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Optimal allocation of wealth for two consuming agents sharing a portfolio

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Abstract

We study the Merton problem of optimal consumption-investment for the case of two investors sharing a final wealth. The typical example would be a husband and wife sharing a portfolio looking to optimize the expected utility of consumption and final wealth. Each agent has different utility function and discount factor. We work in a complete market paradigm and this makes the problem equivalent to maximizing three different utilities separately with separate initial wealths. Consequently, an explicit formulation for the optimal consumptions and portfolio are obtained. We study a numerical example where the market price of risk is assumed to be mean reverting, and provide insights on the influence of risk aversion or discount rates on the initial optimal allocation.

1 Introduction

In the portfolio optimization literature, the single agent framework constitutes the main problem. However, several financial problems involve many agents, e.g., when a household manages a shared portfolio. One can also think of the situation of a portfolio manager working for a pool of clients. The question raised here is thus: how do separate agents consume resources out of a common financial portfolio? Economic theory answers the question, at least partially, by introducing social welfare and a common (utility) function to model the aggregated preferences of several heterogeneous agents, see for example [1]. In the present paper, we investigate the specific form of utilitarian social welfare function composed of the linear combination of individual discounted utility functions. That form is of particular importance for its simplicity, but also because it involves the well-known notions of risk-aversion and individual discount factor (preference for the present).

Imagine the following initial situation. Two agents, having utility functions U_1 and U_2 for consumption streams c^1 and c^2 respectively and discount rates $\beta^1(t)$ and $\beta^2(t)$, share a self-financed portfolio X_t over a period $[0, T]$ with $T > 0$ finite. They also share a third utility function U_3 of terminal wealth X_T discounted with rate $\beta^3(t)$. The goal of the couple is then

to maximize at time 0 the sum of the three expected discounted utilities from consumption and wealth over $[0, T]$ i.e., the quantity

$$\mathbb{E} \left[\int_0^T e^{(-\int_0^t \beta^1(s) ds)} U_1(c_t^1) dt + \int_0^T e^{(-\int_0^t \beta^2(s) ds)} U_2(c_t^2) dt + e^{(-\int_0^T \beta^3(s) ds)} U_3(X_T) \right]. \quad (1.1)$$

This expression can be obviously generalized to any linear combination of a number $n \in \mathbb{N}$ of utility functions for consuming agents. Moreover, the methodology we developed here can be easily extended to the case in which the agents have different utilities of the final wealth.

When the total initial wealth X_0 is given, the portfolio management problem brings up the question of wealth allocation among participants in order to contempt them, additionally to the one of optimal consumption-portfolio strategy for each of them. This initial allocation problem is a one-time static problem at time $t = 0$. It is of fundamental importance to notice that if the criterion is updated at a later date $t > 0$, the solution changes and does not correspond to the wealth obtained by the allocation at date 0 and subsequent optimal portfolio strategies. This means that the problem lacks a time-consistency property, see [3].

The problem with two agents can easily be reduced to a one agent problem by considering the utility function

$$U(s, t, C) := \sup_{c^1 + c^2 = C} \left[\exp \left(- \int_s^t \beta^1(s) ds \right) U_1(c^1) + \exp \left(- \int_s^t \beta^2(s) ds \right) U_2(c^2) \right]. \quad (1.2)$$

It is foreseeable that the optimal behavior of the agent will depend on the initial date s , and therefore be given up at a later date without any commitment device. This is why we reduce ourselves to the problem of initial allocation at date 0, and suppose that this action commits our two agents on the interval $[0, T]$. In a future companion paper, we solve the couple's problem without commitment in a time-consistent manner, by the use of sub game perfect strategies as in [3].

Coming back to (1.1), we actually show that the above problem can be divided in three separate problems involving only one agent at a time. Thus, in a sense, the only real decision on the part of the investor takes place at $t = 0$ with the determination of the initial wealth allocated to each agent. Once the initial allocation is provided, the further evolution of interesting quantities (consumptions and wealth) follows well known solutions provided by [8]. One can draw a parallelism with Pareto-optimality and the representative agent. Imagine that (1.1) is the expected utility of the representative agent. Then we show that maximizing this utility is equivalent to solving individual agent expected utilities problem and then find the Pareto weights, which in our case are the vector of initial wealth allocations.

In order to illustrate the allocation solution, we provide a numerical application with closed form solutions in the framework of [14], i.e., with power utilities and mean reverting market price of risk. We naturally focus on the comparison between the two consuming agents initial wealth as a function of risk aversion and discount rates. We find interesting insights for portfolio managers. As the agent's initial wealth increases, an increasing proportion of the wealth is allocated to finance the portfolio. Furthermore, the less risk averse consumer allocates more money for future consumption for sufficiently large values of the initial wealth. The effects of risk aversion on the allocations are also intuitive. As agents become less risk averse, the allocated fraction of initial wealth increases.

Our main contribution is to show that the aggregate optimal consumption-investment problem for multiple agents is equivalent to multiple optimal single agent consumption-investment problems given an optimal splitting of the initial allocation among the agents. We show how this optimal allocation is computed which in turn leads to explicit formulas for the optimal consumptions and portfolios. Furthermore, in a specific example, by running numerical experiments we show how the vector of optimal wealth allocations is affected by risk aversion and discount rates. The article is organized as follows: Section 2 introduces the model, agents specification, the problem and its solution. The numerical application is discussed in Section 3. Proofs are delegated to an appendix.

2 Market model and portfolio management problem

2.1 Complete market

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a standard d -dimensional Brownian motion $W := (W_t)_{t \in [0, T]} = (W_t^1, \dots, W_t^d)_{t \in [0, T]}^T$. The filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is the augmentation under \mathbb{P} of the natural filtration of W . We consider a complete market composed of $d + 1$ assets (S^0, \dots, S^d) which are continuously traded on $[0, T]$ and evolve according to the differential equations

$$dS_t^0 = r(t)S_t^0 dt, \quad 0 \leq t \leq T,$$

with $S_0^0 = s_0$, and

$$dS_t^i = S_t^i \left(b_i(t) dt + \sum_{j=1}^d \sigma_{ij}(t) dW_t^j \right), \quad 0 \leq t \leq T, \quad (2.1)$$

with $(S_0^1, \dots, S_0^d) = (s_1, \dots, s_d) \in (0, \infty)^d$.

Assumption 2.1. *The interest rate process $(r(t))_{t \in [0, T]}$ is adapted. The vector of mean rates of excess return $b(t) := (b_1(t) \dots b_d(t))_{t \in [0, T]}^T$ and the diffusion matrix $\sigma(t) := (\sigma_{ij}(t))_{1 \leq i, j \leq d, t \in [0, T]}$ are assumed to be adapted. They are such that the SDE (2.1) has a unique strong solution.*

Throughout this paper, we replace all asset prices by the discounted asset prices. The discount factor using the risk-free rate is

$$D_t := \exp \left(- \int_0^t r(u) du \right), \quad 0 \leq t \leq T. \quad (2.2)$$

For a generic process Y_t , we introduce the notation $\tilde{Y}_t := Y_t D(t)$ to denote its discounted counterpart.

The completeness of the market in the sense of [5] implies the existence of a unique \mathbb{P} -equivalent martingale measure $\tilde{\mathbb{P}}$. Define the price of risk process $\theta(t) := \sigma(t)^{-1}(b(t) - r(t)\mathbf{1})$, for $t \in [0, T]$. We can define $\tilde{\mathbb{E}}$ the expectation operator under $\tilde{\mathbb{P}}$. Under some integrability conditions on θ , $\tilde{W}_t := W_t + \int_0^t \theta(s) ds$ is a Brownian motion under $\tilde{\mathbb{P}}$, see [12].

Definition 2.2. The Radon Nikodym derivative process of $\tilde{\mathbb{P}}$ w.r.t. \mathbb{P} is defined by

$$Z(t) := \exp \left\{ - \sum_{i=1}^d \int_0^t \theta_i(s) dW_s^i - \frac{1}{2} \int_0^t \|\theta(s)\|^2 ds \right\}, \quad \text{for } t \in [0, T]. \quad (2.3)$$

Assumption 2.3. The process $Z(t), t \in [0, T]$, is a (true) martingale.

Remark 2.4. Sufficient conditions for the Assumption 2.3 to be satisfied are Novikov and Kazamaki conditions. If the process $\theta(t), t \in [0, T]$ is Markovian, [15] provide finer sufficient conditions.

Remark 2.5. The methodology we developed does not extend naturally to incomplete markets. This is because of the non uniqueness of the martingale measure. The BSDE approach may work in the incomplete markets within our context, but we leave this as topic of future research. The BSDE methodology was pioneered by [7] for the problem of expected utility maximization of terminal wealth with special choice of utilities (power and exponential). It was extended to general utilities by [6] and to allow for intertemporal consumption by [2].

2.2 Specification of agents

The latter framework has been considered by [8] for a single investor and semi-explicit solutions are provided using a martingale approach, see subsection 4.1. [13] considered the special case of a single investor with two different power utilities: one for consumption and one for final wealth. We consider here two consumption streams and one common terminal portfolio value evaluation. We comment the generalization to an arbitrary number of consumption streams and terminal values in subsection 2.4.

