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Composition operators on weighted Bergman spaces of Dirichlet series

Maxime Bailleul

Abstract

We study boundedness and compactness of composition operators on weighted Bergman spaces of Dirichlet series. Particularly, we obtain in some specific cases, upper and lower bounds of the essential norm of these operators and a criterion of compactness on classicals weighted Bergman spaces. Moreover, a sufficient condition of compactness is obtained using the notion of Carleson's measure.

Mathematics Subject Classification.
Primary: 47B33 - Secondary: 46E15.

Keywords.
weighted Bergman spaces - Carleson measure - composition operator - Dirichlet series - essential norm - generalized Nevanlinna counting function

1 Introduction

In [10], the authors defined the Hardy space $\mathcal{H}^2$ of Dirichlet series with square-summable coefficients. Thanks to the Cauchy-Schwarz inequality, it is easy to see that $\mathcal{H}^2$ is a space of analytic functions on $\mathbb{C}_{\frac{1}{2}} := \{s \in \mathbb{C},\ \Re(s) > \frac{1}{2}\}$ and that this domain is maximal. F. Bayart introduced in [2] the more general class of Hardy spaces of Dirichlet series $\mathcal{H}^p$ $(1 \leq p < +\infty)$.

In [9], the bounded composition operators on $\mathcal{H}^2$ are characterized, that is to say the holomorphic functions $\Phi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ such that for any $f \in \mathcal{H}^2$, $f \circ \Phi \in \mathcal{H}^2$. We denote by $\mathcal{D}$ the class of functions f which admit representation by a convergent Dirichlet series in some half-plane and for $\theta \in \mathbb{R}$, $\mathbb{C}_\theta$ will be the following half-plane $\{s \in \mathbb{C},\ \Re(s) > \theta\}$. We shall denote $\mathbb{C}_+$ instead of $\mathbb{C}_0$.

Theorem ([9],[18]). *The function Φ determines a bounded composition operator on $\mathcal{H}^2$ if and only:*

$$\Phi(s) = c_0 s + \varphi(s)$$

where c_0 is a nonnegative integer, $\varphi \in \mathcal{D}$ and φ converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and has the following properties:

a) If $c_0 \geq 1$, $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$.

b) If $c_0 = 0$, $\varphi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$.

Moreover, when $c_0 \geq 1$, C_Φ is a contraction.

This theorem is a strengthening of the original result (see [18],Th3.1). In [2], F. Bayart proved that this result is true on the space $\mathcal{H}^p$ when $p \geq 1$ and $c_0 \neq 0$. When $c_0 = 0$ the condition is necessary for every $p \geq 1$ and sufficient if p is even. The goal of this paper is to study composition operators on the weighted Bergman $\mathcal{A}^p_\mu$ spaces of Dirichlet series defined in [1]. In order to define them, we need to recall the Bohr's point of view principle and the definition of the Hardy spaces $\mathcal{H}^p$.

Let $n \geq 2$ be an integer, we shall denote e_n the function defined by $\mathrm{e}_n(z) = n^{-z}$. The integer n can be written (in a unique way) as a product of prime numbers $n = p_1^{\alpha_1} \cdot \cdot p_k^{\alpha_k}$ where $p_1 = 2$, $p_2 = 3$ etc ... If s is a complex number, we consider $z = (p_1^{-s}, p_2^{-s}, \dots)$. let f be a Dirichlet series of the following form

$$f = \sum_{n=1}^{+\infty} a_n \mathrm{e}_n, \tag{1}$$

then

$$f(s) = \sum_{n=1}^{+\infty} a_n (p_1^{-s})^{\alpha_1} \cdot \cdot (p_k^{-s})^{\alpha_k} = \sum_{n=1}^{+\infty} a_n \, z_1^{\alpha_1} \cdot \cdot z_k^{\alpha_k}.$$

So we can see a Dirichlet series as a Fourier series on the infinite-dimensional polytorus $\mathbb{T}^\infty = \{(z_1, z_2, \cdots), |z_i| = 1, \forall i \geq 1\}$. We shall denote this Fourier series $D(f)$.

Let us fix now $p \geq 1$, the space $H^p(\mathbb{T}^\infty)$ is the closure of the set of analytic polynomials with respect to the norm of $L^p(\mathbb{T}^\infty, m)$ where m is the normalized Haar measure on $\mathbb{T}^\infty$ (see [6] for more details). Let f be a Dirichlet polynomial, by the Bohr's point of view $D(f)$ is an analytic polynomial on $\mathbb{T}^\infty$ and by definition $\|f\|_{\mathcal{H}^p} := \|D(f)\|_{H^p(\mathbb{T}^\infty)}$. The space $\mathcal{H}^p$ is defined by taking the closure of Dirichlet polynomials with respect to this norm. Consequently $\mathcal{H}^p$ and $H^p(\mathbb{T}^\infty)$ are isometrically isomorphic.

For $\sigma > 0$, f_σ will be the translate of f by σ, *ie* $f_\sigma(s) := f(\sigma + s)$. We shall denote by $\mathcal{P}$ the set of Dirichlet polynomials.

Let $p \geq 1$, $P \in \mathcal{P}$ and let μ be a probability measure on $(0, +\infty)$ such that $0 \in Supp(\mu)$. Then

$$\|P\|_{\mathcal{A}^p_\mu} := \left(\int_0^{+\infty} \|P_\sigma\|^p_{\mathcal{H}^p} \, d\mu(\sigma) \right)^{1/p}.$$

$\mathcal{A}^p_\mu$ is the completion of $\mathcal{P}$ with respect to this norm. When $d\mu(\sigma) = 2e^{-2\sigma} \, d\sigma$, these spaces are simply denoted by $\mathcal{A}^p$. It is shown in [1] that they are spaces of convergent Dirichet series on $\mathbb{C}_{1/2}$.

In section 2, we recall some facts about Dirichlet series and precise some notations.

In section 3, we study the boundedness of C_Φ on $\mathcal{A}^p_\mu$. For $\Phi(s) = c_0 s + \varphi(s)$ with $c_0 \geq 1$ and $\varphi \in \mathcal{D}$, Φ induces a bounded composition operator from $\mathcal{A}^p_\mu$ to $\mathcal{A}^p_\mu$ if and only if φ converges uniformly on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$. When $c_0 = 0$, we give sufficient and necessary conditions.

In section 4, we define generalized Nevanlinna counting function in order to obtain some estimates of the essential norm of C_Φ on $\mathcal{H}^2$ and $\mathcal{A}^2_\mu$. Particularly we obtain a criterion of compactness of C_Φ in some specific cases.

In section 5, we obtain a majorization of the generalized Nevanlinna counting function with help of the associated Carleson function (see definition in section 5) and then we get other sufficient conditions of compactness of C_Φ.

2 Background material

Let f be a Dirichlet series of form (1). We do not recall the definition of abscissa of simple (resp. absolute) convergence denoted by σ_c (resp. σ_a), see [17] or [21] for more details. We shall need the two other following abscissas:

$$\begin{aligned}\sigma_u(f) &= \inf\{a \mid \text{The series (1) is uniformly convergent for } \Re(s) > a\}\\ &= \text{abscissa of uniform convergence of } f.\end{aligned}$$

$$\begin{aligned}\sigma_b(f) &= \inf\{a \mid \text{the function } f \text{ has an analytic, bounded extension for } \Re(s) > a\}\\ &= \text{abscissa of boundedness of } f.\end{aligned}$$

It is easy to see that $\sigma_c(f) \leq \sigma_u(f) \leq \sigma_a(f)$. An important result is that $\sigma_u(f)$ and $\sigma_b(f)$ coincide: this is the Bohr's theorem (see [4]), it is really important for the study of $\mathcal{H}^\infty$, the algebra of bounded Dirichlet series on the right half-plane $\mathbb{C}_+$ (see [15]). We shall denote by $\|\cdot\|_\infty$ the norm on this space:

$$\|f\|_\infty := \sup_{\Re(s)>0} |f(s)|.$$

The infinite-dimensional polytorus $\mathbb{T}^\infty$ can be identified with the group of complex-valued characters χ on the positive integers which satisfy the following properties

$$\begin{cases} |\chi(n)| = 1 & \forall n \geq 1,\\ \chi(nm) = \chi(n)\,\chi(m) & \forall n,\, m \geq 1.\end{cases}$$

To obtain this identification for $\chi = (\chi_1, \chi_2, \dots) \in \mathbb{T}^\infty$, it suffices to define χ on the prime numbers by $\chi(p_i) = \chi_i$ and use multiplicativity.

Let f be a Dirichlet series absolutely convergent in a half-plane. For any sequences $(\tau_n) \subset \mathbb{R}$, we consider vertical translations of f:

$$(f_{\tau_n}(s)) := (f(s + i\tau_n)).$$

By Montel's theorem, this sequence is a normal family in the half-plane of absolute convergence of f and so there exists a convergent subsequence, say $(f_{\tau_{n_k}})$, such that $f_{\tau_{n_k}}$ converges uniformly on compact subsets of the domain of absolute convergence of f to a limit function $\tilde{f}$. We say that $\tilde{f}$ is a vertical limit of f.

Proposition ([10]). *Let f be a Dirichlet series of the form (1), absolutely convergent in a half-plane. The vertical limit functions of f are exactly the functions of the form $f_\chi = \sum_{n\geq 1} a_n \chi(n)\, \mathrm{e}_n$ where $\chi \in \mathbb{T}^\infty$.*

When $f \in \mathcal{A}^p_\mu$, the vertical limits have good properties:

Proposition ([1]). *Let $p \geq 1$ and $f \in \mathcal{A}^p_\mu$. For almost all $\chi \in \mathbb{T}^\infty$, f_χ converges on $\mathbb{C}_+$.*

Let $\Phi(s) = c_0 s + \varphi(s)$ where c_0 is a nonnegative integer and $\varphi \in \mathcal{D}$. For $\tau \in \mathbb{R}$, we define Φ_τ by $\Phi_\tau(s) = c_0 s + \varphi(s + i\tau)$, then if $\chi \in \mathbb{T}^\infty$ the function Φ_χ defined by

$$\Phi_\chi(s) = c_0 s + \varphi_\chi(s)$$

is a vertical limit of the functions $(\Phi_\tau)_{\tau\in\mathbb{R}}$.

