



HAL
open science

Variational-based reduced-order model in dynamic substructuring of coupled structures through a dissipative physical interface: Recent advances

Roger Ohayon, Christian Soize, R. Sampaio

► To cite this version:

Roger Ohayon, Christian Soize, R. Sampaio. Variational-based reduced-order model in dynamic substructuring of coupled structures through a dissipative physical interface: Recent advances. Archives of Computational Methods in Engineering, 2014, 21 (3), pp.321-329. 10.1007/s11831-014-9107-y . hal-00940080

HAL Id: hal-00940080

<https://hal.science/hal-00940080>

Submitted on 31 Jan 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Variational-Based Reduced-Order Model in Dynamic Substructuring of Coupled Structures Through a Dissipative Physical Interface: Recent Advances

R. Ohayon · C. Soize · R. Sampaio

Accepted: January 2014

Abstract This paper deals with a variational-based reduced-order model in dynamic substructuring of two coupled structures through a physical dissipative flexible interface. We consider the linear elastodynamic of a dissipative structure composed of two main dissipative substructures perfectly connected through interfaces by a linking substructure. The linking substructure is flexible and is modeled in the context of the general linear viscoelasticity theory, yielding damping and stiffness operators depending on the frequency, while the two main dissipative substructures are modeled in the context of linear elasticity with an additional classical viscous damping modeling which is assumed to be independent of the frequency. We present recent advances adapted to such a situation, which is positioned with respect to an appropriate review that we carry out on the different methods used in dynamic substructuring. It consists in constructing a reduced-order model using the free-interface elastic modes of the two main substructures and, for the linking substructure, an appropriate frequency-independent elastostatic lifting operator and the frequency-dependent fixed-interface vector basis.

R. Ohayon
Structural Mechanics and Coupled Systems Laboratory.
Conservatoire National des Arts et Métiers (CNAM).
2 rue Conté, Paris, 75003, France
E-mail: roger.ohayon@cnam.fr

C. Soize
Laboratoire Modélisation et Simulation Multi Echelle, MSME UMR 8208 CNRS.
Université Paris-Est.
5 boulevard Descartes, Marne-la-Vallée, 77454, France
E-mail: christian.soize@univ-paris-est.fr

R. Sampaio
Mechanical Engineering Department.
PUC-Rio.
225 Rua Marquês de São Vicente, Rio de Janeiro, 22453-900, Brazil
E-mail: rsampaio@puc-rio.br

1 Introduction

The mathematical aspects related to the variational formulation, existence and uniqueness, finite element discretization of boundary value problems for elastodynamics can be found in [12, 23, 37, 61, 64, 76]. General mechanical formulations in computational structural dynamics, vibration and substructuring techniques can be found in [6, 10, 8, 22, 29, 52]. In structural dynamics and coupled problems such as fluid-structure interaction, general computational methods can also be found in [11, 19, 28, 32, 45, 57, 88] and algorithms for solving large eigenvalue problems in [18, 70, 75]. For computational structural dynamics in the low- and medium-frequency ranges and extensions to structural acoustics, we refer the reader to [62, 65] and for uncertainty quantification (UQ) in computational structural dynamics to [78].

The problem considered here is the construction of a reduced-order model for linear vibration of a dissipative structure subjected to prescribed forces and composed of two main linear dissipative substructures connected through a physical flexible viscoelastic interface (linking substructure).

Let us recall that the concept of substructures was first introduced by Argyris and Kelsey in 1959 [5] and by Przemieniecki in 1963 [71] and was extended by Guyan and Irons in 1965 [30, 41]. In 1960 and 1965, Hurty [38, 39] considered the case of two substructures coupled through a geometrical interface, for which the first substructure is represented using its elastic modes with fixed geometrical interface and the second substructure is represented using its elastic modes with free geometrical interface completed by static boundary functions of the first substructure. Finally, Craig and Bampton in 1968 [21] adapted the Hurty method in order to represent each substructure of the same manner consisting in using the elastic modes of the substructure with fixed geometrical interface and the static boundary functions on its geometrical interface. Improvements have been proposed with many variants [1, 7, 9, 15, 17, 27, 31, 35, 47, 53, 54, 83, 87], in particular for complex dynamical systems with many appendages considered as substructures (such as disk with blades) Benfield and Hruda in 1971 [13] proposed a component mode substitution using the Craig and Bampton method for each appendage. Another type of methods has been introduced in order to use the structural modes with free geometrical interface for two coupled substructures instead of the structural modes with fixed geometrical interface (elastic modes) as used in the Craig and Bampton method. In this context, MacNeal in 1971 [50] introduced the concept of residual flexibility which has then been used by Rubin in 1975 [74]. The Lagrange multipliers have also been used to write the coupling on the geometrical interface [51, 67, 68, 73]. Dynamic substructuring in the medium-frequency range has been analyzed [43, 33, 77, 81]. On the other hand, uncertainty quantification is nowadays recognized as playing an important role in order to improve the robustness of the models for the low-frequency range and especially, in the medium-frequency range in the context of substructuring techniques [16, 34, 36, 55, 63, 80, 79]. Reviews have also been performed [20, 24, 45]. It should be noted that all these above dynamic substructuring methodologies, which have been developed for the discrete case (computational model), have also been reanalyzed in the framework of the continuous case (continuum mechanics) by Morand and Ohayon in 1979 [56] for which details can be found in [57] for conservative systems and by Ohayon and Soize in

1998 [62] for the dissipative systems.

A physical flexible interface (the linking substructure) between two coupled substructures, modeled by an elastic medium, has been considered by Kuhar and Stahle [44] considering a static behavior of the junction. In this paper, a generalization of this work is presented in a more general framework for a viscoelastic dynamic behavior of the junction using existing component mode synthesis methods (see [66] for computational details). We consider a structure composed of two dissipative main substructures coupled through a physical flexible viscoelastic linking substructure by two geometrical interfaces. The boundary value problem is written in the frequency domain. The linking substructure is modeled in the context of the general linear viscoelasticity theory [14,84], yielding damping and stiffness sesquilinear forms depending on the frequency, while the two main dissipative substructures are modeled in the context of linear elasticity with an additional classical viscous damping modeling which is assumed to be independent of the frequency.

We present a variational-based reduced-order model in dynamic substructuring, adapted to computational dynamics, for linear elastodynamic of a dissipative structure composed of two main dissipative substructures coupled through a physical flexible viscoelastic interface constituting the linking substructure. A reduced-order model is constructed using the structural modes of the two main substructures with free geometrical interfaces and, for the linking substructure, using an adapted frequency-dependent vector basis with fixed geometrical interfaces and an appropriate static lifting operator with respect to the geometrical interfaces. It should be noted that the linking substructures models which are used, generally correspond to a rough modeling of the real linking systems and consequently, uncertainties induced by modeling errors can be introduced [55]. Another interest of using free structural modes of the two main substructures is also to allow a direct dynamical identification of the main substructures using experimental modal analysis [2, 3, 26, 42, 49, 58–60, 72, 86]. Such a reduced-order model is very useful for sensitivity analysis, design optimization and controller design for vibration active control [25, 40, 46, 69, 85].

2 Displacement Variational Formulation for Two Substructures Connected with a Linking Substructure

2.1 Description of the mechanical system and hypotheses

This paper deals with the linear vibration of a free structure, around a static equilibrium configuration which is taken as a natural state (for the sake of brevity, prestresses are not considered but could be added without changing the presentation), submitted to prescribed external forces which are assumed to be in equilibrium at each instant. The displacement field of the structure is then defined up to an additive rigid body displacement field. In this paper, we are only concerned in the part of the displacement field due to the structural deformation.

