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Robust Multi-image Processing With Optimal Sparse Regularization

Yann Traonmilin · Saïd Ladjal · Andrés Almansa

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Abstract Sparse modeling can be used to characterize outlier type noise. Thanks to sparse recovery theory, it was shown that 1-norm super-resolution is robust to outliers if enough images are captured. Moreover, sparse modeling of signals is a way to overcome ill-posedness of under-determined problems. This naturally leads to the question: does an added sparsity assumption on the signal will improve the robustness to outliers of the 1-norm super-resolution, and if yes, how strong should this assumption be? In this article, we review and extend results of the literature to the robustness to outliers of overdetermined signal recovery problems under sparse regularization, with a convex variational formulation. We then apply them to general random matrices, and show how the regularization parameter acts on the robustness to outliers. Finally, we show that in the case of multi-image processing, the structure of the support of signal and noise must be studied precisely. We show that the sparsity assumption improves robustness if outliers do not overlap with signal jumps, and determine how the regularization parameter can be chosen.

Keywords multi-image processing, super-resolution, outliers, sparse signal, regularization parameter

1 Introduction

Sparse signal approximation is a well known tool to deal with ill-conditioned inverse problems. A wide range of

applications benefit from this tool. From compressive sensing (Candes et al. 2006) to image denoising (Elad and Aharon. 2006; Yu et al. 2012) and texture synthesis (Tartavel et al. 2013), the added knowledge that a signal is sparse allows for the recovery of said signal when the recovery of a non-sparse signal would have been impossible. It was shown that natural images can have sparse representations in some dictionaries (Mairal et al. 2009; Mallat and Yu. 2010; Benoit et al. 2011).

The sparse approximation has mainly been made on signals. It was shown that if an under-determined observation matrix has the required property (Null Space Property, Restricted Isometry Property), then perfect recovery from these observations is possible by solving a convex L^1 norm minimization problem. Candes and Tao (2005) show that the underdetermined sparse recovery problem is equivalent to an overdetermined outlier robustness problem. In this setting, the sparsity assumption is made on the noise which contaminates the data. Such noise is often considered as outlier, as it can represent observations unrelated to the signal. This was used to estimate the robustness to outliers of L^1 multi-image super-resolution (SR) in Traonmilin et al (2013). It was shown that, if enough images are available, then L^1 -super-resolution can forgive a fraction of contaminated images in the acquired low resolution images. In Mitra et al (2013), similar theoretical developments are used and applied to the robustness to outliers of random uniform observations.

Knowing the benefits of sparsity priors, the question of imposing it on both the signal and the noise comes naturally. In this article, we will study the following problem: given an overdetermined outlier robustness problem, does sparse regularization improve the number of outliers that can be removed? This question has been studied in (Studer et al. 2012; Kuppinger et al.

2012; Studer and Baraniuk, 2013) with a different (and underdetermined) observation model. It was shown that a compromise between signal and noise sparsity must be made to guarantee recovery for particular algorithms (approximate L^0 , basis pursuit and $L^1 - L^2$) and observation matrices. However, in the variational formulation, the question of determining the optimal regularization parameter is not treated. Moreover the effect of such methods on multi-image processing is not studied.

Multi-image processing is a recent trend in computational photography. The use of information redundancy permits the design of image processing algorithms that produce images of higher quality. Multi-image denoising (or burst denoising) aims at lowering the noise in the final image. Super-resolution is the process of recovering a high resolution (HR) image from low resolution (LR) versions of it (Farsiu et al. 2004a;b; Milanfar, 2010; Tian and Ma, 2011). The L^1 super-resolution problem is already in use and recognized for its robustness (He et al. 2007; Yap et al. 2009). Moreover, sparse regularization is already in use for mono-image (Ning et al. 2013) super resolution. Even popular regularization methods such as total variation (TV) regularization (Marquina and Osher, 2008) can be interpreted as the convex version of a sparse gradient regularization. By extending results of sparse recovery theory to the case of multi-image processing, this work will give a better understanding of how these methods behave.

1.1 Outline and contributions

In this article, we begin with the introduction of the multi-image acquisition model in **Section 2**. This model is the justification for the more generic set-up that we use for the theoretical content of the paper. We recall the L^1 convex variational framework used to recover the higher quality image. In **Section 3**, we study the problem of outlier robustness (which we call forgiveness) using convex programming with sparse convex regularization. We show that forgiveness on sparse solutions is equivalent to an extension of a Non Concentration Property highlighting the distinct role of the supports of the outliers and the signal. In Studer et al (2012); Kuppinger et al (2012); Studer and Baraniuk (2013), coherency based sufficient conditions of convergence are given for variations of basis pursuit algorithm. We also show an extension of the popular Restricted Isometry Property (RIP) which respects the different roles of signal and noise sparsity. In **Section 4**, we apply our results to Gaussian random observation matrices, with 2 different regularization schemes. We show that the simplest case can be studied with existing results on

compressive sensing. When the regularization parameter varies, checking our extended RIP gives a good tool to predict the qualitative behaviour of forgiveness with respect to sparse regularization. We also point out that the choice of the matrix on which we check the RIP is important, and illustrate that remark by using 2 different constructions. Finally, in **Section 5**, we study the benefits of sparse regularization for forgiveness in the area of multi-image processing. While the RIP was a good tool to study random matrices, we observe in two image processing problems (multi-image denoising and super resolution), that the convex sparse prior (the TV in our case) adds some benefits only if precise requirements on the structure of the support of outliers and signal are met: signal jumps and outliers should not overlap. The structure of sparsity supports has been studied before (Jenatton et al. 2011; Bach et al. 2012) and has led to the creation of dedicated measures of sparsity (mixed L^1, L^q norms) or dedicated algorithms (Yu et al. 2012). In this article, we show that, under a non-structured norm (L^1 norm), when the observation operator and sparsifying transform are structured, perfect recovery is possible for some structured sparsity supports. Such requirements can only be studied with characterizations more precise than the RIP such as the non concentration property. In the case of super-resolution, these requirements can be met only for particular acquisition models (finite support of the translation operator). These theoretical results are illustrated by experiments.

2 Multi-image acquisition Model

2.1 Forward model

We introduce an image generation model to describe the acquisition of a burst of images using a hand-held camera. In a finite-dimensional context, we suppose that images are generated by a linear map A :

$$\begin{aligned} A : \mathbb{R}^{Ml \times Ml} &\rightarrow (\mathbb{R}^{l \times l})^N \\ u &\rightarrow (A_i u)_{i=1, N} = (SQ_i u)_{i=1, N} \end{aligned} \quad (1)$$

where M is the super-resolution factor, N is the number of acquired images, $l \times l$ is the size of acquired images, u is a HR image of size $Ml \times Ml$, the A_i are linear maps generating LR images, S is the sub-sampling operator by a factor M and Q_i are the deformations associated with each image. SR is the process of recovering u_0 from $w = Au_0 + n$ (n is the observation noise). In this paper, we suppose that the Q_i are known. In this setting, for $M > 1$, the inversion of A is called super-resolution interpolation. When $M = 1$, we will simply

talk of multi-image denoising.

It has been shown in Traonmilin et al (2012) that A is almost surely full rank when motions are random compositions of translations and rotations and $N \geq M^2$.

2.2 Variational Formulation and previous results

When A is full rank and $M^2 \leq N$, L^2 -norm minimization guarantees that the energy of the reconstruction noise is bounded by the energy of observation noise times the operator norm of the pseudo-inverse A^\dagger of A . This leads to useful results when observation noise has bounded energy. In the case of outliers, no assumption is made on the power of the noise and L^2 reconstruction does not guarantee a good result (unbounded reconstruction noise). It was shown that L^1 -norm minimization removes outliers if the ratio of images contaminated by outliers is small enough. We write L^1 -norm minimization of the data-fit :

$$\operatorname{argmin}_u \|Au - w\|_1 \quad (2)$$

with $w = Au_0 + n_0$. The problem is to find conditions on A ensuring that u_0 is the unique solution of (2) when n_0 is an outlying noise. Outliers have the form : $n_0 = n.T$ with T a vector of 0 and 1 representing the support of the noise (the \cdot represents the component-by-component vector product). We do not make any hypothesis on n . (Traonmilin et al. 2013) showed that if the number of noisy images N_c fulfills the condition $N_c/N < C$ with C a constant, then u_0 is the unique solution of Equation (2).

In Section 3, matrix A will be a general full rank matrix of an over-determined system. In other sections, A will be an over-determined full rank acquisition operator of size $Nl^2 \times (Ml)^2$ with $N > M^2$.

2.3 Sparse priors on natural images

Given an image u , we say that u is sparse for the particular sparsifying transform Ψ if Ψu is sparse. A wide category of sparse priors (Ψ) have been considered for natural images. Some of these are wavelet decompositions, projection in a dictionary of patches (Geiger et al. 1999; Elad and Aharon. 2006). Another frequent assumption on images is that they minimize the L^1 -norm of the gradient (the total variation). Under the light of the sparse recovery theory, minimizing this L^1 -norm is equivalent to imposing some sparsity on the gradient. In the part on super-resolution, we will use this prior as a practical example. We will study the minimization

$$\operatorname{argmin}_u \|Au - w\|_1 + \lambda \|\Psi u\|_1 \quad (3)$$

where Ψ is the sparsifying transform.

