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Optimal allocation of wealth for two consuming agents sharing a portfolio

O. Mbodji, A. Nguyen Huu, T. Pirvu

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Abstract

We study the Merton problem of optimal consumption-investment for the case of two investors sharing a final wealth. The typical example would be a husband and wife sharing a portfolio looking to optimize the expected utility of consumption and final wealth. Each agent has different utility function and discount factor. An explicit formulation for the optimal consumptions and portfolio can be obtained in the case of a complete market. The problem is shown to be equivalent to maximizing three different utilities separately with separate initial wealths. We study a numerical example where the market price of risk is assumed to be mean reverting, and provide insights on the influence of risk aversion or discount rates on the initial optimal allocation.

1 Introduction

In the portfolio optimization literature, the single agent framework constitutes the main problem. However, several financial problems involve many agents, e.g., when a household manages a shared portfolio. One can also think of the situation of a portfolio manager working for a pool of clients. The question raised here is thus: how do separate agents consume resources out of a common financial portfolio? Economic theory answers the question, at least partially, by introducing social welfare and a common (utility) function to model the aggregated preferences of several heterogeneous agents, see for example [1]. In the present paper, we investigate the specific form of Utilitarian social welfare function composed of the linear combination of individual discounted utility functions. That form is of particular importance for its simplicity, but also because it involves the well-known notions of risk-aversion and individual discount factor (preference for the present).

Imagine the following initial situation. Two agents, having utility functions U_1 and U_2 for consumption c^1 and c^2 respectively and discount rates $\beta^1(t)$ and $\beta^2(t)$, share a self-financed portfolio X_t over a period $[0, T]$ with $T > 0$ finite. They also share a third utility function U_3 of terminal wealth X_T discounted with rate $\beta^3(t)$. The goal of the couple is then to maximize at time 0 the sum of the three expected discounted utilities from consumption and wealth over

$[0, T]$ i.e., the quantity

$$\mathbb{E} \left[\int_0^T e^{-\int_0^t \beta^1(s) ds} U_1(c_t^1) dt + \int_0^T e^{-\int_0^t \beta^1(s) ds} U_2(c_t^2) dt + e^{-\int_0^T \beta^3(s) ds} U_3(X_T) \right]. \quad (1.1)$$

This expression can be obviously generalized to any linear combination of a number $n \in \mathbb{N}$ of utility functions for consuming agents.

When the total initial wealth X_0 is given, the portfolio management problem brings up the question of wealth allocation among participants in order to contempt them, additionally to the one of optimal consumption-portfolio strategy for each of them. This initial allocation problem is a one-time static problem at time $t = 0$. It is of fundamental importance to notice that if the criterion is updated at a later date $t > 0$, the solution changes and does not correspond to the wealth obtained by the allocation at date 0 and subsequent optimal portfolio strategies. This means that the problem lacks a time-consistency property, see [2]. Indeed, the problem with two agents can easily be reduced to a one agent problem by considering the utility function

$$U(s, t, C) := \sup_{c^1 + c^2 = C} \left[\exp \left(- \int_s^t \beta^1(s) ds \right) U_1(c^1) + \exp \left(- \int_s^t \beta^2(s) ds \right) U_2(c^2) \right].$$

It is foreseeable that the optimal behavior of the agent will depend on the initial date s , and therefore be given up at a later date without any commitment device. This is why we reduce ourselves to the problem of initial allocation at date 0, and suppose that this action commits our two agents on the interval $[0, T]$. In a future companion paper, we solve the couple problem without commitment in a time-consistent manner, by the use of sub game perfect strategies as in [2].

Coming back to (1.1), we actually show that the above problem can be divided in three separate problems involving only one agent at a time. Thus, in a sense, the only real decision on the part of the investor takes place at $t = 0$ with the determination of the initial wealth allocated to each agent. Once the initial allocation is provided, the further evolution of interesting quantities (consumptions and wealth) follows well known solutions provided by [5].

In order to illustrate the allocation solution, we provide a numerical application with closed form solutions in the framework of [11], i.e., with power utilities and mean reverting market price of risk. We naturally focus on the comparison between the two consuming agents initial wealth as a function of risk aversion and discount rates. We find interesting insights for portfolio managers. As the agent's initial wealth increases, an increasing proportion of the wealth is allocated to finance the portfolio. Furthermore, the less risk averse consumer allocates more money for future consumption for sufficiently large values of the initial wealth. The effects of risk aversion on the allocations are also intuitive. As agents become less risk averse, the allocated fraction of initial wealth increases.

The article is organized as follows: Section 2 introduces the model, admissibility conditions and discusses complete market properties. In Section 3, we introduce the total value function and the three separated sub-problems. We provide their solution by relying on duality methods. The proof of these results are provided in Section 4. The numerical application is discussed in Section 5.

2 Market model and super-martingale property of portfolios

2.1 Complete market and specification of agents

We place ourselves in a complete financial market with d tradable risky assets with prices driven by Wiener processes, and a riskless asset. This framework has been considered by Karatzas and al. [5] for a single investor and semi-explicit solutions are provided using a martingale approach. Six [10] considered the special case of a single investor with two different power utilities: one for consumption and one for final wealth. We show that the same method applies in our context without difficulty.

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with standard d -dimensional Brownian motion $W := (W_t)_{t \in [0, T]} = (W_t^1, \dots, W_t^d)_{t \in [0, T]}^T$. The filtration $(\mathcal{F}_t)_{t \in [0, T]}$ is the augmentation under \mathbb{P} of the natural filtration of W . We consider a complete market composed of $d + 1$ assets (S^0, \dots, S^d) which are continuously traded on $[0, T]$ and evolve according to the differential equations $dS_t^0 = r(t)S_t^0 dt$ for $t \in [0, T]$ with $S_0^0 = s_0$, and

$$dS_t^i = S_t^i \left(b_i(t) dt + \sum_{j=1}^d \sigma_{ij}(t) dW_t^j \right), \quad 0 \leq t \leq T \quad (2.1)$$

with $(S_0^1, \dots, S_0^d) = (s_1, \dots, s_d) \in (0, \infty)^d$. Let us assume that the SDE (2.1) has a unique strong solution. The interest rate process $(r(t))_{t \in [0, T]}$ is adapted and bounded, uniformly in $(t, \omega) \in [0, T] \times \Omega$. Throughout this paper, we replace all asset prices by the discounted asset prices. The discount factor using the risk-free rate is

$$D_t := \exp \left(- \int_0^t r(u) du \right). \quad (2.2)$$

For a generic process Y_t , we introduce the notation $\tilde{Y}_t := Y_t D(t)$ to denote its discounted counterpart. The vector of mean rates of excess return $b(t) := (b_1(t) \dots b_d(t))_{t \in [0, T]}^T$ and the diffusion matrix $\sigma(t) := (\sigma_{ij}(t))_{1 \leq i, j \leq d, t \in [0, T]}$ are assumed to be adapted and bounded, uniformly in $(t, \omega) \in [0, T] \times \Omega$. We introduce the covariance matrix $a(t) = \sigma(t) \sigma^T(t)$ and assume that for some $\varepsilon > 0$, $\zeta^T a(t, \omega) \zeta \geq \varepsilon \|\zeta\|^2$, for any $(t, \omega) \in [0, T] \times \Omega$ and any $\zeta \in \mathbb{R}^d$.