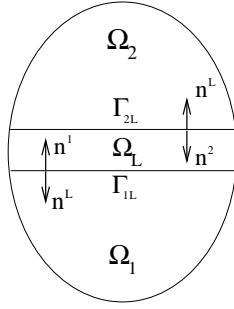


Fig. 1 Two substructures Ω_1 and Ω_2 connected with a linking substructure Ω_L

Let Ω_1 , Ω_2 and Ω_L be three open bounded domains in \mathbb{R}^3 with sufficiently smooth boundaries. The structure Ω is composed of two substructures Ω_1 and Ω_2 perfectly connected through interfaces Γ_{1L} and Γ_{2L} by a linking substructure Ω_L (see Fig. 1). The boundaries are such that $\partial\Omega_1 = \Gamma_{1L} \cup \Gamma_1$, $\partial\Omega_2 = \Gamma_{2L} \cup \Gamma_2$, $\partial\Omega_L = \Gamma_{1L} \cup \Gamma_L \cup \Gamma_{2L}$. We then have $\Omega = \Omega_1 \cup \Gamma_{1L} \cup \Omega_L \cup \Gamma_{2L} \cup \Omega_2$ and $\partial\Omega = \Gamma_1 \cup \Gamma_L \cup \Gamma_2$. The physical space is referred to a cartesian reference system and the generic point is denoted as $\mathbf{x} = (x_1, x_2, x_3)$. A frequency domain formulation is used, the convention for the Fourier transform being $v(\omega) = \int_{\mathbb{R}} e^{-i\omega t} v(t) dt$ where ω denotes the real circular frequency, $v(\omega)$ is in \mathbb{C} and where $\bar{v}(\omega)$ denotes its conjugate. The convention of summation over repeated indices is used and $v_{,j}$ denotes the partial derivative of v with respect to x_j . In $\mathbf{u} \cdot \mathbf{v}$, the dot denotes the usual Euclidean inner product on \mathbb{R}^3 extended to \mathbb{C}^3 .

For r in $\{1, L, 2\}$, the space of admissible displacement fields defined on Ω_r with values in \mathbb{C}^3 (resp. in \mathbb{R}^3) is denoted by \mathcal{C}_{Ω_r} (resp. \mathcal{R}_{Ω_r}). For substructure Ω_r , the test function (weighted function) associated with \mathbf{u}^r is denoted by $\delta\mathbf{u}^r \in \mathcal{C}_{\Omega_r}$ (or in \mathcal{R}_{Ω_r}). The space \mathcal{R}_{Ω_r} is the real Sobolev space $(H^1(\Omega_r))^3$ and the space \mathcal{C}_{Ω_r} is defined as the complexified Hilbert space of \mathcal{R}_{Ω_r} . Let $\mathcal{R}_{\Omega_r}^{rig}$ be the subspace (of dimension 6) of \mathcal{C}_{Ω_r} spanned by all the \mathbb{R}^3 -valued rigid body displacement fields which are written as $\mathbf{u}^r(\mathbf{x}) = \mathbf{t} + \boldsymbol{\theta} \times \mathbf{x}$ for all \mathbf{x} in the closure $\bar{\Omega}_r$ of Ω_r , in which \mathbf{t} and $\boldsymbol{\theta}$ are two arbitrary constant vectors in \mathbb{R}^3 . Let \mathcal{R}_{Ω} be the space of admissible displacement fields defined on Ω with values in \mathbb{R}^3 . Let $\mathcal{R}_{\Omega}^{rig}$ be the subspace (of dimension 6) of \mathcal{R}_{Ω} spanned by all the \mathbb{R}^3 -valued rigid body displacement fields which are written as $\mathbf{u}(\mathbf{x}) = \mathbf{t} + \boldsymbol{\theta} \times \mathbf{x}$ for all \mathbf{x} .

Each substructure is a three-dimensional dissipative elastic medium. The linking substructure Ω_L is modeled in the context of the general linear viscoelasticity theory while the two main dissipative substructures Ω_1 and Ω_2 are modeled in the context of linear elasticity with an additional classical viscous damping modeling which is assumed to be independent of the frequency. For r in $\{1, L, 2\}$, for all fixed ω and for each point \mathbf{x} , the displacement field is denoted by $\mathbf{u}^r(\mathbf{x}, \omega) = (u_1^r(\mathbf{x}, \omega), u_2^r(\mathbf{x}, \omega), u_3^r(\mathbf{x}, \omega))$, the external given body and surface force density fields applied to Ω_r and Γ_r are denoted by $\mathbf{g}^{\Omega_r}(\mathbf{x}, \omega) = (g_1^{\Omega_r}(\mathbf{x}, \omega), g_2^{\Omega_r}(\mathbf{x}, \omega), g_3^{\Omega_r}(\mathbf{x}, \omega))$

and $\mathbf{g}^{\Gamma_r}(\mathbf{x}, \omega) = (g_1^{\Gamma_r}(\mathbf{x}, \omega), g_2^{\Gamma_r}(\mathbf{x}, \omega), g_3^{\Gamma_r}(\mathbf{x}, \omega))$ respectively. Let be

$$\mathbf{g}^{\Omega}(\mathbf{x}, \omega) = \sum_{r=1,L,2} \mathbb{1}_{\Omega_r}(\mathbf{x}) \mathbf{g}^{\Omega_r}(\mathbf{x}, \omega), \quad (1)$$

$$\mathbf{g}^{\partial\Omega}(\mathbf{x}, \omega) = \sum_{r=1,L,2} \mathbb{1}_{\Gamma_r}(\mathbf{x}) \mathbf{g}^{\Gamma_r}(\mathbf{x}, \omega). \quad (2)$$

in which $\mathbb{1}_B$ is the indicator function of set B . Since we are only concerned in the part of the displacement field due to the structural deformation, it will be assumed that the given external forces are such that, for all \mathbf{u} in \mathcal{R}_{rig} ,

$$\int_{\Omega} \mathbf{g}^{\Omega}(\mathbf{x}, \omega) \cdot \mathbf{u}(\mathbf{x}) \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{g}^{\partial\Omega}(\mathbf{x}, \omega) \cdot \mathbf{u}(\mathbf{x}) \, ds(\mathbf{x}) = 0. \quad (3)$$

2.1.1 Constitutive equations

For r in $\{1, L, 2\}$, the linearized strain tensor is defined by

$$\varepsilon_{ij}(\mathbf{u}^r) = \frac{1}{2}(u_{i,j}^r + u_{j,i}^r). \quad (4)$$

(i) For r in $\{1, 2\}$, the constitutive equation for substructure Ω_r , which is assumed to be made up of an elastic material with linear viscous term, is written as

$$\sigma_{\text{tot}}^r = \sigma^r + i\omega s^r, \quad (5)$$

where σ^r is the elastic stress tensor defined by $\sigma_{ij}^r(\mathbf{u}^r) = a_{ijkh}^r(\mathbf{x}) \varepsilon_{kh}(\mathbf{u}^r)$ and where $i\omega s^r$ is the viscous part of the total stress tensor such that $s_{ij}^r(\mathbf{u}^r) = b_{ijkh}^r(\mathbf{x}) \varepsilon_{kh}(\mathbf{u}^r)$. The mechanical coefficients $a_{ijkh}^r(\mathbf{x})$ and $b_{ijkh}^r(\mathbf{x})$ depend on \mathbf{x} but are independent of frequency ω and verify the usual properties of symmetry, positiveness and boundedness (lower and upper).