In section 4, we shall consider the Hilbert spaces of Dirichlet series $\mathcal{H}^2$ and $\mathcal{A}^2_\mu$ where μ is a probability measure on $(0,+\infty)$ such that $0 \in Supp(\mu)$. Moreover we shall assume that $d\mu(\sigma) = h(\sigma)d\sigma$ where h is a nonnegative function such that $\|h\|_{L^1((0,+\infty))} = 1$. We see easily that for an element of $\mathcal{A}^2_\mu$ with form (1):

$$\|f\|_{\mathcal{A}^2_\mu} = \left(\sum_{n=1}^{+\infty} |a_n|^2 w_h(n)\right)^{1/2}$$

where

$$w_h(n) = \int_0^{+\infty} n^{-2\sigma} h(\sigma)d\sigma \qquad \forall n \geq 1.$$

Examples. We follow notations from [1]: let $\alpha > -1$, we denote μ_α the probability measure defined on $(0,+\infty)$ by

$$d\mu_\alpha(\sigma) = \frac{2^{\alpha+1}}{\Gamma(\alpha+1)}\sigma^\alpha \exp(-2\sigma)d\sigma.$$

We denote this space $\mathcal{A}^2_\alpha$ instead of $\mathcal{A}^2_{\mu_\alpha}$ and the corresponding weight by (w^α_n). Then for every $n \geq 1$,

$$w^\alpha_n = \frac{1}{(\log(n)+1)^{\alpha+1}}.$$

We shall denote $(\mathrm{e}^\mu_n)_{n\geq 1}$ the orthonormal basis of $\mathcal{A}^2_\mu$ defined by

$$\mathrm{e}^\mu_n := \frac{\mathrm{e}_n}{\sqrt{w_h(n)}}.$$

3 Boundedness of composition operators

In the sequel μ will be a probability measure on $(0,+\infty)$ such that $0 \in Supp(\mu)$ (as in [1]).

Theorem 1. *Let $\Phi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ be an analytic function of the form $\Phi(s) = c_0 s + \varphi(s)$ where $c_0 \geq 1$ and $\varphi \in \mathcal{D}$. Then C_Φ is bounded on $\mathcal{A}^p_\mu$ if and only φ converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$ and in this case C_Φ is a contraction.*

Remark. The condition $\Phi(s) = c_0 s + \varphi(s)$ where c_0 is a nonnegative integer is an arithmetical condition: Φ has this form if and only if $P \circ \Phi \in \mathcal{D}$ for every $P \in \mathcal{P}$ (see [9],Th.A).

Proof.
$\triangleright$ Assume that φ converges uniformly on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and that $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$. Let P be a Dirichlet polynomial, $P \circ \Phi$ is then a Dirichlet series which is bounded on $\mathbb{C}_+$ (because $\Phi(\mathbb{C}_+) \subset \mathbb{C}_+$) and so $P \circ \Phi$ belongs to $\mathcal{H}^\infty$ and consequently to $\mathcal{H}^p$. By [1],Th.6(ii) we know that

$$\|P \circ \Phi\|^p_{\mathcal{A}^p_\mu} = \int_0^{+\infty} \|(P \circ \Phi)_\sigma\|^p_{\mathcal{H}^p}\, d\mu(\sigma).$$

Now, the key point is to use the boundedness of the composition operators on $\mathcal{H}^p$. We remark that for each $\sigma > 0$,

$$(P \circ \Phi)_\sigma(it) = P(\Phi(\sigma + it)) = P_\sigma(\Phi(\sigma + it) - \sigma).$$

Let $\widetilde{\Phi}$ defined on $\mathbb{C}_+$ by $\widetilde{\Phi}(s) = \Phi(\sigma + s) - \sigma$. We claim that $\widetilde{\Phi}$ verifies the condition of the boundedness of $C_{\widetilde{\Phi}}$ on $\mathcal{H}^p$: it can be written $\widetilde{\Phi}(s) = c_0 s + \widetilde{\varphi}(s)$ where $\widetilde{\varphi}$ converges uniformly on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$. Then it suffices to check that $\widetilde{\varphi}(\mathbb{C}_+) \subset \mathbb{C}_+$ but $\widetilde{\varphi}(s) = \varphi(s + \sigma) + (c_0 - 1)\sigma$ for every $s \in \mathbb{C}_+$ and the conclusion is clear because $c_0 \geq 1$ and $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$.

Now we apply the result on the boundedness on $\mathcal{H}^p$. Since $c_0 \geq 1$, $C_{\widetilde{\Phi}}$ is a contraction on $\mathcal{H}^p$ and we obtain (σ is fixed):

$$\|(P \circ \Phi)_\sigma\|^p_{\mathcal{H}^p} = \|P_\sigma(\widetilde{\Phi})\|^p_{\mathcal{H}^p} \leq \|P_\sigma\|^p_{\mathcal{H}^p}.$$

This is true for every $\sigma > 0$, so

$$\int_0^{+\infty} \|(P \circ \Phi)_\sigma\|^p_{\mathcal{H}^p} \, d\mu(\sigma) \leq \int_0^{+\infty} \|P_\sigma\|^p_{\mathcal{H}^p} \, d\mu(\sigma).$$

Hence $\|P \circ \Phi\|_{\mathcal{A}^p_\mu} \leq \|P\|_{\mathcal{A}^p_\mu}$. Now, by density of $\mathcal{P}$ in $\mathcal{A}^p_\mu$, $C_\Phi : \mathcal{P} \cap \mathcal{A}^p_\mu \to \mathcal{A}^p_\mu$ extends to a bounded operator $T : \mathcal{A}^p_\mu \to \mathcal{A}^p_\mu$ such that $T(P) = P \circ \Phi$ for every $P \in \mathcal{P}$. Let $f \in \mathcal{A}^p_\mu$, there exists a sequence $(P_n) \subset \mathcal{P}$ such that (P_n) converges to f in norm, then by continuity of the point evaluation at $s \in \mathbb{C}_{1/2}$ on $\mathcal{A}^p_\mu$ (see [1],Th.1):

$$\begin{aligned} |T(f)(s) - T(P_n)(s)| &\leq \|\delta_s\|_{(\mathcal{A}^p_\mu)^*} \|T(f) - T(P_n)\|_{\mathcal{A}^p_\mu} \\ &\leq \|\delta_s\|_{(\mathcal{A}^p_\mu)^*} \|T\| \, \|f - P_n\|_{\mathcal{A}^p_\mu} \end{aligned}$$

and then $T(P_n)(s) = P_n(\Phi(s))$ converges to $T(f)(s)$ when n goes to infinity. But by evaluation at $\Phi(s) \in \mathbb{C}_{1/2}$, $P_n(\Phi(s))$ converges to $f(\Phi(s))$, so $T(f)(s) = f(\Phi(s)) = C_\Phi(f)(s)$ and the result is proved.

$\triangleright$ Now assume that Φ induces a bounded composition operator on $\mathcal{A}^p_\mu$. The result follows from ideas similar to the ones of the proof of Th.B from [9]. We only give a sketch of the proof.

Let $f \in \mathcal{A}^p_\mu$. For every $\chi \in \mathbb{T}^\infty$, we can show that (see [9], Prop4.3) for every $s \in \mathbb{C}_{1/2}$,

$$(f \circ \Phi)_\chi(s) = f_{\chi^{c_0}} \circ \Phi_\chi(s). \qquad (\Delta)$$

Since f and $f \circ \Phi$ belong to $\mathcal{A}^p_\mu$, $(f \circ \Phi)_\chi$ and $f_{\chi^{c_0}}$ extend analytically on $\mathbb{C}_+$ almost-surely relatively to χ . As in Prop5.1 from [9], we can show that Φ_χ extends analytically on $\mathbb{C}_+$ for every $\chi \in \mathbb{T}^\infty$.

Now, let $\chi \in \mathbb{T}^\infty$ such that $f_{\chi^{c_0}}$, Φ_χ and $(f \circ \Phi)_\chi$ extend on $\mathbb{C}_+$.

If Φ_χ does not map $\mathbb{C}_+$ to $\mathbb{C}_+$, then there exists $s_0 \in \mathbb{C}_+$ such that $\Phi_\chi(s_0)$ is in the imaginary axis and with an argument of connectedness, we can choose it such that $\Phi'_\chi(s_0) \neq 0$. Then, by (Δ), $f_{\chi^{c_0}}$ has an analytic extension on a small segment on the imaginary axis near $\Phi_\chi(s_0)$. We denote $(p_n)_{n \geq 1}$ the sequence of prime numbers. In [2] it is shown that

$$f = \sum_{n \geq 1} \frac{e_{p_n}}{\sqrt{p_n} \log(p_n)} \in \mathcal{H}^p$$

and that almost surely relatively to χ, $f_{\chi^{c_0}}$ does not extend analytically to any region larger than $\mathbb{C}_+$. But $f \in \mathcal{A}^p_\mu$ because $\mathcal{H}^p \subset \mathcal{A}^p_\mu$. Consequently Φ_χ has to map $\mathbb{C}_+$ to $\mathbb{C}_+$ almost surely relatively to χ and then by Proposition 4.1 from [9], $\Phi(\mathbb{C}_+) \subset \mathbb{C}_+$. Finally, it suffices to apply Th3.1 from [18] to obtain that $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$ and φ converges uniformly on $\mathbb{C}_\varepsilon$ for all $\varepsilon > 0$. □

Remark. In the previous proof we used that $\Phi_u - u : \mathbb{C}_+ \to \mathbb{C}_+$ when $c_0 \geq 1$. We can see this result as an equivalent of the Schwarz's lemma in this framework.

When $c_0 = 0$, we obtain the following result.

Theorem 2. *Let $p \geq 1$ and $\Phi : \mathbb{C}_{\frac{1}{2}} \to \mathbb{C}_{\frac{1}{2}}$ be an analytic function belonging to $\mathcal{D}$. Then*

(i) If C_Φ is bounded on $\mathcal{A}^p_\mu$ then Φ converges uniformly on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$.