3 Forging Matrices and Sparse Regularization

3.1 Definitions

Let \mathcal{T} be a family of supports, A a $N \times M$ full rank matrix. From Traonmilin et al (2013), we define forging matrices :

Definition 1 (Forgiving Matrix.) Let \mathcal{T} be a set of supports in \mathbb{R}^N (subset of $\{0, 1\}^N$). The matrix A is \mathcal{T} -forgiving if for all $T \in \mathcal{T}, n \in \mathbb{R}^N, u_0 \in \mathbb{R}^M$, we have:

$$u_0 = \operatorname{argmin}_u \|Au - (Au_0 + n.T)\|_1 \quad (4)$$

and u_0 is the unique minimizer. We say that A forgives \mathcal{T} errors.

Most of the time, in the literature, \mathcal{T} is the set of supports with a certain cardinal K . They then describe K -sparse outliers. However, preserving arbitrary sets of supports will be crucial to the image processing part of this article. We extend this definition of forgiving matrix to the regularized case. Let the sparsifying transform Ψ be a matrix of size $N' \times M$. We want to study the case when A is \mathcal{T} -forgiving on regular solutions u_0 where Ψu_0 is \mathcal{L} -sparse for some supports of signal sparsity \mathcal{L} . Ideally, we would like to extend the concept as

Definition 2 ((L_1, L_0) Forging Matrix Under a Sparse Hypothesis) Let \mathcal{T} be a set of supports in \mathbb{R}^N and \mathcal{L} a set of supports in $\mathbb{R}^{N'}$. The matrix A is \mathcal{T} -forgiving on Ψ, \mathcal{L} -sparse solutions if for all $T \in \mathcal{T}, n \in \mathbb{R}^N, u_0 \in \mathbb{R}^M$ such that $\Psi u_0 = \alpha_0 \in \mathcal{L} \cdot \mathbb{R}^{N'}$, we have:

$$u_0 = \operatorname{argmin}_u \|Au - (Au_0 + n.T)\|_1 \text{ s.t. } \Psi u \in \mathcal{L} \cdot \mathbb{R}^{N'} \quad (5)$$

and u_0 is the unique minimizer.

However, the lack of practical algorithm to solve exactly such a problem drives us to study a convexified version of it. In this paper we will consider the L^1, L^1 sparse regularization.

Definition 3 ((L_1, L_1) Forging Matrix Under a Sparse Hypothesis.) Let $\lambda \in \mathbb{R}$. Let \mathcal{T} be a set of supports in \mathbb{R}^N and \mathcal{L} a set of supports in $\mathbb{R}^{N'}$. We say that A is \mathcal{T} -forgiving on $\lambda\Psi, \mathcal{L}$ -sparse solutions by convex programming if for all $T \in \mathcal{T}, n \in \mathbb{R}^N, u_0 \in \mathbb{R}^M$ such that $\Psi u_0 = \alpha_0 \in \mathcal{L} \cdot \mathbb{R}^{N'}$, we have:

$$u_0 = \operatorname{argmin}_u \|Au - (Au_0 + n.T)\|_1 + \lambda \|\Psi u\|_1 \quad (6)$$

and u_0 is the unique minimizer.

Remark 1 When performing, minimization (6). We call λ , the regularization parameter. Usually, this parameter is considered as the level of regularization, *i.e.* the amount of regularity we want to impose the on the image.

Remark 2 Equation (6) is equivalent to: u_0 is the unique minimizer of

$$\min_u \|A_r^\lambda u - w_r\|_1 \quad (7)$$

$$\text{where } A_r^\lambda = \begin{pmatrix} A \\ \lambda\Psi \end{pmatrix} \text{ and } w_r = \begin{pmatrix} Au_0 \\ \lambda\Psi u_0 \end{pmatrix} + \begin{pmatrix} n.T \\ -\lambda\Psi u_0 \end{pmatrix} = \begin{pmatrix} Au_0 + n.T \\ 0 \end{pmatrix}$$

Remark 3 From the previous remark, if there is a λ , such that A_r^λ is forgiving outliers with supports in $\mathcal{T}' = \left\{ \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}, \tau_1 \in \mathcal{T}, \tau_2 \in \mathcal{L} \right\}$ (We say then that A_r^λ is \mathcal{T}' -forgiving or \mathcal{T}, \mathcal{L} -forgiving), then A is \mathcal{T} -forgiving on $\lambda\Psi, \mathcal{L}$ sparse solutions. In the next section, we will see that the converse is true.

Definition 4 (Non-Concentration Property.)

Let \mathcal{T} be a set of supports in \mathbb{R}^N and V a subspace of \mathbb{R}^N . We say that V has the \mathcal{T} -Non-Concentration Property (NCP) if for all $v \in V \setminus \{0\}$ and all $T \in \mathcal{T}$,

$$\|v.T\|_1 < \|v.T^c\|_1 \quad (8)$$

where T^c stands for the complement support of T .

We say that a matrix has the \mathcal{T} -Null Space Property (\mathcal{T} -NSP) if its null space has the \mathcal{T} -NCP.

Notice that, given the finite-dimensional setting, the NCP property implies the existence of a constant $\gamma < 1$ such that for all $v \in V$ and all $T \in \mathcal{T}$:

$$\|v.T\|_1 \leq \gamma \|v.T^c\|_1. \quad (9)$$

This constant is called the NSP constant in the area of sparse recovery.

When \mathcal{T} and \mathcal{L} are families of supports of cardinal K and S respectively. We talk about K, S -forgiveness and K, S -NCP. As explained in Candes and Tao (2005), forgiveness of a matrix A is equivalent to the sparse recovery property of matrices B annihilating A .

Definition 5 (Annihilating matrix.) Let A be a $N \times M$ matrix. We say that B is an annihilator of A or annihilates A if B has size $(N - M) \times N$, is full rank and $BA = 0$.

Definition 6 (Sparse Recovery Property.) Let \mathcal{T} be a set of supports in \mathbb{R}^N (subset of $\{0, 1\}^N$). The matrix B has the \mathcal{T} -sparse recovery property if for all $T \in \mathcal{T}, x_0 \in \mathbb{R}^N$, we have:

$$x_0 = \operatorname{argmin}_x \|x\|_1 \text{ s.t. } Bx = Bx_0 \quad (10)$$

and x_0 is the unique minimizer. We say that B is \mathcal{T} -sparse capable.

Moreover forgiveness of a is equivalent to the NCP of the space spanned by A . The NCP of a set is also equivalent to the sparse recovery property of matrices B having this set for kernel:

Theorem 1 For $\lambda = 0$, the forgiveness of A is equivalent to the sparse recovery property of the annihilators of A .

Proof See Candes and Tao (2005) or use Theorem 3. \square

3.2 Exact Recovery Property

This property can be found in Candes and Tao (2006). It is useful to study a particular instance of a forgiveness problem. For completeness of the paper, we produce a direct proof for the forgiveness problem

Theorem 2 Exact Reconstruction Property (ERP)

Let A, T, n, u_0 be an instance of the problem (2). A sufficient condition for u_0 to be the unique minimizer is the existence of a vector $v \in \mathbb{R}^N$ such that :

1. $v \in \ker A^H$ (kernel of A^H)
2. $v.T = \operatorname{sign}(n.T), \|v.T^c\|_\infty < 1$

Proof Because we can always make the change of variables $u' = u - u_0$, it suffices to verify that:

$$\|Au - n.T\|_1 \geq \|n.T\|_1 \quad (11)$$

Using the properties of the infinity norm of v , we have:

$$\|Au - n.T\|_1 \geq \langle v, n.T - Au \rangle \quad (12)$$

which we can decompose:

$$\|Au - n.T\|_1 \geq \langle v, n.T \rangle - \langle v, Au \rangle \quad (13)$$

$$\|Au - n.T\|_1 \geq \|n.T\|_1 - \langle A^H v, u \rangle \quad (14)$$

with $A^H v = 0$ which leads to the desired result. \square

3.3 Characterization of Forgiveness by the Non-Concentration Property

We will extend the following theorem, (a direct demonstration can be found in Traonmilin et al (2013)).

Theorem 3 The two following propositions are equivalent:

1. A is \mathcal{T} -forgiving
2. $\operatorname{Im} A$ (the image of A) has the \mathcal{T} -Non Concentration Property.

Theorem 4 If $N' = M$ and Ψ is invertible, the three following propositions are equivalent (recall that \mathcal{T}' is set of the concatenations of the supports from \mathcal{T} and \mathcal{L}):

1. A is \mathcal{T} -forgiving on $\lambda\Psi, \mathcal{L}$ -sparse solutions.
2. A_r^λ is the \mathcal{T}' -forgiving.
3. $\text{Im}A_r^\lambda$ has the \mathcal{T}' -Non Concentration Property (or \mathcal{T}, \mathcal{L} -NCP).

Proof 2 \Leftrightarrow 3: We use directly Theorem 3.

