M$ be an action of the right) of the Lie group G on the smooth manifold M. With each element $X \in \mathcal{G} \equiv T_e G$ (the tangent space to the Lie group G at the neutral element) we associate the vector field X_M on M defined by

$$X_M(x) = \begin{cases} \dfrac{\mathrm{d}\Phi\big(\exp(sX), x\big)}{ds}\Big|_{s=0} & \text{if } \Phi \text{ is an action on the left,} \\ \dfrac{\mathrm{d}\Psi\big(x, \exp(sX)\big)}{ds}\Big|_{s=0} & \text{if } \Psi \text{ is an action on the right.} \end{cases}$$

The vector field X_M is called the *fundamental vector field* on M associated to X.

Definition 4.1.4. An *action* of a Lie algebra $\mathcal{G}$ on a smooth manifold M is a Lie algebras homomorphism φ of $\mathcal{G}$ into the Lie algebra $A^1(M)$ of smooth vector fields on M (with the Lie bracket of vector fields as composition law). In other words, it is a linear map $\varphi : \mathcal{G} \to A^1(M)$ such that for each pair $(X, Y) \in \mathcal{G} \times \mathcal{G}$,

$$\varphi\big([X, Y]\big) = \big[\varphi(X), \varphi(Y)\big]\,.$$

Remark 4.1.5. Let G be a Lie group. There are two natural ways in which the tangent space $T_eG \equiv \mathcal{G}$ to the Lie group G at the neutral element e can be endowed with a Lie algebra structure.

In the first way, we associate with each element $X \in T_eG$ the *left invariant* vector field X^L on G such that $X^L(e) = X$; its value at a point $g \in G$ is $X^L(g) = TL_g(X)$, where $L_g : G \to G$ is the map $h \mapsto L_g(h) = gh$. We observe that for any pair (X,Y) of elements in $\mathcal{G}$ the Lie bracket $[X^L, Y^L]$ of the vector fields X^L and Y^L on G is left invariant, and we define the bracket $[X,Y]$ by setting $[X,Y] = [X^L, Y^L](e)$. This Lie algebra structure on $\mathcal{G} \equiv T_eG$ will be called the Lie algebra structure of *left invariant vector fields* on G.

In the second way, we choose the *right invariant* vector fields on G X^R and Y^R, instead of the left invariant vector fields X^L and Y^L. Since $[X^R, Y^R](e) = -[X^L, Y^L](e)$, the Lie algebra structure on $\mathcal{G} \equiv T_eG$ obtained in this way, called the Lie algebra structure of *right invariant vector fields*, is the *opposite* of that of left invariant vector fields. We have therefore on T_eG two opposite Lie algebras structures, both equally natural. Fortunately, the choice of one rather than the other as the Lie algebra $\mathcal{G}$ of G does not matter because the map $X \mapsto -X$ is a Lie algebras isomorphism between these two structures.

Proposition 4.1.6. *Let $\Phi : G \times M \to M$ be an action on the left (resp. let $\Psi : M \times G \to M$ be an action on the right) of a Lie group G on a smooth manifold M. We endow $\mathcal{G} \equiv T_eG$ with the Lie algebra structure of right invariant vector fields on G (resp, with the Lie algebra structure of left invariant vector fields on G). The map $\varphi : \mathcal{G} \to A^1(M)$ (resp. $\psi : \mathcal{G} \to A^1(M)$) which associates to each element X of the Lie algebra $\mathcal{G}$ of G the corresponding fundamental vector field X_M, is an action of the Lie algebra $\mathcal{G}$ on the manifold M. This Lie algebra action is said to be* associated *to the Lie group action Φ (resp. Ψ).*

Proof. Let us look at an action on the left Φ. Let $x \in M$, and let $\Phi^x : G \to M$ be the map $g \mapsto \Phi^x(g) = \Phi(g,x)$.For any $X \in T_eG$ and $g \in G$, we have

$$\begin{aligned} X_M\big(\Phi(g,x)\big) &= \frac{\mathrm{d}}{\mathrm{d}s}\Phi\big(\exp(sX), \Phi(g,x)\big) \Big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\Phi\big(\exp(sX)g, x\big) \Big|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Phi\Big(R_g\big(\exp(sX)\big), x\Big) \Big|_{s=0} = T\Phi^x \circ TR_g(X)\,. \end{aligned}$$

We see that for each $X \in T_xG$, the right invariant vector field X^R on G and the fundamental vector field X_M on M are compatible with respect to the map $\Phi^x : G \to M$. Therefore for any pair (X,Y) of elements in T_eG, we have $[X,Y]_M = [X_M, Y_M]$. In other words the map $X \mapsto X_M$ is an action of the Lie algebra $\mathcal{G} = T_eG$ (equipped with the Lie algebra structure of right invariant vector fields on G) on the manifold M.

For an action on the right Ψ, the proof is similar, $\mathcal{G} = T_eG$ being this time endowed with the Lie algebra structure of left invariant vector fields on G. □

Proposition 4.1.7. *Let $\Phi : G \times M \to M$ be an action on the left (resp. let $\Psi : M \times G \to M$ be an action on the right) of a Lie group G on a smooth manifold M. Let X_M be the fundamental vector field associated to an element $X \in \mathcal{G}$. For any $g \in G$, the direct image $(\Phi_g)_*(X_M)$ (resp. $(\Psi_g)_*(X_M)$) of the vector field X_M by the diffeomorphism $\Phi_g : M \to M$ (resp. $\Psi_g : M \to M$) is the fundamnetal vector field $(\mathrm{Ad}_g X)_M$ associated to $\mathrm{Ad}_g X$ (resp. the fundamental vector field $(\mathrm{Ad}_{g^{-1}} X)_M$ associated to $\mathrm{Ad}_{g^{-1}} X$).*

Proof. For each $x \in M$

$$\begin{aligned}(\Phi_g)_*(X_M)(x) &= T\Phi_g\Big(X_M\big(\Phi(g^{-1},x)\big)\Big)\\ &= T\Phi_g\left(\frac{\mathrm{d}}{\mathrm{d}s}\Phi\big(\exp(sX)g^{-1},x\big)\,\Big|_{s=0}\right)\\ &= \frac{\mathrm{d}}{ds}\Big(\Phi\big(g\exp(sX)g^{-1},x\big)\Big)\,\Big|_{s=0} = (\mathrm{Ad}_g X)_M(x)\,,\end{aligned}$$

since $g\exp(sX)g^{-1} = \exp(\mathrm{Ad}_g X)$. The proof for the action on the right Ψ is similar. □

4.2 Linear and affine representations

In this section, after recalling some results about linear and affine transformation groups, we discuss linear and affine representations of a Lie group or of a Lie algebra in a finite-dimensional vector space, which can be seen as special examples of actions.

4.2.1 Linear and affine transformation groups and their Lie algebras

Let E be a finite-dimensional vector space. The set of linear isomorphisms $l : E \to E$ will be denoted by $\mathrm{GL}(E)$. We recall that equipped with the composition of maps

$$(l_1, l_2) \mapsto l_1 \circ l_2$$

as a composition law, $\mathrm{GL}(E)$ is a Lie group whose dimension is $(\dim E)^2$. Its Lie algebra, which will be denoted by $\mathfrak{gl}(E)$, is the set $\mathcal{L}(E,E)$ of linear maps $f : E \to E$, with the commutator

$$(f_1, f_2) \mapsto [f_1, f_2] = f_1 \circ f_2 - f_2 \circ f_1$$

as a composition law.

A map $a : E \to E$ is called an *affine map* if it can be written as

$$a(x) = l(x) + c\,, \quad x \in E\,,$$

the map $l : E \to E$ being linear, and $c \in E$ being a constant. The affine map a is invertible if and only if its linear part l is invertible, in other words if and only if $l \in \mathrm{GL}(E)$; when this condition is satisfied, its inverse is

$$a^{-1}(y) = l^{-1}(y - c)\,, \quad y \in E\,.$$

By identifying the invertible affine map a with the pair (l,c), with $l \in \mathrm{GL}(E)$ and $c \in E$, the set of invertible affine maps of E onto itself becomes identified with $\mathrm{GL}(E) \times E$. The composition law and the inverse map on this product (which is called the *semi-direct product of* $\mathrm{GL}(E)$ *with* E) are

$$(l_1, c_1), (l_2, c_2) \mapsto \big(\big(l_1 \circ l_2, l_1(c_2) + c_1\big)\big)\,, \quad (l,c)^{-1} = \big(l^{-1}, -l^{-1}(c)\big)\,.$$

The semi-direct product $\mathrm{Aff}(E) = \mathrm{GL}(E) \times E$ is a Lie group whose dimension is $(\dim E)^2 + \dim E$; its Lie algebra is the product $\mathfrak{aff}(E) = \mathcal{L}(E,E) \times E$, with the composition law

$$\big((f_1, d_1), (f_2, d_2)\big) \mapsto \big[(f_1, d_1), (f_2, d_2)\big] = \big(f_1 \circ f_2 - f_2 \circ f_1, f_1(d_2) - f_2(d_1)\big)\,.$$

The adjoint representation is given by the formula

$$\mathrm{Ad}_{(l,c)}\big((f,d)\big) = \big(l \circ f \circ l^{-1}, l(d) - l \circ f \circ l^{-1}(d)\big)\,.$$

Remark 4.2.2. The finite-dimensional vector space E can be considered as a smooth manifold on which E itself transitively acts by translations. That action determines a natural trivialization of the tangent bundle TE, the tangent space T_xE at each point $x \in E$ being identified with E. An element $a \in \mathfrak{aff}(E)$, in other words an affine map $a : E \to E$, can therefore be considered as the vector field on E whose value, at each $x \in E$, is $a(x) \in T_xE \equiv E$. A question naturally arises: how the bracket of two elements a_1 and $a_2 \in \mathfrak{aff}(E)$, for the Lie algebra structure of $\mathfrak{aff}(E)$ defined in 4.2.1, compares with the bracket of these two elements when considered as vector fields on E? An easy calculation in local coordinates shows that the bracket $[a_1, a_2]$ defined in 4.2.1 is the *opposite* of the bracket of these two elements when considered as vector fields on E. Remark 4.2.8 below will explain the reason of that change of sign.

Definitions 4.2.3. Let G be a Lie group, $\mathcal{G}$ a Lie algebra and E a finite-dimensional vector space.

1. A *linear representation* (respectively, an *affine representation*) of the Lie group G in the vector space E is a Lie groups homomorphism $R : G \to \mathrm{GL}(E)$ of G in the Lie group $\mathrm{GL}(E)$ of linear transformations of E (respectively, a Lie groups homomorphism $A : G \to \mathrm{Aff}(E)$ of G in the Lie group $\mathrm{Aff}(E)$ of affine transformations of E).

2. A *linear representation* (respectively, an *affine representation*) of the Lie algebra $\mathcal{G}$ in the vector space E is a Lie algebras homomorphism $r : \mathcal{G} \to \mathfrak{gl}(E)$ of the Lie algebra $\mathcal{G}$ in the Lie algebra $\mathfrak{gl}(E)$ of the group of linear transformations of E (resp, a Lie algebras homomorphism $a : \mathcal{G} \to \mathfrak{aff}(E)$ of the Lie algebra $\mathcal{G}$ in the Lie algebra $\mathfrak{aff}(E)$ of the group of affine transformations of E).

Examples 4.2.4. Let G be a Lie group. The *adjoint representation* of G is the linear representation of G in its Lie algebra $\mathcal{G}$ which associates, to each $g \in G$, the linear isomorphism $\mathrm{Ad}_g \in \mathrm{GL}(\mathcal{G})$

$$\mathrm{Ad}_g(X) = TL_g \circ TR_{g^{-1}}(X), \quad (X \in \mathcal{G}).$$

The *coadjoint representation* of G is the contragredient of the adjoint representation. It associates to each $g \in G$ the linear isomorphism $\mathrm{Ad}^*_{g^{-1}} \in \mathrm{GL}(\mathcal{G}^*)$, which satisfies, for each $\zeta \in \mathcal{G}^*$ and $X \in \mathcal{G}$,

$$\left\langle \mathrm{Ad}^*_{g^{-1}}(\zeta), X \right\rangle = \left\langle \zeta, \mathrm{Ad}_{g^{-1}}(X) \right\rangle.$$

The *adjoint representation* of the Lie algebra $\mathcal{G}$ is the linear representation of $\mathcal{G}$ into itself which associates, to each $X \in \mathcal{G}$, the linear map $\mathrm{ad}_X \in \mathfrak{gl}(\mathcal{G})$

$$\mathrm{ad}_X(Y) = [X, Y], \quad (Y \in \mathcal{G}).$$

The *coadjoint representation* of the Lie algebra $\mathcal{G}$ is the contragredient of the adjoint representation. It associates, to each $X \in \mathcal{G}$, the linear map $\mathrm{ad}^*_{-X} \in \mathfrak{gl}(\mathcal{G}^*)$ which satisfies, for each $\zeta \in \mathcal{G}^*$ and $X \in \mathcal{G}$,

$$\left\langle \mathrm{ad}^*_{-X}\, \zeta, Y \right\rangle = \left\langle \zeta, [-X, Y] \right\rangle.$$

The adjoint representation (respectively, the coadjoint representation) of $\mathcal{G}$ is the Lie algebra representation associated to the adjoint representation (respectively, the coadjoint representation) of the Lie group G, in the sense recalled below in the proof of 4.2.7.

Proposition 4.2.5. *Let G be a Lie group and E a finite-dimensional vector space. A map $A : G \to \mathrm{Aff}(E)$ always can be written as*

$$A(g)(x) = R(g)(x) + \theta(g), \quad \text{with } g \in G,\ x \in E,$$

where the maps $R : G \to \mathrm{GL}(E)$ and $\theta : G \to E$ are determined by A. The map A is an affine representation of G in E if and only if the following two properties are satisfied:

- *$R : G \to \mathrm{GL}(E)$ is a linear representation of G in the vector space E,*
- *the map $\theta : G \to E$ is a one-cocycle of G with values in E, for the linear representation R; it means that θ is a smooth map which satisfies, for all g and $h \in G$,*

$$\theta(gh) = R(g)\big(\theta(h)\big) + \theta(g)\,.$$

When these two properties are satisfied, the linear representation R is called the linear part *of the affine representation A, and θ is called the* one-cocycle *of G associated to the affine representation A.*

Proof. Since $\mathrm{Aff}(E) = \mathrm{GL}(E) \times E$, for each $g \in G$ and $x \in E$, we have

$$A(g)(x) = R(g)(x) + \theta(g)\,,$$

where the maps $R : G \to \mathrm{GL}(E)$ and $\theta : G \to E$ are determined by A. By comparing $A(gh)$ and $A(g) \circ A(h)$, for g and $h \in G$, using the composition law of $\mathrm{Aff}(E)$ recalled in subsection 4.2.1, we easily check that A is an affine representation, which means that it is smooth and satisfies, for all g and $h \in G$, $A(gh) = A(g) \circ A(h)$, and $A(e) = \mathrm{id}_E$, if and only if the two above stated properties are satisfied. □

For linear and affine representations of a Lie algebra, we have the following infinitesimal analogue of Proposition 4.2.5.

Proposition 4.2.6. *Let $\mathcal{G}$ be a Lie algebra and E a finite-dimensional vector space. A linear map $a : \mathcal{G} \to \mathfrak{aff}(E)$ always can be written as*

$$a(X)(x) = r(X)(x) + \Theta(X)\,, \quad \text{with } X \in \mathcal{G}\,,\ x \in E\,,$$

where the linear maps $r : \mathcal{G} \to \mathfrak{gl}(E)$ and $\Theta : \mathcal{G} \to E$ are determined by a. The map a is an affine representation of G in E if and only if the following two properties are satisfied:

- *$r : \mathcal{G} \to \mathfrak{gl}(E)$ is a linear representation of the Lie algebra $\mathcal{G}$ in the vector space E,*
- *the linear map $\Theta : \mathcal{G} \to E$ is a one-cocycle of $\mathcal{G}$ with values in E, for the linear representation r; it means that Θ satisfies, for all X and $Y \in \mathcal{G}$,*

$$\Theta\big([X,Y]\big) = r(X)\big(\Theta(Y)\big) - r(Y)\big(\Theta(X)\big)\,.$$

When these two properties are satisfied, the linear representation r is called the linear part *of the affine representation a, and Θ is called the* one-cocycle *of $\mathcal{G}$ associated to the affine representation a.*

Proof. Since $\mathfrak{aff}(E) = \mathfrak{gl}(E) \times E = \mathcal{L}(E,E) \times E$, for each $X \in \mathcal{G}$ and $x \in E$, we have

$$a(X)(x) = r(X)(x) + \Theta(X)\,,$$

where the linear maps $r : \mathcal{G} \to \mathfrak{gl}(E) = \mathcal{L}(E,E)$ and $\Theta : \mathcal{G} \to E$ are determined by a. By comparing $a\big([X,Y]\big)$ and $\big[a(X),a(Y)\big]$, for X and $Y \in \mathcal{G}$, using the expression of the bracket of $\mathfrak{aff}(E)$ recalled in subsection 4.2.1, we easily check that A is an affine representation, which means that it is smooth and satisfies, for all X and $Y \in \mathcal{G}$, $a\big([X,Y]\big) = \big[a(X),a(Y)\big]$ if and only if the two above stated properties are satisfied. □

Proposition 4.2.7. *Let $A : G \to \mathrm{Aff}(E)$ be an affine representation of a Lie group G in a finite-dimensional vector space E, and $\mathcal{G}$ be the Lie algebra of G. Let $R : G \to \mathrm{GL}(E)$ and $\theta : G \to E$ be, respectively, the linear part and the associated cocycle of the affine representation A. Let $a : \mathcal{G} \to \mathfrak{aff}(E)$ be the affine representation of the Lie algebra $\mathcal{G}$ associated (in the sense recalled below in the proof) to the affine representation $A : G \to \mathrm{Aff}(E)$ of the Lie group G. The linear part of a is the linear representation $r : \mathcal{G} \to \mathfrak{gl}(E)$ associated to the linear representation $R : G \to \mathrm{GL}(E)$, and the associated cocycle $\Theta : \mathcal{G} \to E$ is related to the one-cocycle $\theta : G \to E$ by*

$$\Theta(X) = T_e\theta\big(X(e)\big)\,, \quad (X \in \mathcal{G})\,.$$

Proof. We recall that when we have a Lie groups homomorphism $A : G \to H$ of a Lie group G into another Lie group H, the associated Lie algebras homomorphism $a : \mathcal{G} \to \mathcal{H}$ of Lie algebras associates, to each $X \in \mathcal{G}$ (seen as the space of left-invariant vector fields on G) the left-invariant vector field $a(X)$ on H whose value at the neutral element is $T_eA\big(X(e)\big)$. Let $X \in \mathcal{G}$. For each $t \in \mathbb{R}$ and $x \in E$, we have

$$A\big(\exp(tX)\big)(x) = R\big(\exp(tX)\big)(x) + \theta\big(\exp(tX)\big)\,.$$

By taking the derivative of both sides of this equality with respect to t, then setting $t = 0$, we get

$$a(X)(x) = r(X)(x) + T_e\theta(X)\,.$$

Therefore the affine representation a has r as linear part and $\Theta = T_e\theta$ as associated one-cocycle. □

Remark 4.2.8. Let $A : G \to \mathrm{Aff}(E)$ be an affine representation of a Lie group G in a finite-dimensional vector space E. The map $\widetilde{A} : G \times E \to E$,

$$\widetilde{A}(g,x) = A(g)(x)\,, \quad g \in G\,,\ x \in E\,,$$

is an action on the left of G on E. Proposition 4.2.7 shows that $a : \mathcal{G} \to \mathfrak{aff}(E)$ is a Lie algebras homomorphism, the Lie algebra structure of $\mathfrak{aff}(E)$ being the structure defined in 4.2.1. For each $X \in \mathcal{G}$, the element $a(X) \in \mathfrak{aff}(E)$, when considered as an affine vector field on E, is the fundamental vector field associated to X, for the action on the left $\widetilde{A}$ of G on E. We have seen (4.1.6) that for an action on the left of G on E, the map which associates to each $X \in \mathcal{G}$ the corresponding fundamental vector field on E is a Lie algebras homomorphism of the Lie algebra of *right invariant* vector fields on G into the Lie algebra of smooth vector fields on E. This explains why, as was observed in 4.2.2, the Lie algebra structure of $\mathfrak{aff}(E)$ defined in 4.2.1 is the opposite of the Lie algebra structure which exists on the space of affine vector fields on E. Of course, this remark is also valid for a linear representation $R : G \to \mathrm{GL}(E)$, since $\mathrm{GL}(E)$ is a Lie subgroup of $\mathrm{Aff}(E)$.

Definitions 4.2.9.

1. Let $R : G \to \mathrm{GL}(E)$ be a linear representation of a Lie group G in a finite-dimensional vector space E. A *one-coboundary* of G with values in E, for the linear representation R, is a map $\theta : G \to E$ which can be expressed as

$$\theta(g) = R(g)(c) - c\,, \quad (g \in G)\,,$$

where c is a fixed element in E.

2. Ler $r : \mathcal{G} \to \mathfrak{gl}(E)$ be a linear representation of a Lie algebra $\mathcal{G}$ in a finite-dimensional vector space E. A *one-coboundary* of $\mathcal{G}$ with values in E, for the linear representation r, is a linear map $\Theta : \mathcal{G} \to E$ which can be expressed as

$$\Theta(X) = r(X)(c)\,, \quad (X \in \mathcal{G})\,,$$

where c is a fixed element in E.

Remark 4.2.10. The reader will easily check the following properties. A one-coboundary of a Lie group G with values in a finite-dimensional vector space E for a linear representation $R : G \to \mathrm{GL}(E)$, automatically is a one-cocycle in the sense of 4.2.5. Similarly, a one-coboundary of a Lie algebra $\mathcal{G}$ with values in E for the linear representation $r : \mathcal{G} \to \mathfrak{gl}(E)$, automatically is a one-cocycle of $\mathcal{G}$ in the sense of 4.2.6. When a Lie group one-cocycle $\theta : G \to E$ is in fact a one-coboundary, the associated Lie algebra one-cocycle $\Theta = T_e\theta$ is a Lie algebra one-coboundary.