Definition 2.1. *We introduce the following objects:*

1. A portfolio strategy $\pi := \{\pi(t) = (\pi_1(t), \dots, \pi_d(t))^T\}$ is an adapted, \mathbb{R}^d -valued process where $\pi_i(\omega) \in L^2([0, T])$ for \mathbb{P} -almost every $\omega \in \Omega$ and $i = 1, \dots, d$.
2. A consumption process is given by $(c_t^1, c_t^2)_{t \in [0, T]}$, an adapted process with non-negative values such that $C(\omega) := c^1(\omega) + c^2(\omega)$ is in $L^1([0, T])$ for \mathbb{P} -almost every $\omega \in \Omega$.
3. The wealth process $X := (X_t)_{t \in [0, T]}$ is uniquely defined as

$$X_t = \frac{1}{D(t)} \left(x + \int_0^t ((\pi^T(s)b(s) - C_s)) D(s) ds + \pi^T(s) \sigma(s) D(s) dW_s \right) \quad (2.3)$$

or equivalently by the discounted process

$$\tilde{X}_t = \left(x + \int_0^t \left(\tilde{\pi}^T(s)b(s) - \tilde{C}_s \right) ds + \tilde{\pi}^T(s)\sigma(s)dW_s \right). \quad (2.4)$$

Each agent i is endowed with a utility function U_i and a discount rate β^i with corresponding discount factor

$$B_t^i := \exp \left(- \int_0^t \beta^i(s) ds \right). \quad (2.5)$$

The discount rates are assumed to be adapted and bounded for all t uniformly \mathbb{P} -almost surely. We assume the following for the utility functions:

Assumption 2.2. For $i = 1, 2, 3$, we assume that U_i is a strictly increasing, strictly concave real-valued function in $C^2([0, \infty])$ such that U_i' is non decreasing, $U_i(0) \geq -\infty$ and $U_i'(\infty) = 0$. U_i' is defined from $[0, \infty)$ onto $[0, U_i'(0)]$.

Note that we allow for $U_i(0) = -\infty$ or $U_i'(0) = \infty$. This framework encompasses a large class of functions, including CARA and HARA utility functions.

2.2 Super-martingale property of admissible portfolios

The completeness of the market in the sense of [4] implies the existence of a unique \mathbb{P} -equivalent martingale measure $\tilde{\mathbb{P}}$. Define the price of risk process $\theta(t) := \sigma(t)^{-1}(b(t) - r(t)\mathbf{1})$, for $t \in [0, T]$. Next, introduce the Radon Nikodym derivative of $\tilde{\mathbb{P}}$ w.r.t. \mathbb{P} ,

$$Z(t) := \exp \left\{ - \sum_{i=1}^d \int_0^t \theta_i(s) dW_s^i - \frac{1}{2} \int_0^t \|\theta(s)\|^2 ds \right\}, \quad \text{for } t \in [0, T]. \quad (2.6)$$

We can define $\tilde{\mathbb{E}}$ the expectation operator under $\tilde{\mathbb{P}}$. Under some integrability conditions on θ , $\tilde{W}_t := W_t + \int_0^t \theta(s) ds$ is a Brownian motion under $\tilde{\mathbb{P}}$ (see [9]). Thus

$$\tilde{X}_t + \int_0^t \tilde{C}_s ds = x + \int_0^t \tilde{\pi}^T(s)\sigma(s)d\tilde{W}(s). \quad (2.7)$$

We now introduce the key concept of admissibility.

Definition 2.3. A triplet (π, c^1, c^2) of strategy and consumption processes is said to be admissible for the initial endowment $x \geq 0$ if the wealth process X satisfies $X_t \geq 0$ for $[0, T]$ \mathbb{P} -a.s. We call $A(x)$ the class of admissible processes (π, c^1, c^2) for initial wealth x .

For any $(\pi, c^1, c^2) \in A(x)$, the left-hand side of (2.7) is non negative and the right-hand side is a local martingale under $\tilde{\mathbb{P}}$. It follows that the left-hand side, and hence also \tilde{X}_t , is a non

negative super-martingale under $\tilde{\mathbb{P}}$. Now, if $\tau_0 := T \wedge \inf\{0 \leq t \leq T, X(t) = 0\}$, then $X_t = 0$ for all $t \in [\tau_0, T]$ on $\{\tau_0 > -\infty\}$. The super martingale property in (2.7) yields

$$\tilde{\mathbb{E}} \left[\tilde{X}_T + \int_0^T \tilde{C}_t dt \right] \leq x. \quad (2.8)$$

This property allows to express admissibility of strategies and consumptions respectively in a different manner.

Definition 2.4. *We define*

- $C(x)$ (resp. $D(x)$) the class of consumption processes (c^1, c^2) which satisfy

$$\tilde{\mathbb{E}} \left[\int_0^T \tilde{C}_t dt \right] \leq x \quad (\text{resp. } = x); \quad (2.9)$$

- $L(x)$ (resp. $M(x)$) the class of non negative random variables L on $(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}})$ which satisfy

$$\tilde{\mathbb{E}} [L] \leq x \quad (\text{resp. } = x); \quad (2.10)$$

- $P(x)$ the class of portfolio strategies π such that $(\pi, 0, 0) \in A(x)$.

Remark 2.5. *Since $X_T \geq 0$ and $C_t \geq 0$, for all $t \in [0, T]$, $(\pi, c^1, c^2) \in A(x)$ implies $(c^1, c^2) \in C(x)$, and $X_T \in L(x)$ implies inequality conditions (2.9) and (2.10) which turn out to be also sufficient for admissibility. Moreover, according to (2.7), the set $P(x)$ corresponds to strategies such that X_T belongs to $M(x)$.*

We shall show that $C(x)$ consists of exactly those “reasonable” consumption processes, for which the couple of investors, starting out with wealth x at time 0, is able to construct a portfolio that avoids debt (i.e., negative wealth) on $[0, T]$ \mathbb{P} -almost surely.

Proposition 2.6. *For every given $(c^1, c^2) \in C(x)$, there exists a portfolio strategy π such that $(\pi, c^1, c^2) \in A(x)$.*

Proof Let $I := \int_0^T \tilde{C}_t dt$, and define the non-negative process $(N_t)_{t \in [0, T]}$ by

$$N_t := x - \tilde{\mathbb{E}}I + \tilde{\mathbb{E}} \left[\int_t^T \tilde{C}_s ds | \mathcal{F}_t \right] = \left(x + m_t - \int_0^t \tilde{C}_s ds \right)$$

where $m_t := \tilde{\mathbb{E}}[I | \mathcal{F}_t] - \tilde{\mathbb{E}}I$ is a $\tilde{\mathbb{P}}$ -martingale. Note that $N_t \geq 0$ because of (2.9). By the martingale representation theorem [9], we can find an adapted process $\phi(t) \in L^2(\mathbb{R}^d \times [0, T])$ with values in \mathbb{R}^d \mathbb{P} -a.s., so that

$$m_t = \sum_{j=1}^d \int_0^t \phi_j(s) d\tilde{W}_s^j \quad \text{for all } t \in [0, T].$$

By taking $\pi(t) = (D(t)\sigma^T(t))^{-1}\phi(t)$, we then get that $X_t = N_t$ and the consumption process (c^1, c^2) correspond to the wealth process X_t . \square

Remark 2.7. The wealth process X corresponding to any $(c^1, c^2) \in D(x)$, is given by

$$X_t = \tilde{\mathbb{E}} \left[\int_t^T \tilde{C}_s ds | \mathcal{F}_t \right], \quad 0 \leq t \leq T \quad (2.11)$$

In particular, $X_T = 0$ \mathbb{P} -a.s.

Proposition 2.8. For every $L \in L(x)$, there exists a trio $(\pi, c^1, c^2) \in A(x)$ with corresponding wealth process X , for which $X_T = L$ \mathbb{P} -a.s.

Proof Define the non negative process η by

$$\begin{aligned} \eta_t D_t &:= \tilde{\mathbb{E}} \left[\tilde{L} | \mathcal{F}_t \right] + \left(x - \tilde{\mathbb{E}}[\tilde{L}] \right) \left(1 - \frac{t}{T} \right) \\ &= x + m_t - \rho t, \end{aligned} \quad (2.12)$$

where

$$m_t := \tilde{\mathbb{E}} \left[\tilde{L} | \mathcal{F}_t \right] - \tilde{\mathbb{E}}[\tilde{L}] \quad \text{and} \quad \rho := \frac{x - \tilde{\mathbb{E}}[\tilde{L}]}{T}. \quad (2.13)$$

Obviously, $\eta_0 = x$ and $\eta_T = L$ \mathbb{P} -a.s. Furthermore (2.10) implies that $\eta_t \geq 0$. We can obtain a stochastic integral representation of the form (2.13) for the $\tilde{\mathbb{P}}$ -martingale m . Then (2.12) is cast in the form (2.13) once we take π as in Proposition 2.6, $C_t = \rho/D_t$, for $t \in [0, T]$ and $X_t = \eta_t$. Therefore $X_T = L$. \square

Corollary 2.9. For any given $L \in L(x)$, there exists a portfolio strategy $\pi \in P(x)$ with corresponding wealth process

$$X_t = \tilde{\mathbb{E}}[L | \mathcal{F}_t], \quad \text{for } t \in [0, T]. \quad (2.14)$$

This corollary shows that the extreme elements of $L(x)$ are attainable by strategies that mandate zero consumption.