Definition 2.6. We introduce the following objects:

1. A portfolio strategy $\pi := \{\pi(t) = (\pi_1(t), \dots, \pi_d(t))^\top\}$ is an adapted, \mathbb{R}^d -valued process where $\pi_i(\omega) \in L^2([0, T])$ for \mathbb{P} -almost every $\omega \in \Omega$ and $i = 1, \dots, d$; $\pi(t)$ denotes the number of shares of asset i held in the portfolio at time t .
2. A consumption process is given by $(c_t^1, c_t^2)_{t \in [0, T]}$, an adapted process with non-negative values such that $C(\omega) := c^1(\omega) + c^2(\omega)$ is in $L^1([0, T])$ for \mathbb{P} -almost every $\omega \in \Omega$.
3. The wealth process $X := (X_t)_{t \in [0, T]}$ is uniquely defined as

$$X_t = \frac{1}{D(t)} \left(x + \int_0^t ((\pi^T(s)b(s) - C_s)) D(s) ds + \pi^T(s)\sigma(s)D(s)dW_s \right) \quad (2.4)$$

or equivalently by the discounted process

$$\tilde{X}_t = \left(x + \int_0^t \left((\tilde{\pi}^T(s)b(s) - \tilde{C}_s) \right) ds + \tilde{\pi}^T(s)\sigma(s)dW_s \right). \quad (2.5)$$

Equation (2.4) can also write with the $\tilde{\mathbb{P}}$ -Brownian motion:

$$\tilde{X}_t + \int_0^t \tilde{C}_s ds = x + \int_0^t \tilde{\pi}^T(s) \sigma(s) d\tilde{W}(s). \quad (2.6)$$

Definition 2.7. A triplet (π, c^1, c^2) of strategy and consumption processes is said to be admissible for the initial endowment $x \geq 0$ if the wealth process X satisfies $X_t \geq 0$ for $[0, T]$ \mathbb{P} -a.s. We call $A(x)$ the class of admissible processes (π, c^1, c^2) for initial wealth x .

For any $(\pi, c^1, c^2) \in A(x)$, the left-hand side of (2.6) is non negative and the right-hand side is a local martingale under $\tilde{\mathbb{P}}$. It follows that the left-hand side, and hence also \tilde{X}_t , is a non negative super-martingale under $\tilde{\mathbb{P}}$. Now, if $\tau_0 := T \wedge \inf\{0 \leq t \leq T, X(t) = 0\}$, then $X_t = 0$ for all $t \in [\tau_0, T]$ on $\{\tau_0 > -\infty\}$. The super martingale property in (2.6) yields

$$\tilde{\mathbb{E}} \left[\tilde{X}_T + \int_0^T \tilde{C}_t dt \right] \leq x. \quad (2.7)$$

This property allows to express admissibility of strategies and consumptions respectively in a different manner. Additional martingale properties are developed in subsection 4.1.

Each agent i is endowed with a utility function U_i and a discount rate β^i with corresponding discount factor

$$B_t^i := \exp \left(- \int_0^t \beta^i(s) ds \right). \quad (2.8)$$

The discount rates are assumed to be adapted and bounded for all t uniformly \mathbb{P} -almost surely.

Definition 2.8. We define for $i = 1, 2, 3$ the state price process corresponding to the discount factor B^i ,

$$\zeta_t^i := Z_t D_t \exp \left(\int_0^t \beta^i(u) du \right) = \frac{\tilde{Z}_t}{\tilde{B}_t^i}.$$

where the process Z_t is given by (2.3).

Assumption 2.9. For $i = 1, 2, 3$, we assume that U_i is a strictly increasing, strictly concave real-valued function in $C^2([0, \infty])$ such that U_i'' is non decreasing, $U_i(0) \geq -\infty$ and $U_i'(\infty) = 0$. U_i' is defined from $[0, \infty)$ onto $[0, U_i'(0)]$. We define $I_i := (U_i')^{-1}$ be the inverse functions of the marginal utilities for $i = 1, 2, 3$.

Note that we allow for $U_i(0) = -\infty$ or $U_i'(0) = \infty$. This framework encompasses a large class of functions, including CARA and HARA utility functions. Because $U_i' : [0, \infty) \rightarrow [0, U_i'(0)]$ is strictly decreasing, it has a strictly decreasing inverse $I_i : [0, U_i'(0)] \rightarrow [0, \infty]$. We extend I_i to be a continuous function on the entirety of $[0, \infty]$ by setting $I_i(y) = 0$ for $U_i'(0) \leq y \leq \infty$, and note that

$$U_i(I_i(y)) \geq U_i(c) + yI_i(y) - yc, \text{ for } (y, c) \in (0, \infty) \times [0, \infty). \quad (2.9)$$

2.3 The portfolio management problem

We formalize the main problem (1.1).

Definition 2.10. For a given $x \geq 0$, we define the value function at x by

$$V(x) := \sup \left\{ J(x; \pi, c^1, c^2) : (\pi, c^1, c^2) \in \tilde{A}(x) \right\}, \quad (2.10)$$

where

$$J(x; \pi, c^1, c^2) := E \left[\int_0^T (B_t^1 U_1(c_t^1) + B_t^2 U_2(c_t^2)) dt + B_T^3 U_3(X_T) \right]$$

and

$$\tilde{A}(x) := \{(\pi, c^1, c^2) \in A(x) : J(x; \pi, c^1, c^2) < \infty\}.$$

The expectation J is well defined for every pair $(\pi, c^1, c^2) \in \tilde{A}(x)$. The total initial endowment of the couple is x . We can easily see that $\tilde{A}(x) = A(x)$ if $U_i(0) > -\infty$ for $i = 1, 2, 3$. We consider three problems that are sub-problems to the one of Definition 2.10, related to each term in the expression of J .

Definition 2.11. For a given $x_1 > 0$, we define the value function

$$V_1(x_1) := \sup \left\{ J_1(x_1; \pi, c^1, c^2) : (\pi, c^1, c^2) \in A_1(x_1) \right\} \quad (2.11)$$

where

$$J_1(x_1; \pi, c^1, c^2) := E \left[\int_0^T B_t^1 U_1(c_t^1) dt \right]$$

and

$$A_1(x_1) := \{(\pi, c^1, c^2) \in A(x_1) : J_1(x_1; \pi, c^1, c^2) < \infty\}.$$

The expectation J_1 is well defined for every pair $(\pi, c^1, c^2) \in A_1(x_1)$. The value functions V_2, V_3 and the sets A_2, A_3 are defined similarly in an obvious manner. We now turn to elements involved in the solutions of V and V_i , $i = 1, 2, 3$.

2.4 Main result

Definition 2.12. We introduce the following functions on $[0, \infty) \times [0, T]$:

$$\mathcal{H}_i(y, t) := \tilde{\mathbb{E}} \left[\int_t^T D_u I_i(y \zeta_u^i) du \middle| \mathcal{F}_t \right] \text{ for } i = 1, 2, \quad (2.12)$$

and

$$\mathcal{H}_3(y, t) := \tilde{\mathbb{E}} \left[D_T I_3(y \zeta_T^3) \middle| \mathcal{F}_t \right]. \quad (2.13)$$

We will write $\mathcal{H}_i(y) := \mathcal{H}_i(y, 0)$, and call $\mathcal{Y}_i := \mathcal{H}_i^{-1} : [0, \infty] \rightarrow [0, \infty]$ the inverse of the function \mathcal{H}_i for $i = 1, 2, 3$.

Assumption 2.13. For all $y \in (0, \infty)$, $\mathcal{H}_i(y) < \infty$, $i = 1, 2, 3$.

Assumption 2.14. For all $y \in (0, \infty)$,

$$\mathbb{E} \left[\int_0^T B_t^1 |U_1(I_1(y\zeta_t^1))| dt + \int_0^T B_t^2 |U_2(I_2(y\zeta_t^2))| dt + B_T^3 |U_3(I_3(y\zeta_T^3))| \right] < \infty .$$

In the special case of power type utilities these assumptions translate into finiteness of moments for Z defined in (2.3).

The proof of the following Lemma is done in [8].

Lemma 2.15. For $i = 1, 2, 3$, \mathcal{H}_i is a continuous function, strictly decreasing on $(0, \infty)$ with $\mathcal{H}_i(0) = \infty$ and $\mathcal{H}_i(\infty) = 0$

Theorem 2.16. Let $a_1, a_2, a_3 \geq 0$. Then the value functions $V_i(a_i)$ for $i = 1, 2, 3$ are reached with the consumption processes

$$\begin{aligned} \mathbf{c}_t^1(a_1) &:= I_1(\mathcal{Y}_1(a_1)\zeta_t^1) , \\ \mathbf{c}_t^2(a_2) &:= I_2(\mathcal{Y}_2(a_2)\zeta_t^2) , \end{aligned}$$

and the following wealth processes

$$\begin{aligned} \mathbf{X}_t^1(a_1) &:= \tilde{\mathbb{E}} \left[\int_t^T \mathbf{c}_s^1(a_1) D_s ds | \mathcal{F}_t \right] = \mathcal{H}_1(\mathcal{Y}_1(a_1), t) , \\ \mathbf{X}_t^2(a_2) &:= \tilde{\mathbb{E}} \left[\int_t^T \mathbf{c}_s^2(a_2) D_s ds | \mathcal{F}_t \right] = \mathcal{H}_2(\mathcal{Y}_2(a_2), t) , \\ \mathbf{X}_t^3(a_3) &:= \tilde{\mathbb{E}} [I_3(\mathcal{Y}_3(a_3)\zeta_T^3) D_T | \mathcal{F}_t] = \mathcal{H}_3(\mathcal{Y}_3(a_3), t) . \end{aligned}$$

A triplet of financial strategies $(\boldsymbol{\pi}^1(a_1), \boldsymbol{\pi}^2(a_2), \boldsymbol{\pi}^3(a_3))$ corresponds to the agents' wealth processes. It is provided in next section with Proposition 4.3. By linearity we are able to define the total portfolio $\boldsymbol{\pi}$ and total wealth process \mathbf{X} by

$$\boldsymbol{\pi}(a_1, a_2, a_3) := \boldsymbol{\pi}^1(a_1) + \boldsymbol{\pi}^2(a_2) + \boldsymbol{\pi}^3(a_3) , \quad (2.14)$$

$$\mathbf{X}(a_1, a_2, a_3) := \mathbf{X}^1(a_1) + \mathbf{X}^2(a_2) + \mathbf{X}^3(a_3) . \quad (2.15)$$

We now turn to the main result, i.e., the additivity of sub-problems allowing to find the initial allocations for Theorem 2.16.