Proposition 4.2.11. *Let $A : G \to \mathrm{Aff}(E)$ be an affine representation of a Lie group G in a finite-dimensional vector space E, $R : G \to \mathrm{GL}(E)$ be its linear part and $\theta : G \to E$ be the associated Lie group one-cocycle. The following properties are equivalent.*

1. *There exists an element $c \in E$ such that, for all $g \in G$ and $x \in E$,*
$$A(g)(x) = R(g)(x+c) - c\,.$$
2. *The one-cocycle $\theta : G \to E$ is in fact a 1-coboudary, whose expression is*
$$\theta(g) = R(g)(c) - c\,.$$

Proof. Since for each $g \in G$ $R(g)$ is linear, Property 1 can be written
$$A(g)(x) = R(g)(x) + \big(R(g)(c) - c\big)\,.$$
Therefore Property 1 is true if and only if $\theta(g) = R(g)(c) - c$, in other words if and only if Property 2 is true. □

The following Proposition is the infinitesimal analogue, for affine representations of a Lie algebra, of Proposition 4.2.11.

Proposition 4.2.12. *Let $a : \mathcal{G} \to \mathfrak{aff}(E)$ be an affine representation of a Lie algebra $\mathcal{G}$ in a finite-dimensional vector space E, $r : \mathcal{G} \to \mathfrak{gl}(E)$ be its linear part and $\Theta : \mathcal{G} \to E$ be the associated Lie algebra one-cocycle. The following properties are equivalent.*

1. *There exists an element $c \in E$ such that, for all $X \in \mathcal{G}$ and $x \in E$,*
$$a(X)(x) = r(X)(x+c)\,.$$
2. *The one-cocycle $\Theta : \mathcal{G} \to E$ is in fact a 1-coboudary, whose expression is*
$$\Theta(X) = r(X)(c)\,.$$

Proof. Since for each $X \in \mathcal{G}$ $r(X)$ is linear, Property (i) can be written
$$a(X)(x) = r(X)(x) + r(X)(c)\,.$$
Therefore Property 1 is true if and only if $\Theta(X) = r(X)(c)$, in other words if and only if Property 2 is true. □

Remark 4.2.13. Let us say that an affine representation $A : G \to \mathrm{Aff}(E)$ of a Lie group G in a finite-dimensional vector space E is equivalent to its linear part $R : G \to \mathrm{GL}(E)$ if there exists a translation $T : E \to E$ such that, for all $g \in G$ and $x \in E$,
$$A(g)(x) = T^{-1} \circ R(g) \circ T(x)\,.$$
Proposition 4.2.11 expresses the fact that the affine representation A is equivalent to its linear part R if and only if its associated Lie group cocycle θ is a one-coboundary. The reader will easily formulate a similar interpretation of Proposition 4.2.12.

Proposition 4.2.14. *Let G be a connected and simply connected Lie group, $R : G \to \mathrm{GL}(E)$ be a linear representation of G in a finite-dimensional vector space E, and $r : \mathcal{G} \to \mathfrak{gl}(E)$ be the associated linear representation of its Lie algebra $\mathcal{G}$. For any one-cocycle $\Theta : \mathcal{G} \to E$ of the Lie algebra $\mathcal{G}$ for the linear representation r, there exists a unique one-cocycle $\theta : G \to E$ of the Lie group G for the linear representation R such that $\Theta = T_e\theta$, in other words, which has Θ as associated Lie algebra one-cocycle. The Lie group one-cocycle θ is a Lie group one-coboundary if and only if the Lie algrebra one-cocycle Θ is a Lie algebra one-coboundary.*

Proof. If $\theta : G \to E$ is a Lie group one-cocycle such that $T_e\theta = \Theta$ we have, for any $g \in G$ and $X \in \mathcal{G}$,

$$\theta\big(g\exp(tX)\big) = \theta(g) + R(g)\Big(\theta\big(\exp(tX)\big)\Big)\,.$$

By taking the derivative of both sides of this equality with respect to t, then setting $t = 0$, we see that

$$T_g\theta\big(TL_g(X)\big) = R(g)\big(\Theta(x)\big)\,,$$

which proves that if it exists, the Lie group one-cocycle θ such that $T_e\theta = \Theta$ is unique.

For each $g \in G$ let $\eta(g) : T_gG \to E$ be the map

$$\eta(g)(X) = R(g) \circ \Theta \circ TL_{g^{-1}}(X)\,, \quad X \in T_gG\,.$$

The map η is an E-valued differential one-form on G. Let us calculate its exterior differential $\mathrm{d}\eta$, which is an E-valued differential two-form on G (if the reader does not feel at ease with E-valued differential forms on G, he can consider separately the components of η in a basis of E, which are ordinary real-valued one-forms). Let X and Y be two left-invariant vector fields on G. We have, for each $g \in G$,

$$\mathrm{d}\eta(g)\big(X(g),Y(g)\big) = \mathcal{L}(X)\big(\langle\eta,Y\rangle(g)\big) - \mathcal{L}(Y)\big(\langle\eta,X\rangle(g)\big) - \big\langle\eta,[X,Y]\big\rangle(g)\,.$$

But

$$\langle\eta,Y\rangle(g) = R(g)\circ\Theta(Y)\,, \quad \langle\eta,X\rangle(g) = R(g)\circ\Theta(X)\,,$$

therefore

$$\mathcal{L}(X)\big(\langle\eta,Y\rangle(g)\big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(R\big(g\exp(tX)\big)\circ\Theta(Y)\Big)\,\Big|_{t=0} = R(g)\circ r(X)\circ\Theta(Y)\,.$$

Similarly

$$\mathcal{L}(Y)\big(\langle\eta,X\rangle(g)\big) = R(g)\circ r(Y)\circ\Theta(X)\,,$$

and

$$\big\langle\eta,[X,Y]\big\rangle(g) = R(g)\circ\Theta\big([X,Y]\big)\,,$$

Since the condition which expresses that Θ is a Lie algebra one-cocycle for the linear representation r asserts that

$$r(X)\circ\Theta(Y) - r(Y)\circ\Theta(X) - \Theta\big([X,Y]\big) = 0\,,$$

we conclude that the one-form η is closed, *i.e.* satisfies $d\eta = 0$. Since G is assumed to be simply connected, the one-form η is exact, and since G is assumed to be connected, for any g in G, there exists a smooth parametrized curve $\gamma : [0,T] \to G$ such that $\gamma(0) = e$ and $\gamma(T) = g$. Let us set

$$\theta(g) = \int_0^T \eta\left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right) dt\,.$$

Since η is exact, the right hand side of the above equality only depends on the end points $\gamma(0)=e$ and $\gamma(T)=g$ of the parametrized curve γ, which allows us to define $\theta(g)$ by that equality. So defined, $\gamma: G\to E$ is a smooth map. Its very definition shows that $T_e\theta=\Theta$. If g and h are two elements in G, let $\gamma:[0,T_2]\to G$ be a smooth parametrized curve such that $0<T_1<T_2$, $\gamma(0)=e$, $\gamma(T_1)=g$ and $\gamma(t_2)=gh$. We have

$$\theta(gh)=\int_0^{T_2}\eta\left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)dt=\int_0^{T_1}\eta\left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)dt+\int_{T_1}^{T_2}\eta\left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)dt\,.$$

Observe that

$$\int_0^{T_2}\eta\left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)dt=\theta(g)$$

and that

$$\int_{T_1}^{T_2}\eta\left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)dt=R(g)\circ\int_{T_1}^{T_2}\eta\left(\frac{\mathrm{d}\big(L_{g^{-1}}\circ\gamma(t)\big)}{dt}\right)dt=R(g)\big(\theta(h)\big)\,,$$

which proves that θ is a Lie group one-cocycle.

We already know that if θ is a Lie group one-coboundary, $\Theta=T_e\theta$ is a Lie algebra coboundary. Conversely let us assume that Θ is a Lie algebra one-coboundary. We have, for each $X\in\mathcal{G}$,

$$\Theta(X)=r(X)(c)\,,$$

where c is a fixed element in E. Let $g\in G$ and let $\gamma:[0,T]\to G$ be a smooth parametrized curve in G such that $\gamma(0)=e$ and $\gamma(T)=g$. We have

$$\theta(g)=\int_0^T\eta\left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)dt=\int_0^t R\big(\gamma(t)\big)\circ r\left(TL_{\big(\gamma(t)\big)^{-1}}\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)(c)\,dt\,.$$

But by taking the derivative with respect to t of the two sides of the equality

$$R\big(g\exp(tX)\big)=R(g)\circ R\big(\exp(tX)\big)$$

and then setting $t=0$, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}R\big(\gamma(t)\big)=R\big(\gamma(t)\big)\circ r\left(TL_{\big(\gamma(t)\big)^{-1}}\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right).$$

Therefore θ is a Lie group one-coboundary since we have

$$\theta(g)=\int_0^T\frac{\mathrm{d}R\big(\gamma(t)\big)}{\mathrm{d}t}(c)\,dt=R(g)(c)-R(e)(c)=R(g)(c)-c\,.\quad\square$$

4.3 Poisson, symplectic and Hamiltonian actions

Definitions 4.3.1.

1. An action φ of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M,Λ) is called a *Poisson action* if for any $X\in\mathcal{G}$ the corresponding vector field $\varphi(X)$ is a Poisson vector field. When the Poisson manifold is in fact a symplectic manifold (M,ω), Poisson vector fields on M are locally Hamiltonian vector fields and a Poisson action is called a *symplectic action*.

2. An action Φ (either on the left or on the right) of a Lie group G on a Poisson manifold (M,Λ) is called a *Poisson action* when for each $g \in G$,

$$(\Phi_g)_*\Lambda = \Lambda\,.$$

When the Poisson manifold (M,Λ) is in fact a symplectic manifold (M,ω), a Poisson action is called a *symplectic action*; the fibre bundles isomorphism $\Lambda^\sharp : T^*M \to TM$ being the inverse of $\omega^\flat : TM \to T^*M$, we also can say that an action Φ of a Lie group G on a symplectic manifold (M,ω) is called a *symplectic action* when for each $g \in G$,

$$(\Phi_g)^*\omega = \omega\,.$$

Proposition 4.3.2. *We assume that G is a connected Lie group which acts by an action Φ, either on the left or on the right, on a Poisson manifold (M,Λ), in such a way that the corresponding action of its Lie algebra $\mathcal{G}$ is a Poisson action. Then the action Φ itself is a Poisson action.*

Proof. Let $X \in \mathcal{G}$. For each $x \in M$, the parametrized curve $s \mapsto \Phi_{\exp(sX)}(x)$ is the integral curve of the fundamental vector field X_M which takes the value x for $s = 0$. In other words, the reduced flow of the vector field X_M is the map, defined on $\mathbb{R} \times M$ and taking its values in M,

$$(s,x) \mapsto \Phi_{\exp(sX)}(x)\,.$$

According to a formula which relates inverse images of multivectors or differential forms with respect to the flow of a vector field, with their Lie derivatives with respect to that vector field (see for example [44], Appendix 1, section 3.4, page 351), for any $s_0 \in \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\big((\Phi_{\exp(sX)})^*(\Lambda)\big)(x)\Big)\,\Big|_{s=s_0} = \Big((\Phi_{\exp(s_0X)})^*\big(\mathcal{L}(X_M)\Lambda\big)\Big)(x) = 0\,,$$

since $\mathcal{L}(X_M)\Lambda = 0$. Therefore for any $s \in \mathbb{R}$,

$$(\Phi_{\exp(sX)})^*\Lambda = (\Phi_{\exp(-sX)})_*\Lambda = \Lambda\,.$$

The Lie group G being connected, any $g \in G$ is the product of a finite number of exponentials, so $(\Phi_g)_*\Lambda = \Lambda$. □

4.3.3 Other characterizations of Poisson actions

Let Φ be an action, either on the left or on the right, of a Lie group G on a Poisson manifold (M,Λ). The reader will easily prove that the following properties are equivalent. Therefore any of these properties can be used as the definition of a Poisson action.

1. For each $g \in G$,
$$(\Phi_g)_*\Lambda = \Lambda\,.$$
2. For each $g \in G$ and $f \in C^\infty(M,\mathbb{R})$,
$$(\Phi_g)_*(X_f) = X_{(\Phi_g)_*(f)}\,.$$
3. For each $g \in G$, $\Phi_g : M \to M$ is a *Poisson map*, which means that for each pair (f_1,f_2) of smooth functions on M,
$$\big\{(\Phi_g)^*f_1, (\Phi_g)^*f_2\big\} = (\Phi_g)^*\big(\{f_1,f_2\}\big)\,;$$

4. In the special case when the Poisson manifold (M,Λ) is in fact a symplectic manifold (M,ω), for each $g \in G$,
$$(\Phi_g)^*\omega = \omega\,.$$

The reader will easily prove that when these equivalent properties are satisfied, the action of the Lie algebra $\mathcal{G}$ of G which associates, to each $X \in \mathcal{G}$, the fundamental vector field X_M on M, is a Poisson action.

Definitions 4.3.4.
1. An action φ of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M,Λ) is called a *Hamiltonian action* if for every $X \in \mathcal{G}$ the corresponding vector field $\varphi(X)$ is a Hamiltonian vector field on M.

2. An action Φ (either on the left or on the right) of a Lie group G on a Poisson manifold (M,Λ) is called a *Hamiltonian action* if it is a Poisson action (or a symplectic action when the Poisson manifold (M,Λ) is in fact a symplectic manifold(M,ω)) and if, in addition, the associated action φ of its Lie algebra is a Hamiltonian action.

Remarks 4.3.5.
1. A Hamiltonian action of a Lie algebra on a Poisson manifold is automatically a Poisson action.

2. An action Φ of a connected Lie group G on a Poisson manifold such that the corresponding action of its Lie algebra is Hamiltonian, automatially is a Hamiltonian action.

3. Very often, Hamiltonian actions of a Lie algebra (or of a Lie group) on the cotangent bundle T^*N to a smooth manifold N encountered in applications come from an action of this Lie algebra (or of this Lie group) on the manifold N itself. Proposition 4.3.6 explains how an action on N can be lifted to T^*N into a Hamiltonian action.

Proposition 4.3.6. *Let $\varphi : \mathcal{G} \to A^1(N)$ be an action of a finite-dimensional Lie algebra $\mathcal{G}$ on a smooth manifold N. Let $\widehat{\varphi} : \mathcal{G} \to A^1(T^*N)$ be the map wich associates to each $X \in \mathcal{G}$ the canonical lift to T^*N of the vector field $\varphi(X)$ on N (2.4.8). The map $\widehat{\varphi}$ is a Hamiltonian action of $\mathcal{G}$ on $(T^*N, \mathrm{d}\eta_N)$ (where η_N is the Liouville form and $\mathrm{d}\eta_N$ the canonical symplectic form on T^*N). For each $X \in \mathcal{G}$, the smooth function $f_X : T^*N \to \mathbb{R}$*

$$f_X(\xi) = \Big\langle \xi, \varphi(X)\big(\pi_N(\xi)\big)\Big\rangle = \mathrm{i}\big(\widehat{\varphi}(X)\big)\eta_N(\xi)\,, \quad \xi \in T^*N\,,$$

is a Hamiltonian for the vector field $\widehat{\varphi}(X)$. Moreover, for each pair (X,Y) of elements in $\mathcal{G}$,

$$\{f_X, f_Y\} = f_{[X,Y]}\,.$$

Proof. Proposition 2.4.11 proves that for each $X \in \mathcal{G}$ the vector field $\widehat{\varphi}(X)$ is Hamiltonian and admits the function f_X as Hamiltonian. This Proposition also shows that f_X is given by the two equivalent expressions

$$f_X(\xi) = \Big\langle \xi, \varphi(X)\big(\pi_N(\xi)\big)\Big\rangle = \mathrm{i}\big(\widehat{\varphi}(X)\big)\eta_N(\xi)\,, \quad \xi \in T^*N\,.$$

Let (X,Y) be a pair of elements in $\mathcal{G}$. Since the vector fields $\widehat{\varphi}(X)$ and $\widehat{\varphi}(Y)$ admit f_X and f_Y as Hamiltonians, Lemma 2.5.2 shows that $\big[\widehat{\varphi}(X), \widehat{\varphi}(Y)\big]$ admits $\{f_X, f_Y\}$ as Hamiltonian. We have

$$\{f_X, f_Y\} = \mathcal{L}\big(\widehat{\varphi}(X)\big)f_Y = \mathcal{L}\big(\widehat{\varphi}(X)\big) \circ \mathrm{i}\big(\widehat{\varphi}(Y)\big)\eta_N = \mathrm{i}\Big[\widehat{\varphi}(X), \widehat{\varphi}(Y)\Big]\eta_N$$

since, using 2.4.10, we see that $\mathcal{L}\big(\widehat{\varphi}(X)\big)\eta_N = 0$. Therefore, for each $\xi \in T^*N$,

$$\{f_X, f_Y\}(\xi) = \Big\langle \xi, T\pi_N\big([\widehat{\varphi}(X), \widehat{\varphi}(Y)](\xi)\big)\Big\rangle = \big\langle \xi, [X,Y]\circ\pi_N(\xi)\big\rangle = f_{[X,Y]}(\xi)$$

since $T\pi_N\big([\widehat{\varphi}(X), \widehat{\varphi}(Y)](\xi)\big) = [X,Y]\circ\pi_N(\xi)$. Since $\{f_X, f_Y\} = f_{[X,Y]}$, the corresponding Hamiltonian vector fields $\big[\widehat{\varphi}(X), \widehat{\varphi}(Y)\big]$ and $\widehat{\varphi}\big([X,Y]\big)$ are equal. In other words, $\widehat{\varphi}$ is a Lie algebra action of $\mathcal{G}$ on $(T^*N, \mathrm{d}\eta_N)$. □

Proposition 4.3.7. *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M,Λ). Let $\mathcal{G}^*$ be the dual space of $\mathcal{G}$. There exists a smooth map $J : M \to \mathcal{G}^*$ such that for each $X \in \mathcal{G}$ the corresponding Hamiltonian vector field X_M has the function $J_X : M \to \mathbb{R}$, defined by*

$$J_X(x) = \big\langle J(x), X\big\rangle, \quad \textit{with } x \in M,$$

as Hamiltonian.

Such a map $J : M \to \mathcal{G}^*$ is called a momentum map *for the Hamiltonian Lie algebra action φ. When φ is the Lie algebra action associated to a Hamiltonian action Φ of a Lie group G on the Poisson manifold (M,Λ), J is called a* momentum map *for the Hamiltonian Lie group action Φ.*

Proof. Let $(e_1,\ldots,e_p)$ be a basis of the Lie algebra $\mathcal{G}$ and$(\varepsilon^1,\ldots,\varepsilon^p)$ be the dual basis of $\mathcal{G}^*$. Since φ is Hamiltonian, for each i $(1 \le i \le p)$ there exists a Hamiltonian $J_{e_i} : M \to \mathbb{R}$ for the Hamiltonian vector field $\varphi(e_i)$. The map $J : M \to \mathcal{G}$ defined by

$$J(x) = \sum_{i=1}^{p} J_{e_i}\varepsilon^i, \quad x \in M,$$

is a momentum map for φ. □

The momentum map was introduced by Jean-Marie Souriau [60] and, in the Lagrangian formalism, by Stephen Smale [58].

4.4 Some properties of momentum maps

Proposition 4.4.1. *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M,Λ), and $J : M \to \mathcal{G}^*$ be a momentum map for that action. For any pair $(X,Y) \in \mathcal{G}\times\mathcal{G}$, the smooth function $\widetilde{\Theta}(X,Y) : M \to \mathbb{R}$ defined by*

$$\widetilde{\Theta}(X,Y) = \{J_X, J_Y\} - J_{[X,Y]}$$

is a Casimir of the Poisson algebra $C^\infty(M,\mathbb{R})$, which satisfies, for all X, Y and $Z \in \mathcal{G}$,

$$\widetilde{\Theta}\big([X,Y],Z\big) + \widetilde{\Theta}\big([Y,Z],X\big) + \widetilde{\Theta}\big([Z,X],Y\big) = 0. \tag{1}$$

When the Poisson manifold (M,Λ) is in fact a connected symplectic manifold (M,ω), for any pair $(X,Y) \in \mathcal{G}\times\mathcal{G}$ the function $\widetilde{\Theta}(X,Y)$ is constant on M, and the map $\widetilde{\Theta} : \mathcal{G}\times\mathcal{G} \to \mathbb{R}$ is a skew-symmetric bilinear form, which satisfies the above identity (1).

Proof. Since J_X and J_Y are Hamiltonians for the Hamiltonian vector fields $\varphi(X)$ and $\varphi(Y)$, the Poisson bracket $\{J_X, J_Y\}$ is a Hamiltonian for $\big[\varphi(X),\varphi(Y)\big]$. Since $\varphi : \mathcal{G} \to A^1(M)$ is a Lie algebras homomorphism, $\big[\varphi(X),\varphi(Y)\big] = \varphi\big([X,Y]\big)$, and $J_{[X,Y]}$ is a Hamiltonian for this vector

field. We have two different Hamiltonians for the same Hamiltonian vector field. Their difference $\widetilde{\Theta}(X,Y)$ is therefore a Casimir of the Poisson algebra $C^\infty(M,\mathbb{R})$.

Let X, Y and Z be three elements in $\mathcal{G}$. We have

$$\begin{aligned}\widetilde{\Theta}\big([X,Y],Z\big) &= \{J_{[X,Y]},J_Z\} - J_{\big[[X,Y],Z\big]} \\ &= \big\{\{J_X,J_Y\} - \widetilde{\Theta}(X,Y),J_Z\big\} - J_{\big[[X,Y],Z\big]} \\ &= \big\{\{J_X,J_Y\},J_Z\big\} - J_{\big[[X,Y],Z\big]}\end{aligned}$$

since $\widetilde{\Theta}(X,Y)$ is a Casimir of the Poisson algebra $C^\infty(M,\mathbb{R})$. Similarly

$$\begin{aligned}\widetilde{\Theta}\big([Y,Z],X\big) &= \big\{\{J_Y,J_Z\},J_X\big\} - J_{\big[[Y,Z],X\big]}\,, \\ \widetilde{\Theta}\big([Z,X],Y\big) &= \big\{\{J_Z,J_X\},J_Y\big\} - J_{\big[[Z,X],Y\big]}\,.\end{aligned}$$

Adding these three terms and using the fact that the Poisson bracket of functions and the bracket in the Lie algebra $\mathcal{G}$ both satisfy the Jacobi identity, we see that $\widetilde{\Theta}$ satisfies (1).

When (M,Λ) is in fact a connected symplectic manifold (M,ω), the only Casimirs of the Poisson algebra $C^\infty(M,\mathbb{R})$ are the constants, and $\widetilde{\Theta}$ becomes a bilinear skew-symmetric form on $\mathcal{G}$. □

Definition 4.4.2. Under the assumptions of Proposition 4.4.1, the skew-symmetric bilinear map $\widetilde{\Theta}$, defined on $\mathcal{G}\times\mathcal{G}$ and taking its values in the space of Casimirs of the Poisson algebra $C^\infty(M,\mathbb{R})$ (real-valued when the Poisson manifold (M,Λ) is in fact a connected symplectic manifold (M,ω)), is called the *symplectic cocycle of the Lie algebra* $\mathcal{G}$ associated to the momentum map J.