3 The portfolio management problem

Recall that the discount function for agent i is given by (2.5). For a given $x \geq 0$, we define the value function at x by

$$V(x) := \sup \left\{ J(x; \pi, c^1, c^2) : (\pi, c^1, c^2) \in \tilde{A}(x) \right\}, \quad (3.1)$$

where

$$J(x; \pi, c^1, c^2) := E \left[\int_0^T (B_t^1 U_1(c_t^1) + B_t^2 U_2(c_t^2)) dt + B_T^3 U_3(X_T) \right]$$

and

$$\tilde{A}(x) := \left\{ (\pi, c^1, c^2) \in A(x) : E \left[\int_0^T (B_t^1 U_1^-(c_t^1) + B_t^2 U_2^-(c_t^2)) dt + B_T^3 U_3^-(X_T) \right] < \infty \right\}.$$

The expectation J is well defined for every pair $(\pi, c^1, c^2) \in \tilde{A}(x)$. The total initial endowment of the couple is x . We can easily see that $\tilde{A}(x) = A(x)$ if $U_i(0) > -\infty$ for $i = 1, 2, 3$. We consider three problems that are sub-problems to the one above related to each term in the expression of the functional J . For a given $x_1 > 0$, we define the value function

$$V_1(x_1) := \sup \{ J_1(x_1; \pi, c^1, c^2) : (\pi, c^1, c^2) \in A_1(x_1) \} \quad (3.2)$$

where

$$J_1(x_1; \pi, c^1, c^2) := E \left[\int_0^T B_t^1 U_1(c_t^1) dt \right]$$

and

$$A_1(x_1) := \left\{ (\pi, c^1, c^2) \in A(x_1) : E \left[\int_0^T B_t^1 U_1^-(c_t^1) dt \right] < \infty \right\} .$$

The expectation J_1 is well defined for every pair $(\pi, c^1, c^2) \in A_1(x_1)$. The value functions V_2, V_3 and the sets A_2, A_3 are defined similarly in an obvious manner.

We now turn to several definitions to describe the solution to (3.1) and (3.2). Let $I_i := (U_i')^{-1}$ be the inverse functions of the marginal utilities for $i = 1, 2, 3$. Because $U_i' : [0, \infty] \rightarrow [0, U_i'(0)]$ is strictly decreasing, it has a strictly decreasing inverse $I_i : [0, U_i'(0)] \rightarrow [0, \infty]$. We extend I_i to be a continuous function on the entirety of $[0, \infty]$ by setting $I_i(y) = 0$ for $U_i'(0) \leq y \leq \infty$, and note that

$$U_i(I_i(y)) \geq U_i(c) + yI_i(y) - yc, \text{ for } (y, c) \in (0, \infty) \times [0, \infty) . \quad (3.3)$$

Recall that the process Z_t is given by (2.6). Define for $i = 1, 2, 3$ the state price process corresponding to the discount factor B^i ,

$$\zeta_t^i := Z_t D_t \exp\left(\int_0^t \beta^i(u) du\right) = \frac{\tilde{Z}_t}{B_t^i} .$$

We introduce also the following functions:

$$\mathcal{H}_i(y, t) := \tilde{\mathbb{E}} \left[\int_t^T D_u I_i(y \zeta_u^i) du \middle| \mathcal{F}_t \right] \text{ for } i = 1, 2 , \quad (3.4)$$

and

$$\mathcal{H}_3(y, t) := \tilde{\mathbb{E}} [D_T I_3(y \zeta_T^3) | \mathcal{F}_t] . \quad (3.5)$$

We will abuse notation by often writing $\mathcal{H}_i(y) := \mathcal{H}_i(y, 0)$. We now make the following assumptions.

Assumption 3.1. For $i = 1, 2, 3$, we have $\mathcal{H}_i(y) < \infty$, for all $y \in (0, \infty)$.

Assumption 3.2. For all $y \in (0, \infty)$, we have that

$$\mathbb{E} \left[\int_0^T B_t^1 |U_1(I_1(y \zeta_t^1))| dt + \int_0^T B_t^2 |U_2(I_2(y \zeta_t^2))| dt + B_T^3 |U_3(I_3(y \zeta_T^3))| \right] < \infty .$$

Lemma 3.3. For $i = 1, 2, 3$, \mathcal{H}_i is a continuous function, strictly decreasing on $(0, \infty)$ with $\mathcal{H}_i(0) = \infty$ and $\mathcal{H}_i(\infty) = 0$

The proof of Lemma 3.3 is done in [5]. Define the function \mathcal{H} on $[0, \infty)$ by $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3$. We call $\mathcal{Y}_i := \mathcal{H}_i^{-1} : [0, \infty] \rightarrow [0, \infty]$ for $i = 1, 2, 3$ the inverse of the function \mathcal{H}_i . Moreover $\mathcal{Y} := \mathcal{H}^{-1} : [0, \infty] \rightarrow [0, \infty]$.

Definition 3.4. Define for $a_1, a_2, a_3 \geq 0$ the consumption processes

$$\begin{aligned} \mathbf{c}_t^1(a_1) &:= I_1(\mathcal{Y}_1(a_1)\zeta_t^1), \\ \mathbf{c}_t^2(a_2) &:= I_2(\mathcal{Y}_2(a_2)\zeta_t^2), \end{aligned}$$

and the following wealth processes

$$\begin{aligned} \mathbf{X}_t^1(a_1) &:= \tilde{\mathbb{E}} \left[\int_t^T \mathbf{c}_s^1(a_1) D_s ds | \mathcal{F}_t \right] = \mathcal{H}_1(\mathcal{Y}_1(a_1), t), \\ \mathbf{X}_t^2(a_2) &:= \tilde{\mathbb{E}} \left[\int_t^T \mathbf{c}_s^2(a_2) D_s ds | \mathcal{F}_t \right] = \mathcal{H}_2(\mathcal{Y}_2(a_2), t), \\ \mathbf{X}_t^3(a_3) &:= \tilde{\mathbb{E}} [I_3(\mathcal{Y}_3(a_3)\zeta_T^3) D_T | \mathcal{F}_t] = \mathcal{H}_3(\mathcal{Y}_3(a_3), t). \end{aligned}$$

The portfolios corresponding to each agent are given by the triplet $(\boldsymbol{\pi}^1(a_1), \boldsymbol{\pi}^2(a_2), \boldsymbol{\pi}^3(a_3))$ which is found using Proposition 2.6. Finally, by linearity we are able to define the total portfolio $\boldsymbol{\pi}$ and total wealth process \mathbf{X} by

$$\boldsymbol{\pi}(a_1, a_2, a_3) := \boldsymbol{\pi}^1(a_1) + \boldsymbol{\pi}^2(a_2) + \boldsymbol{\pi}^3(a_3), \quad (3.6)$$

$$\mathbf{X}(a_1, a_2, a_3) := \mathbf{X}^1(a_1) + \mathbf{X}^2(a_2) + \mathbf{X}^3(a_3). \quad (3.7)$$

Theorem 3.5. Consider a couple endowed initially with wealth $x > 0$. Then,

$$\begin{aligned} V(x) &= V_*(x) := \sup \{V_1(a_1) + V_2(a_2) + V_3(a_3) : (a_1, a_2, a_3) \in \mathbb{R}_+^3, a_1 + a_2 + a_3 = x\} \\ &= V_1(x_1) + V_2(x_2) + V_3(x_3) \end{aligned}$$

where the initial allocation x_i is given by

$$x_i = \mathcal{H}_i(\mathcal{Y}(x)). \quad (3.8)$$

The optimal consumption processes are given by $\mathbf{c}^1(x_1), \mathbf{c}^2(x_2)$. The optimal wealth processes corresponding to each agent are given by $(\mathbf{X}^1(x_1), \mathbf{X}^2(x_2), \mathbf{X}^3(x_3))$ and the optimal portfolio processes by $(\boldsymbol{\pi}^1(x_1), \boldsymbol{\pi}^2(x_2), \boldsymbol{\pi}^3(x_3))$. The total portfolio process is $\boldsymbol{\pi}(x_1, x_2, x_3)$ and the total wealth process is $\mathbf{X}(x_1, x_2, x_3)$.

