



**HAL**  
open science

# Fractional heat equations with subcritical absorption with initial data measure

Huyuan Chen, Laurent Veron, Ying Wang

► **To cite this version:**

Huyuan Chen, Laurent Veron, Ying Wang. Fractional heat equations with subcritical absorption with initial data measure. 2015. hal-00937420v2

**HAL Id: hal-00937420**

**<https://hal.science/hal-00937420v2>**

Preprint submitted on 25 May 2015 (v2), last revised 9 Sep 2015 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Fractional heat equations with subcritical absorption with initial data measure

Huyuan Chen<sup>1</sup>

Department of Mathematics, Jiangxi Normal University,  
Nanchang 330022, China

Laurent Véron<sup>2</sup>

Laboratoire de Mathématiques et Physique Théorique  
Université François Rabelais, Tours, France

Ying Wang<sup>3</sup>

Departamento de Ingeniería Matemática  
Universidad de Chile, Chile

## Abstract

We study the existence and uniqueness of a weak solutions to (F)  $\partial_t u + (-\Delta)^\alpha u + h(t, u) = 0$  in  $(0, \infty) \times \mathbb{R}^N$ , with initial condition  $u(0, \cdot) = \nu$  in  $\mathbb{R}^N$ , where  $N \geq 2$ , the operator  $(-\Delta)^\alpha$  is the fractional Laplacian with  $\alpha \in (0, 1)$ ,  $\nu$  is a bounded Radon measure and  $h : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function satisfying a subcritical integrability condition.

In particular, if  $h(t, u) = t^\beta u^p$  with  $\beta > -1$  and  $0 < p < p_\beta^* := 1 + \frac{2\alpha(1+\beta)}{N}$ , we prove that there exists a unique solution  $u_k$  to (F) with  $\nu = k\delta_0$ , where  $\delta_0$  is the Dirac mass at the origin. We obtain that  $u_k \rightarrow \infty$  in  $(0, \infty) \times \mathbb{R}^N$  as  $k \rightarrow \infty$  for  $p \in (0, 1]$  and the limit of  $u_k$  exists as  $k \rightarrow \infty$  when  $1 < p < p_c$ , we denote it by  $u_\infty$ . When  $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} := p_\beta^{**} < p < p_\beta^*$ ,  $u_\infty$  is the minimal self-similar solution of  $(F)_\infty \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0$  in  $(0, \infty) \times \mathbb{R}^N$  with the initial condition  $u(0, \cdot) = 0$  in  $\mathbb{R}^N \setminus \{0\}$  and it satisfies  $u_\infty(0, x) = 0$  for  $x \neq 0$ . While if  $1 < p < p_\beta^{**}$ , then  $u_\infty \equiv U_p$  where  $U_p$  is the maximal solution of the differential equation  $y' + t^\beta y^p = 0$  on  $\mathbb{R}_+$ .

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Linear estimates</b>	<b>8</b>
2.1	The Marcinkiewicz spaces . . . . .	8
2.2	The non-homogeneous problem . . . . .	12

---

<sup>1</sup>chenhuyuan@yeah.net

<sup>2</sup>Laurent.Veron@lmpt.univ-tours.fr

<sup>3</sup>yingwang00@126.com

<b>3</b>	<b>Proof of Theorem 1.1</b>	<b>18</b>
<b>4</b>	<b>Dirac mass as initial data</b>	<b>27</b>
<b>5</b>	<b>Self-similar and very singular solutions</b>	<b>30</b>
5.1	The case $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$ . . . . .	32
5.2	The case $1 < p < 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$ . . . . .	39
5.3	The self-similar equation . . . . .	40

**Key words:** Fractional heat equation, Radon measure, Dirac mass, Self-similar solution, Very singular solution

**MSC2010:** 35R06, 35K05, 35R11

## 1 Introduction

Let  $h : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function and  $Q_\infty = (0, \infty) \times \mathbb{R}^N$  with  $N \geq 2$ . The first object of this paper is to consider existence and uniqueness of weak solutions to fractional heat equations

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h(t, u) &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= \nu & \text{in } \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where  $\nu$  belongs to the space  $\mathfrak{M}^b(\mathbb{R}^N)$  of bounded Radon measures in  $\mathbb{R}^N$  and  $(-\Delta)^\alpha$  ( $0 < \alpha < 1$ ) is the fractional Laplacian defined by

$$(-\Delta)^\alpha u(t, x) = \lim_{\epsilon \rightarrow 0^+} (-\Delta)_\epsilon^\alpha u(t, x),$$

where, for  $\epsilon > 0$ ,

$$(-\Delta)_\epsilon^\alpha u(t, x) = \int_{\mathbb{R}^N} \frac{u(t, x) - u(t, z)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz$$

and

$$\chi_\epsilon(r) = \begin{cases} 0 & \text{if } r \in [0, \epsilon], \\ 1 & \text{if } r > \epsilon. \end{cases}$$

In a pioneering work, Brezis and Friedman [5] have studied semilinear the heat equation with measure as initial data

$$\begin{aligned} \partial_t u - \Delta u + u^p &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= k\delta_0 & \text{in } \mathbb{R}^N, \end{aligned} \tag{1.2}$$

where  $k > 0$  and  $\delta_0$  is the Dirac mass at the origin. They proved that if  $1 < p < (N + 2)/N$ , then for every  $k > 0$  there exists a unique solution  $u_k$  to (1.2). When  $p \geq (N + 2)/N$ , problem (1.2) has no solution and even more, they proved that no nontrivial solution of the above equation vanishing on  $\mathbb{R}^N \setminus \{0\}$  at  $t = 0$  exists. When  $1 < p < 1 + \frac{2}{N}$ , Brezis, Peletier and Terman used a dynamical system technique in [6] to prove the existence of a *very singular solution*  $u_s$  to

$$\partial_t u - \Delta u + u^p = 0 \quad \text{in } Q_\infty, \quad (1.3)$$

vanishing at  $t = 0$  on  $\mathbb{R}^N \setminus \{0\}$ . This function  $u_s$  is self-similar, i.e. expressed under the form

$$u_s(t, x) = t^{-\frac{1}{p-1}} f\left(\frac{|x|}{\sqrt{t}}\right), \quad (1.4)$$

and  $f$  is uniquely determined by the following conditions

$$\begin{aligned} f'' + \left(\frac{N-1}{\eta} + \frac{1}{2}\eta\right) f' + \frac{1}{p-1} f - f^p &= 0 \quad \text{on } \mathbb{R}_+ \\ f > 0 \quad \text{and } f \text{ is smooth on } \mathbb{R}_+ & \\ f'(0) = 0 \quad \text{and } \lim_{\eta \rightarrow \infty} \eta^{\frac{2}{p-1}} f(\eta) &= 0. \end{aligned} \quad (1.5)$$

Furthermore it satisfies

$$f(\eta) = c_1 e^{-\eta^2} \eta^{\frac{2}{p-1}-N} \{1 - O(|x|^{-2})\} \quad \text{as } \eta \rightarrow \infty.$$

for some  $c_1 > 0$ . Later on, Kamin and Peletier in [22] proved that the sequence of weak solutions  $u_k$  converges to the very singular solution  $u_s$  as  $k \rightarrow \infty$ . After that, Marcus and Véron in [26] studied the equation in the framework of the *initial trace* theory. They pointed out the role of the very singular solution of (1.3) in the study of the singular set of the initial trace, showing in particular that it is the unique positive solution of (1.3) satisfying

$$\lim_{t \rightarrow 0} \int_{B_\epsilon} u(t, x) dx = \infty \quad \forall \epsilon > 0, B_\epsilon = B_\epsilon(0), \quad (1.6)$$

and

$$\lim_{t \rightarrow 0} \int_K u(t, x) dx = 0 \quad \forall K \subset \mathbb{R}^N \setminus \{0\}, K \text{ compact}. \quad (1.7)$$

If one replaces  $u^p$  by  $t^\beta u^p$  with  $p \in (1, 1 + \frac{2(1+\beta)}{N})$ , these results were extended by Marcus and Véron ( $\beta \geq 0$ ) [26] and then Al Sayed and Véron ( $\beta > -1$ ) [2]. The initial data problem with measure and general absorption term

$$\begin{aligned} \partial_t u - \Delta u + h(t, x, u) &= 0 \quad \text{in } (0, T) \times \Omega, \\ u &= 0 \quad \text{in } (0, T) \times \partial\Omega, \\ u(0, \cdot) &= \nu \quad \text{in } \Omega, \end{aligned} \quad (1.8)$$

in a bounded domain  $\Omega$  is a domain in  $\mathbb{R}^N$ , has been studied by Marcus and Véron in [26] in the framework of the initial trace theory. They proved that the following general integrability condition on  $h$

$$\begin{aligned} 0 \leq |h(t, x, r)| &\leq \tilde{h}(t)f(|r|) \quad \forall (x, t, r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R} \\ \int_0^T \tilde{h}(t)f(\sigma t^{\frac{N}{2}})t^{-\frac{N}{2}} dt &< \infty \quad \forall \sigma > 0 \\ \text{either } \tilde{h}(t) = t^\alpha &\text{ with } \alpha \geq 0 \text{ or } f \text{ is convex.} \end{aligned} \quad (1.9)$$

in order the problem has a unique solution for any bounded measure In the particular case where  $h(t, x, r) = t^\beta |u|^{p-1}u$ , is fulfilled if  $1 < p < 1 + \frac{2(1+\beta)}{N}$  and  $\beta > -1$ , and the very singular solution exists in this range of values.

Motivated by a growing number of applications in physics and by important links on the theory of Lévy process, semilinear fractional equations has been attracted much interest in last few years, (see e.g. [7, 8, 9, 11, 13, 16, 17, 18]). Recently, in [14] we obtained the existence and uniqueness of weak solution to semilinear fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + f(u) &= \nu \quad \text{in } \Omega, \\ u &= 0 \quad \text{in } \Omega^c, \end{aligned} \quad (1.10)$$

when  $\nu$  is Radon measure and  $f$  satisfies a subcritical integrability condition.

One purpose of this paper is to study the existence and uniqueness problem of solutions to semilinear fractional heat equation (1.1) in a measure framework. We first make precise the notion of weak solution of (1.1) that we will use in this note.

**Definition 1.1** *We say that  $u$  is a weak solution of (1.1), if for any  $T > 0$ ,  $u \in L^1(Q_T)$ ,  $h(t, u) \in L^1(Q_T)$  and*

$$\begin{aligned} \int_{Q_T} (u(t, x)[- \partial_t \xi(t, x) + (-\Delta)^\alpha \xi(t, x)] + h(t, u)\xi(t, x)) dx dt \\ = \int_{\mathbb{R}^N} \xi(0, x) d\nu - \int_{\mathbb{R}^N} \xi(T, x)u(T, x) dx \quad \forall \xi \in \mathbb{Y}_{\alpha, T}, \end{aligned} \quad (1.11)$$

where  $Q_T = (0, T) \times \mathbb{R}^N$  and  $\mathbb{Y}_{\alpha, T}$  is a space of functions  $\xi : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$  satisfying

- (i)  $\|\xi\|_{L^1(Q_T)} + \|\xi\|_{L^\infty(Q_T)} + \|\partial_t \xi\|_{L^\infty(Q_T)} + \|(-\Delta)^\alpha \xi\|_{L^\infty(Q_T)} < +\infty$ ;
- (ii) for  $t \in (0, T)$ , there exist  $M > 0$  and  $\epsilon_0 > 0$  such that for all  $\epsilon \in (0, \epsilon_0]$ ,  $\|(-\Delta)_\epsilon^\alpha \xi(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq M$ .

Before stating our main theorems, we introduce the subcritical integrability condition for the nonlinearity  $h$ , that is,

(H) (i) The function  $h : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and for any  $t \in (0, \infty)$ ,  $h(t, 0) = 0$  and  $h(t, r_1) \geq h(t, r_2)$  if  $r_1 \geq r_2$ .

(ii) There exist  $\beta > -1$  and a continuous, nondecreasing function  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$|h(t, r)| \leq t^\beta g(|r|) \quad \forall (t, r) \in (0, \infty) \times \mathbb{R}$$

and

$$\int_1^{+\infty} g(s) s^{-1-p_\beta^*} ds < +\infty, \quad (1.12)$$

where

$$p_\beta^* = 1 + \frac{2\alpha(1+\beta)}{N}. \quad (1.13)$$

We denote by  $H_\alpha : (0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}_+$  the heat kernel for  $(-\Delta)^\alpha$  in  $(0, \infty) \times \mathbb{R}^N$ , by  $\mathbb{H}_\alpha[\nu]$  the associated heat potential of  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ , defined by

$$\mathbb{H}_\alpha[\nu](t, x) = \int_{\mathbb{R}^N} H_\alpha(t, x, y) d\nu(y)$$

and by  $\mathcal{H}_\alpha[\mu]$  the Duhamel operator defined for  $(t, x) \in Q_T$  and any  $\mu \in L^1(Q_T)$  by

$$\mathcal{H}_\alpha[\mu](t, x) = \int_0^t \mathbb{H}_\alpha[\mu(s, \cdot)](t-s, x) ds = \iint_{Q_t} H_\alpha(t-s, x, y) \mu(s, y) dy ds.$$

Now we state our first theorem as follows.

**Theorem 1.1** *Assume that  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$  and the function  $h$  satisfies (H). Then problem (1.1) admits a unique weak solution  $u_\nu$  such that*

$$\mathbb{H}_\alpha[\nu] - \mathcal{H}_\alpha[h(\cdot, \mathbb{H}_\alpha[\nu_+])] \leq u_\nu \leq \mathbb{H}_\alpha[\nu] - \mathcal{H}_\alpha[h(\cdot, -\mathbb{H}_\alpha[\nu_-])] \quad \text{in } Q_\infty, \quad (1.14)$$

where  $\nu_+$  and  $\nu_-$  are respectively the positive and negative part in the Jordan decomposition of  $\nu$ . Furthermore,

- (i) if  $\nu$  is nonnegative, so is  $u_\nu$ ;
- (ii) the mapping:  $\nu \mapsto u_\nu$  is increasing and stable in the sense that if  $\{\nu_n\}$  is a sequence of positive bounded Radon measures converging to  $\nu$  in the weak sense of measures, then  $\{u_{\nu_n}\}$  converges to  $u_\nu$  locally uniformly in  $Q_\infty$ .

According to Theorem 1.1, there exists a unique positive weak solution  $u_k$  to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= k\delta_0 & \text{in } \mathbb{R}^N \end{aligned} \quad (1.15)$$

where  $\beta > -1$ ,  $k > 0$  and  $p \in (0, p_\beta^*)$ . We observe that  $u_k \rightarrow \infty$  in  $(0, \infty) \times \mathbb{R}^N$  as  $k \rightarrow \infty$  for  $p \in (0, 1]$ , see Proposition 4.2 for details. Our next interest of this paper is to study the limit of  $u_k$  as  $k \rightarrow \infty$  for  $p \in (1, p_\beta^*)$ , which exists since  $\{u_k\}_k$  are an increasing sequence of functions, bounded by  $\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}$ , and we set

$$u_\infty = \lim_{k \rightarrow \infty} u_k \quad \text{in } Q_\infty. \quad (1.16)$$

Actually,  $u_\infty$  and  $\{u_k\}_k$  are classical solutions to equation

$$\partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in } Q_\infty, \quad (1.17)$$

see Proposition 4.3 for details.

**Definition 1.2** (i) A solution  $u$  of (1.17) is called a self-similar solution if

$$u(t, x) = t^{-\frac{1+\beta}{p-1}} u(1, t^{-\frac{1}{2\alpha}} x) \quad (t, x) \in Q_\infty.$$

(ii) A solution  $u$  of (1.17) is called a very singular solution if it vanishes on  $\mathbb{R}^N \setminus \{0\}$  at  $t = 0$  and

$$\lim_{t \rightarrow 0^+} \frac{u(t, 0)}{\Gamma_\alpha(t, 0)} = +\infty,$$

where  $\Gamma_\alpha := \mathbb{H}[\delta_0]$  is the fundamental solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= \delta_0 & \text{in } \mathbb{R}^N. \end{aligned} \quad (1.18)$$

We remark that for  $p \in (1, p_\beta^*)$ , a self-similar solution  $u$  of (1.17) is also a very singular solution, since

$$\lim_{t \rightarrow 0^+} \Gamma_\alpha(t, 0) t^{\frac{N}{2\alpha}} = c_2, \quad (1.19)$$

for some  $c_2 > 0$ . For any self-similar solution  $u$  of (1.17),  $v(\eta) := u(1, t^{-\frac{1}{2\alpha}} x)$  with  $\eta = t^{-\frac{1}{2\alpha}} x$  is a solution of the self-similar equation

$$(-\Delta)^\alpha v - \frac{1}{2\alpha} \nabla v \cdot \eta - \frac{1+\beta}{p-1} v + v^p = 0 \quad \text{in } \mathbb{R}^N. \quad (1.20)$$

Since  $\left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}}$  is a constant nonzero solution of (1.20), the function

$$U_p(t) := \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}} \quad t > 0 \quad (1.21)$$

is a flat self-similar solution of (1.17). It is actually the maximal solution of the ODE  $y' + t^\beta y^p = 0$  defined on  $\mathbb{R}_+$ . Our next goal in this paper is to study non-flat self-similar solutions of (1.17).

