Fractional heat equations involving initial measure data and subcritical absorption

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Abstract

We study the existence and uniqueness of weak solution to (F) $\partial_t u + (-\Delta)^\alpha u + h(t,u) = 0$ in $(0,\infty)\times\mathbb{R}^N$, is given by the initial condition $u(0,\cdot)=\nu$ in $\mathbb{R}^N$, where $N\geq 1$, the nonlocal operator $(-\Delta)^\alpha$ denotes the fractional Laplacian with $\alpha\in(0,1)$, ν is a bounded Radon measure and $h:(0,\infty)\times\mathbb{R}\mapsto\mathbb{R}$ is a continuous function satisfying a subcritical integrability condition.

In particular of $h(t,u)=t^\beta u^p$ with $\beta>-1$, there exists a unique solution u_k to (F) with $\nu=k\delta_0$, where δ_0 is the Dirac mass at the origin. We obtain that $u_k\to\infty$ in $(0,\infty)\times\mathbb{R}^N$ as $k\to\infty$ for $p\in(0,1]$ and the limit of u_k exists as $k\to\infty$ when $p\in(1,1+\frac{2\alpha(1+\beta)}{N})$, in this case denoting it by u_∞. When $p\in(1+\frac{2\alpha(1+\beta)}{N+2\alpha},1+\frac{2\alpha(1+\beta)}{N})$, u_∞ is the unique self-similar solution of $(F)_\infty$ $\partial_t u+(-\Delta)^\alpha u+t^\beta u^p=0$ in $(0,\infty)\times\mathbb{R}^N$ with the initial condition $u(0,\cdot)=0$ in $\mathbb{R}^N\setminus\{0\}$. When $p\in(1,1+\frac{2\alpha(1+\beta)}{N+2\alpha})$, u_∞ is a trivial self-similar solution to $(F)_\infty$, precisely,

$$u_\infty(t,x)=(\frac{1+\beta}{p-1})^{\frac{1}{p-1}}t^{-\frac{1+\beta}{p-1}},\quad (t,x)\in(0,\infty)\times\mathbb{R}^N.$$

Contents

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Key words: Fractional heat equation, Radon measure, Dirac mass, Self-similar solution, Very singular solution

MSC2010: 35R06, 35K05, 35R11

1 Introduction

Let $h : (0, \infty) \times \mathbb{R} \mapsto \mathbb{R}$ be a continuous function and $Q_\infty = (0, \infty) \times \mathbb{R}^N$ with $N \geq 1$. The first object of this paper is to consider the existence and uniqueness of weak solution to semilinear fractional heat equation involving initial measure data

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h(t, u) = 0 \quad &\text{in} \quad Q_\infty, \\ u(0, \cdot) = \nu \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ the bounded Radon measure space and the fractional Laplacian $(-\Delta)^\alpha$ with $\alpha \in (0, 1)$ is defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \to 0^+} (-\Delta)^\alpha_\epsilon u(x),$$

here for $\epsilon > 0$,

$$(-\Delta)^\alpha_\epsilon u(x) = -\int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz$$

and

$$\chi_\epsilon(r) = \begin{cases} 0, & \text{if} \quad r \in [0, \epsilon], \\ 1, & \text{if} \quad r > \epsilon. \end{cases}$$

In the pioneering work, Brezis and Friedman [4] have studied parabolic equation involving initial measure data

$$\begin{aligned} \partial_t u - \Delta u + u^p = 0 \quad &\text{in} \quad Q_\infty, \\ u(0, \cdot) = k\delta_0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{1.2}$$

where $k > 0$ is a constant and δ_0 denotes the Dirac mass at the origin. They asserted that if $1 < p < (N+2)/N$, then for every $k > 0$ there exists a unique solution u_k to (1.2). When $p \geq (N+2)/N$, problem (1.2) has no solution. Later on, Brezis, Peletier and Terman in [5] made use of dynamic method to obtain that there exists a unique very singular solution u_s to

$$\begin{aligned} \partial_t u - \Delta u + u^p = 0 \quad &\text{in} \quad \mathrm{Q}_\infty, \\ u(0,\cdot) = 0 \quad &\text{in} \quad \mathbb{R}^{\mathrm{N}} \setminus \{0\} \end{aligned} \tag{1.3}$$

with $1 < p < (N+2)/N$. In addition the asymptotic behavior of $u_s(1,\cdot)$ as $|x| \to \infty$ is given by

$$u_s(1,x) = c_1 e^{-\frac{1}{4}|x|^2} |x|^{\frac{2}{p-1}-N} \{1 - O(|x|^{-2})\},$$

where c_1 is a certain positive constant. From then on, Kamin and Peletier in [18] built the connection that the weak solution u_k convergence to the very singular solution u_s as $k \to \infty$. Marcus and Véron [23], Al Sayed and Véron in [2] obtained the very singular solution of (1.3) replaced u^p by $t^\beta u^p$ with $\beta > -1$ and $p \in (1, 1 + \frac{2(1+\beta)}{N})$. The initial trace of nonlinear parabolic equations with general Radon measure and general nonlinearity

$$\begin{aligned} \partial_t u - \Delta u + h(t,u) = 0 \quad &\text{in} \quad (0,\mathrm{T}) \times \Omega, \\ u = 0 \quad &\text{in} \quad (0,T) \times \partial\Omega, \\ u(0,\cdot) = \nu \quad &\text{in} \quad \Omega, \end{aligned} \tag{1.4}$$

where Ω is a domain in $\mathbb{R}^N$ and $T > 0$, has been well studied by Marcus and Véron. In [21, 22], they proved the existence and uniqueness of solutions to (1.4). Therein, the very singular solution plays a fundamental role in dealing with the initial trace and they used dichotomy argument to make classification to the initial trace of the solutions to (1.4).

Motivated by great applications in physics and by important links on the theory of Lévy process, semilinear fractional equations has been attracted much interest in last few years, and see the references [6, 7, 8, 9, 11, 14, 15, 16]. Recently, in [12] we obtained existence and uniqueness of weak solution to semilinear fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + f(u) = \nu \quad &\text{in} \quad \Omega, \\ u = 0 \quad &\text{in} \quad \Omega^{\mathrm{c}}, \end{aligned} \tag{1.5}$$

when ν is Radon measure and f satisfies a subcritical integrability condition.

Our purpose of this paper is to consider the existence and uniqueness of solutions to semilinear fractional heat equation (1.1) in the measure framework. We observe that most of the techniques used in the case of the Laplacian are not available. First, we make precise the notion of weak solution of (1.1) that we will use in this note.

Definition 1.1 *We say that u is a weak solution of (1.1), if for any $T > 0$, $u \in L^1(Q_T)$, $h(t,u) \in L^1(Q_T)$ and*

$$\begin{aligned} &\int_{Q_T} \{u(t,x)[-\partial_t \xi(t,x) + (-\Delta)^\alpha \xi(t,x)] + h(t,u)\xi(t,x)\}\, dxdt \\ &= \int_{\mathbb{R}^N} \xi(0,x) d\nu - \int_{\mathbb{R}^N} \xi(T,x) u(T,x) dx, \quad \forall \xi \in \mathbb{Y}_{\alpha,T}, \end{aligned} \tag{1.6}$$

where $Q_T = (0,T) \times \mathbb{R}^N$ and $\mathbb{Y}_{\alpha,T}$ is a space of functions $\varsigma : [0,T] \times \mathbb{R}^N \mapsto \mathbb{R}$ satisfying

(i) $\|\varsigma\|_{L^1(Q_T)} + \|\varsigma\|_{L^\infty(Q_T)} + \|\partial_t \varsigma\|_{L^\infty(Q_T)} + \|(-\Delta)^\alpha \varsigma\|_{L^\infty(Q_T)} < +\infty$;

(ii) for $t \in (0,T)$, there exist $M > 0$ and $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, $\|(-\Delta)^\alpha_\epsilon \varsigma(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} < M$.

Before stating our first main theorem, we introduce the subcritical integrability condition for the nonlinearity h, that is,

(H) *(i)* The function $h : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and for any $t \in (0,\infty)$, $h(t,0) = 0$ and $h(t,r_1) \geq h(t,r_2)$ if $r_1 \geq r_2$.

(ii) There exist $\beta > -1$ and a continuous, nondecreasing function $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$|h(t,r)| \leq t^\beta g(|r|), \quad \forall (t,r) \in (0,\infty) \times \mathbb{R}$$

and

$$\int_1^{+\infty} g(s) s^{-1-p^*_\beta} ds < +\infty, \tag{1.7}$$

where

$$p^*_\beta = 1 + \frac{2\alpha(1+\beta)}{N}. \tag{1.8}$$

Denote by $H_\alpha : (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ the heat kernel in $(0,\infty) \times \mathbb{R}^N$ with homogeneous boundary conditions and by $\mathbb{H}_\alpha[\nu]$ the heat potential of $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$, defined by

$$\mathbb{H}_\alpha[\nu](t,x) = \int_{\mathbb{R}^N} H_\alpha(t,x,y) d\nu(y).$$

We state the first theorem as follows.

Theorem 1.1 *Assume that $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ and the function h satisfies (H). Then problem (1.1) admits a unique weak solution u_ν such that*

$$\mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(t,\mathbb{H}_\alpha[\nu_+])] \leq u_\nu \leq \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(t,-\mathbb{H}_\alpha[\nu_-])] \quad \text{in } Q_\infty, \tag{1.9}$$

where ν_+ and ν_- are respectively the positive and negative part in the Jordan decomposition of ν. Furthermore,

(*i*) *if* ν *is nonnegative, so is* u_ν*;*

(*ii*) *the mapping:* $\nu \mapsto u_\nu$ *is increasing.*

According to Theorem 1.1, there exists a unique positive weak solution u_k to fractional problem

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = k\delta_0 \quad &\text{in} \quad \mathbb{R}^N \end{aligned} \tag{1.10}$$

where $\beta > -1$, $k > 0$ and $p \in (0, p^*_\beta)$. It is of interest to investigate the limit of u_k as $k \to \infty$ for $p \in (1, p^*_\beta)$, which exists since $\{u_k\}$ are an increasing sequence functions and bounded by $(\frac{1+\beta}{p-1})^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}$. We denote the limit u_∞ by

$$u_\infty = \lim_{k\to\infty} u_k \quad \text{in} \quad Q_\infty. \tag{1.11}$$

In fact, u_∞ and $\{u_k\}_{k>0}$ are classical solutions to equation

$$\partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty. \tag{1.12}$$

See Proposition 4.3.

Definition 1.2 (*i*) *A solution* u *of (1.12) is called a self-similar solution if*

$$u(t,x) = t^{-\frac{1+\beta}{p-1}} u(1, t^{-\frac{1}{2\alpha}} x), \quad (t,x) \in Q_\infty.$$

(*ii*) *A solution* u *of (1.12) is called a very singular solution if*

$$\lim_{t\to 0^+} \frac{u(t,0)}{\Gamma_\alpha(t,0)} = +\infty,$$

where Γ_α *is the fundamental solution of*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u = 0 \quad &\text{in} \quad Q_\infty, \\ u(0,\cdot) = \delta_0 \quad &\text{in} \quad \mathbb{R}^{\mathrm{N}}. \end{aligned} \tag{1.13}$$

We remark that for $p \in (1, p^*_\beta)$, a self-similar solution u of (1.12) is also a very singular solution, since

$$\lim_{t\to 0^+} \Gamma_\alpha(t,0) t^{\frac{N}{2\alpha}} = c_2, \tag{1.14}$$

for some $c_2 > 0$. For any self-similar solution u of (1.12), $v(\tilde{x}) := u(1, t^{-\frac{1}{2\alpha}} x)$ with $\tilde{x} = t^{-\frac{1}{2\alpha}} x$ is a solution of

$$(-\Delta)^\alpha v - \frac{1}{2\alpha} \nabla v \cdot \tilde{x} - \frac{1+\beta}{p-1} v + v^p = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1.15}$$

It is obvious that $(\frac{1+\beta}{p-1})^{\frac{1}{p-1}}$ is a trivial solution of (1.15) and then the function

$$U_p(t) := (\frac{1+\beta}{p-1})^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}, \quad t > 0 \tag{1.16}$$

is the flat self-similar solution of (1.12). From Proposition 4.1, u_k is not self-similar solution of (1.12). Therefore, our second goal in this paper is to search for non-flat self-similar solution of (1.12).

Theorem 1.2 *Assume that $\beta > -1$, u_∞ is defined by (1.11) and*

$$p \in (p_\beta^{**}, p_\beta^*).$$

where $p_\beta^{**} = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$ and $p_\beta^* = 1 + \frac{2\alpha(1+\beta)}{N}$. Then u_∞ is a positive self-similar solution of (1.12) with initial data $u(0,\cdot) = 0$ in $\mathbb{R}^N \setminus \{0\}$.

Moreover, there exists $c_3 > 1$ such that

$$\frac{c_3^{-1}}{1+|x|^{N+2\alpha}} \le u_\infty(1,x) \le \frac{c_3 \log(2+|x|)}{1+|x|^{N+2\alpha}}, \quad x \in \mathbb{R}^N. \tag{1.17}$$

When $p \in (p_\beta^{**}, p_\beta^*)$ with $\beta > -1$, we observe that u_∞ and U_p are self-similar solutions of (1.12) and u_∞ is not flat. An interest is to obtain the uniqueness of non flat self-similar solution of (1.12). We may propose our uniqueness for self-similar solution to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in} \quad Q_\infty, \\ \lim_{|x|\to\infty} u(1,x) = 0. \end{aligned} \tag{1.18}$$

Theorem 1.3 *Assume that $p \in (p_\beta^{**}, p_\beta^*)$ and u_∞ is defined by (1.11), then u_∞ is the unique positive self-similar solution of (1.18).*

We note when u is self-similar, $\lim_{|x|\to\infty} u(1,x) = 0$ is equivalent to $\lim_{|x|\to\infty} u(t,x) = 0$ for any $t > 0$. Finally, we state the properties of u_∞ when $p \in (1, p_\beta^{**}]$ as follows.