(ii) For $r = L$, the constitutive equation for linking substructure Ω_L , which is assumed to be made up of a linear viscoelastic material, is written as

$$\sigma_{\text{tot}}^L = \sigma^L + i\omega s^L, \quad (6)$$

where σ^L is the elastic part of the total stress tensor defined by $\sigma_{ij}^L(\mathbf{u}^L) = a_{ijkh}^L(\mathbf{x}, \omega) \varepsilon_{kh}(\mathbf{u}^L)$ and where $i\omega s^L$ is the viscous part of the total stress tensor such that $s_{ij}^L(\mathbf{u}^L) = b_{ijkh}^L(\mathbf{x}, \omega) \varepsilon_{kh}(\mathbf{u}^L)$. The mechanical coefficients $a_{ijkh}^L(\mathbf{x}, \omega)$ and $b_{ijkh}^L(\mathbf{x}, \omega)$ depend on \mathbf{x} and on frequency ω and, for all fixed ω , verify the usual properties of symmetry, positiveness and boundedness (lower and upper). At zero frequency, $a_{ijkh}^L(\mathbf{x}, 0)$ is the equilibrium modulus tensor (which differs from the initial elasticity tensor). For more details concerning the properties of tensors $a_{ijkh}^L(\mathbf{x}, \omega)$ and $b_{ijkh}^L(\mathbf{x}, \omega)$, see [23, 62, 84].

2.2 The boundary value problem

2.2.1 Equilibrium equations in the frequency domain for each substructure

For all fixed ω and for r in $\{1, L, 2\}$, the equilibrium equation for substructure Ω_r is written as

$$-\omega^2 \rho^r u_i^r - \{\sigma_{\text{tot}}^r\}_{ij,j} = g_i^{\Omega_r} \quad \text{in } \Omega_r \quad \text{for } i = 1, 2, 3, \quad (7)$$

in which ρ^r is the mass density depending on \mathbf{x} (which is assumed to be strictly positive and bounded), with the boundary condition,

$$\{\sigma_{\text{tot}}^r\}_{ij} n_j^r = g_i^{\Gamma_r} \quad \text{on } \Gamma_r \quad \text{for } i = 1, 2, 3, \quad (8)$$

in which the vector $\mathbf{n}^r = (n_1^r, n_2^r, n_3^r)$ is the unit normal to $\partial\Omega_r$, external to Ω_r .

2.2.2 Coupling conditions

The coupling conditions of the linking substructure Ω_L with substructures Ω_1 and Ω_2 on Γ_{1L} and Γ_{2L} are written as

$$\begin{aligned} \mathbf{u}^1 &= \mathbf{u}^L \quad \text{on } \Gamma_{1L}, \\ \mathbf{u}^2 &= \mathbf{u}^L \quad \text{on } \Gamma_{2L}. \end{aligned} \quad (9)$$

$$\begin{aligned} \sigma_{\text{tot}}^1 \mathbf{n}^1 &= -\sigma_{\text{tot}}^L \mathbf{n}^L \quad \text{on } \Gamma_{1L}, \\ \sigma_{\text{tot}}^2 \mathbf{n}^2 &= -\sigma_{\text{tot}}^L \mathbf{n}^L \quad \text{on } \Gamma_{2L}. \end{aligned} \quad (10)$$

2.2.3 Boundary value problem in displacement

For all fixed ω , the boundary value problem consists in finding the displacement fields \mathbf{u}^1 , \mathbf{u}^L and \mathbf{u}^2 , verifying, for each r equal to 1, L or 2, the equilibrium equation Eq. (7) with the constitutive equation defined by Eq. (5) (for $r = 1, 2$) or by Eq. (6) (for $r = L$), with the Neumann boundary condition defined by Eq. (8) and with the coupling conditions defined by Eqs. (9) and (10).

2.3 Variational formulation of the boundary value problem

2.3.1 Definition of the sesquilinear forms of the problem

For r in $\{1, L, 2\}$, a sesquilinear form on $\mathcal{C}_{\Omega_r} \times \mathcal{C}_{\Omega_r}$ is defined by

$$m^r(\mathbf{u}^r, \delta \mathbf{u}^r) = \int_{\Omega_r} \rho^r \mathbf{u}^r \cdot \overline{\delta \mathbf{u}^r} \, d\mathbf{x}. \quad (11)$$

The sesquilinear form m^r is continuous positive definite Hermitian on $\mathcal{C}_{\Omega_r} \times \mathcal{C}_{\Omega_r}$.

(i) For r in $\{1, 2\}$, two sesquilinear forms on $\mathcal{C}_{\Omega_r} \times \mathcal{C}_{\Omega_r}$, independent of frequency ω are defined by

$$k^r(\mathbf{u}^r, \delta \mathbf{u}^r) = \int_{\Omega_r} a_{ij, kh}^r(\mathbf{x}) \varepsilon_{kh}(\mathbf{u}^r) \varepsilon_{ij}(\overline{\delta \mathbf{u}^r}) \, d\mathbf{x}, \quad (12)$$

$$d^r(\mathbf{u}^r, \delta \mathbf{u}^r) = \int_{\Omega_r} b_{ijkh}^r(\mathbf{x}) \varepsilon_{kh}(\mathbf{u}^r) \varepsilon_{ij}(\overline{\delta \mathbf{u}^r}) d\mathbf{x}. \quad (13)$$

(ii) For $r = L$, two sesquilinear forms on $\mathcal{C}_{\Omega_r} \times \mathcal{C}_{\Omega_r}$, depending on frequency ω , are defined by

$$k^L(\mathbf{u}^L, \delta \mathbf{u}^L; \omega) = \int_{\Omega_L} a_{ijkh}^L(\mathbf{x}, \omega) \varepsilon_{kh}(\mathbf{u}^L) \varepsilon_{ij}(\overline{\delta \mathbf{u}^L}) d\mathbf{x}, \quad (14)$$

$$d^L(\mathbf{u}^L, \delta \mathbf{u}^L; \omega) = \int_{\Omega_L} b_{ijkh}^L(\mathbf{x}, \omega) \varepsilon_{kh}(\mathbf{u}^L) \varepsilon_{ij}(\overline{\delta \mathbf{u}^L}) d\mathbf{x}. \quad (15)$$

Taking into account the usual assumptions related to the constitutive equations, for r in $\{1, L, 2\}$, the sesquilinear forms k^r and d^r are continuous semi-definite positive Hermitian on $\mathcal{C}_{\Omega_r} \times \mathcal{C}_{\Omega_r}$, the semi-definite positiveness being due to the presence of rigid body displacement fields.

(i) For r in $\{1, 2\}$, for all $\delta \mathbf{u}^r$ in \mathcal{C}_{Ω_r} , $k^r(\mathbf{u}^r, \delta \mathbf{u}^r)$ and $d^r(\mathbf{u}^r, \delta \mathbf{u}^r)$ are equal to zero for any \mathbf{u}^r in \mathcal{R}_{rig}^r .

(ii) For $r = L$, for all fixed frequency ω and for all $\delta \mathbf{u}^L$ in \mathcal{C}_{Ω_L} , $k^L(\mathbf{u}^L, \delta \mathbf{u}^L; \omega)$ and $d^L(\mathbf{u}^L, \delta \mathbf{u}^L; \omega)$ are equal to zero for any \mathbf{u}^L in \mathcal{R}_{rig}^L .