**Theorem 1.2** *Assume that  $\beta > -1$ ,  $u_\infty$  is defined by (1.16) and*

$$p_\beta^{**} < p < p_\beta^*,$$

where  $p_\beta^{**} = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$ . Then  $u_\infty$  is a very singular self-similar solution of (1.17) in  $Q_\infty$ . Moreover, there exists  $c_3 > 1$  such that

$$\frac{c_3^{-1}}{1 + |x|^{N+2\alpha}} \leq u_\infty(1, x) \leq \frac{c_3 \ln(2 + |x|)}{1 + |x|^{N+2\alpha}} \quad x \in \mathbb{R}^N. \quad (1.22)$$

When  $p_\beta^{**} < p < p_\beta^*$  with  $\beta > -1$ , we observe that  $u_\infty$  and  $U_p$  are self-similar solutions of (1.17) and  $u_\infty$  is non-flat. Now we are ready to consider the uniqueness of non-flat self-similar solution of (1.17) with decay at infinity, precisely, we study the uniqueness of self-similar solution to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p &= 0 \quad \text{in } Q_\infty, \\ \lim_{|x| \rightarrow \infty} u(1, x) &= 0. \end{aligned} \quad (1.23)$$

We remark if  $u$  is self-similar, then the assumption  $\lim_{|x| \rightarrow \infty} u(1, x) = 0$  is equivalent to  $\lim_{|x| \rightarrow \infty} u(t, x) = 0$  for any  $t > 0$ . Finally, we state the properties of  $u_\infty$  when  $1 < p \leq p_\beta^{**}$  as follows.

**Theorem 1.3** (i) *Assume  $1 < p < p_\beta^{**}$  and  $u_\infty$  is defined by (1.16). Then  $u_\infty = U_p$  where  $U_p$  is given by (1.21).*

(ii) *Assume  $p = p_\beta^{**}$  and  $u_\infty$  is defined by (1.16). Then  $u_\infty$  is a self-similar solution of (1.17) such that*

$$u_\infty(t, x) \geq \frac{c_4 t^{-\frac{N+2\alpha}{2\alpha}}}{1 + |t^{-\frac{1}{2\alpha}} x|^{N+2\alpha}} \quad (t, x) \in (0, 1) \times \mathbb{R}^N, \quad (1.24)$$

for some  $c_4 > 0$ .



We note that Theorem 1.3 indicates that there is no self-similar solution of (1.17) with initial data  $u(0, \cdot) = 0$  in  $\mathbb{R}^N \setminus \{0\}$ , since  $u_\infty$  is the least self-similar solution. In Theorem 1.3 part (ii), we do not know if the self-similar solution is flat or not. From the above theorems, we have the following result.

**Theorem 1.4** (i) Assume  $p_\beta^{**} < p < p_\beta^*$ . Then problem (1.20) admits a minimal positive solution  $v_\infty$  satisfying

$$\lim_{|\eta| \rightarrow \infty} |\eta|^{\frac{2\alpha(1+\beta)}{p-1}} v_\infty(\eta) = 0. \quad (1.25)$$

Furthermore

$$\frac{c_3^{-1}}{1 + |\eta|^{N+2\alpha}} \leq v_\infty(\eta) \leq \frac{c_3 \ln(2 + |\eta|)}{1 + |\eta|^{N+2\alpha}} \quad \forall \eta \in \mathbb{R}^N \quad (1.26)$$

(ii) Assume  $1 < p < p_\beta^{**}$ . Then problem (1.20) admits no positive solution satisfying (1.25).

The question of uniqueness of the very singular solution in the case  $p_\beta^{**} < p < p_\beta^*$  remains an open problem.

This paper is organized as follows. In Section 2 we introduce some properties of Marcinkiewicz spaces and Kato's type inequality for non-homogeneous problem. In Section 3 we prove Theorem 1.1. Section 4 is devoted to investigate the properties of solutions to (1.15). In Section 5 we give the proof of Theorem 1.2 and Theorem 1.3. Finally, we prove Theorem 1.4.

## 2 Linear estimates

### 2.1 The Marcinkiewicz spaces

We recall the definition and basic properties of the Marcinkiewicz spaces.

**Definition 2.1** Let  $\Theta \subset \mathbb{R}^{N+1}$  be an open domain and  $\mu$  be a positive Borel measure in  $\Theta$ . For  $\kappa > 1$ ,  $\kappa' = \kappa/(\kappa - 1)$  and  $u \in L_{loc}^1(\Theta, d\mu)$ , we set

$$\|u\|_{M^\kappa(\Theta, d\mu)} = \inf \left\{ c \in [0, \infty] : \int_E |u| d\mu \leq c \left( \int_E d\mu \right)^{\frac{1}{\kappa'}}, \forall E \subset \Theta \text{ Borel set}, \right\} \quad (2.1)$$

and

$$M^\kappa(\Theta, d\mu) = \{u \in L_{loc}^1(\Theta, d\mu) : \|u\|_{M^\kappa(\Theta, d\mu)} < \infty\}. \quad (2.2)$$

$M^\kappa(\Theta, d\mu)$  is called the Marcinkiewicz space of exponent  $\kappa$  or weak  $L^\kappa$  space and  $\|\cdot\|_{M^\kappa(\Theta, d\mu)}$  is a quasi-norm. The following property holds.

**Proposition 2.1** [3, 14] *Assume that  $1 \leq q < \kappa < \infty$  and  $u \in L^1_{loc}(\Theta, d\mu)$ . Then there exists  $c_6 > 0$  dependent of  $q, \kappa$  such that*

$$\int_E |u|^q d\mu \leq c_6 \|u\|_{M^\kappa(\Theta, d\mu)} \left( \int_E d\mu \right)^{1-q/\kappa},$$

for any Borel set  $E$  of  $\Theta$ .

**Remark 2.1** If  $\Omega$  in  $\mathbb{R}^N$  is a smooth domain, we denote by  $H_\alpha^\Omega : (0, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}_+$  the heat kernel for  $(-\Delta)^\alpha$  and, if  $\nu \in \mathfrak{M}^b(\Omega)$ , by  $\mathbb{H}_\alpha^\Omega[\nu]$  the corresponding heat potential of  $\nu$  defined by

$$\mathbb{H}_\alpha^\Omega[\nu](t, x) = \int_\Omega H_\alpha^\Omega(t, x, y) d\nu(y).$$

When  $\Omega = \mathbb{R}^N$ , by Fourier transform, it is easy clear that

$$H_\alpha(t, x, y) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i(x-y)\cdot\zeta - t|\zeta|^{2\alpha}} d\zeta = H_\alpha(t, x - y, 0).$$

Furthermore  $\|H_\alpha(t, \cdot, 0)\|_{L^1}$  is independent of  $t$  with 1. This implies

$$\|\mathbb{H}_\alpha^\Omega[\nu](t, \cdot)\|_{L^p} \leq \|\nu\|_{L^p} \quad \forall 1 \leq p \leq \infty, \quad \forall \nu \in L^p(\mathbb{R}^N). \quad (2.3)$$

Since  $\mathbb{H}_\alpha^\Omega[\nu](t + s, \cdot) = \mathbb{H}_\alpha^\Omega[\mathbb{H}_\alpha^\Omega[\nu](s, \cdot)](t, \cdot)$  for all  $t, s > 0$  (semigroup property) and  $\nu \geq 0 \implies \mathbb{H}_\alpha^\Omega[\nu](t, \cdot) \geq 0$  the semigroup  $\{\mathbb{H}_\alpha^\Omega[\cdot](t, \cdot)\}_{t \geq 0}$  is sub-Markovian. Furthermore, since in  $L^2(\mathbb{R}^N)$  the operator  $(-\Delta)^\alpha$  is symmetric, the above semigroup is analytic in  $L^p(\mathbb{R}^N)$  for all  $1 \leq p < \infty$ : if  $1 < p < \infty$  it follows from a general result of Sten [31]) and for  $p = 1$  it is a consequence of regularity result from fractional powers of operators theory (see e.g. [23]). For  $1 \leq p < \infty$  generator  $A_p$  of the semigroup in  $L^p(\mathbb{R}^N)$  is the operator  $-(-\Delta)^\alpha$  with domain

$$D(A_p) := \{\nu \in L^p(\mathbb{R}^N) : (-\Delta)^\alpha \nu \in L^p(\mathbb{R}^N)\}. \quad (2.4)$$

and  $D(A_p)$  is dense since it contains  $C_0^\infty(\mathbb{R}^N)$ . If  $p = \infty$ , the natural space is the space  $C_0(\mathbb{R}^N)$  of continuous functions in  $\mathbb{R}^N$  tending to 0 at infinity. The domain of the corresponding operator  $A_{c_0}$  is

$$D(A_{c_0}) := \{\nu \in C_0(\mathbb{R}^N) : (-\Delta)^\alpha \nu \in C_0(\mathbb{R}^N)\}. \quad (2.5)$$

This operator is densely defined in  $C_0(\mathbb{R}^N)$ . In order to avoid confusion,  $C_c(\mathbb{R}^N)$  (resp.  $C_c^\infty(\mathbb{R}^N)$ ) denotes the space of continuous (resp  $C^\infty$ ) functions in  $\mathbb{R}^N$  with compact support. It is a dense subset of  $C_0(\mathbb{R}^N)$ .

**Proposition 2.2** For any  $\beta > -1$  and  $T > 0$ , there exists  $c_5 > 0$  dependent of  $N, \alpha, \beta$  such that for  $\nu \in \mathfrak{M}^b(\Omega)$ ,

$$\|\mathbb{H}_\alpha^\Omega[|\nu|]\|_{M^{p_\beta^*}(Q_T^\Omega, t^\beta dxdt)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\Omega)}, \quad (2.6)$$

where  $p_\beta^*$  is defined by (1.13) and  $Q_T^\Omega = (0, T) \times \Omega$ .

In order to prove this proposition, we introduce some notations. For  $\lambda > 0$  and  $y \in \Omega$ , let us denote

$$A_\lambda^\Omega(y) = \{(t, x) \in Q_T^\Omega : H_\alpha^\Omega(t, x, y) > \lambda\} \text{ and } m_\lambda^\Omega(y) = \int_{A_\lambda(y)} t^\beta dxdt.$$

We also set  $A_\lambda^{\mathbb{R}^N} = A_\lambda$  and  $m_\lambda^\Omega = m_\lambda$ .

**Lemma 2.1** There exists  $c_7 > 0$  such that for any  $\lambda > 1$ ,

$$A_\lambda(y) \subset (0, c_7 \lambda^{-\frac{2\alpha}{N}}] \times B_{c_7 \lambda^{-\frac{1}{N}}}(y), \quad (2.7)$$

where  $B_r(y)$  is the ball with radius  $r$  and center  $y$  in  $\mathbb{R}^N$ .

**Proof.** We observe that  $H_\alpha(t, x, y) = t^{-\frac{N}{2\alpha}} \Gamma_\alpha(1, (x - y)t^{-\frac{1}{2\alpha}})$ , where  $\Gamma_\alpha$  is the fundamental solution of (1.18). From [12], there exists  $c_8 > 0$  such that

$$\Gamma_\alpha(1, z) \leq \frac{c_8}{1 + |z|^{N+2\alpha}}.$$

This implies in particular

$$H_\alpha(t, x, y) \leq \frac{c_8 t^{-\frac{N}{2\alpha}}}{1 + \left(t^{-\frac{1}{2\alpha}} |x - y|\right)^{N+2\alpha}}. \quad (2.8)$$

On the one hand, for  $(t, x) \in A_\lambda(y)$ , we have that

$$t^{-\frac{N}{2\alpha}} \Gamma_\alpha(1, 0) \geq t^{-\frac{N}{2\alpha}} \Gamma_\alpha(1, (x - y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

which implies

$$t < \Gamma_\alpha^{\frac{2\alpha}{N}}(1, 0) \lambda^{-\frac{2\alpha}{N}}. \quad (2.9)$$

On the other hand, letting  $r = |x - y|$ ,

$$\frac{c_8 t}{t^{1+\frac{N}{2\alpha}} + r^{N+2\alpha}} \geq t^{-\frac{N}{2\alpha}} \Gamma_\alpha(1, (x - y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

then

$$r \leq (c_8 t \lambda^{-1})^{\frac{1}{N+2\alpha}}, \quad (2.10)$$

which, together with (2.9), implies

$$r \leq c_9 \lambda^{-\frac{1}{N}},$$

for some  $c_9 > 0$ . □

**Proof of Proposition 2.2.** By Lemma 2.1, there exists  $c_{10} > 0$  such that

$$m_\lambda(y) \leq c_{10} \lambda^{-1 - \frac{2\alpha(1+\beta)}{N}}.$$

Clearly

$$H_\alpha^\Omega(t, x, y) \leq H_\alpha(t, x, y), \quad (2.11)$$

then for any Borel set  $E \subset Q_T^\Omega$  and  $y \in \Omega$ , we have

$$\int_E H_\alpha^\Omega(t, x, y) t^\beta dx dt \leq \lambda \int_E t^\beta dx dt + \int_{A_\lambda(y)} H_\alpha(t, x, y) t^\beta dx dt$$

and

$$\begin{aligned} \int_{A_\lambda(y)} H_\alpha(t, x, y) t^\beta dx dt &= - \int_\lambda^{+\infty} s dm_s(y) = \lambda m_\lambda(y) + \int_\lambda^{+\infty} m_s(y) ds \\ &\leq c_{10} \lambda^{-\frac{2\alpha(1+\beta)}{N}} + c_{10} \int_\lambda^{+\infty} s^{-1 - \frac{2\alpha(1+\beta)}{N}} ds \\ &\leq c_{11} \lambda^{-\frac{2\alpha(1+\beta)}{N}}, \end{aligned}$$

where  $c_{11} = c_{10} \left(1 + \frac{N}{2\alpha(1+\beta)}\right)$ . As a consequence, it follows

$$\int_E H_\alpha^\Omega(t, x, y) t^\beta dx dt \leq \lambda \int_E t^\beta dx dt + c_{11} \lambda^{-\frac{2\alpha(1+\beta)}{N}}.$$

Taking  $\lambda = \left(\int_E t^\beta dx dt\right)^{-\frac{N}{N+2\alpha(1+\beta)}}$ , we obtain

$$\int_E H_\alpha^\Omega(t, x, y) t^\beta dx dt \leq (c_{11} + 1) \left(\int_E t^\beta dx dt\right)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}. \quad (2.12)$$

Since, by Fubini's theorem,

$$\begin{aligned} \int_E \mathbb{H}_\alpha^\Omega[|\nu|](t, x) t^\beta dx dt &= \int_E \int_\Omega H_\alpha^\Omega(t, x, y) d|\nu(y)| t^\beta dx dt \\ &= \int_\Omega \int_E H_\alpha^\Omega(t, x, y) t^\beta dx dt d|\nu(y)|, \end{aligned}$$

together with (2.12), it yields

$$\int_E \mathbb{H}_\alpha^\Omega[|\nu|](t, x) t^\beta dx dt \leq (c_{11} + 1) \|\nu\|_{\mathfrak{M}^b(\Omega)} \left( \int_E t^\beta dx dt \right)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}.$$

Thus,

$$\|\mathbb{H}_\alpha^\Omega[|\nu|]\|_{M^{1+\frac{2\alpha(1+\beta)}{N}}(Q_T^\Omega, t^\beta dx dt)} \leq (c_{11} + 1) \|\nu\|_{\mathfrak{M}^b(\Omega)},$$

which ends the proof.  $\square$

## 2.2 The non-homogeneous problem

The following proposition is the Kato's type estimate which is essential tool to prove the uniqueness of solutions to (1.1). For  $T > 0$ , we denote  $Q_T = (0, T) \times \mathbb{R}^N$ .