(ii) If Φ converges uniformly on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2+\eta}$ (where $\eta > 0$) then C_Φ is Hilbert-Schmidt on $\mathcal{A}^2_\mu$, a fortiori bounded.

(iii) If C_Φ is bounded on $\mathcal{A}^2_\mu$, then it is bounded on $\mathcal{A}^{2k}_\mu$ for any $k \geq 1$. Consequently C_Φ is bounded on $\mathcal{A}^{2k}_\mu$ for any $k \geq 1$ under conditions of (ii).

Proof.

(i) As in the case $c_0 \geq 1$, the proof follows from [9], Th.B and some easy adaptations.

(ii) By Lemma 5 from [2] we have:

$$\|n^{-\Phi_\sigma}\|^2_{\mathcal{H}^2} = \int_{\mathbb{T}^\infty} \int_0^1 |n^{-\Phi_\chi(\sigma+it)}|^2 dt dm(\chi) \qquad \forall n \geq 1.$$

Now $\Phi : \mathbb{C}_+ \to \mathbb{C}_{1/2+\eta}$ and then by Proposition 4.1 from [9], the same holds for Φ_χ for every $\chi \in \mathbb{T}^\infty$. Consequently

$$\|n^{-\Phi_\sigma}\|^2_{\mathcal{H}^2} \leq n^{-1-2\eta} \qquad \forall n \geq 1.$$

Now we show that C_Φ is Hilbert-Schmidt and then obviously bounded. The sequence $(\mathrm{e}^\mu_n)_{n\geq 1}$ is an orthonormal basis of $\mathcal{A}^2_\mu$ and $C_\Phi(\mathrm{e}^\mu_n) \in \mathcal{H}^\infty$ for every $n \geq 1$: indeed $\mathrm{e}^\mu_n \in \mathcal{P}$ and $\Phi(\mathbb{C}_+) \subset \mathbb{C}_+$. Then by [1], Th.6(ii):

$$\sum_{n=1}^{+\infty} \|C_\Phi(\mathrm{e}^\mu_n)\|^2_{\mathcal{A}^2_\mu} = \sum_{n=1}^{+\infty} \int_0^{+\infty} \|(C_\Phi(\mathrm{e}^\mu_n))_\sigma\|^2_{\mathcal{H}^2} d\mu(\sigma)$$

$$= \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{\|n^{-\Phi_\sigma}\|^2_{\mathcal{H}^2}}{w_h(n)} d\mu(\sigma) \leq \sum_{n=1}^{+\infty} \int_0^{+\infty} \frac{n^{-1-2\eta}}{w_h(n)} d\mu(\sigma)$$

$$= \sum_{n=1}^{+\infty} \frac{n^{-1-2\eta}}{w_h(n)}.$$

Now in [16], it is shown that the weight $(w_h(n))$ decreases more slowly than any negative power of n so there exists $C > 0$ such that $w_h(n) \geq Cn^{-\eta}$ for every $n \geq 1$, then

$$\sum_{n=1}^{+\infty} \|C_\Phi(\mathrm{e}^\mu_n)\|^2_{\mathcal{A}^2_\mu} \leq \frac{1}{C} \sum_{n=1}^{+\infty} \frac{1}{n^{1+\eta}} < +\infty$$

and so C_Φ is Hilbert-Schmidt.

(iii) Assume that C_Φ is bounded on $\mathcal{A}^2_\mu$. Let P be a Dirichlet polynomial, we get

$$\|P \circ \Phi\|_{\mathcal{A}^{2k}_\mu} = \|P^k \circ \Phi\|^{1/k}_{\mathcal{A}^2_\mu} \leq \|C_\Phi\|^{1/k} \|P^k\|^{1/k}_{\mathcal{A}^2_\mu} = \|C_\Phi\|^{1/k} \|P\|_{\mathcal{A}^{2k}_\mu}.$$

By density of the polynomials and boundedness of the point evaluation we obtain the result (as in the proof of Theorem 1). □

Questions.

Is it true that C_Φ is actually nuclear?

Can we choose $\eta = 0$ in (ii) to get boundedness?

4 Compactness and Nevanlinna counting function

Let $p \geq 1$ and $X = \mathcal{H}^p$ or $\mathcal{A}^p_\mu$. We begin this section with the following criterion of compactness:

Proposition 1. *Let $\Phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ be an analytic function which induces a bounded composition operator on X. Then C_Φ is compact on X if and only if for every bounded sequence (f_n) in X which converges to 0 uniformly on $\mathbb{C}_{1/2+\varepsilon}$ for every $\varepsilon > 0$, one has $\|C_\Phi(f_n)\|_X \to 0$.*

Proof. The point-evaluation at $s \in \mathbb{C}_{1/2}$ is bounded on X, X has the Fatou-property (in the sense of [13], Prop 4.8) and there exists a Montel's theorem for $\mathcal{H}^\infty$ (see [3]). Then it suffices to apply the same reasoning than in the proof of Proposition 4.8 from [13]. □

Remark. With this criterion, it is easy to see that if C_Φ is compact on $\mathcal{H}^2$ (resp. $\mathcal{A}^2_\mu$) then C_Φ is compact on $\mathcal{H}^{2k}$ (resp. $\mathcal{A}^{2k}_\mu$) for any $k \geq 1$.

A Sufficient conditions of compactness

In this section, we follow ideas from [3]: we define the generalized Nevanlinna counting function associated to the spaces $\mathcal{A}^2_\mu$ in order to study the compactness of composition operators on $\mathcal{A}^2_\mu$. We need some preliminaries from [1].

For $\sigma > 0$, we define

$$\beta_h(\sigma) := \int_0^\sigma (\sigma - u) h(u)\, du = \int_0^\sigma \int_0^t h(u)\, dudt.$$

We point out that if h is continuous, the two first derivatives of β_h are

$$\begin{cases} \beta'_h(\sigma) = \int_0^\sigma h(u) du, \\ \beta''_h(\sigma) = h(\sigma). \end{cases}$$

Remark. If we denote β_α instead of β_h for the spaces $\mathcal{A}^2_\alpha$, it is easy to see that $\beta_\alpha(\sigma) \approx \sigma^{\alpha+2}$ when σ is close to 0.

Theorem ("Littlewood-Paley formula" [1]). *Let η be a Borel probability measure on $\mathbb{R}$. Then*

$$\|f\|^2_{\mathcal{A}^2_\mu} = |f(+\infty)|^2 + 4\int_{\mathbb{T}^\infty}\int_0^{+\infty}\int_{\mathbb{R}} \beta_h(\sigma)|f'_\chi(\sigma+it)|^2\, d\eta(t)d\sigma dm(\chi).$$

Remark. For a Dirichlet series f with expansion (1), $f(+\infty)$ stands for a_1, the constant coefficient.

Notation. We shall call a c_0-symbol any analytic function $\Phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ of the form $\Phi(s) = c_0 s + \varphi(s)$ where c_0 is a nonnegative integer and $\varphi \in \mathcal{D}$ such that C_Φ is bounded on $\mathcal{A}^2_\mu$. When $c_0 \geq 1$ it is equivalent to the fact that φ converges uniformly on $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and $\varphi(\mathbb{C}_+) \subset \mathbb{C}_+$. We point out that in the sequel, the hypothesis "$c_0 \geq 1$" will be crucial.

Definition 1. We define the generalized Nevanlinna counting function associated to a function $\beta : (0,+\infty) \to (0,+\infty)$ by

$$N_{\beta,\Phi}(s) = \begin{cases} \displaystyle\sum_{\substack{a\in\mathbb{C}_+\\ \Phi(a)=s}} \beta(\Re(a)) & \text{if} \qquad s \in \Phi(\mathbb{C}_+), \\ 0 & \text{else.} \end{cases}$$

When $\beta(\sigma) = \sigma$, we just denote N_Φ this function.

Remarks.

(i) N_Φ is associated to the space $\mathcal{H}^2$, it was already defined in [3].

(ii) $N_{\beta,\Phi}$ is the equivalent in our framework of Dirichlet series of the generalized Nevanlinna counting function defined in [11] in the framework of analytic functions on the unit disk.

Lemma 1. *Let Φ be a c_0-symbol with $c_0 \geq 1$. For every $s \in \mathbb{C}_+$, we have*

$$N_{\beta_h,\Phi}(s) = \int_0^{\Re(s)} N_{\Phi_u}(s)h(u)du$$

where Φ_u is defined by $\Phi_u(s) := \Phi(s+u)$.

Proof. Let $s \in \mathbb{C}_+$.

$$\begin{aligned} N_{\beta_h,\Phi}(s) &= \sum_{\substack{a\in\mathbb{C}_+\\ \Phi(a)=s}} \beta_h(\Re(a)) \\ &= \sum_{\substack{a\in\mathbb{C}_+\\ \Phi(a)=s}} \int_0^{\Re(a)} (\Re(a)-u)h(u)du \\ &= \sum_{\substack{\Re(a)\leq\Re(s)\\ \Phi(a)=s}} \int_0^{\Re(a)} (\Re(a)-u)h(u)du \end{aligned}$$

thanks to the "Schwarz's lemma": indeed $\Phi(\mathbb{C}_{\Re(a)}) \subset \mathbb{C}_{\Re(a)}$ and then $\Re(a) > \Re(s)$ implies $\Phi(a) \neq s$. Now by the Fubini's theorem:

$$N_{\beta_h,\Phi}(s) = \int_0^{\Re(s)} \sum_{\substack{u<\Re(a)\leq\Re(s)\\ \Phi(a)=s}} (\Re(a)-u)h(u)\, du.$$

Now it suffices to remark that

$$\begin{aligned}\sum_{\substack{u<\Re(a)\leq\Re(s)\\ \Phi(a)=s}}(\Re(a)-u) &= \sum_{\substack{u<\Re(a)\\ \Phi_u(a-u)=s}}(\Re(a)-u)\\ &= \sum_{\substack{0<\Re(a')\\ \Phi_u(a')=s}}\Re(a')\\ &= N_{\Phi_u}(s).\end{aligned}$$

□

Proposition 2. *Let Φ be a c_0-symbol with $c_0 \geq 1$. For every $s \in \mathbb{C}_+$, we have*

$$N_{\beta_h,\Phi}(s) \leq \frac{\beta_h(\Re(s))}{c_0}.$$

Proof. By the previous lemma, we know that

$$N_{\beta_h,\Phi}(s) = \int_0^{\Re(s)} N_{\Phi_u}(s)h(u)du.$$

We already pointed out that $\Phi_u - u : \mathbb{C}_+ \to \mathbb{C}_+$ and then by Littlewood's inequality ([3],Prop.3):

$$N_{\Phi_u-u}(s') \leq \frac{\Re(s')}{c_0} \quad \forall s' \in \mathbb{C}_+.$$

But $N_{\Phi_u}(s) = N_{\Phi_u-u}(s-u)$ when $\Re(s) > u > 0$. Finally

$$N_{\beta_h,\Phi}(s) \leq \int_0^{\Re(s)} \frac{(\Re(s)-u)h(u)}{c_0}\,du = \frac{\beta_h(\Re(s))}{c_0}.$$

□

In order to obtain an upper bound for the essential norm of C_Φ, we need to obtain some estimates on $N_{\beta_h,\Phi}$. We follow ideas from [3].