2.3.2 Definition of the antilinear forms of the problem

For r in $\{1, L, 2\}$ and for all fixed frequency ω , it is assumed that \mathbf{g}^{Ω_r} and \mathbf{g}^{Γ_r} are such that the antilinear form $\delta \mathbf{u}^r \mapsto f^r(\delta \mathbf{u}^r; \omega)$ on \mathcal{C}_{Ω_r} , defined by

$$\begin{aligned} f^r(\delta \mathbf{u}^r; \omega) &= \int_{\Omega_r} \mathbf{g}^{\Omega_r}(\mathbf{x}, \omega) \cdot \overline{\delta \mathbf{u}^r(\mathbf{x})} d\mathbf{x} \\ &+ \int_{\Gamma_r} \mathbf{g}^{\Gamma_r}(\mathbf{x}, \omega) \cdot \overline{\delta \mathbf{u}^r(\mathbf{x})} ds(\mathbf{x}), \end{aligned} \quad (16)$$

is continuous.

2.3.3 Definition of the complex bilinear forms of the problem

(i) For r in $\{1, 2\}$ and for all fixed ω , the complex bilinear form z^r on $\mathcal{C}_{\Omega_r} \times \mathcal{C}_{\Omega_r}$ is defined by

$$\begin{aligned} z^r(\mathbf{u}^r, \delta \mathbf{u}^r; \omega) &= -\omega^2 m^r(\mathbf{u}^r, \delta \mathbf{u}^r) \\ &+ i\omega d^r(\mathbf{u}^r, \delta \mathbf{u}^r) + k^r(\mathbf{u}^r, \delta \mathbf{u}^r). \end{aligned} \quad (17)$$

(ii) For $r = L$ and for all fixed frequency ω , the complex bilinear form z^L on $\mathcal{C}_{\Omega_L} \times \mathcal{C}_{\Omega_L}$ is defined by

$$\begin{aligned} z^L(\mathbf{u}^L, \delta \mathbf{u}^L; \omega) &= -\omega^2 m^L(\mathbf{u}^L, \delta \mathbf{u}^L) \\ &+ i\omega d^r(\mathbf{u}^L, \delta \mathbf{u}^L; \omega) + k^L(\mathbf{u}^L, \delta \mathbf{u}^L; \omega). \end{aligned} \quad (18)$$

2.3.4 Variational formulation of the boundary value problem

The variational formulation of the boundary value problem in \mathbf{u}^1 , \mathbf{u}^L and \mathbf{u}^2 is defined as follows. For all fixed real ω , find $(\mathbf{u}^1(\omega), \mathbf{u}^L(\omega), \mathbf{u}^2(\omega))$ in $\mathcal{C}_{\Omega_1} \times \mathcal{C}_{\Omega_L} \times \mathcal{C}_{\Omega_2}$ verifying the linear constraints $\mathbf{u}^1(\omega) = \mathbf{u}^L(\omega)$ on Γ_{1L} and $\mathbf{u}^2(\omega) = \mathbf{u}^L(\omega)$ on Γ_{2L} , such that, for all $(\delta\mathbf{u}^1, \delta\mathbf{u}^L, \delta\mathbf{u}^2)$ in $\mathcal{C}_{\Omega_1} \times \mathcal{C}_{\Omega_L} \times \mathcal{C}_{\Omega_2}$ verifying the linear constraints $\delta\mathbf{u}^1 = \delta\mathbf{u}^L$ on Γ_{1L} and $\delta\mathbf{u}^2 = \delta\mathbf{u}^L$ on Γ_{2L} , we have

$$\begin{aligned} z^1(\mathbf{u}^1(\omega), \delta\mathbf{u}^1; \omega) + z^L(\mathbf{u}^L(\omega), \delta\mathbf{u}^L; \omega) \\ + z^2(\mathbf{u}^2(\omega), \delta\mathbf{u}^2; \omega) = f^1(\delta\mathbf{u}^1; \omega) \\ + f^L(\delta\mathbf{u}^L; \omega) + f^2(\delta\mathbf{u}^2; \omega). \end{aligned} \quad (19)$$

Under the given hypotheses relative to the constitutive equations and the hypothesis on the external given forces defined by Eq. (3), the existence and uniqueness of a solution can be proven for all real ω . This solution corresponds only to deformation of the structure (i.e. without rigid body displacements).

3 Reduced-Order Model

The method is based on the use of the variational formulation defined by Eq. (19). The reduced-order model is carried out using (i) for $r = 1, 2$, the eigenmodes of substructure Ω_r with free interface Γ_{rL} , (ii) a frequency-dependent family of eigenfunctions for linking substructure Ω_L with fixed interface $\Gamma_{1L} \cup \Gamma_{2L}$ and (iii) the elastostatic lifting operator of Ω_L with respect to interface $\Gamma_{1L} \cup \Gamma_{2L}$ at zero frequency.

3.1 Eigenmodes of substructure Ω_r with free interface Γ_{rL} for $r = 1, 2$

For $r = 1, 2$, a free-interface mode of substructure Ω_r is defined as an eigenmode of the conservative problem associated with free substructure Ω_r , subject to zero forces on $\partial\Omega_r$. The real eigenvalues $\lambda^r \geq 0$ and the corresponding eigenmodes \mathbf{u}^r in \mathcal{R}_{Ω_r} are then the solutions of the following spectral problem: find $\lambda^r \geq 0$, $\mathbf{u}^r \in \mathcal{R}_{\Omega_r}(\mathbf{u}^r \neq \mathbf{0})$ such that for all $\delta\mathbf{u}^r \in \mathcal{R}_{\Omega_r}$, one has

$$k^r(\mathbf{u}^r, \delta\mathbf{u}^r) = \lambda^r m^r(\mathbf{u}^r, \delta\mathbf{u}^r). \quad (20)$$

It can be shown that there exist six zero eigenvalues $0 = \lambda_{-5}^r = \dots = \lambda_0^r$ (associated with the rigid body displacement fields) and that the strictly positive eigenvalues (associated with the displacement field due to structural deformation) constitute the increasing sequence $0 < \lambda_1^r \leq \lambda_2^r, \dots$. The six eigenvectors $\{\mathbf{u}_{-5}^r, \dots, \mathbf{u}_0^r\}$ associated with zero eigenvalues span \mathcal{R}_{rig} (space of the rigid body displacement fields). The family $\{\mathbf{u}_{-5}^r, \dots, \mathbf{u}_0^r; \mathbf{u}_1^r, \dots\}$ of all the eigenmodes forms a complete family in \mathcal{R}_{Ω_r} . For α and β in $\{-5, \dots, 0; 1, \dots\}$, we have the orthogonality conditions

$$m^r(\mathbf{u}_\alpha^r, \mathbf{u}_\beta^r) = \delta_{\alpha\beta}, \quad (21)$$

$$k^r(\mathbf{u}_\alpha^r, \mathbf{u}_\beta^r) = \delta_{\alpha\beta} \lambda_\alpha^r, \quad (22)$$

in which each eigenmode \mathbf{u}_α^r is normalized to 1 with respect to m^r and where $\omega_\alpha^r = \sqrt{\lambda_\alpha^r}$ is the eigenfrequency of mode α .