**Proposition 2.3** *Assume  $\mu \in L^1(Q_T)$  and  $\nu \in L^1(\mathbb{R}^N)$ . Then there exists a unique weak solution  $u$  to the problem*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u &= \mu & \text{in } Q_T, \\ u(0, \cdot) &= \nu & \text{in } \mathbb{R}^N, \end{aligned} \tag{2.13}$$

and there exists  $c_{12} > 0$  such that

$$\int_{Q_T} |u| dx dt \leq c_{12} \int_{Q_T} |\mu| dx dt + c_{12} \int_{\mathbb{R}^N} |\nu| dx. \tag{2.14}$$

Moreover, for any  $\xi \in \mathbb{Y}_{\alpha, T}$ ,  $\xi \geq 0$ , we have

$$\begin{aligned} \int_{Q_T} |u| (-\partial_t \xi + (-\Delta)^\alpha \xi) dx dt + \int_{\mathbb{R}^N} |u(T, x)| \xi(T, x) dx \\ \leq \int_{Q_T} \xi \text{sign}(u) \mu dx dt + \int_{\mathbb{R}^N} \xi(0, x) |\nu| dx, \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \int_{Q_T} u_+ (-\partial_t \xi + (-\Delta)^\alpha \xi) dx dt + \int_{\mathbb{R}^N} u_+(T, x) \xi(T, x) dx \\ \leq \int_{Q_T} \xi \text{sign}_+(u) \mu dx dt + \int_{\mathbb{R}^N} \xi(0, x) \nu_+ dx, \end{aligned} \tag{2.16}$$

In order to prove Proposition 2.3, we introduce the following notations. We say that  $u : Q_T \rightarrow \mathbb{R}$  is in  $C_{t,x}^{\sigma,\sigma'}(Q_T)$  for  $\sigma, \sigma' \in (0, 1)$  if

$$\|u\|_{C_{t,x}^{\sigma,\sigma'}(Q_T)} := \|u\|_{L^\infty(Q_T)} + \sup_{Q_T} \frac{|u(t,x) - u(s,y)|}{|t-s|^\sigma + |x-y|^{\sigma'}} < +\infty$$

and  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma'}(Q_T)$  if

$$\|u\|_{C_{t,x}^{1+\sigma, 2\alpha+\sigma'}(Q_T)} := \|u\|_{L^\infty(Q_T)} + \|\partial_t u\|_{C_{t,x}^{\sigma,\sigma'}(Q_T)} + \|(-\Delta)^\alpha u\|_{C_{t,x}^{\sigma,\sigma'}(Q_T)} < +\infty.$$

**Lemma 2.2** *Let  $\mu \in C^1(Q_T) \cap L^\infty(Q_T)$ ,  $\nu \in L^\infty(\mathbb{R}^N)$  and  $u$  be a solution of problem (2.13), then there exists  $\sigma \in (0, 1)$  such that  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$ . In particular, if  $\|D^2\nu\|_{L^\infty(\mathbb{R}^N)} + \|(-\Delta)^\alpha \nu\|_{C_x^{1-\alpha}(\mathbb{R}^N)} < \infty$ , then  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ .*

**Proof.** *Step 1.* When  $\|D^2\nu\|_{L^\infty(\mathbb{R}^N)} + \|(-\Delta)^\alpha \nu\|_{C_x^{1-\alpha}(\mathbb{R}^N)} < \infty$ , it follows directly by [8, (A.1)] that  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ .

*Step 2.* When  $\nu \in L^\infty(\mathbb{R}^N)$ , we use [9, Theorem 6.1] to obtain that  $u \in C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}(Q_T)$  for some  $\sigma > 0$ . For any  $T_0 \in (0, T)$ , let  $\eta : [0, T] \rightarrow [0, 1]$  be a  $C^2$  functions such that  $\eta = 0$  in  $[0, \frac{T_0}{4}]$  and  $\eta = 1$  in  $[T_0, T]$  and  $v = \eta u$  in  $Q_T$ . Then we obtain that for  $t \in [\frac{T_0}{4}, T]$  and  $x \in \mathbb{R}^N$ ,

$$\partial_t v + (-\Delta)^\alpha v = \eta \mu + \eta'(t)u,$$

where  $\eta \mu + \eta'(t)u \in C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}(Q_T)$  and  $v(0, \cdot) = 0$  in  $\mathbb{R}^N$ , Then we apply the argument in Step 1 to obtain that  $v \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ . Therefore,  $u$  is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$ . The proof is complete.  $\square$

**Lemma 2.3** (i) *Let  $\mu \in C^1(Q_T) \cap L^\infty(Q_T)$  and  $\nu \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , then problem (2.13) admits a unique solution  $u$  and for some  $\sigma \in (0, 1)$ ,  $u$  is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$ .*

(ii) *Let  $\mu \in C^1(Q_T) \cap L^\infty(Q_T) \cap L^1(Q_T)$ ,  $\nu \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  and  $u$  be the solution of (2.13), then  $u \in L^1(Q_T)$ , is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$  and for any  $\xi \in \mathbb{Y}_{\alpha, T}$ ,*

$$\begin{aligned} & \int_{Q_T} u(t,x)[- \partial_t \xi(t,x) + (-\Delta)^\alpha \xi(t,x)] dx dt \\ &= \int_{Q_T} \mu(t,x) \xi(t,x) dx dt + \int_{\mathbb{R}^N} \xi(0,x) \nu dx - \int_{\mathbb{R}^N} \xi(T,x) u(T,x) dx. \end{aligned} \quad (2.17)$$

(iii) Let  $\mu \in C^1(Q_T) \cap L^\infty(Q_T)$  and  $\nu \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , then problem

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u &= \mu \quad \text{in } Q_T, \\ u(T, \cdot) &= \nu \quad \text{in } \mathbb{R}^N \end{aligned} \quad (2.18)$$

admits a unique solution  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$  for some  $\sigma \in (0, 1)$ . Moreover, if  $\mu \in C^1(Q_T) \cap L^\infty(Q_T) \cap L^1(Q_T)$  and  $\nu \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ , then  $u \in \mathbb{Y}_{\alpha, T}$ .

**Proof.** (i) By [9, Theorem 2.6, Theorem 6.1], there exists a unique viscosity solution  $u \in C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}(Q_T)$  with  $\sigma > 0$  to problem (2.13), and then it follows by Lemma 2.2 that  $u$  is  $C_{t,x}^{1+\sigma', 2\alpha+\sigma'}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$  and some  $\sigma' \in (0, \min\{\frac{\sigma}{2\alpha}, \sigma\})$ . Then  $u$  is a classical solution of (2.13).

(ii) We claim that  $u \in L^1(Q_T)$  and  $u(t, \cdot) \in L^1(\mathbb{R}^N)$  for  $t \in (0, T)$ . By Duhamel formula, we have

$$\begin{aligned} \|u(t, \cdot)\|_{L^1(\mathbb{R}^N)} &\leq \int_{\mathbb{R}^N} \left( \iint_{Q_t} H_\alpha(t-s, x, y) |\mu(s, y)| dy ds \right) dx \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} H_\alpha(t, x, y) |\nu(y)| dy dx \\ &\leq \|\mu\|_{L^1(Q_T)} + \|\nu\|_{L^1(\mathbb{R}^N)} \end{aligned}$$

and

$$\|u\|_{L^1(Q_T)} = \int_0^T \|u(t, \cdot)\|_{L^1(\mathbb{R}^N)} dt \leq T(\|\mu\|_{L^1(Q_T)} + \|\nu\|_{L^1(\mathbb{R}^N)}).$$

In the sequel we denote by  $\mathcal{H}_\alpha$  the operator of  $L^1(Q_T)$  defined for all  $(x, t) \in Q_T$  by

$$\mathcal{H}_\alpha[\mu](x, t) = \int_0^t \mathbb{H}_\alpha[\mu(\cdot, s)](x, t-s) ds = \iint_{Q_t} H_\alpha(t-s, x, y) \mu(s, y) dy ds. \quad (2.19)$$

We claim that  $\|(-\Delta)_\epsilon^\alpha u(t, \cdot)\|_{L^\infty(\mathbb{R}^N)}$  is uniformly bounded with respect to  $\epsilon \in (0, \epsilon_0)$ . Since  $u(t, \cdot) \in C_x^{2\alpha+\sigma}(\mathbb{R}^N)$  for some  $\sigma \in (0, \min\{2-2\alpha, 1\})$ , then for  $x \in \mathbb{R}^N$  and  $y \in B_1(0)$ ,  $|u(x+y) + u(x-y) - 2u(x)| \leq \|u(t, \cdot)\|_{C_x^{2\alpha+\sigma}(\mathbb{R}^N)} |y|^{2\alpha+\sigma}$ . Thus,

$$\begin{aligned} \|(-\Delta)_\epsilon^\alpha u(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} &\leq \sup_{x \in \mathbb{R}^N} \left[ \int_{\mathbb{R}^N \setminus B_1(0)} \frac{|u(x+y) - u(x)|}{|y|^{N+2\alpha}} dy \right. \\ &\quad \left. + \frac{1}{2} \int_{B_1(0) \setminus B_\epsilon(0)} \frac{|u(x+y) + u(x-y) - 2u(x)|}{|y|^{N+2\alpha}} dy \right] \\ &\leq 2\|u\|_{L^1(\mathbb{R}^N)} + \int_{B_1(0)} |y|^{\sigma-N} dy \|u(t, \cdot)\|_{C_x^{2\alpha+\sigma}(\mathbb{R}^N)}. \end{aligned}$$

Next we claim that

$$\int_{Q_T} \xi(-\Delta)_\epsilon^\alpha u dx dt = \int_{Q_T} u(-\Delta)_\epsilon^\alpha \xi dx dt \quad \forall \xi \in \mathbb{Y}_{\alpha, T}. \quad (2.20)$$

Indeed, using the fact that for any  $t > 0$  there holds

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(t, z) - u(t, x)]\xi(t, x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz dx \\ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(t, x) - u(t, z)]\xi(t, z)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz dx, \end{aligned}$$

then we have

$$\begin{aligned} \int_{\mathbb{R}^N} \xi(t, x) (-\Delta)_\epsilon^\alpha u(t, x) dx \\ = -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[ \frac{(u(t, z) - u(t, x))\xi(t, x)}{|z - x|^{N+2\alpha}} + \frac{(u(t, x) - u(t, z))\xi(t, z)}{|z - x|^{N+2\alpha}} \right] \chi_\epsilon(|x - z|) dz dx \\ = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(t, z) - u(t, x)][\xi(t, z) - \xi(t, x)]}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz dx. \end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^N} u(t, x) (-\Delta)_\epsilon^\alpha \xi(t, x) dx = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(t, z) - u(t, x)][\xi(t, z) - \xi(t, x)]}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz dx.$$

Then (2.20) holds. Since  $u$  is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$  and  $\xi$  belongs to  $\mathbb{Y}_{\alpha, T}$ ,  $(-\Delta)_\epsilon^\alpha \xi(t, \cdot) \rightarrow (-\Delta)^\alpha \xi(t, \cdot)$  and  $(-\Delta)_\epsilon^\alpha u(t, \cdot) \rightarrow (-\Delta)^\alpha u(t, \cdot)$  as  $\epsilon \rightarrow 0$  in  $\mathbb{R}^N$  and  $(-\Delta)_\epsilon^\alpha \xi(t, \cdot), (-\Delta)_\epsilon^\alpha u(t, \cdot) \in L^\infty(\mathbb{R}^N)$  and  $\xi(t, \cdot), u(t, \cdot) \in L^1(\mathbb{R}^N)$ , then it follows by the Dominated Convergence Theorem that

$$\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N} \xi(t, x) (-\Delta)_\epsilon^\alpha u(t, x) dx = \int_{\mathbb{R}^N} \xi(t, x) (-\Delta)^\alpha u(t, x) dx$$

and

$$\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N} (-\Delta)_\epsilon^\alpha \xi(t, x) u(t, x) dx = \int_{\mathbb{R}^N} (-\Delta)^\alpha \xi(t, x) u(t, x) dx.$$

Combining this with (2.20), and letting  $\epsilon \rightarrow 0^+$ , we have

$$\int_{\mathbb{R}^N} \xi(t, x) (-\Delta)^\alpha u(t, x) dx = \int_{\mathbb{R}^N} (-\Delta)^\alpha \xi(t, x) u(t, x) dx,$$

integrating over  $[0, T]$  and by (2.13), we conclude that (2.17) holds.



(iii) *End of the proof.* Let  $u$  be the solution of problem (2.13) and

$$w(t, x) = u(T - t, x) \quad (t, x) \in [0, T] \times \mathbb{R}^N.$$

Then  $w$  is a solution of (2.18) and for some  $\sigma \in (0, 1)$ ,  $w$  is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ . On the contrary, if  $w$  is a solution of (2.18), then  $u(t, x) = w(T - t, x)$  for  $(t, x) \in [0, T] \times \mathbb{R}^N$  is a solution of (2.13), then the uniqueness holds since the solution of (2.13) is unique. Since  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ , then  $(-\Delta)^\alpha u(t, \cdot) \in C_x^\sigma$  and then  $(-\Delta)_\epsilon^\alpha u(t, \cdot)$  is bounded, which implies  $u \in \mathbb{Y}_{\alpha, T}$ .  $\square$

**Proof of Proposition 2.3. Uniqueness.** Let  $v$  be a weak solution of

$$\begin{aligned} \partial_t v + (-\Delta)^\alpha v &= 0 & \text{in } Q_T, \\ v(0, \cdot) &= 0 & \text{in } \mathbb{R}^N. \end{aligned} \tag{2.21}$$

We claim that  $v = 0$  a.e. in  $Q_T$ .

In fact, let  $\omega$  be a Borel subset of  $Q_T$  and  $\eta_{\omega, n}$  be the solution of

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u &= \zeta_n & \text{in } Q_T, \\ u(T, \cdot) &= 0 & \text{in } \mathbb{R}^N, \end{aligned} \tag{2.22}$$

where  $\zeta_n : \bar{Q}_T \rightarrow [0, 1]$  is a function  $C_c^1(Q_T)$  such that

$$\zeta_n \rightarrow \chi_\omega \quad \text{in } L^\infty(\bar{Q}_T) \quad \text{as } n \rightarrow \infty.$$

Then  $\eta_{\omega, n} \in \mathbb{Y}_{\alpha, T}$  by Lemma 2.3, and

$$\int_{Q_T} v \zeta_n dx dt = 0.$$

Passing to the limit when  $n \rightarrow \infty$ , we derive

$$\int_\omega v dx dt = 0.$$

This implies  $v = 0$  a.e. in  $Q_T$ .

*Existence and estimate (2.15).* For  $\delta > 0$ , we define an even convex function  $\phi_\delta$  by

$$\phi_\delta(t) = \begin{cases} |t| - \frac{\delta}{2} & \text{if } |t| \geq \delta, \\ \frac{t^2}{2\delta} & \text{if } |t| < \delta/2. \end{cases} \tag{2.23}$$

Then for any  $t, s \in \mathbb{R}$ ,  $|\phi'_\delta(t)| \leq 1$ ,  $\phi_\delta(t) \rightarrow |t|$  and  $\phi'_\delta(t) \rightarrow \text{sign}(t)$  when  $\delta \rightarrow 0^+$ . Moreover,

$$\phi_\delta(s) - \phi_\delta(t) \geq \phi'_\delta(t)(s - t). \tag{2.24}$$

Let  $\{\mu_n\}, \{\nu_n\}$  be two sequences of functions in  $C_0^2(Q_T), C_0^2(\mathbb{R}^N)$ , respectively, such that

$$\lim_{n \rightarrow \infty} \int_{Q_T} |\mu_n - \mu| dx dt = 0, \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nu_n - \nu| dx = 0.$$

We denote by  $u_n$  the corresponding solution to (2.13) where  $\mu, \nu$  are replaced by  $\mu_n, \nu_n$ , respectively. By Lemma 2.2 and Lemma 2.3(ii),  $u_n \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T) \cap L^1(Q_T)$  and then we use Lemma 2.3 in [14] and Lemma 2.3 (ii) to obtain that for any  $\delta > 0$  and  $\xi \in \mathbb{Y}_{\alpha, T}$ ,  $\xi \geq 0$ ,

$$\begin{aligned} & \int_{Q_T} \phi_\delta(u_n) [-\partial_t \xi + (-\Delta)^\alpha \xi] dx dt + \int_{\mathbb{R}^N} \xi(T, x) \phi_\delta(u_n(T, x)) dx \\ &= \int_{Q_T} \xi [\partial_t \phi_\delta(u_n) + (-\Delta)^\alpha \phi_\delta(u_n)] dx dt + \int_{\mathbb{R}^N} \xi(0, x) \phi_\delta(\nu_n) dx \\ &\leq \int_{Q_T} \xi \phi'_\delta(u_n) [\partial_t u_n + (-\Delta)^\alpha u_n] dx dt + \int_{\mathbb{R}^N} \xi(0, x) \phi_\delta(\nu_n) dx \\ &= \int_{Q_T} \xi \phi'_\delta(u_n) \mu_n dx dt + \int_{\mathbb{R}^N} \xi(0, x) \phi_\delta(\nu_n) dx. \end{aligned}$$