3 \Rightarrow 1: Using Theorem 3, $\text{Im}A_r^\lambda$ has the \mathcal{T}' -NCP implies that A_r^λ is \mathcal{T}' -forgiving. In particular, for every $w_r = \begin{pmatrix} Au_0 + n.T \\ 0 \end{pmatrix}$ defined as in Remark 2. Consequently, problem (6) has u_0 as a unique solution for each u_0, \mathcal{T}' which is equivalent to the forgiveness on $\lambda\Psi, \mathcal{L}$ -sparse solutions.

1 \Rightarrow 2: Let A be \mathcal{T} -forgiving on $\lambda\Psi, \mathcal{L}$ -sparse solutions, $L \in \mathcal{L}$ and $T \in \mathcal{T}$. Let $u_0 \in \mathbb{R}^N$, let $n_1 \in \mathbb{R}^N, n_2 \in \mathbb{R}^{N'}$. Let $u_1 = \Psi^{-1}(n_2.L)$. Using the forgiveness of A on $\lambda\Psi, \mathcal{L}$ -sparse solutions, u_1 is the unique minimizer of

$$u_1 = \underset{u}{\text{argmin}} \|Au - (Au_1 + n_1.T)\|_1 + \lambda\|\Psi u\|_1. \quad (15)$$

We make the change of variable $u = v + u_1 - u_0$. Then u_0 is the unique minimizer:

$$\begin{aligned} u_0 &= \underset{v}{\text{argmin}} \|Av - (Au_0 + n_1.T)\|_1 + \lambda\|\Psi(v + u_1 - u_0)\|_1 \\ u_0 &= \underset{v}{\text{argmin}} \|Av - (Au_0 + n_1.T)\|_1 + \lambda\|\Psi v - \Psi u_0 - n_2.L\|_1 \end{aligned} \quad (16)$$

Consequently, A_r^λ is \mathcal{T}' -forgiving. \square

Remark 4 If Ψ is a generic decomposition in an overdetermined dictionary, only the sufficient condition 3 \Leftrightarrow 2 \Rightarrow 1 holds.

Remark 5 From this theorem, the forgiveness of A_r^λ is guaranteed when A is \mathcal{T} -forgiving and Ψ is \mathcal{L} -forgiving.

A simple application of this theorem is the behaviour of the regularization when $\lambda \rightarrow 0$.

Proposition 1 *Let us suppose that A is K -forgiving. Then for λ sufficiently small A_r^λ is K forgiving on sparse solutions for any sparsity.*

Proof Let $T' = \begin{pmatrix} T \\ L \end{pmatrix} \in \mathcal{T}'$. Let $\lambda > 0$ and u with $\|u\|_1 = 1$. Because $\text{Im}A$ has the K -NCP with constant γ , we have:

$$\begin{aligned} \|(Au).T\|_1 + \lambda\|(\Psi u).L\|_1 &< \gamma\|(Au).T^c\|_1 + \lambda\|(\Psi u).L\|_1 \\ \|(A_r^\lambda u).T'\|_1 &< \gamma\|(Au).T^c\|_1 + \lambda(\|(\Psi u).L\|_1 \\ &\quad - \|(\Psi u).L^c\|_1) + \lambda\|(\Psi u).L^c\|_1 \end{aligned} \quad (17)$$

If $(\|(\Psi u).L\|_1 - \|(\Psi u).L^c\|_1) < 0$, $\text{Im}A_r^\lambda$ has the T, L -NCP. If not, for λ sufficiently small and independent from u (Ψ is a finite dimensional linear operator), we have

$$\gamma\|(Au).T^c\|_1 + \lambda(\|(\Psi u).L\|_1 - \|(\Psi u).L^c\|_1) < \|(Au).T^c\|_1 \quad (18)$$

and $\text{Im}A_r^\lambda$ has the T, L -NCP. The conclusion follows from Theorem 4. \square

With the non concentration property, we can reject cases where sparse regularization will not improve forgiveness.

Proposition 2 *If Ψ does not have the S -NCP. Then for λ large enough, A_r^λ does not have the K, S -NCP.*

Proof Let T, L be supports of cardinal K and S respectively. There is a u such that $\|(\Psi u).L\|_1 > \|\Psi u.L^c\|_1$. Let $0 < a < \|(\Psi u).L\|_1 - \|\Psi u.L^c\|_1$. Then,

$$\begin{aligned} \|(Au).T\|_1 + \lambda\|(\Psi u).L\|_1 &> \|(Au).T\|_1 + \lambda a \\ &\quad + \lambda\|(\Psi u).L^c\|_1. \end{aligned} \quad (19)$$

For λ large enough, $\|(Au).T\|_1 + \lambda a > \|(Au).T^c\|_1$ and

$$\|(Au).T\|_1 + \lambda\|(\Psi u).L\|_1 > \|(Au).T^c\|_1 + \lambda\|\Psi u.L^c\|_1 \quad (20)$$

and A_r^λ does not have the K, S -NCP. \square

3.4 A sufficient condition of forgiveness: the Restricted Isometry Property

The Restricted Isometry Property introduced by (Candes and Tao. 2005; Candès. 2008) can be a convenient way to ensure forgiveness of some matrices. This sufficient property can be verified for a matrix B annihilating A . It will then guarantee that B is sparse capable, and give the forgiveness of A with Theorem 1 The RIP is designed for supports with a particular cardinality. In order to study the regularized problem and its robustness to outliers, we need the RIP property for supports of the form $T = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$ with cardinality $|\tau_1| = K$ on the first N coordinates and $|\tau_2| = S$ on the last N' coordinates. We say that such T has cardinal K, S . Following definition 5, the matrix B (annihilating A) has size $N \times (N + N')$. We will write B_T the restriction of any matrix B to the columns matching the support T ($B_T x = B(x_T)$) where x_T is x zeroed outside of T .

3.4.1 Extension of the RIP to the regularized case

Definition 7 B has the restricted isometry property of order K, S and constant $\delta > 0$ if for all $x \in \mathbb{R}^{N+N'}$, for all supports T with cardinal K, S

$$(1 - \delta)\|x_T\|_2 \leq \|B(x_T)\|_2 \leq (1 + \delta)\|x_T\|_2. \quad (21)$$

The usual demonstration (Candes and Tao. 2005; Cohen et al. 2009) proves that the RIP implies the non concentration property, which implies the forgiveness. Here, we show that it is important to distinguish the sparsity of the signal and of the outliers. We kept the simplest demonstration to obtain the qualitative result. Better constant can be obtained with slightly more complicated arguments.

Theorem 5 Let B be an annihilator of A . If B has RIP of order $3K, 3S$ with constant $\delta \leq \frac{\sqrt{2}-1}{\sqrt{2}+1}$, then $\ker B$ has NCP of order K, S and A is K, S forgiving.

Proof We adapt the proof from Cohen et al (2009). We prove the NCP of $\ker B$. Let $x \in \ker B$. Let T_0 be the support with cardinal K, S selecting the K biggest absolute values of x on the first N coordinates and S biggest absolute values of x on the last N' coordinates. Let $T_1 \dots T_r$ be the sequence of support of cardinal $2K, 2S$ having the next biggest elements. Let $x_0 = x_{T_0} + x_{T_1}$, then $Bx_0 = -B(\sum_{i=2}^r x_{T_i})$ because $x \in \ker B$. It is sufficient to verify the NCP for $T = T_0$ because it concentrates the most energy of x . Using Cauchy-Schwartz inequality (or norm equivalence) $\|x_T\|_1 \leq K^{1/2}\|x_{T,1}\|_2 + S^{1/2}\|x_{T,2}\|_2$ where $x_{T,1}$ is x restricted to first N coordinates, and $x_{T,2}$ is x restricted to the last N' coordinates

Because $x_{0,1}$ and $x_{0,2}$ are $3K, 3S$ -sparse, we use the definition of x_0 followed by the RIP hypothesis and the triangle inequality :

$$\begin{aligned} \|x_T\|_2 &\leq K^{1/2}\|x_{T,1}\|_2 + S^{1/2}\|x_{T,2}\|_2 \\ &\leq K^{1/2}\|x_{0,1}\|_2 + S^{1/2}\|x_{0,2}\|_2 \\ &\leq (1 - \delta)^{-1}(K^{1/2}\|Bx_{0,1}\|_2 + S^{1/2}\|Bx_{0,2}\|_2) \\ &\leq (1 - \delta)^{-1} \left(K^{1/2}\|B \left(\sum_{i=2}^r x_{T_i,1} \right)\|_2 \right. \\ &\quad \left. + S^{1/2}\|B \left(\sum_{i=2}^r x_{T_i,2} \right)\|_2 \right) \\ &\leq (1 - \delta)^{-1} \left(\sum_{i=2}^r (K^{1/2}\|Bx_{T_i,1}\|_2 + S^{1/2}\|Bx_{T_i,2}\|_2) \right) \end{aligned} \quad (22)$$

We use the RIP again:

$$\|x_T\|_2 \leq \frac{1 + \delta}{1 - \delta} \left(\sum_{i=2}^r (K^{1/2}\|x_{T_i,1}\|_2 + S^{1/2}\|x_{T_i,2}\|_2) \right)$$

(23)

Now, we bound the right side with $\|x\|_1$. Let $j \geq 1$ and y (respectively y') be one of the first N coordinate of $x_{T_{j+1}}$ (respectively x_{T_j}). Then $|y| \leq |y'|$. Let z (respectively z') be one of the last N' coordinate of $x_{T_{j+1}}$ (respectively x_{T_j}). Then $|z| \leq |z'|$. From this observation,

$$\begin{aligned} 2Ky &\leq \|x_{T_j,1}\|_1 \\ 2Sz &\leq \|x_{T_j,2}\|_1. \end{aligned} \quad (24)$$