Remark 4.4.3. Under the assumptions of Proposition 4.4.1, let us assume in addition that the Poisson manifold (M,Λ) is in fact a connected symplectic manifold (M,ω). The symplectic cocycle $\widetilde{\Theta}$ is then a real-valued skew-symmetric bilinear form on $\mathcal{G}$. Therefore it is a symplectic cocycle in the sense of 3.4.3. Two different interpretations of this cocycle can be given.

1. Let $\Theta : \mathcal{G}\to\mathcal{G}^*$ be the map such that, for all X and $Y\in\mathcal{G}$

$$\big\langle\Theta(X),Y\big\rangle = \widetilde{\Theta}(X,Y)\,.$$

Written for Θ, Equation (1) of 4.4.1 becomes

$$\Theta\big([X,Y]\big) = \mathrm{ad}^*_{-X}\big(\Theta(Y)\big) - \mathrm{ad}^*_{-Y}\big(\Theta(X)\big)\,,\quad X \text{ and } Y\in\mathcal{G}\,.$$

The map Θ is therefore the one-cocycle of the Lie algebra $\mathcal{G}$ with values in $\mathcal{G}^*$, for the coadjoint representation (4.2.4) $X\mapsto\mathrm{ad}^*_{-X}$ of $\mathcal{G}$, associated to the affine action of $\mathcal{G}$ on its dual

$$a_\Theta(X)(\zeta) = \mathrm{ad}^*_{-X}(\zeta) + \Theta(X)\,,\quad X\in\mathcal{G}\,,\ \zeta\in\mathcal{G}^*\,,$$

in the sense of 4.2.6. The reader is referred to the book [32] for a more thorough discussion of the cohomology theories of Lie groups and Lie algebras.

2. Let G be a Lie group whose Lie algebra is $\mathcal{G}$. The skew-symmetric bilinear form $\widetilde{\Theta}$ on $\mathcal{G}=T_eG$ can be extended, either by left translations or by right translations, into a left invariant (or a right invariant) closed differential two-form on G, since the identity (1) of 4.4.1 means that its exterior differential $\mathrm{d}\widetilde{\Theta}$ vanishes. In other words, $\widetilde{\Theta}$ is a 2-cocycle for the restriction of the de Rham cohomology of G to left (or right) invariant differential forms.

Proposition 4.4.4. *Let $\varphi : \mathcal{G} \to A^1(N)$ be an action of a finite-dimensional Lie algebra $\mathcal{G}$ on a smooth manifold N, and let $\widehat{\varphi} : \mathcal{G} \to A^1(T^*N)$ be the Hamiltonian action of $\mathcal{G}$ on $(T^*N, \mathrm{d}\eta_N)$ introduced in Proposition 4.3.6. The map $J : T^*N \to \mathcal{G}^*$ defined by*

$$\big\langle J(\xi), X \big\rangle = \mathrm{i}\big(\widehat{\varphi}(X)\big)\eta_N(\xi)\,, \quad X \in \mathcal{G}\,,\ \xi \in T^*N\,,$$

is a momentum map for the action $\widehat{\varphi}$ which satisfies, for all X and $Y \in \mathcal{G}$,

$$\big\{J_X, J_Y\big\} = J_{[X,Y]}\,.$$

In other words, the symplectic cocycle of $\mathcal{G}$ associated to J, in the sense of 4.4.2, identically vanishes.

Proof. These properties immediately follow from 4.3.6. □

Theorem 4.4.5 (First Emmy Noether's theorem in Hamiltonian form)**.** *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M, Λ), $J : M \to \mathcal{G}^*$ be a momentum map for φ and $H : M \to \mathbb{R}$ be a smooth Hamiltonian. If the action φ leaves H invariant, that means if*

$$\mathcal{L}\big(\varphi(X)\big)H = 0 \quad \textit{for any } X \in \mathcal{G}\,,$$

the momentum map J is a $\mathcal{G}^*$-valued first integral (3.3.4) of the Hamiltonian vector field $\Lambda^\sharp(\mathrm{d}H)$, which means that it keeps a constant value along each integral curve of that vector field.

Proof. For any $X \in \mathcal{G}$, let $J_X : M \to \mathbb{R}$ be the function $x \mapsto \big\langle J(x), X\big\rangle$. Let $t \mapsto \psi(t)$ be an integral curve of the Hamiltonian vector field $\Lambda^\sharp(\mathrm{d}H)$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(J_X\big(\psi(t)\big)\Big) &= \mathcal{L}\big(\Lambda^\sharp(\mathrm{d}H)\big)\big(J_X\big)\big(\psi(t)\big) = \Lambda\big(\mathrm{d}H, \mathrm{d}J_X\big)\big(\psi(t)\big)\\ &= -\mathcal{L}\Big(\Lambda^\sharp\big(\mathrm{d}J_X\big)\Big)H = -\mathcal{L}\big(\varphi(X)\big)H = 0\,.\end{aligned}$$

Therefore, for any $X \in \mathcal{G}$, the derivative of $\langle J, X\rangle\big(\psi(t)\big)$ with respect to the parameter t of the parametrized curve $t \mapsto \psi(t)$ vanishes identically, which means that J keeps a constant value along that curve. □

The reader will find in the book by Yvette Kosmann-Schwarzbach [37] a very nice exposition of the history and scientific applications of the Noether's theorems.

Proposition 4.4.6. *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a Poisson manifold (M, Λ) and $J : M \to \mathcal{G}^*$ be a momentum map for that action. Let S be a symplectic leaf of (M, Λ) and ω_S be its symplectic form.*

1. *For each $x \in S$, in the symplectic vector space $\big(T_xS, \omega_S(x)\big)$, each of the two vector subspaces $T_xS \cap \ker(T_xJ)$ and $\big\{\,\varphi(X)(x)\,;\, X \in \mathcal{G}\,\big\}$ is the symplectic orthogonal of the other.*

2. *For each $x \in S$, $T_xJ(T_xS)$ is the annihilator of the isotropy subalgebra $\mathcal{G}_x = \big\{X \in \mathcal{G}; \phi(X)(x) = 0\big\}$ of x.*

Proof. Let $v \in T_xS$. For each $X \in \mathcal{G}$ we have

$$\omega_S\big(v, \varphi(X)(x)\big) = \big\langle\, d\langle J, X\rangle(x), v\big\rangle = \big\langle T_xJ(v), X\big\rangle\,.$$

Therefore a vector $v \in T_xS$ belongs to $\mathrm{orth}\big\{\,\varphi(X)(x)\,;\, X \in \mathcal{G}\,\big\}$ if and only if $T_xJ(v) = 0$. In other words, in the symplectic vector space $\big(T_xS, \omega_S(x)\big)$, $T_xS \cap \ker(T_xJ)$ is the symplectic orthogonal of $\big\{\,\varphi(X)(x)\,;\, X \in \mathcal{G}\,\big\}$. Of course, conversely $\big\{\,\varphi(X)(x)\,;\, X \in \mathcal{G}\,\big\}$ is the symplectic orthogonal of $T_xS \cap \ker(T_xJ)$.

The same formula shows that $\big\langle T_xJ(v), X\big\rangle = 0$ for all $v \in T_xS$ if and only if $X \in \mathcal{G}_x$. □

Remark 4.4.7. Under the assumptions of 4.4.6, when φ is the Lie algebra action associated to a Hamiltonian action Φ of a Lie group G, the vector space $\{\,\varphi(X)(x)\,;X\in\mathcal{G}\,\}$ is the space tangent at x to the G-orbit of this point.

Corollary 4.4.8. *Let φ be a Hamiltonian action of a Lie algebra $\mathcal{G}$ on a symplectic manifold (M,ω) and $J:M\to\mathcal{G}^*$ be a momentum map for that action.*

1. *For each $x\in M$, in the symplectic vector space $\big(T_xM,\omega(x)\big)$ each of the two vector subspaces $\ker(T_xJ)$ and $\{\,\varphi(X)(x)\,;X\in\mathcal{G}\,\}$ is the symplectic orthogonal of the other.*

2. *For each $x\in M$, $T_xJ(T_xM)$ is the annihilator of the isotropy subalgebra $\mathcal{G}_x=\{X\in\mathcal{G};\varphi(X)(x)=0\}$ of x.*

Proof. These assertions both follow immediately from 4.4.6 since the symplectic leaves of (M,ω) are its connected components. □

Proposition 4.4.9. *Let Φ be a Hamiltonian action of a Lie group G on a connected symplectic manifold (M,ω) and $J:M\to\mathcal{G}^*$ be a momentum map for that action. There exists a unique action A of the Lie group G on the dual $\mathcal{G}^*$ of its Lie algebra for which the momentum map J is equivariant, that means satisfies for each $x\in M$ and $g\in G$*

$$J\big(\Phi_g(x)\big)=A_g\big(J(x)\big)\,.$$

The action A is an action on the left (respectively, on the right) if Φ is an action on the left (respectively, on the right), and its expression is

$$\begin{cases}A(g,\xi)=\mathrm{Ad}^*_{g^{-1}}(\xi)+\theta(g) & \text{if } \Phi \text{ is an action on the left,}\\ A(\xi,g)=\mathrm{Ad}^*_{g}(\xi)-\theta(g^{-1}) & \text{if } \Phi \text{ is an action on the right,}\end{cases}\qquad g\in G,\ \xi\in\mathcal{G}^*\,.$$

The map $\theta:G\to\mathcal{G}^*$ is called the symplectic cocycle of the Lie group G *associated to the momentum map J.*

Proof. Let us first assume that Φ is an action on the left. For each $X\in\mathcal{G}$ the associated fundamental vector field X_M is Hamiltonian and the function $J_X:M\to\mathbb{R}$ defined by

$$J_X(x)=\big\langle J(x),X\big\rangle,\quad x\in M\,,$$

is a Hamiltonian for X_M. We know by the characterizations 4.3.3 of Poisson actions that $(\Phi_{g^{-1}})_*(X_M)$, the direct image of X_M by the diffeomorphism $\Phi_{g^{-1}}$, is a Hamiltonian vector field for which the function $J_X\circ\Phi_g$ is a Hamiltonian. Proposition 4.1.7 shows that $(\Phi_{g^{-1}})_*(X_M)$ is the fundamental vector field associated to $\mathrm{Ad}_{g^{-1}}(X)$, therefore has the function

$$x\mapsto\big\langle J(x),\mathrm{Ad}_{g^{-1}}(X)\big\rangle=\big\langle\mathrm{Ad}^*_{g^{-1}}\circ J(x),X\big\rangle$$

as a Hamiltonian. The difference between these two Hamiltonians for the same Hamiltonian vector field is a constant since M is assumed to be connected. Therefore the expression

$$\big\langle J\circ\Phi_g(x)-\mathrm{Ad}^*_{g^{-1}}\circ J(x),X\big\rangle$$

does not depend on $x\in M$, and depends linearly on $X\in\mathcal{G}$ (and of course smoothly depends on $g\in G$). We can therefore define a smooth map $\theta:G\to\mathcal{G}^*$ by setting

$$\theta(g)=J\circ\Phi_g-\mathrm{Ad}^*_{g^{-1}}\circ J\,,\quad g\in G\,.$$

It follows that the map $a : G \times \mathcal{G}^* \to \mathcal{G}^*$,

$$a(g,\xi) = \mathrm{Ad}^*_{g^{-1}}(\xi) + \theta(g)$$

is an action on the the left of the Lie group G on the dual $\mathcal{G}^*$ of its Lie algebra, which renders the momentum map J equivariant.

The case when Φ is an action on the right easily follows by observing that $(g,x) \mapsto \Phi(x,g^{-1})$ is a Hamiltonian action on the left whose momentum map is the opposite of that of Φ. □

Proposition 4.4.10. *Under the same assumptions as those of Proposition 4.4.9, the map $\theta : G \to \mathcal{G}^*$ satisfies, for all g and $h \in G$,*

$$\theta(gh) = \theta(g) + \mathrm{Ad}^*_{g^{-1}}\big(\theta(h)\big)\,.$$

Proof. In Proposition 4.4.9, the cocycle θ introduced for an action on the right $\Psi : M \times G \to M$ was the cocycle of the corresponding action on the left $\Phi : G \times M \to M$ defined by $\Phi(g,x) = \Psi(x,g^{-1})$. We can therefore consider only the case when Φ is an action on the left.

Let g and $h \in G$. We have

$$\begin{aligned}
\theta(gh) &= J\big(\Phi(gh,x)\big) - \mathrm{Ad}^*_{(gh)^{-1}} J(x) \\
&= J\Big(\Phi\big(g,\Phi(h,x)\big)\Big) - \mathrm{Ad}^*_{g^{-1}} \circ \mathrm{Ad}^*_{h^{-1}} J(x) \\
&= \theta(g) + \mathrm{Ad}^*_{g^{-1}}\Big(J\big(\Phi(h,x)\big) - \mathrm{Ad}^*_{h^{-1}} J(x)\Big) \\
&= \theta(g) + \mathrm{Ad}^*_{g^{-1}}\,\theta(h)\,.
\end{aligned}$$

□

Proposition 4.4.11. *Let Φ be a Hamiltonian action of a Lie group G on a connected symplectic manifold (M,ω) and $J : M \to \mathcal{G}^*$ be a momentum map for that action. The symplectic cocycle $\theta : G \to \mathcal{G}^*$ of the Lie group G introduced in Proposition 4.4.9 and the symplectic cocycle $\Theta : \mathcal{G} \to \mathcal{G}^*$ of its Lie algebra $\mathcal{G}$ introduced in Definition 4.4.2 and Remark 4.4.3 are related by*

$$\Theta = T_e\theta\,,$$

where e is the neutral element of G, the Lie algebra $\mathcal{G}$ being identified with T_eG and the tangent space at $\mathcal{G}^*$ at its origin being identified with $\mathcal{G}^*$. Moreover J is a Poisson map when $\mathcal{G}^*$ is endowed with

- *its canonical Poisson structure modified by the symplectic cocycle Θ (defined in 3.4.3) if Φ is an action on the right,*
- *the opposite of this Poisson structure if Φ is an action on the left.*

Proof. As in the proof of Proposition 4.4.10, whe have only to consider the case when Φ is an action on the left. The map which associates to each $X \in \mathcal{G}$ the fundamental vector field X_M is a Lie algebras homomorphism when $\mathcal{G}$ is endowed with the Lie algebra structure of *right invariant vector fields* on the Lie group G. We will follow here the more common convention, in which $\mathcal{G}$ is endowed with the Lie algebra structure of *left invariant vector fields* on G. With this convention the map $X \mapsto X_M$ is a Lie algebras antihomomorphism and we must change a sign in the definition of $\widetilde{\Theta}$ given in Proposition 4.4.1 and take

$$\widetilde{\Theta}(X,Y) = \big\langle \Theta(X), Y\big\rangle = \{J_X, J_Y\} + J_{[X,Y]}\,, \quad X \text{ and } Y \in \mathcal{G}\,.$$

We have, for any $x \in M$,

$$\begin{aligned}\{J_X, J_Y\}(x) &= \omega(X_M, Y_M)(x) = i(X_M)d\big(\langle J, Y\rangle\big)(x)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle J\big(\Phi(\exp(tX), x\big), Y\Big\rangle\Big|_{t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \mathrm{Ad}^*_{\exp(-tX)} J(x) + \theta\big(\exp(tX)\big), Y\Big\rangle\Big|_{t=0}\\ &= \big\langle J(x), -[X,Y]\big\rangle + \big\langle T_e\theta(X), Y\big\rangle\\ &= -J_{[X,Y]}(x) + \big\langle T_e\theta(X), Y\big\rangle\,.\end{aligned}$$

We see that $\Theta = T_e\theta$. Moreover, the elements X and Y in $\mathcal{G}$ can be considered as linear functions on $\mathcal{G}^*$. Their Poisson bracket, when $\mathcal{G}^*$ is equipped with its canonical Poisson structure modified by $\widetilde{\Theta}$, is

$$\{X, Y\}_{\widetilde{\Theta}}(\xi) = \big\langle \xi, [X,Y]\big\rangle - \widetilde{\Theta}(X,Y)\,.$$

The formula $\{J_X, J_Y\}(x) = -J_{[X,Y]}(x) + \widetilde{\Theta}(X,Y)$ can be read as

$$\{X \circ J, Y \circ J\}(x) = -\{X,Y\}_{\widetilde{\Theta}} \circ J(x)\,.$$

Since the value taken at a point by the Poisson bracket of two functions only depends on the values of the differentials of these two functions at that point, this result proves that J is a Poisson map when $\mathcal{G}^*$ is equipped with the opposite of the Poisson bracket $\{\ ,\ \}_{\widetilde{\Theta}}$. □

Remarks 4.4.12. Let Φ be a Hamiltonian action on the left of a Lie group G on a connected symplectic manifold (M, ω), $J : M \to \mathcal{G}^*$ be a momentum map for that action and $\theta : G \to \mathcal{G}^*$ be the symplectic cocycle of the Lie group G introduced in Proposition 4.4.9.

1. The symplectic cocycle $\theta : G \to \mathcal{G}^*$ is the Lie group one-cocycle with values in $\mathcal{G}^*$, for the coadjoint representation, associated to the affine representation $A : G \to \mathrm{Aff}(\mathcal{G}^*)$,

$$A(g)(\zeta) = \mathrm{Ad}^*_{g^{-1}}(\zeta) + \theta(g)\,, \quad \zeta \in \mathcal{G}^*\,,$$

in the sense of 4.2.5.

2. If instead of J we take for momentum map

$$J'(x) = J(x) - c\,, \quad x \in M\,,$$

where $c \in \mathcal{G}^*$ is constant, the symplectic cocycle θ is replaced by

$$\theta'(g) = \theta(g) + \mathrm{Ad}^*_{g^{-1}}(c) - c\,.$$

The map $\theta' - \theta$ is a one-coboundary of G with values in $\mathcal{G}^*$ for the coadjoint representation (4.2.9). Therefore the cohomology class of the symplectic cocycle θ only depends on the Hamiltonian action Φ, not on the choice of its momentum map J. This property is used by Jean-Marie Souriau ([60], chapter III, p. 153) to offer a very nice cohomological interpretation of the total mass of a classical (non-relativistic) isolated mechanical system. He proves that the space of all possible motions of the system is a symplectic manifold on which the Galilean group acts by a Hamiltonian action. The dimension of the symplectic cohomology space of the Galilean group (the quotient of the space of symplectic one-cocycles by the space of symplectic one-coboundaries) is equal to 1. The cohomology class of the symplectic cocycle associated to a momentum map of the action of the Galilean group on the space of motions of the system is interpreted as the *total mass* of the system.

4.4.13 Other properties of the momentum map

The momentum map has several other very remarkable properties. Michael Atiyah [8], Victor Guillemin and Shlomo Sternberg [24, 25] have shown that the image of the momentum map of a Hamiltonian action of a torus on a compact symplectic manifold is a convex polytope. Frances Kirwan [36] adapted this result when the torus is replaced by any compact Lie group. Thomas Delzant [16] has shown that the convex polytope which is the image of a Hamiltonian action of a torus on a compact symplectic manifold determines this manifold.

4.5 Actions of a Lie group on its cotangent bundle

In this section G is a Lie group, $\mathcal{G}$ is its Lie algebra and $\mathcal{G}^*$ is the dual space of $\mathcal{G}$. The Liouville one-form on T^*G is denoted by η_G.

The group composition law $m : G \times G \to G$, $m(g,h) = gh$, can be seen as an action of G on itself either on the left, or on the right. For each $g \in G$ we will denote by $L_g : G \to G$ and $R_g : G \to G$ the diffeomorphisms

$$L_g(h) = gh\,, \quad R_g(h) = hg\,, \quad h \in G\,.$$

called, respectively, the *left translation* and the *right translation* of G by g.

Definitions 4.5.1. The *canonical lifts* to the tangent bundle TG of the actions of G on itself by left translations (respectively, by right translations) are, repectively, the maps $\overline{L} : G \times TG \to TG$ and $\overline{R} : TG \times G \to TG$

$$\overline{L}(g,v) = TL_g(v)\,, \quad \overline{R}(v,g) = TR_g(v)\,, \quad g \in G, \quad v \in TG\,.$$

The *canonical lifts* to the cotangent bundle T^*G of the actions of G on itself by left translations (respectively, by right translations) are, respectively, the maps $\widehat{L} : G \times T^*G \to T^*G$ and $\widehat{R} : T^*G \times G \to T^*G$

$$\widehat{L}(g,\xi) = \left(TL_{g^{-1}}\right)^T(\xi)\,, \quad \widehat{R}(\xi,g) = \left(TR_{g^{-1}}\right)^T(\xi)\,, \quad g \in G, \quad \xi \in T^*G\,.$$

We have denoted by $\left(TL_{g^{-1}}\right)^T$ and $\left(TR_{g^{-1}}\right)^T$ the transposes of the vector bundles morphisms $TL_{g^{-1}}$ and $TR_{g^{-1}}$, respectively.

Proposition 4.5.2. *The canonical lifts to the tangent bundle and to the cotangent bundle of the actions of the Lie group G on itself by left translations (respectively, by right translations) are actions on the left (respectively, on the right) of G on its tangent bundle and on its cotangent bundle, which project onto the actions of G on itself by left translations (respectively, by right translations). It means that for all $g \in G$ and $v \in TG$*

$$\tau_G\big(\overline{L}(g,v)\big) = L_g\big(\tau_G(v)\big)\,, \quad \tau_G\big(\overline{R}(v,g)\big) = R_g\big(\tau_G(v)\big)\,,$$

and that for all $g \in G$ and $\xi \in T^*G$

$$\pi_G\big(\widehat{L}(g,\xi)\big) = L_g\big(\pi_G(\xi)\big)\,, \quad \pi_G\big(\widehat{R}(\xi,g)\big) = R_g\big(\pi_G(\xi)\big)\,.$$

Proof. It is an easy verification that the properties of actions are indeed satisfied by the maps $\overline{L}$, $\overline{R}$, $\widehat{L}$ and $\widehat{R}$, which is left to the reader. □

Theorem 4.5.3. *The canonical lifts to the cotangent bundle $\widehat{L}$ and $\widehat{R}$ of the actions of the Lie group G on itself by translations on the left and on the right are two Hamiltonian actions of G on the symplectic manifold $(T^*G, \mathrm{d}\eta_G)$. The maps $J^L : T^*G \to \mathcal{G}^*$ and $J^R : T^*G \to \mathcal{G}^*$ defined, for each $\xi \in T^*G$, by*

$$J^L(\xi) = \widehat{R}\big(\xi, \pi_G(\xi)^{-1}\big), \quad J^R(\xi) = \widehat{L}\big(\pi_G(\xi)^{-1}, \xi\big)$$

are momentum maps for the actions $\widehat{L}$ and $\widehat{R}$, respectively.