Letting  $\delta \rightarrow 0^+$ , we obtain

$$\begin{aligned} & \int_{Q_T} |u_n| [-\partial_t \xi + (-\Delta)^\alpha \xi] dx dt + \int_{\mathbb{R}^N} \xi(T, x) |u_n(T, x)| dx \\ &\leq \int_{Q_T} \xi \text{sign}(u_n) \mu_n dx dt + \int_{\mathbb{R}^N} \xi(0, x) |\nu_n| dx. \end{aligned} \tag{2.25}$$

Let  $\eta_k$  be the solution of

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u &= \varsigma_k \quad \text{in } Q_T, \\ u(T, \cdot) &= 0 \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{2.26}$$

where  $\varsigma_k : Q_T \rightarrow [0, 1]$  is a  $C_0^2$  function such that  $\varsigma_k = 1$  in  $(0, T) \times B_k(0)$ . From the proof of Lemma 2.3,  $\tilde{\eta}_k(t, x) := \eta_k(T - t, x)$  satisfies with  $\tilde{\varsigma}_k(t, x) = \varsigma_k(T - t, x)$

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u &= \tilde{\varsigma}_k \quad \text{in } Q_T, \\ u(0, \cdot) &= 0 \quad \text{in } \mathbb{R}^N. \end{aligned}$$

By Lemma 2.2,  $\tilde{\eta}_k \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$  with some  $\sigma \in (0, 1)$  and

$$\begin{aligned} 0 \leq \tilde{\eta}_k(t, x) &\leq c_8 \int_t^T \int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}}}{1 + |(s-t)^{-\frac{1}{2\alpha}}(y-x)|^{N+2\alpha}} dy ds \\ &\leq c_8 \int_t^T \int_{\mathbb{R}^N} \frac{dz}{1 + |z|^{N+2\alpha}} ds \\ &= c_{13}(T-t). \end{aligned}$$

Taking  $\xi = \eta_k$  in (2.25), we derive that

$$\int_{Q_T} |u_n| \chi_{(0,T) \times B_k(0)} dx dt \leq c_{13} T \int_{Q_T} |\mu_n| dx dt + c_{13} T \int_{\mathbb{R}^N} |\nu_n| dx.$$

Then, letting  $k \rightarrow \infty$ , we have

$$\int_{Q_T} |u_n| dx dt \leq c_{13} T \int_{Q_T} |\mu_n| dx dt + c_{13} T \int_{\mathbb{R}^N} |\nu_n| dx. \quad (2.27)$$

Similarly,

$$\int_{Q_T} |u_n - u_m| dx \leq c_{13} T \int_{Q_T} |\mu_n - \mu_m| dx dt + c_{13} T \int_{\mathbb{R}^N} |\nu_n - \nu_m| dx. \quad (2.28)$$

Therefore,  $\{u_n\}_n$  is a Cauchy sequence in  $L^1(Q_T)$  and its limit  $u$  is a weak solution of (2.13). Letting  $n \rightarrow \infty$ , (2.15) and (2.14) follow by (2.25) and (2.27), respectively. The proof of (2.16) is similar.  $\square$

### 3 Proof of Theorem 1.1

If  $h(t, \cdot)$  is monotone nondecreasing, for any  $\lambda > 0$ ,  $I + \lambda h(t, \cdot)$  is an homeomorphism of  $\mathbb{R}$  and the inverse function  $J_\lambda(t, \cdot) = (I + \lambda h(t, \cdot))^{-1}$  is a contraction. We define the Yosida approximation by

$$h_\lambda(t, \cdot) = \frac{I - J_\lambda(t, \cdot)}{\lambda}. \quad (3.1)$$

The function  $h_\lambda(t, \cdot)$  is monotone nondecreasing, vanishes at 0 as  $h$  does it and it is  $\frac{1}{\lambda}$ -Lipschitz continuous. Furthermore

$$r h_\lambda(t, r) \uparrow r h(t, r) \quad \text{as } \lambda \rightarrow 0 \quad \forall r \in \mathbb{R}, \quad (3.2)$$

see [4, Chap 2, Prop. 2.6] and . If  $u$  is a real valued function we will denote by  $h \circ u$  and  $h_\lambda \circ u$  respectively the functions  $(t, x) \mapsto h(t, u(t, x))$  and  $(t, x) \mapsto h_\lambda(t, u(t, x))$

**Lemma 3.1** *Assume  $h$  satisfy (H)-(i),  $\lambda > 0$  and  $\phi \in L^1(\mathbb{R}^N)$ . Then there exists a unique solution  $u_\phi$  of*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h_\lambda \circ u &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= \phi & \text{in } \mathbb{R}^N, \end{aligned} \quad (3.3)$$

Moreover

$$\mathbb{H}_\alpha[\phi] - \mathcal{H}_\alpha[h_\lambda \circ \mathbb{H}_\alpha[\phi_+]] \leq u_\phi \leq \mathbb{H}_\alpha[\phi] - \mathcal{H}_\alpha[h_\lambda \circ (-\mathbb{H}_\alpha[\phi_-])] \quad \text{in } Q_T, \quad (3.4)$$

where  $\phi_{\pm} = \max\{0, \pm\phi\}$  and

$$\|u_{\phi}(t, \cdot) - u_{\psi}(t, \cdot)\|_{L^1} \leq \|\phi - \psi\|_{L^1} \quad \forall 1 \leq p \leq \infty. \quad (3.5)$$

(i)  $u_{\phi} \geq 0$  if  $\phi \geq 0$  in  $\Omega$ ;

(ii) the mapping  $\phi \mapsto u_{\phi}$  is increasing.

**Proof.** Existence is a consequence of the Cauchy-Lipschitz-Picard theorem (see [10, Chap 4]): we write (3.3) under the integral form  $u = \mathcal{T}[u] = \mathbb{H}_{\alpha}[\phi] - \mathcal{H}_{\alpha}[h_{\lambda} \circ u]$ , i.e.

$$\mathcal{T}[u](t, \cdot) = \mathbb{H}_{\alpha}[\phi](t, \cdot) - \int_0^t \mathbb{H}_{\alpha}[h_{\lambda} \circ u](t - s, \cdot) ds \quad (3.6)$$

The space  $C([0, \infty); L^1(\mathbb{R}^N))$  endowed with the norm

$$\|w\|_{C-L^1} = \sup \left\{ e^{-kt} \|w(t, \cdot)\|_{L^1} : t \geq 0 \right\},$$

( $k > \lambda^{-1}$ ) is a Banach space. Since  $u \mapsto h_{\lambda}(t, u)$  is  $\frac{1}{\lambda}$ -Lipschitz continuous the mapping  $\mathcal{T}$  is  $\frac{1}{\lambda k}$ -Lipschitz continuous in  $X_p$ . Thus it admits a unique fixed point  $u_{\phi}$  which is an integral solution of (3.3).

$$u_{\phi}(t, \cdot) = \mathbb{H}_{\alpha}[\phi](t, \cdot) - \int_0^t \mathbb{H}_{\alpha}[h_{\lambda} \circ u_{\phi}](t - s, \cdot) ds. \quad (3.7)$$

The semigroup  $\{\mathbb{H}_{\alpha}[\cdot](t, \cdot)\}_{t \geq 0}$  is analytic in  $L^1(\mathbb{R}^N)$  since generated by the fractional power of a closed operator. It follows from the classical regularity theory for analytic semigroups as it exposed in [21, Sec 6] that that  $u_{\phi}$  is a strong solution of (3.3). Since it is continuous, it is also a weak solution in the sense that

$$\begin{aligned} & \int_{Q_T} (u_{\phi}[-\partial_t \xi + (-\Delta)^{\alpha} \xi] + \xi h_{\lambda} \circ u_{\phi}) dx dt \\ &= \int_{\mathbb{R}^N} \xi(0, x) \phi(x) dx - \int_{\mathbb{R}^N} \xi(T, x) u_{\phi}(T, x) dx \quad \forall \xi \in \mathbb{Y}_{\alpha, T}. \end{aligned} \quad (3.8)$$

If  $\phi_1, \phi_2 \in L^1(\mathbb{R}^N)$  and  $u_{\phi_j}$  are the corresponding solutions of (3.3), it follows from the positivity of  $H_{\alpha}$  that

$$(u_{\phi_2} - u_{\phi_1})_+ \leq (\mathcal{H}_{\alpha}[h_{\lambda} \circ u_{\phi_2} - h_{\lambda} \circ u_{\phi_1}])_+ \leq \frac{1}{\lambda} \mathcal{H}_{\alpha}[(u_{\phi_2} - u_{\phi_1})_+].$$

Therefore

$$\|(u_{\phi_2}(t, \cdot) - u_{\phi_1}(t, \cdot))_+\|_{L^p} \leq \frac{1}{\lambda} \int_0^t \|(u_{\phi_2}(t - s) - u_{\phi_1}(t - s))_+\|_{L^p} ds,$$

and by Gronwall inequality

$$\|(u_{\phi_2}(t) - u_{\phi_1}(t))_+\|_{L^p} \leq e^{\frac{t}{\lambda}} \|(\phi_2 - \phi_1)_+\|_{L^p}.$$

This implies (i) and (ii). As a consequence,

$$-\mathbb{H}_\alpha[\phi_-] \leq -u_{\phi_-} \leq u_\phi \leq u_{\phi_+} \leq \mathbb{H}_\alpha[\phi_+]$$

and thus

$$h_\lambda \circ (-\mathbb{H}_\alpha[\phi_-]) \leq h_\lambda \circ (-u_{\phi_-}) \leq h_\lambda \circ u_\phi \leq h_\lambda \circ u_{\phi_+} \leq h_\lambda \circ \mathbb{H}_\alpha[\phi_+].$$

Jointly with (3.7) it yields (3.4).  $\square$

**Notation.** In the sequel, if  $\eta \in L^1(Q_\tau)$  and  $\tau \geq T$ , we denote by  $\xi_{\eta,\tau}$  the solution of

$$\begin{aligned} -\partial_t \xi_\eta + (-\Delta)^\alpha \xi_\eta &= \eta & \text{in } Q_\tau \\ \xi_\eta(\tau, \cdot) &= 0 \end{aligned} \quad (3.9)$$

If  $\eta \geq 0$ , then  $\xi_{\eta,\tau} \geq 0$ ; if  $\eta \in C_0^\infty(\mathbb{R}^{N+1})$ , then  $\eta \in \mathbb{Y}_{\alpha,\tau}$ ; if  $\eta_n = \eta(\frac{\cdot}{n})$ , where  $n \in \mathbb{N}_*$  and  $\eta \in C_0^\infty(\mathbb{R}^{N+1})$  is nonnegative,  $0 \leq \eta \leq 1$ , with value 1 on  $B_1$  and 0 on  $B_2^c$ , then  $\xi_{\eta_n,\tau} \uparrow \tau - t$  as  $n \rightarrow \infty$ .

In the next lemma we prove that we can replace  $h_\lambda$  by  $h$ .

**Lemma 3.2** *Assume  $h$  satisfy (H)-(i) and  $\phi \in L^1(\mathbb{R}^N)$ . Then there exists a unique solution  $u_\phi \in C([0, \infty); L^1(\mathbb{R}^N))$  of*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h \circ u &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= \phi & \text{in } \mathbb{R}^N, \end{aligned} \quad (3.10)$$

Moreover inequality (3.5) and statements (i) and (ii) in Lemma 3.1 hold.

**Proof.** We denote by  $u_{\lambda,\phi}$  the solution of (3.3).

*Step 1- A priori estimate.* Let  $\phi \geq 0$ . If we take  $\xi = \xi_{\eta_n,\tau}$  in (3.8) and let  $n \rightarrow \infty$ , we derive

$$\int_{Q_T} (u_{\lambda,\phi} + (\tau - t)h_\lambda \circ u_{\lambda,\phi}) dxdt + (\tau - T) \int_{\mathbb{R}^N} u_{\lambda,\phi}(T, \cdot) dx = \tau \int_{\mathbb{R}^N} \phi(x) dx. \quad (3.11)$$

For  $0 < \lambda < \lambda'$  we set  $w = u_{\lambda,\phi} - u_{\lambda',\phi}$ . It follows from (2.16) and inequality  $h_{\lambda'} \circ u_{\lambda,\phi} \leq h_\lambda \circ u_{\lambda,\phi}$ , that for any nonnegative  $\xi$  in  $\mathbb{Y}_{\alpha,T}$ ,

$$\begin{aligned} & \int_{Q_T} (w_+ [-\partial_t \xi + (-\Delta)^\alpha \xi] + \xi (h_\lambda \circ u_{\lambda,\phi} - h_\lambda \circ u_{\lambda',\phi}) \text{sign}_+(w)) dxdt \\ & \leq \int_{Q_T} w_+ (h_{\lambda'} \circ u_{\lambda',\phi} - h_\lambda \circ u_{\lambda',\phi}) dxdt - \int_{\mathbb{R}^N} \xi(T, x) w_+(T, x) dx, \end{aligned}$$

Since  $h_\lambda(t, \cdot)$  is nondecreasing, we derive

$$\int_{Q_T} w_+ [-\partial_t \xi + (-\Delta)^\alpha \xi] dx dt \leq 0 \quad \forall \xi \in \mathbb{Y}_{\alpha, T}, \xi \geq 0.$$

If  $\eta \in C_0^\infty(\mathbb{R}^{N+1})$  is nonnegative, than  $\xi_\eta \in \mathbb{Y}_{\alpha, T}$ ,  $\xi_\eta \geq 0$  and

$$\int_{Q_T} w_+ \eta dx dt = 0.$$

This implies  $u_{\lambda, \phi} \leq u_{\lambda', \phi}$ .

*Step 2- Truncation.* We replace  $\phi$  by  $\phi_n = \inf\{\phi, n\}$  for  $n \in \mathbb{N}_*$  and denote by  $u_{\lambda, \phi_n}$  the corresponding solution of (3.3). By Step 1, the sequence  $\{u_{\lambda, \phi_n}\}_{\lambda > 0}$  is decreasing and it converges to some nonnegative  $u_{\phi_n}$  when  $\lambda \downarrow 0$ . Therefore  $h_\lambda \circ u_{\lambda, \phi_n} \rightarrow h \circ u_{\phi_n}$  a.e. in  $Q_T$ . It follows from (3.11) and Fatou's lemma that

$$\int_{Q_T} (u_{\phi_n} + (\tau - t)h \circ u_{\phi_n}) dx dt + (\tau - T) \int_{\mathbb{R}^N} u_{\phi_n}(\cdot, T) dx = \tau \int_{\mathbb{R}^N} \phi_n(x) dx. \quad (3.12)$$

Since  $0 \leq u_{\lambda, \phi_n} \leq n$ , then  $0 \leq h_\lambda \circ u_{\lambda, \phi_n} \leq h \circ u_{\lambda, \phi_n} \leq h(n)$  by (3.5). If  $E \subset Q_T$  is a Borel set,

$$\int_E h_\lambda \circ u_{\lambda, \phi_n} dx dt \leq h(n) |E|.$$

By Vitali convergence theorem  $h_\lambda \circ u_{\lambda, \phi_n} \rightarrow h \circ u_{\phi_n}$  in  $L^1(Q_T)$ . Therefore, we can let  $\lambda \rightarrow 0$  in identity (3.8) and conclude that  $u_{\phi_n}$  is a weak solution of (3.10) with initial data  $\phi_n$ .