4 Proofs

In this section we provide the proof of Theorem 3.5. For this purpose, we introduce the solution for each sub-problem.

Proposition 4.1 (Consumption problem). *Let $a_1 \in \mathbb{R}_+$. Then $V_1(a_1) = J_1(a_1; \boldsymbol{\pi}^1(a_1), \mathbf{c}^1(a_1), 0)$ where \mathbf{c}^1 is defined in Definition 3.4 and $\boldsymbol{\pi}^1$ follows from Proposition 2.6.*

Remark 4.2. *The same result holds for problem V_2 in an obvious manner. Since $(\pi, c^1, c^2) \in A_1(a_1)$ implies that $(\pi, c^1 + c^2, 0) \in A_1(a_1)$,*

$$J_1(a_1; \pi, c^1, c^2) \leq J_1(a_1; \pi, c^1 + c^2, 0).$$

Thus, since U_1 is an increasing function, the functional J_1 is maximized when $c^2 = 0$. The reciprocal holds for J_2 .

Proof We take

$$c^i = \hat{c}_t^i := \frac{a_i}{\tilde{\mathbb{E}} \int_0^T D_t dt} \in D(a_i),$$

so that

$$\mathbb{E} \left[\int_0^T B_t^i U_i(\hat{c}_t^i) dt \right] = U_i(\hat{c}^i) \mathbb{E} \left[\int_0^T B_t^i dt \right] < \infty.$$

Notice that $\hat{c}^i \in D(a_i)$ and since $\tilde{\mathbb{E}} \left[\int_0^T D_t \mathbf{c}_t^i dt \right] = \mathcal{H}_i(\mathcal{Y}_i(a_i)) = a_i$, $\mathbf{c}_t^i \in D(a_i)$. Inequality (3.3) implies that for any $c^i \in C(a_i)$ and $t \in [0, T]$,

$$U_i(\mathbf{c}_t^i) \geq U_i(c_t^i) + \mathcal{Y}_i(a_i) \zeta_t^i \mathbf{c}_t^i - \mathcal{Y}_i(a_i) \zeta_t^i c_t^i, \quad \mathbb{P} - \text{a.s.}$$

Therefore,

$$\begin{aligned} \mathbb{E} \left[\int_0^T B_t^i U_i^-(\mathbf{c}_t^i) dt \right] &\leq \mathbb{E} \left[\int_0^T B_t^i \left(U_i(\hat{c}_t^i) + \mathcal{Y}_i(a_i) \zeta_t^i \mathbf{c}_t^i - \mathcal{Y}_i(a_i) \zeta_t^i \hat{c}_t^i \right)^- dt \right] \\ &\leq \mathbb{E} \left[\int_0^T B_t^i \left(U_i^-(\hat{c}_t^i) + \mathcal{Y}_i(a_i) \zeta_t^i \hat{c}_t^i \right) dt \right] < \infty. \end{aligned}$$

Consider the measure on $[0, T] \times \Omega$ defined by $d\nu^i(t, \omega) = B_t^i dt \mathbb{P}(d\omega)$. For any other consumption process $c^i \in D(a_i)$, we have

$$\iint_{[0, T] \times \Omega} U_i(\mathbf{c}_t^i) d\nu^i \geq \iint_{[0, T] \times \Omega} U_i(\hat{c}_t^i) d\nu^i + \iint_{[0, T] \times \Omega} \mathcal{Y}_i(a_i) \zeta_t^i \mathbf{c}_t^i d\nu^i - \iint_{[0, T] \times \Omega} \mathcal{Y}_i(a_i) \zeta_t^i \hat{c}_t^i d\nu^i$$

By using the fact that $\mathbf{c}^i, \hat{c}^i \in D(a_i)$,

$$\mathbb{E} \left[\int_0^T B_t^i U_i(\mathbf{c}_t^i) dt \right] \geq \mathbb{E} \left[\int_0^T B_t^i U_i(\hat{c}_t^i) dt \right], \quad i = 1, 2.$$

□

Proposition 4.3 (Final wealth problem). *Let $a_3 \in \mathbb{R}_+$. Then*

$$V_3(a_3) = J_3(a_3; \boldsymbol{\pi}^3(a_3), 0, 0) . \quad (4.1)$$

The corresponding final wealth is given by

$$\mathbf{X}_T^3(a_3) = I_3(\mathcal{Y}_3(a_3)\zeta_T^3) \in M(a_3) . \quad (4.2)$$

Proof 1. First, we show that the strategy $\boldsymbol{\pi}^3(a_3) \in P(a_3)$ and that the generated portfolio process $\mathbf{X}^3(a_3)$ belongs to $M(a_3)$. According to (4.2), we have

$$\tilde{\mathbb{E}}[\mathbf{X}_T^3(a_3)] = \tilde{\mathbb{E}}[I_3(\mathcal{Y}_3(a_3)\zeta_T)] = \mathcal{H}_3(\mathcal{Y}_3(a_3)) = a_3 .$$

Considering the constant final wealth $b := a_3/\tilde{\mathbb{E}}[D_T] \in D(a_3)$, we get

$$U_3(\mathbf{X}_T^3(a_3)) \geq U_3(b) + \mathcal{Y}_3(a_3)\zeta_T^3\mathbf{X}_T^3(a_3) - \mathcal{Y}_3(a_3)\zeta_T^3b \quad \mathbb{P} - a.s.$$

Therefore,

$$\mathbb{E}[B_T^3 U_3^-(\mathbf{X}_T^3(a_3))] \leq \mathbb{E}[B_T^3 (U_3^-(b) + \mathcal{Y}_3(a_3)\zeta_T^3 b)] < \infty .$$

2. Let's show that the optimal strategy requires zero consumption. Let $(\pi, c^1, c^2) \in A(a_3)$ with wealth process X be given. Define the random variable

$$B := \begin{cases} \frac{a_3}{\tilde{\mathbb{E}}[D_T X_T]} X_T & \text{if } \tilde{\mathbb{E}}[X_T] > 0 \\ b & \text{if } \tilde{\mathbb{E}}[X_T] = 0 \end{cases} .$$

Since $X_T \in A(a_3)$, $\tilde{\mathbb{E}}[D_T X_T] \leq a_3$. Then $B \in M(a_3)$ and $B \geq X_T$ \mathbb{P} -a.s. From Proposition 2.8 and Corollary 2.9, there exists a portfolio $\hat{\pi} \in P(a_3)$ with corresponding terminal wealth $\hat{X}_T = B \geq X_T$ \mathbb{P} -a.s. Thus $(\hat{\pi}, 0, 0) \in A_3(a_3)$ and $J_3(a_3, \pi, c^1, c^2) \leq J_3(a_3, \hat{\pi}, 0, 0)$.

3. To obtain (4.1), it suffices to proceed as in Proposition 4.1:

$$\mathbb{E}[B_T^3 U_3(\mathbf{X}_T^3(x_3))] \geq \mathbb{E}[B_T^3 (U_3(\mathbf{X}_T) + \mathcal{Y}_3(x_3)\zeta_T^3(\mathbf{X}_T^3(x_3) - X_T))] \geq \mathbb{E}[B_T^3 U_3(\mathbf{X}_T)] .$$

□

Having the solution to sub-problems, we turn to the solution of problem (3.1). We start with a preliminary lemma.