Moreover, the map J^L is constant on each orbit of the action $\widehat{R}$, the map J^R is constant on each orbit of the action $\widehat{L}$ and for each $\xi \in T^*G$ each of the tangent spaces at ξ to the orbits $\widehat{L}(G, \xi)$ and $\widehat{R}(\xi, G)$ is the symplectic orthogonal of the other. The maps $J^L : T^*G \to \mathcal{G}^*$ and $J^R : T^*G \to \mathcal{G}^*$ are Poisson maps when T^*G is equipped with the Poisson structure associated to its canonical symplectic structure and when $\mathcal{G}^*$ is equipped, respectively, with its canonical Poisson structure (3.4.2) and with the opposite of its canonical Poisson structure.

Proof. For each $X \in \mathcal{G}$, let X^L_G and X^R_G be the fundamental vector fields on G associated to X for the actions of G on itself, respectively by left and by right translations. Similarly, let $X^L_{T^*G}$ and $X^R_{T^*G}$ be the fundamental vector fields on T^*G associated to X for the actions $\widehat{L}$ and $\widehat{R}$ of G on T^*G defined in 4.5.1. The reduced flows of X^L and of X^R are the maps

$$\Phi^{X^L}(t, g) = \exp(tX)g, \quad \Phi^{X^R}(t, g) = g\exp(tX), \quad t \in \mathbb{R},\ g \in G.$$

Therefore

$$X^L(g) = TR_g(X), \quad X^R(g) = TL_g(X), \quad g \in G,$$

and we see that the fundamental vector fields $X^L_{T^*G}$ and $X^R_{T^*G}$ on T^*G are the canonical lifts to the cotangent bundle of the vector fields X^L_G and X^R_G on the Lie group G. Proposition 2.4.11 proves that $X^L_{T^*G}$ and $X^R_{T^*G}$ are Hamiltonian vector fields which admit as Hamiltonians, respectively, the maps

$$J^L_X(\xi) = \Big\langle \xi, X^L_G\big(\pi_G(\xi)\big)\Big\rangle, \quad J^R_X(\xi) = \Big\langle \xi, X^R_G\big(\pi_G(\xi)\big)\Big\rangle, \quad \xi \in T^*G.$$

Replacing X^L_G and X^R_G by their expressions given above and using the definitions of $\widehat{R}$ and $\widehat{L}$, we easily get the stated expressions for J^L and J^R. These expressions prove that J^L is constant on each orbit of the action $\widehat{R}$, and that J^R is constant on each orbit of the action $\widehat{L}$.

The actions $\widehat{L}$ and $\widehat{R}$ being free, each of their orbits is a smooth submanifold of T^*G of dimension $\dim G$. The ranks of the maps J^L and J^R are everywhere equal to $\dim G$ since their restrictions to each fibre of T^*G is a diffeomorphism of that fibre onto $\mathcal{G}^*$. Therefore, for each $\xi \in T^*G$,

$$\ker T_\xi J^L = T_\xi\big(\widehat{R}(\xi, G)\big), \quad \ker T_\xi J^R = T_\xi\big(\widehat{L}(\xi, G)\big).$$

Corollary 4.4.8 proves that for each $\xi \in T^*G$ each of the two vector subspaces of $T_\xi(T^*G)$:

$$T_\xi\big(\widehat{L}(G, \xi)\big) \quad \text{and} \quad T_\xi\big(\widehat{R}(\xi, G)\big)$$

is the symplectic orthogonal of the other.

Finally, the fact that J^L and J^R are Poisson maps when $\mathcal{G}$ is equipped with its canonical Poisson structure or its opposite is an easy consequence of Proposition 3.3.7. □

In [44], Chapter IV, Section 4, we proposed a generalization of Proposition 4.5.3 taking into account a symplectic cocycle $\theta : G \to \mathcal{G}^*$ in which the action $\widehat{L} : G \times T^*G \to T^*G$ remained

unchanged while the action $\widehat{R} : T^*G \times G \to T^*G$ was modified. Below we propose a more general and more symmetrical generalization. The symplectic form on T^*G will be the sum of its canonical symplectic form $d\eta_G$ and of the pull-back by the canonical projection $\pi_G : T^*G \to G$ of a suitable closed two-form on G, deduced from θ. The actions $\widehat{L} : G \times T^*G \to T^*G$ and $\widehat{R} : T^*G \times G \to T^*G$ will be modified in the following way: for each $g \in G$, the map $\widehat{L}_g : T^*G \to T^*G$ will be composed with a translation in the fibres of T^*G, determined by addition of a *right-invariant* one-form on G depending of the element $g \in G$, deduced from θ; similarly, the map $\widehat{R}_g : T^*G \to T^*G$ will be composed with a translation in the fibres of T^*G, determined by addition of a *left-invariant* one-form on G depending of the element $g \in G$, deduced from θ. As the reader will see, it is possible to modify the action $\widehat{L}$ and to keep $\widehat{R}$ unchanged, or to modify the action $\widehat{R}$ and to keep $\widehat{L}$ unchanged; in the first case, the momentum map $J^L : T^*G \to \mathcal{G}^*$ remains unchanged, while $J^R : T^*G \to \mathcal{G}^*$ must be modified; in the second case, it is $J^R : T^*G \to \mathcal{G}^*$ which remains unchanged while $J^L : T^*G \to \mathcal{G}^*$ must be modified. It is even possible to simultaneously modify both the actions $\widehat{L}$ and $\widehat{R}$; then we get a pair of actions of G on T^*G depending on two real parameters.

Theorem 4.5.4. *Let G be a Lie group, $\theta : G \to \mathcal{G}^*$ be a symplectic cocycle of G, $\Theta = T_e\theta : \mathcal{G} \to \mathcal{G}^*$ be the associated symplectic cocycle of its Lie algebra $\mathcal{G}$, and $\widetilde{\Theta} : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ be the skew-symmetric bilinear form $\widetilde{\Theta}(X,Y) = \big\langle \Theta(X), Y \big\rangle$. Let $\widetilde{\Theta}_L$ and $\widetilde{\Theta}_R$ be the differential two-forms on G, respectively left-invariant and right-invariant, whose value at the neutral element is $\widetilde{\Theta}$. The differential two-form on T^*G*

$$\omega_{T^*G} = \mathrm{d}\eta_G + \pi_G^*(\lambda_L \widetilde{\Theta}_L - \lambda_R \widetilde{\Theta}_R)\,,$$

where λ_L and λ_R are real constants and where η_G is the Liouville form on T^*G, is a symplectic form on T^*G. The formulae, in which $g \in G$, $\xi \in T^*G$,

$$\Phi^L(g,\xi) = \widehat{L}_g(\xi) + \lambda_R \widehat{R}_{g\pi_G(\xi)}\big(\theta(g)\big)\,,$$
$$\Phi^R(\xi,g) = \widehat{R}_g(\xi) + \lambda_L \widehat{L}_{\pi_G(\xi)g}\big(\theta(g^{-1})\big)$$

define two Hamiltonian actions $\Phi^L : G \times T^*G \to T^*G$ and $\Phi^R : T^*G \times G \to T^*G$ of G on the symplectic manifold (T^*G, ω_{T^*G}), respectively on the left and on the right. The maps $J^{L,\lambda_L} : T^*G \to \mathcal{G}^*$ and $J^{R,\lambda_R} : T^*G \to \mathcal{G}^*$ defined, for each $\xi \in T^*G$, by

$$J^{L,\lambda_L}(\xi) = \widehat{R}_{\left(\pi_G(\xi)\right)^{-1}}(\xi) + \lambda_L \theta\big(\pi_G(\xi)\big)\,,$$
$$J^{R,\lambda_R}(\xi) = \widehat{L}_{\left(\pi_G(\xi)\right)^{-1}}(\xi) + \lambda_R \theta\Big(\big(\pi_G(\xi)\big)^{-1}\Big)$$

are momentum maps for the actions Φ^L and Φ^R, respectively.

Moreover, the map J^{L,λ_L} is constant on each orbit of the action Φ^R, the map J^{R,λ_R} is constant on each orbit of the action Φ^L and for each $\xi \in T^*G$ each of the tangent spaces at ξ to the orbits $\Phi^L(G,\xi)$ and $\Phi^R(\xi,G)$ is the symplectic orthogonal of the other (with respect to the symplectic form ω_{T^*G}). The maps $J^{L,\lambda_L} : T^*G \to \mathcal{G}^*$ and $J^{R,\lambda_R} : T^*G \to \mathcal{G}^*$ are Poisson maps when T^*G is equipped with the Poisson structure associated to the symplectic form ω_{T^*G} and when $\mathcal{G}^*$ is equipped, respectively, with its canonical Poisson structure modified by the cocycle $(\lambda_L + \lambda_R)\widetilde{\Theta}$ (3.4.3)

$$\{f,g\}_{(\lambda_L+\lambda_R)\widetilde{\Theta}}(\zeta) = \Big\langle \zeta, \big[\mathrm{d}f(\zeta), \mathrm{d}g(\zeta)\big] \Big\rangle - (\lambda_L + \lambda_R)\widetilde{\Theta}\big(\mathrm{d}f(\zeta), \mathrm{d}g(\zeta)\big)$$

and with the opposite of this Poisson structure.

Proof. The sum of the canonical symplectic form on T^*G with the pull-back of any closed two-form on G always is nondegenerate, therefore symplectic. So ω_{T^*G} is symplectic. For g and $h \in G$, $\xi \in T^*G$, let us calculate

$$\Phi^L\big(g,\Phi^L(h,\xi)\big) - \Phi^L(gh,\xi) \quad \text{and} \quad \Phi^R\big(\Phi^R(\xi,g),h\big) - \Phi^R(\xi,gh)\,.$$

We get

$$\begin{aligned}\Phi^L\big(g,\Phi^L(h,\xi)\big) - \Phi^L(gh,\xi) &= \lambda_R \widehat{R}_{gh\pi_G(\xi)}\Big(\mathrm{Ad}^*_{g^{-1}}\big(\theta(h)\big) + \theta(g) - \theta(gh)\Big)\\ &= 0\end{aligned}$$

since θ is a one-cocycle. The map Φ^L is therefore an action on the left of G on T^*G. Similarly

$$\begin{aligned}\Phi^R\big(\Phi^R(\xi,g),h\big) - \Phi^R(\xi,gh) = \lambda_L \widehat{L}_{\pi_G(\xi)gh}\Big(&\mathrm{Ad}^*_h\,\theta(g^{-1}) + \theta(h^{-1})\\ &- \theta(h^{-1}g^{-1})\Big)\\ = 0\end{aligned}$$

for the same reason. The map Φ^R is therefore an action on the right of G on T^*G.

Let $X \in \mathcal{G}$ and $\xi = T^*G$. By calculating the derivative with respect to t of $\Phi^L\big(\exp(tX),\xi\big)$ and of $\Phi^R\big(\xi,\exp(tX)\big)$, then setting $t = 0$, we get the following expressions for the fundamental vector fields on T^*G associated to the actions Φ^L and Φ^R:

$$\begin{aligned}X^{L,\lambda_R}_{T^*G}(\xi) &= X^L_{T^*G}(\xi) + \lambda_R T\widehat{R}_{\pi_G(\xi)}\Theta(X)\,,\\ X^{R,\lambda_L}_{T^*G}(\xi) &= X^R_{T^*G}(\xi) - \lambda_L T\widehat{L}_{\pi_G(\xi)}\Theta(X)\,,\end{aligned}$$

the vector fields $X^L_{T^*G}$ and $X^R_{T^*G}$ being, as in the proof of 4.5.3, the canonical lifts to T^*G of the fundamental vector fields X^L and X^R on G, for the actions of G on itself by translations on the left and on the right, respectively. Using these expressions, we easily check that

$$\mathrm{i}(X^{L,\lambda_R}_{T^*G})\omega_{T^*G} = -\mathrm{d}J^{L,\lambda_L}_X\,, \quad \mathrm{i}(X^{R,\lambda_L}_{T^*G})\omega_{T^*G} = -\mathrm{d}J^{R,\lambda_R}_X\,,$$

which means that the actions Φ^L and Φ^R are Hamiltonian and have, respectively, J^{L,λ_L} and J^{R,λ_R} as momentum maps.

The facts that J^{R,λ_R} is constant on each orbit of Φ^L and that J^{L,λ_L} is constant on each orbit of Φ^R directly follow from the expressions of Φ^L, Φ^R, J^{L,λ_L} and J^{R,λ_R}.

Finally, let X and $Y \in \mathcal{G}$. When considered as linear functions on $\mathcal{G}^*$, their Poisson bracket for the Poisson structure on $\mathcal{G}^*$ for which J^{L,λ_L} is a Poisson map is easily determined by calculating the Poisson bracket $\{J^{L,\lambda_L}\circ X, J^{L,\lambda_L}\circ Y\} = \{J^{L,\lambda_L}_X, J^{L,\lambda_L}_Y\}$, for the Poisson structure on T^*G associated to the symplectic form ω_{T^*G}. This calculation fully determines the Poisson structure on $\mathcal{G}^*$ for which J^{L,λ_L} is a Poisson map, and proves that it is indeed the canonical Poisson structure on T^*G modified by the symplectic cocycle $(\lambda_L+\lambda_R)\widetilde{\Theta}$, in the sense of 3.4.3. A similar calculation shows that J^{R,λ_R} is a Poisson map when $\mathcal{G}^*$ is equipped with the opposite Poisson structure. □

Proposition 4.5.5. *Under the assumptions and with the notations of 4.5.4, the momentum map $J^{L,\lambda_L} : T^*G \to \mathcal{G}^*$ is equivariant when G acts on the left on T^*G by the action Φ^L and on $\mathcal{G}^*$ by the action*

$$(g,\zeta) \mapsto \mathrm{Ad}^*_{g^{-1}}(\zeta) + \theta(g)\,, \quad (g,\zeta) \in G\times\mathcal{G}^*\,.$$

Similarly, the momentum map $J^{R,\lambda_R} : T^*G \to \mathcal{G}^*$ is equivariant when G acts on the right on T^*G by the action Φ^R and on $\mathcal{G}^*$ by the action

$$(\zeta,g) \mapsto \mathrm{Ad}^*_g(\zeta) + \theta(g^{-1})\,, \quad (\zeta,g) \in \mathcal{G}^*\times G\,.$$

Proof. Let $g \in G$ and $\xi \in T^*G$. Using the expressions of J^{L,λ_L} and of Φ^L, we obtain

$$J^{L,\lambda_L}\big(\Phi^L(g,\xi)\big) = \mathrm{Ad}^*_{g^{-1}}\big(J^{L,\lambda_L}(\xi)\big) + (\lambda_L+\lambda_R)\theta(g)\,,$$
$$J^{R,\lambda_R}\big(\Phi^R(\xi,g)\big) = \mathrm{Ad}^*_{g}\big(J^{R,\lambda_R}(\xi)\big) + (\lambda_L+\lambda_R)\theta(g^{-1})\,,$$

which proves that J^{L,λ_L} and J^{R,λ_R} are equivariant with respect to the indicated actions, respectively on the left and on the right, of G on T^*G and on $\mathcal{G}^*$. □

5 Reduction of Hamiltonian systems with symmetries

Very early, many scientists (Lagrange, Jacobi, Poincaré, ...) used first integrals to facilitate the determination of integral curves of Hamiltonian systems. It was observed that the knowledge of *one* real-valued first integral often allows the reduction by *two* units of the dimension of the phase space in which solutions are searched for.

J. Sniatycki and W. Tulczyjew [59] and, when first integrals come from the momentum map of a Lie group action, K. Meyer [53], J. Marsden and A. Weinstein [52], developed a geometric presentation of this reduction procedure, widely known now under the name "Marsden-Weinstein reduction".

Another way in which symmetries of a Hamiltonian system can be used to facilitate the determination of its integral curves was discovered around 1750 by Leonard Euler (1707–1783) when he derived the equations of motion of a rigid body around a fixed point. In a short Note published in 1901 [57], Henri Poincaré formalized and generalized this reduction procedure, often called today, rather improperly, "Lagrangian reduction" while the equations obtained by its application are called the "Euler-Poincaré equations" [11, 12].

We present in the following sections these two reduction procedures.

5.1 The Marsden-Weinstein reduction procedure

Theorem 5.1.1. *Let (M,ω) be a connected symplectic manifold on which a Lie group G acts by a Hamiltonian action Φ, with a momentum map $J : M \to \mathcal{G}^*$. Let $\xi \in J(M) \subset \mathcal{G}^*$ be a possible value of J. The subset G_ξ of elements $g \in G$ such that $\Phi_g\big(J^{-1}(\xi)\big) = J^{-1}(\xi)$ is a closed Lie subgroup of G.*

If in addition ξ is a weakly regular value of J in the sense of Bott [10], $J^{-1}(\xi)$ is a submanifold of M on which G_ξ acts, by the action Φ restricted to G_ξ and to $J^{-1}(\xi)$, in such a way that all orbits are of the same dimension. For each $x \in J^{-1}(\xi)$ the kernel of the two-form induced by ω on $J^{-1}(\xi)$ is the space tangent at this point to its G_ξ-orbit. Let $M_\xi = J^{-1}(\xi)/G_\xi$ be the set of all these orbits. When M_ξ has a smooth manifold structure for which the canonical projection $\pi_\xi : J^{-1}(\xi) \to M_\xi$ is a submersion, there exists on M_ξ a unique symplectic form ω_ξ such that $\pi_\xi^*\omega_\xi$ is the two-form induced on $J^{-1}(\xi)$ by ω. The symplectic manifold (M_ξ,ω_ξ) is called the reduced symplectic manifold *(in the sense of Marsden an Weinstein) for the value ξ of the momentum map.*

Proof. Proposition 4.4.9 shows that there exists an affine action a of G on $\mathcal{G}^*$ for which the momentum map J is equivariant. The subset G_ξ of G is therefore the isotropy subgroup of ξ for the action a, which proves that it is indeed a closed subgroup of G. A well known theorem due to Élie Cartan allows us to state that G_ξ is a Lie subgroup of G.

When ξ is a weakly regular value of J, $J^{-1}(\xi)$ is a submanifold of M and, for each $x \in J^{-1}(\xi)$, the tangent space at x to this submanifold is $\ker T_x J$ (it is the definition of a weakly regular value in the sense of Bott). Let $N = J^{-1}(\xi)$ and let $i_N : N \to M$ be the canonical injection. For all $x \in N$, the vector spaces $\ker T_x J$ all are of the same dimension $\dim N$, and $\dim\big(T_x J(T_x M)\big) = \dim M - \dim N$. Corollary 4.4.8 shows that $T_x J\big(T_x M\big)$ is the annihilator of $\mathcal{G}_x$. Therefore for all $x \in N$ the isotropy subalgebras $\mathcal{G}_x$ are of the same dimension $\dim G - \dim M + \dim N$. The G_ξ-orbits of all points $x \in N$ are all of the same dimension $\dim G_\xi - \dim G_x$.

Corollary 4.4.8 also shows that $\operatorname{orth}(\ker T_x J) = \operatorname{orth}(T_x N) = T_x\big(\Phi(G,x)\big)$. Therefore, for each $x \in N$,

$$\ker(i_N^*\omega)(x) = T_x N \cap \operatorname{orth}(T_x N) = T_x N \cap T_x\big(\Phi(G,x)\big) = T_x\big(\Phi(G_\xi,x)\big)\,.$$

It is indeed the space tangent at this point to its G_ξ-orbit. When $M_\xi = N/G_\xi$ has a smooth manifold structure such that the canonical projection $\pi_\xi : N \to M_\xi$ is a submersion, for each $x \in N$ the kernel of $T_x\pi_\xi$ is $\ker(i_N^*\omega)(x)$, and the existence on M_ξ of a symplectic form ω_ξ such that $\pi_\xi^*(\omega_\xi) = i_N^*\omega$ easily follows. □

Proposition 5.1.2. *The assumptions made here are the strongest of those made in Theorem 5.1.1: the set $J^{-1}(\xi)/G_\xi$ has a smooth manifold structure such that the canonical projection $\pi_\xi : J^{-1}(\xi) \to J^{-1}(\xi)/G_\xi$ is a submersion. Let $H : M \to \mathbb{R}$ be a smooth Hamiltonian, invariant under the action Φ. There exists an unique smooth function $H_\xi : M_\xi \to \mathbb{R}$ such that $H_\xi \circ \pi_\xi$ is equal to the restricton of H to $J^{-1}(\xi)$. Each integral curve $t \mapsto \varphi(t)$ of the Hamiltonian vector field X_H which meets $J^{-1}(\xi)$ is entirely contained in $J^{-1}(\xi)$, and in the reduced symplectic manifold (M_ξ, ω_ξ) the parametrized curve $t \mapsto \pi_\xi \circ \varphi(t)$ is an integral curve of X_{H_ξ}.*

Proof. As in the proof of Theorem 5.1.1, we set $N = J^{-1}(\xi)$ and denote by $i_N : N \to M$ the canonical injection. Let $\omega_N = i_N^*\omega$. Since H is invariant under the action Φ, it keeps a constant value on each orbit of G_ξ contained in N, so there exists on M_ξ an unique function H_ξ such that $H_\xi \circ \pi_\xi = H \circ i_N$. The projection π_ξ being a surjective submersion, H_ξ is smooth. Noether's theorem (4.4.5) proves that the momentum map J remains constant on each integral curve of the Hamiltonian vector field X_H. So if one of these integral curves meets N it is entirely contained in N, and we see that the Hamiltonian vector field X_H is tangent to N. We have, for each $x \in N$,

$$\begin{aligned}\pi_\xi^*\Big(\mathrm{i}\Big(T_x\pi_\xi\big(X_H(x)\big)\Big)\omega_\xi\big(\pi_\xi(x)\big)\Big) &= i\big(X_H(x)\big)\big(i_N^*\omega(x)\big) = -\mathrm{d}(i_N^*H)(x)\\ &= -\pi_\xi^*\big(\mathrm{d}H_\xi\big)(x) = \pi_\xi^*\big(\mathrm{i}(X_{H_\xi})\omega_\xi\big)(x)\,.\end{aligned}$$

Since π_ξ is a submersion and ω_ξ a non-degenerate two-form, this implies that for each $x \in N$, $T_x\pi_\xi\big(X_H(x)\big) = X_{H_\xi}\big(\pi_\xi(x)\big)$. The restriction of X_H to N and X_{H_ξ} are therefore two vector fields compatible with respect to the map $\pi_\xi : N \to M_\xi$, which implies the stated result. □

Remark 5.1.3. Theorem 5.1.1 and Proposition 5.1.2 still hold when instead of the Lie group action Φ we have an action φ of a finite-dimensional Lie algebra. The proof of the fact that the G_ξ-orbits in $J^{-1}(\xi)$ all are of the same dimension can easily be adapted to prove that for all $x \in J^{-1}(\xi)$, the vector spaces $\{\varphi(X)(x); X \in \mathcal{G}_\xi\}$ all are of the same dimension and determine a foliation of $J^{-1}(\xi)$. We have then only to replace the G_ξ-orbits by the leaves of this foliation.