3.2 Frequency-dependent family of eigenfunctions for linking substructure Ω^L with fixed interface $\Gamma_{1L} \cup \Gamma_{2L}$

Let us introduce the following admissible space,

$$\mathcal{R}_{\Omega_L}^0 = \left\{ \mathbf{u}^L \in \mathcal{R}_{\Omega_L} \mid \mathbf{u}^L = 0 \quad \text{on} \quad \Gamma_{1L} \cup \Gamma_{2L} \right\}. \quad (23)$$

At given frequency ω , a fixed-interface vector basis of linking substructure Ω_L is defined as an eigenfunction of the conservative problem associated with Ω_L with fixed interface $\Gamma_{1L} \cup \Gamma_{2L}$, subject to zero forces on $\partial\Omega_L$. A real eigenvalue $\lambda^L(\omega) > 0$ and the corresponding eigenfunction $\mathbf{u}^L(\omega)$ in \mathcal{R}_{Ω_L} are then the solution of the following spectral problem: find $\lambda^L(\omega) > 0$ and nonzero $\mathbf{u}^L(\omega)$ in $\mathcal{R}_{\Omega_L}^0$ such that for all $\delta\mathbf{u}^L \in \mathcal{R}_{\Omega_L}^0$, we have

$$k^L(\mathbf{u}^L(\omega), \delta\mathbf{u}^L; \omega) = \lambda^L(\omega) m^L(\mathbf{u}^L(\omega), \delta\mathbf{u}^L). \quad (24)$$

It can be shown that there exist a strictly positive increasing sequence of eigenvalues, $0 < \lambda_1^L(\omega) \leq \lambda_2^L(\omega), \dots$, and a corresponding family $\{\mathbf{u}_1^L(\omega), \mathbf{u}_2^L(\omega), \dots\}$ of eigenfunctions which constitutes a complete family in $\mathcal{R}_{\Omega_L}^0$. For α and β in $\{1, 2, \dots\}$, we have the orthogonality conditions

$$m^L(\mathbf{u}_\alpha^L(\omega), \mathbf{u}_\beta^L(\omega)) = \delta_{\alpha\beta}, \quad (25)$$

$$k^L(\mathbf{u}_\alpha^L(\omega), \mathbf{u}_\beta^L(\omega); \omega) = \delta_{\alpha\beta} \lambda_\alpha^L(\omega), \quad (26)$$

in which each eigenfunction $\mathbf{u}_\alpha^L(\omega)$ is normalized to 1 with respect to m^L .

Remarks. The reduced-order model will be constructed in using a finite family $\mathcal{U}^L(\omega) = \{\mathbf{u}_1^L(\omega), \dots, \mathbf{u}_{N_L}^L(\omega)\}$ of the sequence $\{\mathbf{u}_1^L(\omega), \mathbf{u}_2^L(\omega), \dots\}$ of the eigenfunctions associated with the first N_L eigenvalues $0 < \lambda_1^L(\omega) \leq \dots \leq \lambda_{N_L}^L(\omega)$. In addition, such a reduced-order model will be constructed for analyzing the response of the structure for frequency ω belonging to a given frequency band of analysis, $B = [\omega_{\min}, \omega_{\max}]$, with $0 \leq \omega_{\min} < \omega_{\max}$. In practice, the response is calculated for the frequencies belonging to the finite subset $\mathcal{B} = \{\omega_1, \omega_2, \dots, \omega_\mu\}$ of μ sampling frequencies of band B , for which μ can be of several hundreds or of the order of one thousand.

(i) In the above formulation, the generalized eigenvalue problem defined by Eq. (24) must be solved for all ω in \mathcal{B} . If the number μ of frequencies is not too high (that is generally the case for the linking substructure), such a computation remains feasible. It should be noted that the use of massively parallel computers facilitates the analysis of such numerical problem depending on continuous parameter ω .

(ii) For solving the generalized eigenvalue problem defined by Eq. (24) for all ω in \mathcal{B} , the numerical cost can be reduced using the following procedure. First, Eq. (24) is solved for ω belonging to the subset $\mathcal{B}' = \{\omega'_1, \dots, \omega'_{\mu'}\}$ of chosen frequency points in \mathcal{B} with $\mu' \ll \mu$, called master frequency points. Then, an approximation $\tilde{\mathcal{U}}^L(\omega)$ of $\mathcal{U}^L(\omega)$ is calculated for all ω in $\mathcal{B} \setminus \mathcal{B}'$ by using an interpolation procedure based on the values $\mathcal{U}^L(\omega'_1), \dots, \mathcal{U}^L(\omega'_{\mu'})$ at the μ' frequency points. Finally, for each ω in $\mathcal{B} \setminus \mathcal{B}'$, the generalized eigenvalue problem defined by Eq. (24) is projected on the subspace spanned by the finite family $\tilde{\mathcal{U}}^L(\omega)$ which yields a

reduced-order generalized eigenvalue problem of dimension N_L and which allows an approximation $\tilde{\lambda}_1^L(\omega), \dots, \tilde{\lambda}_{N_L}^L(\omega)$ of $\lambda_1^L(\omega), \dots, \lambda_{N_L}^L(\omega)$ to be calculated. Such a procedure can be found in [4].

(iii) Another way consists in replacing the above interpolation procedure by the following construction of a frequency-independent basis adapted to band B. It consists in extracting the larger family of linearly independent functions from the family $\mathcal{U}^L(\omega'_1), \dots, \mathcal{U}^L(\omega'_{\mu'})$ at the μ' master frequency points.

(iv) If $k^L(\cdot, \cdot; \omega)$ slowly varies for ω in band B, a well adapted frequency-independent basis for all ω in B, consists in choosing $\mathcal{U}^L(\omega)$ for an arbitrary ω in B but in such a case convergence with respect to N_L must be carefully checked.

(v) More generally, any subset of N_L functions extracted from a Hilbertian basis of the admissible space $\mathcal{R}_{\Omega_L}^0$ can be used.

3.3 Elastostatic lifting operator of Ω_L with respect to interface $\Gamma_{1L} \cup \Gamma_{2L}$ at zero frequency

We consider the solution \mathbf{u}_S^L of the elastostatic problem at zero frequency for linking substructure Ω_L subjected to prescribed displacement fields $\mathbf{u}_{\Gamma_{1L}}$ on Γ_{1L} and $\mathbf{u}_{\Gamma_{2L}}$ on Γ_{2L} , and zero force on Γ_L . We introduce the following sets of functions,

$$\mathcal{R}_{\Gamma_{1L}, \Gamma_{2L}} = (H^{1/2}(\Gamma_{1L}))^3 \times (H^{1/2}(\Gamma_{2L}))^3, \quad (27)$$

$$\begin{aligned} \mathcal{R}_{\Omega_L}^{\mathbf{u}_{\Gamma_{1L}}, \mathbf{u}_{\Gamma_{2L}}} &= \{ \mathbf{u}^L \in \mathcal{R}_{\Omega_L} \mid \mathbf{u}^L = \mathbf{u}_{\Gamma_{1L}} \text{ on } \Gamma_{1L}; \\ &\mathbf{u}^L = \mathbf{u}_{\Gamma_{2L}} \text{ on } \Gamma_{2L} \text{ with } (\mathbf{u}_{\Gamma_{1L}}, \mathbf{u}_{\Gamma_{2L}}) \in \mathcal{R}_{\Gamma_{1L}, \Gamma_{2L}} \}. \end{aligned} \quad (28)$$

Displacement field \mathbf{u}_S^L satisfies the following variational formulation,

$$\begin{aligned} k^L(\mathbf{u}_S^L, \delta \mathbf{u}_S^L; 0) &= 0 \quad , \quad \mathbf{u}_S^L \in \mathcal{R}_{\Omega_L}^{\mathbf{u}_{\Gamma_{1L}}, \mathbf{u}_{\Gamma_{2L}}}, \\ &\forall \delta \mathbf{u}_S^L \in \mathcal{R}_{\Omega_L}^0, \end{aligned} \quad (29)$$

corresponding to the following boundary value problem,

$$\{a_{ijkh}^L(\mathbf{x}, 0) \varepsilon_{kh}(\mathbf{u}^L)\}_{,j} = 0 \quad \text{in } \Omega_L \quad \text{for } i = 1, 2, 3, \quad (30)$$

with the Neumann and Dirichlet boundary conditions,

$$a_{ijkh}^L(\mathbf{x}, 0) \varepsilon_{kh}(\mathbf{u}^L) n_j^L = 0 \quad \text{on } \Gamma_L \quad \text{for } i = 1, 2, 3, \quad (31)$$

$$\begin{aligned} \mathbf{u}^L &= \mathbf{u}_{\Gamma_{1L}} \text{ on } \Gamma_{1L}; \\ \mathbf{u}^L &= \mathbf{u}_{\Gamma_{2L}} \text{ on } \Gamma_{2L}. \end{aligned} \quad (32)$$