*Step 3- Existence with  $\phi$  bounded.* If  $\phi = \phi_+ - \phi_- \in L^1(\mathbb{R}^N)$ , set  $\phi_{+, n} = \inf\{\phi_+, n\}$  and  $\phi_{-, n} = \inf\{\phi_-, n\}$ . We denote by  $u_{\lambda, \phi_{+, n}}$ ,  $u_{\phi_{+, n}}$ ,  $u_{\lambda, -\phi_{-, n}}$  and  $u_{-\phi_{-, n}}$  the corresponding solutions of (3.3) and (3.10). Then

$$u_{\lambda, -\phi_{-, n}} \leq u_{\lambda, \phi_{+, n} - \phi_{-, n}} \leq u_{\lambda, \phi_{+, n}} \quad (3.13)$$

which implies

$$h_\lambda \circ u_{\lambda, -\phi_{-, n}} \leq h_\lambda \circ u_{\lambda, \phi_{+, n} - \phi_{-, n}} \leq h_\lambda \circ u_{\lambda, \phi_{+, n}}.$$

Estimate (3.11) is valid under the form

$$\begin{aligned} \int_{Q_T} (u_{\lambda, \phi_{+, n}} + (\tau - t)h_\lambda \circ u_{\lambda, \phi_{+, n}}) dx dt \\ + (\tau - T) \int_{\mathbb{R}^N} u_{\lambda, \phi_{+, n}}(\cdot, T) dx = \tau \int_{\mathbb{R}^N} \phi_{+, n}(x) dx. \end{aligned} \quad (3.14)$$

and

$$\begin{aligned} & \int_{Q_T} (u_{\lambda, -\phi_{-,n}} + (\tau - t)h_\lambda \circ u_{\lambda, -\phi_{-,n}}) dxdt \\ & + (\tau - T) \int_{\mathbb{R}^N} u_{\lambda, -\phi_{-,n}}(\cdot, T) dx = -\tau \int_{\mathbb{R}^N} \phi_{-,n}(x) dx. \end{aligned} \quad (3.15)$$

Since  $h_\lambda \circ u_{\lambda, \phi_{+,n}}$  and  $h_\lambda \circ u_{\lambda, -\phi_{-,n}}$  are bounded in  $L^1(Q_T)$  independently of  $\lambda$  and  $n$ ,  $h_\lambda \circ u_{\lambda, \phi_{+,n} - \phi_{-,n}}$  endows the same property. Since

$$u_{\lambda, \phi_{+,n} - \phi_{-,n}} = \mathbb{H}_\alpha[\phi_{+,n} - \phi_{-,n}] - \mathcal{H}_\alpha[h_\lambda \circ u_{\lambda, \phi_{+,n} - \phi_{-,n}}]$$

it follows from [21, Sec 6] that  $u_{\lambda, \phi_{+,n} - \phi_{-,n}}$  remains bounded in the interpolation space  $Y_1 := L^1(0, T; D(A_1)(\mathbb{R}^N)) \cap W^{s,1}(0, T; L^1(\mathbb{R}^N))$  for any  $s \in (0, 1)$  where  $D(A_1)$  is defined in (2.4). Although a bounded subset  $K$  of  $Y_1$  is not a relatively compact subset of  $L^1(Q_T)$ , for any ball  $B \subset \mathbb{R}^N$ , the set of restriction to  $B$  of functions belonging to  $K$  is relatively compact in  $L^1((0, T) \times B)$ . Thus, there exists a subsequence  $\{\lambda_k\}$  such that  $\{u_{\lambda_k, \phi_{+,n} - \phi_{-,n}}\}$  converges a.e. to some function  $U_n$ . Furthermore  $\{h_{\lambda_k} \circ u_{\lambda_k, \phi_{+,n} - \phi_{-,n}}\}$  converges a.e. to  $h \circ U_n$ . Since the sequences  $\{u_{\lambda_k, -\phi_{-,n}}\}_{\lambda_k}$ ,  $\{u_{\lambda_k, \phi_{+,n}}\}_{\lambda_k}$ ,  $\{h_{\lambda_k} \circ u_{\lambda_k, -\phi_{-,n}}\}_{\lambda_k}$  and  $\{h_{\lambda_k} \circ u_{\lambda_k, \phi_{+,n}}\}_{\lambda_k}$  are convergent in  $L^1(Q_T)$  they are uniformly integrable. Because of (3.13) the same property is shared by the two sequences  $\{u_{\lambda_k, \phi_{+,n} - \phi_{-,n}}\}_{\lambda_k}$  and  $\{h_{\lambda_k} \circ u_{\lambda_k, \phi_{+,n} - \phi_{-,n}}\}_{\lambda_k}$ . Letting  $\lambda_k$  to 0 in the identity

$$u_{\lambda_k, \phi_{+,n} - \phi_{-,n}}(t, \cdot) = \mathbb{H}_\alpha[\phi_{+,n} - \phi_{-,n}](t, \cdot) - \int_0^t \mathbb{H}_\alpha[h_{\lambda_k} \circ u_{\lambda_k, \phi_{+,n} - \phi_{-,n}}](t - s, \cdot) ds. \quad (3.16)$$

yields

$$U_n(t, \cdot) = \mathbb{H}_\alpha[\phi_{+,n} - \phi_{-,n}](t, \cdot) - \int_0^t \mathbb{H}_\alpha[h \circ U_n](t - s, \cdot) ds. \quad (3.17)$$

This implies that  $U_n$  is an integral solution, thus a weak solution of (3.10) with initial data  $\phi_{+,n} - \phi_{-,n} = \text{sgn}(\phi) \inf\{n, |\phi|\}$  and then  $U_n = u_{\phi_n}$ .

*Step 4- Existence with  $\phi \in L^1(\mathbb{R}^N)$ .* By Kato's inequality (2.15) we obtain,

$$\begin{aligned} & \int_{Q_T} (|u_{\phi_k} - u_{\phi_m}|(-\partial_t \xi + (-\Delta)^\alpha \xi) + \xi |h \circ u_{\phi_k} - h \circ u_{\phi_m}|) dxdt \\ & + \int_{\mathbb{R}^N} |u_{\phi_k}(T, x) - u_{\phi_m}(T, x)| \xi(T, x) dx \leq \int_{\mathbb{R}^N} \xi(0, x) |\phi_k - \phi_m| dx, \end{aligned}$$

for  $m, k \in \mathbb{N}_*$  and  $\xi \in \mathbb{Y}_{\alpha, T}$ ,  $\xi > 0$ . Taking  $\xi = \xi_{\eta_n, \tau}$  as in (3.9) and letting  $n \rightarrow \infty$  yields

$$\begin{aligned} \int_{Q_T} (|u_{\phi_k} - u_{\phi_m}| + (\tau - t)|h \circ u_{\phi_k} - h \circ u_{\phi_m}|) dxdt \\ + (\tau - T) \int_{\mathbb{R}^N} |u_{\phi_k}(T, \cdot) - u_{\phi_m}(T, \cdot)| dx \leq \tau \int_{\mathbb{R}^N} |\phi_k - \phi_m| dx, \end{aligned} \quad (3.18)$$

Since  $\{\phi_m\}$  is a Cauchy sequence in  $L^1(\mathbb{R}^N)$ ,  $\{u_{\phi_m}\}$  and  $\{h \circ u_{\phi_m}\}$  are also Cauchy sequences in  $C(0, T; L^1(\mathbb{R}^N))$  and  $L^1(Q_T)$  respectively. Set  $U = \lim_{m \rightarrow \infty} u_{\phi_m}$ , then it satisfies

$$\begin{aligned} \int_{Q_T} (U[-\partial_t \xi + (-\Delta)^\alpha \xi] + \xi h \circ U) dxdt \\ = \int_{\mathbb{R}^N} \xi(0, x) \phi(x) dx - \int_{\mathbb{R}^N} \xi(T, x) U(T, x) dx \quad \forall \xi \in \mathbb{Y}_{\alpha, T}. \end{aligned} \quad (3.19)$$

and it is also an integral solution of (3.10). Thus  $u_\phi \in C([0, \infty); L^1(\mathbb{R}^N))$ .

Finally, we end the proof with uniqueness which is a consequence of the inequality below

$$\begin{aligned} \int_{Q_T} (|U - U'| + (\tau - t)|h \circ U - h \circ U'|) dxdt \\ + (\tau - T) \int_{\mathbb{R}^N} |U(T, \cdot) - U'(T, \cdot)| dx \leq \tau \int_{\mathbb{R}^N} |\phi - \phi'| dx, \end{aligned} \quad (3.20)$$

valid for two solutions  $U$  and  $U'$  of problem (3.10) with respective initial data  $\phi$  and  $\phi'$ , the proof of which is the same as the one of (3.18). Notice also that statement (i) and (ii) as well as inequality (3.5) follows by the above approximations.  $\square$

**Remark 3.1** *By the same method it can be proved that for any  $p \in (1, \infty)$  and  $\phi \in L^p(\mathbb{R}^N)$  (resp.  $\phi \in C_0(\mathbb{R}^N)$ ) there exists a unique solution  $u_\phi \in C[0, \infty; L^p(\mathbb{R}^N)$  (resp.  $u_\phi \in C[0, \infty; C_0(\mathbb{R}^N))$ ) solution of (3.10). Furthermore (3.5) holds.*

**Proof of Theorem 1.1.** *Existence for  $\nu \geq 0$ .* We consider a sequence of nonnegative functions  $\{\nu_n\}_n \subset C_0^2(\mathbb{R}^N)$  such that  $\nu_n \rightarrow \nu$  as  $n \rightarrow \infty$  in the weak sense of bounded measures, i.e.

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \zeta \nu_n dx = \int_{\mathbb{R}^N} \zeta d\nu \quad \forall \zeta \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \quad (3.21)$$

It follows from the Banach-Steinhaus theorem that  $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)}$  is bounded independently of  $n$  and we assume that  $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)} \leq 2\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}$ . By Lemma 3.1,



we denote by  $u_{\nu_n}$  the corresponding solution of (3.10) initial data  $\nu_n$ . Then  $u_n$  is nonnegative and satisfies that

$$0 \leq u_{\nu_n} = \mathbb{H}_\alpha[\nu_n] - \mathcal{H}_\alpha[h \circ u_{\nu_n}] \leq \mathbb{H}_\alpha[\nu_n] \quad \text{in } Q_T. \quad (3.22)$$

Jointly with (2.6) it implies

$$\|u_{\nu_n}\|_{M^{p,\beta}_*(Q_T, t^\beta dxdt)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}. \quad (3.23)$$

We have also the following estimates from (2.8) and (3.12)

$$u_{\nu_n}(t, x) \leq \mathbb{H}_\alpha[\nu_n](t, x) \leq 2c_8 t^{-\frac{N}{2\alpha}} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)} \quad \forall (t, x) \in Q_T. \quad (3.24)$$

and

$$\begin{aligned} \int_{Q_T} (u_{\nu_n} + (\tau - t)h \circ u_{\nu_n}) dxdt + (\tau - T) \int_{\mathbb{R}^N} u_{\nu_n}(\cdot, T) dx &= \tau \int_{\mathbb{R}^N} \nu_n(x) dx \\ &\leq 2\tau \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}. \end{aligned} \quad (3.25)$$

As in the proof of Lemma 3.2-Step 3, using the regularizing properties of the semi-group  $\mathbb{H}_\alpha[\cdot](t)$  (see [21, Sec 6]) infer that there exists a subsequence  $\{u_{\nu_{n_k}}\}$  which converges a.e. in  $Q_T$  to some function  $U$  and  $\{h \circ u_{\nu_{n_k}}\}$  converges a.e. to  $h \circ U$ .

For  $\kappa > 0$ , we denote  $S_\kappa = \{(t, x) \in Q_T : |u_{\nu_{n_k}}(t, x)| > \kappa\}$  and  $\omega(\kappa) = \int_{S_\kappa} t^\beta dxdt$ . Then for any Borel set  $E \subset Q_T$

$$\begin{aligned} \iint_E h \circ u_{\nu_{n_k}} dxdt &\leq \iint_{E \cap \{u_{\nu_{n_k}} \leq \kappa\}} h \circ u_{\nu_{n_k}} dxdt + \iint_{E \cap S_\kappa} h \circ u_{\nu_{n_k}} dxdt \\ &\leq g(\kappa) \iint_E t^\beta dxdt + \iint_{\{S_\kappa\}} t^\beta g(u_{\nu_{n_k}}) dxdt \\ &\leq g(\kappa) \iint_E t^\beta dxdt - \int_\kappa^\infty g(s) d\omega(s) \end{aligned}$$

where

$$\int_\kappa^\infty g(s) d\omega(s) = \lim_{M \rightarrow \infty} \int_\kappa^M g(s) d\omega(s).$$

By (2.1) and (3.23),  $\omega(s) \leq c_{14}s^{-p_\beta^*}$ , thus

$$\begin{aligned} - \int_\kappa^M g(s)d\omega(s) &= - \left[ g(s)\omega(s) \right]_{s=\kappa}^{s=M} + \int_\kappa^M \omega(s)dg(s) \\ &\leq g(\kappa)\omega(\kappa) - g(M)\omega(M) + c_{14} \int_\kappa^M s^{-p_\beta^*} dg(s) \\ &\leq g(\kappa)\omega(\kappa) - g(M)\omega(M) + c_{14} \left( M^{-p_\beta^*}g(M) - \kappa^{-p_\beta^*}g(\kappa) \right) \\ &\quad + \frac{c_{14}}{p_\beta^* + 1} \int_\kappa^M s^{-1-p_\beta^*}g(s)ds. \end{aligned}$$

Since  $\lim_{M \rightarrow \infty} M^{-p_\beta^*}g(M) = 0$  by (1.12) and [14, Lemma 4.1] and  $\omega(s) \leq c_{14}s^{-p_\beta^*}$ , we derive  $g(\kappa)\omega(\kappa) \leq c_{14}\kappa^{-p_\beta^*}g(\kappa)$  and then

$$- \int_\kappa^\infty g(s)d\omega(s) \leq \frac{c_{14}}{p_\beta^* + 1} \int_\kappa^\infty s^{-1-p_\beta^*}g(s)ds.$$

The above quantity on the right-hand side tends to 0 when  $\kappa \rightarrow \infty$ . The conclusion follows: for any  $\epsilon > 0$  there exists  $\kappa > 0$  such that

$$\frac{c_{14}}{p_\beta^* + 1} \int_\kappa^\infty s^{-1-p_\beta^*}g(s)ds \leq \frac{\epsilon}{2}$$

and there exists  $\delta > 0$  such that

$$\int_E t^\beta dxdt \leq \delta \implies g(\kappa) \int_E t^\beta dxdt \leq \frac{\epsilon}{2}.$$

This means that  $\{h_{n_k} \circ u_{\nu_{n_k}}\}$  is uniformly integrable in  $L^1(Q_T)$  and by Vitali convergence theorem  $h_{n_k} \circ u_{\nu_{n_k}} \rightarrow h \circ U$  in  $L^1(Q_T)$ . Letting  $n_k \rightarrow \infty$  in the identity

$$u_{\nu_{n_k}}(t, \cdot) = \mathbb{H}_\alpha[\nu_{n_k}](t, \cdot) - \int_0^t \mathbb{H}_\alpha[h \circ u_{\nu_{n_k}}(s, \cdot)](t-s, \cdot)ds$$

for some  $t > 0$  such that  $u_{\nu_{n_k}}(t, \cdot) \rightarrow U(t, \cdot)$  a.e. in  $\mathbb{R}^N$  yields

$$U(t, \cdot) = \mathbb{H}_\alpha[\nu](t, \cdot) - \int_0^t \mathbb{H}_\alpha[h \circ U(s, \cdot)](t-s, \cdot)ds.$$

This is valid for almost all  $t > 0$  and implies that  $U \in C(0, T; L^1(\mathbb{R}^N))$ , up to a modification on a set of  $t > 0$  with zero measure. Moreover

$$\begin{aligned} \int_{Q_T} \left( u_{\nu_{n_k}}(-\partial_t \xi + (-\Delta)^\alpha \xi) + \xi h \circ u_{\nu_{n_k}} \right) dxdt \\ = \int_{\mathbb{R}^N} \xi(0, x)\nu_{n_k} dx - \int_{\mathbb{R}^N} u_{\nu_{n_k}}(T, x)\xi(T, x)dx. \end{aligned}$$

where  $\xi \in \mathbb{Y}_{\alpha, T}$  is arbitrary. Thus, using the continuity of  $t \mapsto U(t, \cdot)$  in  $L^1(\mathbb{R}^N)$ , we derive

$$\begin{aligned} \int_{Q_T} (U(-\partial_t \xi + (-\Delta)^\alpha \xi) + \xi h \circ U) dx dt \\ = \int_{\mathbb{R}^N} \xi(0, x) d\nu(x) - \int_{\mathbb{R}^N} U(T, x) \xi(T, x) dx. \end{aligned}$$

From this infers that  $U$  is a weak solution of (1.1).

*Existence for general  $\nu$ .* For  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ , a sequence  $\{\nu_n\}$  in  $C_0^2(\mathbb{R}^N)$  converge to  $\nu$  in the weak sense of bounded measures. Because of the monotonicity of  $h(t, \cdot)$

$$-\mathbb{H}_\alpha[|\nu_n|] \leq u_{-|\nu_n|} \leq u_{\nu_n} \leq u_{|\nu_n|} \leq \mathbb{H}_\alpha[|\nu_n|].$$

Then by above analysis, the sequence  $\{h \circ u_{-|\nu_n|}\}$  and  $\{h \circ u_{|\nu_n|}\}$  are relatively compact in  $L^1(Q_T^B)$  for any  $T > 0$  and ball  $B$  and (3.23) holds for  $\{u_{\nu_n}\}$ . Therefore  $\{u_{\nu_n}\}$  is relatively locally compact in  $L^1(Q_T^B)$  and there exist some subsequence  $\{u_{\nu_{n_k}}\}$  and  $U \in L^1(Q_T)$  such that

$$u_{\nu_{n_k}} \rightarrow U \implies h \circ u_{\nu_{n_k}} \rightarrow h \circ U \quad \text{as } k \rightarrow \infty \quad \text{a.e. in } Q_T.$$

As in the previous case it implies that  $U$  is a weak solution of (1.1) and also an integral solution.