Theorem 1.4 *(i) Assume that $p \in (1, p_\beta^{**})$ and u_∞ is defined by (1.11). Then u_∞ is the flat self-similar solution of (1.12), that is,*

$$u_\infty = U_p,$$

where U_p is given by (1.16).

(ii) Assume that $p = p_\beta^{**}$ and u_∞ is defined by (1.11). Then u_∞ is a self-similar solution of (1.12) such that

$$u_\infty(t,x) \ge \frac{c_4 t^{-\frac{N+2\alpha}{2\alpha}}}{1+|t^{-\frac{1}{2\alpha}}x|^{N+2\alpha}}, \quad (t,x) \in (0,1) \times \mathbb{R}^N, \tag{1.19}$$

for some $c_4 > 0$.

We note that Theorem 1.4 indicates that there is no self-similar solution of (1.12) with initial data $u(0,\cdot)=0$ in $\mathbb{R}^N\setminus\{0\}$, since u_∞ is the least self-similar solution. In Theorem 1.4 part (ii), we don't know if the self-similar solution is flat or not.

According to theorems above, we make a summary for problem (1.15) as follows:

Remark 1.1 (i) *When $p\in(p^{**}_\beta,p^*_\beta)$, problem (1.15) under the condition $\lim_{|x|\to\infty}v(x)=0$ admits a unique positive solution.*

(ii) *When $p\in(1,p^{**}_\beta)$, problem (1.15) under the condition $\lim_{|x|\to\infty}v(x)=0$ has no any positive solution.*

This paper is organized as follows. In Section 2 we obtain that

$$\|\mathbb{H}^\Omega_\alpha[|\nu|]\|_{M^{p^*_\beta}(Q^\Omega_T,t^\beta dxdt)}\le c_5\|\nu\|_{\mathfrak{M}^b(\Omega)}$$

and the Kato's type inequality for the non-homogeneous problem. In Section 3 we prove Theorem 1.1. Section 4 is devoted to investigate the properties of solutions to (1.10). In Section 5 we give the proof of Theorem 1.2 and Theorem 1.4. Finally, we put the uniqueness of self-similar solution when $p\in(p^{**}_\beta,p^*_\beta)$ in the section 6.

2 Linear estimates

2.1 The Marcinkiewicz spaces

We recall the definition and basic properties of the Marcinkiewicz spaces.

Definition 2.1 *Let $\Theta\subset\mathbb{R}^{N+1}$ be an open domain and μ be a positive Borel measure in Θ. For $\kappa>1$, $\kappa'=\kappa/(\kappa-1)$ and $u\in L^1_{loc}(\Theta,d\mu)$, we set*

$$\|u\|_{M^\kappa(\Theta,d\mu)}=\inf\{c\in[0,\infty]:\int_E|u|d\mu\le c\left(\int_E d\mu\right)^{\frac{1}{\kappa'}},\ \forall E\subset\Theta\text{ Borel set}\}\tag{2.1}$$

and

$$M^\kappa(\Theta,d\mu)=\{u\in L^1_{loc}(\Theta,d\mu):\|u\|_{M^\kappa(\Theta,d\mu)}<\infty\}.\tag{2.2}$$

$M^\kappa(\Theta,d\mu)$ is called the Marcinkiewicz space of exponent κ or weak L^κ space and $\|.\|_{M^\kappa(\Theta,d\mu)}$ is a quasi-norm. The following property holds.

Proposition 2.1 *[3, 12] Assume $1\le q<\kappa<\infty$ and $u\in L^1_{loc}(\Theta,d\mu)$. Then there exists $c_6>0$ dependent of q,κ such that*

$$\int_E|u|^q d\mu\le c_6\|u\|_{M^\kappa(\Theta,d\mu)}\left(\int_E d\mu\right)^{1-q/\kappa},$$

for any Borel set E of Θ.

For an open regular domain Ω in $\mathbb{R}^N$, we denote $H_\alpha^\Omega : (0,\infty)\times\Omega\times\Omega \to \mathbb{R}_+$ the heat kernel in $(0,\infty)\times\Omega$ with homogeneous boundary conditions and by $\mathbb{H}_\alpha^\Omega[\nu]$ the heat potential of $\nu \in \mathfrak{M}^b(\Omega)$, defined by

$$\mathbb{H}_\alpha^\Omega[\nu](t,x) = \int_\Omega H_\alpha^\Omega(t,x,y)d\nu(y).$$

When $\Omega = \mathbb{R}^N$, by Fourier transform, it is obvious that

$$H_\alpha(t,x,y) = \frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N} e^{i(x-y)\cdot\zeta - t|\zeta|^{2\alpha}}d\zeta.$$

Proposition 2.2 *For any $\beta > -1$ and $T > 0$, there exists a positive constant $c_5 > 0$ dependent of N, α, β such that for $\nu \in \mathfrak{M}^b(\Omega)$,*

$$\|\mathbb{H}_\alpha^\Omega[|\nu|]\|_{M^{p_\beta^*}(Q_T^\Omega, t^\beta dxdt)} \le c_5\|\nu\|_{\mathfrak{M}^b(\Omega)}, \tag{2.3}$$

where p_β^* is defined by (1.8) and $Q_T^\Omega = (0,T)\times\Omega$.

In order to prove this proposition, we introduce some normal notations. For $\lambda > 0$ and $y \in \Omega$, let us denote

$$A_\lambda(y) = \{(t,x) \in Q_T^\Omega : H_\alpha(t,x,y) > \lambda\} \quad \text{and} \quad m_\lambda(y) = \int_{A_\lambda(y)} t^\beta dxdt.$$

Lemma 2.1 *There exists $c_7 > 0$ such that for any $\lambda > 1$,*

$$A_\lambda(y) \subset (0,\ c_7\lambda^{-\frac{2\alpha}{N}}] \times B_{c_7\lambda^{-\frac{1}{N}}}(y), \tag{2.4}$$

where $B_r(y)$ is the ball with radius r and center y in $\mathbb{R}^N$.

Proof. We observe that $H_\alpha(t,x,y) = t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,(x-y)t^{-\frac{1}{2\alpha}})$, where Γ_α is the fundamental solution of (1.13). From (1.1) in [10], there exists $c_8 > 0$ such that

$$\Gamma_\alpha(1,z) \le \frac{c_8}{1+|z|^{N+2\alpha}}.$$

On the one hand, for $(t,x) \in A_\lambda(y)$, we have that

$$t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,0) \ge t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,(x-y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

which implies that

$$t < \Gamma_\alpha^{\frac{2\alpha}{N}}(1,0)\lambda^{-\frac{2\alpha}{N}}. \tag{2.5}$$

On the other hand, letting $r = |x-y|$,

$$\frac{c_8 t}{t^{1+\frac{N}{2\alpha}} + r^{N+2\alpha}} \ge t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1,(x-y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

then

$$r \le (c_8 t\lambda^{-1})^{\frac{1}{N+2\alpha}}, \tag{2.6}$$

which, together with (2.5), implies that

$$r \le c_9 \lambda^{-\frac{1}{N}},$$

for some $c_9 > 0$. □

Proof of Proposition 2.2. By Lemma 2.1, there exists $c_{10} > 0$ such that

$$m_\lambda(y) \le c_{10}\lambda^{-1-\frac{2\alpha(1+\beta)}{N}}.$$

We observe that

$$H_\alpha^\Omega(t,x,y) \le H_\alpha(t,x,y). \tag{2.7}$$

Then for any Borel set $E \subset Q_T^\Omega$ and $y \in \Omega$, we have that

$$\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdt \le \lambda \int_E t^\beta dxdt + \int_{A_\lambda(y)} H_\alpha(t,x,y)t^\beta dxdt$$

and

$$\begin{aligned}\textstyle\int_{A_\lambda(y)} H_\alpha(t,x,y)t^\beta dxdt &= -\textstyle\int_\lambda^{+\infty} s dm_s(y)\\ &= \lambda m_\lambda(y) + \textstyle\int_\lambda^{+\infty} m_s(y)ds\\ &\le c_{10}\lambda^{-\frac{2\alpha(1+\beta)}{N}} + c_{10}\textstyle\int_\lambda^{+\infty} s^{-1-\frac{2\alpha(1+\beta)}{N}}ds\\ &\le c_{11}\lambda^{-\frac{2\alpha(1+\beta)}{N}},\end{aligned}$$

where $c_{11} = c_{10}[1 + \frac{N}{2\alpha(1+\beta)}]$. As a consequence, we have that

$$\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdt \le \lambda \int_E t^\beta dxdt + c_{11}\lambda^{-\frac{2\alpha(1+\beta)}{N}}.$$

Taking $\lambda = (\int_E t^\beta dxdt)^{-\frac{N}{N+2\alpha(1+\beta)}}$, we obtain that

$$\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdt \le (c_{11}+1)(\int_E t^\beta dxdt)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}. \tag{2.8}$$

By Fubini's theorem, we have that

$$\begin{aligned}\int_E \mathbb{H}_\alpha^\Omega[|\nu|](t,x)t^\beta dxdt &= \int_E\int_\Omega H_\alpha^\Omega(t,x,y)d|\nu(y)|t^\beta dxdt\\ &= \int_\Omega\int_E H_\alpha^\Omega(t,x,y)t^\beta dxdtd|\nu(y)|,\end{aligned}$$

which, together with (2.8), implies that

$$\int_E \mathbb{H}_\alpha^\Omega[|\nu|](t,x)t^\beta dxdt \le (c_{11}+1)\|\nu\|_{\mathfrak{M}^b(\Omega)}(\int_E t^\beta dxdt)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}.$$

Thus,

$$\|\mathbb{H}^{\Omega}_{\alpha}[|\nu|]\|_{M^{1+\frac{2\alpha(1+\beta)}{N}}(Q^{\Omega}_{T},t^{\beta}dxdt)} \leq (c_{11}+1)\|\nu\|_{\mathfrak{M}^{b}(\Omega)},$$

which ends the proof. □

2.2 Non-homogeneous problem

The following proposition is the Kato's type estimate which is essential tool to prove the uniqueness of solution to (1.1). For $T > 0$, we denote $Q_T = (0,T) \times \mathbb{R}^N$.

Proposition 2.3 *Assume that $\mu \in L^1(Q_T)$ and $\nu \in L^1(\mathbb{R}^N)$. Then there exists a unique weak solution u of the problem*

$$\begin{aligned} \partial_t u + (-\Delta)^{\alpha} u = \mu \quad &\text{in} \quad Q_T, \\ u(0,\cdot) = \nu \quad &\text{in} \quad \mathbb{R}^N \end{aligned} \tag{2.9}$$

and there exists $c_{12} > 0$ such that

$$\int_{Q_T} |u| dxdt \leq c_{12} \int_{Q_T} |\mu| dxdt + c_{12} \int_{\mathbb{R}^N} |\nu| dx. \tag{2.10}$$

Moreover, for any $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi \geq 0$, we have

$$\begin{aligned} &\textstyle\int_{Q_T} |u|\{-\partial_t \xi + (-\Delta)^{\alpha}\xi\} dxdt + \int_{\mathbb{R}^N} |u(T,x)|\xi(T,x)dx \\ &\quad \leq \textstyle\int_{Q_T} \xi \mathrm{sign}(u)\mu dxdt + \int_{\mathbb{R}^N} \xi(0,x)|\nu| dx. \end{aligned} \tag{2.11}$$

To prove Proposition 2.3, we introduce following notations. We say that $u : Q_T \mapsto \mathbb{R}$ is in $C^{\sigma,\sigma'}_{t,x}(Q_T)$ for $\sigma, \sigma' \in (0,1)$ if

$$\|u\|_{C^{\sigma,\sigma'}_{t,x}(Q_T)} := \|u\|_{L^{\infty}(Q_T)} + \sup_{Q_T} \frac{|u(t,x) - u(s,y)|}{|t-s|^{\sigma} + |x-y|^{\sigma'}} < +\infty$$

and $u \in C^{1+\sigma,2\alpha+\sigma'}_{t,x}(Q_T)$ if

$$\|u\|_{C^{1+\sigma,2\alpha+\sigma'}_{t,x}(Q_T)} := \|u\|_{L^{\infty}(Q_T)} + \|\partial_t u\|_{C^{\sigma,\sigma'}_{t,x}(Q_T)} + \|(-\Delta)^{\alpha} u\|_{C^{\sigma,\sigma'}_{t,x}(Q_T)} < +\infty.$$

Lemma 2.2 *Let $\mu \in C^1(Q_T) \cap L^{\infty}(Q_T)$, $\nu \in L^{\infty}(\mathbb{R}^N)$ and u is a solution of problem (2.9), then there exists $\sigma \in (0,1)$ such that $u \in C^{1+\sigma,2\alpha+\sigma}_{t,x}$ in $(T_0,T) \times \mathbb{R}^N$ for any $T_0 \in (0,T)$.*

In particular, if $\|D^2\nu\|_{L^{\infty}(\mathbb{R}^N)} + \|(-\Delta)^{\alpha}\nu\|_{C^{1-\alpha}_{x}(\mathbb{R}^N)} < \infty$, we have $u \in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$.

Proof. *Step 1.* When $\|D^2\nu\|_{L^\infty(\mathbb{R}^N)} + \|(-\Delta)^\alpha\nu\|_{C_x^{1-\alpha}(\mathbb{R}^N)} < \infty$, it follows directly by [7, $(A.1)$] that $u \in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$.