We square and sum over y and z :

$$\begin{aligned} 2K\|x_{T_{j+1},1}\|_2^2 &\leq \|x_{T_j,1}\|_1^2 \\ 2S\|x_{T_{j+1},2}\|_2^2 &\leq \|x_{T_j,2}\|_1^2 \end{aligned} \quad (25)$$

We take the square root and add these inequalities:

$$\sqrt{2}(K^{1/2}\|x_{T_{j+1},1}\|_2 + S^{1/2}\|x_{T_{j+1},2}\|_2) \leq \|x_{T_j}\|_1 \quad (26)$$

We use this result with equation (23) :

$$\|x_T\|_1 \leq \frac{1 + \delta}{1 - \delta} \frac{1}{\sqrt{2}} \|x_{T^c}\|_1 \quad (27)$$

which is the NCP if $\frac{1}{\sqrt{2}} \frac{1+\delta}{1-\delta} \leq 1$. This is equivalent to $\delta \leq \frac{\sqrt{2}-1}{1+\sqrt{2}}$. \square

Remark 6 This theorem could be easily extended to matrices B having more than 2 blocks, *i.e.* for signal corrupted by sparse noise in a union of dictionaries A_r .

3.4.2 The choice of the annihilating matrix

The quality of the RIP is driven by the conditioning of sub-matrices of B . To our knowledge, the choice of the B such that $\ker B = \text{Im} A$ and B has the RIP with best constant and/or order, is an open problem. This leads to some other questions. Given a matrix C , C might not be the best candidate to verify the RIP for its own sparse recovery property. However, we can easily construct a B from A by taking the orthogonal projection on the orthogonal of the image of A ($P_{(\text{Im} A)^\perp}$) and restricting it to its image. Then B is an annihilator of A . Also, we can construct a family of annihilators of A_r using an annihilator of A .

Proposition 3 Let $\epsilon \Rightarrow 0$. Let $\Phi = (\Psi^H \Psi)^{-1} \Psi$. Let $B_r = (B_1, -\lambda^{-1} A \Phi)$, with $B_1 = \epsilon P_{(\text{Im} A)^\perp} + I$ (I is the identity matrix). Then B_r is an annihilator of A_r^λ .

Proof First, we remark that $B_2 = -\lambda^{-1} B_1 A \Phi$ implies $B_r A_r^\lambda = 0$. Thus $B_r = (B_1, -\lambda^{-1} B_1 A \Phi)$ is sufficient. Moreover B_1 is full rank and $B_1 A = A$. \square

Remark 7 We wanted to construct B_r from $P_{(\text{Im} A)^\perp}$. We added I to ensure that B_1 is full rank.

In Section 4.3, we show with an example that different classes of annihilating matrices lead to different RIP constants.

3.4.3 A sufficient condition for the RIP of the annihilating matrices

Here we show that, with our particular construction of annihilating matrices, we can find sufficient conditions for the K, S -RIP of A_r^λ . Let $B_r = (B_1 \ B_2)$ annihilating A_r^λ , with B_1 a $N \times N$ matrix and B_2 a $N \times N'$ matrix. We ask ourselves when B_r has RIP.

Definition 8 (Operator bounds on sparse vectors.) B is bounded by σ_m, σ_M on K sparse vectors if :

$$\sigma_M = \sup_{x, |T| \leq K} \frac{\|Bx_T\|_2}{\|x_T\|_2} \quad (28)$$

$$\sigma_m = \inf_{x, |T| \leq K} \frac{\|Bx_T\|_2}{\|x_T\|_2} \quad (29)$$

Proposition 4 (Normalization.) If $\sigma_m < \frac{\|Bx\|_2}{\|x\|_2} < \sigma_M$ for x K, S sparse, there is an η such that ηB has RIP with constant $\delta = \frac{\sigma_M - \sigma_m}{\sigma_M + \sigma_m}$. We call ηB normalized B .

Proof If we let $\eta = 2/(\sigma_M + \sigma_m)$ and $\delta = \frac{\sigma_M - \sigma_m}{\sigma_M + \sigma_m}$, then $\eta\sigma_m = 1 - \delta$ and $\eta\sigma_M = 1 + \delta$. \square

The mutual coherence between B_1 and B_2 (Similarly as in Candès and Romberg (2007)) is useful to produce a deterministic sufficient condition.

Proposition 5 (Sufficient condition for the RIP of the annihilator of A_r^λ .) Let B_1 bounded by $\sigma_{m_1}, \sigma_{M_1}$ on K -sparse vectors and B_2 bounded by $\sigma_{m_2}, \sigma_{M_2}$ on S -sparse vectors. Let $\sigma_m = \min(\sigma_{m_1}, \sigma_{m_2})$ and $\sigma_M = \max(\sigma_{M_1}, \sigma_{M_2})$. Let $\mu = \sup_{|T| \leq K, |L| \leq S} \|B_{1,T}^H B_{2,L}\|_{op}$. If $\mu < \sigma_m^2$, then normalized B_r has RIP of order K, S and constant $\delta' = \frac{\sqrt{\sigma_M^2 + \mu} - \sqrt{\sigma_m^2 - \mu}}{\sqrt{\sigma_M^2 + \mu} + \sqrt{\sigma_m^2 - \mu}} < 1$.

Proof Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ (x_1 has size N) be a vector supported on T with cardinal K, S .

$$\begin{aligned} \|Bx\|_2^2 &= \|B_1x_1 + B_2x_2\|_2^2 \\ &= \|B_1x_1\|_2^2 + \|B_2x_2\|_2^2 + 2 \langle B_1x_1, B_2x_2 \rangle \end{aligned} \quad (30)$$

The mutual coherence μ between B_1 and B_2 plays an important role in this equation:

$$|\langle B_1x_1, B_2x_2 \rangle| \leq \mu \|x_1\|_2 \|x_2\|_2 \leq \frac{\mu}{2} \|x\|_2^2 \quad (31)$$

because $ab \leq (a^2 + b^2)/2$. The first two terms are controlled by the bounds of B_1 and B_2 . We have :

$$\begin{aligned} \min(\sigma_{m_1}, \sigma_{m_2})^2 - \mu &\leq \frac{\|Bx\|_2^2}{\|x\|_2^2} \leq \max(\sigma_{M_1}, \sigma_{M_2})^2 + \mu \\ \sqrt{\sigma_m^2 - \mu} &\leq \frac{\|Bx\|_2}{\|x\|_2} \leq \sqrt{\sigma_M^2 + \mu} \end{aligned}$$

The conclusion follows from proposition 4. \square

3.4.4 On the choice of λ

The parameter λ represents the level of a priori knowledge on the signal that we want to use in the minimization of the considered problem. The question of choosing λ in different settings (e.g. L^2, L^2 minimization L^2, L^1 minimization) is studied intensively in the area of image processing. A way to perform this choice is to set an objective function (generally a risk function) and find the λ which minimize the risk (Vaiter et al. 2012). Here, the case of noiseless recovery is studied. Consequently, two questions come naturally:

- Given λ , which couples K, S lead to to a perfect recovery.
- Given K, S , can we find λ leading to a perfect recovery (i.e. the problem is K, S -forgiving for some λ).

Ideally, we would need to choose λ to optimize the NCP constant or to guarantee the ERP. Because the NCP is an equivalent characterization of forgiveness, this would associate each couple K, S with an optimal NCP constant (when λ varies). If this constant is lower than 1 then recovery would be possible. However, this is difficult to determine in general as L^1 conditioning of sub-matrices of A_r^λ will intervene. In specific image processing problems, it might be possible to estimate directly this constant. In some simple set-up, we can study exactly the NCP and ERP with respect to λ (Section 5).

A first step is to try to optimize the RIP, which will give a sufficient conditioning for K, S forgiveness. We show that we can guarantee the RIP for some λ with our construction of annihilating matrix for a certain class of A_r^λ . However, as said before, because we do not know the optimal annihilator (in terms of RIP), this result has limited practical use, but highlights how blocks of B interact. Later experiments will show that testing directly the RIP of some annihilators lead to the qualitative behaviour of some random matrices.

Next, we study how, from A_r^λ we can calculate the RIP constant δ' for a particular B_r . Which in turn, permits to derive sufficient condition on λ to guarantee the K, S -forgiveness of A_r^λ .

Theorem 6 We consider a regularized A_r^λ and the corresponding B_r (constructed as in Proposition 3) with $\epsilon = 0$: $B_r = (I, -\lambda^{-1}A\Phi)$. Let $A' = A\Phi$. We suppose that A' is bounded by $\sigma_m(A'), \sigma_M(A')$ on S -sparse vectors. Let $\mu_1 = \sup_{|T| < K, |L| < S} \|I_T^H A'_L\|_2$.

If $\mu_1^2 < \sigma_m(A')^2$, then we can find λ such that $\frac{\mu_1}{\sigma_m(A')^2} < \lambda^{-1} < \frac{1}{\mu_1}$. With such λ , matrix B_r (normalized) has RIP of order K, S and constant $\delta' < 1$.