The problem defined by Eq. (29) has a unique solution \mathbf{u}_S^L which defines the linear continuous operator S^L from $\mathcal{R}_{\Gamma_{1L}, \Gamma_{2L}}$ into $\mathcal{R}_{\Omega_L}^{\mathbf{u}_{\Gamma_{1L}}, \mathbf{u}_{\Gamma_{2L}}}$ (called the elastostatic lifting operator at zero frequency),

$$(\mathbf{u}_{\Gamma_{1L}}, \mathbf{u}_{\Gamma_{2L}}) \mapsto \mathbf{u}_S^L = S^L(\mathbf{u}_{\Gamma_{1L}}, \mathbf{u}_{\Gamma_{2L}}). \quad (33)$$

Let $\mathcal{R}_{\Omega_L}^{\text{stat}}$ be the subspace of $\mathcal{R}_{\Omega_L}^{\mathbf{u}_{\Gamma_{1L}}, \mathbf{u}_{\Gamma_{2L}}}$ constituted of all the solutions of Eq. (29) (*i.e.* the range of operator S^L). It can then be proven that $\mathcal{R}_{\Omega_L} = \mathcal{R}_{\Omega_L}^{\text{stat}} \oplus \mathcal{R}_{\Omega_L}^0$. Finally, introducing the complexified spaces $\mathcal{C}_{\Omega_L}^{\text{stat}}$ and $\mathcal{C}_{\Omega_L}^0$ of $\mathcal{R}_{\Omega_L}^{\text{stat}}$ and $\mathcal{R}_{\Omega_L}^0$, it can be proven that

$$\mathcal{C}_{\Omega_L} = \mathcal{C}_{\Omega_L}^{\text{stat}} \oplus \mathcal{C}_{\Omega_L}^0. \quad (34)$$

3.4 Construction of a reduced-order model

The following reduced-order model can then be constructed using the elastostatic lifting operator and performing a Ritz-Galerkin projection with the free-interface modes of substructures Ω_1 and Ω_2 , and the fixed interface modes of linking substructure Ω_L . Let N_1 , N_L and N_2 be finite integers. For all fixed ω , the following finite projections $\mathbf{u}^{1,N_1}(\omega)$, $\mathbf{u}^{L,N_L}(\omega)$ and $\mathbf{u}^{2,N_2}(\omega)$ of $\mathbf{u}^1(\omega)$, $\mathbf{u}^L(\omega)$ and $\mathbf{u}^2(\omega)$ are introduced as follows,

$$\mathbf{u}^{1,N_1}(\omega) = \sum_{\alpha=-5}^{N_1} q_{\alpha}^1(\omega) \mathbf{u}_{\alpha}^1, \quad (35)$$

$$\mathbf{u}^{L,N_L}(\omega) = S^L(\mathbf{u}_{\Gamma_{1L}}^{N_1}(\omega), \mathbf{u}_{\Gamma_{2L}}^{N_2}(\omega)) + \sum_{\alpha=1}^{N_L} q_{\alpha}^L(\omega) \mathbf{u}_{\alpha}^L(\omega), \quad (36)$$

$$\mathbf{u}^{2,N_2}(\omega) = \sum_{\alpha=-5}^{N_2} q_{\alpha}^2(\omega) \mathbf{u}_{\alpha}^2, \quad (37)$$

in which $\mathbf{u}_{\Gamma_{1L}}^{N_1}(\omega)$ and $\mathbf{u}_{\Gamma_{2L}}^{N_2}(\omega)$ are such that

$$\mathbf{u}_{\Gamma_{1L}}^{N_1}(\omega) = \sum_{\alpha=-5}^{N_1} q_{\alpha}^1(\omega) \mathbf{u}_{\alpha|_{\Gamma_{1L}}}^1, \quad (38)$$

$$\mathbf{u}_{\Gamma_{2L}}^{N_2}(\omega) = \sum_{\alpha=-5}^{N_2} q_{\alpha}^2(\omega) \mathbf{u}_{\alpha|_{\Gamma_{2L}}}^2. \quad (39)$$

Note that Eq. (36) is due to the property defined by Eq. (34). The corresponding test functions are then written as,

$$\delta \mathbf{u}^{1,N_1} = \sum_{\alpha=-5}^{N_1} \delta q_{\alpha}^1 \mathbf{u}_{\alpha}^1, \quad (40)$$

$$\delta \mathbf{u}^{L,N_L}(\omega) = S^L(\delta \mathbf{u}_{\Gamma_{1L}}^{N_1}, \delta \mathbf{u}_{\Gamma_{2L}}^{N_2}) + \sum_{\alpha=1}^{N_L} \delta q_{\alpha}^L \mathbf{u}_{\alpha}^L(\omega), \quad (41)$$

$$\delta \mathbf{u}^{2,N_2} = \sum_{\alpha=-5}^{N_2} \delta q_{\alpha}^2 \mathbf{u}_{\alpha}^2, \quad (42)$$

in which $\delta \mathbf{u}_{\Gamma_{1L}}^{N_1}$ and $\delta \mathbf{u}_{\Gamma_{2L}}^{N_2}$ are such that

$$\delta \mathbf{u}_{\Gamma_{1L}}^{N_1} = \sum_{\alpha=-5}^{N_1} \delta q_{\alpha}^1 \mathbf{u}_{\alpha|_{\Gamma_{1L}}}^1, \quad (43)$$

$$\delta \mathbf{u}_{\Gamma_{2L}}^{N_2} = \sum_{\alpha=-5}^{N_2} \delta q_{\alpha}^2 \mathbf{u}_{\alpha|_{\Gamma_{2L}}}^2. \quad (44)$$

Substituting Eqs. (35) to (44) into Eq. (19) yields the following variational reduced-order model of order $N = N_1 + 6 + N_L + N_2 + 6$. For all fixed Ω , find $(\mathbf{q}^1(\omega), \mathbf{q}^L(\omega))$,

$\mathbf{q}^2(\omega)$ in $\mathbb{C}^{N_1+6} \times \mathbb{C}^{N_L} \times \mathbb{C}^{N_2+6}$ such that, for all $(\delta\mathbf{q}^1, \delta\mathbf{q}^L, \delta\mathbf{q}^2)$ in $\mathbb{C}^{N_1+6} \times \mathbb{C}^{N_L} \times \mathbb{C}^{N_2+6}$, we have

$$\begin{aligned} z_N^{\text{red}}(\mathbf{q}^1(\omega), \mathbf{q}^L(\omega), \mathbf{q}^2(\omega), \delta\mathbf{q}^1, \delta\mathbf{q}^L, \delta\mathbf{q}^2; \omega) \\ = f^N(\delta\mathbf{q}^1, \delta\mathbf{q}^L, \delta\mathbf{q}^2; \omega), \end{aligned} \quad (45)$$

in which $\mathbf{q}^1 = (q_{-5}^1, \dots, q_0^1, q_1^1 \dots q_{N_1}^1)$, $\mathbf{q}^L = (q_1^L, \dots, q_{N_L}^L)$, $\mathbf{q}^2 = (q_{-5}^2, \dots, q_0^2, q_1^2 \dots q_{N_2}^2)$, $\delta\mathbf{q}^1 = (\delta q_{-5}^1, \dots, \delta q_0^1, \delta q_1^1 \dots \delta q_{N_1}^1)$, $\delta\mathbf{q}^L = (\delta q_1^L, \dots, \delta q_{N_L}^L)$ and $\delta\mathbf{q}^2 = (\delta q_{-5}^2, \dots, \delta q_0^2, \delta q_1^2 \dots \delta q_{N_2}^2)$.