Step 2. When $\nu \in L^\infty(\mathbb{R}^N)$, we use [8, Theorem 6.1] to obtain that $u \in C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}(Q_T)$ for some $\sigma > 0$. For any $T_0 \in (0,T)$, let $\eta : [0,T] \to [0,1]$ be a C^2 functions such that $\eta = 0$ in $[0, \frac{T_0}{4}]$ and $\eta = 1$ in $[T_0, T]$ and $v = \eta u$ in Q_T. Then we obtain that for $t \in [\frac{T_0}{4}, T]$ and $x \in \mathbb{R}^N$,

$$\partial_t v + (-\Delta)^\alpha v = \eta\mu + \eta'(t)u,$$

where $\eta\mu + \eta'(t)u \in C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}(Q_T)$ and $v(0,\cdot) = 0$ in $\mathbb{R}^N$, Then we apply argument in Step 1 to obtain that $v \in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$. Therefore, u is $C_{t,x}^{1+\sigma,2\alpha+\sigma}$ in $(T_0,T)\times\mathbb{R}^N$. The proof is complete. □

Lemma 2.3 *(i) Let $\mu \in C^1(Q_T) \cap L^\infty(Q_T)$ and $\nu \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then problem (2.9) admits a unique solution u and for some $\sigma \in (0,1)$, u is $C_{t,x}^{1+\sigma,2\alpha+\sigma}$ in $(T_0,T)\times\mathbb{R}^N$ for any $T_0 \in (0,T)$.*

(ii) Assume that $\mu \in C^1(Q_T)\cap L^\infty(Q_T)\cap L^1(Q_T)$, $\nu \in C^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$ and u be the solution of (2.9), then $u \in L^1(Q_T)$, is $C_{t,x}^{1+\sigma,2\alpha+\sigma}$ in $(T_0,T)\times\mathbb{R}^N$ for any $T_0 \in (0,T)$ and for any $\xi \in \mathbb{Y}_{\alpha,T}$,

$$\begin{aligned} &\textstyle\int_{Q_T} u(t,x)[-\partial_t\xi(t,x) + (-\Delta)^\alpha\xi(t,x)]dxdt \\ &= \textstyle\int_{Q_T} \mu(t,x)\xi(t,x)dxdt + \int_{\mathbb{R}^N} \xi(0,x)\nu dx - \int_{\mathbb{R}^N} \xi(T,x)u(T,x)dx. \end{aligned} \tag{2.12}$$

(iii) Let $\mu \in C^1(Q_T) \cap L^\infty(Q_T)$ and $\nu \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then problem

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u = \mu \quad &\text{in} \quad Q_T, \\ u(T,\cdot) = \nu \quad &\text{in} \quad \mathbb{R}^N \end{aligned} \tag{2.13}$$

admits a unique solution $u \in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$ for some $\sigma \in (0,1)$.

Moreover, $u \in \mathbb{Y}_{\alpha,T}$ if $\mu \in C^1(Q_T)\cap L^\infty(Q_T)\cap L^1(Q_T)$ and $\nu \in C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N)$.

Proof. (i) By [8, Theorem 2.6, Theorem 6.1], there exists a unique viscosity solution $u \in C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}(Q_T)$ with $\sigma > 0$ of (2.9), and then it follows by Lemma 2.2 that u is $C_{t,x}^{1+\sigma',2\alpha+\sigma'}$ in $(T_0,T)\times\mathbb{R}^N$ for any $T_0 \in (0,T)$ and some $\sigma' \in (0, \min\{\frac{\sigma}{2\alpha}, \sigma\})$. Then u is a classical solution of (2.9).

(ii) *To prove $u \in L^1(Q_T)$ and $u(t,\cdot) \in L^1(\mathbb{R}^N)$ for $t \in (0,T)$.* By Duhamel formula, we have

$$\begin{aligned} \|u(t,\cdot)\|_{L^1(\mathbb{R}^N)} &\le \int_{\mathbb{R}^N}\int_{Q_T} H_\alpha(t,s,x,y)|\mu(s,y)|dydsdx \\ &\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H_\alpha(t,x,y)|\nu(y)|dydx \\ &\le \|\mu\|_{L^1(Q_T)} + \|\nu\|_{L^1(\mathbb{R}^N)} \end{aligned}$$

and

$$\|u\|_{L^1(Q_T)} = \int_0^T \|u(t,\cdot)\|_{L^1(\mathbb{R}^N)}dt \le T(\|\mu\|_{L^1(Q_T)} + \|\nu\|_{L^1(\mathbb{R}^N)}).$$

To prove that $\|(-\Delta)^\alpha_\epsilon u(t,\cdot)\|_{L^\infty(\mathbb{R}^N)}$ *is uniformly bounded with to* $\epsilon \in (0,\epsilon_0)$. Since $u(t,\cdot) \in C^{2\alpha+\sigma}_x(\mathbb{R}^N)$ for some $\sigma \in (0, \min\{2-2\alpha, 1\})$, then for $x \in \mathbb{R}^N$ and $y \in B_1(0)$, $|u(x+y)+u(x-y)-2u(x)| \le \|u(t,\cdot)\|_{C^{2\alpha+\sigma}_x(\mathbb{R}^N)}|y|^{2\alpha+\sigma}$. Thus

$$\begin{aligned}
\|\|(-\Delta)^\alpha_\epsilon u(t,\cdot)\|\|_{L^\infty(\mathbb{R}^N)} &\le \sup_{x\in\mathbb{R}^N}[\int_{\mathbb{R}^N\setminus B_1(0)} \frac{|u(x+y)-u(x)|}{|y|^{N+2\alpha}}dy \\
&\quad + \frac{1}{2}\int_{B_1(0)\setminus B_\epsilon(0)} \frac{|u(x+y)+u(x-y)-2u(x)|}{|y|^{N+2\alpha}}dy] \\
&\le 2\|u\|_{L^1(\mathbb{R}^N)} + \int_{B_1(0)} |y|^{\sigma-N}dy \|u(t,\cdot)\|_{C^{2\alpha+\sigma}_x(\mathbb{R}^N)}.
\end{aligned}$$

To prove (2.12). We claim that

$$\int_{Q_T} \xi(-\Delta)^\alpha_\epsilon u dxdt = \int_{Q_T} u(-\Delta)^\alpha_\epsilon \xi dxdt, \qquad \forall \xi \in \mathbb{Y}_{\alpha,T}. \tag{2.14}$$

Indeed, by using the fact that for any $t > 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(t,z)-u(t,x)]\xi(t,x)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx \\
&\qquad = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(t,x)-u(t,z)]\xi(t,z)}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx,
\end{aligned}$$

then we have

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}^N} \xi(t,x)(-\Delta)^\alpha_\epsilon u(t,x)dx \\
&\quad = -\frac{1}{2}\textstyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[\frac{(u(t,z)-u(t,x))\xi(t,x)}{|z-x|^{N+2\alpha}} + \frac{(u(t,x)-u(t,z))\xi(t,z)}{|z-x|^{N+2\alpha}}]\chi_\epsilon(|x-z|)dzdx \\
&\quad = \frac{1}{2}\textstyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{[u(t,z)-u(t,x)][\xi(t,z)-\xi(t,x)]}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx.
\end{aligned}$$

Similarly,

$$\textstyle\int_{\mathbb{R}^N} u(t,x)(-\Delta)^\alpha_\epsilon \xi(t,x)dx = \frac{1}{2}\int_{\mathbb{R}^{2N}} \frac{[u(t,z)-u(t,x)][\xi(t,z)-\xi(t,x)]}{|z-x|^{N+2\alpha}}\chi_\epsilon(|x-z|)dzdx.$$

Then (2.14) holds. Since u is $C^{1+\sigma,2\alpha+\sigma}_{t,x}$ in $(T_0,T)\times\mathbb{R}^N$ for any $T_0 \in (0,T)$ and ξ belongs to $\mathbb{Y}_{\alpha,T}$, $(-\Delta)^\alpha_\epsilon \xi(t,\cdot) \to (-\Delta)^\alpha \xi(t,\cdot)$ and $(-\Delta)^\alpha_\epsilon u(t,\cdot) \to (-\Delta)^\alpha u(t,\cdot)$ as $\epsilon \to 0$ in $\mathbb{R}^N$ and $(-\Delta)^\alpha_\epsilon \xi(t,\cdot)$, $(-\Delta)^\alpha_\epsilon u(t,\cdot) \in L^\infty(\mathbb{R}^N)$ and $\xi(t,\cdot), u(t,\cdot) \in L^1(\mathbb{R}^N)$, then it follows by the Dominated Convergence Theorem

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N} \xi(t,x)(-\Delta)^\alpha_\epsilon u(t,x)dx = \int_{\mathbb{R}^N} \xi(t,x)(-\Delta)^\alpha u(t,x)dx$$

and

$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N}(-\Delta)_\epsilon^\alpha\xi(t,x)u(t,x)dx=\int_{\mathbb{R}^N}(-\Delta)^\alpha\xi(t,x)u(t,x)dx.$$

Combining with (2.14), taking $\epsilon\to 0^+$, we have that

$$\int_{\mathbb{R}^N}\xi(t,x)(-\Delta)^\alpha u(t,x)dx=\int_{\mathbb{R}^N}(-\Delta)^\alpha\xi(t,x)u(t,x)dx,$$

integrating over $[0,T]$ and by (2.9), we conclude that (2.12) holds.

(iii) Let u be the solution of problem (2.9) and

$$w(t,x)=u(T-t,x),\quad (t,x)\in[0,T]\times\mathbb{R}^N.$$

Then w is a solution of (2.13) and for some $\sigma\in(0,1)$, w is $C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$. On the contrary, if w is a solution of (2.13), then $u(t,x)=w(T-t,x)$ for $(t,x)\in[0,T]\times\mathbb{R}^N$ is a solution of (2.9), then the uniqueness holds since the solution of (2.9) is unique. Since $u\in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$, then $(-\Delta)^\alpha u(t,\cdot)\in C_x^\sigma$ and then $(-\Delta)_\epsilon^\alpha u(t,\cdot)$ is bounded, which implies $u\in\mathbb{Y}_{\alpha,T}$. □

Proof of Proposition 2.3. *Uniqueness.* Let v be a weak solution of

$$\begin{aligned}\partial_t v+(-\Delta)^\alpha v&=0\quad\text{in}\quad Q_T,\\ v(0,\cdot)&=0\quad\text{in}\quad\mathbb{R}^N.\end{aligned}\tag{2.15}$$

We claim that $v=0$ a.e. in Q_T.

In fact, let ω be a Borel subset of Q_T and $\eta_{\omega,n}$ be the solution of

$$\begin{aligned}-\partial_t u+(-\Delta)^\alpha u&=\zeta_n\quad\text{in}\quad Q_T,\\ u(T,\cdot)&=0\quad\text{in}\quad\mathbb{R}^N,\end{aligned}\tag{2.16}$$

where $\zeta_n:\bar{Q}_T\mapsto[0,1]$ is a function $C_0^1(Q_T)$ such that

$$\zeta_n\to\chi_\omega\quad\text{in } L^\infty(\bar{Q}_T)\quad\text{as } n\to\infty.$$

Then by Lemma 2.3, we have that $\eta_{\omega,n}\in\mathbb{Y}_{\alpha,T}$ and

$$\int_{Q_T}v\zeta_n dxdt=0.$$

Thus passing the limit of $n\to\infty$, we have

$$\int_\omega vdxdt=0.$$

This implies $v=0$ a.e. in Q_T.

Existence and estimate (2.11). For $\delta > 0$, we define an even convex function ϕ_δ by

$$\phi_\delta(t) = \begin{cases} |t| - \frac{\delta}{2}, & \text{if} \quad |\mathrm{t}| \geq \delta, \\ \frac{t^2}{2\delta}, & \text{if} \quad |\mathrm{t}| < \delta/2. \end{cases} \tag{2.17}$$

Then for any $t, s \in \mathbb{R}$, $|\phi'_\delta(t)| \leq 1$, $\phi_\delta(t) \to |t|$ and $\phi'_\delta(t) \to \mathrm{sign(t)}$ when $\delta \to 0^+$. Moreover

$$\phi_\delta(s) - \phi_\delta(t) \geq \phi'_\delta(t)(s-t). \tag{2.18}$$

Let $\{\mu_n\}$, $\{\nu_n\}$ be two sequences functions in $C_0^2(Q_T)$, $C_0^2(\mathbb{R}^N)$ respectively such that

$$\lim_{n\to\infty} \int_{Q_T} |\mu_n - \mu| dxdt = 0, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |\nu_n - \nu| dx = 0.$$

We denote u_n be the corresponding solution to (2.9) replaced μ, ν by μ_n, ν_n, respectively. By Lemma 2.2 and Lemma 2.3(ii), $u_n \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T) \cap L^1(Q_T)$ and then we use Lemma 2.3 in [12] and Lemma 2.3 (ii) to obtain that for any $\delta > 0$ and $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi \geq 0$,

$$\begin{aligned} &\textstyle\int_{Q_T} \phi_\delta(u_n)[-\partial_t \xi + (-\Delta)^\alpha \xi] dxdt + \int_{\mathbb{R}^N} \xi(T,x)\phi_\delta(u_n(T,x))dx \\ &\quad = \textstyle\int_{Q_T} \xi[\partial_t \phi_\delta(u_n) + (-\Delta)^\alpha \phi_\delta(u_n)] dxdt + \int_{\mathbb{R}^N} \xi(0,x)\phi_\delta(\nu_n)dx \\ &\quad\quad \leq \textstyle\int_{Q_T} \xi \phi'_\delta(u_n)[\partial_t u_n + (-\Delta)^\alpha u_n] dxdt + \int_{\mathbb{R}^N} \xi(0,x)\phi_\delta(\nu_n)dx \\ &\quad\quad\quad = \textstyle\int_{Q_T} \xi \phi'_\delta(u_n)\mu_n dxdt + \int_{\mathbb{R}^N} \xi(0,x)\phi_\delta(\nu_n)dx. \end{aligned}$$

Taking $\delta \to 0^+$, we obtain

$$\begin{aligned} &\int_{Q_T} |u_n|[-\partial_t \xi + (-\Delta)^\alpha \xi] dxdt + \int_{\mathbb{R}^N} \xi(T,x)|u_n(T,x)|dx \\ &\qquad \leq \int_{Q_T} \xi \mathrm{sign}(u_n)\mu_n dxdt + \int_{\mathbb{R}^N} \xi(0,x)|\nu_n|dx. \end{aligned} \tag{2.19}$$

Denote by η_k the solution of

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u = \varsigma_k \quad &\text{in} \quad Q_T, \\ u(T,\cdot) = 0 \quad &\text{in} \quad \mathbb{R}^N, \end{aligned} \tag{2.20}$$

where $\varsigma_k : Q_T \mapsto [0,1]$ is a C_0^2 function such that $\varsigma_k = 1$ in $(0,T) \times B_k(0)$. From the proof of Lemma 2.3, $\tilde{\eta}_k(t,x) := \eta_k(T-t,x)$ solves