5.1.4 Use of the Marsden-Weinstein reduction procedure

Theorem 5.1.1 and Proposition 5.1.2 are used to determine the integral curves of the Hamiltonian vector field X_H contained in $J^{-1}(\xi)$ in two steps:

- their projections on M_ξ are first determined: they are integral curves of the Hamiltonian vector field X_{H_ξ}; this step is often much easier than the full determination of the integral curves of X_H, since the dimension of the reduced symplectic manifold M_ξ is smaller than the dimension of M;
- then these curves themselves are determined; this second step, called *reconstruction*, involves the resolution of a differential equation on the Lie group G_ξ.

Many scientists (T. Ratiu, R. Cushman, J. Sniatycki, L. Bates, J.-P. Ortega, ...) generalized this reduction procedure in several ways: when M is a Poisson manifold instead of a symplectic manifold, when ξ is not a weakly regular value of J, ... The reader will find more results on the subject in the recent book *Momentum maps and Hamiltonian reduction* by J.-P. Ortega and T.S. Ratiu [56].

Reduced symplectic manifolds occur in many applications other than the determination of integral curves of Hamiltonian systems. The reader will find such applications in the book *Symplectic techniques in Physics* by V. Guillemin and S. Sternberg [26] and in the papers on the phase space of a particle in a Yang-Mills field [62, 65]).

5.2 The Euler-Poincaré equation

In his Note [57], Henri Poincaré writes the equations of motion of a Lagrangian mechanical system when a finite-dimensional Lie algebra acts on its configuration space by a locally transitive action. Below we adapt his results to the Hamiltonian formalism.

Proposition 5.2.1. *Let $\mathcal{G}$ be a finite-dimensional Lie algebra which acts, by an action $\varphi : \mathcal{G} \to A^1(N)$, on a smooth manifold N. The action φ is assumed to be locally transitive, which means that for each $x \in N$, $\{\varphi(X)(x); X \in \mathcal{G}\} = T_xN$. Let $\widehat{\varphi} : \mathcal{G} \to A^1(T^*N)$ be the Hamiltonian action of $\mathcal{G}$ on $(T^*N, \mathrm{d}\eta_N)$ which associates, to each $X \in \mathcal{G}$, the canonical lift to T^*N of the vector field $\varphi(X)$ on N (4.3.6), and let $J : T^*N \to \mathcal{G}^*$ be the the momentum map of $\widehat{\varphi}$ given by the formula (4.4.4)*

$$\big\langle J(\xi), X \big\rangle = \mathrm{i}\big(\widehat{\varphi}(X)\big)\eta_N(\xi), \quad X \in \mathcal{G},\ \xi \in T^*N.$$

Let $H : T^*N \to \mathbb{R}$ be a smooth Hamiltonian, which comes from a hyper-regular Lagrangian $L : TN \to \mathbb{R}$ (hyper-regular means that the associated Legendre map $\mathcal{L} : TN \to T^*N$is a diffeomorphism). Let $\psi : I \to T^*N$ be an integral curve of the Hamiltonian vector field X_H defined on an open interval I and $V : I \to \mathcal{G}$ be a smooth parametrized curve in $\mathcal{G}$ which satisfies, for each $t \in I$,

$$\varphi\big(V(t)\big)\big(\pi_N \circ \psi(t)\big) = \frac{d\big(\pi_N \circ \psi(t)\big)}{\mathrm{d}t}. \tag{1}$$

The curve $J \circ \psi : I \to \mathcal{G}^*$, obtained by composition with J of the integral curve ψ of the Hamiltonian vector field X_H, satisfies the differential equation in $\mathcal{G}^*$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{ad}^*_{V(t)}\right)\big(J \circ \psi(t)\big) = J\Big(d_1\overline{L}\big(\pi_N \circ \psi(t), V(t)\big)\Big). \tag{2}$$

We have denoted by $\overline{L} : N \times \mathcal{G} \to \mathbb{R}$ the map

$$(x, X) \mapsto \overline{L}(x, X) = L\big(\varphi(X)(x)\big), \quad x \in N,\ X \in \mathcal{G},$$

and by $d_1\overline{L} : N \times \mathcal{G} \to T^*N$ the partial differential of $\overline{L}$ with respect to its first variable.

Equation (2) *is called the* Euler-Poincaré equation*, while Equation* (1) *is called the* compatibility condition*.*

Proof. For each $\xi \in T^*N$ and each $X \in \mathcal{G}$

$$\big\langle J(\xi), X \big\rangle = \varphi(X)\big(\pi_N(\xi)\big)\,,$$

therefore

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\big(J\circ\psi(t)\big), X \Big\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big\langle \psi(t), \varphi(X)\big(\pi_N\circ\psi(t)\big)\Big\rangle\,.$$

Let $(x^1,\ldots,x^n)$ be local coordinates on N, and $(x^1,\ldots,x^n,p_1,\ldots,p_n)$ be the associated local coordinates on T^*N. The smooth curves ψ and $\pi_N\circ\psi$ can be expressed as

$$t \mapsto \big(y^i(t), \varpi_i(t)\big) \quad \text{and} \quad t \mapsto \big(y^i(t)\big)\,, \quad (1 \le i \le n)\,,$$

so we can write

$$\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\big(J\circ\psi(t)\big), X \Big\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n} \varpi_i(t)\big(\varphi(X)\big)^i\big(y^1(t),\ldots,y^n(t)\big)\right).$$

We have denoted by $\big(\varphi(X)\big)^i(x^1,\ldots,x^n)$ the value of the i-th component of the vector field $\varphi(X)$, expressed as a function of the local coordinates x^i $(1 \le i \le n)$.

The compatibility condition (1) becomes

$$\frac{\mathrm{d}y^k(t)}{\mathrm{d}t} = \Big(\varphi\big(V(t)\big)\Big)^k\big(y^1(t),\ldots,y^n(t)\big)\,.$$

In what follows we write y^i for $y^i(t)$, ϖ_i for $\varpi_i(t)$, (y) for $\big(y^1(t),\ldots,y^n(t)\big)$ and (y,ϖ) for $\big(y^1(t),\ldots,y^n(t),\varpi_1(t),\varpi_n(t)\big)$. We have

$$\begin{aligned}\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\big(J\circ\psi(t)\big), X \Big\rangle = {} & \sum_{i=1}^{n} \frac{\mathrm{d}\varpi_i}{\mathrm{d}t}\big(\varphi(X)\big)^i(y) \\ & + \sum_{(i,k)=(1,1)}^{(n,n)} \varpi_i \frac{\partial\big(\varphi(X)\big)^i(y)}{\partial x^k}\Big(\varphi\big(V(t)\big)\Big)^k(y)\,.\end{aligned}$$

By using the local expression of the bracket of vector fields

$$\begin{aligned}\Big[\varphi\big(V(t)\big), \varphi(X)\Big]^i(x) = {} & \sum_{k=1}^{n}\Big(\varphi\big(V(t)\big)\Big)^k(x)\frac{\partial\big(\varphi(X)\big)^i(x)}{\partial x^k} \\ & - \sum_{k=1}^{n}\big(\varphi(X)\big)^k(x)\frac{\partial\Big(\varphi\big(V(t)\big)\Big)^i(x)}{\partial x^k}\end{aligned}$$

and taking into account the fact that, φ being a Lie algebras homomorphism,

$$\Big[\varphi\big(V(t)\big), \varphi(X)\Big] = \varphi\Big(\big[V(t), X\big]\Big)\,,$$

we get

$$\begin{aligned}\Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\big(J\circ\psi(t)\big), X \Big\rangle = {} & \Big\langle \psi(t), \varphi\Big(\big[V(t), X\big]\Big)\Big\rangle \\ & + \sum_{i=1}^{n}\big(\varphi(X)\big)^i(y)\left(\frac{\mathrm{d}\varpi_i}{\mathrm{d}t} + \sum_{k=1}^{n}\varpi_k\frac{\partial\Big(\varphi\big(V(t)\big)\Big)^k(y)}{\partial x^i}\right).\end{aligned}$$

The first term in the right hand side can be written

$$\Big\langle \psi(t), \varphi\Big(\big[V(t),X\big]\Big)\Big\rangle = \Big\langle J\circ\psi(t), \big[V(t),X\big]\Big\rangle = \Big\langle \mathrm{ad}^*_{V(t)}\big(J\circ\psi(t)\big), X\Big\rangle .$$

For all $(x,X)\in N\times\mathcal{G}$ we have

$$\overline{L}(x,X) = L\big(\varphi(X)(x)\big) .$$

For any point $x\in N$ and any vector $w\in T_xN$, there exists a smooth curve $s\mapsto z(s)$ in N such that $z(0)=x$ and $\dfrac{\mathrm{d}z(s)}{\mathrm{d}s}\Big|_{s=0} = w$. We easily obtain $\Big\langle \mathrm{d}_1\overline{L}\big(x,V(t)\big), w\Big\rangle$ by taking the derivative of $\overline{L}\big(x(s),V(t)\big)$ with respect to s (t remaining fixed), then making $s=0$. We obtain

$$\Big\langle \mathrm{d}_1\overline{L}\big(x,V(t)\big), w\Big\rangle = \sum_{i=1}^{n} w^i \left(\frac{\partial L(x,v)}{\partial x^i} + \sum_{k=1}^{n} \frac{\partial L(x,v)}{\partial v^k} \frac{\partial\Big(\varphi\big(V(t)\big)\Big)^k(x)}{\partial x^i} \right) .$$

Let us set $x=\pi_N\circ\psi(t)$, $w=\varphi(X)\big(\pi_N\circ\psi(t)\big)$. We observe that

$$\Big\langle \mathrm{d}_1\overline{L}\big(\pi_N\circ\psi(t),V(t)\big), \varphi(X)\big(\pi_N\circ\psi(t)\big)\Big\rangle = \Big\langle J\Big(d_1\overline{L}\big(\pi_N\circ\psi(t),V(t)\big)\Big), X\Big\rangle .$$

Now we take into account the well known relations which exist between the partial derivatives of the Lagrangian and of the Hamiltonian expressed in local coordinates

$$\frac{\partial L(x,v)}{\partial x^i} = -\frac{\partial H(x,p)}{\partial x^i} = \frac{d\varpi_i}{dt}, \qquad \frac{\partial L(x,v)}{\partial v^k} = \varpi_k ,$$

and we obtain

$$\Big\langle J\Big(d_1\overline{L}\big(\pi_N\circ\psi(t),V(t)\big)\Big), X\Big\rangle = \sum_{i=1}^{n} \big(\varphi(X)\big)^i(y) \left(\frac{\mathrm{d}\varpi_i}{\mathrm{d}t} + \sum_{k=1}^{n} \varpi_k \frac{\partial\Big(\varphi\big(V(t)\big)\Big)^k(y)}{\partial x^i} \right) .$$

Since X can be any element in $\mathcal{G}$, the Euler-Poincaré equation follows. □

Remark 5.2.2. The assumptions made by Poincaré in [57] are less restrictive than those made in 5.2.1: he uses the Lagrangian formalism for a smooth Lagrangian $L : TN\to\mathbb{R}$ which is not assumed to be hyper-regular. The associated Legendre map $\mathcal{L} : TN\to T^*N$ still exists as a smooth map (it is the vertical differential of L, see for example [19]), but may not be a diffeomorphism. Of course the momentum map $J : T^*N\to\mathcal{G}^*$ still exists and can be used, together with the Legendre map, to express Poincaré's results intrinsically [51], independently of any choice of local coordinates. Poincaré proves that if a smooth parametrized curve $\gamma : [t_0,t_1]\to N$ is an extremal of the action functional

$$I(\gamma) = \int_{t_0}^{t_1} L\left(\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right) \mathrm{d}t$$

for infinitesimal variations of γ with fixed end points, and if $V : [t_0,t_1]\to\mathcal{G}$ is a smooth parametrized curve which satisfies, for each $t\in[t_0,t_1]$, the compatibility condition

$$\varphi\big(V(t)\big)\big(\gamma(t)\big) = \frac{d\big(\gamma(t)\big)}{\mathrm{d}t} , \tag{1}$$

the parametrized curve $t \mapsto J \circ \mathcal{L} \circ \dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t}$ satisfies the Euler-Poincaré equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{ad}^*_{V(t)}\right)\left(J \circ \mathcal{L} \circ \frac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right) = J\Big(d_1\overline{L}\big(\gamma(t), V(t)\big)\Big)\,. \tag{2}$$

The Euler-Poincaré equation can be written under a slightly different form in which, instead of the Legendre map $\mathcal{L} : TN \to T^*N$, the partial differential $\mathrm{d}_2\overline{L} : N \times \mathcal{G} \to \mathcal{G}^*$ of the map $\overline{L} : N \times \mathcal{G} \to \mathbb{R}$ with respect to its second variable is used. We have indeed, for all $x \in N$ and $X \in \mathcal{G}$,

$$\mathrm{d}_2\overline{L}(x, X) = J \circ \mathcal{L}\big(\varphi(X)(x)\big)\,,$$

which allows to write the Euler-Poincaré equation under the form

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{ad}^*_{V(t)}\right)\Big(\mathrm{d}_2\overline{L}\big(\gamma(t), V(t)\big)\Big) = J\Big(d_1\overline{L}\big(\gamma(t), V(t)\big)\Big)\,. \tag{3}$$

5.2.3 Use of the Euler-Poincaré equation for reduction

Poincaré observes in his Note [57] that the Euler-Poincaré equation can be useful mainly when its right hand side vanishes and when it reduces to an autonomous differential equation on $\mathcal{G}^*$ for the parametrized curve $t \mapsto J \circ \psi(t)$. We will see in Section 6.4 that the first condition is satisfied when the Hamiltonian system under consideration describes the motion of a rigid body around a fixed point in the absence of external forces (Euler-Poinsot problem). The second condition generally is not satisfied, since the Euler-Poincaré equation involves the parametrized curve $t \mapsto V(t)$ in $\mathcal{G}$, whose dependence on $J \circ \psi(t)$ is complicated.

This simplification occurs when there exists a smooth function $h : \mathcal{G}^* \to \mathbb{R}$ such that

$$H = h \circ J\,,$$

which implies that H is constant on each level set of J. Then it can be shown that the Euler-Poincaré equation becomes the Hamilton equation on $\mathcal{G}^*$ for the Hamiltonian h and its canonical Poisson structure.

If we assume that the manifold N is a Lie group G and that the action $\varphi : \mathcal{G} \to A^1(G)$ of its Lie algebra is the action associated to the action of G on itself by translations on the left (respectively, on the right), $\widehat{\varphi}$ is the Lie algebra action associated to the canonical lift to T^*G of the canonical action of G on itself by translations on the left (respectively, on the right). The conditions under which the Euler-Poincaré equation can be used for reduction are exactly the same as those under which the Marsden-Weinstein reduction method can be applied, but for the canonical lift to T^*G of the action of G on itself by translations *on the right* (respectively, on the left). Moreover, applications of these two reduction methods lead to essentially the same equations: the only difference is that the Euler-Poincaré reduction method leads to a differential equation on $\mathcal{G}^*$, while the Marsden-Weinstein reduction method leads, for each value of the momentum map, to the same differential equation restricted to a coadjoint orbit of $\mathcal{G}^*$. The reader will find the proof of these assertions in [48, 51].

6 Examples of Hamiltonian dynamical systems

We present in this section three classical examples of Hamiltonian dynamical systems in which the previously discussed concepts (symmetry groups, momentum maps and first integrals, reduction methods) are illustrated. The configuration space of the first system (the spherical pendulum)

is a sphere embedded in physical space; each point of the sphere is a possible position of a material point which moves on that sphere. The third example (the Kepler problem) deals with the motion of a material point in the acceleration field created by an attracting centre; the configuration space is the physical space minus one point (the attractive centre). In the second example (the motion of a rigid body around a fixed point) the configuration space is a little more complicated: it is the set of all maps which send the material body onto one of its possible positions in space.

6.1 The mathematical description of space and time

The framework in which the motions of material bodies occur is the physical space-time. It will be mathematically described here as it is usually done in classical (non-relativistic) Mechanics. In a Galilean reference frame, once units of length and of time are chosen, the physical space and the physical time are mathematically described by affine Euclidean spaces E and T, respectively three-dimensional and one-dimensional. We will consider E and T as *oriented*: T has a natural orientation (towards the future), while by convention, an arbitrary orientation of E is chosen. The choice of a particular element of T as origin will allow us to identify T with the real line $\mathbb{R}$.

In the three examples treated below there exists a privileged element of E (the centre of the sphere, the fixed point and the attractive centre, respectively in the first, second and third examples) which will be taken as origin. The space E will therefore be considered as an Euclidean three-dimensional *vector* space. For the same reason the abstract space S of material points used in the second example will be considered too as an Euclidean three-dimensional *vector* space.

In our three examples, the configuration space of the system will be denoted by N: therefore in the first example N is the sphere embedded in E centered on the origin on which the material point is moving; in the third example $N = E\backslash\{O\}$, where O is the attractive centre; and we will see that in the second example, $N = \mathrm{Isom}(S,E)$ is the space of orientation preserving linear isometries of an abstract three-dimensional Euclidean vector space S (the space of material points) onto the physical space E.

6.2 Vector calculus in a three-dimensional oriented Euclidean vector space

The group $\mathrm{SO}(E)$ of orientation preserving linear isometries of E, isomorphic to $\mathrm{SO}(3)$, acts on the space E, and so does its Lie algebra $\mathfrak{so}(E)$, isomorphic to $\mathfrak{so}(3)$, by the associated action. The Euclidean vector space E being three-dimensional and oriented, there exists an isomorphism of $\mathfrak{so}(E)$ onto the space E itself widely used in elementary vector calculus, in which an element $X \in \mathfrak{so}(E)$, which is a linear map $E \to E$ represented, in some orthonormal positively oriented basis $(\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3)$ of E, by the skew-symmetric 3×3-matrix

$$\begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

is identified with the vector $\overrightarrow{X} = a\overrightarrow{e}_1 + b\overrightarrow{e}_2 + c\overrightarrow{e}_3$. With this identification, the bracket in $\mathfrak{so}(E)$, in other words the map $(X,Y) \mapsto [X,Y] = X\circ Y - Y\circ X$, corresponds to the *vector product* $(\overrightarrow{X}, \overrightarrow{Y}) \mapsto \overrightarrow{X} \times \overrightarrow{Y}$. Expressed in terms of the vector product, the Jacobi identity becomes

$$\overrightarrow{X} \times (\overrightarrow{Y} \times \overrightarrow{Z}) + \overrightarrow{Y} \times (\overrightarrow{Z} \times \overrightarrow{X}) + \overrightarrow{Z} \times (\overrightarrow{X} \times \overrightarrow{Y}) = 0\,. \tag{1}$$

Let us recall another very useful formula which expresses, in terms of the scalar and vector products, the ad-invariance of the pairing between $\mathfrak{so}(E)$ and its dual by means of the scalar product. For any triple $(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}) \in E \times E \times E$, we have

$$\overrightarrow{u}.(\overrightarrow{v} \times \overrightarrow{w}) = \overrightarrow{v}.(\overrightarrow{w} \times \overrightarrow{u}) = \overrightarrow{w}.(\overrightarrow{u} \times \overrightarrow{v}). \tag{2}$$

The map $(\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}) \mapsto \overrightarrow{u}.(\overrightarrow{v} \times \overrightarrow{w})$ is therefore a skew-symmetric trilinear form on E sometimes called the *mixed product*.

The dual E^* of E will be identified with E, with the scalar product $(\overrightarrow{u}, \overrightarrow{v}) \mapsto \overrightarrow{u}.\overrightarrow{v}$ as pairing by duality. The tangent and cotangent bundles TE and T^*E will therefore both be identified with $E \times E$, the canonical projections $\tau_E : TE \to E$ and $\pi_E : T^*E \to E$ both being the projection of $E \times E$ onto its first factor. The Lie algebra action of $\mathfrak{so}(E)$ on E associates, to each $\overrightarrow{X} \in \mathfrak{so}(E) \equiv E$, the vector field $\overrightarrow{X_E}$ on E whose value at an element $\overrightarrow{x} \in E$ is

$$\overrightarrow{X_E}(\overrightarrow{x}) = (\overrightarrow{x}, \overrightarrow{X} \times \overrightarrow{x}). \tag{3}$$

Since we have identified $\mathfrak{so}(E)$ with E, its dual space $\mathfrak{so}(E)^*$ is identified with E^*, which we have identified with E by means of the scalar product. Therefore $\mathfrak{so}(E)^*$ too will be identified with E.

The canonical lift to the cotangent bundle of the action of $\mathrm{SO}(E)$ on E is a Hamiltonian action (4.3.6) whose momentum map $J_E : T^*E \equiv E \times E \to \mathfrak{so}(E)^* \equiv E$ can easily be expressed in terms of the vector product. Indeed the map J_E must satisfy, for each $\overrightarrow{X} \in \mathfrak{so}(E) \equiv E$ and each $(\overrightarrow{x}, \overrightarrow{p}) \in T^*E \equiv E \times E$,

$$\langle J_E(\overrightarrow{x}, \overrightarrow{p}), X \rangle = \langle (\overrightarrow{x}, \overrightarrow{p}), \overrightarrow{X_E}(\overrightarrow{x}) \rangle = \overrightarrow{p}.(\overrightarrow{X} \times \overrightarrow{x}) = \overrightarrow{X}.(\overrightarrow{x} \times \overrightarrow{p}),$$

the last equality being obtained by using the above formula (2). We therefore see that

$$J_E(\overrightarrow{x}, \overrightarrow{p}) = \overrightarrow{x} \times \overrightarrow{p}. \tag{4}$$

Expressed in terms of the vector product, the adjoint and coadjoint actions become

$$\mathrm{ad}_{\overrightarrow{X}} \overrightarrow{Y} = \overrightarrow{X} \times \overrightarrow{Y}, \quad \mathrm{ad}^*_{\overrightarrow{X}} \overrightarrow{\xi} = -\overrightarrow{X} \times \overrightarrow{\xi} = \overrightarrow{\xi} \times \overrightarrow{X}, \tag{5}$$

where $\overrightarrow{X}$ and $\overrightarrow{Y} \in \mathfrak{so}(E) \equiv E$ and $\overrightarrow{\xi} \in \mathfrak{so}(E)^* \equiv E$.