Definition 2.17. Let \mathcal{H} be defined on $[0, \infty)$ by $\mathcal{H} := \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$, and its inverse $\mathcal{Y} := \mathcal{H}^{-1} : [0, \infty] \rightarrow [0, \infty]$.

Theorem 2.18. Consider a couple endowed initially with wealth $x > 0$. Then,

$$\begin{aligned} V(x) &= V_*(x) := \sup \{ V_1(a_1) + V_2(a_2) + V_3(a_3) : (a_1, a_2, a_3) \in \mathbb{R}_+^3, a_1 + a_2 + a_3 = x \} \\ &= V_1(x_1) + V_2(x_2) + V_3(x_3) \end{aligned}$$

where the initial allocation x_i is given by

$$x_i = \mathcal{H}_i(\mathcal{Y}(x)) . \quad (2.16)$$

Let us give some intuition for this result. It says that the couple value function is the optimal aggregation of individual value functions. This is a Pareto-optimality type result; the novelty is that the Pareto weights are the initial wealth allocation.

Remark 2.19. *Definition 2.10 easily extends to more than two consuming agents. The extension to more than one evaluation of the terminal wealth is however a matter of definition. The function J is indeed defined with only one portfolio strategy and one terminal value: the linearity of X_t in the financial strategy π_t allows to separate into three sub-problems as asserted by Theorem 2.18. However, changing the third term in the objective function J for a term like*

$$\mathbb{E} \left[e^{-\int_0^T \beta^1(s) ds} U_4(X_T^1) + e^{-\int_0^T \beta^2(s) ds} U_5(X_T^2) \right],$$

implies to define how agents 1 and 2 proceed. If they share an initial wealth, a common portfolio, and decide at T to split the final wealth, then the problem is strictly equivalent to Problem (2.10) by using the artifact that the one suggested in (1.2) with $\beta^3 = 0$ and

$$U_3(x) := \sup_{\epsilon \in [0,1]} e^{-\int_0^T \beta^4(s) ds} U_4(\epsilon x) + e^{-\int_0^T \beta^5(s) ds} U_5((1-\epsilon)x).$$

If one wants to distinguish agents portfolios, then he shall redefine J in order to separate trading portfolios and consumption portfolios from $t = 0$. The problem can be solved by using V_ of Theorem 2.18, and the splitting provided by Lemma 4.11 in Section 4.*

We will also discuss the quantitative consequences of the agents specification on the splitting of the initial wealth suggested by Theorem 2.18. As it is foreseeable and proved below in subsection 4.1, the wealth attributed to consuming agents is integrally consumed by the end T . We thus introduce the following quantity.

Definition 2.20. *Let $x > 0$ represent the total initial wealth and x_1, x_2 be given by (2.16). The consumption satisfaction proportion (CSP) is then defined by $(x_1 + x_2)/x$.*

We also introduce a well-known quantity regarding risk aversion for the consuming couple.

Definition 2.21. *The relative risk-aversion for the couple as $R(x) := -xV''(x)/V'(x)$, with V of (2.10).*

3 CRRA utilities and mean reverting market price of risk

3.1 Configuration of the market

We provide here an explicit model of the previously studied framework. The three agents share a common initial wealth x and have CRRA type utilities

$$U_i(x) = \frac{x^{\gamma_i}}{\gamma_i} \quad \text{for } i = 1, 2, 3.$$

Here $1 - \gamma_i \in (0, \infty)$ is the risk aversion of agent i . Notice that U_i satisfy Assumptions 2.9, and $I_i(x) = x^{\frac{1}{\gamma_i-1}}$. Each agent has his own constant discount rate ρ_i .

Next take $d = 1$ as in [14] (the extension to multiple stocks is straightforward). The asset price follows a geometric Brownian motion. In order to isolate the effects of time variation on expected returns, the risk-free rate is assumed to be constant and equal to $r \geq 0$ but this assumption can be relaxed. We fix the volatility $\sigma := \sigma_{11} \in (0, \infty)$ for (2.1), but we do not specify the drift $b_1 \in \mathbb{R}$. Instead, we model the price of risk θ by

$$d\theta_t = -\lambda_\theta(\theta_t - \bar{\theta})dt - \sigma_\theta dW_t, \quad t \geq 0,$$

where $(\lambda_\theta, \sigma_\theta, \bar{\theta}) \in (0, \infty)^3$. We assume $W = W^1$, so that the stock price S_t^1 and the state variable θ_t are perfectly negatively correlated. These assumptions are like those in [9], except that the latter allows for imperfect correlation, and thus incomplete markets. The extension of our results to incomplete markets is a non-trivial issue (see Remark 2.5).

The body of academic literature on long term mean reversion is more tractable than that on short term mean reversion. A comprehensive study on the existence of mean reversion in Equity Prices has been done in [10]. The primary case for the existence of long term mean reversion was made in two papers published in 1988, one by [11], the other by [4]. In summary, these papers conclude that for period lengths between 3 and 5 years, long term mean reversion was present in stock market returns between 1926 and 1985.

3.2 Semi-explicit solutions

In this framework, the modeling assumptions of Section 2 are satisfied. We now seek for explicit formulations in Theorem 2.18: we aim at providing the initial repartition x_1, x_2, x_3 such that $x_1 + x_2 + x_3 = x$, and the corresponding optimal strategies $(\boldsymbol{\pi}^1, \boldsymbol{\pi}^2, \boldsymbol{\pi}^3)$ and consumption processes $(\mathbf{c}^1, \mathbf{c}^2)$. We provide first the formulation of wealth processes for consuming agents.

Proposition 3.1. *Let $i = 1$ or 2 . Then*

$$\mathbf{X}_t^i = Y_t^{\frac{1}{1-\gamma_i}} e^{\frac{(r+\rho_i)t}{1-\gamma_i}} \int_0^{T-t} \exp\left(-\frac{r(1-\gamma_1) + \rho_i}{1-\gamma_i}s + A_{1i}(s)\frac{\theta_t^2}{2} + A_{2i}(s)\theta_t + A_{3i}(s)\right) ds \quad (3.1)$$

where the process Y_t is given by

$$Y_t := (yZ_t)^{-1} e^{rt} \quad (3.2)$$

$y = \mathcal{Y}(x)$ is given by Definition 2.12, Z by Definition 2.2, and A_{1i}, A_{2i}, A_{3i} are continuous functions on $\{s \in [0, T-t]\}$ verifying

$$A_{1i}(0) = A_{2i}(0) = A_{3i}(0) = 0, \quad 0 \leq s \leq T-t, \quad (3.3)$$

and satisfying the following system of ODEs

$$\begin{cases} -A'_{1i}(s) - 2\lambda_\theta A_{1i}(s) + \sigma_\theta^2 A_{1i}(s)^2 + \frac{\gamma_1}{(1-\gamma_1)^2} = 0 \\ -A'_{2i}(s) + \lambda_\theta(\bar{\theta}A_{1i}(s) - A_{2i}(s)) + \sigma_\theta^2 A_{1i}(s)A_{2i}(s) = 0 \\ -A'_{3i}(s) + \lambda_\theta\bar{\theta}A_{2i}(s) + \frac{\sigma_\theta^2}{2}(A_{1i}(s) + A_{2i}(s))^2 = 0 \end{cases} \quad (3.4)$$

Proof Following Theorem 2.18, the optimal initial allocation is given by $x_i = \mathcal{H}_i(\mathcal{Y}(x))$. Denoting $y := \mathcal{Y}(x) = \mathcal{Y}_i(x_i)$, the theorem gives also the optimal consumption

$$\mathbf{c}_t^i(x_i) = I_i(y\zeta_t^i) = (y \exp(\rho_i t) Z_t)^{\frac{1}{\gamma_i-1}}, \quad 0 \leq t \leq T,$$

where Z_t is the state density process defined by (2.3). By Ito's formula,

$$dY_t = (r + \theta_t^2)Y_t dt + \theta_t Y_t dW_t \quad (3.5)$$

The optimal total wealth process of agent i is thus given by

$$\mathbf{X}_t^i = \tilde{\mathbb{E}} \left[\int_t^T D_s \mathbf{c}_s^i ds \mid \mathcal{F}_t \right].$$