Lemma 4.4. *For $y > 0$, define*

$$G_1(y) := \mathbb{E} \left[\int_0^T B_t^1 U_1(I_1(y\zeta_t^1)) dt \right] , \quad (4.3)$$

$$G_2(y) := \mathbb{E} \left[\int_0^T B_t^2 U_2(I_2(y\zeta_t^2)) dt \right] , \quad (4.4)$$

$$G_3(y) := \mathbb{E} [B_T^3 U_3(I_3(y\zeta_T^3))] . \quad (4.5)$$

Then

$$G'_i(y) = y \mathcal{H}'_i(y) \quad i = 1, 2, 3 \quad (4.6)$$

and $V_i \in C^2((0, \infty))$ with

$$V'_i(x) = \mathcal{Y}_i(x) \quad i = 1, 2, 3, \quad x \geq 0 . \quad (4.7)$$

Proof According to Assumption 3.2, we can take derivatives under the expectation and integral signs to obtain

$$\begin{aligned} G'_1(y) &= \mathbb{E} \int_0^T B_t^1 \zeta_t^1 I'_1(y \zeta_t^1) U'_1(I_1(y \zeta_t^1)) dt = \mathbb{E} \int_0^T B_t^1 \zeta_t^1 y \zeta_t^1 I'_1(y \zeta_t^1) dt \\ &= \tilde{\mathbb{E}} \int_0^T y \zeta_t^1 I'_1(y \zeta_t^1) dt = y \mathcal{H}'_1(y) . \end{aligned}$$

Therefore,

$$V'_1(x) = \frac{d}{dx} G_1(\mathcal{Y}_1(x)) = Y'_1(x) G'_1(\mathcal{Y}_1(x)) = \mathcal{Y}'_1(x) \mathcal{Y}_1(x) \mathcal{H}'_1(\mathcal{Y}_1(x)) = \mathcal{Y}_1(x) .$$

The other derivatives are computed in the same manner. \square

Proposition 4.5. For $x \geq 0$,

$$V(x) = V_*(x) := \max \{V_1(a_1) + V_2(a_2) + V_3(a_3) | a_1, a_2, a_3 \in [0, \infty); a_1 + a_2 + a_3 = x\} . \quad (4.8)$$

Proof For $x \geq 0$, we are given an arbitrary triplet $(\pi, c^1, c^2) \in \tilde{A}(x)$ with corresponding wealth process X_t . Recall that

$$a_1 := \tilde{\mathbb{E}} \left[\int_0^T B_t^1 c_t^1 dt \right] , \quad a_2 := \tilde{\mathbb{E}} \left[\int_0^T B_t^2 c_t^2 dt \right] \quad \text{and} \quad a_3 := \tilde{\mathbb{E}} [B_T^3 X_T] .$$

By the super martingale property, $a := a_1 + a_2 + a_3 \leq x$ and by Propositions 4.1 and 4.3,

$$\begin{cases} \mathbb{E} [B_t^1 U_1(c_t^1) dt] \leq J_1(a_1; \boldsymbol{\pi}^1(a_1), \mathbf{c}^1(a_1), 0) = V_1(a_1), \\ \mathbb{E} [B_t^2 U_2(c_t^2) dt] \leq J_2(a_2; \boldsymbol{\pi}^2(a_2), 0, \mathbf{c}^2(a_2)) = V_2(a_2), \\ \mathbb{E} [B_T^3 U_3(X_T)] \leq J_3(a_3; \boldsymbol{\pi}^3(a_3), 0, 0) = V_3(a_3). \end{cases}$$

Adding the three terms, we get

$$V_1(a_1) + V_2(a_2) + V_3(a_3) = J(a; \boldsymbol{\pi}, \mathbf{c}^1(a_1), \mathbf{c}^2(a_2)) \geq J(a; \pi, c^1, c^2) .$$

Taking the supremum over $a_1 + a_2 + a_3 \leq x$ and over $(\pi, c^1, c^2) \in \tilde{A}(x)$, we get

$$\begin{aligned} V(x) &\leq \sup \{V_1(a_1) + V_2(a_2) + V_3(a_3) : a_1, a_2, a_3 \in [0, \infty); a_1 + a_2 + a_3 \leq x\} \\ &= \sup \{V_1(a_1) + V_2(a_2) + V_3(a_3) : a_1, a_2, a_3 \in [0, \infty); a_1 + a_2 + a_3 = x\} := V_*(x) \end{aligned}$$

from the non-decreasing characteristic of V_i for $i = 1, 2, 3$. Furthermore, by continuity of the function $(a_1, a_2, a_3) \mapsto V_1(a_1) + V_2(a_2) + V_3(a_3)$, the supremum above is attained at a point (x_1, x_2, x_3) and

$$V(x) \leq V_*(x) = V_1(x_1) + V_2(x_2) + V_3(x_3) = J(x; \boldsymbol{\pi}(x_1, x_2, x_3), \mathbf{c}^1(x_1), \mathbf{c}^2(x_2)) \leq V(x) .$$

The processes \mathbf{X}^1 , \mathbf{X}^2 and \mathbf{X}^3 are nonnegative, so \mathbf{X} is nonnegative. \mathbf{X} is clearly in $\tilde{A}(x)$. We conclude by saying that the x_i are found by using the envelope theorem: $V'_1(x_1) = V'_2(x_2) = V'_3(x_3)$. Equivalently by (4.7),

$$\mathcal{Y}_1(x_1) = \mathcal{Y}_2(x_2) = \mathcal{Y}_3(x_3) = y ,$$

i.e., $x_i = \mathcal{H}_i(y) = \mathcal{H}_i(\mathcal{Y}(x))$. This concludes the proof of Theorem 3.5. \square

5 CRRA utilities and mean reverting market price of risk

5.1 Configuration of the market

In this section, we provide an explicit model of the previously studied framework. The three agents share a common initial wealth x and have CRRA type utilities

$$U_i(x) = \frac{x^{\gamma_i}}{\gamma_i} \quad \text{for } i = 1, 2, 3.$$

Here $1 - \gamma_i \in (0, \infty)$ is the risk aversion of agent i . Notice that U_i satisfy Assumptions 2.2, and $I_i(x) = x^{\frac{1}{\gamma_i-1}}$. Each agent has his own constant discount rate ρ_i .

Next take $d = 1$ as in Wachter [11] (the extension to multiple stocks is straightforward). The asset price follows a geometrical Brownian motion. In order to isolate the effects of time variation on expected returns, the risk-free rate is assumed to be constant and equal to $r \geq 0$ but this assumption can be relaxed. We fix the volatility $\sigma := \sigma_{11} \in (0, \infty)$ for (2.1), but we do not specify the drift $b_1 \in \mathbb{R}$. Instead, we model the price of risk θ by

$$d\theta_t = -\lambda_\theta(\theta_t - \bar{\theta})dt - \sigma_\theta dW_t, \quad t \geq 0,$$

where $(\lambda_\theta, \sigma_\theta, \bar{\theta}) \in (0, \infty)^3$. We assume $W = W^1$, so that the stock price S_t^1 and the state variable θ_t are perfectly negatively correlated. These assumptions are like those in [6], except that the latter allows for imperfect correlation, and thus incomplete markets.

The body of academic literature on long term mean reversion is more tractable than that on short term mean reversion. A comprehensive study on the existence of mean reversion in Equity Prices has been done in [7]. The primary case for the existence of long term mean reversion was made in two papers published in 1988, one by [8], the other by [3]. In summary, these papers conclude that for period lengths between 3 and 5 years, long term mean reversion was present in stock market returns between 1926 and 1985.