Of course all the above properties hold for the three-dimensional Euclidean oriented vector space S of material points which is used in the second example, for the group $\mathrm{SO}(S)$ of its linear orientation preserving isometries and for its Lie algebra $\mathfrak{so}(S)$.

6.3 The spherical pendulum

6.3.1 Mathematical description of the problem

Let us consider a heavy material point of mass m constrained, by an ideal constraint, on the surface of a sphere N of centre O and radius R embedded in the physical space E. Since the action of $\mathrm{SO}(E)$ on E maps N onto itself, $\mathrm{SO}(E)$ acts on N on the left, and so does its Lie algebra $\mathfrak{so}(E)$ by the associated action, which is locally (and globally) transitive. The configuration space

N is the set of vectors $\overrightarrow{x} \in E$ which satisfy $\overrightarrow{x} \cdot \overrightarrow{x} = R^2$ and its tangent bundle TN is the subset of $TE \equiv E \times E$ of pairs $(\overrightarrow{x}, \overrightarrow{v})$ of vectors which satisfy

$$\overrightarrow{x} \cdot \overrightarrow{x} = R^2, \quad \overrightarrow{x} \cdot \overrightarrow{v} = 0.$$

We assume that the material point is submitted to a constant acceleration field $\overrightarrow{g}$ (which, in most applications, will be the vertical gravity field directed downwards). The Lagrangian of the system is

$$L(\overrightarrow{x}, \overrightarrow{v}) = \frac{m\|\overrightarrow{v}\|^2}{2} + m\overrightarrow{g} \cdot \overrightarrow{x}.$$

The Legendre map $\mathcal{L} : TN \to T^*N$ is expressed as

$$\mathcal{L}(\overrightarrow{x}, \overrightarrow{v}) = (\overrightarrow{x}, \overrightarrow{p}) \quad \text{with } \overrightarrow{p} = m\overrightarrow{v}.$$

The Hamiltonian of the system is therefore

$$H(\overrightarrow{x}, \overrightarrow{p}) = \frac{\|\overrightarrow{p}\|^2}{2m} - m\overrightarrow{g} \cdot \overrightarrow{x}.$$

The momentum map J_E of the canonical lift to the cotangent bundle of the Lie algebra action φ, expressed in terms of the vector product, is given by Formula (4) in Section 6.2.

6.3.2 The Euler-Poincaré equation

The map $\widetilde{\varphi} : N \times \mathfrak{so}(E) \to TN$ defined by $\widetilde{\varphi}(\overrightarrow{x}, \overrightarrow{X}) = \overrightarrow{X_N}(\overrightarrow{x})$, expressed, in terms of the vector product, is $\widetilde{\varphi}(\overrightarrow{x}, \overrightarrow{X}) = (\overrightarrow{x}, \overrightarrow{X} \times \overrightarrow{x})$. Using Formula (2) of 6.2, we easily obtain the expression of $\overline{L} = L \circ \widetilde{\varphi} : N \times \mathfrak{so}(E) \to \mathbb{R}$:

$$\overline{L}(\overrightarrow{x}, \overrightarrow{X}) = \frac{mR^2}{2}\left(\|\overrightarrow{X}\|^2 - \frac{(\overrightarrow{X} \cdot \overrightarrow{x})^2}{R^2}\right) + m\overrightarrow{g} \cdot \overrightarrow{x}.$$

The partial differentals of $\overline{L}$ with respect to its first and second variables are

$$\mathrm{d}_1\overline{L}(\overrightarrow{x}, \overrightarrow{X}) = \left(\overrightarrow{x}, m\left(\overrightarrow{g} - (\overrightarrow{X} \cdot \overrightarrow{x})\overrightarrow{X} + \frac{(\overrightarrow{x} \cdot \overrightarrow{X})^2 - \overrightarrow{g} \cdot \overrightarrow{x}}{R^2}\overrightarrow{x}\right)\right),$$
$$\mathrm{d}_2\overline{L}(\overrightarrow{x}, \overrightarrow{X}) = m\left(R^2\overrightarrow{X} - (\overrightarrow{X} \cdot \overrightarrow{x})\overrightarrow{x}\right).$$

Let $t \mapsto \overrightarrow{x(t)}$ be a smooth curve in N, parametrized by the time t, solution of the Euler-Lagrange equation for the Lagrangian L. The compatibility condition (1) of 5.2.2 becomes, for a smooth map $t \mapsto V(t)$ in $\mathfrak{so}(E)$,

$$\frac{\mathrm{d}\overrightarrow{x(t)}}{\mathrm{d}t} = \overrightarrow{V}(t) \times \overrightarrow{x(t)},$$

and the Euler-Poincaré equation (3) of 5.2.2 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mR^2\overrightarrow{V}(t) - m\left(\overrightarrow{x(t)} \cdot \overrightarrow{V}(t)\right)\overrightarrow{x(t)}\right) = m\overrightarrow{x(t)} \times \overrightarrow{g}.$$

This equation can easily be obtained by much more elementary methods: it expresses the fact that the time derivative of the angular momentum at the origin is equal to the moment at that

point of the gravity force (since the moment at the origin of the constraint force which binds the material point to the surface of the sphere vanishes).

The Euler-Poincaré equation allows a reduction of the problem if and only if its right hand side vanishes, which occurs if and only if $\overrightarrow{g} = 0$. When that condition is satisfied, it can be written as

$$m\frac{\mathrm{d}}{\mathrm{d}t}\left(\overrightarrow{x(t)} \times \frac{\mathrm{d}\overrightarrow{x(t)}}{\mathrm{d}t}\right) = 0\,,$$

which implies that the material point moves on a great circle of the sphere N, in the plane through its centre orthogonal to the constant vector $\overrightarrow{x(t)} \times \dfrac{\mathrm{d}\overrightarrow{x(t)}}{\mathrm{d}t}$. Using the conservation of energy H, we see that $\| \overrightarrow{v}(t) \|$ remains constant during the motion.

6.3.3 Reduction by the use of first integrals

Equation (4) of Section 6.2 shows that the map

$$J(\overrightarrow{x}, \overrightarrow{p}) = \overrightarrow{x} \times \overrightarrow{p}\,, \quad \text{with } (\overrightarrow{x}, \overrightarrow{p}) \in T^*N \equiv N \times E\,,$$

is a momentum map for the canonical lift to T^*N of the action of $\mathrm{SO}(E)$ on N. When $\overrightarrow{g} \neq 0$ that action does not leave invariant the Hamiltonian H, but its restriction to the subgroup G_1 of rotations around the vertical line through the centre of the sphere N does leave H invariant. The Lie algebra of G_1 and its dual being identified with $\mathbb{R}$, the momentum map of this restricted action is

$$J_1(\overrightarrow{x}, \overrightarrow{p}) = \overrightarrow{e}_g.(\overrightarrow{x} \times \overrightarrow{p})$$

where $\overrightarrow{e}_g$ is the unit vector such that $\overrightarrow{g} = g\,\overrightarrow{e}_g$, with $g > 0$. The only singular value of J_1 is 0. It is reached when the three vectors $\overrightarrow{x}$, $\overrightarrow{p}$ and $\overrightarrow{e}_g$ lie in the same vertical plane. Therefore, for any $\zeta \neq 0$, $J_1^{-1}(\zeta)$ is a three-dimensional submanifold of T^*N which does not contain $T^*_{R\overrightarrow{e}_g} N \cup T^*_{-R\overrightarrow{e}_g} N$ and which remains invariant under the action of G_1. The set of orbits of this action is the Marsden-Weinstein reduced symplectic manifold for the value ζ of the momentum map. On this two-dimensional reduced symplectic manifold all the integral curves of the Hamiltonian vector field associated to the reduced Hamiltonian H_ζ are periodic.

6.4 The motion of a rigid body around a fixed point

6.4.1 Mathematical description of the problem

We consider the motion of a rigid body containing at least three non-aligned material points. A configuration of the body in space is mathematically represented by an affine, isometric and orientation preserving map defined on an abstract Euclidean three-dimensional oriented affine space S (called the *space of material points*), with values in E, the three-dimensional Euclidean oriented affine space which mathematically describes the physical space. When the configuration of the body is represented by the map $x : S \to E$, the position in space of the material point of the body represented by $z \in S$ is $x(z)$.

We assume that one geometric point of the rigid body is constrained, by an ideal constraint, to keep a fixed position in the physical space. By using this fixed point as origin, both for S and for E, we can now consider these spaces as *vector* spaces. Each configuration of the body in space

is therefore represented by a *linear* isometry. The set N of all possible configurations of the material body in space is therefore $\mathrm{Isom}(S,E)$, the set of linear orientation-preserving isometries of S onto E.

The Lie groups $\mathrm{SO}(S)$ and $\mathrm{SO}(E)$ of linear orientation-preserving isometries, respectively of S and of E, both isomorphic to $\mathrm{SO}(3)$, act on N, respectively on the left and on the right, by the two commuting actions Φ_S and Φ_E

$$\Phi_S(x,g_S) = x \circ g_S\,,\ \Phi_E(g_E,x) = g_E \circ x\,, \quad g_E \in \mathrm{SO}(E)\,,\ g_S \in \mathrm{SO}(S)\,,\ x \in N\,.$$

The values at $x \in N$ of the fundamental vector fields on N associated to $X \in \mathfrak{so}(S)$ and $Y \in \mathfrak{so}(E)$ are

$$X_N(x) = \frac{\mathrm{d}\big(x \circ \exp(sX)\big)}{ds}\Big|_{s=0}\,, \quad Y_N(x) = \frac{\mathrm{d}(\exp(sY) \circ x)}{ds}\Big|_{s=0}\,.$$

The Lie algebra actions $\varphi_S : \mathfrak{so}(S) \to A^1(N)$ and $\varphi_E : \mathfrak{so}(E) \to A^1(N)$ associated to the Lie group actions Φ_S and Φ_E are, respectively, the maps

$$\varphi_S(X) = X_N\,, \quad \varphi_E(Y) = Y_N\,, \quad X \in \mathfrak{so}(S)\,, \quad Y \in \mathfrak{so}(E)\,.$$

One should be careful with signs: since Φ_S is an action of $\mathrm{SO}(S)$ on the right, the bracket of elements in the Lie algebra $\mathfrak{so}(S)$ for which φ_S is a Lie algebras homomorphism is the bracket of *left-invariant* vector fields on the Lie group $\mathrm{SO}(S)$; similarly, since Φ_E is an action of $\mathrm{SO}(E)$ on the left, the bracket of elements in the Lie algebra $\mathfrak{so}(E)$ for which φ_E is a Lie algebras homomorphism is the bracket of *right-invariant* vector fields on the Lie group $\mathrm{SO}(E)$.

Let $\widetilde{\varphi}_S : N \times \mathfrak{so}(S) \to TN$ and $\widetilde{\varphi}_E : N \times \mathfrak{so}(E) \to TN$ be the vector bundles isomorphisms

$$\widetilde{\varphi}_S(x,X) = \varphi_S(X)(x)\,, \quad \widetilde{\varphi}_E(x,Y) = \varphi_E(Y)(x)\,.$$

Let $\Omega_S : TN \to \mathfrak{so}(S)$ and $\Omega_E : TN \to \mathfrak{so}(E)$ be the vector bundles maps ($\mathfrak{so}(S)$ and $\mathfrak{so}(E)$ being considered as trivial vector bundles over a base reduced to a singleton)

$$\Omega_S(v) = \pi_{\mathfrak{so}(S)} \circ \widetilde{\varphi}_S^{-1}(v)\,, \quad \Omega_E(v) = \pi_{\mathfrak{so}(E)} \circ \widetilde{\varphi}_E^{-1}(v)\,, \quad v \in TN\,,$$

where $\pi_{\mathfrak{so}(S)} : N \times \mathfrak{so}(S) \to \mathfrak{so}(S)$ and $\pi_{\mathfrak{so}(E)} : N \times \mathfrak{so}(E) \to \mathfrak{so}(E)$ are the projections of these two products on their respective second factor.

A *motion* of the rigid body during a time interval $[t_0,t_1]$ is mathematically described by a smooth parametrized curve $\gamma : [t_0,t_1] \to N$. In his beautiful paper [3], Vladimir Arnold clearly explained the physical meaning, for each $t \in [t_0,t_1]$, of $\dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t}$, $\Omega_S\left(\dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)$ and $\Omega_E\left(\dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)$:

- $\dfrac{d\gamma(t)}{\mathrm{d}t} \in T_{\gamma(t)}N$ is the value, at time t, of the *true angular velocity* of the body,
- $\Omega_S\left(\dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)$ is the value, at time t, of the *angular velocity of the body seen by an observer bound to the moving body* and moving with it,
- and $\Omega_E\left(\dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t}\right)$ is the value, at time t, of the *angular velocity of the body seen by an observer bound to the Galilean reference frame in which the motion is studied* and at rest with respect to that reference frame.

The following comments may be useful to explain Arnold's assertions. To shorten the notations, let us state, for some time $t \in [t_0, t_1]$, $x = \gamma(t) \in N$, $v = \dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \in T_xN$, $X = \Omega_S(v) \in \mathfrak{so}(S)$ and $Y = \Omega_E(v) \in \mathfrak{so}(E)$. We have

$$\widetilde{\varphi}_S(x,X) = \widetilde{\varphi}_E(s,Y) = v\,.$$

Let $z \in S$ be some material point of the moving body. Its position at time t is $x(z) \in E$ and its velocity is $\dfrac{\mathrm{d}}{\mathrm{d}t}\big(\gamma(t)(z)\big) \in T_{x(z)}E$. It depends only of $v = \dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t} \in T_xN$, not of the whole curve γ. We can therefore replace γ by the parametrized curve $s \mapsto \exp(sY) \circ x$, since we have

$$\frac{\mathrm{d}\big(\exp(sY)\circ x\big)}{\mathrm{d}s}\,\Big|_{s=0} = Y_N(x) = v\,.$$

Therefore the velocity at time t of the material point $z \in S$ is

$$\frac{\mathrm{d}\big(\exp(sY)\circ x(z)\big)}{\mathrm{d}s}\,\Big|_{s=0} = Y_E\big(x(z)\big) = \big(\overrightarrow{x(z)},\, \overrightarrow{Y} \times \overrightarrow{x(z)}\big) \in TE \equiv E \times E\,,$$

where we have denoted by Y_E the fundamental vector field on E associated to $Y \in \mathfrak{so}(E)$for the action Φ_E, and used Formula (3) of Section 6.2. This proves that the fundamental vector field Y_E is the velocity field of the rigid body as it appears in space E at time t, and explains why Arnold calls Y the angular velocity of the body seen by an observer bound to the Galilean frame in which the motion is studied.

Since $\dfrac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = \dfrac{\mathrm{d}\big(x\circ\exp(sX)\big)}{\mathrm{d}s}\,\Big|_{s=0}$, the value at $x(z) \in E$ of the fundamental vector field Y_E is also given by $\dfrac{\mathrm{d}\big(x\circ\exp(sX)(z)\big)}{\mathrm{d}s}\,\Big|_{s=0}$. The pull-back $x^*(Y_E)$ by the isomorphism $x : S \to E$ of the fundamental vector field Y_E, *i.e.* of the velocity field of the moving body in space E at time t, is the vector field on S whose value at $z \in S$ is

$$x^*(Y_E)(z) = \frac{\mathrm{d}\big(\exp(sX)(z)\big)}{\mathrm{d}s}\,\Big|_{s=0} = X_S(z) = (\,\overrightarrow{z}\,,\, \overrightarrow{X} \times \overrightarrow{z}\,) \in TS \equiv S \times S\,,$$

where we have again used Formula (3) of Section 6.2. The pull-back $x^*(Y_E)$ of the velocity field of the moving body in space E at time t, by the isomorphism $x : S \to E$, is therefore the fundamental vector field X_S associated to $X \in \mathfrak{so}(S)$, for the action Φ_S. That explains why Arnold calls X the angular velocity of the body seen by an observer bound to the moving body.

The above observations prove that for any $x \in N$ and $v \in T_xN$,

$$x\big(\overrightarrow{\Omega_S(v)}\big) = \overrightarrow{\Omega_E(v)}\,, \tag{1}$$

the arrows over $\Omega_S(v) \in \mathfrak{so}(S)$ and $\Omega_E(v) \in \mathfrak{so}(E)$ indicating that they are here considered as vectors in S and in E, respectively.

When the true angular velocity of the body is $v \in TN$, its *kinetic energy* is

$$\mathbb{T}(v) = \frac{1}{2} I\big(\Omega_S(v), \Omega_S(v)\big)\,,$$

where $I : \mathfrak{so}(S) \times \mathfrak{so}(S) \to \mathbb{R}$ is a symmetric, positive definite bilinear form which describes the *inertia properties* of the body. The assumed rigidity of the body is mathematically described by

the fact that the bilinear form I does not depend on time, nor on the configuration $\tau_N(v)$ of the body. Let us set, for each pair (v,w) of vectors in TN such that $\tau_N(v)=\tau_N(w)$,

$$\widetilde{\mathbb{T}}(v,w)=\frac{1}{2}I\big(\Omega_S(v),\Omega_S(w)\big)\,,\quad \text{so we can write } \mathbb{T}(v)=\widetilde{\mathbb{T}}(v,v)\,.$$

The symmetric bilinear form $\widetilde{\mathbb{T}}$ is a Riemannian metric on the manifold N. Let us consider the effects on $\widetilde{\mathbb{T}}$ of the canonical lifts to TN of the actions Φ_E and Φ_S on the manifold N. For each g_E in $\mathrm{SO}(E)$, $g_S\in\mathrm{SO}(S)$, we denote by $\Phi_{E\,g_E}:N\to N$ and by $\Phi_{S\,g_S}:N\to N$ the diffeomorphisms

$$\Phi_{E\,g_E}(x)=\Phi_E(g_E,x)=g_E\circ x\,,\quad \Phi_{S\,g_S}(x)=\Phi_S(x,g_S)=x\circ g_S\,,\quad x\in N\,.$$

For each $v\in TN$, with $\tau_N(v)=x\in N$, we have of course

$$\widetilde{\varphi}_S\big(x,\Omega_S(v)\big)=v\,.$$

Since the actions Φ_E and Φ_S commute we have, for any $g_E\in\mathrm{SO}(E)$, $v\in TN$, $t\in\mathbb{R}$ and $x=\tau_N(v)\in N$,

$$\Phi_E\Big(g_E,x\circ\exp\big(t\Omega_S(v)\big)\Big)=g_E\circ x\circ\exp\big(t\Omega_S(v)\big)=\Phi_S\Big(g_E\circ x,\exp\big(t\Omega_S(v)\big)\Big)\,.$$

By taking the derivative with respect to t, then setting $t=0$, we get

$$T\Phi_{E\,g_E}(v)=\widetilde{\varphi}_S\big(g_E\circ x,\Omega_S(v)\big)\,,$$

which means that

$$\Omega_S\big(T\Phi_{E\,g_E}(v)\big)=\Omega_S(v)\,.$$

The Riemannian metric $\widetilde{\mathbb{T}}$ therefore satisfies, for each $g_E\in\mathrm{SO}(E)$ and each pair (v,w) of vectors in TN which satisfy $\tau_N(v)=\tau_N(w)$,

$$\widetilde{\mathbb{T}}\big(T\Phi_{E\,g_E}(v),T\Phi_{E\,g_E}(w)\big)=\widetilde{\mathbb{T}}(v,w)\,.$$

This result means that the Riemannian metric $\widetilde{\mathbb{T}}$ remains invariant under the canonical lift to TN of the action Φ_E.

A similar calculation, in which $g_E\in\mathrm{SO}(E)$ is replaced by $g_S\in\mathrm{SO}(S)$, proves that, for each $v\in TN$,

$$\Omega_S\big(T\Phi_{S\,g_S}(v)\big)=\mathrm{Ad}_{g_S^{-1}}\big(\Omega_S(v)\big)\,,$$

so we have, for v and $w\in TN$ satisfying $\tau_N(v)=\tau_N(w)$,

$$\widetilde{\mathbb{T}}\big(T\Phi_{S\,g_S}(v),T\Phi_{S\,g_S}(w)\big)=\frac{1}{2}I\big(\mathrm{Ad}_{g_S^{-1}}\circ\Omega_S(v),\mathrm{Ad}_{g_S^{-1}}\circ\Omega_S(w)\big)\,.$$

For a general rigid body, the kinetic energy $\mathbb{T}$ and the Riemannian metric $\widetilde{\mathbb{T}}$ do not remain invariant under the canonical lift to TN of the action Φ_S. However, let us define an action on the left of G_S on the vector space of bilinear forms forms on $\mathfrak{so}(S)$ by setting, for each such bilinear form B and each $g_S\in G_S$

$$(g_S.B)(X_S,Y_S)=B\big(\mathrm{Ad}_{g_S^{-1}}(X_S),\mathrm{Ad}_{g_S^{-1}}(Y_S)\big)\,,\quad X_S \text{ and } Y_S\in\mathfrak{so}(S)\,.$$

We see that the kinetic energy $\mathbb{T}$ and the Riemannian metric $\widetilde{\mathbb{T}}$ remain invariant under the action of an element $g_S\in G_S$ if and only if $g_S.I=I$, *i.e.* if and only if g_S is an element of the isotropy

subgroup if I for the above defined action of G_S on the space of bilinear forms on $\mathfrak{so}(S)$. This happens, for example, when the body has a symmetry axis, the isotropy subgroup of I being the group of rotations around that axis.

When the configuration of the body is $x \in N$, its *potential energy* is

$$U(x) = -\left\langle P, x(\overrightarrow{a})\right\rangle,$$

where $\overrightarrow{a} \in S$ is the vector whose origin is the fixed point O_S and extremity the centre of mass of the body, and $P \in E^*$ is the gravity force. Since E is identified with its dual E^*, the pairing by duality being the scalar product, P can be seen as a fixed vertical vector $\overrightarrow{P} \in E$ directed downwards, equal to the weight of the body (product of its mass with the gravity acceleration), and the potential energy can be written

$$U(x) = -\overrightarrow{P}.\overrightarrow{x(a)} = -\overrightarrow{x^{-1}(P)}.\overrightarrow{a},$$

where we have written $\overrightarrow{x(a)}$ for $x(\overrightarrow{a})$ and $\overrightarrow{x^{-1}(P)}$ for $x^{-1}(\overrightarrow{P})$ to stress the fact that they are vectors, elements of E and of S, respectively. We also used the fact that the transpose $x^T : E^* \to S^*$ of the orthogonal linear map $x : S \to E$ is expressed, when S and E are identified with their dual spaces by means of the scalar product, as $x^{-1} : E \to S$.