4 Concluding remarks and research perspectives

A continuum-based substructuring techniques has been presented for the linear dynamic analysis of two substructures connected with a physical flexible viscoelastic interface, for which each substructure is reduced using its free-interface elastic modes. Concerning the research perspectives, for such a dynamical system, the physical flexible viscoelastic interface generally presents model uncertainties induced by imperfect coupling boundary conditions with the substructures, the viscoelastic model used for of the material and the geometrical parameters. The complexity of such physical interface model can require advanced dynamic multiscale methods in micro-macro mechanics for materials [48] and requires to take into account model uncertainties induced by modeling errors. Such implementation using the nonparametric probabilistic approach of model uncertainties, coupling recent advanced research concerning uncertainty quantification for viscoelastic structures [65,82] and uncertain coupling interface methodologies [55], is in progress. In addition, the introduction of smart materials in the physical interface would be of prime interest for micromechanical systems.

Acknowledgment

This research was partially supported by Brazil-France project CAPES-COFECUB Ph672/10.

References

1. Agrawal BN (1976) Mode synthesis technique for dynamic analysis of structures. *Journal of the Acoustical Society of America* 59:1329-1338
2. Allen MS. Mayes RL. Bergman EJ (2010) Experimental modal substructuring to couple and uncouple substructures with flexible fixtures and multi-point connections. *Journal of Sound and Vibration* 329:4891-4906
3. Allen MS. Gindlin HM. Mayes RL (2011) Experimental modal substructuring to estimate fixed-base modes from tests on a flexible fixture. *Journal of Sound and Vibration* 330:4413-4428
4. Amsallem D. Farhat C (2011) An online method for interpolating linear parametric reduced-order models. *SIAM Journal on Scientific Computing* 33:2169-2198
5. Argyris JH. Kelsey S (1959) The analysis of fuselages of arbitrary cross-section and taper: A DSIR sponsored reserach program on the development and application of the matrix force method and the digital computer. *Aircraft Engineering and Aerospace technology* 31:272-83
6. Argyris J. Mlejnek HP (1991) *Dynamics of structures*. North-Holland. Amsterdam

7. Balmes E (1996) Optimal Ritz vectors for component mode synthesis using the singular value decomposition. *AIAA Journal* 34:1256-1260
8. Bathe KJ (1996) *Finite element procedures*. Prentice-Hall. New York
9. Bathe KJ. and Gracewski S (1981) On non-linear dynamic analysis using substructuring and mode superposition. *Computers and Structures* 13:699-707
10. Bathe KJ. Wilson EL (1976) *Numerical methods in finite element analysis*. Prentice-Hall. New York
11. Bazilevs Y. Takizawa K. Tezduyar TE (2013) *Computational fluid-structure interaction*. Wiley. Chichester
12. Belytschko TB. Liu WK. Moran B (2000) *Nonlinear finite element for continua and structures*. Wiley. Chichester
13. Benfield WA. Hruda RF (1971) Vibration analysis of structures by component mode substitution. *AIAA Journal* 9:1255-1261
14. Bland DR (1960) *The theory of linear viscoelasticity*. Pergamon. London
15. Bourquin F. d'Hennezel F (1992) Numerical study of an intrinsic component mode synthesis method. *Computer Methods in Applied Mechanics and Engineering* 97:49-76
16. Brown AM. Ferri AA (1996) Probabilistic component mode synthesis of nondeterministic substructures. *AIAA Journal* 34:830-834
17. Castanier MP. Tan YC. Pierre C (2001) Characteristic constraint modes for component mode synthesis. *AIAA Journal* 39:1182-1187
18. Chatelin F (2012) *Eigenvalues of matrices - Revised edition*. Society for Industrial and Applied Mathematics (SIAM). Philadelphia
19. Clough RW. Penzien J (1975) *Dynamics of structures*. McGraw-Hill. New York
20. Craig RR (1985) A review of time domain and frequency domain component mode synthesis method in Combined experimental-analytical modeling of dynamic structural systems. Edited by D.R. Martinez and A.K. Miller 67 ASME-AMD. New York
21. Craig RR. Bampton MCC (1968) Coupling of substructures for dynamic analyses. *AIAA Journal* 6:1313-1322
22. Craig RR. Kurdila A (2006) *Fundamentals of structural dynamics*. Wiley. Chichester
23. Dautray R. Lions JL (1992) *Mathematical analysis and numerical methods for science and technology*. Springer-Verlag. Berlin
24. de Klerk D. Rixen DJ. Voormeeren SN (2008) General framework for dynamic substructuring: History, review, and classification of techniques. *AIAA Journal* 46:1169-1181
25. El-Khoury O. Adeli H (2013) Recent advances on vibration control of structures and their dynamic loading. *Archive for Computational Methods in Engineering* 20:353-360
26. Ewins DJ (2000) *Modal testing: theory, practice and applications*. Second edition. Research Studies Press Ltd. Baldock.
27. Farhat C. Geradin M (1994) On a component mode method and its application to incompatible substructures. *Computers and Structures* 51:459-473
28. Felippa CA. Park KC. Farhat C (2001) Partitioned analysis of coupled mechanical systems. *Computer Methods in Applied Mechanics and Engineering* 190:3247-3270
29. Geradin M. Rixen D (1997) *Mechanical vibrations: Theory and applications to structural dynamics*. Second edition. Wiley. Chichester
30. Guyan RJ (1965) Reduction of stiffness and mass matrices. *AIAA Journal* 3:380-380
31. Hale AL. Meirovitch L (1982) A procedure for improving discrete substructure representation in dynamic synthesis. *AIAA Journal* 20:1128-1136
32. Har J. Tamma K (2012) *Advances in computational dynamics of particles, materials and structures*. John Wiley & Sons. Chichester
33. Herran M. Nelias D. Combescure A. Chalons H (2011) Optimal component mode synthesis for medium frequency problem. *International Journal For Numerical Methods In Engineering* 86:301-315
34. Hinke L. Dohnal F. Mace BR. Waters TP. Ferguson NS (2009) Component mode synthesis as a framework for uncertainty analysis. *Journal of Sound and Vibration* 324:161-178
35. Hintz RM (1975) Analytical methods in component modal synthesis. *AIAA Journal* 13:1007-1016
36. Hong SK. Epureanu BI. Castanier MP. Gorsich DJ (2011) Parametric reduced-order models for predicting the vibration response of complex structures with component damage and uncertainties. *Journal of Sound and Vibration* 330:1091-1110
37. Hughes TJR (2000) *The finite element method: Linear static and dynamic finite element analysis*. Dover Publications. New York