*Uniqueness.* Let  $u_1, u_2$  be two weak solutions of (1.1) with the same initial  $\nu$  and  $w = u_1 - u_2$ . Then

$$\partial_t w + (-\Delta)^\alpha w = h \circ u_2 - h \circ u_1 \quad \text{in } Q_T.$$

Since  $h \circ u_2 - h \circ u_1 \in L^1(Q_T)$ , then by (2.15), for  $\xi \in \mathbb{Y}_{\alpha, T}$ ,  $\xi \geq 0$ , we have that

$$\begin{aligned} \int_{Q_T} |w|[-\partial_t \xi + (-\Delta)^\alpha \xi] dx dt + \int_{\mathbb{R}^N} |w(T, x)| \xi(T, x) dx \\ + \int_{Q_T} (h \circ u_2 - h \circ u_1) \text{sign}(w) \xi dx dt \leq 0. \end{aligned}$$

This implies  $w = 0$  by monotonicity.

Statements (i) and (ii) and inequality (1.14) follows from the fact that the same relations holds for  $u_{\nu_n}$  by Lemma 3.2.

Stability is proved by the same approach that existence. If  $\{\nu_n\}$  converges to  $\nu$  in the weak sense of measures, then  $\|\nu_n\|_{\mathfrak{M}^b}$  is bounded independently of  $n$ . Since the distribution function of  $h \circ u_{\nu_n}$  depends only on the supremum of  $\|\nu_n\|_{\mathfrak{M}^b}$ , this set of functions is uniformly integrable in  $Q_T$ . This, combined with local compactness of the set  $\{u_{\nu_n}\}$  in  $L^1(Q_T)$ , implies the convergence of a subsequence  $(u_{\nu_{n_k}}, h \circ u_{\nu_{n_k}})$  to  $(u_\nu, h \circ u_\nu)$  where  $u_\nu$  is the solution of (1.1). Because of uniqueness, all converging subsequence have the same limit which imply the convergence of the whole sequence and stability.  $\square$

## 4 Dirac mass as initial data

In this section, we study the properties of solutions to (1.1) when  $h(t, r) = t^\beta r^p$  with  $\beta > -1$  and  $0 < p < p_\beta^*$  and the initial data is  $\nu = k\delta_0$  with  $k > 0$ .

**Proposition 4.1** *Assume  $0 < p < p_\beta^*$  and  $u_k$  is the solution of (1.15), then there exists  $c_{15} > 0$  such that*

$$\lim_{t \rightarrow 0^+} t^{\frac{N}{2\alpha}} u_k(t, 0) = c_{15} k. \quad (4.1)$$

**Proof.** By (1.14) it follows that

$$u_k(t, 0) \leq k\mathbb{H}_\alpha[\delta_0](t, 0) = k\Gamma_\alpha(t, 0) \quad t > 0. \quad (4.2)$$

We claim that there exists  $c_{16} > 0$  independent of  $k$  such that

$$u_k(t, 0) \geq k\Gamma_\alpha(t, 0) - c_{16} k^p t^{-\frac{N}{2\alpha} p + 1 + \beta}, \quad t \in (0, 1/2). \quad (4.3)$$

Indeed, from (1.14), it infers that

$$u_k(t, 0) \geq k\Gamma_\alpha(t, 0) - k^p W(t, 0) \quad t \in (0, 1/2),$$

where

$$W(t, x) = \int_0^t \mathbb{H}_\alpha[s^\beta (\mathbb{H}_\alpha^p[\delta_0])](t-s, x) ds \quad (t, x) \in Q_\infty.$$

For  $t \in (0, 1/4)$ , there exists  $c_{17}, c_{18} > 0$  such that

$$\begin{aligned} W(t, 0) &\leq c_{17} \int_0^t \int_{\mathbb{R}^N} \frac{(t-s)^{-\frac{N}{2\alpha}} s^\beta}{1 + ((t-s)^{-\frac{1}{2\alpha}} |y|)^{N+2\alpha}} \left( \frac{s^{-\frac{N}{2\alpha}}}{1 + (s^{-\frac{1}{2\alpha}} |y|)^{N+2\alpha}} \right)^p dy ds \\ &\leq c_{17} \int_0^t \int_{\mathbb{R}^N} \frac{s^{\beta - \frac{N}{2\alpha} p} dz ds}{\left( 1 + \left( \left( \frac{t-s}{s} \right)^{\frac{1}{2\alpha}} |z| \right)^{(N+2\alpha)p} \right) (1 + |z|^{N+2\alpha})} \\ &\leq c_{17} t^{\beta+1 - \frac{Np}{2\alpha}} \int_0^1 \int_{\mathbb{R}^N} \frac{d\tau dZ}{\left( 1 + \left( \frac{1-\tau}{\tau} \right)^{\frac{(N+2\alpha)p}{2\alpha}} |Z|^{(N+2\alpha)p} \right) (1 + |Z|^{N+2\alpha})} \\ &\leq c_{18} t^{\beta+1 - \frac{Np}{2\alpha}}. \end{aligned}$$

Combining (1.19) and  $-\frac{N}{2\alpha} p + 1 + \beta > -\frac{N}{2\alpha}$ , we obtain that

$$\lim_{t \rightarrow 0^+} t^{\frac{N}{2\alpha}} W(t, 0) = 0.$$

Therefore, (4.1) holds.  $\square$

In what follows we consider the limit of the solution  $\{u_k\}$  of (1.15) as  $k \rightarrow \infty$  for  $p \in (0, 1]$ .

**Proposition 4.2** *Assume  $0 < p \leq 1$  and that  $u_k$  is the solution of (1.15), then*

$$\lim_{k \rightarrow \infty} u_k = \infty \quad \text{in } Q_\infty.$$

**Proof.** We observe that  $\mathbb{H}_\alpha[\delta_0]$  and  $\mathbb{H}_\alpha[t^\beta(\mathbb{H}_\alpha[\delta_0])^p]$  are positive in  $(0, \infty) \times \mathbb{R}^N$ . By (1.14), for  $p \in (0, 1)$  and  $(t, x) \in (0, \infty) \times \mathbb{R}^N$ , we have that

$$u_k \geq k\mathbb{H}_\alpha[\delta_0] - k^p W \implies \lim_{k \rightarrow \infty} u_k = \infty.$$

For  $p = 1$ , it is obvious that  $u_k = ku_1$  and  $u_1 > 0$  in  $(0, \infty) \times \mathbb{R}^N$ , then

$$\lim_{k \rightarrow \infty} u_k = \infty \quad \text{in } Q_\infty.$$

The proof is complete. □

Now we deal with the range  $p \in (1, p_\beta^*)$ .

**Lemma 4.1** *Assume  $1 < p < p_\beta^*$  and that  $u_k$  is the solution of (1.15). Then for any  $k > 0$ ,*

$$0 \leq u_k \leq U_p \quad \text{in } Q_\infty, \tag{4.4}$$

where  $U_p$  is given by (1.21).

**Proof.** Let  $\{f_{n,k}\}$  be a sequence of nonnegative functions in  $C_c^1(\mathbb{R}^N)$  which converges to  $k\delta_0$  as  $n \rightarrow \infty$ . We denote by  $u_{n,k}$  the corresponding solution of (1.17) with initial data by  $f_{n,k}$ .

We claim that

$$u_{n,k} \leq U_p \quad \text{in } Q_\infty, \tag{4.5}$$

where, we recall it,  $U_p$  is the maximal solution of the ODE  $y' + t^\beta y^p = 0$  on  $\mathbb{R}_+$ . Indeed this implies (4.4).

*Step 1. We claim that*

$$\lim_{|x| \rightarrow \infty} u_{n,k}(t, x) = 0 \quad \forall t > 0. \tag{4.6}$$

From [12, 16], there exists  $c_8 > 0$  such that for any  $x, y \in \mathbb{R}^N$  and  $t \in (0, \infty)$ ,

$$0 < \Gamma_\alpha(t, x - y) \leq \frac{c_8 t^{-\frac{N}{2\alpha}}}{1 + (|x - y| t^{-\frac{1}{2\alpha}})^{N+2\alpha}}.$$

Then for  $|x| > 1$ ,

$$\begin{aligned}
0 \leq \mathbb{H}_\alpha[f_{n,k}](t, x) &\leq c_8 t^{-\frac{N}{2\alpha}} \int_{\mathbb{R}^N} \frac{f_{n,k}(y)}{1 + (|x - y|t^{-\frac{1}{2\alpha}})^{N+2\alpha}} dy \\
&= c_8 \int_{\mathbb{R}^N} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \\
&= c_8 \left( \int_{\mathbb{R}^N \setminus B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz + \int_{B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \right),
\end{aligned}$$

where  $R = \frac{1}{2}|x|t^{-\frac{1}{2\alpha}}$  and  $B_R = \{z \in \mathbb{R}^N : |z| < R\}$ . It is obvious that

$$|x - zt^{\frac{1}{2\alpha}}| \geq |x| - |z|t^{\frac{1}{2\alpha}} \geq |x|/2 \quad \text{for all } z \in B_R.$$

Then

$$\begin{aligned}
\int_{B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz &\leq \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) \int_{B_R} \frac{1}{1 + |z|^{N+2\alpha}} dz \\
&\leq \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) \int_{\mathbb{R}^N} \frac{1}{1 + |z|^{N+2\alpha}} dz \\
&= c_{16} \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y)
\end{aligned}$$

and

$$\int_{\mathbb{R}^N \setminus B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \leq \int_{\mathbb{R}^N \setminus B_R} \frac{\|f_{n,k}\|_{L^\infty(\mathbb{R}^N)}}{1 + |z|^{N+2\alpha}} dz \leq c_{18} R^{-2\alpha} = \frac{c_{18}t}{|x|^{2\alpha}},$$

for some  $c_{18} > 0$  independent of  $x, t$  and  $R$ . Since  $f_{n,k} \in C_0^1(\mathbb{R}^N)$ , we have that

$$\lim_{|x| \rightarrow \infty} \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) = 0$$

and then for any  $t > 0$ ,  $0 \leq u_{n,k}(t, x) \leq \mathbb{H}_\alpha[f_{n,k}](t, x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

*Step 2.* We claim that (4.5) holds. By contradiction, if (4.5) is not verified, there exists  $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^N$  such that

$$(U_p - u_{n,k})(t_0, x_0) = \min_{(t,x) \in (0,\infty) \times \mathbb{R}^N} (U_p - u_{n,k})(t, x) < 0,$$

since  $U_p(t) > 0 = \lim_{|x| \rightarrow \infty} u_{n,k}(t, x)$  for any  $t \in (0, \infty)$ ,  $U_p(0) = \infty > f_{n,k}(x) = u_{n,k}(0, x)$  for  $x \in \mathbb{R}^N$  and  $\lim_{t \rightarrow \infty} U_p(t) = \lim_{t \rightarrow \infty} u_{n,k}(t, x) = 0$  for  $x \in \mathbb{R}^N$ . Then

$\partial_t(U_p - u_{n,k})(t_0, x_0) = 0$ . Moreover since

$$\begin{aligned}(U_p - u_{n,k})(t_0, x_0) &= \min\{U_p(t_0) - u_{n,k}(t_0, x) : x \in \mathbb{R}^N\} \\ &= U_p(t_0) - \max\{u_{n,k}(t_0, x) : x \in \mathbb{R}^N\}\end{aligned}$$

and

$$u_{n,k}(t_0, x_0) = \max\{u_{n,k}(t_0, x) : x \in \mathbb{R}^N\} \implies (-\Delta)^\alpha u_{n,k}(t_0, x_0) \geq 0$$

and

$$0 = \partial_t(U_p - u_{n,k})(t_0, x_0) - (-\Delta)^\alpha u_{n,k}(t_0, x_0) + t_0^\beta U_p^p(t_0) - t_0^\beta u_{n,k}^p(t_0, x_0) < 0,$$

which is impossible. Thus (4.5) holds.  $\square$

**Proposition 4.3** (i) Assume  $0 < p < p_\beta^*$  and that  $u_k$  is the solution of (1.15). Then  $u_k$  is a classical solution of (1.17).

(ii) Assume  $1 < p < p_\beta^*$  and that  $u_\infty$  is defined by (1.16). Then  $u_\infty$  is a classical solution of (1.17).

**Proof.** (i) Since  $u_k \leq k\mathbb{H}_\alpha[\delta_0]$ , it infers that  $u_k$  is bounded in  $(T_0, \infty) \times \mathbb{R}^N$  for  $T_0 > 0$ . Let  $\{g_{n,k}\}$  be a sequence of nonnegative functions in  $C_0^1(\mathbb{R}^N)$  which converges to  $k\delta_0$  as  $n \rightarrow \infty$  and  $u_{n,k}$  the corresponding solution of (1.17) with initial data  $g_{n,k}$ . Then  $\mathbb{H}_\alpha[g_{n,k}] \rightarrow k\mathbb{H}_\alpha[\delta_0]$  as  $n \rightarrow \infty$  uniformly in  $[T_0, \infty) \times \mathbb{R}^N$  for any  $T_0 > 0$  and by the Comparison Principle, there exists  $c_{19} > 1$  such that

$$0 \leq u_{n,k}(t, x) \leq k\mathbb{H}_\alpha[g_{n,k}] \leq c_{19}k\mathbb{H}_\alpha[\delta_0] \quad \text{in } [T_0, \infty) \times \mathbb{R}^N$$

and there exists  $\sigma \in (0, 1)$  such that  $\{u_{n,k}\}$  are uniformly bounded with respect to  $n$  in  $C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}((T_0, \infty) \times \mathbb{R}^N)$  with  $T_0 > 0$ . Therefore, by the Arzela-Ascoli theorem,  $u_{n,k}$  converges to  $u_k$  in  $C_{t,x}^{\frac{\sigma'}{2\alpha}, \sigma'}((T_0, \infty) \times \mathbb{R}^N)$  with  $\sigma' \in (0, \sigma)$  and then  $u_k$  is a viscosity solution of (1.17) in  $(T_0, \infty) \times \mathbb{R}^N$ . By estimate (A.1) in [8],  $u_k$  is in  $C_{t,x}^{1+\sigma', 2\alpha+\sigma'}((T_0, \infty) \times \mathbb{R}^N)$  and  $u_k$  is a classical solution of (1.17) in  $(T_0, \infty) \times \mathbb{R}^N$ . (ii) The proof is the same as part (i), just replacing  $u_k \leq k\mathbb{H}_\alpha[\delta_0]$  by  $u_\infty \leq U_p$ .  $\square$

## 5 Self-similar and very singular solutions

By Theorem 1.1 and (4.4), we see that  $\{u_k\}$  is an increasing sequence of nonnegative functions bounded from above by  $U_p$ . Then for  $p \in (1, p_\beta^*)$ , there exists  $u_\infty = \lim_{k \rightarrow \infty} u_k$ , which is a classical solution of (1.17) by Proposition 4.3 (ii) and satisfies

$$u_\infty \leq U_p \quad \text{in } Q_\infty. \quad (5.1)$$

**Proposition 5.1** *Assume  $1 < p < p_\beta^*$ , then  $u_\infty$  is a self-similar solution of (1.17).*

**Proof.** For  $\lambda > 0$ , we set

$$T_\lambda[u](t, x) = \lambda^{\frac{2\alpha(1+\beta)}{p-1}} u(\lambda^{2\alpha} t, \lambda x) \quad (t, x) \in Q_\infty.$$

It is straightforward to verify that  $T_\lambda[u_k]$  is the solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p &= 0 && \text{in } Q_\infty \\ u(0, \cdot) &= \lambda^{\frac{2\alpha(1+\beta)}{p-1} - N} k \delta_0 && \text{in } \mathbb{R}^N. \end{aligned} \quad (5.2)$$

Because of uniqueness,  $T_\lambda[u_k] = u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1} - N}}$ . Letting  $k \rightarrow \infty$  and using the continuity of  $u \mapsto T_\lambda[u]$ , we have that

$$\lim_{k \rightarrow \infty} T_\lambda[u_k] = T_\lambda[u_\infty] = u_\infty$$

which implies that  $u_\infty$  is a self-similar solution (1.17).  $\square$

Let us denote

$$U_\infty(z) = u_\infty(1, z), \quad z \in \mathbb{R}^N$$

and we observe that  $U_\infty$  is a classical solution of (1.20). It is obvious that the constant  $(\frac{1+\beta}{p-1})^{\frac{1}{p-1}}$  is a constant positive solution of the self-similar equation (1.20).

We observe that  $N < \frac{2\alpha(1+\beta)}{p-1} < N + 2\alpha$  when  $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$ .

We prove below this fundamental result that  $u_\infty$  is the minimal self similar solution.