5.2 Explicit formulas

The goal of the computation is to provide the initial repartition x_1, x_2, x_3 such that $x_1 + x_2 + x_3 = x$. According to Theorem 3.5, the optimal allocation is given by $x_i = \mathcal{H}_i(\mathcal{Y}(x))$. Denoting $y := \mathcal{Y}(x) = \mathcal{Y}_i(x_i)$, the theorem gives also the optimal consumption

$$\begin{cases} \mathbf{c}_t^1(x_1) &= I_1(y\zeta_t^1) = (y \exp(\rho_1 t) Z_t)^{\frac{1}{\gamma_1-1}} \\ \mathbf{c}_t^2(x_2) &= I_2(y\zeta_t^2) = (y \exp(\rho_2 t) Z_t)^{\frac{1}{\gamma_2-1}} \end{cases}, \quad 0 \leq t \leq T,$$

where Z_t is the state density process defined by (2.6), and the optimal total wealth process is given by

$$\mathbf{X}_t = \tilde{\mathbb{E}} \left[\int_t^T D_s(\mathbf{c}_s^1 + \mathbf{c}_s^2) | \mathcal{F}_t \right] + \tilde{\mathbb{E}} [D_T I_3(y\zeta_T^3) | \mathcal{F}_t], \quad t \geq 0.$$

The process $Y_t := (yZ_tD_t)^{-1}$ satisfies the SDE

$$dY_t = Y_t(r_t + \theta_t^2)dt + Y_t\theta_t dW_t, \quad t \geq 0.$$

The the optimal consumption for first agent can be expressed as $\mathbf{c}_t^1 = I_1(y\zeta_t^1) = (Y_tD_tB_t^1)^{\frac{1}{1-\gamma_1}}$ and his/her wealth process by

$$\begin{aligned} \mathbf{X}_t^1 &= \tilde{\mathbb{E}} \left[\int_t^T \mathbf{c}_s^1 ds | \mathcal{F}_t \right] = \frac{1}{Z_t} \mathbb{E} \left[\int_t^T Z_s (Y_s D_s B_s^1)^{\frac{1}{1-\gamma_1}} | \mathcal{F}_t \right] \\ &= Y_t \mathbb{E} \left[\int_t^T Y_s^{\frac{1}{1-\gamma_1}-1} e^{r(s-t)} e^{-\frac{(r+\rho_1)s}{1-\gamma_1}} | \mathcal{F}_t \right] \\ &= Y_t^{\frac{1}{1-\gamma_1}} \int_t^T e^{r(s-t) - \frac{(r+\rho_1)(s-t)}{1-\gamma_1}} f^1(t, s, \theta_t) ds \end{aligned}$$

where

$$f^1(t, \tau, \theta) := \mathbb{E} \left[\exp \left(\frac{\gamma_1}{2(1-\gamma_1)} \int_t^\tau \theta_s^2 ds + \frac{\gamma_1}{1-\gamma_1} \int_t^\tau \theta_s dW_s \right) | \theta_t = \theta \right].$$

We can obviously define the same quantities for the second agent. The process

$$f^1(t, \tau, \theta_t) \exp \left(\frac{\gamma_1}{2(1-\gamma_1)} \int_0^t \theta_s^2 ds + \frac{\gamma_1}{1-\gamma_1} \int_0^t \theta_s dW_s \right), \quad 0 \leq t \leq \tau,$$

is a \mathbb{P} -martingale for a given $\tau \leq T$. Given that $f^1(\cdot, \tau, \cdot)$ is $\mathcal{C}^{1,2}$, it follows by Feynman-Kac formula that

$$f_t^1 - \lambda_\theta(\theta - \bar{\theta})f_\theta^1 + \frac{\sigma_\theta^2}{2}f_{\theta\theta}^1 + \left(\frac{\gamma_1}{2(1-\gamma_1)} + \frac{\gamma_1^2}{2(1-\gamma_1)^2} \right) \theta^2 f^1 = 0. \quad (5.1)$$

As in [11] we search for f^1 of the form

$$f^1(t, \tau, \theta) = \exp \left(A_1(t, \tau) \frac{\theta^2}{2} + A_2(t, \tau) \theta + A_3(t, \tau) \right).$$

The terminal condition implies that in the latter expression, $A_1(\tau, \tau) = A_2(\tau, \tau) = A_3(\tau, \tau) = 0$. In the sequel, we omit τ in the notation of A_j , $j = 1, 2, 3$. Plugging the expression of f^1 in (5.1), we get

$$\begin{aligned} \frac{A_1'(t)\theta^2}{2} + A_2'(t)\theta + A_3'(t) + \lambda_\theta(\bar{\theta} - \theta)(A_1(t)\theta + A_2(t)) \\ + \frac{\sigma_\theta^2}{2} (A_1(t) + (A_1(t)\theta + A_2(t))^2) + \frac{\gamma_1}{2(1-\gamma_1)^2} \theta^2 = 0. \end{aligned}$$

We obtain a second-order polynomial in θ . Since the equation holds for any θ , we separate the coefficients in θ^2 , θ and constant. We then shall have

$$\begin{cases} A_1'(t) - 2\lambda_\theta A_1(t) + \sigma_\theta^2 A_1(t)^2 + \frac{\gamma_1}{(1-\gamma_1)^2} = 0 \\ A_2'(t) + \lambda_\theta(\bar{\theta} A_1(t) - A_2(t)) + \sigma_\theta^2 A_1(t) A_2(t) = 0 \\ A_3'(t) + \lambda_\theta \bar{\theta} A_2(t) + \frac{\sigma_\theta^2}{2} (A_1(t) + A_2(t)^2) = 0 \end{cases} \quad (5.2)$$

Suppose that a C^1 function $A_1(t), t \in [0, \tau]$ has been found. Then A_2 is given by a linear ODE, which finally allows to retrieve A_3 :

$$A_3(t) = \int_t^\tau \lambda_\theta \bar{\theta} A_2(u) + \frac{\sigma_\theta^2}{2} (A_1(u) + A_2(u)^2) du .$$

We therefore seek for a solution $A_1(t)$ first. According to the first equation, we define the determinant

$$\Delta' = \lambda_\theta^2 - \frac{\gamma_i \sigma_\theta^2}{(1 - \gamma_i)^2}$$

and

$$\gamma_{lim} := \frac{2}{2 + b + \sqrt{(2 + b)^2 - 4}} \text{ for } b := \frac{\sigma_\theta^2}{\lambda_\theta^2} .$$

This is the biggest γ for which $\Delta' > 0$. We then have three cases to consider.

- If $\Delta' > 0$, i.e., $\gamma_i < \gamma_{lim}$, $\gamma_i \neq 0$, then there are two distinct roots to the characteristic polynomial of the first ODE of (5.2), given by $m_\pm := (\lambda_\theta \pm \sqrt{\Delta'})/\sigma_\theta^2$. A general solution $A_1(t)$ shall verify

$$\tau - t = \frac{1}{\sigma_\theta^2(m_- - m_+)} \int_t^\tau \frac{dA_1}{A_1 - m_+} - \frac{dA_1}{A_1 - m_-} , \quad \forall t \leq \tau .$$

It follows that

$$\exp(\sigma_\theta^2(m_- - m_+)(\tau - t)) = \left| \frac{m_+ A_1(t) - m_-}{m_- A_1(t) - m_+} \right|$$

and finally on $[0, \tau]$

$$A_1(t) = m_+ + \frac{m_+ - m_-}{\frac{m_-}{m_+} \exp(\sigma_\theta^2(m_- - m_+)(\tau - t)) - 1} .$$

- If $\Delta' = 0$, we have a double root $m := \lambda_\theta/\sigma_\theta^2$. As above,

$$\tau - t = \frac{1}{\sigma_\theta^2} \int_t^\tau \frac{dA_1}{(A_1 - m)^2} = \frac{1}{\sigma_\theta^2} \left(\frac{1}{m} - \frac{1}{m - A_1(t)} \right) .$$

As long as $\tau < 1/\lambda_\theta$, there is a solution on $[0, \tau]$ given by

$$A_1(t) = \frac{\lambda_\theta^2(\tau - t)}{\sigma_\theta^2 \lambda_\theta(\tau - t) - \sigma_\theta^2} .$$

- If $\Delta' < 0$, then

$$A_1'(t) + \left(\sigma_\theta A_1(t) - \frac{\lambda_\theta}{\sigma_\theta^2} \right)^2 - \frac{\Delta'}{\sigma_\theta^2} = 0 .$$

Taking $y(t) := (\sigma_\theta^2 A_1(t) - \lambda_\theta)/\sqrt{-\Delta'}$, we get

$$\arctan(y(t)) - \arctan\left(-\frac{\lambda_\theta}{\sqrt{-\Delta'}}\right) = \sqrt{-\Delta'}(\tau - t)$$

so that

$$A_1(t) = \left(y(t) + \frac{\lambda_\theta}{\sqrt{-\Delta'}}\right) \frac{\sqrt{-\Delta'}}{\sigma_\theta^2}.$$

This equation does not always have a continuous solution on $[0, \tau]$.