In particular for $\lambda^{-1} = 1/\sigma_m(A')$, we have $\delta' \leq \frac{\sqrt{\kappa^2 + \mu_1} - \sqrt{1 - \mu_1}}{\sqrt{\kappa^2 + \mu_1} + \sqrt{1 - \mu_1}}$ where $\kappa = \sigma_M(A')/\sigma_m(A')$

Proof Let $\sigma_m = \min(\sigma_{m_1}, \sigma_{m_2})$, $\sigma_M = \max(\sigma_{M_1}, \sigma_{M_2})$ as defined in Proposition 5. Because $B_1 = I$, $\sigma_{m_1} = \sigma_{M_1} = 1$:

$$\sigma_m = \min(1, \lambda^{-1}\sigma_m(A')) \quad (33)$$

and

$$\sigma_M = \max(1, \lambda^{-1}\sigma_M(A')) \quad (34)$$

To meet the hypothesis of Proposition 5, we need $\mu = \lambda^{-1}\mu_1 < \sigma_m^2$. This is true if the following inequalities are true :

$$\begin{aligned} \lambda^{-1}\mu_1 &< 1 \\ \lambda^{-1}\mu_1 &< \lambda^{-2}\sigma_m(A')^2 \end{aligned} \quad (35)$$

which are equivalent to

$$\begin{aligned} \lambda^{-1} &< \frac{1}{\mu_1} \\ \frac{\mu_1}{\sigma_m(A')^2} &< \lambda^{-1} \end{aligned} \quad (36)$$

Such λ^{-1} exist if:

$$\mu_1^2 < \sigma_m(A')^2 \quad (37)$$

Then, with Proposition 5, B_r has RIP of order K, S and constant $\delta' < 1$.

If $\lambda^{-1} = 1/\sigma_m(A')$, we have $\sigma_m = 1$, $\sigma_M = \kappa$ and μ . Using these values :

$$\delta' \leq \frac{\sqrt{\kappa^2 + \mu_1} - \sqrt{1 - \mu_1}}{\sqrt{\kappa^2 + \mu_1} + \sqrt{1 - \mu_1}} \quad (38)$$

□

Remark 8 We showed that given a particular family of B_r , we can sometimes determine which one gives a RIP.

Remark 9 The condition $\mu_1^2 < \sigma_m(A')^2$ is strong. Some matrices verify this : take for example the matrix $B_2 = \begin{pmatrix} I \\ I \end{pmatrix}$.

Using the explicit value of λ from this theorem. Its hypotheses require that A' has a well conditioned submatrices, and small norm (*i.e.* small mutual coherence between A and Φ) if $\kappa = 1.25$ and $\mu_1 = 0.1$, $\delta' \leq \frac{\sqrt{2}-1}{\sqrt{2}+1}$.

4 The case of random matrices

We study the case of convex sparse regularization and robustness for random matrices. First, we suppose that $A_r^\lambda = \begin{pmatrix} A \\ \lambda\Psi \end{pmatrix}$ where A and Ψ are a $N \times M$ and $N' \times M$ Gaussian random matrices. Let \mathcal{T} be the set of supports $T \subset \mathbb{R}^N$ with $|T| = K$ and $\mathcal{L} \subset \mathbb{R}^{N'}$ the set of supports L with $|L| = S$. In Section 4.3, the regularization part Ψ will be identity.

4.1 Review of existing bounds for $\lambda = 1$

Let B_r be an annihilator of A_r^λ . In Candes and Tao (2005), Candes and Tao argue that we can think of B as a random Gaussian matrix. Matrix B has size $N - M \times N$ ($\dim \text{Im} A = \dim \ker B = M$). Then B has the K -sparse recovery capability and A is K -forgiving with overwhelming probability if :

$$K < C \frac{N - M}{\log \frac{N}{N - M}} \quad (39)$$

with C a constant. We want to see how this condition evolves when we add regularization. We use the following proposition:

Proposition 6 *If A_r^λ is $K + S$ -forgiving for any sparsity, then A is \mathcal{T} -forgiving on $\lambda\Psi, \mathcal{L}$ -sparse vectors.*

Proof We show the NCP. $\text{Im} A_r^\lambda$ has the $K + S$ -NCP. Moreover the set of supports of size $K + S$ contains $(\mathcal{T}, \mathcal{L})$. Consequently, $\text{Im} A_r^\lambda$ has the $(\mathcal{T}, \mathcal{L})$ -NCP and A is \mathcal{T} -forgiving on $\lambda\Psi, \mathcal{L}$ -sparse vectors. □

A_r^λ has size $N + N' \times M$. We see how the bound for random matrix is changed by convex sparse regularization.

Proposition 7 *A is \mathcal{T} -forgiving on $\lambda\Psi, \mathcal{L}$ -sparse vectors (or A_r^λ is K, S forgiving) with overwhelming probability if:*

$$K + S < C \frac{N + N' - M}{\log \frac{N + N'}{N + N' - M}} \quad (40)$$

Proof We use equation (39) with the dimensions of A_r^λ and apply Proposition 6. □

This study is only theoretical as taking the same kind of basis for observation and regularization is not realistic. However, it shows an example where adding regularization adds forgiveness power. The main point is that $K + S$ is what drives the forgiveness power, *i.e.* when there is more outliers, the signal needs to be the more sparse according to this bound. This is confirmed by the experiment from Figure 1, where we generated

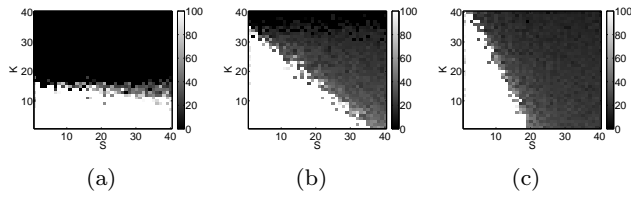


Fig. 1 *A* random, Ψ random. Worst reconstruction PSNR when A_r^λ has size $N = 100$, $M = 50$ w.r.t to signal and noise sparsity K, S for (a) $\lambda = 0.1$, (b) $\lambda = 1$ and (c) $\lambda = 2$.

random forgiveness experiments for different values of K, S . We show the worst reconstruction Peak Signal to Noise Ratio (PSNR) in decibels (dB) of the result of the minimization of problem (7) (using $\ell_1 - MAGIC$, the software used in Candes and Tao (2006)).

This review showed that $K + S$ is the limiting factor for $\lambda = 1$. However, each line of A_r^λ in this case has the same energy. Thus checking $K + S$ in this non-uniform RIP leads to the same result as checking the K, S RIP.

4.2 K, S -forgiveness with respect to λ

We generate K, S -forgiveness in Figure 1. 100 random forgiveness experiments using for each experiment a Gaussian random matrix A of size 100×50 and a Gaussian random matrix Ψ of size 50×50 . The PSNR of the worst experiment for each K, S is displayed. We observe that for each λ , we have a constraint of the type $K + c(\lambda)S < C_0$ where $c(\lambda)$ is a constant depending on λ . Qualitatively, for large λ much more outliers are removed, but this works for very sparse signals. For small λ , improvement with respect to the non regularized case is less strong but works for signals that are less sparse.

This is confirmed by our theoretical analysis with the K, S -RIP. We show experimentally that the K, S -RIP gives a qualitative way to check if the matrix is K, S -forgiving. Let A and Ψ be Gaussian random matrices. We show an estimation of the RIP of $P_{(\text{Im}A_r^\lambda)^\perp}$. In Figure 2, we generate the same number of experiments, and check if the worst conditioning of one sub-matrix per experiment has the required RIP constant. We show when the RIP is verified for different λ . The union of all the couple K, S shows where we can have a perfect reconstruction with sparse regularization (provided we can find λ). The shape of the zone of the K, S couples leading to a perfect reconstruction follows the same behaviour as in Figure 1. As expected, a quantitative gap exists between the characterization by the RIP and the recovery experiments.

We studied the case of purely random matrices to make a link with results from the literature. Next, we

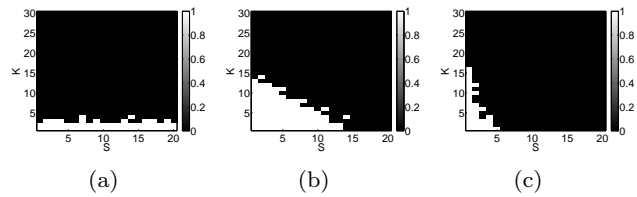


Fig. 2 *A* random, Ψ random. Signal and noise sparsity K, S leading to the RIP when A_r^λ is a random observation with random regularization matrix with $N = 100$, $M = 50$ for (a) $\lambda = 0.1$, (b) $\lambda = 1$ and (c) $\lambda = 2$.

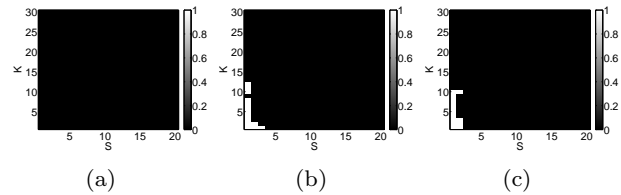


Fig. 3 *A* random, $\Psi = I$. Signal and noise sparsity K, S leading to the K, S -RIP of $B_r = (I, -\lambda^{-1}A)$ with $N = 100$, $M = 50$ for (a) $\lambda = 0.1\sqrt{M}$, (a) $\lambda = \sqrt{M}$ and (c) $\lambda = 2\sqrt{M}$.