**Proposition 5.2** *Assume that  $1 < p < 1 + \frac{2\alpha(1+\beta)}{N}$  and  $\tilde{u}$  is a positive self-similar solution of (1.23). Then  $u_\infty \leq \tilde{u}$ .*

**Proof.** For any  $r > 0$ , we have that

$$\begin{aligned} \int_{B_r(0)} \tilde{u}(t, x) dx &= t^{-\frac{1+\beta}{p-1}} \int_{B_r(0)} \tilde{u}(1, t^{-\frac{1}{2\alpha}} x) dx \\ &= t^{-\frac{1+\beta}{p-1} + \frac{N}{2\alpha}} \int_{B_{t^{-\frac{1}{2\alpha}} r}(0)} \tilde{u}(1, z) dz \\ &\geq t^{-\frac{1+\beta}{p-1} + \frac{N}{2\alpha}} \int_{B_1(0)} \tilde{u}(1, z) dz \\ &\rightarrow +\infty \quad \text{as } t \rightarrow 0^+, \end{aligned}$$



where last inequality holds for  $t \in (0, r^{2\alpha}]$ . Let  $\{\epsilon_n\}$  be a sequence positive decreasing numbers converging to 0 as  $n \rightarrow \infty$ . For  $\epsilon_n$  and  $k > 0$ , there exists  $t_{n,k} > 0$  such that

$$\int_{B_{\epsilon_n}(0)} \tilde{u}(t_{n,k}, x) dx = k.$$

We observe that for any fixed  $k$ ,  $t_{n,k} \rightarrow 0$  as  $n \rightarrow \infty$  since  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ . Let  $\eta_0 : \mathbb{R}^N \rightarrow [0, 1]$  be a  $C^2$  function such that  $\text{supp } \eta_0 \subset \bar{B}_2(0)$ ,  $\eta_0 = 1$  in  $B_1(0)$  and  $\eta_n(x) = \eta_0(\epsilon_n^{-1}x)$  for  $x \in \mathbb{R}^N$ . Choosing  $\{f_{n,k}\}$  be a sequence of  $C^2$  functions such that

$$0 \leq f_{n,k}(x) \leq \eta_n(x)\tilde{u}(t_{n,k}, x) \quad \forall x \in \mathbb{R}^N$$

and

$$f_{n,k} \rightarrow k\delta_0 \quad \text{as } n \rightarrow \infty.$$

Let  $u_{n,k}$  be the solution of (1.1) with initial data  $f_{n,k}$ , then

$$u_{n,k}(t, x) \leq u(t_{n,k} + t, x) \quad \forall (t, x) \in Q_\infty$$

and by uniqueness of  $u_k$ ,  $\lim_{n \rightarrow \infty} u_{n,k} = u_k$ , where  $u_k$  is the solution of (1.1) with initial data  $k\delta_0$ . Then for any  $k$ , we have  $u_k \leq \tilde{u}$  in  $Q_\infty$ , which implies that

$$u_\infty \leq \tilde{u} \quad \text{in } Q_\infty.$$

□

### 5.1 The case $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$

We define the function  $w_\lambda$  by

$$w_\lambda(t, x) = \lambda t^{-\frac{1+\beta}{p-1}} w(t^{-\frac{1}{2\alpha}}|x|) \quad (t, x) \in Q_\infty, \quad (5.3)$$

where  $w(s) = \frac{\ln(\epsilon+s^2)}{1+s^{N+2\alpha}}$ .

**Lemma 5.1** *Assume  $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$ , then there exists  $\Lambda_0 > 0$  such that for  $\lambda \geq \Lambda_0$ ,*

$$\partial_t w_\lambda(t, x) + (-\Delta)^\alpha w_\lambda(t, x) + t^\beta w_\lambda^p(t, x) \geq 0 \quad \forall (t, x) \in Q_\infty. \quad (5.4)$$

**Proof.** By direct computation, we have

$$\partial_t w_\lambda(t, x) = -\frac{\lambda(1+\beta)}{p-1} t^{-\frac{1+\beta}{p-1}-1} w(t^{-\frac{1}{2\alpha}}|x|) - \frac{\lambda}{2\alpha} t^{-\frac{1+\beta}{p-1}-\frac{1}{2\alpha}-1} |x| w'(t^{-\frac{1}{2\alpha}}|x|)$$

and

$$(-\Delta)^\alpha w_\lambda(t, x) = \lambda t^{-\frac{1+\beta}{p-1}-1} (-\Delta)^\alpha w(t^{-\frac{1}{2\alpha}} |x|),$$

which implies

$$\begin{aligned} & \partial_t w_\lambda(t, x) + (-\Delta)^\alpha w_\lambda(t, x) + t^\beta w_\lambda^p(t, x) \\ &= \lambda t^{-\frac{1+\beta}{p-1}-1} \left[ (-\Delta)^\alpha w(s) - \frac{1}{2\alpha} w'(s) s - \frac{1+\beta}{p-1} w(s) + \lambda^{p-1} w^p(s) \right], \end{aligned} \quad (5.5)$$

where  $s = |z|$  with  $z = t^{-\frac{1}{2\alpha}} x$ . Next, for  $s > 0$ , we have

$$-\frac{1}{2\alpha} w'(s) s - \frac{1+\beta}{p-1} w(s) = \left[ \frac{N+2\alpha}{2\alpha} \frac{s^{N+2\alpha}}{1+s^{N+2\alpha}} - \frac{1+\beta}{p-1} - \frac{s^2(e+s^2)^{-1}}{\alpha \ln(e+s^2)} \right] w(s).$$

Since  $\frac{N+2\alpha}{2\alpha} > \frac{1+\beta}{p-1}$ ,  $\lim_{s \rightarrow \infty} \frac{s^{N+2\alpha}}{1+s^{N+2\alpha}} = 1$  and  $\lim_{s \rightarrow \infty} \frac{1}{\ln(e+s^2)} = 0$ , there exists  $R_0 > 0$  and  $\sigma_0 > 0$  such that

$$-\frac{1}{2\alpha} w'(s) s - \frac{1+\beta}{p-1} w(s) \geq \sigma_0 w(s) \quad \forall s \geq R_0. \quad (5.6)$$

For  $|z| > 2$ , and using the definition of the fractional Laplacian, we have

$$\begin{aligned} -(-\Delta)^\alpha w(|z|) &= \frac{1}{2} \int_{\mathbb{R}^N} \left( \frac{\ln(e + |z + \tilde{y}|^2)}{1 + |z + \tilde{y}|^{N+2\alpha}} + \frac{\ln(e + |z - \tilde{y}|^2)}{1 + |z - \tilde{y}|^{N+2\alpha}} - \frac{2 \ln(e + |z|^2)}{1 + |z|^{N+2\alpha}} \right) \frac{d\tilde{y}}{|\tilde{y}|^{N+2\alpha}} \\ &= \frac{w(|z|)}{2|z|^{2\alpha}} \int_{\mathbb{R}^N} \frac{I_z(y)}{|y|^{N+2\alpha}} dy, \end{aligned} \quad (5.7)$$

where

$$\begin{aligned} I_z(y) &= \frac{1 + |z|^{N+2\alpha}}{1 + |z|^{N+2\alpha} |e_z + y|^{N+2\alpha}} \frac{\ln(e + |z|^2 |e_z + y|^2)}{\ln(e + |z|^2)} \\ &\quad + \frac{1 + |z|^{N+2\alpha}}{1 + |z|^{N+2\alpha} |e_z - y|^{N+2\alpha}} \frac{\ln(e + |z|^2 |e_z - y|^2)}{\ln(e + |z|^2)} - 2 \end{aligned}$$

and  $e_z = \frac{z}{|z|}$ .

We claim that there exists  $c_{20} > 0$  such that

$$\int_{B_{\frac{1}{2}}(-e_z) \cup B_{\frac{1}{2}}(e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq \frac{c_{20}}{w(|z|) |z|^N}. \quad (5.8)$$

In fact, for  $y \in B_{\frac{1}{2}}(-e_z)$ , there exists  $c_{21} > 0$  such that

$$\frac{1 + |z|^{N+2\alpha}}{1 + |z|^{N+2\alpha} |e_z - y|^{N+2\alpha}} \frac{\ln(e + |z|^2 |e_z - y|^2)}{\ln(e + |z|^2)} \leq c_{21}$$

and then

$$\begin{aligned}
\int_{B_{\frac{1}{2}}(-e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy &\leq \omega_N \int_0^{\frac{1}{2}} \frac{1 + |z|^{N+2\alpha}}{1 + (|z|r)^{N+2\alpha}} \frac{\ln(e + |z|^2 r^2)}{\ln(e + |z|^2)} r^{N-1} dr + c_{22} \\
&\leq \frac{\omega_N}{w(|z|)|z|^N} \int_0^\infty \frac{t^{N-1} \ln(e + t^2)}{1 + t^{N+2\alpha}} dt + c_{22} \\
&\leq \frac{c_{23}}{w(|z|)|z|^N},
\end{aligned}$$

where  $c_{22}, c_{23} > 0$  and the last inequality holds since  $w(|z|)|z|^N \rightarrow 0$  as  $|z| \rightarrow \infty$ . Thus,

$$\int_{B_{\frac{1}{2}}(e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy = \int_{B_{\frac{1}{2}}(-e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq \frac{c_{23}}{w(|z|)|z|^N}.$$

We claim that there exists  $c_{24} > 0$  such that

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq c_{24}. \tag{5.9}$$

Indeed, since the function  $I_z$  is  $C^2$  in  $\bar{B}_{\frac{1}{2}}(0)$ ,  $I_z(0) = 0$  and  $I_z(y) = I_z(-y)$ , then  $\nabla I_z(0) = 0$  and there exists  $c_{34} > 0$  such that

$$|D^2 I_z(y)| \leq c_{25} \quad \forall y \in B_{\frac{1}{2}}(0).$$

Then we have

$$I_z(y) \leq c_{25}|y|^2 \quad \forall y \in B_{\frac{1}{2}}(0),$$

which implies

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq c_{25} \int_{B_{\frac{1}{2}}(0)} \frac{|y|^2}{|y|^{N+2\alpha}} dy \leq c_{24}.$$

We claim that there exists  $c_{26} > 0$  such that

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq c_{26}, \tag{5.10}$$

where  $A = \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(e_z) \cup B_{\frac{1}{2}}(-e_z))$ . In fact, for  $y \in A$ , we observe that there exists  $c_{27} > 0$  such that  $I_z(y) \leq c_{27}$  and

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} \frac{c_{27}}{|y|^{N+2\alpha}} \leq c_{28},$$

for some  $c_{28} > 0$ . Therefore, by (5.5)-(5.10), there exists  $c_{29} > 0$  such that

$$(-\Delta)^\alpha w(|z|) \geq -\frac{c_{29}}{1 + |z|^{N+2\alpha}}, \quad |z| \geq 2. \quad (5.11)$$

By (5.6) and (5.11), there exists  $R_1 \geq R_0 + 2$  such that for  $|z| > R_1$ ,

$$\begin{aligned} (-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) &\geq \sigma_0 w(|z|) - \frac{c_{29}}{1 + |z|^{N+2\alpha}} \\ &= w(|z|) \left( \sigma_0 - \frac{c_{29}}{\ln(e + |z|^2)} \right) \\ &\geq 0. \end{aligned}$$

When  $|z| \leq R_1$ , it is clear that there exists  $c_{30} > 0$  such that

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) \geq -c_{30}.$$

Then there exists  $\Lambda_0 > 0$  such that for  $\lambda \geq \Lambda_0$ ,

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) + \lambda^{p-1} w^p(|z|) \geq 0 \quad \forall z \in \mathbb{R}^N, \quad (5.12)$$

which, together with (5.5), implies that (5.4) holds.  $\square$

Next we prove that  $u_\infty$  is not a trivial flat solution when  $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < p_\beta^*$ .

**Lemma 5.2** *Assume  $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$ , that  $w_{\Lambda_0}$  is given in (5.3) and  $u_\infty$  is given in (1.16). Then*

$$u_\infty(t, x) \leq w_{\Lambda_0}(t, x) \quad \forall (t, x) \in Q_\infty. \quad (5.13)$$

Moreover,

$$\lim_{t \rightarrow 0} u_\infty(t, x) = 0 \quad \text{uniformly on } B_\epsilon^c \quad \forall \epsilon > 0. \quad (5.14)$$

**Proof.** Let us denote

$$f_0(r) = \frac{k_0 \ln(e + r^2)}{1 + r^{N+2\alpha}} \quad \forall r \geq 0 \quad \text{and} \quad f_{n,k}(x) = kn^N f_0(n|x|) \quad \forall x \in \mathbb{R}^N,$$

where

$$k_0 = \left[ \omega_N \int_0^\infty \frac{\ln(e + r^2)}{1 + r^{N+2\alpha}} r^{N-1} dr \right]^{-1}.$$

Then for any  $\eta \in C_c(\mathbb{R}^N)$ , we have that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f_{n,k} \eta dx = k \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f_0(|x|) \eta \left( \frac{x}{n} \right) dx = k \eta(0).$$

Let  $t_n = n^{-2\alpha}$  and then

$$\begin{aligned} w_{\Lambda_0}(t_n, x) &= \Lambda_0 t_n^{-\frac{1+\beta}{p-1}} \frac{\ln(e + (t_n^{-\frac{1}{2\alpha}} |x|)^2)}{1 + (t_n^{-\frac{1}{2\alpha}} |x|)^{N+2\alpha}} = \Lambda_0 n^{\frac{2\alpha(1+\beta)}{p-1}} \frac{\ln(e + (n|x|)^2)}{1 + (n|x|)^{N+2\alpha}} \\ &= \frac{\Lambda_0}{k_0} n^{\frac{2\alpha(1+\beta)}{p-1} - N} n^N f_0(n|x|) \\ &\geq \frac{\Lambda_0}{k_0} \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N} n^N f_0(n|x|) = f_{n, k_{\tilde{n}}}(x), \end{aligned}$$

where  $\tilde{n} \leq n$  and  $k_{\tilde{n}} = \Lambda_0 \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N}$ . We see that  $k_{\tilde{n}} = \Lambda_0 \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N} \rightarrow \infty$  as  $\tilde{n} \rightarrow \infty$ , since  $\frac{2\alpha(1+\beta)}{p-1} - N > 0$ . Let  $u_{n, k_{\tilde{n}}}$  be the solution of (1.17) with initial data  $f_{n, k_{\tilde{n}}}$ . By Lemma 5.1,  $w_{\Lambda_0}(\cdot + t_n, \cdot)$  is a super-solution of (1.17) with initial data  $w_{\Lambda_0}(t_n, \cdot)$ , that is, for  $(t, x) \in Q_\infty$ ,

$$\partial_t w_\lambda(t + t_n, x) + (-\Delta)^\alpha w_\lambda(t + t_n, x) + (t + t_n)^\beta w_\lambda^p(t + t_n, x) \geq 0.$$

By the Comparison Principle,

$$u_{n, k_{\tilde{n}}}(t, x) \leq w_{\Lambda_0}(t + t_n, x) \quad \forall (t, x) \in Q_\infty,$$

for any  $\tilde{n} \leq n$ . Letting  $n \rightarrow \infty$  infers

$$u_{k_{\tilde{n}}}(t, x) \leq w_{\Lambda_0}(t, x) \quad \forall (t, x) \in Q_\infty, \quad (5.15)$$

where  $u_{k_{\tilde{n}}}$  is the solution of (1.17) with  $k_{\tilde{n}} \delta_0$  initial data. Thus (5.13) is obtained by letting  $\tilde{n} \rightarrow \infty$ . Finally (5.14) follows by the fact that

$$\lim_{t \rightarrow 0^+} w_{\Lambda_0}(t, x) = 0 \quad \forall x \in \mathbb{R}^N \setminus \{0\},$$

which completes the proof.  $\square$

**Lemma 5.3** *Assume  $1 < p < p_\beta^*$ , then there exists  $c_{31} > 0$  such that*

$$u_\infty(t, x) \geq \frac{c_{31} t^{-\frac{1+\beta}{p-1}}}{1 + |t^{-\frac{1}{2\alpha}} x|^{N+2\alpha}} \quad \forall (t, x) \in (0, 1) \times \mathbb{R}^N. \quad (5.16)$$

**Proof.** We divide the proof into two steps.

*Step 1.* Let  $\sigma_0 = 1 + \beta - \frac{N}{2\alpha}(p-1) > 0$ ,  $\eta(t) = 2 - t^{\sigma_0}$  for  $t > 0$  and denote

$$v_\epsilon(t, x) = \epsilon\eta(t)\Gamma_\alpha(t, x),$$

where  $\Gamma_\alpha$  is the fundamental solution of (1.17). In this step we prove that there exists  $\epsilon_0 > 0$  such that

$$u_{k_0} \geq v_{\epsilon_0} \quad \text{in } (0, 1) \times \mathbb{R}^N, \quad (5.17)$$

where  $k_0 = 2\epsilon_0$  and  $u_{k_0}$  is the solution of (1.17) with initial data  $k_0\delta_0$ . Indeed,

$$\partial_t v_\epsilon(t, x) = \epsilon\eta'(t)\Gamma_\alpha(t, x) + \epsilon\eta(t)\partial_t\Gamma_\alpha(t, x)$$

and

$$(-\Delta)^\alpha v_\epsilon(t, x) = \epsilon\eta(t)(-\Delta)^\alpha\Gamma_\alpha(t, x),$$