Proposition 3. *Let $\Phi : \mathbb{C}_+ \to \mathbb{C}_+$ and $\Phi_k : \mathbb{C}_+ \to \mathbb{C}_+$ where $k \geq 0$, some analytic functions. Assume that $(\Phi_k)_{k\geq 0}$ converges uniformly to Φ on every compact subset of $\mathbb{C}_+$, then for every $s \in \mathbb{C}_+$,*

$$N_{\beta_h,\Phi}(s) \leq \liminf_{k\to+\infty} N_{\beta_h,\Phi_k}(s).$$

Proof. Let us fix $s \in \Phi(\mathbb{C}_+)$ (if $s \notin \Phi(\mathbb{C}_+)$ there is nothing to prove) and $\varepsilon > 0$. Let $a_1, \ldots, a_n \in \mathbb{C}_+$ satisfying $\Phi(a_i) = s$ for every $i = 1, \ldots, n$ and $\delta > 0$ such that $2n\delta < \varepsilon$. $(\Phi_k)_{k\geq 0}$ converges uniformly to Φ on every compact set of $\mathbb{C}_+$ and so particularly on each disk $D(a_i, \delta)$ of center a_i and radius δ. Then by the Hurwitz's lemma, there exists $K \geq 0$ such that for any $k \geq K$, $s \in \Phi_k(D(a_i, \delta))$ for every $i = 1, \ldots, n$.

Now let us fix $k \geq K$, there exist $a_1^k \in D(a_1, \delta)$, ... , $a_n^k \in D(a_n, \delta)$ such that $\Phi(a_i^k) = s$ for every $i = 1, \ldots, n$. By definition of β_h, we have

$$\begin{aligned}
\beta_h(\Re(a_i^k)) &= \int_0^{\Re(a_i^k)} (\Re(a_i^k) - u)h(u)du \\
&\geq \int_0^{\Re(a_i)-\delta} (\Re(a_i) - \delta - u)h(u)du \\
&= \beta_h(\Re(a_i)) - \left(\delta \int_0^{\Re(a_i)-\delta} h(u)du + \int_{\Re(a_i)-\delta}^{\Re(a_i)} (\Re(a_i) - u)h(u)du\right) \\
&\geq \beta_h(\Re(a_i)) - 2\delta
\end{aligned}$$

because $\|h\|_1 = 1$. Summing up all the terms

$$\beta_h(\Re(a_1^k)) + \cdots + \beta_h(\Re(a_n^k)) \geq \beta(\Re(a_1)) + \cdots + \beta(\Re(a_n)) - 2n\delta.$$

Then for every $k \geq K$, we get

$$\begin{aligned}
\beta(\Re(a_1)) + \cdots + \beta(\Re(a_n)) &\leq 2n\delta + \beta_h(\Re(a_1^k)) + \cdots + \beta_h(\Re(a_n^k)) \\
&\leq \varepsilon + N_{\Phi_k}(s).
\end{aligned}$$

So

$$\beta(\Re(a_1)) + \cdots + \beta(\Re(a_n)) \leq \varepsilon + \liminf_{k \to +\infty} N_{\Phi_k}(s)$$

and since n and ε are arbitrary, the result is proved. □

Corollary 1. *Let Φ a c_0-symbol with $c_0 \geq 1$. Assume that*

$$\sup_{\Re(s)>0} \frac{N_{\beta,\Phi}(s)}{\beta(\Re(s))} = C < +\infty,$$

then

$$\sup_{\chi \in \mathbb{T}^\infty} \sup_{\Re(s)>0} \frac{N_{\beta,\Phi_\chi}(s)}{\beta(\Re(s))} = C.$$

Proof. For every $\tau \in \mathbb{R}$, it is easy to see that $N_{\beta_h,\Phi_\tau}(s) = N_{\beta_h,\Phi}(s + ic_0\tau)$ and so for every $\tau \in \mathbb{R}$ we have

$$\sup_{\Re(s)>0} \frac{N_{\beta,\Phi_\tau}(s)}{\beta(\Re(s))} = C.$$

Now for $\chi \in \mathbb{T}^\infty$, Φ_χ is a vertical limit of a sequence $(\Phi_{\tau_n})_{n\geq 0}$ where $(\tau_n)_{n\geq 0} \subset \mathbb{R}$ and then it suffices to apply Proposition 3 to obtain that

$$\sup_{\Re(s)>0} \frac{N_{\beta,\Phi_\chi}(s)}{\beta(\Re(s))} \leq C$$

The other inequality is true because $\Phi = \Phi_\chi$ with $\chi = (1, 1, \dots)$. □

Remark. In the previous corollary, we can take the supremum on a band of the form $\{s \in \mathbb{C}_+, \Re(s) < \theta\}$ where $\theta \in \mathbb{R}$.

Let H a Hilbert space and $\mathcal{K}(H)$ the space of all compact operators on H. Let T be a bounded operator on H, we recall that its essential norm is defined by

$$\|T\|_{H,e} = \inf\{\|T - K\|,\ K \in \mathcal{K}(H)\}.$$

Clearly, the essential norm of T is zero if and only if $T \in \mathcal{K}(H)$. The next result will be very useful.

Proposition. *([20],Prop5.1) Let T be a bounded operator on H and $(K_n)_{n\geq 1}$ a sequence of self-adjoint compact operators on H. Let $R_n = I - K_n$. Assume that $\|R_n\| = 1$ for every $n \geq 1$ and that $(R_n)_{n\geq 1}$ converges pointwise to zero. Then*

$$\|T\|_{H,e} = \lim_{n\to+\infty} \|TR_n\|.$$

Theorem 3. *Let Φ a c_0-symbol with $c_0 \geq 1$. Assume that $Im(\varphi)$ is bounded on $\mathbb{C}_+$, then*

$$\|C_\varphi\|_{\mathcal{A}^2_\mu,e} \leq \left(2\sup_{\mathbb{C}_+} |Im(\varphi)| + c_0\right) \cdot \limsup_{\Re(s)\to 0} \frac{N_{\beta_h,\Phi}(s)}{\beta_h(Re(s))}$$

and

$$\|C_\varphi\|_{\mathcal{H}^2,e} \leq \left(2\sup_{\mathbb{C}_+} |Im(\varphi)| + c_0\right) \cdot \limsup_{\Re(s)\to 0} \frac{N_\Phi(s)}{\Re(s)}.$$

Corollary 2. *Assume that $Im(\varphi)$ is bounded on $\mathbb{C}_+$ then:*

(i) If $N_{\beta_h,\Phi}(s) = o(\beta(\Re(s))$ when $\Re(s) \to 0$ then C_Φ is compact on $\mathcal{A}^2_\mu$.

(ii) If $N_\Phi(s) = o(\Re(s))$ when $\Re(s) \to 0$ then C_Φ is compact on $\mathcal{H}^2$.

Remark. $Cor2.(ii)$ is already known (see [3], Th2).

Proof of Th.3. We only give the proof in the case of $\mathcal{A}^2_\mu$, the proof in the case $\mathcal{H}^2$ is essentially the same.

For $n \geq 1$ and $f \in \mathcal{A}^2_\mu$ which has form (1) we define

$$\begin{cases} K_n(f) &= \displaystyle\sum_{k=1}^{n} a_k e_k \\ R_n(f) &= (I - K_n)(f) \quad = \displaystyle\sum_{k=n+1}^{+\infty} a_k e_k. \end{cases}$$

Each K_n is a self-adjoint compact operator on $\mathcal{A}^2_\mu$ with norm 1 because K_n is a projection on a finite linear subspace and (R_n) clearly converges pointwise to zero. Then by the previous proposition

$$\|C_\Phi\|_{A^2_\mu,e} = \lim_{n\to+\infty} \|C_\Phi R_n\| = \lim_{n\to+\infty} \left\{ \sup_{\|f\|_{\mathcal{A}^2_\mu}\leq 1} \|C_\Phi(R_n(f))\|_{\mathcal{A}^2_\mu} \right\}.$$

In the sequel, we shall work on $\Omega = (0,+\infty) \times (0,1)$ with the area measure: we apply the Littlewood-Paley formula for $\mathcal{A}^2_\mu$ with the Lebesgue measure on $(0,1)$:

$$\begin{aligned} \|C_\Phi(R_n(f))\|^2_{\mathcal{A}^2_\mu} &= |R_n(f) \circ \Phi(+\infty)|^2 \\ &+ 4\int_{\mathbb{T}^\infty} \iint_\Omega \beta_h(\Re(s)) |(R_n(f) \circ \Phi)'_\chi(s)|^2 \, ds dm(\chi). \end{aligned}$$

We can point out that $R_n(f) \circ \Phi(+\infty) = 0$ for every $n \geq 2$ because $c_0 \geq 1$ and then $\Phi(+\infty) = +\infty$. Now by Proposition 4.3 from [9]:

$$\|C_\Phi(R_n(f))\|^2_{\mathcal{A}^2_\mu} = 4\int_{\mathbb{T}^\infty}\iint_\Omega \beta_h(\Re(s))|R_n(f)'_{\chi^{c_0}}(\Phi_\chi(s))|^2|\Phi'_\chi(s)|^2\,dsdm(\chi).$$