The consumption process is $\mathbf{c}_s^i = y e^{\rho_i s} Z_s = e^{\rho_i s} e^{r s} Y_s^{-1}$ and $\frac{Z_s}{Z_t} = e^{r(s-t)} \frac{Y_t}{Y_s}$. Therefore,

$$\begin{aligned} \mathbf{X}_t^i &= Y_t^{1+\frac{1}{1-\gamma_i}-1} \mathbb{E}_t \left[\int_t^T e^{-rs} e^{r(s-t)} e^{(r+\rho_i)\frac{s}{\gamma_i-1}} \left(\frac{Y_s}{Y_t} \right)^{\frac{1}{1-\gamma_i}-1} ds \right] \\ &= Y_t^{\frac{1}{1-\gamma_i}} \int_t^T e^{-rs} e^{r(s-t)} e^{(r+\rho_i)\frac{s}{\gamma_i-1}} e^{\frac{\gamma_i r(s-t)}{1-\gamma_i}} f^i(t, s, \theta_t) ds \\ &= Y_t^{\frac{1}{1-\gamma_i}} \int_t^T e^{\frac{\gamma_i r(s-t) - (r+\rho_i)s}{1-\gamma_i}} f^i(t, s, \theta_t) ds \\ &= Y_t^{\frac{1}{1-\gamma_i}} \int_t^T e^{\frac{\gamma_i r(s-t) - (r+\rho_i)s}{1-\gamma_i}} f^i(t, s, \theta_t) ds \\ &= Y_t^{\frac{1}{1-\gamma_i}} e^{\frac{(r+\rho_i)t}{1-\gamma_i}} \int_t^T e^{\frac{((\gamma_i-1)r - \rho_i)(s-t)}{1-\gamma_i}} f^i(t, s, \theta_t) ds \end{aligned}$$

where the function f^i is defined for $t \leq \tau \leq T$ and $\theta \in \mathbb{R}$ by

$$f^i(t, \tau, \theta) = \mathbb{E} \left[\exp \left(\frac{\gamma_i}{2(1-\gamma_i)} \int_t^\tau \theta_s^2 ds + \frac{\gamma_i}{1-\gamma_i} \int_t^\tau \theta_s dW_s \right) \mid \theta_t = \theta \right].$$

For, $0 \leq t \leq \tau$, the process

$$f^i(t, \tau, \theta_t) \exp \left(\frac{\gamma_i}{2(1-\gamma_i)} \int_0^t \theta_s^2 ds + \frac{\gamma_i}{1-\gamma_i} \int_0^t \theta_s dW_s \right),$$

is a conditional expectation of a \mathcal{F}_T -measurable random variable for any fixed θ_0 . It is then a \mathbb{P} -martingale on time $\tau \leq T$. Notice that in the definition of f^i , the coefficients of the exponential are independent of t . Therefore we look for f^i of the form $f^i(t, \tau, \theta_t) := g^i(\tau - t, \theta_t)$. We make the change of variables $\tau - t := s$. Given that g^i is $\mathcal{C}^{1,2}$, it follows by Ito's formula that

$$-g_s^i - \lambda_\theta(\theta - \bar{\theta})g_\theta^i + \frac{\sigma_\theta^2}{2}g_{\theta\theta}^i + \left(\frac{\gamma_i}{2(1-\gamma_i)} + \frac{\gamma_i^2}{2(1-\gamma_i)^2} \right) \theta^2 g^i = 0. \quad (3.6)$$

with the condition $g^i(0, \theta) = 1$. We follow [14] and search for g^i of the form

$$g^i(s, \theta) = \exp \left(A_{1i}(s) \frac{\theta^2}{2} + A_{2i}(s)\theta + A_{3i}(s) \right),$$

where A_{1i}, A_{2i}, A_{3i} are three continuous functions of s . The terminal condition in the latter expression implies condition (3.3). Plugging the expression of g^i in (3.6), we get a second-order polynomial in θ

$$\begin{aligned} & \theta^2 \left[-A'_{1i}(s) - 2\lambda_\theta A_{1i}(s) + \sigma_\theta^2 A_{1i}(s)^2 + \frac{\gamma_i}{(1-\gamma_i)^2} \right] \\ & + \theta \left[-A'_{2i}(s) + \lambda_\theta (\bar{\theta} A_{1i}(s) - A_{2i}(s)) + \sigma_\theta^2 A_{1i}(s) A_{2i}(s) \right] \\ & - A'_{3i}(s) + \lambda_\theta \bar{\theta} A_{2i}(s) + \frac{\sigma_\theta^2}{2} (A_{1i}(s) + A_{2i}(s))^2 = 0 \end{aligned}$$

Since the equation holds for any $\theta \in \mathbb{R}$, we separate the coefficients in θ^2, θ and constant to obtain (3.4). \square

If a C^1 function $A_{1i}(s), s \in [0, T-t]$ has been found, then A_{2i} is given by a linear ODE, which finally allows to retrieve A_{3i} :

$$A_{3i}(s) = \int_0^s \lambda_\theta \bar{\theta} A_{2i}(u) + \frac{\sigma_\theta^2}{2} (A_{1i}(u) + A_{2i}(u))^2 du.$$

This allows for the detailed numerical analysis we present in subsection 3. The following provides the missing part.

Proposition 3.2. *Let $A_{1i}(s)$ be a solution on $\{0 \leq t \leq T-t\}$ of the ODE*

$$-A'_{1i}(s) - 2\lambda_\theta A_{1i}(s) + \sigma_\theta^2 A_{1i}(s)^2 + \frac{\gamma_i}{(1-\gamma_i)^2} = 0 \quad (3.7)$$

such that $A_{1i}(0) = 0$. Then, denoting $\Delta = \lambda_\theta^2 - \frac{\gamma_i \sigma_\theta^2}{(1-\gamma_i)^2}$, A_{1i} is defined on $[0, T-t]$ by

$$A_{1i}(s) = \begin{cases} \frac{\gamma_i}{(1-\gamma_i)^2 \sigma_\theta} \frac{1 - \exp(-2\sqrt{\Delta} \sigma_\theta^2 s)}{\lambda_\theta + \sqrt{\Delta} - (\lambda_\theta - \sqrt{\Delta}) \exp(-2\sqrt{\Delta} \sigma_\theta^2 s)} & \text{if } \Delta > 0 \\ \frac{\lambda_\theta^2 s}{\sigma_\theta^2 \lambda_\theta s + \sigma_\theta^2} & \text{if } \Delta = 0 \\ \frac{1}{\sigma_\theta^2} \left(\sqrt{-\Delta} \tan \left(-\sqrt{-\Delta} s + \arctan \left(\frac{-\lambda_\theta}{\sqrt{-\Delta}} \right) \right) + \lambda_\theta \right) & \text{if } \Delta < 0 \end{cases} \quad (3.8)$$

Proof *Case 1:* $\Delta > 0$. There are two distinct roots to the characteristic polynomial of the

ODE, given by $m_{\pm} := \frac{\lambda_{\theta}}{\sigma_{\theta}} \pm \sqrt{\Delta}$. A general solution $A_1(t)$ to (3.7) shall verify

$$\begin{aligned}
\sigma_{\theta} &= \frac{\sigma_{\theta} A'_{1i}(\tau)}{(\sigma_{\theta} A_{1i}(\tau) - \frac{\lambda_{\theta}}{\sigma_{\theta}})^2 - \sigma_{\theta}^2 \Delta} \\
\sigma_{\theta} s &= \int_0^s \frac{\sigma_{\theta} A'_{1i}(\tau) d\tau}{(\sigma_{\theta} A_{1i}(\tau) - \frac{\lambda_{\theta}}{\sigma_{\theta}} - \sigma_{\theta} \sqrt{\Delta})(\sigma_{\theta} A_{1i}(\tau) - \frac{\lambda_{\theta}}{\sigma_{\theta}} + \sigma_{\theta} \sqrt{\Delta})} \\
&= \int_0^{A_{1i}(s)} \frac{\sigma_{\theta} dx}{(\sigma_{\theta} x - \sigma_{\theta} m_+)(\sigma_{\theta} x - \sigma_{\theta} m_-)} \\
&= \frac{1}{\sigma_{\theta}(m_+ - m_-)} \int_0^{A_{1i}(s)} \frac{dx}{x - m_+} - \frac{dx}{x - m_-}, \\
&= \frac{1}{2\sigma_{\theta}\sqrt{\Delta}} \log \left(\frac{(A_{1i}(s) - m_+)m_-}{(A_{1i}(s) - m_-)m_+} \right),
\end{aligned}$$

Therefore,

$$\frac{A_{1i}(s) - m_+}{A_{1i}(s) - m_-} = \frac{m_+}{m_-} \exp \left(2\sqrt{\Delta}\sigma_{\theta}^2 s \right)$$

$$A_{1i}(s) = m_+ + \frac{2\sigma_{\theta}\sqrt{\Delta}}{1 - \frac{m_+}{m_-} \exp \left(2\sqrt{\Delta}\sigma_{\theta}^2 s \right)} \quad \forall t \leq \tau.$$

Case 2: $\Delta = 0$. With the double root $m := \lambda_{\theta}/\sigma_{\theta}^2$, the same operation provides

$$\begin{aligned}
\sigma_{\theta} &= \frac{A'_{1i}(\tau)}{(A_{1i}(\tau) - m)^2} \\
\sigma_{\theta} s &= \int_0^s \frac{A'_{1i}(\tau)}{(A_{1i}(\tau) - m)^2} d\tau \\
\sigma_{\theta} s &= \left(-\frac{1}{m} + \frac{1}{m - A_{1i}(s)} \right).
\end{aligned}$$

The solution then follows:

$$A_{1i}(s) = \frac{\sigma_{\theta} m^2 s}{1 + m\sigma_{\theta} s}$$

Case 3: $\Delta < 0$. We can write the ODE as

$$-A'_{1i}(s) + \left(\sigma_{\theta} A_{1i}(s) - \frac{\lambda_{\theta}}{\sigma_{\theta}} \right)^2 - \frac{\Delta}{\sigma_{\theta}^2} = 0.$$

Taking $y(s) := (\sigma_{\theta}^2 A_{1i}(s) - \lambda_{\theta})/\sqrt{-\Delta}$, we get

$$\arctan(y(s)) - \arctan \left(-\frac{\lambda_{\theta}}{\sqrt{-\Delta}} \right) = -\sqrt{-\Delta} s$$

providing the solution. \square

Notice that A_{1i} is not continuous nor well defined for all s , if $\Delta \leq 0$. The condition $\Delta > 0$ can also write

$$\gamma_1 < \gamma_{lim} := \frac{2}{2 + b + \sqrt{(2 + b)^2 - 4}} \text{ for } b := \frac{\sigma_\theta^2}{\lambda_\theta^2}.$$

Proposition 3.1 provides the portfolio process value for a consuming agent. For the third agent, Theorem 2.18 provides for $0 \leq t$

$$\mathbf{X}_t^3 = \tilde{\mathbb{E}} [D_T I_3(y \zeta_T^3) | \mathcal{F}_t]$$

which can be solved by Proposition 3.1 and equation (3.3).