38. Hurty, WC (1960) Vibrations of structural systems by component mode synthesis. *Journal of Engineering Mechanics / American Society of Civil Engineers* 86:51-69.
39. Hurty WC (1965) Dynamic analysis of structural systems using component modes. *AIAA Journal* 3:678-685
40. Inman DJ (2006) *Vibration with control*. Wiley. Chichester
41. Irons B(1965) Structural eigenvalue problems - elimination of unwanted variables. *AIAA Journal* 3:961-962
42. Jezequel L (1985) A hybrid method of modal synthesis using vibration tests. *Journal of Sound and Vibration* 100:191-210
43. Kassem M. Soize C. Gagliardini L (2011) Structural partitioning of complex structures in the medium-frequency range - An application to an automotive vehicle. *Journal of Sound and Vibration* 330:937-946
44. Kuhar EJ. Stahle CV (1974) Dynamic transformation method for modal synthesis. *AIAA Journal* 12:672-678
45. Leung AYT (1993) *Dynamic stiffness and substructures*. Springer-Verlag. Berlin
46. Lim CN. Neild SA. Stoten DP. Drury D. Taylor CA (2007) Adaptive control strategy for dynamic substructuring tests. *Journal of Engineering Mechanics-ASCE* 133:864-873
47. Lindberg E. Horlin NE. Goransson P (2013) Component mode synthesis using undeformed interface coupling modes to connect soft and stiff substructures. *Shock and Vibration* 20:157-170
48. Liu WK. Karpov EG. Park HS (2006) *Nanomechanics and materials -Theory, multiscale methods and applications*. Wiley. Chichester
49. Liu W. Ewins DJ (2000) Substructure synthesis via elastic media Part I: Joint identification in Proceedings of the 18th IMAC Conference on Computational Challenges in Structural Dynamics (IMAC-XVIII). Book Series: Proceedings of The Society of Photo-Optical Instrumentation Engineers (SPIE) Bellingham 4062:1153-1159
50. MacNeal RH (1971) A hybrid method of component mode synthesis. *Computers and Structures* 1:581-601
51. Markovic D. Park KC. Ibrahimbegovic A (2007) Reduction of substructural interface degrees of freedom in flexibility-based component mode synthesis. *International Journal for Numerical Methods in Engineering* 70:163-180
52. Meirovitch L (1980) *Computational methods in structural dynamics*. Sijthoff and Noordhoff
53. Meirovitch L. Hale AL (1981) On the substructure synthesis method. *AIAA Journal* 19:940-947
54. Meirovitch L. Kwak MK (1991) Rayleigh-Ritz based substructure synthesis for flexible multibody systems. *AIAA Journal* 29:1709-1719
55. Mignolet MP. Soize C. Avalos J (2013) Nonparametric stochastic modeling of structures with uncertain boundary conditions / coupling between substructures. *AIAA Journal* 51:1296-1308
56. Morand HJP. Ohayon R (1979) Substructure variational analysis for the vibrations of coupled. *International Journal for Numerical Methods in Engineering* 14:741-755
57. Morand HJP. Ohayon R (1995) *Fluid structure interaction*. Wiley. Chichester
58. Morgan JA. Pierre C. Hulbert GM (1998) Calculation of component mode synthesis matrices from measured frequency response functions, part 1: Theory. *Journal of Vibration and Acoustics-Transactions of the ASME* 120:503-508
59. Morgan JA. Pierre C. Hulbert GM (1998) Calculation of component made synthesis matrices from measured frequency response functions, part 2: Application. *Journal of Vibration and Acoustics-Transactions of the ASME* 120:509-516
60. Nobari AS. Robb DA. Ewins DJ (1995) A new approach to modal-based structural dynamic-model updating and joint identification. *Mechanical Systems and Signal Processing* 9:85-100
61. Oden JT. Reddy JN (2011) *An introduction to the mathematical theory of finite elements*. Dover Publications. New York
62. Ohayon R. Soize C (1998) *Structural acoustics and vibration*. Academic Press. London
63. Ohayon R. Soize C (2012) Advanced computational dissipative structural acoustics and fluid-structure interaction in low- and medium-frequency domains - Reduced-order models and uncertainty quantification. *International Journal of Aeronautical and Space Sciences* 13:127-153
64. Ohayon R. Soize C (2013) *Structural dynamics in Encyclopedia of Applied and Computational Mathematics (EACM)*. B. Engquist (Editor) and J.T. Oden (Field editor for Mechanics. Springer Verlag. Heidelberg

65. Ohayon R. Soize C (2014) *Advanced computational vibroacoustics*. Cambridge University Press. New York
66. Ohayon R. Soize C (2014) Clarification about component mode synthesis methods for substructures with physical flexible interfaces. *AIAA Journal*. Submitted in September 2013
67. Ohayon R. Sampaio R. Soize C (1997) Dynamic substructuring of damped structures using singular value decomposition. *Journal of Applied Mechanics-Transactions of the ASME* 64:292-298
68. Park KC. Park YH (2004) Partitioned component mode synthesis via a flexibility approach. *AIAA Journal* 42: 1236-1245
69. Perdahcioglu DA. Geijselaers HJM. Ellenbroek MHM. de Boer A (2012) Dynamic substructuring and reanalysis methods in a surrogate-based design optimization environment. *Structural and Multidisciplinary Optimization* 45:129-138
70. Philippe B. Sameh A (2011) Eigenvalue and singular value problems 608-615 *Encyclopedia of parallel computing*. Edited by D. Padua. Springer. Berlin
71. Przemieniecki JS (1963) Matrix structural analysis of substructures. *AIAA Journal* 1:138-147
72. Reynders E (2012) System identification methods for (operational) modal analysis: Review and comparison. *Archives of Computational Methods in Engineering* 19:51-124
73. Rixen DJ (2004) A dual Craig-Bampton method for dynamic substructuring. *Journal of Computational and Applied Mathematics* 168:383-391
74. Rubin S (1975) Improved component-mode representation for structural dynamic analysis. *AIAA Journal* 13:995-1006
75. Saad Y (2011) *Numerical methods for large eigenvalue problems*. Society for Industrial and Applied Mathematics (SIAM). Philadelphia
76. Sanchez-Hubert J. Sanchez-Palencia E (1989) *Vibration and coupling of continuous systems - Asymptotic methods*. Springer-Verlag. Berlin
77. Sarkar A. Ghanem R (2003) A substructure approach for the midfrequency vibration of stochastic systems. *Journal of the Acoustical Society of America* 113:1922-1934
78. Soize C (2012) Stochastic models of uncertainties in computational mechanics. *American Society of Civil Engineers*. Reston
79. Soize C. Batou A (2011) Stochastic reduced-order model in low-frequency dynamics in presence of numerous local elastic modes. *Journal of Applied Mechanics - Transactions of the ASME* 78:061003-1-061003-9
80. Soize C. Chebli H (2003) Random uncertainties model in dynamic substructuring using a nonparametric probabilistic model. *Journal of Engineering Mechanics* 129:449-457
81. Soize C. Mziou S (2003) Dynamic substructuring in the medium-frequency range. *AIAA Journal* 41:1113-1118
82. Soize C. Poloskov IE (2012) Time-domain formulation in computational dynamics for linear viscoelastic media with model uncertainties and stochastic excitation. *Computers and Mathematics with Applications* 64:3594-3612
83. Suarez LE. and Singh MP (1992) Improved fixed interface method for modal synthesis. *AIAA Journal* 30:2952-2958
84. Truesdell C (1984) (ed.) *Mechanics of solids, Vol III, Theory of viscoelasticity, plasticity, elastic waves and elastic stability*. Springer-Verlag. Berlin
85. Tu JY. Yang HT. Lin PY. Chen PC (2013) Dynamics, control and real-time issues related to substructuring techniques: Application to the testing of isolated structure systems. *Journal of Systems and Control Engineering* 227:507-522
86. Urgueira APV (1989) *Dynamic analysis of coupled structures using experimental data*. Thesis of the University of London for the Diploma of Imperial College of Science, Technology and Medicine. London
87. Voormeeren SN. van der Valk PL. Rixen DJ (2011) Generalized methodology for assembly and reduction of component models for dynamic substructuring. *AIAA Journal* 49:1010-1020
88. Zienkiewicz OC. Taylor RL (2005) *The finite element method for solid and structural mechanics*. 6th ed. Elsevier, Butterworth-Heinemann. Amsterdam