When either $\overrightarrow{a} = 0$ or $\overrightarrow{P} = 0$ the potential energy vanishes, therefore remains invariant under the actions Φ_E of G_E and Φ_S of G_S on the manifold N. When both $\overrightarrow{a} \neq 0$ and $\overrightarrow{P} \neq 0$, the above formulae show that the potential energy remains invariant by the action of an element $g_E \in \mathrm{SO}(E)$ if and only if $\overrightarrow{g_E(P)} = \overrightarrow{P}$, which means if and only if g_E is an element of the isotropy group of $\overrightarrow{P}$ for the natural action of G_E on E. This isotropy subgroup is the group of rotations of E around the vertical straight line through the fixed point. Simlilarly, the potential energy remains invariant by the action of an element $g_S \in \mathrm{SO}(S)$ if and only if $\overrightarrow{g_S(a)} = \overrightarrow{a}$, which means if and only if g_S is an element of the isotropy group of $\overrightarrow{a}$ for the natural action of G_S on S. This isotropy subgroup is the group of rotations of S around the straight line which joins the fixed point and the centre of mass of the body.

The motion of the rigid body can be mathematically described by a Lagrangian system whose Lagrangian $L : TN \to \mathbb{R}$ is given, for $v \in TN$, by

$$L(v) = \widetilde{\mathbb{T}}(v,v) + \overrightarrow{P}.\overrightarrow{\tau_N(v)(a)} = \widetilde{\mathbb{T}}(v,v) + \overrightarrow{\left(\tau_N(v)\right)^{-1}(P)}.\overrightarrow{a}.$$

We denote by $\widetilde{\mathbb{T}}^\flat : TN \to T^*N$ the map determined by the equality, in which v and $w \in TN$ satisfy $\tau_N(v) = \tau_N(w)$,

$$\left\langle \widetilde{\mathbb{T}}^\flat(v), w\right\rangle = \widetilde{\mathbb{T}}(v,w).$$

The Legendre map $\mathcal{L} : TN \to T^*N$ determined by the Lagrangian L is

$$\mathcal{L} = 2\widetilde{\mathbb{T}}^\flat.$$

Its linearity and the positive definiteness of I ensure that it is a vector bundles isomorphism. The motion of the rigid body can therefore be described by a Hamiltonian system whose Hamiltonian $H : T^*N \to \mathbb{R}$ is given, for $p \in T^*N$, by

$$H(p) = \frac{1}{4}\left\langle p, (\widetilde{\mathbb{T}}^\flat)^{-1}(p)\right\rangle - \overrightarrow{P}.\overrightarrow{\pi_N(p)(a)} = \frac{1}{4}\left\langle p, (\widetilde{\mathbb{T}}^\flat)^{-1}(p)\right\rangle - \overrightarrow{\left(\pi_N(p)\right)^{-1}(P)}.\overrightarrow{a}.$$

6.4.2 The Hamiltonian in terms of momentum maps

Let $x \in N$ be fixed. The maps

$$\Omega_{Sx} = \Omega_S \big|_{T_xN} : T_xN \to \mathfrak{so}(S) \quad \text{and} \quad \Omega_{Ex} = \Omega_E \big|_{T_xN} : T_xN \to \mathfrak{so}(E)$$

are vector spaces isomorphisms. Their transpose

$$\Omega_{Sx}^T : \mathfrak{so}(S)^* \to T_x^*N \quad \text{and} \quad \Omega_{Ex}^T : \mathfrak{so}(E)^* \to T_x^*N$$

are too vector spaces isomorphisms. Their inverses are closely linked to the momentum maps $J_S : T^*N \to \mathfrak{so}(S)^*$ and $J_E : T^*N \to \mathfrak{so}(E)^*$ of the canonical lifts to T^*N of the actions Φ_S of G_S and Φ_E of G_E, respectively, on the manifold N. We have indeed, for any $x \in N$,

$$J_S \big|_{T_x^*N} = (\Omega_{Sx}^T)^{-1}, \quad J_E \big|_{T_x^*N} = (\Omega_{Ex}^T)^{-1}.$$

As above, let $x \in E$ be fixed and let v and $w \in T_xN$. The Legendre map $\mathcal{L} : TN \to T^*N$ satisfies

$$\begin{aligned} \big\langle \mathcal{L}(v), w \big\rangle &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s} I\big(\Omega_S(v+sw), \Omega_S(v+sw)\big) \big|_{s=0} \\ &= I\big(\Omega_S(v), \Omega_S(w)\big) \\ &= \big\langle I^\flat \circ \Omega_S(v), \Omega_S(w) \big\rangle \\ &= \big\langle \Omega_{Sx}^T \circ I^\flat \circ \Omega_S(v), w \big\rangle, \end{aligned}$$

where $I^\flat : \mathfrak{so}(S) \to \mathfrak{so}(S)^*$ is the map defined by

$$\big\langle I^\flat(X_S), Y_S \big\rangle = I(X_S, Y_S), \quad X_S \text{ and } Y_S \in \mathfrak{so}(S).$$

So we can write

$$\mathcal{L} \big|_{T_xN} = \Omega_{Sx}^T \circ I^\flat \circ \Omega_S \big|_{T_xN},$$

which shows that the momentum map J_S composed with the Legendre map $\mathcal{L}$ has the very simple expression

$$J_S \circ \mathcal{L} = I^\flat \circ \Omega_S.$$

The momentum map J_E composed with $\mathcal{L}$ has a slightly more complicated expression, valid for each $x \in N$,

$$J_E \circ \mathcal{L} \big|_{T_xN} = (\Omega_{Ex}^T)^{-1} \circ \Omega_{Sx}^T \circ I^\flat \circ \Omega_S \big|_{T_xM}.$$

Let $I^* : \mathfrak{so}(S)^* \times \mathfrak{so}(S)^* \to \mathbb{R}$ be the symmetric, positive definite bilinear form on $\mathcal{G}^*$

$$I^*(\xi, \eta) = I\big((I^\flat)^{-1}(\xi), (I^\flat)^{-1}(\eta)\big) = \big\langle \xi, (I^\flat)^{-1}(\eta) \big\rangle = \big\langle \eta, (I^\flat)^{-1}(\xi) \big\rangle.$$

The above expression of $J_S \circ \mathcal{L}$ and the bilinear form I^* allow us to write the Hamiltonian H as

$$H(p) = \frac{1}{2} I^*\big(J_S(p), J_S(p)\big) - \overrightarrow{\pi_N(p)^{-1}(P).\, a}, \quad p \in T^*N.$$

Although the kinetic energy remains invariant under the canonical lift to T^*N of the action Φ_E, the expression of H in terms of the other momentum map J_E is too complicated to be useful.

6.4.3 The Euler-Poincaré equation

We use the vector bundles isomorphism $\widetilde{\varphi}_S : N \times \mathfrak{so}(S) \to TN$ to derive the Euler-Poincaré equation. The map $\overline{L} = L \circ \widetilde{\varphi}_S : N \times \mathfrak{so}(S) \to \mathbb{R}$ is

$$\overline{L}(x,X) = \frac{1}{2} I(X,X) + \overrightarrow{P} \cdot \overrightarrow{x(a)}, \quad X \in \mathfrak{so}(S),\ x \in N.$$

Its partial differential $\mathrm{d}_2\overline{L}$ with respect to its second variable is

$$\mathrm{d}_2\overline{L}(x,X) = I^\flat(X) \in \mathfrak{so}(S)^* \equiv S.$$

A calculation similar to those of Section 6.4.2 leads to the following expression of J_S composed with the partial differential of $\overline{L}$ with respect to its first variable:

$$J_S \circ \mathrm{d}_1\overline{L}(x,X) = \overrightarrow{a} \times \overrightarrow{x^{-1}(P)} \in \mathfrak{so}(S)^* \equiv S.$$

Let $t \mapsto x(t)$ be a smooth curve in N solution of the Euler-Lagrange equation for the Lagrangian L, and $t \mapsto X(t)$ a smooth curve in $\mathfrak{so}(S)$ which satisfies the compatibility condition (1) of 5.2.2

$$\frac{\mathrm{d}x(t)}{dt} = \widetilde{\varphi}_S\big(x(t), X(t)\big). \tag{1}$$

The Euler-Poincaré equation (3) of 5.2.2, satisfied by the smooth curve $t \mapsto \big(x(t), X(t)\big)$ in $N \times \mathfrak{so}(S)$, is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{ad}^*_{X(t)}\right)\Big(I^\flat\big(X(t)\big)\Big) = \overrightarrow{a} \times \overrightarrow{x^{-1}(P)}.$$

Using the expression of ad^* given by Formula (4) of 6.2, we can write the Euler-Poincaré equation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{I^\flat\big(X(t)\big)} - \overrightarrow{I^\flat\big(X(t)\big)} \times \overrightarrow{X(t)} = \overrightarrow{a} \times \overrightarrow{P_S(t)}, \tag{2}$$

where we have set $\overrightarrow{P_S(t)} = \overrightarrow{x(t)^{-1}(P)}$. The physical meaning of the quantities which appear in this equation is the following: $\overrightarrow{X(t)}$ is the angular velocity, $\overrightarrow{I^\flat\big(X(t)\big)}$ the angular momentum and $\overrightarrow{P_S(t)}$ the weight of the moving body, all three a time t and seen by an observer bound to the body, therefore considered as vectors in S. We recognize the classical *Euler equation* for the motion of a rigid body around a fixed point.

Of course $x(t)\big(\overrightarrow{P_S(t)}\big) = \overrightarrow{P}$ is a constant vector in E, therefore

$$\frac{\mathrm{d}\Big(x(t)\big(\overrightarrow{P_S(t)}\big)\Big)}{\mathrm{d}t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\big(\overrightarrow{P_S(t)}\big) + x(t)\left(\frac{\mathrm{d}\overrightarrow{P_S(t)}}{\mathrm{d}t}\right) = 0.$$

The first term in the right hand side, $\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}\big(\overrightarrow{P_S(t)}\big)$, is the value at $\overrightarrow{P} \in E$ of the velocity field in E of the moving body. Therefore

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}\big(\overrightarrow{P_S(t)}\big) = \overrightarrow{\Omega_E(v)} \times \overrightarrow{P}, \quad \text{with } v = \frac{\mathrm{d}x(t)}{\mathrm{d}t} \in T_{x(t)}N.$$

Therefore we have

$$\frac{\mathrm{d}\overrightarrow{P_S(t)}}{\mathrm{d}t} = -x(t)^{-1}\Big(\overrightarrow{\Omega_E(v)} \times \overrightarrow{P}\Big) = -\overrightarrow{X(t)} \times \overrightarrow{P_S(t)},$$

since, by Formula (1) of 6.4.1, $x(t)^{-1}\left(\overrightarrow{\Omega_E(v)}\right) = \overrightarrow{\Omega_S(v)} = \overrightarrow{X(t)}$. The compatibility condition and the Euler-Poincaré equation (Equations (1) and (2) of this Section) have lead us to the differential equation on $S \times S$, for the unknown parametrized curve $t \mapsto \left(\overrightarrow{X(t)}, \overrightarrow{P_S(t)}\right)$,

$$\begin{cases} \dfrac{\mathrm{d}\overrightarrow{I^\flat\left(X(t)\right)}}{\mathrm{d}t} = \overrightarrow{I^\flat\left(X(t)\right)} \times \overrightarrow{X(t)} + \overrightarrow{a} \times \overrightarrow{P_S(t)}, \\ \dfrac{\mathrm{d}\overrightarrow{P_S(t)}}{\mathrm{d}t} = -\overrightarrow{X(t)} \times \overrightarrow{P_S(t)}. \end{cases} \tag{3}$$

When the right hand side of equation (2) vanishes, which occurs when the fixed point is the centre of mass of the body ($\overrightarrow{a} = 0$) or when there is no gravity field ($\overrightarrow{P} = 0$), the Euler-Poincaré equation yields an important reduction, since the first equation of (3) becomes an autonomous differential equation on the three-dimensional vector space S for the smooth curve $t \mapsto \overrightarrow{I^\flat\left(X_S(t)\right)}$, while the Euler-Lagrange equation or the Hamilton equation live on the six-dimensional manifolds TN or T^*N, respectively. Under these assumptions, the study of all possible motions of the rigid body is is known in Mechanics as the *Euler-Poinsot problem*. The reader will find in [14] a very nice and thorough geometric presentation of the phase portrait of this problem.

When $\overrightarrow{a} \neq 0$ and $\overrightarrow{P} \neq 0$, the first equation of (3) is no more autonomous: one has to solve (3) on the six-dimensional vector space $S \times S$. The use of the Euler-Poincaré equation does not allow a reduction of the dimension of the phase space, but (3) may be easier to solve than the Euler-Lagrange equation or the Hamilton equation, because it lives on a vector space instead of on the tangent or cotangent bundle to a manifold.

6.4.4 Use of the Lie algebra of Euclidean displacements

As explained for example in Theorem 4.1 of [48] or in Proposition 13 and Example 14 of [51], there exists a Hamiltonian action on T^*N of the semi-direct product $G_S \times S$, (the group of Euclidean displacements, generated by rotations and translations, of the Euclidean affine space S) which extends the canonical lift to T^*N of the action Φ_S, such that the Hamiltonian H can be expressed as composed of the momentum map of that action with a smooth function $h : \mathfrak{so}(S)^* \times S^* \to \mathbb{R}$. We briefly explain below the construction of that action.

For each $\overrightarrow{b} \in S$, let $f_{\overrightarrow{b}} : N \to \mathbb{R}$ be the smooth function

$$f_{\overrightarrow{b}}(x) = \left\langle x^T(P), \overrightarrow{b} \right\rangle = \overrightarrow{b} . \overrightarrow{x^{-1}(P)}, \quad x \in N.$$

The map $\Psi : T^*N \times S \to T^*N$ defined by

$$\Psi(p, \overrightarrow{b}) = p - \mathrm{d}f_{\overrightarrow{b}} \circ \pi_N(p), \quad p \in T^*N, \ \overrightarrow{b} \in S,$$

is a Hamiltonian action of S on the symplectic manifold $(T^*N, \mathrm{d}\eta_N)$: the Lie algebra of S can indeed be identified with S, the exponential map becoming the identity of S, and for each $\overrightarrow{b} \in S$, the vector field on T^*N whose flow is the one-parameter group of transformations of T^*N

$$\left\{ p \mapsto p - t\mathrm{d}f_{\overrightarrow{b}}\left(\pi_N(p)\right) ; t \in \mathbb{R} \right\}$$

is Hamiltonian an admits as Hamiltonian the function

$$p \mapsto f_{\overrightarrow{b}} \circ \pi_N(p) == \overrightarrow{b} . \overrightarrow{\pi_N(p)^{-1}(P)}, \quad p \in T^*N.$$

This formula proves that Ψ is a Hamiltonian action which admits

$$J_\Psi : T^*N \to S^* \equiv S\,, \quad J_\Psi(p) = \overrightarrow{\pi_N(p)^{-1}(P)}$$

as a momentum map. Gluing together Ψ with the canonical lift $\widehat{\Phi}_S$ of Φ_S to the cotangent bundle, we obtain a Hamiltonian action on the right Ξ of the semi-direct product $G_S \times S$ on the symplectic manifold $(T^*N, \mathrm{d}\eta_N)$:

$$\Xi\big(p, (g_S, \overrightarrow{b})\big) = \Psi(\widehat{\Phi}_S(p, g_S), \overrightarrow{b})$$

with $(J_S, J_\psi) : T^*N \to \mathfrak{so}(S)^* \times S^* \equiv S \times S$ as a momentum map. The function $h : \mathfrak{so}(S)^* \times S^* \equiv S \times S \to \mathbb{R}$

$$h(\overrightarrow{\eta}, \overrightarrow{\zeta}) = \frac{1}{2} I^*(\overrightarrow{\eta}, \overrightarrow{\eta}) - \overrightarrow{\zeta}\,\overrightarrow{\eta}$$

is such that the Hamiltonian $H : T^*N \to \mathbb{R}$ can be written as $H = h \circ (J_S, J_\Psi)$, and Equation (3) of Section 6.4.3 is the Hamilton equation on $\mathfrak{so}(S)^* \times S^* \equiv S \times S$ (endowed with its canonical Poisson structure) for the Hamiltonian h. This result is in agreement with the fact that (J_S, J_Ψ) is an ad^*-invariant Poisson map (4.4.11).

6.4.5 Reduction by the use of first integrals

The effects on the kinetic and potential energies of the Hamiltonian actions $\widehat{\Phi}_E$ and $\widehat{\Phi}_S$ were discussed in Section 6.4.1. When $\overrightarrow{a} \neq 0$ and $\overrightarrow{P} \neq 0$, the Hamiltonian H remains invariant under the restriction of the action $\widehat{\Phi}_E$ to the subgroup of rotations around the vertical axis through the fixed point. The corresponding momentum map, which is the orthogonal projection of the momentum map J_E on the vertical direction, is therefore a first integral. Another first integral is the total energy, *i.e.* the Hamiltonian H itself. For a general rigid body, no other independent first integrals are known. However, in two special cases of particular rigid bodies, there exists another independent first integral.

The first case, known as the *Euler-Lagrange problem* in Mechanics, is when the straight line which joins the fixed point and the centre of mass of the body is an axis of symmetry for the inertia properties of the body. The Hamiltonian H remains then invariant under the restriction of the action $\widehat{\Phi}_S$ to the subgroup of $\mathrm{SO}(S)$ of rotations around this straight line. The corresponding momentum map is the orthogonal projection of the momentum map J_S on the direction of the symmetry axis.

The second case, discovered by the Russian mathematican Sonya Kovalevskaya (1850–1891) [41] is when two of the principal moments of inertia of the body are equal to twice the third and when the centre of mass of the body lies in the plane of the two equal moments of inertia. The explanation of the existence, in this very special case, of an additional integral is much more complicated than that of the existence of an additional integral for the Euler-Lagrange problem, and involves mathematical tools which are not discussed in the present paper. The reader is referred to the book by Michèle Audin [6] for a discussion of these tools and to the beautiful other book by the same author [7] for a very moving presentation of the life of Sonya Kovalevskaya.

When $\overrightarrow{a} = 0$ or $\overrightarrow{P} = 0$ (the Euler-Poinsot problem) the Hamiltonian H remains invariant under the action $\widehat{\Phi}_E$ of the full group $SO(E)$, so the corresponding momentum map J_E is (as already seen in Section 6.4.3) a (vector valued) first integral.

6.5 The Kepler problem

6.5.1 Mathematical description of the problem

We consider the motion in space of a material point of mass m submitted to the gravitational field created by an attractive centre O. Taking O as origin allows us to consider E as a *vector* Euclidean three-dimensional oriented space. The configuration space, *i.e.* the set of all possible positions of the material point, is $N = E\backslash\{O\}$. The tangent bundle TN and the cotangent bundle T^*N will both be identified with $N \times E$. An element of TN is therefore a pair $(\overrightarrow{x}, \overrightarrow{v}) \in E \times E$ satisfying $\overrightarrow{x} \neq 0$. Similarly an element of T^*N is a pair $(\overrightarrow{x}, \overrightarrow{p}) \in E \times E$ satisfying $\overrightarrow{x} \neq 0$.

The kinetic energy $\mathbb{T} : TN \equiv N \times E \to \mathbb{R}$ and the potential energy $U : N \to \mathbb{R}$ are

$$\mathbb{T}(\overrightarrow{x}, \overrightarrow{v}) = \frac{1}{2} m \| \overrightarrow{v} \|^2, \quad U(\overrightarrow{x}) = \frac{mk}{\| \overrightarrow{x} \|}.$$

The Lagrangian $L : TN \equiv N \times E \to \mathbb{R}$ of the Kepler problem is therefore

$$L(\overrightarrow{x}, \overrightarrow{v}) = \frac{1}{2} m \| \overrightarrow{v} \|^2 + \frac{mk}{\| \overrightarrow{x} \|}.$$

The Legendre map $\mathcal{L} : TN \equiv N \times E \to T^*N \equiv N \times E$ is

$$\mathcal{L}(\overrightarrow{x}, \overrightarrow{v}) = (\overrightarrow{x}, \overrightarrow{p}), \quad \text{with } \overrightarrow{p} = m \overrightarrow{v}.$$

The Kepler problem can therefore be mathematically formulated as a Hamiltonian dynamical system on $T^*N \equiv N \times E$, with the Hamiltonian

$$H(\overrightarrow{x}, \overrightarrow{p}) = \frac{1}{2m} \| \overrightarrow{p} \|^2 - \frac{mk}{\| \overrightarrow{x} \|}.$$

The natural action Φ_E of $\mathrm{SO}(E)$ on E leaves invariant $N = E\backslash\{O\}$, therefore is an action of $\mathrm{SO}(E)$ on N. With the identifications of TN and T^*N with $N \times E$ which we have made, the canonical lifts $\overline{\Phi}_E$ and $\widehat{\Phi}_E$ of that action to the tangent and cotangent bundles, respectively, are expressed as

$$\overline{\Phi}_E\big(g_E, (\overrightarrow{x}, \overrightarrow{v})\big) = \big(g_E(\overrightarrow{x}), g_E(\overrightarrow{v})\big), \; \widehat{\Phi}_E\big(g_E, (\overrightarrow{x}, \overrightarrow{p})\big) = \big(g_E(\overrightarrow{x}), g_E(\overrightarrow{p})\big).$$

Since the norm of a vector in E remains invariant under the action Φ_E, the Lagrangian L and the Hamiltonian H remain invariant under the actions $\overline{\Phi}_E$ and $\widehat{\Phi}_E$, respectively. The action $\widehat{\Phi}_E$ is Hamiltonian, and we know (Formula (4) of 6.2) that its momentum map $J_E : T^*N \equiv N \times E \to \mathfrak{so}(E)^* \equiv E$ is

$$J_E(\overrightarrow{x}, \overrightarrow{p}) = \overrightarrow{x} \times \overrightarrow{p}.$$

The map J_E is the *angular momentum* of the moving material point with respect to the attractive centre. Noether's theorem (4.4.5) shows that it is a first integral of the Kepler problem. Another first integral of the Kepler problem is the total energy H, as shown by 3.3.4.