Corollary 3.3. *Define for $i = 1, 2$ the function $H_i : \mathbb{R} \times [0, T] \rightarrow (0, \infty)$ by*

$$H_i(\theta, \tau) := \exp \left(A_{1i}(0, \tau) \frac{\theta^2}{2} + A_{2i}(0, \tau) \theta + A_{3i}(0, \tau) - \frac{r(1 - \gamma_i) + \rho_i}{1 - \gamma_i} \tau \right),$$

and $s_i = \int_0^T H_i(\theta_0, s) ds$. *The initial allocations for the three agents are:*

$$\begin{cases} x_1 &= y^{\frac{1}{\gamma_1 - 1}} \int_0^T H_1(\theta_0, s) ds = y^{\frac{1}{\gamma_1 - 1}} s_1 \\ x_2 &= y^{\frac{1}{\gamma_2 - 1}} \int_0^T H_2(\theta_0, s) ds = y^{\frac{1}{\gamma_2 - 1}} s_2 \\ x_3 &= y^{\frac{1}{\gamma_3 - 1}} \exp \left(rT - \frac{(r + \rho_3)T}{1 - \gamma_3} \right) =: y^{\frac{1}{\gamma_3 - 1}} s_3 \end{cases} \quad (3.9)$$

where $y = \mathcal{Y}(x)$ is uniquely defined such that $x_1 + x_2 + x_3 = x$. Define the density function $p_i(\theta_t, t, \tau) := H_i(\theta_t, \tau) \left(\int_0^{T-t} H_i(\theta_t, s) ds \right)^{-1}$ for $i = 1, 2$. Then the portfolio strategies $\boldsymbol{\pi}_t^i$ are thus determined by

$$\begin{cases} \boldsymbol{\pi}_t^1 &= \frac{1}{1 - \gamma_1} \frac{\mu_t - r}{\sigma^2} - \frac{\sigma_\theta}{(1 - \gamma_1)\sigma} \int_0^{T-t} p_1(\theta_t, t, \tau) (A_{11}(0, \tau) \theta_t + A_{21}(0, \tau)) d\tau \\ \boldsymbol{\pi}_t^2 &= \frac{1}{1 - \gamma_2} \frac{\mu_t - r}{\sigma^2} - \frac{\sigma_\theta}{(1 - \gamma_2)\sigma} \int_0^{T-t} p_2(\theta_t, t, \tau) (A_{12}(0, \tau) \theta_t + A_{22}(0, \tau)) d\tau \\ \boldsymbol{\pi}_t^3 &= \frac{1}{1 - \gamma_3} \frac{\mu_t - r}{\sigma^2} - \frac{\sigma_\theta}{(1 - \gamma_3)\sigma} (A_{13}(0, T - t) \theta_t + A_{23}(0, T - t)) \end{cases} \quad (3.10)$$

Together, equations of (3.10) and (2.14) solve the couple of investors optimal consumption and portfolio choice problem. The economic consequences of these equations are explored in the next subsection. We continue here to explore the analytical results.

Proposition 3.4. *Assume that $\theta_t > 0$. (positive MPR). If $\gamma_i < 0$, $i = 1, 2$, the CSP decreases with θ . On the other hand if $\gamma_i \geq 0$, $i = 1, 2$, the CSP increases with θ .*

Proof From direct computations one gets

$$\frac{dy}{d\theta} = \frac{y^{\frac{1}{\gamma_1 - 1}} ds_1 + y^{\frac{1}{\gamma_2 - 1}} ds_2}{\left(\frac{s_1}{1 - \gamma_1} y^{\frac{1}{\gamma_1 - 1} - 1} + \frac{s_2}{1 - \gamma_2} y^{\frac{1}{\gamma_2 - 1} - 1} + \frac{s_3}{1 - \gamma_3} y^{\frac{1}{\gamma_3 - 1} - 1} \right) d\theta}.$$

Moreover

$$\frac{d}{d\theta} \left(\frac{x_1 + x_2}{x} \right) = \frac{d}{d\theta} \left(1 - \frac{x_3}{x} \right) = -\frac{dx_3}{xd\theta} = \frac{y^{\frac{1}{\gamma_3-1}-1} dy}{(1-\gamma_3)xd\theta}.$$

If $\theta_t > 0$ and $\gamma_i < 0$, it follows from the monotonicity of A_{1i} and A_{2i} that

$$A_{1i}(0, T-t)\theta_t + A_{2i}(0, T-t) \leq \int_0^{T-t} p_i(\theta_t, t, \tau)(A_{1i}(0, \tau)\theta_t + A_{2i}(0, \tau))d\tau \leq 0.$$

On the other hand if $\gamma_i \geq 0$,

$$0 \leq \int_0^{T-t} p_i(\theta_t, t, \tau)(A_{1i}(0, \tau)\theta_t + A_{2i}(0, \tau))d\tau \leq A_{1i}(0, T-t)\theta_t + A_{2i}(0, T-t).$$

□

Remark 3.5. *During favorably market conditions, i.e., when θ is increasing, the agents behave differently according to their risk aversion. Thus, if they are more risk averse they will use a higher fraction of the initial wealth to finance investment; else if they are less risk averse they will use a higher fraction of the initial wealth to finance consumption.*

Proposition 3.6. *Assume that $\gamma_1 < \gamma_2 < \gamma_3$. Then recalling $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ from (2.12), (2.13), we get*

$$\lim_{x \rightarrow \infty} R(x) = (1 - \gamma_1) \frac{\mathcal{H}_3(1)}{\mathcal{H}_1(1)} \quad \text{and} \quad \lim_{x \rightarrow 0} R(x) = (1 - \gamma_3) \frac{\mathcal{H}_1(1)}{\mathcal{H}_3(1)}.$$

Proof Recall that $V(x) = G(\mathcal{Y}(x))$ and $V'(x) = \mathcal{Y}(x)$. Thus $V''(x) = \mathcal{Y}'(x) = 1/\mathcal{H}'(y)$ (with $y := \mathcal{Y}(x)$) and $R(x) = -x\mathcal{Y}'(x)/\mathcal{Y}(x) = -\mathcal{H}(y)/(y\mathcal{H}'(y))$. In light of

$$\mathcal{H}(y) = y^{\frac{1}{\gamma_1-1}} \mathcal{H}_1(1) + y^{\frac{1}{\gamma_2-1}} \mathcal{H}_2(1) + y^{\frac{1}{\gamma_3-1}} \mathcal{H}_3(1),$$

it follows that

$$R(x) = \frac{y^{\frac{1}{\gamma_1-1}} \mathcal{H}_1(1) + y^{\frac{1}{\gamma_2-1}} \mathcal{H}_2(1) + y^{\frac{1}{\gamma_3-1}} \mathcal{H}_3(1)}{y^{\frac{1}{\gamma_1-1}} \frac{\mathcal{H}_1(1)}{1-\gamma_1} + y^{\frac{1}{\gamma_2-1}} \frac{\mathcal{H}_2(1)}{1-\gamma_2} + y^{\frac{1}{\gamma_3-1}} \frac{\mathcal{H}_3(1)}{1-\gamma_3}},$$

whence the claim. □

Remark 3.7. *For small initial wealth or high initial wealth the couple risk aversion is driven by one of the agents. Thus, the less risk averse agent determines the couple's utility for little initial wealth. This is in accordance with risk seeking agents behavior when the latter are poor.*

3.3 Numerical results

For the numerical applications, we have chosen the following fixed parameters for the market

$$(y, r, \sigma_\theta, \lambda_\theta, \bar{\theta}) = (3, 0.048, 0.0655, 0.2712, 0.9456).$$

We first plot the fraction of initial wealth for each agent as a function of total wealth x in figure 1. In this situation, we assume that discount rates ρ_i are all equal to 0.01, but that risk aversions differ and are given by $(\gamma_1, \gamma_2, \gamma_3) = (-9, -3, -2)$.

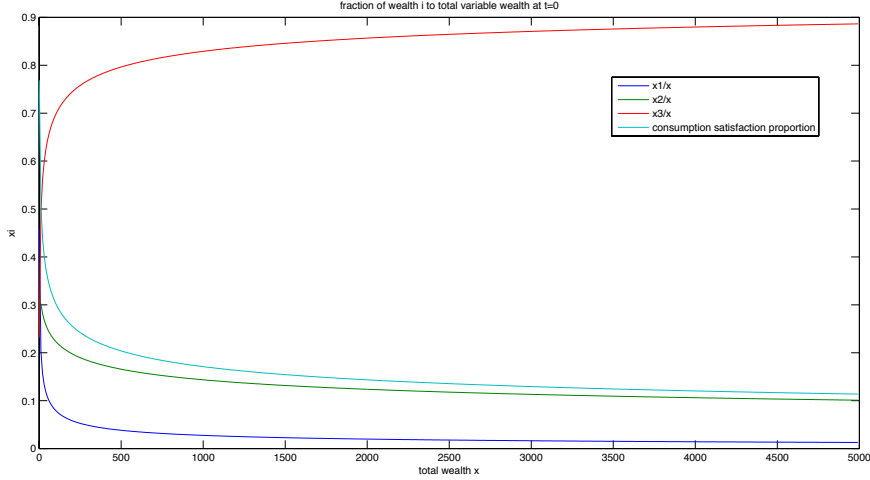


Figure 1: Fraction of initial wealth as a function of total wealth for $T = 1$.