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u = \varsigma_k \quad &\text{in} \quad Q_T, \\ u(0,\cdot) = 0 \quad &\text{in} \quad \mathbb{R}^N. \end{aligned}$$

By Lemma 2.2, then $\tilde{\eta}_k \in C^{1+\sigma,2\alpha+\sigma}_{t,x}(Q_T)$ with some $\sigma \in (0,1)$ and

$$\begin{aligned} 0 \le \tilde{\eta}_k(t,x) &\le c_8 \int_t^T \int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}}}{1+|(s-t)^{-\frac{1}{2\alpha}}(y-x)|^{N+2\alpha}} dyds \\ &\le c_8 \int_t^T \int_{\mathbb{R}^N} \frac{dz}{1+|z|^{N+2\alpha}} ds \\ &= c_{13}(T-t). \end{aligned}$$

Taking $\xi = \eta_k$ in (2.19), we derive

$$\int_{Q_T} |u_n| \chi_{(0,T)\times B_k(0)} dxdt \le c_{13} T \int_{Q_T} |\mu_n| dxdt + c_{13} T \int_{\mathbb{R}^N} |\nu_n| dx.$$

Then taking $k \to \infty$,

$$\int_{Q_T} |u_n| dxdt \le c_{13} T \int_{Q_T} |\mu_n| dxdt + c_{13} T \int_{\mathbb{R}^N} |\nu_n| dx. \tag{2.21}$$

Similarly,

$$\int_{Q_T} |u_n - u_m| dx \le c_{13} T \int_{Q_T} |\mu_n - \mu_m| dxdt + c_{13} T \int_{\mathbb{R}^N} |\nu_n - \nu_m| dx. \tag{2.22}$$

Therefore, $\{u_n\}$ is a Cauchy sequence in $L^1(Q_T)$ and its limit u is a weak solution of (2.9). Letting $n \to \infty$, (2.11) and (2.10) follow by (2.19) and (2.21), respectively. □

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first consider the classical solution of (1.1) in domain Q_T with regular initial data for $T > 0$. Assume that

$\widetilde{(H)}$ The function h satisfies (H) part (i) and $h \in C^\gamma([0,T] \times \mathbb{R})$ for some $\gamma \in (0,1)$.

Lemma 3.1 *Let h satisfy $\widetilde{(H)}$, $\phi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $T > 0$. Then there exists a unique solution u_ϕ of (1.1) in Q_T with $\nu = \phi$ such that*

$$\mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, \mathbb{H}_\alpha[\phi_+])] \le u_\phi \le \mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, -\mathbb{H}_\alpha[\phi_-])] \quad \text{in } Q_T, \tag{3.1}$$

where $\phi_\pm = \max\{0, \pm\phi\}$. Moreover,
(i) $u_\phi \ge 0$ if $\phi \ge 0$ in Ω;
(ii) the mapping $\phi \mapsto u_\phi$ is increasing.

Proof. *To prove that there exists $T'>0$ such that problem*

$$\begin{aligned} \partial_t u+(-\Delta)^\alpha u+h(t,u)\chi_{(0,T')}(t)=0 \quad &\text{in} \quad \mathrm{Q}_\infty,\\ u(0,\cdot)=\phi \quad &\text{in} \quad \mathbb{R}^{\mathrm{N}} \end{aligned} \tag{3.2}$$

admits a solution $u_{0,\phi}$.

For parameter $T'>0$, we denote the operator Υ by

$$\Upsilon(u)=\mathbb{H}_\alpha[\phi-h(t,u)\chi_{[0,T']}], \quad u\in L^\infty(Q_\infty).$$

We observe that $\mathbb{H}_\alpha[\phi](0,\cdot)=\phi$ in $\mathbb{R}^N$, then

$$\begin{aligned} \|\phi\|_{L^\infty(\mathbb{R}^N)} &\le \|\mathbb{H}_\alpha[\phi]\|_{L^\infty(Q_\infty)}\\ &\le c_{14}\|\phi\|_{L^\infty(\mathbb{R}^N)}\|\textstyle\int_{\mathbb{R}^N}\frac{t^{-\frac{N}{2\alpha}}}{1+|t^{-\frac{1}{2\alpha}}(x-y)|^{N+2\alpha}}dy\|_{L^\infty(Q_\infty)}\\ &\le c_{15}\|\phi\|_{L^\infty(\mathbb{R}^N)}, \end{aligned} \tag{3.3}$$

where $c_{14}>0$ and $c_{15}=c_{14}\int_{\mathbb{R}^N}\frac{1}{1+|z|^{N+2\alpha}}dz$. Let $\mathcal{B}_r=\{u\in L^\infty(Q_\infty): \|u\|_{L^\infty(Q_\infty)}\le r\}$ with $r=2c_{15}\|\phi\|_{L^\infty(\mathbb{R}^N)}$ and $A=[0,T]\times[-r,r]$. For $T'>0$ small enough and $u\in\mathcal{B}_r$, we derive that

$$\begin{aligned} &\|\mathbb{H}_\alpha[h(\cdot,u)\chi_{(0,T')}(\cdot)]\|_{L^\infty(Q_\infty)}\\ &\quad\le c_{14}\max_{(t,s)\in A}|h(t,s)|\|\textstyle\int_t^{T'}\int_{\mathbb{R}^N}\frac{(s-t)^{-\frac{N}{2\alpha}}}{1+|(s-t)^{-\frac{1}{2\alpha}}(x-y)|^{N+2\alpha}}dyds\|_{L^\infty(Q_{T'})}\\ &\quad\le c_{15}T'\max_{(t,s)\in A}|h(t,s)|. \end{aligned}$$

Therefore, choosing $T'>0$ such that $T'\max_{(t,s)\in A}|h(t,s)|\le\frac{r}{2}$, we derive that

$$\begin{aligned} \|\Upsilon(u)\|_{L^\infty(Q_\infty)} &\le c_{15}\left(\|\phi\|_{L^\infty(\mathbb{R}^N)}+T'\max_{(t,s)\in A}|h(t,s)|\right)\\ &\le r. \end{aligned} \tag{3.4}$$

For our choosing r and T', we have that $\Upsilon(\mathcal{B}_r)\subset\mathcal{B}_r$.

Next we prove that the operator Υ is compact in $\mathcal{B}_r$. We observe that for $u\in\mathcal{B}_r$,

$$\|u(t,\cdot)\|_{L^1(\mathbb{R}^N,\,\tilde{\omega}dx)}\le c_{16}\|u\|_{L^\infty(Q_\infty)}\le c_{16}r, \tag{3.5}$$

where $c_{16}>0$ and $\tilde{\omega}(x)=(1+|x|^{N+2\alpha})^{-1}$. By [8, Theorem 5.1], there exist $\sigma\in(0,1)$ and $c_{17},c_{18}>0$ only dependent of α,N such that

$$\begin{aligned} \|\Upsilon(u)\|_{C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}(Q_\infty)} &\le c_{17}\{\|u\|_{L^\infty(Q_\infty)}+\sup_{\tau>0}\|u(\tau,\cdot)\|_{L^1(\mathbb{R}^N,\,\tilde{\omega}dx)}\\ &\qquad+\|h(\cdot,u)\|_{L^\infty(Q_{T'})}\}\\ &\le c_{18}\{\|\phi\|_{L^\infty(\mathbb{R}^N)}+\max_{(t,s)\in A}|h(t,s)|\}. \end{aligned} \tag{3.6}$$

Therefore, Υ is compact. It follows by Schauder's fixed point theorem, that there exists $u_{0,\phi} \in \mathcal{B}_r$ such that $\Upsilon(u_{0,\phi}) = u_{0,\phi}$. It infers by (3.6) that $u_{0,\phi} \in C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}(Q_\infty)$. By Lemma 2.2, $u_{0,\phi}$ is in $C_{t,x}^{1+\sigma',2\alpha+\sigma'}(Q_\infty)$ for some $\sigma' > 0$ and $u_{0,\phi}$ is a classical solution of (3.2).

Moreover, since $\mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, -\mathbb{H}_\alpha[\phi_-])]$ and $\mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, \mathbb{H}_\alpha[\phi_+])]$ are super and sub solution of (3.2) respectively, then by Comparison Principle, we have that

$$\mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, \mathbb{H}_\alpha[\phi_+])] \leq u_{0,\phi} \leq \mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, -\mathbb{H}_\alpha[\phi_-])] \quad \text{in } Q_\infty. \tag{3.7}$$

The mapping $\phi \mapsto u_{0,\phi}$ is increasing. Let v_1, v_2 be solutions of (3.2) with initial data ϕ_1 and ϕ_2 respectively, where $\phi_1 \geq \phi_2$ and denote $w = v_1 - v_2$. We prove this argument by contradiction. Assume that there exists $(t_0, x_0) \in Q_\infty$ satisfying $w(t_0, x_0) < 0$. Since

$$w(0, x) = \phi_1(x) - \phi_2(x) \geq 0, \quad x \in \mathbb{R}^N,$$

then there exist $t_1 \in (0, t_0]$ and $x_1 \in \mathbb{R}^N$ such that

$$w(t_1, x_1) = \min_{(t,x)\in[0,t_1]\times\mathbb{R}^N} w(t, x) < 0,$$

thus, $\partial_t w(t_1, x_1) \leq 0$ and $(-\Delta)^\alpha w(t_1, x_1) < 0$ and by monotonicity of h, we have $h(t_1, v_1(t_1, x_1)) \leq h(t_1, v_2(t_1, x_1))$. Therefore,

$$\partial_t w(t_1, x_1) + (-\Delta)^\alpha w(t_1, x_1) + h(t_1, v_1(t_1, x_1)) - h(t_1, v_2(t_1, x_1)) < 0.$$

which contradicts that v_1 and v_2 are solutions of (3.2).

Moveover, the uniqueness of solution to (3.2) follows by the fact that the mapping $\phi \mapsto u_{0,\phi}$ is increasing. Then 0 is the unique solution of (3.2) with 0 initial data. Therefore, $u_{0,\phi} \geq 0$ if $\phi \geq 0$.

To prove that (1.1) in Q_T with $T > 0$ admits a solution u_ϕ.
Since $\int_{\mathbb{R}^N} H_\alpha(t, x - y)dy = 1$ for any $t > 0$ and $x \in \mathbb{R}^N$, then

$$\begin{aligned}
\|\phi\|_{L^\infty(\mathbb{R}^N)} &\leq \|\mathbb{H}_\alpha[\phi]\|_{L^\infty(Q_\infty)} \\
&\leq \|\phi\|_{L^\infty(\mathbb{R}^N)} \| \textstyle\int_{\mathbb{R}^N} H_\alpha(t, x - y)dy\|_{L^\infty(Q_\infty)} \\
&= \|\phi\|_{L^\infty(\mathbb{R}^N)},
\end{aligned}$$

which, combining (3.7) with $|\phi|$, infers that

$$\|u_{0,\phi}(T', \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \|\mathbb{H}_\alpha[|\phi|](T', \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \|\phi\|_{L^\infty(\mathbb{R}^N)}. \tag{3.8}$$

Thus, there exists a unique solution $u_{1,\phi}$ to (3.2) with initial data $u_{1,\phi}(0, \cdot) = u_{0,\phi}(T', \cdot)$ and we note that it is also able to choose the same T' by (3.8) in this step and it could iterate this process by times k and we obtain a unique

solution $u_{k,\phi}$ of (3.2) with $u_{k,\phi}(0,\cdot) = u_{k-1,\phi}(T',\cdot)$ initial data for the same $T' > 0$. Since for any $T > 0$, there exists $m \in \mathbb{N}$ such that $mT' \geq T$ and $(m-1)T' < T$, letting

$$u_\phi(t,x) = \sum_{i=0}^{m-1} u_{i,\phi}(t - iT', x)\chi_{(iT',(i+1)T']}(t), \quad (t,x) \in (0,T) \times \mathbb{R}^N,$$

then u_ϕ is the unique solution of (1.1) in Q_T with $\nu = \phi$ and (3.1) follows by (3.7). □

Proof of Theorem 1.1. *Existence for $\nu \geq 0$.* We consider a sequence nonnegative functions $\{\nu_n\} \subset C_0^2(\mathbb{R}^N)$ such that $\nu_n \to \nu$ in the duality sense of

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \zeta \nu_n dx = \int_{\mathbb{R}^N} \zeta d\nu, \quad \forall \zeta \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{3.9}$$

It follows from the Banach-Steinhaus theorem that $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)}$ is bounded independently of n and we assume that $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)} \leq 2\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}$. For any $T > 0$, we consider a sequence functions $\{h_n\}$ satisfying $\widetilde{(H)}$,

$$|h_n| \leq |h| \quad \text{and} \quad \lim_{n\to\infty} \|h_n - h\|_{L^\infty([0,T]\times\mathbb{R})} = 0.$$

By Lemma 3.1, we denote by u_n the corresponding solution of (1.1) with nonlinearity h_n and initial ν_n, then u_n is nonnegative and satisfies that

$$0 \leq u_n = \mathbb{H}_\alpha[\nu_n - h_n(t,u_n)] \leq \mathbb{H}_\alpha[\nu_n] \quad \text{in } Q_T. \tag{3.10}$$

By (2.3) it follows that

$$\|u_n\|_{M^{p^*_\beta}(Q_T, t^\beta dxdt)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}. \tag{3.11}$$