Having study the existence and the form of solution $A_1(t)$, we assume that we have computed explicitly the solution to (5.2) and the function f_1 . The precedent calculations apply for agents 1 and 2. Let A_{1i}, A_{2i}, A_{3i} be the functions corresponding to agent i (there is a dependence on γ_i). The initial allocations are:

$$\begin{cases} x_1 &= y^{\frac{1}{\gamma_1-1}} \int_0^T \exp\left(A_{11}(0)\theta_0^2/2 + A_{21}(0)\theta_0 + A_{31}(0) + r\tau - \frac{(r+\rho_1)\tau}{1-\gamma_1}\right) d\tau =: y^{\frac{1}{\gamma_1-1}} s_1 \\ x_2 &= y^{\frac{1}{\gamma_2-1}} \int_0^T \exp\left(A_{12}(0)\theta_0^2/2 + A_{22}(0)\theta_0 + A_{32}(0) + r\tau - \frac{(r+\rho_2)\tau}{1-\gamma_2}\right) d\tau =: y^{\frac{1}{\gamma_2-1}} s_2 \\ x_3 &= y^{\frac{1}{\gamma_3-1}} \exp\left(rT - \frac{(r+\rho_3)T}{1-\gamma_3}\right) =: y^{\frac{1}{\gamma_3-1}} s_3 \end{cases} \quad (5.3)$$

We choose $y = \mathcal{Y}(x)$ uniquely such that $x_1 + x_2 + x_3 = x$. Notice that

$$\mathcal{Y}(x) \in \left[\max_{i=1,2,3} ((x/s_i)^{\gamma_i-1}), \max_{i=1,2,3} ((x/(s_1 + s_2 + s_3))^{\gamma_i-1}) \right].$$

Define the density function $p_i(\theta_t, t, \tau) := H_i(\theta_t, \tau) \left(\int_0^{T-t} H_i(\theta_t, s) ds\right)^{-1}$ for $i = 1, 2$ where

$$H_i(\theta, \tau) := \exp\left(A_{1i}(0, \tau)\frac{\theta^2}{2} + A_{2i}(0, \tau)\theta + A_{3i}(0, \tau) + r\tau - \frac{(r + \rho_i)\tau}{1 - \gamma_i}\right).$$

The portfolio strategies π_t^i are thus determined by

$$\begin{cases} \pi_t^1 &= \frac{1}{1-\gamma_1} \frac{\mu_t - r}{\sigma^2} - \frac{\sigma_\theta}{(1-\gamma_1)\sigma} \int_0^{T-t} p_1(\theta_t, t, \tau)(A_{11}(\tau)\theta_t + A_{21}(\tau))d\tau \\ \pi_t^2 &= \frac{1}{1-\gamma_2} \frac{\mu_t - r}{\sigma^2} - \frac{\sigma_\theta}{(1-\gamma_2)\sigma} \int_0^{T-t} p_2(\theta_t, t, \tau)(A_{12}(\tau)\theta_t + A_{22}(\tau))d\tau \\ \pi_t^3 &= \frac{1}{1-\gamma_3} \frac{\mu_t - r}{\sigma^2} - \frac{\sigma_\theta}{(1-\gamma_3)\sigma} (A_{13}(T-t)\theta_t + A_{23}(T-t)) \end{cases} \quad (5.4)$$

Together, equations of (5.4) and (3.6) solve the couple of investors optimal consumption and portfolio choice problem. The economic consequences of these equations are explored in the next subsection. We continue here to explore the analytical results.

Proposition 5.1. Assume that $\theta_t > 0$. If $\gamma_i < 0$, it follows that

$$A_{1i}(T-t)\theta_t + A_{2i}(T-t) \leq \int_0^{T-t} p_i(\theta_t, t, \tau)(A_{1i}(\tau)\theta_t + A_{2i}(\tau))d\tau \leq 0.$$

On the other hand if $\gamma_i \geq 0$,

$$0 \leq \int_0^{T-t} p_i(\theta_t, t, \tau)(A_{1i}(\tau)\theta_t + A_{2i}(\tau))d\tau \leq A_{1i}(T-t)\theta_t + A_{2i}(T-t).$$

Proof These inequalities follow from the monotonicity of A_{1i} and A_{2i} . \square

Proposition 5.2. Assume that $\theta_t > 0$. If $\gamma_i < 0$, $i = 1, 2$, the consumption satisfaction proportion (CSP) defined by $(x_1 + x_2)/x$ is decreasing in θ . On the other hand if $\gamma_i \geq 0$, $i = 1, 2$, CSP is increasing in θ .

Remark 5.3. It is interesting to see that during favourably market conditions, i.e., θ is increasing, the agents behave differently according to their risk aversion. Thus, if they are more risk averse they will use a higher fraction of the initial wealth to finance investment; else if they are less risk averse they will use a higher fraction of the initial wealth to finance consumption.

Proof From direct computations one gets

$$\frac{dy}{d\theta} = \frac{y^{\frac{1}{\gamma_1-1}} \frac{ds_1}{d\theta} + y^{\frac{1}{\gamma_2-1}} \frac{ds_2}{d\theta}}{\frac{s_1}{1-\gamma_1} y^{\frac{1}{\gamma_1-1}-1} + \frac{s_2}{1-\gamma_2} y^{\frac{1}{\gamma_2-1}-1} + \frac{s_3}{1-\gamma_3} y^{\frac{1}{\gamma_3-1}-1}}.$$

Moreover

$$\frac{d}{d\theta} \left(\frac{x_1 + x_2}{x} \right) = \frac{d}{d\theta} \left(1 - \frac{x_3}{x} \right) = -\frac{dx_3}{d\theta} = \frac{dy}{d\theta} y^{\frac{1}{\gamma_3-1}-1}.$$

Thus, by (5.3) and Proposition 5.1 the claims yield. \square

Proposition 5.4. Let us define the relative risk-aversion for the couple as $R(x) := -xV''(x)/V'(x)$, with V of (3.1). Assume that $\gamma_1 < \gamma_2 < \gamma_3$. Then with $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ of (3.4), (3.5)

$$\lim_{x \rightarrow \infty} R(x) = (1 - \gamma_1) \frac{\mathcal{H}_3(1)}{\mathcal{H}_1(1)} \quad \text{and} \quad \lim_{x \rightarrow 0} R(x) = (1 - \gamma_3) \frac{\mathcal{H}_1(1)}{\mathcal{H}_3(1)}.$$

Remark 5.5. It is interesting enough to point out that for small initial wealth or high initial wealth the couple risk aversion is driven by one of the agents. Thus, the less risk averse agent determines the couple utility for little initial wealth. This is in accordance with risk seeking agents behaviour when the latter are poor.