Now we make the following "change of variables": $w=\Phi_\chi(s)$, Φ_χ is not necessarily injective but we use the generalized formula of change of variable which involves the generalized Nevanlinna counting function (see [8] for example). We get:

$$\begin{aligned}&\iint_\Omega \beta_h(\Re(s))|R_n(f)'_{\chi^{c_0}}(\Phi_\chi(s))|^2|\Phi'_\chi(s)|^2\,ds\\ =\ &\iint_{\Phi_\chi(\Omega)}|R_n(f)'_{\chi^{c_0}}(s)|^2 N_{\beta_h,\Phi_\chi}(s)\,ds.\end{aligned}$$

$Im(\varphi)$ is bounded then there exists $A>0$ such that $|Im(\varphi)|<A$ and the same inequality holds for φ_χ for every $\chi\in\mathbb{T}^\infty$ instead of φ: indeed φ_χ is a vertical limit of φ. Now, when $s\in\Omega$, we have $Im(s)\in(0,1)$ and $w=\Phi_\chi(s)$ verifies $Im(w)=c_0Im(s)+Im(\varphi_\chi(s))$ so $Im(w)\in[-A,A+c_0]$. Finally if we denote $Q=(0,+\infty)\times(-A,A+c_0)$, we get

$$\begin{aligned}&\int_\Omega \beta_h(\Re(s))|R_n(f)'_{\chi^{c_0}}(\Phi_\chi(s))|^2|\Phi'_\chi(s)|^2\,ds\\ \le\ &\iint_Q |R_n(f)'_{\chi^{c_0}}(s)|^2 N_{\beta_h,\Phi_\chi}(s)\,ds.\end{aligned}$$

Let us fix $\theta>0$, we write $Q=Q_\theta\cup\tilde{Q}_\theta$ where $Q_\theta=(0,\theta)\times(-A,A+c_0)$ and $\tilde{Q}_\theta=(\theta,+\infty)\times(-A,A+c_0)$. We shall cut the previous integral in two parts. Let γ_θ defined by

$$\gamma_\theta=\sup_{\chi\in\mathbb{T}^\infty}\left\{\sup_{0<Re(s)<\theta}\frac{N_{\beta_h,\Phi_\chi}(s)}{\beta_h(\Re(s))}\right\}=\sup_{0<Re(s)<\theta}\frac{N_{\beta_h,\Phi}(s)}{\beta_h(\Re(s))}.$$

The second equality is true by Corollary 1. We have

$$\begin{aligned}&\int_{\mathbb{T}^\infty}\iint_{Q_\theta}|R_n(f)'_{\chi^{c_0}}(s)|^2 N_{\beta_h,\Phi_\chi}(s)\,dsdm(\chi)\\ \le\ &\gamma_\theta\int_{\mathbb{T}^\infty}\iint_{Q_\theta}|R_n(f)'_{\chi^{c_0}}(s)|^2\beta_h(\Re(s))\,dsdm(\chi).\end{aligned}$$

Now we take the supremum:

$$\begin{aligned}&\sup_{\|f\|_{\mathcal{A}^2_\mu}\le 1}\left\{\int_{\mathbb{T}^\infty}\iint_{Q_\theta}|R_n(f)'_{\chi^{c_0}}(s)|^2 N_{\beta_h,\Phi_\chi}(s)\,dsdm(\chi)\right\}\\ \le\ &\gamma_\theta\sup_{\|f\|_{\mathcal{A}^2_\mu}\le 1}\left\{\int_{\mathbb{T}^\infty}\iint_{Q_\theta}|R_n(f)'_{\chi^{c_0}}(s)|^2\beta_h(\Re(s))\,dsdm(\chi)\right\}\\ \le\ &\gamma_\theta\sup_{\|f\|_{\mathcal{A}^2_\mu}\le 1}\left\{\int_{\mathbb{T}^\infty}\iint_{Q_\theta}|f'_{\chi^{c_0}}(s)|^2\beta_h(\Re(s))\,dsdm(\chi)\right\}.\end{aligned}$$

In the last inequality, we use the fact that if f belongs to the unit ball of $\mathcal{A}^2_\mu$, $R_n(f)$ too. Finally, by using again the Littlewood-Paley formula with

the normalized Lebesgue measure on $(-A, A+c_0)$ and the fact that the map $\chi \to \chi^{c_0}$ is measure preserving we obtain:

$$\sup_{\|f\|_{\mathcal{A}^2_\mu} \leq 1} \left\{ 4 \int_{\mathbb{T}^\infty} \iint_{Q_\theta} |R_n(f)'_{\chi^{c_0}}(s)|^2 N_{\beta_h, \Phi_\chi}(s)\, ds dm(\chi) \right\} \leq (2A + c_0)\gamma_\theta.$$

Now we obtain an upper bound for the second integral. By Proposition 2:

$$\begin{aligned} & \int_{\mathbb{T}^\infty} \iint_{\tilde{Q}_\theta} |R_n(f)'_{\chi^{c_0}}(s)|^2 N_{\beta_h, \Phi_\chi}(s)\, ds dm(\chi) \\ \leq & \int_{\mathbb{T}^\infty} \iint_{\tilde{Q}_\theta} |R_n(f)'_{\chi^{c_0}}(s)|^2 \frac{\beta_h(\Re(s))}{c_0}\, ds dm(\chi) \qquad (\Delta) \end{aligned}$$

Recall (see [3], lem2) that for every $f \in \mathcal{H}^2$ with form (1) and every probability measure η on $\mathbb{R}$ we have for every $\sigma > 0$:

$$\int_{\mathbb{T}^\infty} \int_{\mathbb{R}} |f'_\chi(\sigma + it)|^2 d\eta(t) dm(\chi) = \sum_{k=2}^{+\infty} |a_k|^2 k^{-2\sigma} log^2(k), \qquad \forall \sigma > 0.$$

Hence, we get:

$$(\Delta) = \frac{2A + c_0}{c_0} \int_\theta^{+\infty} \sum_{k=n+1}^{+\infty} \beta_h(\sigma) |a_k|^2 k^{-2\sigma} log^2(k) d\sigma.$$

Let us fix $\varepsilon > 0$. We claim that there exists $K = K_\theta > 0$ such that for every $k \geq K$:

$$\int_\theta^{+\infty} \beta_h(\sigma) k^{-2\sigma} log^2(k) d\sigma \leq \varepsilon w_h(k).$$

Indeed, we know by [16] that $(w_h(k))$ is a decreasing sequence that decays more slowly than any negative power, so there exists $C = C_\theta > 0$ such that:

$$w_h(k) \geq C k^{-\theta} \quad \forall k \geq 1.$$

Then

$$\int_\theta^{+\infty} \beta_h(\sigma) \frac{k^{-2\sigma} log^2(k)}{w_h(k)} d\sigma \leq \int_\theta^{+\infty} \sigma k^{\theta - 2\sigma} C^{-1} log^2(k) d\sigma$$

because $\beta_h(\sigma) \leq \sigma$ by definition of β_h. Clearly the right-hand side of the inequality goes to 0 when k goes to infinity.

Finally when $n \geq K$, we obtain

$$\frac{2A + c_0}{c_0} \int_\theta^{+\infty} \sum_{k=n+1}^{+\infty} \beta_h(\sigma) |a_k|^2 k^{-2\sigma} log^2(k) d\sigma \leq \frac{2A + c_0}{c_0} \times \varepsilon$$

because we work with functions in $\mathcal{A}^2_\mu$ with norm less than 1. Now summing up the two integrals, we obtain that for $n \geq K$,

$$\|C_\Phi R_n\| \leq (2A + c_0)\gamma_\theta + \frac{2A + c_0}{c_0} \times \varepsilon \qquad \forall n \geq K.$$

At last,

$$\begin{aligned}\|C_\Phi\|_{A^2_\mu,e} &= \lim_{n\to+\infty}\|C_\Phi R_n\| \\ &\leq (2A+c_0)\gamma_\theta + \frac{2A+c_0}{c_0}\times\varepsilon.\end{aligned}$$

Now since ε and θ are arbitrary, we obtain the result. □

B Necessary conditions of compactness

First, we recall some facts from [1]. The reproducing kernel at $s \in \mathbb{C}_{1/2}$ associated to $\mathcal{A}^2_\mu$ is defined for every $w \in \mathbb{C}_{1/2}$ by

$$K_{\mu,s}(w) := \sum_{n=1}^{+\infty} \frac{n^{-\overline{s}-w}}{w_h(n)}.$$

Example. Let $\alpha > -1$. The reproducing kernel at $s \in \mathbb{C}_{1/2}$ associated to the space $\mathcal{A}^2_\alpha$ is denoted by $K_{\alpha,s}$ instead of $K_{\mu_\alpha,s}$ and we have for every $w \in \mathbb{C}_{1/2}$,

$$K_{\alpha,s}(w) = \sum_{n=1}^{+\infty}(1+\log(n))^{\alpha+1}n^{-\overline{s}-w}.$$

We saw in the previous section that the compactness is linked with the behaviour of Φ near to the imaginary axis. Then, as in [3], we shall work with partial reproducing kernels which are defined on $\mathbb{C}_+$ and not only on $\mathbb{C}_{1/2}$.

Definition. Let $s \in \mathbb{C}_+$ and $l \geq 1$. On $\mathcal{A}^2_\mu$, we define the partial reproducing kernel of order l at $s \in \mathbb{C}_+$ by

$$K^l_{\mu,s}(w) := \sum_{\substack{n\geq 1\\ p^+(n)\leq p_l}} \frac{n^{-\overline{s}-w}}{w_h(n)}$$

where $p^+(n)$ is the greatest prime divisor of n.