For $\epsilon > 0$ and $\sigma \in (0,\alpha)$, set $\xi_\epsilon = (\eta_k + \epsilon)^{\frac{\sigma}{\alpha}} - \epsilon^{\frac{\sigma}{\alpha}}$, where η_k is given by (2.20), then by [12, Lemma 2.3 part (ii)],

$$\begin{aligned} -\partial_t \xi_\epsilon + (-\Delta)^\alpha \xi_\epsilon &= -\frac{\sigma}{\alpha}(\eta_k + \epsilon)^{\frac{\sigma-\alpha}{\alpha}} \partial_t \eta_k + \frac{\sigma}{\alpha}(\eta_k + \epsilon)^{\frac{\sigma-\alpha}{\alpha}} (-\Delta)^\alpha \eta_k \\ &\quad - \frac{\sigma(\sigma-\alpha)}{\alpha^2}(\eta_k + \epsilon)^{\frac{\sigma-2\alpha}{\alpha}} \int_{\mathbb{R}^N} \frac{(\eta_k(y) - \eta_k(x))^2}{|y-x|^{N+2\alpha}} dy \\ &\geq \frac{\sigma}{\alpha}(\eta_k + \epsilon)^{\frac{\sigma-\alpha}{\alpha}}, \end{aligned}$$

and $\xi_\epsilon \in \mathbb{Y}_{\alpha,T}$. By (2.11)

$$\begin{aligned} \int_{Q_T} (|u_n|[-\partial_t \xi_\epsilon + (-\Delta)^\alpha \xi_\epsilon] + |h_n(t,u_n)|\xi_\epsilon)\, dxdt + \int_{\mathbb{R}^N} |u(T,x)|\xi_\epsilon(T,x) dx \\ \leq \int_{\mathbb{R}^N} \xi_\epsilon(0,x)|\nu_n(x)| dx, \end{aligned}$$

then we obtain

$$\int_{Q_T} |h_n(t,u_n)|\xi_\epsilon(t,x)dxdt \le \int_{\mathbb{R}^N} \xi_\epsilon(0,x)|\nu_n(x)|dx.$$

If we let $\epsilon \to 0$, we obtain

$$\int_{Q_T} |h_n(t,u_n)|\eta_k^{\frac{\sigma}{\alpha}}(t,x)dxdt \le \int_{\mathbb{R}^N} \eta_k^{\frac{\sigma}{\alpha}}(0,x)|\nu_n(x)|dx.$$

Taking $k \to \infty$ and then $\sigma \to 0^+$, we obtain that

$$\|h_n(t,u_n)\|_{L^1(Q_T)} \le \|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)} \le 2\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}. \tag{3.12}$$

Since

$$H_\alpha(t,x,y) = t^{-\frac{N}{2\alpha}}\Gamma_\alpha(1, t^{-\frac{1}{2\alpha}}(x-y)),$$

then combining with (3.10), we obtain that for $t > 0$

$$\begin{aligned}
\|u_n(t,\cdot)\|_{L^\infty(\mathbb{R}^N)} &\le c_8 t^{-\frac{N}{2\alpha}} \sup_{x\in\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\nu_n(y)}{1+|t^{-\frac{1}{2\alpha}}(x-y)|^{N+2\alpha}}dy \\
&\le c_8 t^{-\frac{N}{2\alpha}} \|\nu_n\|_{L^1(\mathbb{R}^N)} \\
&\le 2c_8 t^{-\frac{N}{2\alpha}} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathbb{H}_\alpha[\nu_n](t,\cdot)\|_{L^1(\mathbb{R}^N,\,\tilde{\omega}dx)} &\le c_8 t^{-\frac{N}{2\alpha}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\nu_n(y)}{1+|t^{-\frac{1}{2\alpha}}(x-y)|^{N+2\alpha}} \frac{dydx}{1+|x|^{N+2\alpha}} \\
&\le c_{15} t^{-\frac{N}{2\alpha}} \|\nu_n\|_{L^1(\mathbb{R}^N)} \\
&\le 2c_{15} t^{-\frac{N}{2\alpha}} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)},
\end{aligned}$$

then

$$\begin{aligned}
\|\mathbb{H}_\alpha[h_n(t,u_n)]\|_{L^1(\mathbb{R}^N,\,\tilde{\omega}dx)} &\le c_8 t^{-\frac{N}{2\alpha}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|h_n(t,u_n)|}{1+|t^{-\frac{1}{2\alpha}}(x-y)|^{N+2\alpha}} \frac{dydx}{1+|x|^{N+2\alpha}} \\
&\le c_{15}\|h_n(t,u_n)\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{1+|x|^{N+2\alpha}} \frac{1}{1+|y|^{N+2\alpha}} dydx \\
&\le c_{20} t^\beta g(2c_8 t^{-N/2\alpha}\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)})
\end{aligned}$$

and then

$$\begin{aligned}
\|u_n(t,\cdot)\|_{L^1(\mathbb{R}^N,\,\tilde{\omega}dx)} &\le \|\mathbb{H}_\alpha[\nu_n]\|_{L^1(\mathbb{R}^N,\,\tilde{\omega}dx)} + \|\mathbb{H}_\alpha[h_n(t,u_n)]\|_{L^1(\mathbb{R}^N,\,\tilde{\omega}dx)} \\
&\le 2c_{15} t^{-\frac{N}{2\alpha}} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)} + c_{20} t^\beta g(2c_8 t^{-N/2\alpha}\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}).
\end{aligned}$$

Then by [8, Theorem 5.1], there exists $\sigma \in (0,1)$ such that for any $T_0 \in (0,T)$,

$$\begin{aligned}\|u_n\|_{C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}((T_0,T)\times\mathbb{R}^N)} &\le c_{18}\{\|u_n\|_{L^\infty((T_0,T)\times\mathbb{R}^N)} + \sup_{\tau\in(T_0,T)}\|u_n(\tau,\cdot)\|_{L^1(\mathbb{R}^N,\,\tilde{\omega}dx)} \\ &\qquad + \|h_n(\cdot,u_n)\|_{L^\infty((T_0,T)\times\mathbb{R}^N)}\} \\ &\le c_{21}T_0^{-\frac{N}{2\alpha}}\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)} + c_{22}(T^\beta + T_0^\beta)g(2c_8T_0^{-N/2\alpha}\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}).\end{aligned} \tag{3.13}$$

Therefore, there exist a sub-sequence $\{u_{n_k}\}$ and some $u \in C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}$ locally in Q_T such that $u_{n_k} \to u$ in Q_T and then $h_{n_k}(\cdot,u_{n_k}) \to h(\cdot,u)$ in Q_T.

For $\lambda > 0$, we denote $S_\lambda = \{(t,x) \in Q_T : |u_{n_k}(t,x)| > \lambda\}$ and $\omega(\lambda) = \int_{S_\lambda} t^\beta dxdt$. Then for any Borel set $E \subset Q_T$, we use (H) to obtain

$$\begin{aligned}\int_E |h_{n_k}(t,u_{n_k})|dxdt &\le \int_{E\cap S_\lambda^c} g(|u_{n_k}|)t^\beta dxdt + \int_{E\cap S_\lambda} g(|u_{n_k}|)t^\beta dxdt \\ &\le g(\lambda)\int_E t^\beta dxdt + \int_{S_\lambda} g(u_{n_k})t^\beta dxdt \\ &\le g(\lambda)\int_E t^\beta dxdt - \int_\lambda^\infty g(s)d\omega(s),\end{aligned}$$

where

$$\int_\lambda^\infty g(s)d\omega(s) = \lim_{M\to\infty}\int_\lambda^M g(s)d\omega(s).$$

By (3.11), we have $\omega(s) \le c_{23}s^{-p_\beta^*}$ and then

$$\begin{aligned}-\int_\lambda^M g(s)d\omega(s) &= -\Big[g(s)\omega(s)\Big]_{s=\lambda}^{s=M} + \int_\lambda^M \omega(s)dg(s) \\ &\le g(\lambda)\omega(\lambda) - g(M)\omega(M) + c_{23}\int_\lambda^M s^{-p_\beta^*}dg(s) \\ &\le g(\lambda)\omega(\lambda) - g(M)\omega(M) + c_{23}\left(M^{-p_\beta^*}g(M) - \lambda^{-p_\beta^*}g(\lambda)\right) \\ &\qquad + \frac{c_{23}}{p_\beta^*+1}\int_\lambda^M s^{-1-p_\beta^*}g(s)ds.\end{aligned}$$

By (1.7) and Lemma 4.1 in [12], we have $\lim_{M\to\infty} M^{-p_\beta^*}g(M) = 0$. Since $\omega(s) \le c_{23}s^{-p_\beta^*}$, we obtain $g(\lambda)\omega(\lambda) \le c_{23}\lambda^{-p_\beta^*}g(\lambda)$ and then

$$-\int_\lambda^\infty g(s)d\omega(s) \le \frac{c_{23}}{p_\beta^*+1}\int_\lambda^\infty s^{-1-p_\beta^*}g(s)ds.$$

Notice that the above quantity on the right-hand side tends to 0 when $\lambda \to \infty$. The conclusion follows: for any $\epsilon > 0$ there exists $\lambda > 0$ such that

$$\frac{c_{23}}{p_\beta^*+1}\int_\lambda^\infty s^{-1-p_\beta^*}g(s)ds \le \frac{\epsilon}{2}$$

and there exists $\delta > 0$ such that

$$\int_E t^\beta dxdt \le \delta \Longrightarrow g(\lambda) \int_E t^\beta dxdt \le \frac{\epsilon}{2}.$$

This means that $\{h_{n_k}(\cdot, u_{n_k})\}$ is uniformly integrable in $L^1(Q_T)$. Combining (3.12), we have $h_{n_k}(\cdot, u_{n_k}) \to h(\cdot, u)$ in $L^1(Q_T)$ by Vitali convergence theorem. Letting $n_k \to \infty$ in the identity

$$\begin{aligned}&\int_{Q_T} [-u_{n_k}\partial_t\xi + u_{n_k}(-\Delta)^\alpha\xi + \xi h_{n_k}(t, u_{n_k})]\, dxdt\\ &= \int_{\mathbb{R}^N} \xi(0,x)\nu_{n_k} dx - \int_{\mathbb{R}^N} u_{n_k}(T,x)\xi(T,x)dx, \quad \forall \xi \in \mathbb{Y}_{\alpha,T},\end{aligned}$$

which infers that u is a weak solution of (1.1).

Existence for general ν. For $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$, a sequence $\{\nu_n\}$ in $C_0^2(\mathbb{R}^N)$ converge to ν in the dual sense of (3.9). From the monotonicity of $h_n(t,\cdot)$

$$-\mathbb{H}_\alpha[|\nu_n|] \le u_{-|\nu_n|} \le u_{\nu_n} \le u_{|\nu_n|} \le \mathbb{H}_\alpha[|\nu_n|].$$

Then by above analysis, the sequence $\{h_n(\cdot, u_{-|\nu_n|})\}$ and $\{h_n(\cdot, u_{|\nu_n|})\}$ are compact in $L^1(Q_T)$ for any $T > 0$ and (3.11) holds for $\{u_{\nu_n}\}$. Therefore $\{u_{\nu_n}\}$ is compact in $L^1(Q_T^\Omega)$ and there exist some subsequence $\{u_{\nu_{n_k}}\}$ and $u_\nu \in L^1(Q_T)$ such that

$$u_{\nu_{n_k}} \to u_\nu \quad \text{as} \quad k \to \infty \quad \text{in} \quad Q_T$$

and

$$\begin{aligned}&\int_{Q_T} [-u_\nu\partial_t\xi + u_\nu(-\Delta)^\alpha\xi + \xi h(t, u_\nu)]\, dxdt\\ &\qquad = \int_{\mathbb{R}^N} \xi(0,x)d\nu - \int_{\mathbb{R}^N} u_\nu(T,x)\xi(T,x)dx, \quad \forall \xi \in \mathbb{Y}_{\alpha,T}.\end{aligned}$$

Uniqueness. Let u_1, u_2 be two weak solutions of (1.1) with the same initial ν and $w = u_1 - u_2$. Then

$$\partial_t w + (-\Delta)^\alpha w = h(t, u_2) - h(t, u_1) \quad \text{in} \quad Q_T.$$

Since $h(t, u_2) - h(t, u_1) \in L^1(Q_T)$, then by (2.11), for $\xi \in \mathbb{Y}_{\alpha,T}$, $\xi \ge 0$, we have

$$\begin{aligned}&\int_{Q_T} |w|[-\partial_t\xi + (-\Delta)^\alpha\xi]dxdt + \int_{\mathbb{R}^N} |w(T,x)|\xi(T,x)dxdt\\ &\qquad + \int_{Q_T} [h(t,u_1) - h(t,u_2)]\mathrm{sign}(w)\xi dxdt \le 0.\end{aligned}$$

Together with $\int_{Q_T}[h(t,u_1)-h(t,u_2)]\text{sign}(w)\xi dxdt \geq 0$, then

$$w = 0 \quad \text{a.e. in} \quad Q_T.$$

To prove that the mapping $\nu \mapsto u_\nu$ is increasing. Let $\nu_1, \nu_2 \in \mathfrak{M}^b(\mathbb{R}^N)$ and $\nu_1 \geq \nu_2$, then there exist two sequences $\{\nu_{1,n}\}$ and $\{\nu_{2,n}\}$ in $C_0^2(\mathbb{R}^N)$ such that $\nu_{1,n} \geq \nu_{2,n}$, converge to ν_1 and ν_2 respectively in the sense of (3.9). Let $u_{i,n}$ be the corresponding unique solution of (1.1) with initial $\nu_{i,n}$ and u_{ν_i} be the unique solution of (1.1) with initial ν_i for $i = 1, 2$. Then $u_{1,n} \geq u_{2,n}$. Moveover, by uniqueness, $u_{i,n}$ convergence to u_{ν_i} in $L^1(Q_T)$ for $i = 1, 2$. Then we have $u_{\nu_1} \geq u_{\nu_2}$.