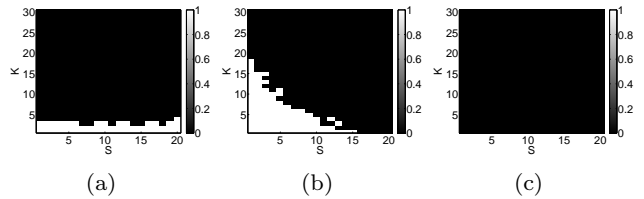


Fig. 4 *A* random, $\Psi = I$. Same experiment as Figure 3 using a different annihilator. Signal and noise sparsity K, S leading to the K, S -RIP with $N = 100$, $M = 50$ for (a) $\lambda = 0.1\sqrt{M}$, (a) $\lambda = \sqrt{M}$ and (c) $\lambda = 2\sqrt{M}$.

study another type of random matrices where only the sparsifying transform of the signal is random.

4.3 The case of random observation from a sparse regularization space

Here we consider $A_r^\lambda = \begin{pmatrix} A \\ \lambda I \end{pmatrix}$. We perform the same experiments as in the previous section. Using the construction of B_r from Section 3.4.3 for $\epsilon = 0$, we have $B_1 = I$ and $B_2 = -\lambda^{-1}A$. For each λ , we check the couples, K, S , for which, B_r has RIP in Figure 3. In Figure 4, we make the same experiment using $P_{(\text{Im}A_r^\lambda)^\perp}$. It illustrates the fact that different matrices having the same kernel lead to different RIP. In conclusion, the RIP should only be used as a qualitative measure of how the recovery behaves, and the annihilator should be chosen carefully.

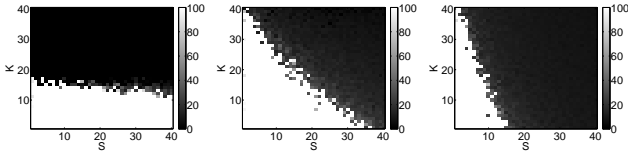


Fig. 5 A random, $\Psi = I$. Worst PSNR of the reconstruction w.r.t. signal and noise sparsity K, S , for A_r^λ with $N = 100$, $M = 50$ for (a) $\lambda = 0.1\sqrt{M}$, (b) $\lambda = \sqrt{M}$ (c) $\lambda = 2\sqrt{M}$

In Figure 5, the perfect recovery experiment is made using A_r^λ . Variations of λ give the same behaviour of the shape of the zone leading to K, S -forgiveness.

5 Application to multi-image processing

In this section, we first show in the case of multi-image processing (multi image denoising), that for supports having a fixed value of sparsity K, L , TV regularization does not improve forgiveness. However, if we make the assumption that supports of signal gradient and outliers are disjoint, we show that forgiveness is improved. We use this result to show that some particular SR problems can be better solved using $L^1 - TV$ minimization. These considerations lead to the determination of the value of the optimal regularization parameter for 1D $L^1 - TV$ multi-signal denoising and super-resolution problem.

5.1 Multi-image denoising

Multi-image denoising (also called burst denoising) is the process of using several images of the same scene to produce an image with reduced noise. We consider here the very simple problem of removing outliers from a collection of registered images. Using the multi-image acquisition model from Section 2 with sub-sampling factor $M = 1$ and no motion ($Q_i = Id$). The L^1 minimization from Equation 2 will lead to the selection of the median of the observations of each pixel. In this case, we can directly study the non-concentration property of A and A_r^λ . Because it is a particular case of super-resolution (with $M = 1$), it will allow us to draw some limits for the behaviour of $L^1 - TV$ SR.

Let N be odd. In 1D, A is of size $Nl \times l$ and made of $l \times l$ diagonal blocks of 1. Let $T \in \mathcal{T}$,

Proposition 8 *A is \mathcal{T} forgiving if and only if each pixel is contaminated at most $N/2$ times*

Proof We check the equivalent \mathcal{T} NCP for A . First, if a pixel i is contaminated $K = \lfloor N/2 \rfloor + 1$ times. We

consider the problem where only this pixel is not 0. Then

$$\|(Au).T\|_1 = K|u_i| > \lfloor N/2 \rfloor |u_i| = \|(Au).T^c\|_1 \quad (41)$$

and the NCP is not verified. It is necessary that any pixel is not contaminated more than $\lfloor N/2 \rfloor$ times for A to be \mathcal{T} forgiving.

Conversely, if every pixel is not contaminated more than $\lfloor N/2 \rfloor$, we look at the worst case when every pixel is contaminated exactly $K = \lfloor N/2 \rfloor$ times. We have

$$\begin{aligned} \|(Au).T\|_1 &= \\ K \sum_i |u_i| &< (\lfloor N/2 \rfloor + 1) \sum_i |u_i| = \|(Au).T^c\|_1 \end{aligned} \quad (42)$$

Then $\text{Im}A$ has the \mathcal{T} -NCP. \square

We now consider Ψ the discrete gradient function : $(\Psi u)_1 = u_1$, $(\Psi u)_i = u_i - u_{i-1}$. Because Ψ is invertible, for any support of sparsity L , we can find u such that :

$$\|(\Psi u).L\|_1 > \|(\Psi u).L^c\|_1 \quad (43)$$

If we consider the worst case for the NCP of A_r^λ , we find that Ψ cannot improve it. Given a support of sparsity L , we can always find a signal u such that $\|(\Psi u).L\|_1 - \|(\Psi u).L^c\|_1 > 0$. With such u , we would need to ensure the NCP :

$$\begin{aligned} \|(Au).T\|_1 + \lambda \|(\Psi u).L\|_1 &= K \sum_{i=1, S} |u_i| + \lambda \|(\Psi u).L\|_1 \\ &< (N - K) \sum_{i=1, S} |u_i| + \lambda \|(\Psi u).L^c\|_1 \end{aligned} \quad (44)$$

which is equivalent to

$$\begin{aligned} K \sum_{i=1, S} |u_i| + \lambda (\|(\Psi u).L\|_1 - \|(\Psi u).L^c\|_1) \\ &< (N - K) \sum_{i=1, S} |u_i| \end{aligned} \quad (45)$$

The left side is minimized for $\lambda = 0$. This is the weakest inequality that we can obtain. Consequently, in the worst case, it is better to avoid TV regularization. In practice, outliers might not contaminate jumps in the signal. Using the ERP, we can show that the system becomes very forgiving, we can even determine an optimal λ .

Let us suppose that no support of outlier coincides with jumps in the signal, *i.e.* if the gradient at a some position is non 0, the 2 samples used for its calculation are never contaminated by outliers.

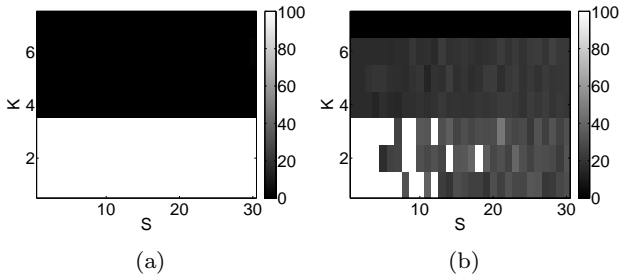


Fig. 6 A multi-image observation, Ψ gradient, overlap. Worst PSNR of the reconstruction w.r.t. signal and noise sparsity K, S when outliers can overlap with signal jumps with $N = 7$ and $l = 50$ for (a) $\lambda = 0$, (b) $\lambda = 0.95N/2$

Proposition 9 Let \mathcal{T}' be the sparsity support of outliers and signal, such that outliers and signal jumps be disjoint. Then \mathcal{T}' is not a concatenation of two sets of supports of signal and noise. Let us suppose that at most K outliers contaminate each pixel and $\lambda > 0$. If $(2K - N)/2 < \lambda < N/2$, A_r^λ is \mathcal{T}' -forgiving.

Proof We look for a vector $v \in \mathbb{R}^{(N+1)l}$ which verifies properties of Theorem 2. We index the values of v by indices $(v_{j,k})_{j=1..l, k=1..N+1}$. Index j represents the pixel number. Index k represents the acquired image number for $k \leq N$ and the image gradient for $k = N + 1$. Let $T' = (T, L)$ be a support in \mathcal{T}' . Let v such that $v.T = \text{sign}(n).T$ $v.L = \text{sign}(\Psi u_0).L$. We need $(A_r^\lambda)^H v = 0$ which translates for each pixel (values indexed by j of $(A_r^\lambda)^H v$):

$$\sum_{k=0, N-1} v_{j,k} + \lambda(v_{j, N+1} - v_{j+1, N+1}) = 0 \quad (46)$$

Given a pixel, we can distinguish two cases :

- pixel j is not contaminated by outliers. Then, for $k = 1..N$ each variable $v_{j,k}$ is free and we can choose $v_{j,k} = -\lambda(v_{j, N+1} - v_{j+1, N+1})/N$. Because the maximum L^1 norm of the right side is 2, we guarantee that $\|v.T^c\|_\infty < 1$ if $\lambda < N/2$.
- pixel j is contaminated by outliers. Then we need $\sum_{k \in T} v_{j,k} = -\sum_{k \in T^c} v_{j,k} - \lambda(v_{j, N+1} - v_{j+1, N+1})$, with the right side having free variables. Because the maximum L^1 norm of the left side is K , we can find the right v with $\|v.T^c\|_\infty < 1$ if $K < N - K + 2\lambda$, which is rewritten $\lambda > (2K - N)/2$.