We see from the plot that the fraction of initial wealth allocated to each of the agents is monotonous in wealth and it is higher for the less risk averse agent. Moreover, as the agents' initial wealth increases, initial wealth allocation for financing investment increases. The model with one agent only also mentions this fact, see [13].

Next we explore the effect of varying risk aversion. In figure 2, we vary γ_1 while holding γ_2, γ_3 constant. As expected, when agent 1 becomes more risk-averse his/her initial wealth allocation decreases and the initial wealth allocation for financing investment increases.

In figure 3, we vary γ_3 and fix γ_1 and γ_2 to fall below the range of γ_3 . The initial wealth allocation for financing investment increases in γ_3 .

In figure 4 we observe the effect on the initial wealth allocation of the market price of risk θ . Here $(\gamma_1, \gamma_2, \gamma_3) = (-9, -3, -2)$. The findings are in accordance with Proposition 5.2.

3.3.1 Comparison with the single-agent solution

In this subsection we perform a comparison between consumption satisfaction proportion (CSP) in single agent and multiple agent models. The effects of time discounting and risk aversion are studied separately. In figure 5 we plot CSP given $\gamma_1 = \gamma_2 = \gamma_3 = -3$ and $\rho_i \in \{0.0052, 0.3\}$, $i = 1, 2, 3$. We denote by $CSP(1, 1, 1)$ and $CSP(2, 2, 2)$ the optimal consumption satisfaction proportion in a single agent model with $\rho = 0.0052$ ($CSP(1, 1, 1)$) and $\rho = 0.3$ ($CSP(2, 2, 2)$). Moreover $CSP(1, 2, 1), CSP(1, 2, 2), CSP(1, 1, 2), CSP(2, 1, 1), CSP(2, 1, 2), CSP(2, 2, 1)$ denote the optimal consumption satisfaction proportion in a two agents model as follows:

- $CSP(1, 2, 1)$ corresponds to $(\rho_1, \rho_2, \rho_3) = (0.0052, 0.3, 0.0052)$;
- $CSP(1, 2, 2)$ corresponds to $(\rho_1, \rho_2, \rho_3) = (0.0052, 0.3, 0.3)$;
- $CSP(1, 1, 2)$ corresponds to $(\rho_1, \rho_2, \rho_3) = (0.0052, 0.0052, 0.3)$;

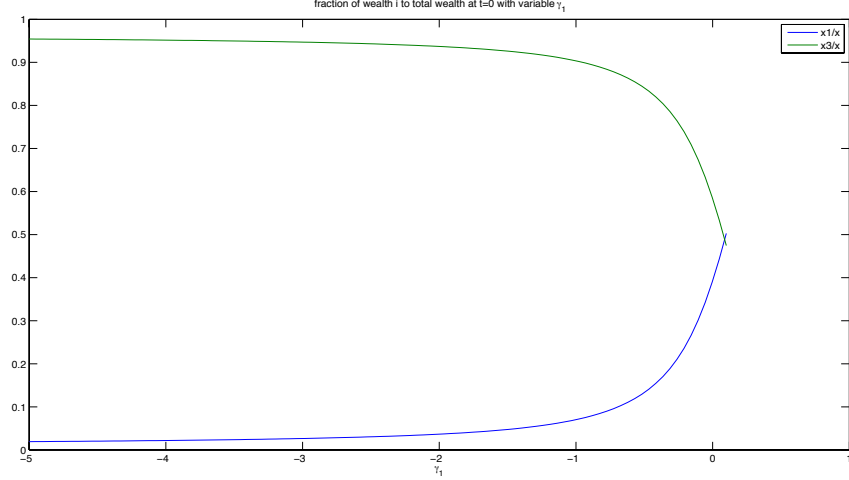


Figure 2: Fraction of initial wealth as a function of risk aversion γ_1 , for $\gamma_1 \leq \gamma_3$.

- $CSP(2, 1, 1)$ corresponds to $(\rho_1, \rho_2, \rho_3) = (0.3, 0.0052, 0.0052)$;
- $CSP(2, 1, 2)$ corresponds to $(\rho_1, \rho_2, \rho_3) = (0.3, 0.0052, 0.3)$;
- $CSP(2, 2, 1)$ corresponds to $(\rho_1, \rho_2, \rho_3) = (0.3, 0.3, 0.0052)$;

By symmetry notice that $CSP(1, 2, 2) = CSP(2, 1, 2)$ and $CSP(1, 2, 1) = CSP(2, 1, 1)$. Since $\gamma_1 = \gamma_2 = \gamma_3$, CSP does not depend on the initial wealth (see (3.9)). We observe three effects from the plot:

- when it comes to discounting utility of intertemporal consumption a higher discount rate leads to higher CSP (see $CSP(1, 2, 1) = CSP(2, 1, 1) > CSP(1, 1, 1)$, $CSP(1, 2, 2) = CSP(2, 1, 2) > CSP(1, 1, 2)$)
- when it comes to discounting utility of final wealth a higher discount rate leads to lower CSP (see $CSP(2, 2, 1) > CSP(2, 2, 2)$)
- discounting has a higher impact on the utility of final wealth (see $CSP(1, 1, 1) > CSP(2, 2, 2)$)

In figure 6 we plot CSP given $\rho_1 = \rho_2 = \rho_3 = 0.0052$ and $\gamma_i \in \{-3, -9\}$, $i = 1, 2, 3$. We denote by $CSP(1, 1, 1)$ and $CSP(2, 2, 2)$ the optimal consumption satisfaction proportion in a single agent model with $\gamma = -3$ ($CSP(1, 1, 1)$) and $\gamma = -9$ ($CSP(2, 2, 2)$). Moreover $CSP(1, 2, 1)$, $CSP(1, 2, 2)$, $CSP(1, 1, 2)$, $CSP(2, 1, 1)$, $CSP(2, 1, 2)$, $CSP(2, 2, 1)$ denote the optimal consumption satisfaction proportion in a two agents model as follows:

- $CSP(1, 2, 1)$ corresponds to $(\gamma_1, \gamma_2, \gamma_3) = (-3, -9, -3)$;
- $CSP(1, 2, 2)$ corresponds to $(\gamma_1, \gamma_2, \gamma_3) = (-3, -9, -9)$;
- $CSP(1, 1, 2)$ corresponds to $(\gamma_1, \gamma_2, \gamma_3) = (-3, -3, -9)$;

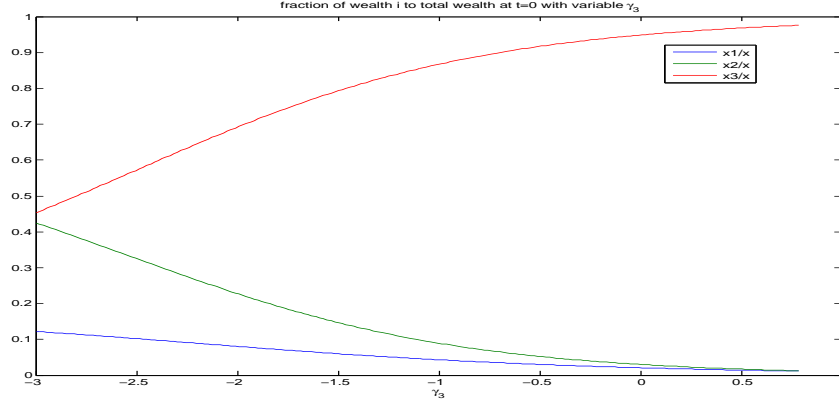


Figure 3: Fraction of initial wealth as a function of γ_3 .

- $CSP(2, 1, 1)$ corresponds to $(\gamma_1, \gamma_2, \gamma_3) = (-9, -3, -3)$;
- $CSP(2, 1, 2)$ corresponds to $(\gamma_1, \gamma_2, \gamma_3) = (-9, -3, -9)$;
- $CSP(2, 2, 1)$ corresponds to $(\gamma_1, \gamma_2, \gamma_3) = (-9, -9, -3)$;

By symmetry notice that $CSP(1, 2, 2) = CSP(2, 1, 2)$ and $CSP(1, 2, 1) = CSP(2, 1, 1)$. We observe two effects from the plot:

- when it comes to utility of intertemporal consumption a higher γ leads to lower CSP (see $CSP(1, 2, 1) = CSP(2, 1, 1) > CSP(1, 1, 1)$, $CSP(2, 2, 2) > CSP(2, 1, 2)$)
- when it comes to utility of final wealth a higher γ leads to higher CSP (see $CSP(2, 2, 1) > CSP(2, 2, 2)$)

4 Appendix: Proof of Theorem 2.18

For the purpose of this section, we first start with basic properties of portfolio processes in the complete market setting. We then introduce the solution for each sub-problem in order to compare the value functions as in Theorem 2.18.

4.1 Super-martingale property of admissible portfolios

We shall show that $C(x)$ consists of exactly those “reasonable” consumption processes, for which the couple of investors, starting out with wealth x at time 0, is able to construct a portfolio that avoids debt (i.e., negative wealth) on $[0, T]$ \mathbb{P} -almost surely.