To prove (1.9). We make Jordan decomposition of $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ by the positive part ν_+ and the negative part ν_-. For $\nu_\pm$, it is able to find two sequences of nonnegative functions $\{\nu_{n_\pm}\}$ in $C_0^2(\mathbb{R}^N)$ which converge to $\nu_\pm$ in the dual sense of (3.9) respectively. Then the sequence functions $\nu_n = \nu_{n_+} - \nu_{n_-}$ converge to ν in the dual sense of (3.9). From the monotonicity of $h(t,\cdot)$,

$$-\mathbb{H}_\alpha[\nu_{n_-}] \leq u_{-\nu_{n_-}} \leq u_{\nu_n} \leq u_{\nu_{n_+}} \leq \mathbb{H}_\alpha[\nu_{n_+}],$$

which, together with the uniqueness of u_ν, implies that

$$-\mathbb{H}_\alpha[\nu_-] \leq u_\nu \leq \mathbb{H}_\alpha[\nu_+]. \tag{3.14}$$

Therefore, by monotonicity of $h(t,\cdot)$,

$$\begin{aligned} u_\nu &= \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(\cdot,u_\nu)] \\ &\leq \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(\cdot,-\mathbb{H}_\alpha[\nu_-])] \end{aligned}$$

and

$$\begin{aligned} u_\nu &= \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(\cdot,u_\nu)] \\ &\geq \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(\cdot,\mathbb{H}_\alpha[\nu_+])], \end{aligned}$$

which imply (1.9). The proof is complete. □

4 Initial Dirac mass

In this section, we will consider the properties of solutions to (1.1) in the particular case of $h(t,r) = t^\beta r^p$ and the initial data $\nu = k\delta_0$, where $\beta > -1$ and $p \in (0, p^*_\beta)$.

Proposition 4.1 *Let $p \in (0, p^*_\beta)$ and u_k be the solution of (1.10), then there exists $c_{24} > 0$ such that*

$$\lim_{t\to 0^+} u_k(t,0)t^{\frac{N}{2\alpha}} = c_{24}k. \tag{4.1}$$

Proof. By (1.9) it follows that

$$u_k(t,0) \le k\mathbb{H}_\alpha[\delta_0](t,0) = k\Gamma_\alpha(t,0), \quad t>0. \tag{4.2}$$

We claim that there exists $c_{25}>0$ independent of k such that

$$u_k(t,0) \ge k\Gamma_\alpha(t,0) - c_{25}k^p t^{-\frac{N}{2\alpha}p+1+\beta}, \quad t\in(0,1/2). \tag{4.3}$$

Indeed, from (1.9), it infers that

$$u_k(t,0) \ge k\Gamma_\alpha(t,0) - k^p W(t,0), \quad t\in(0,1/2),$$

where

$$W(t,x) = \mathbb{H}_\alpha[t^\beta(\mathbb{H}_\alpha[\delta_0])^p](t,x), \quad (t,x)\in Q_\infty.$$

For $t\in(0,1/4)$, there exists $c_{26}, c_{27}>0$ such that

$$\begin{aligned}
W(t,0) &\le c_{26}\int_t^1\int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}}s^\beta}{1+|(s-t)^{-\frac{1}{2\alpha}}y|^{N+2\alpha}}\left(\frac{s^{-\frac{N}{2\alpha}}}{1+|s^{-\frac{1}{2\alpha}}y|^{N+2\alpha}}\right)^p dyds + c_{27}\\
&\le c_{26}\int_t^1\int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}}s^{-\frac{N}{2\alpha}p+\beta}}{1+|(\frac{s-t}{s})^{-\frac{1}{2\alpha}}y|^{N+2\alpha}}dyds + c_{27}\\
&\le c_{26}\int_t^1 (s-t)^{-\frac{N}{2\alpha}}s^{-\frac{N}{2\alpha}p+\beta}(\frac{s-t}{s})^{\frac{N}{2\alpha}}ds + c_{27}\\
&\le c_{26}t^{-\frac{N}{2\alpha}p+1+\beta} + c_{27}.
\end{aligned}$$

Combining (1.14) and $-\frac{N}{2\alpha}p+1+\beta > -\frac{N}{2\alpha}$, we obtain that

$$\lim_{t\to0^+} t^{\frac{N}{2\alpha}}W(t,0) = 0.$$

Therefore, (4.1) holds. □

In what follows we consider the limit of the solution $\{u_k\}$ of (1.10) as $k\to\infty$ for $p\in(0,1]$.

Proposition 4.2 *Assume that $p\in(0,1]$ and u_k is the solution of (1.10), then*

$$\lim_{k\to\infty} u_k = \infty \quad \text{in} \quad Q_\infty.$$

Proof. We observe that $\mathbb{H}_\alpha[\delta_0]$ and $\mathbb{H}_\alpha[t^\beta(\mathbb{H}_\alpha[\delta_0])^p]$ are positive in $(0,\infty)\times\mathbb{R}^N$. By (1.9), for $p\in(0,1)$ and $(t,x)\in(0,\infty)\times\mathbb{R}^N$, we have that

$$\begin{aligned}
u_k &\ge k\mathbb{H}_\alpha[\delta_0] - k^p\mathbb{H}_\alpha[t^\beta(\mathbb{H}_\alpha[\delta_0])^p]\\
&\to \infty \quad \text{as} \quad k\to\infty.
\end{aligned}$$

For $p=1$, it is obvious that $u_k = ku_1$ and $u_1>0$ in $(0,\infty)\times\mathbb{R}^N$, then

$$\lim_{k\to\infty} u_k = \infty \quad \text{in} \quad Q_\infty.$$

The proof is complete. □

Now we deal with the range $p\in(1,p^*_\beta)$.

Lemma 4.1 *Let $p \in (1, p^*_\beta)$ and u_k be the solution of (1.10), then for any $k > 0$,*

$$0 \le u_k \le U_p \quad \text{in} \quad Q_\infty, \tag{4.4}$$

where U_p is given by (1.16).

Proof. Let $\{f_{n,k}\}$ be a sequence of nonnegative functions in $C_0^1(\mathbb{R}^N)$ that converge to $k\delta_0$ in the distribution sense as $n \to \infty$, denote by $u_{n,k}$ the corresponding solution of (1.12) with initial data by $f_{n,k}$.

We claim that

$$u_{n,k} \le U_p \quad \text{in} \quad Q_\infty. \tag{4.5}$$

Now we assume this claim holds at this moment, then it follows that

$$u_k \le U_p \quad \text{in} \quad Q_\infty,$$

where u_k is the solution of (1.10), since $\lim_{n\to\infty} u_{n,k} = u_k$ in $(0,\infty) \times \mathbb{R}^N$.

Step 1. To prove $\lim_{|x|\to\infty} u_{n,k}(t,x) = 0$ for any $t > 0$. From [10, 14], there exists $c_8 > 0$ such that for any $x, y \in \mathbb{R}^N$ and $t \in (0,\infty)$,

$$0 < \Gamma_\alpha(t, x-y) \le \frac{c_8 t^{-\frac{N}{2\alpha}}}{1 + (|x-y|t^{-\frac{1}{2\alpha}})^{N+2\alpha}}.$$

Then for $|x| > 1$,

$$\begin{aligned} 0 \le \mathbb{H}_\alpha[f_{n,k}](t,x) &\le c_8 t^{-\frac{N}{2\alpha}} \int_{\mathbb{R}^N} \frac{f_{n,k}(y)}{1 + (|x-y|t^{-\frac{1}{2\alpha}})^{N+2\alpha}} dy \\ &= c_8 \int_{\mathbb{R}^N} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1+|z|^{N+2\alpha}} dz \\ &= c_8 \left(\int_{\mathbb{R}^N \setminus B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1+|z|^{N+2\alpha}} dz + \int_{B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1+|z|^{N+2\alpha}} dz \right), \end{aligned}$$

where $R = \frac{1}{2}|x|t^{-\frac{1}{2\alpha}}$ and $B_R = \{z \in \mathbb{R}^N : |z| < R\}$. It is obvious that

$$|x - zt^{\frac{1}{2\alpha}}| \ge |x| - |z|t^{\frac{1}{2\alpha}} \ge |x|/2 \quad \text{for all } z \in B_R.$$

Then

$$\begin{aligned} \int_{B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1+|z|^{N+2\alpha}} dz &\le \sup_{|y| \ge \frac{|x|}{2}} f_{n,k}(y) \int_{B_R} \frac{1}{1+|z|^{N+2\alpha}} dz \\ &\le \sup_{|y| \ge \frac{|x|}{2}} f_{n,k}(y) \int_{\mathbb{R}^N} \frac{1}{1+|z|^{N+2\alpha}} dz \\ &= c_{25} \sup_{|y| \ge \frac{|x|}{2}} f_{n,k}(y) \end{aligned}$$

and

$$\int_{\mathbb{R}^N\setminus B_R}\frac{f_{n,k}(x-zt^{\frac{1}{2\alpha}})}{1+|z|^{N+2\alpha}}dz\le\int_{\mathbb{R}^N\setminus B_R}\frac{\|f_{n,k}\|_{L^\infty(\mathbb{R}^N)}}{1+|z|^{N+2\alpha}}dz\le c_{26}R^{-2\alpha}=\frac{c_{27}t}{|x|^{2\alpha}},$$

for some $c_{27}>0$ independent of x,t and R. Since $f_{n,k}\in C_0^1(\mathbb{R}^N)$, we have $\lim_{|x|\to\infty}\sup_{|y|\ge\frac{|x|}{2}}f_{n,k}(y)=0$ and then $0\le u_{n,k}(t,x)\le\mathbb{H}_\alpha[f_{n,k}](t,x)\to 0$ as $|x|\to\infty$ for $t>0$.

Step 2. To prove (4.5). If (4.5) fails, there exists $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^N$ such that

$$(U_p-u_{n,k})(t_0,x_0)=\min_{(t,x)\in(0,\infty)\times\mathbb{R}^N}(U_p-u_{n,k})(t,x)<0,$$

since $U_p(t)>0=\lim_{|x|\to\infty}u_{n,k}(t,x)$ for any $t\in(0,\infty)$, $U_p(0)=\infty>f_{n,k}(x)=u_{n,k}(0,x)$ for $x\in\mathbb{R}^N$ and $\lim_{t\to\infty}U_p(t)=\lim_{t\to\infty}u_{n,k}(t,x)=0$ for $x\in\mathbb{R}^N$. Then $\partial_t(U_p-u_{n,k})(t_0,x_0)=0$, $(-\Delta)^\alpha(U_p-u_{n,k})(t_0,x_0)\le 0$ and

$$\partial_t(U_p-u_{n,k})(t_0,x_0)+(-\Delta)^\alpha(U_p-u_{n,k})(t_0,x_0)+t_0^\beta U_p^p(t_0)-t_0^\beta u_{n,k}^p(t_0,x_0)<0,$$

which is impossible with

$$\begin{aligned}&\partial_t(U_p-u_{n,k})(t_0,x_0)+(-\Delta)^\alpha(U_p-u_{n,k})(t_0,x_0)+t_0^\beta U_p^p(t_0)-t_0^\beta u_{n,k}^p(t_0,x_0)\\ =\;&\partial_t U_p(t_0)+t_0^\beta U_p^p(t_0)-\{\partial_t u_{n,k}(t_0,x_0)+(-\Delta)^\alpha u_{n,k}(t_0,x_0)+t_0^\beta u_{n,k}^p(t_0,x_0)\}\\ =\;&0.\end{aligned}$$

The proof is complete. □

Proposition 4.3 (*i*) *Assume that $p\in(0,p_\beta^*)$ and u_k is the solution of (1.10). Then u_k is a classical solution of (1.12).*

(*ii*) *Assume that $p\in(1,p_\beta^*)$ and u_∞ is defined by (1.11). Then u_∞ is a classical solution of (1.12).*

Proof. (*i*) Since $u_k\le k\mathbb{H}_\alpha[\delta_0]$, then it infers that u_k is bounded in $(T_0,\infty)\times\mathbb{R}^N$ with $T_0>0$. Let $\{g_{n,k}\}$ be a sequence of nonnegative functions in $C_0^1(\mathbb{R}^N)$ that converge to $k\delta_0$ in the distribution sense as $n\to\infty$ and $u_{n,k}$ the corresponding solution of (1.12) with initial data $g_{n,k}$. Then $\mathbb{H}_\alpha[g_{n,k}]\to k\mathbb{H}_\alpha[\delta_0]$ as $n\to\infty$ uniformly in $[T_0,\infty)\times\mathbb{R}^N$ for any $T_0>0$ and by Comparison Principle, there exists $c_{28}>1$ such that

$$0\le u_{n,k}(t,x)\le k\mathbb{H}_\alpha[g_{n,k}]\le c_{28}k\mathbb{H}_\alpha[\delta_0]\quad\text{in }\ [T_0,\infty)\times\mathbb{R}^N$$

and there exists $\sigma\in(0,1)$ such that $\{u_{n,k}\}$ are uniformly bounded with respect to n in $C_{t,x}^{\frac{\sigma}{2\alpha},\sigma}((T_0,\infty)\times\mathbb{R}^N)$ with $T_0>0$. Therefore, by the Arzela-Ascoli theorem, $u_{n,k}$ converges to u_k in $C_{t,x}^{\frac{\sigma'}{2\alpha},\sigma'}((T_0,\infty)\times\mathbb{R}^N)$ with $\sigma'\in(0,\sigma)$

and then u_k is a viscosity solution of (1.12) in $(T_0,\infty)\times\mathbb{R}^N$. By estimate $(A.1)$ in [7], u_k is in $C_{t,x}^{1+\sigma',2\alpha+\sigma'}((T_0,\infty)\times\mathbb{R}^N)$ and u_k is a classical solution of (1.12) in $(T_0,\infty)\times\mathbb{R}^N$.