□

The same proposition is true in 2D with $(2K - N)/4 < \lambda < N/4$.

The result of experiments with overlapping outliers and signal jumps are displayed in Figure 6. They show that when we allow outliers and signal jumps to overlap, multi-image denoising with $L^1 - TV$ is not forgiving.

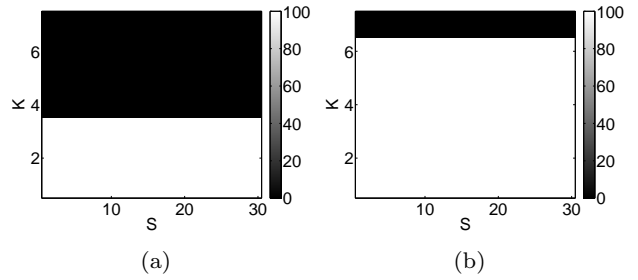


Fig. 7 $A =$ multi-image observation, $\Psi =$ gradient. No overlap Worst PSNR of the reconstruction w.r.t. signal and noise sparsity K, S when outliers do not overlap with signal jumps with $N = 7$ and $l = 50$ for (a) $\lambda = 0$, (b) $\lambda = 0.95N/2$

In Figure 7, we show that with no overlap between jumps and outliers and $(N - 2)/2 < \lambda < N/2$, A_r^λ is $N - 1$ -forgiving., for $N = 7$. In other words, only one clean observation of each pixel is needed when there is no overlap.

5.2 Super-Resolution

5.2.1 The limits of $L^1 - TV$ super-resolution for outlier removal

We now consider A as a 1D translational SR matrix and Ψ is the same discrete gradient calculation as in the previous section. The study in the worst case scenario for multi-image denoising shows that without a precise hypothesis on the support of signal and outliers, sparse L^1 regularization does not enhance forgiveness.

For large number of images, L^1 norm super-resolution is forgiving N_c contaminated images if the number of total images N is greater than CN with C a constant. In Figure 8, we show experiments of the forgiveness of A and A_r^λ for an acquisition setting having poor forgiveness. We observe the trade-off between the number of outliers and the sparsity of the signal for a perfect reconstruction.

The main conclusion that we can draw from this experiment is that enforcing a regularity condition on the image allows to be robust to more outliers, but that the improvement is limited compared to the strength of the assumption on the signal.

Similarly to the multi-image denoising case, we would like to know if outliers not coinciding with signal jumps would be removed with TV regularization. The main problem with exact super-resolution is that the lines of A are sine cardinal. Consequently, the contamination by one outlier will concern all pixels in the desired HR image.

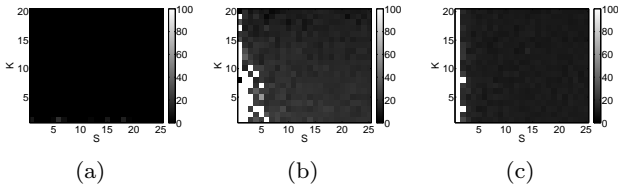


Fig. 8 $A = 1D$ SR operator, $\Psi =$ gradient. Worst PSNR of the reconstruction for TV regularized 1D super-resolution w.r.t to signal and noise sparsity with $N = 10$, $M = 2$ and $l = 25$ for (a) $\lambda = 0$, (b) $\lambda = 1$ and (c) $\lambda = 10$

5.2.2 Finite impulse response L^1 SR algorithm benefits from TV regularization

In practice, some SR methods make the approximation that the shift operator is of finite length, such model can be found in Champagnat et al (2009). We propose to replace the shift by convolution by a sine cardinal by the convolution by a truncated Gaussian. Such model can be justified by the knowledge of real Point Spread Function estimation (which can be found in Delbracio et al (2012)). Here, for 1D SR, A is defined by

$$A : \mathbb{R}^{Ml} \rightarrow (\mathbb{R}^l)^N \quad (47)$$

$$u \rightarrow (A_i u)_{i=1,N} = (SG_i u)_{i=1,N}$$

where G_i is the convolution by a shifted truncated Gaussian $(g_q)_{q=-p,p}$ of size $2p + 1$. We choose the Gaussian parameter a such that $|g_p|$ is small. We consequently suppose that G_i is 0 outside of the first $p + 3$ diagonals (the amplitude of translations in SR can be kept less than $M = 2$ without loss of generality). We keep the same definition for Ψ . To avoid complicated calculations, we also suppose that signal jumps are separated by the length of the filter.

Proposition 10 *Let \mathcal{T}' be the supports of sparsity such that outliers and signal jumps are separated by $p + 1$ and that jumps are separated by at least $2p + 1$ pixel. Let us suppose that at most K outliers contaminate each pixel. let $G_+ = \max_{i,j} \|G_{i,j}\|_1$ and $G_- = \min_{i,j} \|G_{i,j}\|_1$ where $G_{i,j}$ is the j th column of SG_i . If $(K(G_+ + G_-) - NG_-)/2 < \lambda < NG_-/2$, A_r^λ is K -forgiving.*

Proof We look for a vector $v \in \mathbb{R}^{Nl+Ml}$ which verifies properties of theorem 2. Let $T' = (T, L)$ be a support in \mathcal{T}' . Let v such that $v.T = \text{sign}(n).T$ $v.L = \text{sign}(\Psi u_0).L$. We write $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ where v_1 has size Nl and v_2 has size Ml . We need $(A_r^\lambda)^H v = 0$ which, for each HR pixel (each column j of A_r^λ), translates to:

$$A_j^H v_1 + \lambda \Psi_j^H v_2 = 0, \quad (48)$$

with A_j the j th column of A and Ψ_j the j th column of Ψ . Given a pixel, we can distinguish two cases :

- pixel j is not contaminated by outliers. Then each variable in v_1 is free. We need $A_j^H v_1 > 2\lambda$ because the maximum value of $\Psi_j^H v_2$ is 2. The restriction $|v_1|_\infty < 1$ imposes that $\lambda < \|A_j^H\|_1/2$.
- pixel j is contaminated by outliers. Then we need

$$A_j^H(v_1.T) = -A_j^H(v_2.T^c) - \lambda \Psi_j^H v_2 \quad (49)$$

with the right side being free variables. Similarly as before, it is necessary that $\|A_{T,j}^H\|_1 < \|A_{T^c,j}^H\|_1 + 2\lambda$, which is equivalent to $\lambda > (\|A_{T,j}^H\|_1 - \|A_{T^c,j}^H\|_1)/2$.

These two conditions are sufficient, and using the definition of G_+ and G_- , the condition $(K(G_+ + G_-) - NG_-)/2 < \lambda < NG_-/2$ is sufficient for the ERP. \square

Remark 10 We can find a λ if $K < K_{max} = N/(1 + G_+/G_-)/2$. Typically, $G_+/G_- \sim 2$. If consecutive jumps were allowed in the signal, the constants would change and would require the use of the L^1 norm of inverse matrices of restrictions of A . Because estimating the extremum of these norms is difficult, we restrict ourselves to this simplified version of the problem.

Experiments for 1D super-resolution. In Figure 9, we show the influence of TV regularization on 1D finite length SR with zoom $M = 2$. The HR image has size $l = 210$. For each K, S , we generated 100 experiments meeting hypotheses of Proposition 10, contaminating the same pixel in the 10 LR images, in an area without jumps. We solved the regularized SR problem using our calculation of λ . We see that after regularization, the number of contaminated pixels can be greater than half the number of images.

Experiments for 2D super-resolution. We use the same model for 2D super-resolution (a 2D Gaussian is used instead). We use the image Shepp Logan phantom which is sparse in the gradient domain. Ψ is the 2D discrete gradient. We perform a 2D L^1 super-resolution with total variation regularization with $M = 2$. We use an iteratively reweighted least squares algorithm (Daubechies et al. 2010). We contaminate the same region of the image on the different LR images. In Figure 10, we show the ideal image, the sparsity level of the image, a clean low resolution image and a contaminated low resolution image.