Definition 4.1. *We define*

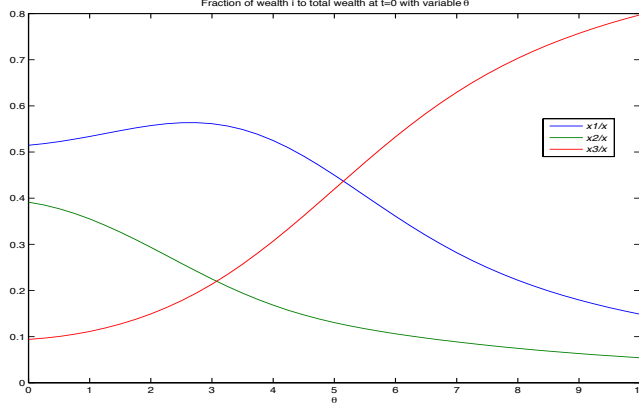


Figure 4: Fraction of initial wealth as a function of θ .

- $C(x)$ (resp. $D(x)$) the class of consumption processes (c^1, c^2) which satisfy

$$\tilde{\mathbb{E}} \left[\int_0^T \tilde{C}_t dt \right] \leq x \quad (\text{resp. } = x) ; \quad (4.1)$$

- $L(x)$ (resp. $M(x)$) the class of non negative random variables L on $(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}})$ which satisfy

$$\tilde{\mathbb{E}} [L] \leq x \quad (\text{resp. } = x) ; \quad (4.2)$$

- $P(x)$ the class of portfolio strategies π such that $(\pi, 0, 0) \in A(x)$.

Remark 4.2. Since $X_T \geq 0$ and $C_t \geq 0$, for all $t \in [0, T]$, $(\pi, c^1, c^2) \in A(x)$ implies $(c^1, c^2) \in C(x)$, and $X_T \in L(x)$ implies inequality conditions (4.1) and (4.2) which turn out to be also sufficient for admissibility. Moreover, according to (2.6), the set $P(x)$ corresponds to strategies such that X_T belongs to $M(x)$.

Proposition 4.3. For every given $(c^1, c^2) \in C(x)$, there exists a portfolio strategy π such that $(\pi, c^1, c^2) \in A(x)$.

Proof It is a straightforward application of martingale representation theorem; for more details see [8]. \square

Remark 4.4. The wealth process X corresponding to any $(c^1, c^2) \in D(x)$ satisfies

$$X_t = \tilde{\mathbb{E}} \left[\int_t^T \tilde{C}_s ds \mid \mathcal{F}_t \right], \quad 0 \leq t \leq T \quad (4.3)$$

In particular, $X_T = 0$ \mathbb{P} -a.s.

Proposition 4.5. For every $L \in L(x)$, there exists a trio $(\pi, c^1, c^2) \in A(x)$ with corresponding wealth process X , for which $X_T = L$ \mathbb{P} -a.s.

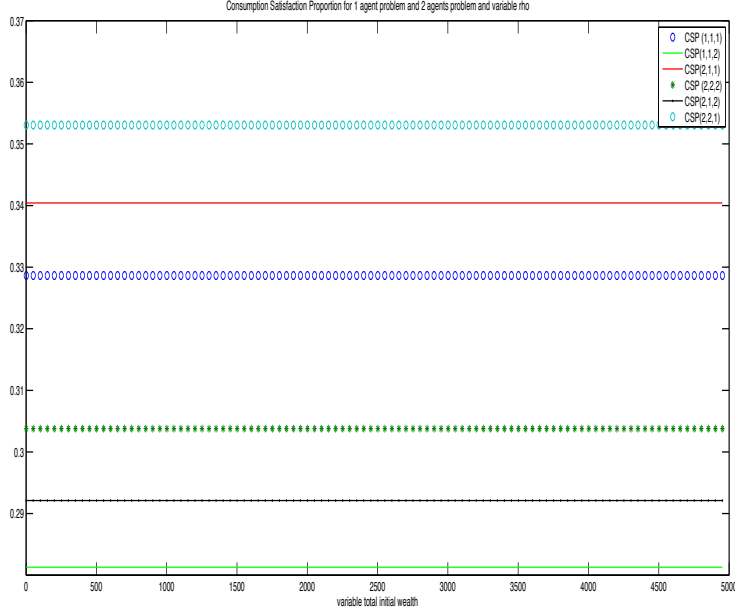


Figure 5: Fraction of consumption satisfaction proportion (CSP) as a function of ρ and initial wealth.

Proof It is a straightforward application of martingale representation theorem; for more details see [8]. \square

Corollary 4.6. *For any given $L \in L(x)$, there exists a portfolio strategy $\pi \in P(x)$ with corresponding wealth process*

$$X_t = \tilde{\mathbb{E}}[L|\mathcal{F}_t], \quad \text{for } t \in [0, T]. \quad (4.4)$$

This corollary shows that the extreme elements of $L(x)$ are attainable by strategies that mandate zero consumption.

4.2 Proof of Theorem 2.16

Proposition 4.7 (Consumption problem). *Let $a_1 \geq 0$. Then $V_1(a_1) = J_1(a_1; \boldsymbol{\pi}^1(a_1), \mathbf{c}^1(a_1), 0)$ where \mathbf{c}^1 is defined in Theorem 2.16 and $\boldsymbol{\pi}^1$ follows from Proposition 4.3.*

Proof We take

$$c^i = \hat{c}_t^i := \frac{a_i}{\tilde{\mathbb{E}} \int_0^T D_t dt} \in D(a_i),$$

so that

$$\mathbb{E} \left[\int_0^T B_t^i U_i(\hat{c}_t^i) dt \right] = U_i(\hat{c}^i) \mathbb{E} \left[\int_0^T B_t^i dt \right] < \infty.$$

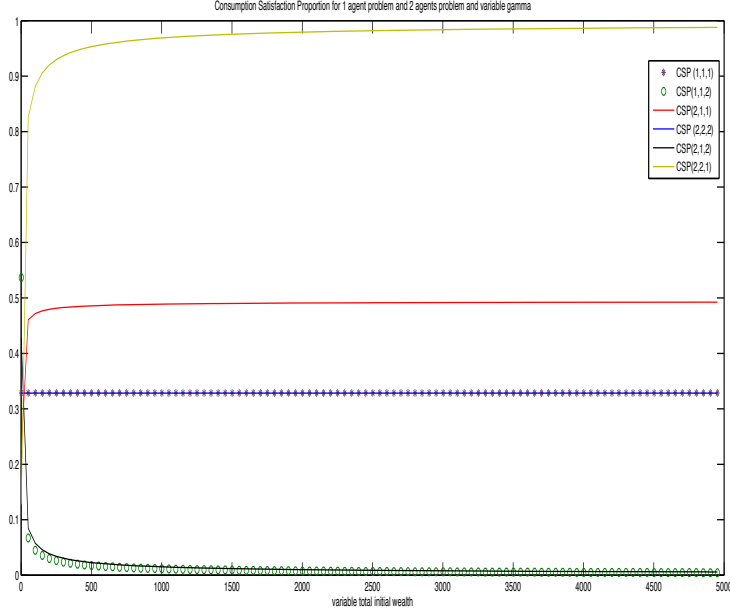


Figure 6: Fraction of consumption satisfaction proportion (CSP) as a function of γ and initial wealth.

Notice that $\hat{c}^i \in D(a_i)$ and since $\tilde{\mathbb{E}} \left[\int_0^T D_t \mathbf{c}_t^i dt \right] = \mathcal{H}_i(\mathcal{Y}_i(a_i)) = a_i$, $\mathbf{c}_t^i \in D(a_i)$. Inequality (2.9) implies that for any $c^i \in C(a_i)$ and $t \in [0, T]$,

$$U_i(\mathbf{c}_t^i) \geq U_i(c_t^i) + \mathcal{Y}_i(a_i) \zeta_t^i \mathbf{c}_t^i - \mathcal{Y}_i(a_i) \zeta_t^i c_t^i, \quad \mathbb{P} - \text{a.s.}$$

Therefore,

$$\begin{aligned} \mathbb{E} \left[\int_0^T B_t^i U_i^-(\mathbf{c}_t^i) dt \right] &\leq \mathbb{E} \left[\int_0^T B_t^i \left(U_i(\hat{c}_t^i) + \mathcal{Y}_i(a_i) \zeta_t^i \mathbf{c}_t^i - \mathcal{Y}_i(a_i) \zeta_t^i \hat{c}_t^i \right)^- dt \right] \\ &\leq \mathbb{E} \left[\int_0^T B_t^i \left(U_i^-(\hat{c}_t^i) + \mathcal{Y}_i(a_i) \zeta_t^i \hat{c}_t^i \right) \right] < \infty. \end{aligned}$$

Consider the measure on $[0, T] \times \Omega$ defined by $d\nu^i(t, \omega) = B_t^i dt \mathbb{P}(d\omega)$. For any other consumption process $c^i \in D(a_i)$, we have

$$\int \int_{[0, T] \times \Omega} U_i(\mathbf{c}_t^i) d\nu^i \geq \int \int_{[0, T] \times \Omega} U_i(c_t^i) d\nu^i + \int \int_{[0, T] \times \Omega} \mathcal{Y}_i(a_i) \zeta_t^i \mathbf{c}_t^i d\nu^i - \int \int_{[0, T] \times \Omega} \mathcal{Y}_i(a_i) \zeta_t^i c_t^i d\nu^i$$

By using the fact that $\mathbf{c}^i, c^i \in D(a_i)$,

$$\mathbb{E} \left[\int_0^T B_t^i U_i(\mathbf{c}_t^i) dt \right] \geq \mathbb{E} \left[\int_0^T B_t^i U_i(c_t^i) dt \right], \quad i = 1, 2.$$

□

Remark 4.8. *The same result holds for problem V_2 in an obvious manner. Since $(\pi, c^1, c^2) \in A_1(a_1)$ implies that $(\pi, c^1 + c^2, 0) \in A_1(a_1)$,*

$$J_1(a_1; \pi, c^1, c^2) \leq J_1(a_1; \pi, c^1 + c^2, 0).$$

Thus, since U_1 is an increasing function, the functional J_1 is maximized when $c^2 = 0$. The reciprocal holds for J_2 .