6.5.2 The Euler-Poincaré equation

The Lie group $\mathrm{SO}(E)$ does not act transitively on N by the action Φ_E, since the orbits of this action are spheres centered on O. However, extending this action by homotheties of strictly positive ratio, we obtain a transitive action on N of the direct product $\mathrm{SO}(E) \times]0, +\infty[$

$$\Psi_E\big((g_E, r), \overrightarrow{x}\big) = r g_E(\overrightarrow{x}), \quad g_E \in \mathrm{SO}(E), \; r \in]0, +\infty[, \; \overrightarrow{x} \in N.$$

Let $\psi_E : \mathfrak{so}(E) \times \mathbb{R} \to A^1(N)$ be the associated action of the Lie algebra $\mathfrak{so}(E) \times \mathbb{R}$. The map $\widetilde{\psi}_E : N \times \big(\mathfrak{so}(E) \times \mathbb{R}\big) \to TN$, $\widetilde{\psi}_E\big(\overrightarrow{x}, (\overrightarrow{X}, \lambda)\big) = \psi_E(\overrightarrow{X}, \lambda)(\overrightarrow{x})$, can be written, with the identifications of $\mathfrak{so}(E)$ with E and of TN with $N \times E$,

$$\widetilde{\psi}_E\big(\overrightarrow{x}, (\overrightarrow{X}, \lambda)\big) = \big(\overrightarrow{x}, \overrightarrow{X} \times \overrightarrow{x} + \lambda \overrightarrow{x}\big).$$

The function $\overline{L} = L \circ \widetilde{\psi}_E$ is therefore

$$\overline{L}\big(\overrightarrow{x}, (\overrightarrow{X}, \lambda)\big) = \frac{1}{2} m \| \overrightarrow{x} \|^2 \big(\| \overrightarrow{X} \|^2 + \lambda^2 \big) - \frac{1}{2} m (\overrightarrow{X} . \overrightarrow{x})^2 + \frac{mk}{\| \overrightarrow{x} \|}.$$

Its partial differentials $\mathrm{d}_1 \overline{L}$ and $\mathrm{d}_2 \overline{L}$ with respect to its first variable $\overrightarrow{x}$ and to its second variable $(\overrightarrow{X}, \lambda)$ are, with the identifications of E^* and $\big(\mathfrak{so}(E) \times \mathbb{R}\big)^*$ with, respectively, E and $E \times \mathbb{R}$,

$$\mathrm{d}_1 \overline{L}\big(\overrightarrow{x}, (\overrightarrow{X}, \lambda)\big) = \left(m(\| \overrightarrow{X} \|^2 + \lambda^2) - \frac{mk}{\| \overrightarrow{x} \|^3} \right) \overrightarrow{x} - m(\overrightarrow{X} . \overrightarrow{x}) \overrightarrow{X},$$
$$\mathrm{d}_2 \overline{L}\big(\overrightarrow{x}, (\overrightarrow{X}, \lambda)\big) = \big(m \| \overrightarrow{x} \|^2 \overrightarrow{X} - m(\overrightarrow{X} . \overrightarrow{x}) \overrightarrow{x}, m \| \overrightarrow{x} \|^2 \lambda \big).$$

The canonical lift $\widehat{\Psi}_E$ of Ψ_E to the cotangent bundle is a Hamiltonian action, whose momentum map $(J_E, K_E) : T^*N \to \mathfrak{so}(E)^* \times \mathbb{R}$ has J_E as first component. Its second component is

$$K_E(\overrightarrow{x}, \overrightarrow{p}) = \overrightarrow{x} . \overrightarrow{p}.$$

Let $t \mapsto \overrightarrow{x(t)}$ be a smooth curve in N, parametrized by the time t, solution of the Euler-Lagrange equation for the Lagrangian L. The compatibility condition (1) of 5.2.2, for a smooth map $t \mapsto \big(\overrightarrow{x(t)}, \lambda(t)\big)$ in $\mathfrak{so}(E) \times \mathbb{R}$, can be written as

$$\frac{\mathrm{d}\overrightarrow{x(t)}}{\mathrm{d}t} = \overrightarrow{x(t)} \times \overrightarrow{x(t)} + \lambda(t) \overrightarrow{x(t)}. \tag{1}$$

This equation does not involve the component of $\overrightarrow{x(t)}$ parallel to $\overrightarrow{x(t)}$, since the vector product of this component with $\overrightarrow{x(t)}$ vanishes.

The Euler-Poincaré equation (3) of 5.2.2 has now two components, on $\mathfrak{so}(E)^*$ and on $\mathbb{R}^*$ identified, respectively, with E and with $\mathbb{R}$. With the above expressions of $\mathrm{d}_1 \overline{L}$, $\mathrm{d}_2 \overline{L}$, J_E and K_E, we obtain for its first component

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(m \| \overrightarrow{x(t)} \|^2 \left(\overrightarrow{x(t)} - \frac{\overrightarrow{x(t)} . \overrightarrow{x(t)}}{\| \overrightarrow{x} \|^2} \overrightarrow{x(t)} \right) \right)$$
$$= m\big(\overrightarrow{x(t)} . \overrightarrow{x(t)}\big) \overrightarrow{x(t)} \times \overrightarrow{x(t)} - m\big(\overrightarrow{x(t)} . \overrightarrow{x(t)}\big) \operatorname{ad}^*_{\overrightarrow{x(t)}} \overrightarrow{x(t)} = 0,$$

where we have used Formula (4) of 6.2. Its second component is

$$\frac{\mathrm{d}}{\mathrm{d}t} \big(m \| \overrightarrow{x(t)} \|^2 \lambda \big) = m \left(\| \overrightarrow{x(t)} \|^2 + \lambda^2 - \frac{\big(\overrightarrow{x(t)} . \overrightarrow{x(t)} \big)^2}{\| \overrightarrow{x} \|^2} \right) \| \overrightarrow{x(t)} \|^2 - \frac{mk}{\| \overrightarrow{x(t)} \|}.$$

The vector $\overrightarrow{X(t)}$ is the sum of two components $\overrightarrow{X_1(t)}$ orthogonal to $\overrightarrow{x(t)}$ and $\overrightarrow{X_2(t)}$ parallel to $\overrightarrow{x(t)}$. Since

$$\overrightarrow{X_1(t)} = \overrightarrow{X(t)} - \frac{\overrightarrow{X(t)} . \overrightarrow{x(t)}}{\| \overrightarrow{x(t)} \|^2} \overrightarrow{x(t)},$$

the two components of the Euler-Poincaré equation become

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(m\|\overrightarrow{x(t)}\|^2\overrightarrow{X_1(t)}\right) = 0\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(m\|\overrightarrow{x(t)}\|^2\lambda\right) = m\left(\|\overrightarrow{X_1(t)}\|^2+\lambda^2\right)\|\overrightarrow{x(t)}\|^2 - \dfrac{mk}{\|\overrightarrow{x(t)}\|}\,. \end{cases}$$

The first equation expresses the fact that J_E is a first integral of the Kepler problem, since we have

$$m\|\overrightarrow{x(t)}\|^2\overrightarrow{X_1(t)} = \overrightarrow{x(t)}\times\overrightarrow{p(t)} = J_E\left(\overrightarrow{x(t)},\overrightarrow{p(t)}\right).$$

Similarly, the second equation can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\overrightarrow{p(t)}.\overrightarrow{x(t)}\right) = \frac{\|\overrightarrow{p(t)}\|^2}{m} - \frac{km}{\|\overrightarrow{x}\|}\,,$$

which is a direct consequence of Hamilton's equations for the Hamiltonian H of the Kepler problem.

Neither the Euler-Poincaré equation nor the compatibility condition involve the component $\overrightarrow{X_2(t)}$ of $\overrightarrow{X(t)}$ parallel to $\overrightarrow{x(t)}$. This illustrates the fact that the system made by these equations in underdetermined when the dimension of the Lie algebra which acts on the configuration space is strictly larger than the dimension of this space.

6.5.3 Hamilton's method of solving the Kepler problem

The Hamiltonian H of the Kepler problem remains invariant under the canonical lift to T^*N of the action of $\mathrm{SO}(E)$. Noether's theorem (4.4.5) shows that the corresponding momentum map J_E is a first integral. Of course the total energy, *i.e.* the Hamiltonian H, is too a first integral (3.3.4). Following the method due to Hamilton [29], we explain below how the three Kepler laws can easily be deduced from the first integrals J_E and H.

Let us assume that at a particular time t_0, $\overrightarrow{x(t_0)}$ and $\overrightarrow{p(t_0)}$ are not collinear. The vector $\overrightarrow{\Omega} = J_E\left(\overrightarrow{x(t)},\overrightarrow{p(t)}\right) = \overrightarrow{x(t)}\times\overrightarrow{p(t)}$ does not depend on t since J_E is a first integral, and is $\neq 0$ since for $t=t_0$, $\overrightarrow{x(t)}$ and $\overrightarrow{p(t)}$ are not collinear. We choose an orthonormal positively oriented basis $(\overrightarrow{e_x},\overrightarrow{e_y},\overrightarrow{e_z})$ of E such that $\overrightarrow{\Omega} = \Omega\overrightarrow{e_z}$, with $\Omega>0$. The vectors $\overrightarrow{x(t)}$ and $\overrightarrow{p(t)}$ remain for all times in the two-dimensional vector subspace F spanned by $(\overrightarrow{e_x},\overrightarrow{e_y})$. Let $\theta(t)$ be the polar angle made by $\overrightarrow{x(t)}$ with $\overrightarrow{e_x}$. We have

$$\begin{aligned} \overrightarrow{x(t)} &= r(t)\cos\theta(t)\overrightarrow{e_x} + r(t)\sin\theta(t)\overrightarrow{e_y}\,, \\ \overrightarrow{p(t)} &= m\left(\frac{\mathrm{d}r(t)}{\mathrm{d}t}\cos\theta(t) - r(t)\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\sin\theta(t)\right)\overrightarrow{e_x} \\ &\quad + m\left(\frac{\mathrm{d}r(t)}{\mathrm{d}t}\sin\theta(t) + r(t)\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\cos\theta(t)\right)\overrightarrow{e_y} \\ \overrightarrow{\Omega} &= mr^2\frac{\mathrm{d}\theta(t)}{\mathrm{d}t}\overrightarrow{e_z}\,. \end{aligned}$$

Therefore

$$mr^2\frac{d\theta}{dt} = \Omega = \text{Constant}\,.$$

This is the *second Kepler law*, also called *law of areas*, since $\frac{\Omega}{2m}$ is the area swept by the straight line segment joining the moving material point to the attractive centre during an unit time. Since $t \mapsto \theta(t)$ is a strictly increasing function whose derivative never vanishes, we can take θ instead of time t as independent variable. Using Hamilton's equation (or Newton's equation), we can write

$$\frac{\mathrm{d}\overrightarrow{p}(\theta)}{\mathrm{d}\theta} = \frac{\mathrm{d}\overrightarrow{p(t)}}{dt}\frac{\mathrm{d}t}{d\theta} = \frac{mr(\theta)^2}{\Omega}\left(-\frac{mk}{r(\theta)^3}\overrightarrow{x(\theta)}\right) = -\frac{m^2k}{\Omega}(\cos\theta\,\overrightarrow{e_x} + \sin\theta\,\overrightarrow{e_y})\,.$$

This ordinary differential equation for the unknown $\overrightarrow{p}(\theta)$, which no more involves $\overrightarrow{x(\theta)}$, can be readily integrated:

$$\overrightarrow{p}(\theta) = \frac{m^2k}{\Omega}(-\sin\theta\,\overrightarrow{e_x} + \cos\theta\,\overrightarrow{e_y}) + \overrightarrow{c}\,,$$

where $\overrightarrow{c}$ is a (vector) integrating constant. We will choose $\overrightarrow{e_y}$ such that $\overrightarrow{c} = c\overrightarrow{e_y}$, where c is a numeric constant which satisfy $c \geq 0$.

With O as origin let us draw two vectors in the plane xOy, the first one (constant) being equal to $\overrightarrow{c}$, and the second one (which varies with θ) equal to $\overrightarrow{p}$. The end point of that second vector moves on a circle whose centre is the end point of the vector equal to $\overrightarrow{c}$, and whose radius is $\mathcal{R} = \frac{m^2k}{\Omega}$. The part of this circle swept by the end point of this second vector is (up to multiplication by m) the *hodograph* of the Kepler problem. A short calculation leads to the following very simple relation between the energy H of a motion, the radius $\mathcal{R}$ of its hodograph and the distance c from the attracting centre O to the centre of the hodograph:

$$2mH = c^2 - \mathcal{R}^2\,.$$

The right-hand side $c^2 - \mathcal{R}^2$ is the *power*[1] of O with respect to the hodograph.

We also obtain $r = \|\overrightarrow{x}\|$ as a function of θ

$$r(\theta) = \frac{\Omega^2}{m^2k + \Omega c\cos\theta} = \frac{\Lambda}{1+\varepsilon\cos\theta}\,, \quad \text{with } \Lambda = \frac{\Omega^2}{m^2k}\,,\ \varepsilon = \frac{\Omega c}{m^2k}\,.$$

It is the polar equation of a conic section with O as focus point and ε as eccentricity. This conic section (or, when $\varepsilon > 1$, the arc of this conic swept by the moving material point) is the orbit in E of the moving material point. This result is the *first Kepler law*.

The modulus Ω of the angular momentum, the total energy H and the eccentricity ε satisfy

$$\varepsilon^2 - 1 = \frac{2\Omega^2 H}{m^3k^2}\,.$$

This formula shows that the orbit in E of the moving material point is an ellipse ($0 \leq \varepsilon < 1$) if $H < 0$, a parabola ($\varepsilon = 1$) if $E = 0$ and a connected component of a hyperbola ($\varepsilon > 1$) if $H > 0$.

When $H < 0$, the orbit in E of the moving point is an ellipse and its motion is periodic. The period T is easily obtained by writing that the area swept in a time T by the straight line segment which joins the moving point to the attractive centre is the area A delimited by the orbit:

$$T = \frac{2mA}{\Omega} = \frac{2\pi ma^2\sqrt{1-\varepsilon^2}}{\Omega}$$

[1] In plane Euclidean geometry, the *power* of a point O with respect to a circle $\mathcal{C}$ is the real number $\overrightarrow{OA}.\overrightarrow{OB}$, where A and B are the two intersection points of $\mathcal{C}$ with a straight line $\mathcal{D}$ through O. That number does not depend on $\mathcal{D}$ and is equal to $\|\overrightarrow{OC}\|^2 - \mathcal{R}^2$, where C is the centre and $\mathcal{R}$ the radius of $\mathcal{C}$.

where a is the length of the half major axis of the orbit. By using the formula

$$\Omega^2 = m^2 k a \sqrt{1-\varepsilon^2}$$

we obtain

$$T^2 = \frac{4\pi^2}{k} a^3 .$$

We conclude that the square of the period is proportional to the third power of the length of the half major axis. This result is the *third Kepler law*.

Hamilton's method of solving the Kepler problem is much easier than the Marsden-Weinstein reduction procedure, to which it is only very loosely related. A non-zero vector $\overrightarrow{\Omega}$ is a regular value of J_E, so $J_E^{-1}(\overrightarrow{\Omega})$ is a smooth three-dimensional submanifold of T^*N: it is the set of pairs of vectors $(\overrightarrow{x}, \overrightarrow{p}) \in F \times F$ such that $\overrightarrow{x} \times \overrightarrow{p} = \overrightarrow{\Omega}$, where F is the two-dimensional vector subspace of E orthogonal to $\overrightarrow{\Omega}$. This submanifold remains invariant under the action on T^*N of the one-dimensional subgroup of $\mathrm{SO}(E)$, isomorphic to the circle S^1, of rotations around the straight line through O parallel to $\overrightarrow{\Omega}$. The reduced Marsden-Weinstein symplectic manifold is the set of orbits of this action. It is isomorphic to the open half-plane $\{(r,\lambda) \in \mathbb{R}^2; r > 0\}$, and the projection of $J^{-1}(\overrightarrow{\Omega})$ onto the reduced symplectic manifold is the map $(\overrightarrow{x}, \overrightarrow{p}) \mapsto (r,\lambda)$, with $r = \| \overrightarrow{x} \|$, $\lambda = \overrightarrow{x} . \overrightarrow{p}$. The reduced symplectic form and Hamiltonian are, respectively,

$$\omega_{\overrightarrow{\Omega}} = \frac{1}{r} \mathrm{d}\lambda \wedge \mathrm{d}r , \quad H_{\overrightarrow{\Omega}} = \frac{m(\Omega^2 + \lambda^2)}{2r^2} - \frac{mk}{r} .$$

Instead of using this reduced symplectic manifold and this reduced Hamiltonian, Hamilton's method uses a clever choice of independent and dependent variables on $J_E^{-1}(\overrightarrow{\Omega})$ which leads to an easy to solve autonomous differential equation for $\overrightarrow{p}$ as a function of the polar angle θ of $\overrightarrow{x}$. It is successful essentially because the hodograph of the Kepler problem is a circle (or, when $H \geq 0$, a part of a circle).

6.5.4 The eccentricity vector

There exists still another vector valued first integral $\overrightarrow{\varepsilon}$ of the Kepler problem called the *eccentricity vector*, discovered by Jakob Hermann (1678–1753) three centuries ago [9, 31], often improperly called the *Laplace vector* or the *Ruge-Lenz vector*, whose expression is

$$\overrightarrow{\varepsilon} = -\frac{\overrightarrow{x}}{\| \overrightarrow{x} \|} + \frac{\overrightarrow{p} \times (\overrightarrow{x} \times \overrightarrow{p})}{m^2 k} = \left(\frac{\| \overrightarrow{p} \|^2}{m^2 k} - \frac{1}{\| \overrightarrow{x} \|} \right) \overrightarrow{x} - \frac{\overrightarrow{p} . \overrightarrow{x}}{m^2 k} \overrightarrow{p} .$$

For each motion of the moving material point, the eccentricity vector $\overrightarrow{\varepsilon}$ is a dimensionless vector parallel to the straight line segment which joins the attractive centre O to the perihelion of the orbit (*i.e.* the point of the orbit which is the nearest to the attractive centre), of length numerically equal to the eccentricity ε of the orbit. When the orbit is a circle, the perihelion is undetermined and $\overrightarrow{\varepsilon} = 0$. We briefly explain below the group theoretical origin of the eccentricity vector. A more detailed explanation can be found for example in [50]. Many other interesting results about the Kepler problem can be found in the excellent books [13, 27, 14, 20, 23].

Motions $t \mapsto (\overrightarrow{x(t)}, \overrightarrow{p(t)})$ of the Kepler problem in which $\overrightarrow{x(t)}$ and $\overrightarrow{p(t)}$ are parallel are not defined for all values of the time t: the curves drawn in E by the vectors $\overrightarrow{x(t)}$ and $\overrightarrow{p(t)}$ both

are supported by the same straigh line through the attractive centre O, so the motion finishes, or begins, at a finite time, when $\overrightarrow{x(t)}$ reaches 0, *i.e.* when the moving point collides with the attractive centre or is expelled by it. When t tends towards that final (or initial) instant, $\|\overrightarrow{p}(t)\|$ tends towards $+\infty$. This fact complicates the study of the global topological properties of the set of all possible motions of the Kepler problem.

For any motion $t \mapsto (\overrightarrow{x(t)}, \overrightarrow{p(t)})$, the curves drawn in E by the vectors $\overrightarrow{x(t)}$ and $\overrightarrow{p(t)}$ are, respectively, the orbit and the hodograph of the motion. The exchange $(\overrightarrow{x}, \overrightarrow{p}) \mapsto (\overrightarrow{p}, \overrightarrow{x})$ is an anti-symplectic map, which allows us, at the price of a change of sign of the symplectic form, to consider the curve drawn by $\overrightarrow{p(t)}$ as the orbit of some Hamiltonian dynamical system and the curve drawn by $\overrightarrow{x(t)}$ as the corresponding hodograph. This remark offers a way of studying the global properties of the set of all possible motions: for a motion $t \mapsto (\overrightarrow{x(t)}, \overrightarrow{p(t)})$ which starts or ends at finite instant by a collision with the attractive centre or an ejection by that point, the curve drawn by $\overrightarrow{p(t)}$, now considered as an orbit rather than a hodograph, goes to infiniy when t tends towards this limit instant. By a inverse stereographic projection, E can be mapped on a three-dimensional sphere Q minus a point (the pole of the stereographic projection), and the curve drawn by $\overrightarrow{p(t)}$ is mapped onto a curve which tends towards the pole P of the stereographic projection. The canonical prolongation of the inverse stereographic projection to the cotangent bundles allows us to map the phase space of the Kepler problem onto the open subset of T^*Q complementary to the fibre $T^*_P Q$ over the pole of the stereographic projection. On T^*Q, motions which reach $T^*_P Q$ can be prolongated and no more appear as starting, or ending, at a finite instant of time. This idea, due to Fock [21] who applied it to the study of the hydrogen atom in quantum mechanics, was used by Moser [55] for the regularization of the Kepler problem for negative values of the Hamioltonian H. Györgyi [28] used a similar idea. Since the inverse stereographic projection maps circles onto circles, the image of $\overrightarrow{p(t)}$ draws a circle on the three-dimensional sphere Q and, for a particular value of the total energy H, this circle is a great circle, *i.e.* a geodesic of Q. Györgyi [28] proved that the cylindrical projection onto E of these great circles are ellipses centered on O whose eccentricity is the same as those of the orbits drawn on E by the corresponding vector $\overrightarrow{x(t)}$. The group $\mathrm{SO}(4)$ acts on the three-dimensional sphere Q and, by the canonical lift to the cotangent bundle, on T^*Q by a Hamiltonian action. The transformed Hamiltonian is not really invariant under that action and some more work (a reparametrization of time) is still needed, but finally Noether's theorem can be used. The eccentricity vector $\overrightarrow{\varepsilon}$ is (modulo the identification of the phase space of the Kepler problem with an open subset of T^*Q) the momentum map of that action, which explains why it is a first integral.

For motions with a positive value of the total energy, there exists a similar construction in which instead of a three-dimensional sphere, Q is a two-sheeted revolution three-dimensional hyperboloid. The symmetry group is the Lorentz group $\mathrm{SO}(3,1)$; the eccentricity vector $\overrightarrow{\varepsilon}$ still is the momentum map of its action on T^*Q [54, 2]. For a motion with a zero value of H, the circle drawn in E by the vector $\overrightarrow{p(t)}$ contains the attractive centre O, so an inversion with O as pole transforms this circle into a straight line, *i.e.* a geodesic of E. The symmetry group is then the group of Euclidean displacements in E (generated by rotations and translations); the eccentricity vector $\overrightarrow{\varepsilon}$ still is the momentum map of its action on T^*E.

Ligon ans Schaaf [47] used these results to construct a global symplectic diffeomorphism of the phase space of the Kepler problem (for negative values of H) onto an open subset of the cotangent bundle to a three-dimensional sphere. Györgyi had done that earlier [28] but it seems that his work was not known by mathematicians. Later several other authors pursued these studies [15, 30].

Souriau [61] used a totally different approach. He built the regularized manifold of motions of the Kepler problem in a single step, for all values of the energy, by successive derivations of the equations of motion and analytic prolongation, calculated its symplectic form and directly determined its symmetry groups. The eccentricity vector appears again as a momentum map for the Hamiltonian actions of these groups.

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