(ii) The proof is the same as part (i) just replacing $u_k\le k\mathbb{H}_\alpha[\delta_0]$ by $u_\infty\le U_p$. □

5 Self-similar solution when $p\in(1,1+\frac{2\alpha(1+\beta)}{N})$

5.1 Non-flat self-similar solution for $p\in(1+\frac{2\alpha(1+\beta)}{N+2\alpha},1+\frac{2\alpha(1+\beta)}{N})$

By Theorem 1.1 and (4.4), we see that $\{u_k\}$ are a sequence of nonnegative and increasing functions and controlled by function U_p defined in (1.16),then for $p\in(1,p^*_\beta)$, the limit of u_k as $k\to\infty$ exists and unique, then we denote it by u_∞, see (1.11). By Proposition 4.3 (ii) and (4.4), u_∞ is a classical solution of (1.12) and

$$u_\infty\le U_p\quad\text{in}\quad Q_\infty. \tag{5.1}$$

Proposition 5.1 *Assume that $p\in(1,p^*_\beta)$ and u_∞ is defined in (1.16). Then u_∞ is a self-similar solution of (1.12).*

Proof. For $\lambda>0$, we denote

$$\tilde{u}_\lambda(t,x)=\lambda^{\frac{2\alpha(1+\beta)}{p-1}}u_k(\lambda^{2\alpha}t,\lambda x),\quad (t,x)\in Q_\infty.$$

By direct computation, we have

$$\begin{aligned}&\partial_t\tilde{u}_\lambda(t,x)+(-\Delta)^\alpha\tilde{u}_\lambda(t,x)+t^\beta\tilde{u}^p_\lambda(t,x)\\&=\lambda^{\frac{2\alpha p(1+\beta)}{p-1}}[\partial_t u_k(\lambda^{2\alpha}t,\lambda x)+(-\Delta)^\alpha u_k(\lambda^{2\alpha}t,\lambda x)+t^\beta u^p_k(\lambda^{2\alpha}t,\lambda x)]\\&=0.\end{aligned}\tag{5.2}$$

Moreover, for $f\in C_0(\mathbb{R}^N)$

$$\begin{aligned}\langle\tilde{u}_\lambda(0,\cdot),f\rangle&=\lambda^{\frac{2\alpha(1+\beta)}{p-1}}\lim_{t\to0^+}\int_{\mathbb{R}^N}u_k(\lambda^{2\alpha}t,\lambda x)f(x)dx\\&=\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}\lim_{t\to0^+}\int_{\mathbb{R}^N}u_k(\lambda^{2\alpha}t,z)f(\frac{z}{\lambda})dz\\&=\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}kf(0),\end{aligned}\tag{5.3}$$

where $\frac{2\alpha(1+\beta)}{p-1}-N>0$ by the fact of $p\in(1,p^*_\beta)$. Thus,

$$\tilde{u}_\lambda(0,\cdot)=\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}k\delta_0\quad\text{in }\mathbb{R}^N.$$

By (5.3), $u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}$ is a unique solution of (1.12) with initial data $\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}k\delta_0$, then by (5.2) it infers that for $(t,x)\in Q_\infty$,

$$\lambda^{\frac{2\alpha(1+\beta)}{p-1}}u_k(\lambda^{2\alpha}t,\lambda x)=\tilde{u}_\lambda(t,x)=u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}(t,x) \tag{5.4}$$

and letting $k\to\infty$ we have that

$$u_\infty(t,x)=\lambda^{\frac{2\alpha(1+\beta)}{p-1}}u_\infty(\lambda^{2\alpha}t,\lambda x),\qquad (t,x)\in Q_\infty,$$

which implies that u_∞ is a self-similar solution (1.12). □

Let us denote

$$U_\infty(z)=u_\infty(1,z),\qquad z\in\mathbb{R}^N$$

and we observe that U_∞ is a classical solution of (1.15). It is obvious that the constant $(\frac{1+\beta}{p-1})^{\frac{1}{p-1}}$ is a trivial positive classical solution of (1.15). We observe that $N<\frac{2\alpha(1+\beta)}{p-1}<N+2\alpha$ for $p\in(1+\frac{2\alpha(1+\beta)}{N+2\alpha},p^*_\beta)$. To be convenient, we introduce the auxiliary function

$$w_\lambda(t,x)=\lambda t^{-\frac{1+\beta}{p-1}}w(t^{-\frac{1}{2\alpha}}|x|),,\qquad (t,x)\in Q_\infty, \tag{5.5}$$

where $w(s)=\frac{\log(e+s^2)}{1+s^{N+2\alpha}}$ with e is the natural number.

Lemma 5.1 *Assume that $p\in(1+\frac{2\alpha(1+\beta)}{N+2\alpha},p^*_\beta)$ and w_λ is defined by (5.5). Then there exists $\Lambda_0>0$ such that for $\lambda\geq\Lambda_0$,*

$$\partial_t w_\lambda(t,x)+(-\Delta)^\alpha w_\lambda(t,x)+t^\beta w^p_\lambda(t,x)\geq 0,\quad \forall(t,x)\in Q_\infty. \tag{5.6}$$

Proof. By direct computation,

$$\partial_t w_\lambda(t,x)=-\frac{\lambda(1+\beta)}{p-1}t^{-\frac{1+\beta}{p-1}-1}w(t^{-\frac{1}{2\alpha}}|x|)-\frac{\lambda}{2\alpha}t^{-\frac{1+\beta}{p-1}-1}w'(t^{-\frac{1}{2\alpha}}|x|)|t^{-\frac{1}{2\alpha}}x|$$

and

$$(-\Delta)^\alpha w_\lambda(t,x)=\lambda t^{-\frac{1+\beta}{p-1}-1}(-\Delta)^\alpha w(t^{-\frac{1}{2\alpha}}|x|),$$

which implies that

$$\begin{aligned}&\partial_t w_\lambda(t,x)+(-\Delta)^\alpha w_\lambda(t,x)+t^\beta w^p_\lambda(t,x) \qquad (5.7)\\ =\;&\lambda t^{-\frac{1+\beta}{p-1}-1}\left[(-\Delta)^\alpha w(s)-\frac{1}{2\alpha}w'(s)s-\frac{1+\beta}{p-1}w(s)+\lambda^{p-1}w^p(s)\right],\end{aligned}$$

where $s=|z|$ with $z=t^{-\frac{1}{2\alpha}}x$.

For $s>0$, by the direct computation, we obtain that

$$-\frac{1}{2\alpha}w'(s)s-\frac{1+\beta}{p-1}w(s)=\left[\frac{N+2\alpha}{2\alpha}\frac{s^{N+2\alpha}}{1+s^{N+2\alpha}}-\frac{1+\beta}{p-1}-\frac{s^2(e+s^2)^{-1}}{\alpha\log(e+s^2)}\right]w(s).$$

Since $\frac{N+2\alpha}{2\alpha} > \frac{1+\beta}{p-1}$, $\lim_{s\to\infty}\frac{s^{N+2\alpha}}{1+s^{N+2\alpha}} = 1$ and $\lim_{s\to\infty}\frac{1}{\log(e+s^2)} = 0$, then there exists $R_0 > 0$ and $\sigma_0 > 0$ such that

$$-\frac{1}{2\alpha}w'(s)s - \frac{1+\beta}{p-1}w(s) \geq \sigma_0 w(s), \qquad s \geq R_0. \tag{5.8}$$

For $|z| > 2$, by the definition of fractional Laplacian, we have that

$$\begin{aligned} -(-\Delta)^\alpha w(|z|) &= \frac{1}{2}\int_{\mathbb{R}^N} \frac{\frac{\log(e+|z+\tilde{y}|^2)}{1+|z+\tilde{y}|^{N+2\alpha}} + \frac{\log(e+|z-\tilde{y}|^2)}{1+|z-\tilde{y}|^{N+2\alpha}} - \frac{2\log(e+|z|^2)}{1+|z|^{N+2\alpha}}}{|\tilde{y}|^{N+2\alpha}} d\tilde{y} \\ &= \frac{w(|z|)}{2|z|^{2\alpha}}\int_{\mathbb{R}^N} \frac{I_z(y)}{|y|^{N+2\alpha}} dy, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} I_z(y) &= \frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z+y|^{N+2\alpha}} \frac{\log(e+|z|^2|e_z+y|^2)}{\log(e+|z|^2)} \\ &\quad + \frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z-y|^{N+2\alpha}} \frac{\log(e+|z|^2|e_z-y|^2)}{\log(e+|z|^2)} - 2 \end{aligned}$$

and $e_z = \frac{z}{|z|}$.

To estimate that there exists $c_{29} > 0$ such that

$$\int_{B_{\frac{1}{2}}(-e_z)\cup B_{\frac{1}{2}}(e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq \frac{c_{29}}{w(|z|)|z|^N}. \tag{5.10}$$

In fact, for $y \in B_{\frac{1}{2}}(-e_z)$, there exists $c_{30} > 0$ such that

$$\frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z-y|^{N+2\alpha}} \frac{\log(e+|z|^2|e_z-y|^2)}{\log(e+|z|^2)} \leq c_{30}$$

and then

$$\begin{aligned} \int_{B_{\frac{1}{2}}(-e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy &\leq \omega_N \int_0^{\frac{1}{2}} \frac{1+|z|^{N+2\alpha}}{1+(|z|r)^{N+2\alpha}} \frac{\log(e+|z|^2r^2)}{\log(e+|z|^2)} r^{N-1} dr + c_{31} \\ &\leq \frac{\omega_N}{w(|z|)|z|^N}\int_0^\infty \frac{t^{N-1}\log(e+t^2)}{1+t^{N+2\alpha}} dt + c_{31} \\ &\leq \frac{c_{32}}{w(|z|)|z|^N}, \end{aligned}$$

where $c_{31}, c_{32} > 0$ and the last inequality holds since $w(|z|)|z|^N \to 0$ as $|z| \to \infty$.

$$\int_{B_{\frac{1}{2}}(e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy = \int_{B_{\frac{1}{2}}(-e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq \frac{c_{32}}{w(|z|)|z|^N}.$$

To estimate that there exists $c_{33} > 0$ such that

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le c_{33}. \tag{5.11}$$

Indeed, since function I_z is C^2 in $\bar{B}_{\frac{1}{2}}(0)$ such that

$$I_z(0) = 0, \quad I_z(y) = I_z(-y),$$

then $\nabla I_z(0) = 0$ and there exists $c_{34} > 0$ such that

$$|D^2 I_z(y)| \le c_{34}, \quad y \in B_{\frac{1}{2}}(0).$$

Then we have

$$I_z(y) \le c_{34}|y|^2, \quad y \in B_{\frac{1}{2}}(0),$$

which implies that

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le c_{34} \int_{B_{\frac{1}{2}}(0)} \frac{|y|^2}{|y|^{N+2\alpha}} dy \le c_{33}.$$

To estimate that there exists $c > 0$ such that

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le c, \tag{5.12}$$

where $A = \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(e_z) \cup B_{\frac{1}{2}}(-e_z))$. In fact, for $y \in A$, we observe that there exists $c_{35} > 0$ such that $I_z(y) \le c_{35}$ and

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \le \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} \frac{c_{35}}{|y|^{N+2\alpha}} \le c_{36},$$

for some $c_{36} > 0$. Therefore, by (5.7)-(5.12), there exists $c_{37} > 0$ such that

$$(-\Delta)^\alpha w(|z|) \ge -\frac{c_{37}}{1 + |z|^{N+2\alpha}}, \qquad |z| \ge 2. \tag{5.13}$$

By (5.8) and (5.13), there exists $R_1 \ge R_0 + 2$ such that for $|z| > R_1$,

$$\begin{aligned}(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) &\ge \sigma_0 w(|z|) - \frac{c_{37}}{1 + |z|^{N+2\alpha}} \\ &= w(|z|) \left(\sigma_0 - \frac{c_{37}}{\log(e + |z|^2)} \right) \\ &\ge 0.\end{aligned}$$

For $|z| \le R_1$, it is obvious that there exists $c_{38} > 0$ such that

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) \ge -c_{38}.$$

Then there exists $\Lambda_0 > 0$ such that for $\lambda \geq \Lambda_0$,

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) + \lambda^{p-1} w^p(|z|) \geq 0, \quad z \in \mathbb{R}^N, \tag{5.14}$$

which together with (5.7), implies that (5.6) holds. □

Lemma 5.2 *Assume that $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, p^*_\beta)$, w_{Λ_0} is given in (5.5) and u_∞ is given in (1.11). Then*

$$u_\infty(t,x) \leq w_{\Lambda_0}(t,x), \quad (t,x) \in Q_\infty. \tag{5.15}$$

In particular,

$$u_\infty(0,x) = 0, \qquad x \in \mathbb{R}^N \setminus \{0\}. \tag{5.16}$$

Proof. Let us denote

$$f_0(r) = \frac{k_0 \log(e + r^2)}{1 + r^{N+2\alpha}}, \ r \geq 0 \quad \text{and} \quad f_{n,k}(x) = k n^N f_0(n|x|), \ x \in \mathbb{R}^N,$$

where

$$k_0 = \left[\omega_N \int_0^\infty \frac{\log(e + r^2)}{1 + r^{N+2\alpha}} r^{N-1} dr \right]^{-1}.$$

Then for any $\eta \in C_0(\mathbb{R}^N)$, we have

$$\begin{aligned} \lim_{n\to\infty} \langle f_{n,k}, \eta \rangle &= k \lim_{n\to\infty} n^N \int_{\mathbb{R}^N} f_0(n|x|)\eta(x) dx \\ &= k \lim_{n\to\infty} \int_{\mathbb{R}^N} f_0(|x|)\eta(\frac{x}{n}) dx \\ &= k\eta(0). \end{aligned}$$

Let $t_n = n^{-2\alpha}$ and then

$$\begin{aligned} w_{\Lambda_0}(t_n, x) &= \Lambda_0 t_n^{-\frac{1+\beta}{p-1}} \frac{\log(e + (t_n^{-\frac{1}{2\alpha}}|x|)^2)}{1 + (t_n^{-\frac{1}{2\alpha}}|x|)^{N+2\alpha}} = \Lambda_0 n^{\frac{2\alpha(1+\beta)}{p-1}} \frac{\log(e + (n|x|)^2)}{1 + (n|x|)^{N+2\alpha}} \\ &= \frac{\Lambda_0}{k_0} n^{\frac{2\alpha(1+\beta)}{p-1} - N} n^N f_0(n|x|) \\ &\geq \frac{\Lambda_0}{k_0} \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N} n^N f_0(n|x|) = f_{n,k_{\tilde{n}}}(x), \end{aligned}$$

where $\tilde{n} \leq n$ and $k_{\tilde{n}} = \Lambda_0 \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N}$. We see that $k_{\tilde{n}} = \Lambda_0 \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N} \to \infty$ as $\tilde{n} \to \infty$, since $\frac{2\alpha(1+\beta)}{p-1} - N > 0$. Let $u_{n,k_{\tilde{n}}}$ be the solution of (1.12) with the initial data $f_{n,k_{\tilde{n}}}$. By Lemma 5.1, $w_{\Lambda_0}(\cdot + t_n, \cdot)$ is a super solution of (1.12) with the initial data $w_{\Lambda_0}(t_n, \cdot)$, that is, for $(t,x) \in Q_\infty$,

$$\partial_t w_\lambda(t + t_n, x) + (-\Delta)^\alpha w_\lambda(t + t_n, x) + (t + t_n)^\beta w^p_\lambda(t + t_n, x) \geq 0.$$

By Comparison Principle,

$$u_{n,k_{\tilde{n}}}(t,x) \le w_{\Lambda_0}(t+t_n,x), \quad (t,x) \in Q_\infty,$$

for any $\tilde{n} \le n$. Then taking $n \to \infty$, implies that

$$u_{k_{\tilde{n}}}(t,x) \le w_{\Lambda_0}(t,x), \quad (t,x) \in Q_\infty, \tag{5.17}$$

where $u_{k_{\tilde{n}}}$ is the solution of (1.12) with $k_{\tilde{n}}\delta_0$ initial data. We derive (5.15) by taking $\tilde{n} \to \infty$.