We claim that these partial reproducing kernels are defined on $\mathbb{C}_+$: indeed we know by [16] that the weight $(w_h(n))$ decreases more slowly than any negative power of n so if $\varepsilon > 0$, there exists $C > 0$ such that $w_h(n) > Cn^{-\varepsilon}$ for every $n \geq 1$. Then

$$\sum_{\substack{n\geq 1\\ p^+(n)\leq l}} \frac{n^{-\Re(s)-\Re(w)}}{w_h(n)} \leq \frac{1}{C}\sum_{\substack{n\geq 1\\ p^+(n)\leq l}} \frac{1}{n^{\Re(s)+\Re(w)-\varepsilon}} = \frac{1}{C}\prod_{i=1}^{l}\left(1-p_i^{-\Re(s)-\Re(w)+\varepsilon}\right)^{-1}.$$

For $s \in \mathbb{C}_+$, this quantity is then bounded for any $w \in \mathbb{C}_\varepsilon$ and any $\varepsilon > 0$ and so the claim is proved.

We shall need the following proposition which follows easily from [3],Prop.5.

Proposition 4. *Let Φ a c_0-symbol with $c_0 \geq 1$ and $l \geq 1$. Assume that $\Phi(s) = c_0 s + \sum_{n=1}^{+\infty} c_n n^{-s}$ where $c_n = 0$ when $p^+(n) > l$. Then $C^*_\Phi(K^l_{\mu,s}) = K^l_{\mu,\Phi(s)}$.*

Lemma 2. *On the unit ball $B_{\mathcal{A}^2_\mu}$ of $\mathcal{A}^2_\mu$, the weak topology is the topology of uniform convergence on every half-plane $\mathbb{C}_{1/2+\varepsilon}$ with $\varepsilon > 0$.*

Proof. Let $(f_n) \subset B_{\mathcal{A}^2_\mu}$ which converges weakly to 0. Let $\varepsilon > 0$, by the result on the point evaluation from [1], there exists a constant C_ε such that

$$|f_n(s)| \leq C_\varepsilon \|f_n\| \quad \text{for every s } \in \mathbb{C}_{1/2+\varepsilon}$$

and then the sequence (f_n) is uniformly bounded on $\mathbb{C}_{1/2+\varepsilon}$. Now by the Montel's theorem for $\mathcal{H}^\infty$ (see [2],Lem18), there exists a subsequence which converges uniformly on $\mathbb{C}_{1/2+\varepsilon}$. It suffices to show now that 0 is the unique cluster point, but the point evaluation is bounded and (f_n) converges weakly to 0 and so 0 is the unique cluster point.

Conversely let $(f_n) \subset B_{\mathcal{A}^2_\mu}$ which converges uniformly to 0 on $\mathbb{C}_{1/2+\varepsilon}$ for every $\varepsilon > 0$. There exists at least a subsequence which converges weakly to $f \in \mathcal{A}^2_\mu$. But the point evaluation is bounded for every $s \in \mathbb{C}_{1/2}$ and (f_n) converges uniformly to 0 on $\mathbb{C}_{1/2+\varepsilon}$ for every $\varepsilon > 0$ so $f = 0$. □

Theorem 4. *Let Φ a c_0-symbol with $c_0 \geq 1$ and $l \geq 1$. Assume that $\Phi(s) = c_0 s + \sum_{n=1}^{+\infty} c_n n^{-s}$ where $c_n = 0$ when $p^+(n) > l$. We have:*

$$\|C_\Phi\|_{\mathcal{A}^2_\mu, e} \geq \limsup_{\Re(s)\to 0} \frac{\|K^l_{\mu,\Phi(s)}\|_{\mathcal{A}^2_\mu}}{\|K^l_{\mu,s}\|_{\mathcal{A}^2_\mu}}.$$

Proof. Let $l \geq 1$. For $s \in \mathbb{C}_+$, we define the following family of functions:

$$k_s(w) := \frac{K^l_{\mu,s}(w)}{\|K^l_{\mu,s}\|_{\mathcal{A}^2_\mu}}, \qquad \text{where } w \in \mathbb{C}_+.$$

This family is contained in $B_{\mathcal{A}^2_\mu}$ and converges uniformly to 0 on $\mathbb{C}_{1/2+\varepsilon}$ for every $\varepsilon > 0$: indeed we have

$$\lim_{\Re(s)\to 0} \|K^l_{\mu,s}\|^2_{\mathcal{A}^2_\mu} = \lim_{\Re(s)\to 0} K^l_{\mu,s}(s) = +\infty$$

and for every $w \in \mathbb{C}_{1/2+\varepsilon}$:

$$|K^l_{\mu,s}(w)| \leq \sum_{\substack{n\geq 1 \\ p^+(n)\leq l}} \frac{n^{-1/2-\varepsilon}}{w_h(n)} < +\infty$$

because the partial reproducing kernels are defined on $\mathbb{C}_+$.

Let $K \in \mathcal{K}(\mathcal{A}^2_\mu)$, we point out that $K^* \in \mathcal{K}(\mathcal{A}^2_\mu)$ as well and by the previous lemma we know that (k_s) converges weakly to 0 so $K^*(k_s)$ is norm convergent to 0. Then

$$\|C_\Phi - K\| = \|C^*_\Phi - K^*\| \geq \|C^*_\Phi(k_s) - K^*(k_s)\|_{\mathcal{A}^2_\mu} \geq \|C^*_\Phi(k_s)\|_{\mathcal{A}^2_\mu} - \|K^*(k_s)\|_{\mathcal{A}^2_\mu}$$

and then

$$\|C_\Phi - K\| \geq \limsup_{\Re(s)\to 0} \|C^*_\Phi(k_s)\|_{\mathcal{A}^2_\mu}.$$

Now with Proposition 4, and the fact that the inequality is true for every $K \in \mathcal{K}(\mathcal{A}^2_\mu)$, we obtain the result. □

Corollary 3. *Let Φ a c_0-symbol with $c_0 \geq 1$ and $l \geq 1$. Assume that $\Phi(s) = c_0 s + \sum_{n=1}^{+\infty} c_n n^{-s}$ where $c_n = 0$ when $p^+(n) > l$. Assume that $C_\Phi \in \mathcal{K}(\mathcal{A}^2_\mu)$, then:*

$$\lim_{\Re(s)\to 0} \frac{\|K^l_{\mu,\Phi(s)}\|_{\mathcal{A}^2_\mu}}{\|K^l_{\mu,s}\|_{\mathcal{A}^2_\mu}} = 0.$$

In particular if $C_\Phi \in \mathcal{A}^2_\alpha$ with $\alpha > -1$, it implies that

$$\lim_{\Re(s)\to 0} \frac{\Re(s)}{\Re(\Phi(s))} = 0.$$

Remark. The result concerning $\mathcal{A}^2_\alpha$ is already known (see [3]).

C A criterion of compactness

In this section, we still work on $\mathcal{A}^2_\mu$ where μ is a probability measure such that $d\mu(\sigma) = hd\sigma$ with h a positive continuous function. Recall that

$$\beta_h(\sigma) = \int_0^\sigma (\sigma - u)h(u)du.$$

In the spirit of [11] (see p12) we introduce a new condition: we say that a weight β verifies condition (κ) if the function $G = G_\beta$ defined by $G(\sigma) = \beta(\sigma)/\sigma$ (where $\sigma > 0$) is such that

$$\lim_{\eta\to 0^+} \limsup_{\sigma\to 0^+} \frac{G(\eta\sigma)}{G(\sigma)} = 0.$$

Examples.

(i) When h is a non decreasing function then (κ) is fullfilled for β_h: indeed in this case we extend continuously G at 0 by $G(0) = 0$ and it suffices to check that G is a convex function but it is easy to see that

$$G'(\sigma) = \frac{\int_0^\sigma uh(u)du}{\sigma^2}$$

and

$$G''(\sigma) = \frac{2\int_0^\sigma u(h(\sigma) - h(u))du}{\sigma^3}.$$

(ii) Let $\alpha > -1$, β_α verifies (κ) because $G(\sigma) \approx \sigma^{\alpha+1}$ when σ is small enough.

Proposition 5. *Let Φ a c_0-symbol with $c_0 \geq 1$. Assume that β_h verifies (κ) then if*

$$\lim_{\Re(s)\to 0} \frac{\Re(s)}{\Re(\Phi(s))} = 0$$

then

$$N_{\beta_h,\Phi}(s) = o(\beta_h(\Re(s))) \text{ when } \Re(s) \to 0,$$

hence C_Φ is compact on $\mathcal{A}^2_\mu$ if $Im(\varphi)$ is bounded.

Proof. Let $s \in \mathbb{C}_+$, we have

$$\begin{aligned} N_{\beta_h,\Phi}(s) &= \sum_{\substack{a\in\mathbb{C}_+\\ \Phi(a)=s}} \beta_h(\Re(a)) \\ &= \sum_{\substack{a\in\mathbb{C}_+\\ \Phi(a)=s}} G(\Re(a))\Re(a). \end{aligned}$$

Let $\varepsilon > 0$, we know that

$$\lim_{\Re(s)\to 0} \frac{\Re(s)}{\Re(\Phi(s))} = 0$$

and then by Condition (κ), there exists $\delta > 0$ such that

$$G(\Re(s)) \leq \varepsilon G(\Re(\Phi(s)))$$

when $\Re(s) < \delta$. Now, thanks to the Schwarz's lemma if $\Phi(a) = s$ and $\Re(s) < \delta$ then $\Re(a) < \delta$ too. Consequently

$$\begin{aligned} N_{\beta_h,\Phi}(s) &= \sum_{\substack{a\in\mathbb{C}_+\\ \Phi(a)=s}} G(\Re(a))\Re(a) \\ &\leq \varepsilon \sum_{\substack{a\in\mathbb{C}_+\\ \Phi(a)=s}} G(\Re(\Phi(a)))\Re(a) \\ &= \varepsilon G(\Re(s)) \sum_{\substack{a\in\mathbb{C}_+\\ \Phi(a)=s}} \Re(a) \\ &= \varepsilon G(\Re(s)) N_\Phi(s) \end{aligned}$$

for every $s \in \mathbb{C}_+$ such that $\Re(s) < \delta$. Now by Littlewood's inequality we get:

$$N_{\beta_h,\Phi}(s) \leq \frac{\varepsilon G(\Re(s))\Re(s)}{c_0} = \frac{\varepsilon \beta_h(\Re(s))}{c_0}$$

when $\Re(s) < \delta$ and this proves the result. □

Theorem 5. *Let $\alpha > -1$, Φ a c_0-symbol with $c_0 \geq 1$ and $l \geq 1$. Assume that $\Phi(s) = c_0 s + \sum_{n=1}^{+\infty} c_n n^{-s}$ where $c_n = 0$ when $p^+(n) > l$ and that $Im(\varphi)$ is bounded, then the following assertions are equivalent:*

(i) C_Φ is compact on $\mathcal{A}^2_\alpha$.