From Proposition 10, we can infer that a range of λ will make the 2D $L^1 - TV$ SR problem forgiving. We show in Figure 11, the result of 2D L^1 SR without regularization, and with TV regularization. The SR was performed using 6 LR images, with 2 contaminated images (recall that the minimum number of images for a perfect reconstruction is 4 for $M = 2$). While SR without regularization fails TV regularization gives a

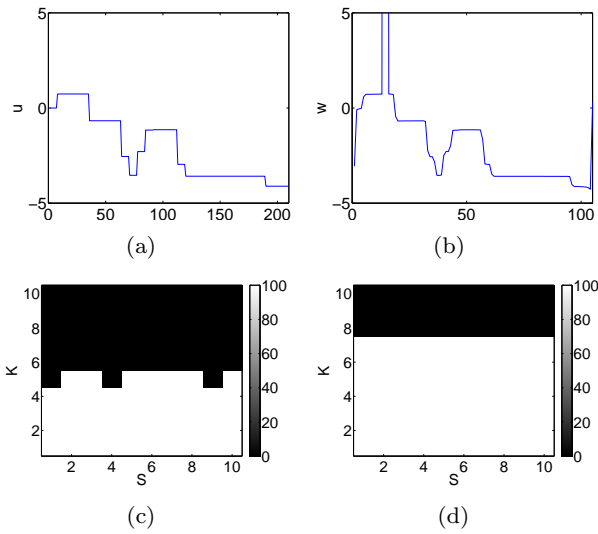


Fig. 9 $A =$ finite length 1D SR operator, $\Psi =$ gradient. Worst reconstruction PSNR for different signal and noise sparsity K, S with no overlap ($N = 10, M = 2, l = 20$). (a) Example of an HR image, (b) example of a contaminated LR image, (c) SR without regularization (d) SR with TV regularization with optimal λ .

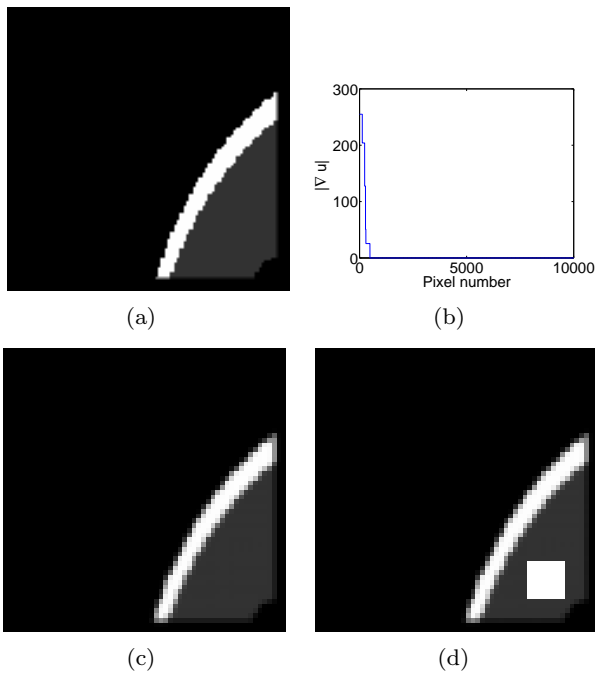


Fig. 10 2D SR problem. (a) HR image, (b) sorted values of the 2D discrete gradient (HR image gradient sparsity), (c) one clean LR image and (d) one contaminated LR image

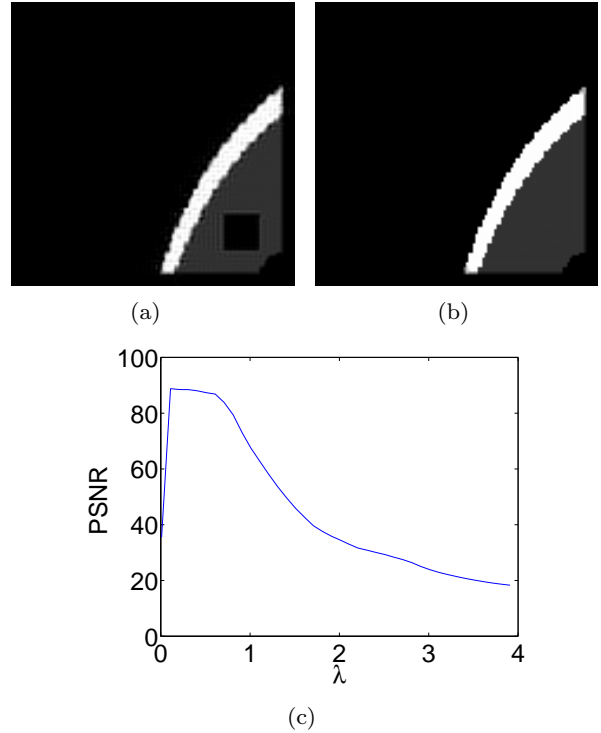


Fig. 11 2D SR results, experiment 1 (a) reconstruction without regularization, PSNR=25.49dB, (b) reconstruction with regularization PSNR=88.27dB, (c) PSNR with respect to λ .

perfect reconstruction. We also show how the PSNR behaves with respect to λ . A range of λ values lead to a perfect reconstruction (from 0.1 to 0.8), outside of the range the effect of oversmoothing (from 0.8 to 4) and failure (small λ) can be observed.

In Figure 12, we show another experiment with $N = 10$ and $M = 6$. Solving the unregularized problem is not forgiving the outliers and fails completely. TV regularization leads to a perfect reconstruction. In this experiment, only the critical number of images $M = 4$ (for the system to be invertible) is necessary to recover the HR image and there is less clean images than contaminated images. The behaviour of the reconstruction with respect to λ shows a similar behaviour as before. A range of λ (from 0.9 to 1.05) yields a perfect reconstruction.

Remark 11 In this article, we supposed that only sparse outlier noise was corrupting the signal. This hypothesis was made to keep the exposition of the main concepts clear. However, because the theoretical concepts are direct extensions of sparse recovery theory, stability with respect to additive noise (*e.g.* Gaussian noise) should be obtained by extending the corresponding results in the literature (Candès et al. 2006; Cohen et al. 2009).

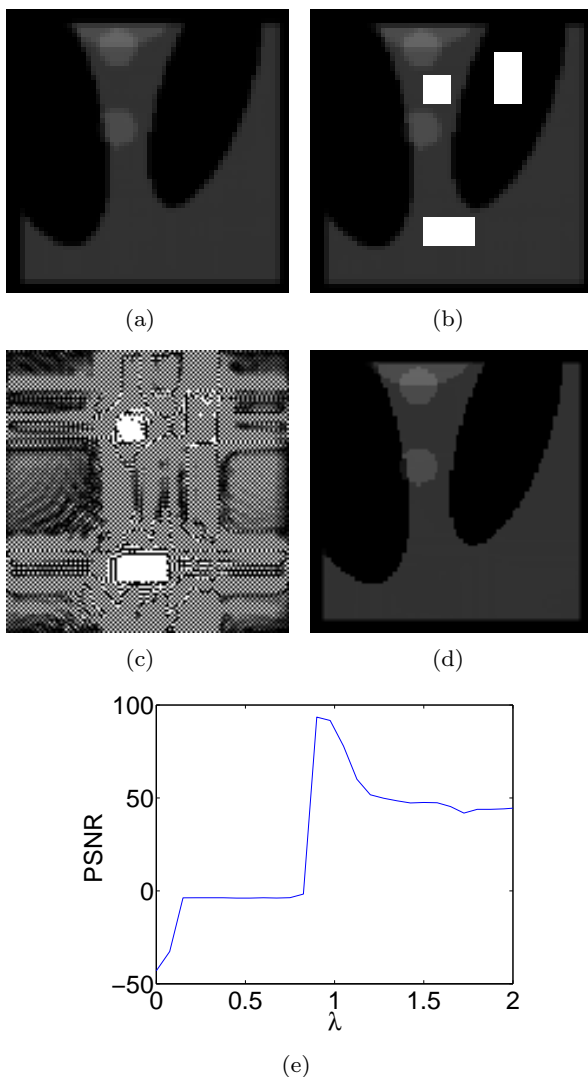


Fig. 12 2D SR results, experiment 2 (a) HR image, (b) one contaminated LR image, (c) reconstruction without regularization, PSNR=-46.84dB, (d) reconstruction with regularization PSNR=93.31dB, (e) PSNR with respect to λ .

5.2.3 Other regularization schemes, L_0 and outlier detection

As previously noticed, the precise *sinc* SR method does not benefit a lot from the L^1-TV method. Two possible ways could be explored.

First, a way to ensure the robustness of the problem using sparse regularization is to make sure the overdetermined matrix Ψ is \mathcal{L} -forgiving with good *NCP* constant. Popular dictionary based or non-local approaches for image modeling can lead to a Ψ being a projection of the signal into a high dimensional space where the signal is sparse. If this projection has the right *NCP* with respect to signal sparsity supports, outlier removal

power will be increased with any type of observation (universal encoding).

Secondly, we can make the following observation : very energetic and concentrated outliers are easy to detect. While L^1-TV minimization will not be able to remove them in critical cases (*e.g.* for 2D super-resolution $N = M^2 + 1$). Then detecting the outlier by thresholding, and removing contaminated equations will lead to an invertible system and the right solution. In this case this, process can be thought as an L^0 minimization, because the (informed) outlier suppression is a way to try to maximize the number of equations which are met by the data.

6 Conclusion

In this article, we showed how the theory of sparse recovery can be used and extended to the case of outlier robustness with sparsity priors on the signal for multi-image processing. We illustrated how some tools are more adapted to study matrices without too much structure (random matrices and the RIP) and others to the multi-image problems (Exact Recovery Property, Non Concentration Property). We showed that considering particular classes of supports allows for an understanding of how the L^1-TV scheme behaves with outliers. In simple cases, we could determine the regularization parameter leading to the best robustness. The intuition gained was verified experimentally in the more theoretically challenging 2D SR case.

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