Proof Recall that $V(x) = G(\mathcal{Y}(x))$ and $V'(x) = \mathcal{Y}(x)$. Thus $V''(x) = \mathcal{Y}'(x) = \frac{1}{\mathcal{H}'(y)}$ (with $y := \mathcal{Y}(x)$) and $R(x) = -\frac{x\mathcal{Y}'(x)}{\mathcal{Y}(x)} = -\frac{\mathcal{H}(y)}{y\mathcal{H}'(y)}$. In light of

$$\mathcal{H}(y) = y^{\frac{1}{\gamma_1-1}} \mathcal{H}_1(1) + y^{\frac{1}{\gamma_2-1}} \mathcal{H}_2(1) + y^{\frac{1}{\gamma_3-1}} \mathcal{H}_3(1),$$

it follows that

$$R(x) = \frac{y^{\frac{1}{\gamma_1-1}} \mathcal{H}_1(1) + y^{\frac{1}{\gamma_2-1}} \mathcal{H}_2(1) + y^{\frac{1}{\gamma_3-1}} \mathcal{H}_3(1)}{y^{\frac{1}{\gamma_1-1}} \frac{\mathcal{H}_1(1)}{1-\gamma_1} + y^{\frac{1}{\gamma_2-1}} \frac{\mathcal{H}_2(1)}{1-\gamma_2} + y^{\frac{1}{\gamma_3-1}} \frac{\mathcal{H}_3(1)}{1-\gamma_3}},$$

whence the claim. □

5.3 Numerical results

For the numerical applications, following [11], we have chosen the following fixed parameters for the market

$$(y, r, \sigma_\theta, \lambda_\theta, \bar{\theta}) = (3, 0.048, 0.0655, 0.2712, 0.9456).$$

We first plot the fraction of initial wealth for each agent as a function of total wealth x in figure 1. In this situation, we assume that discount rates ρ_i are all equal to 0.01, but that risk aversions differ and are given by $(\gamma_1, \gamma_2, \gamma_3) = (-9, -3, -2)$.

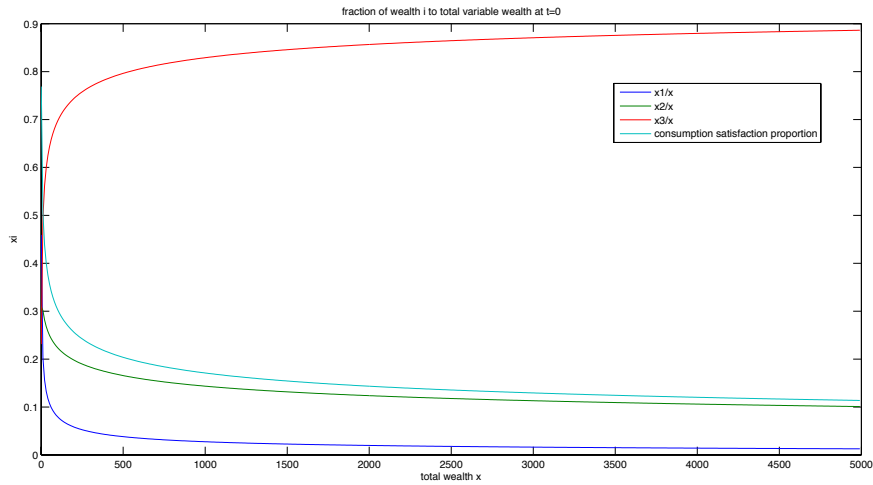


Figure 1: Fraction of Initial Wealth as a function of total wealth for $T = 1$.

We see from the plot that the fraction of initial wealth allocated to each of the agents is monotonous in wealth and it is higher for the less risk averse agent. Moreover, as the agents' initial wealth increases, initial wealth allocation for financing investment increases. The model with one agent only also mentions this fact, see [10].

Next we explore the effect of varying risk aversion. In figure 2, we vary γ_1 while holding γ_2, γ_3 constant. As expected, when agent 1 becomes more risk-averse his/her initial wealth allocation decreases and the initial wealth allocation for financing investment increases.

In figure 3, we vary γ_3 and fix γ_1 and γ_2 to fall below the range of γ_3 . The initial wealth allocation for financing investment increases in γ_3 .

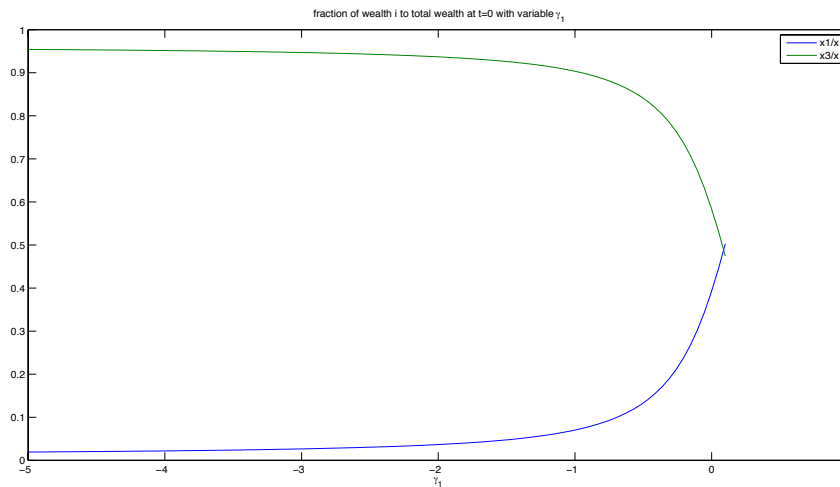


Figure 2: Fraction of Initial Wealth as a function of risk aversion γ_1 , for $\gamma_1 \leq \gamma_3$.

In figure 4 we observe the effect on the initial wealth allocation of the market price of risk θ . Here $(\gamma_1, \gamma_2, \gamma_3) = (-9, -3, -2)$. The findings are in accordance with Proposition 5.2.

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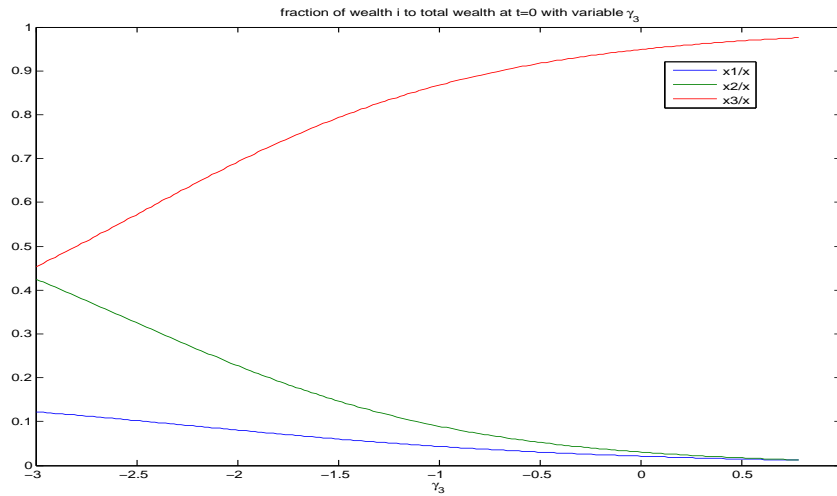


Figure 3: Fraction of Initial Wealth as a function of total wealth for variable γ_3 .

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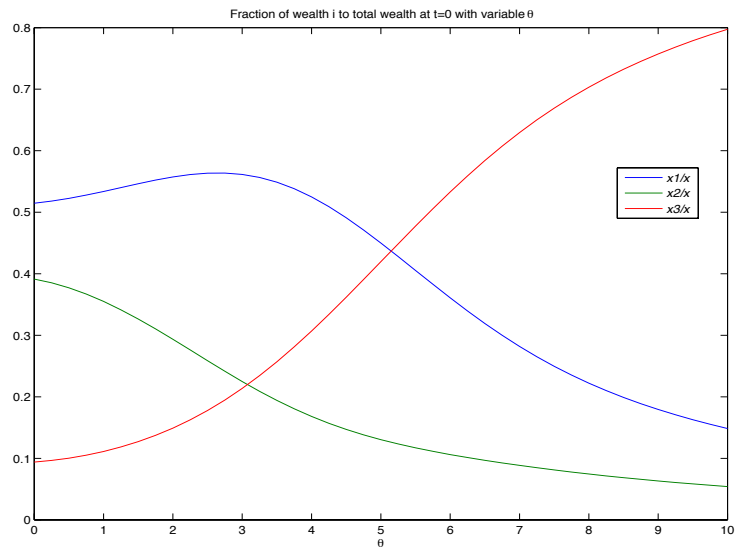


Figure 4: Fraction of Initial Wealth as a function of total wealth for variable θ .