(ii) $\lim_{\Re(s)\to 0} \frac{\Re(s)}{\Re(\Phi(s))} = 0.$

(iii) $N_{\alpha,\Phi}(s) = o((\Re(s))^{\alpha+2})$ when $\Re(s)$ goes to 0.

Proof. It suffices to use Corollary 3, Proposition 5 and Corollary 2. □

Remark. We point out that the condition (ii) does not depend on α.

Definition. Let $\Phi : \mathbb{C}_+ \to \mathbb{C}_+$ be an analytic function and $k \in \mathbb{N}$, we say that Φ is $k-$valent if for every $w \in \mathbb{C}_+$, there exists at most k solutions to $\Phi(s) = w$. If Φ is $k-$valent for some $k \in \mathbb{N}$ then we shall say that Φ is finite-valent.

Theorem 6. *Let $l \geq 1$ and Φ be a c_0-symbol with $c_0 \geq 1$ such that $\Phi(s) = c_0 s + \sum_{n=1}^{+\infty} c_n n^{-s}$ where $c_n = 0$ if $p^+(n) > l$. Assume that $Im(\varphi)$ is bounded and that Φ is finite-valent, then the following assertions are equivalent:*

(i) C_Φ *is compact on* $\mathcal{H}^2$.

(ii) $\lim\limits_{\Re(s)\to 0} \dfrac{\Re(s)}{\Re(\Phi(s))} = 0.$

(iii) $N_\Phi(s) = o(\Re(s))$ *when* $\Re(s)$ *goes to* 0.

Proof. We only have to proof that $(ii) \Rightarrow (iii)$: the other implications have been proven in [3]. We assume that Φ is k-valent for some $k \geq 0$. Let $\varepsilon > 0$ and $s \in \mathbb{C}_+$, as in the proof of Proposition 5 we can show that there exists $\delta > 0$ such that for every $\in \mathbb{C}_+$ with $\Re(s) < \delta$:

$$N_\Phi(s) \leq \varepsilon \Re(s) \sum_{\substack{a\in\mathbb{C}_+ \\ \Phi(a)=s}} 1 \leq \varepsilon k \Re(s).$$

□

Remark. Let $l \geq 1$ and Φ a c_0-symbol with $c_0 \geq 1$ such that $\Phi(s) = c_0 s + \sum_{n=1}^{+\infty} c_n n^{-s}$ where $c_n = 0$ if $p^+(n) > l$. If $Im(\varphi)$ is bounded and C_φ is compact on $\mathcal{H}^2$ then

$$\lim_{\Re(s)\to 0} \frac{\Re(s)}{\Re(\Phi(s))} = 0$$

and consequently C_Φ is compact on $\mathcal{A}^2_\alpha$ for any $\alpha > -1$.

Questions. Is the previous implication true for general c_0-symbols?

5 Compactness and Carleson measures

In this section, we are going to obtain a sufficient condition of compactness for C_Φ with a "Carleson-measure" condition. We still work on $\mathcal{A}^2_\mu$ where μ is a probability measure on $(0, +\infty)$ such that $0 \in Supp(\mu)$.

When Φ is a c_0-symbol with $c_0 \geq 1$, it is shown in [2] that Φ admits radial limits and so if λ is the Lebesgue measure on $\mathbb{R}$,

$$\Phi^*(it) := \lim_{\sigma\to 0^+} \Phi(\sigma + it) \text{ exists } \lambda\text{-almost everywhere.}$$

Notation. Let $t \in \mathbb{R}$ and $h > 0$. We define the Carleson window centered at it and of size h by $H(t,h) := \{s \in \mathbb{C}_+,\ |s - it| < h\}$.

Definition 2.

(i) We denote λ_Φ the pullback measure of λ by Φ^*, *ie*, for every open set $\Omega \subset \mathbb{C}_+$:

$$\lambda_\Phi(\Omega) := \lambda(\{t \in \mathbb{R}, \Phi^*(it) \in \Omega\}).$$

The Carleson function associated to λ_Φ is then defined by

$$\rho_\Phi(h) := \sup_{t\in\mathbb{R}} \lambda_\Phi(H(t,h)).$$

(ii) Let $\lambda_\mu = \lambda \otimes \mu$. We denote by $\lambda_{\mu,\Phi}$ the pullback measure of λ_μ by Φ, *ie*, for every open set $\Omega \subset \mathbb{C}_+$:

$$\lambda_{\mu,\Phi}(\Omega) := \lambda_\mu(\{s \in \mathbb{C}_+, \Phi(s) \in \Omega\}).$$

The Carleson function associated to λ_Φ is then defined by

$$\rho_{\mu,\Phi}(h) := \sup_{t\in\mathbb{R}} \lambda_{\mu,\Phi}(H(t,h)).$$

In the sequel we shall denote β instead of β_h. The next theorem is the main result of this section.

Theorem 7. *Let Φ be a c_0-symbol with $c_0 \geq 1$. There exists $K > 0$ (independent of Φ) such that:*

(i)

$$\sup_{s\in H(t,h/2)} N_\Phi(s) \leq K \cdot \lambda_\Phi\big(H(t, 2c_0 h)\big) \quad \textit{for every } h \textit{ small enough.}$$

In particular:

$$\sup_{\substack{s\in\mathbb{C}_+\\ \Re(s)<h/2}} N_\Phi(s) \leq K \cdot \rho_\Phi\big(2c_0 h\big) \quad \textit{for every } h \textit{ small enough.}$$

(ii)

$$\sup_{s\in H(t,h/2)} N_{\beta,\Phi}(s) \leq K \cdot \lambda_{\mu,\Phi}\big(H(t, 2c_0 h)\big) \quad \textit{for every } h \textit{ small enough.}$$

In particular:

$$\sup_{\substack{s\in\mathbb{C}_+\\ \Re(s)<h/2}} N_{\beta,\Phi}(s) \leq K \cdot \rho_{\mu,\Phi}\big(2c_0 h\big) \quad \textit{for every } h \textit{ small enough.}$$

Corollary 4. *Let Φ be a c_0-symbol with $c_0 \geq 1$ such that $Im(\varphi)$ is bounded on $\mathbb{C}_+$. Then there exists $C > 0$ such that:*

(i)

$$\|C_\Phi\|_{\mathcal{H}^2,e} \leq C \limsup_{h\to 0} \frac{\rho_\Phi(h)}{h}.$$

In particular if $\rho_\Phi(h) = o(h)$ when $h \to 0$ then C_Φ is compact on $\mathcal{H}^2$.

(ii)

$$\|C_\Phi\|_{\mathcal{A}^2_\mu,e} \leq C \limsup_{h\to 0} \frac{\rho_{\mu,\Phi}(h)}{\beta(h)}.$$

In particular if $\rho_{\mu,\Phi}(h) = o(\beta(h))$ when $h \to 0$ then C_Φ is compact on $\mathcal{A}^2_\mu$.

Example. When $\mu = \mu_\alpha$, we denote N_α (resp. $\rho_{\alpha,\Phi}$) the corresponding generalized Nevanlinna counting function (resp. the Carleson function). We know that in that case, $\beta_\alpha(\sigma) \approx \sigma^{\alpha+2}$ when σ is small and then if $Im(\varphi)$ is bounded:

$$\|C_\Phi\|_{e,\mathcal{A}^2_\alpha} \leq C \limsup_{h\to 0} \frac{\rho_{\alpha,\Phi}(h)}{h^{\alpha+2}}.$$

Proof of Cor.4. We only give the proof of (i). Let $s \in \mathbb{C}_+$ with $\Re(s)$ small enough, we have

$$N_\Phi(s) \leq \sup_{s' \in H(Im(s), 2\Re(s))} N_\Phi(s') \leq K \cdot \rho_\Phi\big(8c_0\Re(s)\big)$$

by theorem 7. Then we obtain

$$\frac{N_\Phi(s)}{\Re(s)} \leq 8c_0 K \cdot \frac{\rho_\Phi\big(8c_0\Re(s)\big)}{8c_0\Re(s)}$$

and so the result. □

In order to prove Theorem 7, we need to recall some facts about Carleson measures and the classical Nevanlinna counting function on the unit disk.

Let $\phi : \mathbb{D} \to \mathbb{D}$ be an analytic self-map. The Nevanlinna counting function of ϕ is defined for $w \in \phi(\mathbb{D})$ by

$$N_\phi(w) = \sum_{\substack{z \in \mathbb{D} \\ \phi(z) = w}} \log(1/|z|).$$

When $w \notin \phi(\mathbb{D})$, $N_\phi(w) = 0$ by convention.

We denote by m_ϕ the pullback measure of the Lebesgue measure on the unit circle by ϕ^*, the boundary values function of ϕ (which exists because $\phi \in H^\infty(\mathbb{D})$). By abusing the notation, we still denote ϕ instead of ϕ^*.

For $\eta \in \mathbb{T}$ and $h > 0$, the Carleson window $S(\eta, h)$ is defined by

$$S(\eta, h) := \{z \in \mathbb{D},\ |z - \eta| < h\}.$$

We shall make a crucial use of the following result (see [14],Th1.1):

Let $\phi : \mathbb{D} \to \mathbb{D}$ an analytic self-map. For every $\beta > 1$, there exists a universal constant $C_\beta > 0$ such that:

$$(1/C_\beta) m_\phi(S(\xi, h)) \leq \sup_{z \in S(\xi, h)} N_\phi(z) \leq C_\beta m_\phi(S(\xi, \beta h)) \qquad \circledast\circledast$$

for every $0 < h < (1 - |\phi(0)|)/\beta$ and every $\xi \in \mathbb{T}$.