Proposition 4.9 (Final wealth problem). *Let $a_3 \in \mathbb{R}_+$. Then*

$$V_3(a_3) = J_3(a_3; \boldsymbol{\pi}^3(a_3), 0, 0). \quad (4.5)$$

The corresponding final wealth is given by

$$\mathbf{X}_T^3(a_3) = I_3(\mathcal{Y}_3(a_3)\zeta_T^3) \in M(a_3). \quad (4.6)$$

Proof 1. First, we show that the strategy $\boldsymbol{\pi}^3(a_3) \in P(a_3)$ and that the generated portfolio process $\mathbf{X}^3(a_3)$ belongs to $M(a_3)$. According to (4.6), we have

$$\tilde{\mathbb{E}}[\mathbf{X}_T^3(a_3)] = \tilde{\mathbb{E}}[I_3(\mathcal{Y}_3(a_3)\zeta_T)] = \mathcal{H}_3(\mathcal{Y}_3(a_3)) = a_3.$$

Considering the constant final wealth $b := a_3/\tilde{\mathbb{E}}[D_T] \in D(a_3)$, we get

$$U_3(\mathbf{X}_T^3(a_3)) \geq U_3(b) + \mathcal{Y}_3(a_3)\zeta_T^3\mathbf{X}_T^3(a_3) - \mathcal{Y}_3(a_3)\zeta_T^3b \quad \mathbb{P} - a.s.$$

Therefore,

$$\mathbb{E}[B_T^3 U_3^-(\mathbf{X}_T^3(a_3))] \leq \mathbb{E}[B_T^3 (U_3^-(b) + \mathcal{Y}_3(a_3)\zeta_T^3b)] < \infty.$$

2. Let's show that the optimal strategy requires zero consumption. Let $(\pi, c^1, c^2) \in A(a_3)$ with wealth process X be given. Define the random variable

$$B := \begin{cases} \frac{a_3}{\tilde{\mathbb{E}}[D_T X_T]} X_T & \text{if } \tilde{\mathbb{E}}[X_T] > 0 \\ b & \text{if } \tilde{\mathbb{E}}[X_T] = 0 \end{cases}.$$

Since $X_T \in A(a_3)$, $\tilde{\mathbb{E}}[D_T X_T] \leq a_3$. Then $B \in M(a_3)$ and $B \geq X_T$ \mathbb{P} -a.s. From Proposition 4.5 and Corollary 4.6, there exists a portfolio $\hat{\pi} \in P(a_3)$ with corresponding terminal wealth $\hat{X}_T = B \geq X_T$ \mathbb{P} -a.s. Thus $(\hat{\pi}, 0, 0) \in A_3(a_3)$ and $J_3(a_3, \pi, c^1, c^2) \leq J_3(a_3, \hat{\pi}, 0, 0)$.

3. To obtain (4.5), it suffices to proceed as in Proposition 4.7:

$$\mathbb{E}[B_T^3 U_3(\mathbf{X}_T^3(x_3))] \geq \mathbb{E}[B_T^3 (U_3(\mathbf{X}_T) + \mathcal{Y}_3(x_3)\zeta_T^3(\mathbf{X}_T^3(x_3) - X_T))] \geq \mathbb{E}[B_T^3 U_3(\mathbf{X}_T)].$$

□

4.3 Proof of Theorem 2.18

Having the solution to sub-problems, we turn to the solution of problem (2.10).

Proposition 4.10. *For $x \geq 0$,*

$$V(x) = V_*(x) := \max \{V_1(a_1) + V_2(a_2) + V_3(a_3) | a_1, a_2, a_3 \in [0, \infty); a_1 + a_2 + a_3 = x\} . \quad (4.7)$$

Proof For $x \geq 0$, we are given an arbitrary triplet $(\pi, c^1, c^2) \in \tilde{A}(x)$ with corresponding wealth process X_t . Recall that

$$a_1 := \tilde{\mathbb{E}} \left[\int_0^T B_t^1 c_t^1 dt \right] , \quad a_2 := \tilde{\mathbb{E}} \left[\int_0^T B_t^2 c_t^2 dt \right] \quad \text{and} \quad a_3 := \tilde{\mathbb{E}} [B_T^3 X_T] .$$

By the super martingale property, $a := a_1 + a_2 + a_3 \leq x$ and by Propositions 4.7 and 4.9,

$$\begin{cases} \mathbb{E} \left[B_t^1 U_1(c_t^1) dt \right] \leq J_1(a_1; \boldsymbol{\pi}^1(a_1), \mathbf{c}^1(a_1), 0) = V_1(a_1), \\ \mathbb{E} \left[B_t^2 U_2(c_t^2) dt \right] \leq J_2(a_2; \boldsymbol{\pi}^2(a_2), 0, \mathbf{c}^2(a_2)) = V_2(a_2), \\ \mathbb{E} \left[B_T^3 U_3(X_T) \right] \leq J_3(a_3; \boldsymbol{\pi}^3(a_3), 0, 0) = V_3(a_3). \end{cases}$$

Adding the three terms, we get

$$V_1(a_1) + V_2(a_2) + V_3(a_3) = J(a; \boldsymbol{\pi}, \mathbf{c}^1(a_1), \mathbf{c}^2(a_2)) \geq J(a; \pi, c^1, c^2) .$$

Taking the supremum over $a_1 + a_2 + a_3 \leq x$ and over $(\pi, c^1, c^2) \in \tilde{A}(x)$, we get

$$\begin{aligned} V(x) &\leq \sup \{V_1(a_1) + V_2(a_2) + V_3(a_3) : a_1, a_2, a_3 \in [0, \infty); a_1 + a_2 + a_3 \leq x\} \\ &= \sup \{V_1(a_1) + V_2(a_2) + V_3(a_3) : a_1, a_2, a_3 \in [0, \infty); a_1 + a_2 + a_3 = x\} := V_*(x) \end{aligned}$$

from the non-decreasing characteristic of V_i for $i = 1, 2, 3$. Furthermore, by continuity of the function $(a_1, a_2, a_3) \mapsto V_1(a_1) + V_2(a_2) + V_3(a_3)$, the supremum above is attained at a point (x_1, x_2, x_3) and

$$V(x) \leq V_*(x) = V_1(x_1) + V_2(x_2) + V_3(x_3) = J(x; \boldsymbol{\pi}(x_1, x_2, x_3), \mathbf{c}^1(x_1), \mathbf{c}^2(x_2)) \leq V(x) .$$

The processes \mathbf{X}^1 , \mathbf{X}^2 and \mathbf{X}^3 are nonnegative, so \mathbf{X} is nonnegative: \mathbf{X} is clearly in $\tilde{A}(x)$. \square

We conclude the proof of Theorem 2.18 by saying that the x_i are found by using the envelope theorem, together with Lemma 4.11 below, which implies that

$$V_1'(x_1) = V_2'(x_2) = V_3'(x_3) = \mathcal{Y}_1(x_1) = \mathcal{Y}_2(x_2) = \mathcal{Y}_3(x_3) = y ,$$

i.e., $x_i = \mathcal{H}_i(y) = \mathcal{H}_i(\mathcal{Y}(x))$.

Lemma 4.11. *For $y > 0$, define*

$$G_1(y) := \mathbb{E} \left[\int_0^T B_t^1 U_1(I_1(y \zeta_t^1)) dt \right] , \quad (4.8)$$

$$G_2(y) := \mathbb{E} \left[\int_0^T B_t^2 U_2(I_2(y \zeta_t^2)) dt \right] , \quad (4.9)$$

$$G_3(y) := \mathbb{E} [B_T^3 U_3(I_3(y \zeta_T^3))] . \quad (4.10)$$

Then

$$G'_i(y) = y\mathcal{H}'_i(y) \quad i = 1, 2, 3 \quad (4.11)$$

and $V_i \in C^2((0, \infty))$ with

$$V'_i(x) = \mathcal{Y}_i(x) \quad i = 1, 2, 3, \quad x \geq 0. \quad (4.12)$$

Proof According to Assumption 2.14, we can take derivatives under the expectation and integral signs to obtain

$$\begin{aligned} G'_1(y) &= \mathbb{E} \int_0^T B_t^1 \zeta_t^1 I'_1(y\zeta_t^1) U'_1(I_1(y\zeta_t^1)) dt = \mathbb{E} \int_0^T B_t^1 \zeta_t^1 y \zeta_t^1 I'_1(y\zeta_t^1) dt \\ &= \tilde{\mathbb{E}} \int_0^T y \zeta_t^1 I'_1(y\zeta_t^1) dt = y\mathcal{H}'_1(y). \end{aligned}$$

Therefore,

$$V'_1(x) = \frac{d}{dx} G_1(\mathcal{Y}_1(x)) = Y'_1(x) G'_1(\mathcal{Y}_1(x)) = \mathcal{Y}'_1(x) \mathcal{Y}_1(x) \mathcal{H}'_1(\mathcal{Y}_1(x)) = \mathcal{Y}_1(x).$$

The other derivatives are computed in the same manner. □

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