Let  $\Gamma_1(t^{-\frac{1}{2\alpha}}x) = \Gamma_\alpha(1, t^{-\frac{1}{2\alpha}}x)$ , then there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \leq \epsilon_0$  and  $(t, x) \in (0, 1) \times \mathbb{R}^N$ , we have that

$$\begin{aligned} & \partial_t v_\epsilon(t, x) + (-\Delta)^\alpha v_\epsilon(t, x) + t^\beta v_\epsilon^p(t, x) \\ &= \epsilon\eta'(t)t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + \epsilon^p\eta^p(t)t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x) \\ &\leq -\epsilon\sigma_0 t^{-\frac{N}{2\alpha}-1+\sigma_0}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + 2^p\epsilon^p t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x) \leq 0, \end{aligned}$$

the last inequality holds since  $-\frac{N}{2\alpha} - 1 + \sigma_0 = -\frac{N}{2\alpha}p + \beta$  and  $\Gamma_1$  is bounded. In particular, there holds

$$\partial_t v_{\epsilon_0}(t, x) + (-\Delta)^\alpha v_{\epsilon_0}(t, x) + t^\beta v_{\epsilon_0}^p(t, x) \leq 0 \quad \forall (t, x) \in (0, 1) \times \mathbb{R}^N. \quad (5.18)$$

Let  $f_n(x) = v_{\epsilon_0}(t_n, x)$  with  $t_n = n^{-2\alpha}$ . Since  $\lim_{t \rightarrow 0^+} \eta(t) = 2$ , then we have that  $f_n \rightarrow 2\epsilon_0\delta_0$  as  $n \rightarrow \infty$  in the weak sense of measures. There exists  $N_0 > 0$  such that  $t_n \in (0, \frac{1}{8})$  for  $n \geq N_0$ . Let  $w_n$  be the solution of (1.17) with initial data  $f_n$ , then it infers that

$$w_n(t, x) \geq v_{\epsilon_0}(t + t_n, x) \quad (t, x) \in (0, 1 - t_n) \times \mathbb{R}^N.$$

Because  $u_{k_0}$  is uniquely defined, there holds

$$w_n \rightarrow u_{k_0} \quad \text{as } n \rightarrow \infty \quad \text{in } (0, 1) \times \mathbb{R}^N$$

and

$$\lim_{n \rightarrow \infty} v_{\epsilon_0}(t + t_n, x) = v_{\epsilon_0}(t, x) \quad \forall (t, x) \in (0, 1) \times \mathbb{R}^N,$$

which imply (5.17).

Step 2. We claim that (5.16) holds. Since

$$v_{\epsilon_0}(t, x) \geq \epsilon_0 t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x) \quad (t, x) \in (0, 1) \times \mathbb{R}^N,$$

then, along with the relation  $T_\lambda[u_k] = u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}$ , we observe that for any  $\lambda > 0$ ,

$$\begin{aligned} u_{k_0\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}(t, x) &= \lambda^{\frac{2\alpha(1+\beta)}{p-1}} u_{k_0}(\lambda^{2\alpha} t, \lambda x) \\ &\geq \lambda^{\frac{2\alpha(1+\beta)}{p-1}} v_{\epsilon_0}(\lambda^{2\alpha} t, \lambda x) \\ &\geq \epsilon_0 \lambda^{\frac{2\alpha(1+\beta)}{p-1}-N} t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x). \end{aligned}$$

Let  $\varrho = \lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}$ ,  $t_\varrho = (2\varrho)^{\frac{1}{\frac{N}{2\alpha}-\frac{1+\beta}{p-1}}}$  and  $T_\varrho = \varrho^{\frac{1}{\frac{N}{2\alpha}-\frac{1+\beta}{p-1}}}$ , then

$$0 < t_\varrho < T_\varrho \rightarrow 0 \quad \text{as } \varrho \rightarrow \infty.$$

For  $(t, x) \in (t_\varrho, T_\varrho) \times \mathbb{R}^N$ , we have that

$$u_{k_0\varrho}(t, x) \geq \epsilon_0 \varrho t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x) \geq \frac{\epsilon_0}{2} t^{-\frac{1+\beta}{p-1}} \Gamma_1(t^{-\frac{1}{2\alpha}} x),$$

then

$$u_\infty(t, x) \geq \frac{\epsilon_0}{2} t^{-\frac{1+\beta}{p-1}} \Gamma_1(t^{-\frac{1}{2\alpha}} x) \quad \forall (t, x) \in (t_\varrho, T_\varrho) \times \mathbb{R}^N.$$

which implies (5.16) and completes the proof.  $\square$

**Proof of Theorem 1.2.** It follows from Proposition 5.1 and Lemma 5.2 that  $u_\infty$  is a nontrivial self-similar solution of (1.17) and (1.22) follows by (5.13), (5.16) and  $\ln(e + |t^{-\frac{1}{2\alpha}} x|^2) \leq 2 \ln(2 + |t^{-\frac{1}{2\alpha}} x|)$ , which ends the proof.  $\square$

We have actually a stronger result which is a consequence of Theorem 1.4-(i) proved in next section:

**Corollary 5.1** Assume  $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$ . Then either

$$\tilde{u} > u_\infty \quad \text{in } Q_\infty \tag{5.19}$$

or

$$\tilde{u} \equiv u_\infty \quad \text{in } Q_\infty. \tag{5.20}$$

## 5.2 The case $1 < p < 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$

For  $1 < p < 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$ , it follows from Lemma 5.3 that

$$\lim_{t \rightarrow 0^+} u_\infty(t, x) = \infty \quad \forall x \in \mathbb{R}^N. \quad (5.21)$$

**Proof of Theorem 1.3 (i).** Let  $f_0 \in C_c(\mathbb{R}^N)$  be a nonnegative function such that

$$\text{supp} f_0 \subset B_1(0) \quad \text{and} \quad \max_{x \in B_1(0)} f_0 = 1.$$

Denote

$$f_{n,k}(x) = kn^{\theta N} f_0(n^\theta(x - x_0)),$$

where  $k \leq n^\tau$  with  $\tau = \frac{1}{2}(\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha) > 0$ ,  $\theta = \frac{\tau}{N}$  and  $x_0 \in \mathbb{R}^N$ . Since  $f_{n,k}(x) \leq n^\tau$  for  $x \in B_1(x_0)$ ,  $f_n(x) = 0$  for  $x \in B_1^c(x_0)$  and

$$v_{\epsilon_0}(t_n, x) \geq \frac{c_{39} n^{\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha}}{(2 + |x_0|)^{N+2\alpha}} \quad \forall x \in B_1(x_0),$$

where  $t_n = n^{-2\alpha}$ . Then there exists  $N_0 > 0$  such that for any  $n \geq N_0$ ,

$$f_{n,k}(x) \leq v_{\epsilon_0}(t_n, x) \quad \forall x \in B_1(x_0).$$

Since  $n^{\theta N} f_0(n^\theta(x - x_0)) \rightarrow c_{41} \delta_{x_0}$ , as  $n \rightarrow \infty$  in weak sense of measures, for some  $c_{41} > 0$ .

Let  $w_{n,k}$  be the solution of (1.17) with initial data  $f_{n,k}$ , then

$$w_{n,k}(0, x) = f_{n,k}(x) \leq v_{\epsilon_0}(t_n, x) \leq u_\infty(t_n, x) \quad \forall x \in \mathbb{R}^N.$$

Therefore, by the Comparison Principle

$$w_{n,k}(t, x) \leq u_\infty(t + t_n, x) \quad \forall (t, x) \in Q_\infty.$$

We observe that

$$\lim_{k \rightarrow \infty} [\lim_{n \rightarrow \infty} w_{n,k}(t, x)] = u_\infty(t, x - x_0) \quad \forall (t, x) \in Q_\infty.$$

Thus, we derive that

$$u_\infty(t, x - x_0) \leq u_\infty(t, x) \quad \forall (t, x) \in Q_\infty. \quad (5.22)$$

Then  $u_\infty(t, x - x_0) = u_\infty(t, x)$  for all  $(t, x) \in Q_\infty$ , which implies that  $u_\infty$  is independent of  $x$ . Combining (5.1) and (5.16), implies that

$$u_\infty = \left( \frac{1 + \beta}{p - 1} \right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}.$$



The proof is complete.  $\square$

In the case of  $p = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$ , it derive from Lemma 5.3 that

$$\liminf_{t \rightarrow 0^+} u_\infty(t, x) \geq \lim_{t \rightarrow 0^+} \frac{c_{40} t^{-\frac{1+\beta}{p-1}}}{1 + |t^{-\frac{1}{2\alpha}} x|^{N+2\alpha}} = \frac{c_{40}}{|x|^{N+2\alpha}} \quad \forall x \in \mathbb{R}^N.$$

**Proof of Theorem 1.3 (ii).** We note that  $u_\infty$  is a self-similar solution of (1.17). Moreover, we derive (1.24) by (5.16), which ends the proof.  $\square$

### 5.3 The self-similar equation

In this section we prove Theorem 1.4.

**Proof of (i)** We set  $v_\infty(\eta) = t^{\frac{1+\beta}{p-1}} u_\infty(1, \eta)$ . Then relations (1.25) and (1.26) hold from Lemmas 5.2 and 5.3. Assume  $\tilde{v}$  is another positive solution of (1.20). Then  $(t, x) \mapsto t^{-\frac{1+\beta}{p-1}} \tilde{v}(t^{-\frac{1}{2\alpha}} x)$  is a positive self-similar solution of (1.23). By Proposition 5.2 it is larger than  $u_\infty$ . Thus  $v_\infty \leq \tilde{v}$ . Assume now that there exists  $\eta_0 \in \mathbb{R}^N$  such that  $v_\infty(\eta_0) = \tilde{v}(\eta_0)$ . and set  $w = \tilde{v} - v_\infty$ . Then

$$\begin{aligned} (-\Delta)^\alpha w(\eta_0) &= \lim_{\epsilon \rightarrow 0} (-\Delta)_\epsilon^\alpha w(\eta_0) \\ &= \lim_{\epsilon \rightarrow 0} \int_{B_\epsilon(\eta_0)} \frac{w(\eta_0) - w(\eta)}{|\eta - \eta_0|^{N+2\alpha}} d\eta \\ &< 0 \end{aligned}$$

Since  $\nabla w(\eta_0)$  we reach a contradiction.  $\square$

**Proof of (ii)** It is a consequence of the equality

$$u_\infty = U_p \iff v_\infty = \left( \frac{1+\beta}{p-1} \right)^{\frac{1}{p-1}}$$

**Open problem** We conjecture that in the case  $1 + \frac{2\alpha(1+\beta)}{N+2\alpha} < p < 1 + \frac{2\alpha(1+\beta)}{N}$ ,  $v_\infty$  is the unique positive solution of the self-similar equation satisfying (1.25). One step could be to prove that any positive solution  $\tilde{v}$  satisfying (1.25) satisfies, for some  $K > 1$ ,

$$\tilde{v} \leq K v_\infty \quad \text{in } \mathbb{R}^N. \quad (5.23)$$

We also conjecture that  $v_\infty$  satisfies the following asymptotic behavior

$$v_\infty(\eta) = c_{N,p,\alpha,\beta} |\eta|^{-N-2\alpha} \quad \text{as } |\eta| \rightarrow \infty. \quad (5.24)$$

Thus if any positive solution  $\tilde{v}$  endows the same property, the conclusion (and the uniqueness) follows.

**Acknowledge:** H. Chen is supported by NSFC of China grant 11401270 and L. Véron is supported by the MathAmsud collaboration program 13MATH-02 QUESP.

## References

- [1] D. R. Adams and M. Pierre, Capacity strong type estimates in semilinear problems, *Ann. Inst. Fourier Grenoble* 41, 117-135 (1991).
- [2] W. Al Sayed and L. Véron, Initial trace of solutions of semilinear heat equations with absorption, *Nonlinear Anal.* 93, 197-225 (2013).
- [3] Ph. Bénilan, H. Brezis and M. Crandall, A semilinear elliptic equation in  $L^1(\mathbb{R}^N)$ , *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* 2, 523-555 (1975).
- [4] H. Brezis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. *Notas de matematicas* 50, North-Holland (1973)
- [5] H. Brezis and A. Friedman, Nonlinear parabolic equations involving measures as initial conditions, *J. Math. Pures Appl.* 62(9), 73-97 (1983).
- [6] H. Brezis, L. A. Peletier and D. Terman, A very singular solution of the heat equation with absorption, *Arch. Rat. Mech. Anal.* 95, 185-209 (1986).
- [7] L. Caffarelli, C. Chan and A. Vasseur, Regularity theory for nonlinear integral operators, *J. Amer. Math. Soc.* 24(3), 849-869 (2011).
- [8] L. Caffarelli and A. Figalli, Regularity of solutions to the parabolic fractional obstacle problem, *J. für die Reine und Angewandte Mat.* 680, 191-233 (2013).
- [9] H. Chang Lara and G. Dávila, Regularity for solutions of non local parabolic equations, *Calc. of Var. and Part. Dif. Eq.* 49, 1-34 (2011).
- [10] Th. Cazenave and A. Haraux, An introduction to semilinear evolution equations, *Oxford Lecture Series in Mathematics and its Applications* 13. The Clarendon Press, Oxford University Press, New York (1998).
- [11] H. Chen, P. Felmer and A. Quaas, Large solution to elliptic equations involving fractional Laplacian, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, doi:10.1016/j.anihpc.2014.08.001.
- [12] Z. Chen, P. Kim and R. Song, Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc.* 12, 1307-1329 (2010).

- [13] H. Chen and L. Véron, Semilinear fractional elliptic equations involving measures, *J. Dif. Eq.* 257(5), 1457-1486 (2014).
- [14] H. Chen and L. Véron, Semilinear fractional elliptic equations with gradient nonlinearity involving measures, *J. Funct. Anal.* 266(8), 5467-5492 (2014).
- [15] H. Chen and L. Véron, Weakly and strongly singular solutions of semilinear fractional elliptic equations, *Asymptotic Anal.* 88, 165-184 (2014).
- [16] Z. Chen and J. Tokle, Global heat kernel estimates for fractional laplacians in unbounded open sets, *Probab. Theory Relat. Field* 149, 373-395 (2011).
- [17] J. Droniou and C. Imbert, Fractal first-order partial differential equations, *Arch. Rat. Mech. Anal.* 182, 299-331 (2006).
- [18] P. Felmer and Y. Wang, Radial symmetry of positive solutions to equations involving the fractional laplacian, *Comm. Contem. Math.* 16, No. 01 (2014).
- [19] A. Fino and G. Karch, Decay of mass for nonlinear equation with fractional laplacian, *Monatsh. Math.* 160, 375-384 (2010).
- [20] R. Frank, E. Lanzmann and L. Silvestre , Uniqueness of radial solutions for the fractional Laplacian, *Comm. Pure Appl. Math.* to appear. Arxiv: 1302.2652v1 (2013).
- [21] P. Grisvard, Équations différentielles abstraites, *Ann. Sci. Ecole Norm. Sup. 4ème sér., t. 2*, 311-395 (1969).
- [22] S. Kamin and L.A. Peletier, Singular solutions of the heat equation with absorption, *Proc. Am. Math. Soc.* 95, 205-210 (1985).
- [23] H. Komatsu, Fractional powers of operaors, *Pacific J. Math.* 19, 285-346 (1966).
- [24] M. Marcus and L. Véron, Initial trace of positive solutions of some nonlinear parabolic equations, *Comm. Part. Dif. Eq.* 24, 1445-1499 (1999).
- [25] M. Marcus and L. Véron, Semilinear parabolic equations with measure boundary data and isolated singularities, *J. Anal. Math.* 85, 245-290 (2001).
- [26] M. Marcus and L. Véron, Initial trace of positive solutions to semilinear parabolic inequalities, *Adv. Nonlinear Stud.* 2, 395-436 (2002).
- [27] I. Moutoussamy and L. Véron, Isolated singularities and asymptotic behaviour of the solutions of a semi-linear heat equation, *Asymptotic Anal.* 9, 259-289 (1994).

- [28] Y. Naito and T. Suzuki, Radial symmetry of self-similar solutions for semilinear heat equations, *J. Dif. Eq.* 163, 407-428 (2000).
- [29] T. Nguyen-Phuoc and L. Véron, Local and global properties of solutions of heat equation with superlinear absorption., *Adv. Diff. Eq.* 5-6, 487-522 (2011).
- [30] L. Oswald, Isolated positive singularities for a nonlinear heat equation, *Houston J. Math.* 14, 543-572 (1988).
- [31] E. Stein, Harmonic Analysis: Real-variable methods, orthogonality and oscillatory integrals. Princeton University Press, Princeton NJ. (1993).