Proof of Th.7(i).

▷ Step one: We prove the result when $t = 0$.

We shall use a family of functions $(\psi_\xi)_{\xi > 0}$ where ψ_ξ is defined by

$$\begin{cases} \psi_\xi : & \mathbb{C}_+ & \longrightarrow & \mathbb{D} \\ & s & \longmapsto & \dfrac{s - \xi}{s + \xi}. \end{cases}$$

Claim. For every $h > 0$, $H(0, h/2) \subset \psi_{c_0\xi}^{-1}(S(-1, h/\xi))$.

Let us prove this fact:

$$s \in \psi_{c_0\xi}^{-1}(S(-1, h/\xi)) \quad \Leftrightarrow \quad \psi_{c_0\xi}(s) \in S(-1, h/\xi) \quad \Leftrightarrow \quad \left|\frac{s - c_0\xi}{s + c_0\xi} + 1\right| < \frac{h}{\xi}$$

$$\Leftrightarrow \quad \left|\frac{2s}{s + c_0\xi}\right| < \frac{h}{\xi} \quad \Leftrightarrow \quad |s| < \frac{h}{2}\left|\frac{s + c_0\xi}{\xi}\right|.$$

Now if $s \in H(0, h/2)$ then $|s| < \frac{h}{2}$ and it suffices to remark that for every $\xi > 0$,

$$\frac{|s + c_0\xi|}{\xi} \geq \frac{\Re(s + c_0\xi)}{\xi} \geq c_0 \geq 1,$$

and this justifies the claim.

Let $h > 0$ and $s \in H(0, h/2)$. Then $w_\xi := \psi_{c_0\xi}(s) \in S(-1, h/\xi)$. Assume that $s \in \Phi(\mathbb{C}_+)$ (if it is not the case, there is nothing to prove) and let $s_1, \ldots, s_n \in \mathbb{C}_+$ such that $\Phi(s_i) = s$ for every $i = 1, \ldots, n$.

For $\xi > 0$, we define the analytic self-map $\Theta_\xi : \mathbb{D} \to \mathbb{D}$ by $\Theta_\xi = \psi_{c_0\xi} \circ \Phi \circ \psi_\xi^{-1}$. We note that $\Theta_\xi(\psi_\xi(s_i)) = \psi_{c_0\xi}(\Phi(s_i)) = \psi_{c_0\xi}(s) = w_\xi$ and so

$$\sum_{i=1}^{n} \log(1/|\psi_\xi(s_i)|) \leq N_{\Theta_\xi}(w_\xi).$$

For $i \in \{1, \ldots, n\}$ we have

$$\log(1/|\psi_\xi(s_i)|) = \frac{1}{2} \log \left| \frac{|s_i|^2 + 2\xi\Re(s_i) + \xi^2}{|s_i|^2 - 2\xi\Re(s_i) + \xi^2} \right| = \frac{1}{2} \log \left| 1 + \frac{4\xi\Re(s_i)}{|s_i|^2 - 2\xi\Re(s_i) + \xi^2} \right|.$$

Let $\varepsilon > 0$. For ξ large enough we obtain that for every $i \in \{1, \ldots, n\}$:

$$2(1 - \varepsilon)\frac{\Re(s_i)}{\xi} \leq \log(1/|\psi_\xi(s_i)|).$$

And so

$$2(1 - \varepsilon)\sum_{i=1}^{n} \frac{\Re(s_i)}{\xi} \leq N_{\Theta_\xi}(w_\xi).$$

Now using $\circledast\circledast$ with $\beta = 2$ and remembering that $w_\xi \in S(-1, h/\xi)$ for every $\xi > 0$, we obtain

$$N_{\Theta_\xi}(w_\xi) \leq C_2 m_{\Theta_\xi}(S(-1, 2h/\xi))$$

for every $0 < h/\xi < (1 - |\Theta_\xi(0)|)/2$. Point out that

$$\Theta_\xi(0) = \psi_{c_0\xi}(\Phi(\xi)) = \frac{\varphi(\xi)}{2c_0\xi + \varphi(\xi)}$$

and this quantity goes to 0 when ξ goes to infinity. So we define

$$m_0 = \inf_{\xi > 1} \xi(1 - |\Theta_\xi(0)|)/2 > 0.$$

It is clear that $m_0 \leq 1/2$. Let us point out that $m_0 = m_0(\varphi) = m_0(\varphi(\cdot + i\tau))$ for any $\tau \in \mathbb{R}$ (this remark will be useful in step 2). When $h < m_0$, we get

$$N_{\Theta_\xi}(w_\xi) \leq C_2 m_{\Theta_\xi}(S(-1, 2h/\xi))$$

and

$$\begin{aligned} m_{\Theta_\xi}(S(-1, 2h/\xi)) &= m(\{\eta \in \mathbb{T}, \Theta_\xi(\eta) \in S(-1, 2h/\xi)\}) \\ &= m(\{\eta \in \mathbb{T}, \Psi_{c_0\xi} \circ \Phi^* \circ \psi_\xi^{-1}(\eta) \in S(-1, 2h/\xi)\}) \\ &= \int_{\{\Psi_{c_0\xi} \circ \Phi^* \circ \psi_\xi^{-1}(e^{i\theta}) \in S(-1, 2h/\xi)\}} \frac{d\theta}{2\pi}. \end{aligned}$$

Now we make the following change of variables $iu = \psi_\xi^{-1}(e^{i\theta})$:

$$\begin{aligned} m_{\Theta_\xi}(S(-1,2h/\xi)) &= \int_{\{\Phi^*(iu)\in\Psi_{c_0\xi}^{-1}(S(-1,2h/\xi))\}} \frac{\xi}{\pi(\xi^2+u^2)}du \\ &= \int_{\{|\Phi^*(iu)|<h\frac{|\Phi^*(iu)+c_0\xi|}{\xi}\}} \frac{\xi}{\pi(\xi^2+u^2)}du. \end{aligned}$$

Then for $\xi > 1$ large enough we obtain

$$2(1-\varepsilon)\sum_{i=1}^n \frac{Re(s_i)}{\xi} \leq C_2 \int_{\{|\Phi^*(iu)|<h\frac{|\Phi^*(iu)+c_0\xi|}{\xi}\}} \frac{\xi}{\pi(\xi^2+u^2)}du$$

and then

$$2(1-\varepsilon)\sum_{i=1}^n \Re(s_i) \leq C_2 \int_{\{|\Phi^*(iu)|<h\frac{|\Phi^*(iu)+c_0\xi|}{\xi}\}} \frac{du}{\pi}.$$

But if $|\Phi^*(iu)| < h\frac{|\Phi^*(iu)+c_0\xi|}{\xi}$ then $|\Phi^*(iu)| < \frac{c_0h}{(1-h/\xi)} < \frac{c_0h}{(1-h)} < 2c_0h$ since $h < m_0 \leq 1/2$. Then

$$2(1-\varepsilon)\sum_{i=1}^n Re(s_i) \leq C_2 m_\Phi(H(0,2c_0h)).$$

Since n is arbitrary we get:

$$2(1-\varepsilon)N_\Phi(s) \leq C_2 m_\Phi(H(0,2c_0h)).$$

for any $\varepsilon > 0$ and the first step is proved.

$\triangleright$ Step two: we prove now the result when $t \neq 0$.
We have

$$\sup_{s\in H(c_0t,h/2)} N_\Phi(s) = \sup_{s\in H(0,h/2)} N_\Phi(s+ic_0t).$$

But if $\tilde{\Phi}(s) := c_0s + \varphi(s+it) = \Phi(s+it) - ic_ot$, we obtain for $s \in H(0,h/2)$:

$$\begin{aligned} N_\Phi(s+ic_0t) &= \sum_{\substack{a\in\mathbb{C}_+ \\ \Phi(a)=s+ic_0t}} \Re(a) = \sum_{\substack{a\in\mathbb{C}_+ \\ c_0(a-it)+\varphi(it+a-it)=s}} \Re(a-it) \\ &= \sum_{\substack{a'\in\mathbb{C}_+ \\ \tilde{\Phi}(a')=s}} \Re(a') = N_{\tilde{\Phi}}(s). \end{aligned}$$

Applying step one to $\tilde{\Phi}$, we get

$$\sup_{s\in H(0,h/2)} N_\Phi(s+ic_0t) = \sup_{s\in H(0,h/2)} N_{\tilde{\Phi}}(s) \leq K\lambda_{\tilde{\Phi}}(H(0,2c_0h))$$

for every h small enough. Now it suffices to remark that

$$\begin{aligned} \lambda_{\tilde{\Phi}}(H(0,2c_0h)) &= \lambda(\{t'\in\mathbb{R},\ \tilde{\Phi}^*(it')\in H(0,2c_0h)\}) \\ &= \lambda(\{t'\in\mathbb{R},\ |\Phi^*(it+it')-ic_0t| < 2c_0h\}) \\ &= \lambda(\{t''\in\mathbb{R},\ |\Phi^*(it'')-ic_0t| < 2c_0h\}) \\ &= \lambda_\Phi(H(c_0t,2c_0h)). \end{aligned}$$

$\square$

Proof of Th.7(ii). By Lemma 1, we know that

$$N_{\beta_h,\Phi}(s) = \int_0^{\Re(s)} N_{\Phi_u}(s)h(u)du.$$

Now in the proof of (i), we point out that we can choose the same m_0 for Φ_u for every $u \geq 0$ and so for $s \in H(t, \tilde{h}/2)$ with $\tilde{h} < m_0$ we have

$$N_{\beta_h,\Phi}(s) \leq K \int_0^{\Re(s)} \lambda_{\Phi_u}(H(t, 2c_0\tilde{h}))h(u)du$$

$$\leq K \int_0^{+\infty} \lambda(\{t' \in \mathbb{R}, \Phi(u+it') \in H(t, 2c_0\tilde{h})\})h(u)du = K\lambda_{\mu,\Phi}(H(t, 2c_0\tilde{h})$$

thanks to the Fubini's theorem. □

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