In particular, the argument (5.16) follows by the fact of

$$\lim_{t\to 0^+} w_{\Lambda_0}(t,x) = 0, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

The proof is complete. □

Lemma 5.3 *Assume that $p \in (1, p^*_\beta)$, then there exists $c_{39} > 0$ such that*

$$u_\infty(t,x) \ge \frac{c_{39} t^{-\frac{1+\beta}{p-1}}}{1+|t^{-\frac{1}{2\alpha}}x|^{N+2\alpha}}, \quad \forall (t,x) \in (0,1)\times\mathbb{R}^N. \tag{5.18}$$

Proof. We divide the proof into several steps.
Step 1. Let $\sigma_0 = 1+\beta-\frac{N}{2\alpha}(p-1) > 0$,

$$\eta(t) = 2 - t^{\sigma_0}, \quad t > 0$$

and denote

$$v_\epsilon(t,x) = \epsilon\eta(t)\Gamma_\alpha(t,x),$$

where Γ_α is the solution of (1.13).

In this step we prove that there exists $\epsilon_0 > 0$ such that

$$u_{k_0} \ge v_{\epsilon_0} \quad \text{in } (0,1)\times\mathbb{R}^N, \tag{5.19}$$

where $k_0 = 2\epsilon_0$ and u_{k_0} is the solution of (1.12) with initial data $k_0\delta_0$. Indeed,

$$\partial_t v_\epsilon(t,x) = \epsilon\eta'(t)\Gamma_\alpha(t,x) + \epsilon\eta(t)\partial_t\Gamma_\alpha(t,x)$$

and

$$(-\Delta)^\alpha v_\epsilon(t,x) = \epsilon\eta(t)(-\Delta)^\alpha\Gamma_\alpha(t,x),$$

Let $\Gamma_1(t^{-\frac{1}{2\alpha}}x) = \Gamma_\alpha(1, t^{-\frac{1}{2\alpha}}x)$, then there exists $\epsilon_0 > 0$ such that for any $\epsilon \le \epsilon_0$ and $(t,x) \in (0,1)\times\mathbb{R}^N$

$$\begin{aligned}
&\partial_t v_\epsilon(t,x) + (-\Delta)^\alpha v_\epsilon(t,x) + t^\beta v_\epsilon^p(t,x)\\
&\quad = \epsilon\eta'(t)t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + \epsilon^p\eta^p(t)t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x)\\
&\quad \le -\epsilon\sigma_0 t^{-\frac{N}{2\alpha}-1+\sigma_0}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + 2^p\epsilon^p t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x)\\
&\quad \le 0,
\end{aligned}$$

the last inequality holds since $-\frac{N}{2\alpha}-1+\sigma_0=-\frac{N}{2\alpha}p+\beta$ and Γ_1 is bounded. In particular, we have that

$$\partial_t v_{\epsilon_0}(t,x)+(-\Delta)^\alpha v_{\epsilon_0}(t,x)+t^\beta v_{\epsilon_0}^p(t,x)\le 0,\quad \forall (t,x)\in(0,1)\times\mathbb{R}^N. \tag{5.20}$$

Let $f_n(x)=v_{\epsilon_0}(t_n,x)$ with $t_n=n^{-2\alpha}$. Since $\lim_{t\to 0^+}\eta(t)=2$, then we have that $f_n\rightharpoonup 2\epsilon_0\delta_0$ as $n\to\infty$ in the distribution sense. There exists $N_0>0$ such that $t_n\in(0,\frac{1}{8})$ for $n\ge N_0$. Let w_n be the solution of (1.12) with initial data f_n, then it infers that

$$w_n(t,x)\ge v_{\epsilon_0}(t+t_n,x),\quad (t,x)\in(0,1-t_n)\times\mathbb{R}^N$$

By the uniqueness of u_{k_0}, we have

$$w_n\to u_{k_0}\quad \text{as}\quad n\to\infty\quad \text{in}\quad (0,1)\times\mathbb{R}^N$$

and

$$\lim_{n\to\infty} v_{\epsilon_0}(t+t_n,x)=v_{\epsilon_0}(t,x),\quad (t,x)\in(0,1)\times\mathbb{R}^N,$$

which imply (5.19).

Step 2. To prove (5.18). Since

$$v_{\epsilon_0}(t,x)\ge \epsilon_0 t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x),\quad (t,x)\in(0,1)\times\mathbb{R}^N,$$

then, along with (5.4), we observe that for any $\lambda>0$,

$$\begin{aligned} u_{k_0\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}(t,x) &= \lambda^{\frac{2\alpha(1+\beta)}{p-1}}u_{k_0}(\lambda^{2\alpha}t,\lambda x)\\ &\ge \lambda^{\frac{2\alpha(1+\beta)}{p-1}}v_{\epsilon_0}(\lambda^{2\alpha}t,\lambda x)\\ &\ge \epsilon_0\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x). \end{aligned}$$

Let $\varrho=\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}$, $t_\varrho=(2\varrho)^{\frac{1}{\frac{N}{2\alpha}-\frac{1+\beta}{p-1}}}$ and $T_\varrho=\varrho^{\frac{1}{\frac{N}{2\alpha}-\frac{1+\beta}{p-1}}}$, then

$$0<t_\varrho<T_\varrho\to 0\quad \text{as}\quad \varrho\to\infty.$$

For $(t,x)\in(t_\varrho,T_\varrho)\times\mathbb{R}^N$,

$$u_{k_0\varrho}(t,x)\ge\epsilon_0\varrho t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x)\ge\frac{\epsilon_0}{2}t^{-\frac{1+\beta}{p-1}}\Gamma_1(t^{-\frac{1}{2\alpha}}x),$$

then

$$u_\infty(t,x)\ge\frac{\epsilon_0}{2}t^{-\frac{1+\beta}{p-1}}\Gamma_1(t^{-\frac{1}{2\alpha}}x),\quad (t,x)\in(t_\varrho,T_\varrho)\times\mathbb{R}^N.$$

which, choosing a sequence $\{\varrho_n\}$ such that $(0,1)\subset\cup(t_{\varrho_n},T_{\varrho_n})$, infers (5.18). The proof is complete. □

Proof of Theorem 1.2. By Proposition 5.1 and Lemma 5.2, we have that u_∞ is a nontrivial self-similar solution of (1.12) and (1.17) follows by (5.15), (5.18) and $\log(e+|t^{-\frac{1}{2\alpha}}x|^2)\le 2\log(2+|t^{-\frac{1}{2\alpha}}x|)$. The proof is complete. □

5.2 Trivial self-similar solution for $p \in (1, 1+\frac{2\alpha(1+\beta)}{N+2\alpha})$

For $p \in (1, 1+\frac{2\alpha(1+\beta)}{N+2\alpha})$, it derive from Lemma 5.3 that

$$\lim_{t\to 0^+} u_\infty(t,x) = \infty, \quad \forall x \in \mathbb{R}^N. \tag{5.21}$$

Proof of Theorem 1.4 (i)**.** Let $f_0 : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous function such that

$$\mathrm{supp} f_0 \subset B_1(0) \quad \text{and} \quad \max_{x\in B_1(0)} f_0 = 1.$$

Denote

$$f_{n,k}(x) = kn^{\theta N} f_0(n^\theta(x - x_0)),$$

where $k \le n^\tau$ with $\tau = \frac{1}{2}(\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha) > 0$, $\theta = \frac{\tau}{N}$ and $x_0 \in \mathbb{R}^N$. Since $f_{n,k}(x) \le n^\tau$ for $x \in B_1(x_0)$, $f_n(x) = 0$ for $x \in B_1^c(x_0)$ and

$$v_{\epsilon_0}(t_n, x) \ge \frac{c_{38} n^{\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha}}{(2 + |x_0|)^{N+2\alpha}}, \quad x \in B_1(x_0),$$

where $t_n = n^{-2\alpha}$. Then there exists $N_0 > 0$ such that for any $n \ge N_0$

$$f_{n,k}(x) \le v_{\epsilon_0}(t_n, x), \quad x \in B_1(x_0).$$

Since $n^{\theta N} f_0(n^\theta(x - x_0)) \rightharpoonup c_{41}\delta_{x_0}$, as $n \to \infty$ in the distribution sense, for some $c_{41} > 0$.

Let $w_{n,k}$ be the solution of (1.12) with initial data f_n and then

$$w_{n,k}(0,x) = f_{n,k}(x) \le v_{\epsilon_0}(t_n, x) \le u_\infty(t_n, x), \quad x \in \mathbb{R}^N.$$

Therefore, by comparison principle

$$w_{n,k}(t,x) \le u_\infty(t + t_n, x), \quad (t,x) \in Q_\infty.$$

We observe that

$$\lim_{k\to\infty} [\lim_{n\to\infty} w_{n,k}(t,x)] = u_\infty(t, x - x_0), \quad (t,x) \in Q_\infty.$$

Thus, we derive that

$$u_\infty(t, x - x_0) \le u_\infty(t,x), \quad (t,x) \in Q_\infty. \tag{5.22}$$

By changing the role of x_0 and 0 in (5.22), we have that

$$u_\infty(t, x - x_0) = u_\infty(t,x), \quad (t,x) \in Q_\infty,$$

which implies that u_∞ is independent of x. Combining (5.1) and (5.18), implies that

$$u_\infty = (\frac{1+\beta}{p-1})^{\frac{1+\beta}{p-1}} t^{-\frac{1+\beta}{p-1}}.$$

The proof is complete. □

In the case of $p = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$, it derive from Lemma 5.3 that

$$\liminf_{t\to 0^+} u_\infty(t,x) \geq \lim_{t\to 0^+} \frac{c_{39} t^{-\frac{1+\beta}{p-1}}}{1+|t^{-\frac{1}{2\alpha}}x|^{N+2\alpha}} = \frac{c_{39}}{|x|^{N+2\alpha}}, \quad \forall x \in \mathbb{R}^N.$$

Proof of Theorem 1.4 (ii)**.** We note that u_∞ is a self-similar solution of (1.12). Moreover, we derive (1.19) by (5.18), ends the proof. □

6 Uniqueness of self-similar solution when $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, 1 + \frac{2\alpha(1+\beta)}{N})$

Theorem 1.3 will be proved by contradiction. We first introduce some auxiliary lemmas.

Lemma 6.1 *Assume that $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, 1 + \frac{2\alpha(1+\beta)}{N})$ and $\tilde{u}$ is a positive self-similar solution of (1.18). Then either*

$$\tilde{u} > u_\infty \quad \text{in} \quad Q_\infty \tag{6.1}$$

or

$$\tilde{u} \equiv u_\infty \quad \text{in} \quad Q_\infty. \tag{6.2}$$

Proof. For any $r > 0$, we have that

$$\begin{aligned}
\int_{B_r(0)} \tilde{u}(t,x)dx &= t^{-\frac{1+\beta}{p-1}} \int_{B_r(0)} \tilde{u}(1, t^{-\frac{1}{2\alpha}}x)dx \\
&= t^{-\frac{1+\beta}{p-1}+\frac{N}{2\alpha}} \int_{B_{t^{-\frac{1}{2\alpha}}r}(0)} \tilde{u}(1,z)dz \\
&\geq t^{-\frac{1+\beta}{p-1}+\frac{N}{2\alpha}} \int_{B_1(0)} \tilde{u}(1,z)dz \\
&\to +\infty \quad \text{as } t \to 0^+,
\end{aligned}$$

where last inequality holds for $t \in (0, r^{2\alpha}]$. Let $\{\epsilon_n\}$ be a sequence positive decreasing numbers converging to 0 as $n \to \infty$. For ϵ_n and $k > 0$, there exists $t_{n,k} > 0$ such that

$$\int_{B_{\epsilon_n}(0)} \tilde{u}(t_{n,k}, x)dx = k.$$

We observe that for any fixed k, $t_{n,k} \to 0$ as $n \to \infty$ since $\lim_{n\to\infty} \epsilon_n = 0$. Let $\eta_0 : \mathbb{R}^N \to [0,1]$ be a C^2 function such that $\text{supp}\eta_0 \subset \bar{B}_2(0)$, $\eta_0 = 1$ in $B_1(0)$ and $\eta_n(x) = \eta_0(\epsilon_n^{-1}x)$ for $x \in \mathbb{R}^N$. Choosing $\{f_{n,k}\}$ be a sequence of C^2 functions such that

$$0 \le f_{n,k}(x) \le \eta_n(x)\tilde{u}(t_{n,k}, x), \quad x \in \mathbb{R}^N$$

and

$$f_{n,k} \rightharpoonup k\delta_0 \quad \text{as} \quad n \to \infty.$$

Let $u_{n,k}$ be the solution of (1.1) with initial data $f_{n,k}$, then

$$u_{n,k}(t,x) \le u(t_{n,k} + t, x), \quad (t,x) \in Q_\infty$$

and by uniqueness of u_k, $\lim_{n\to\infty} u_{n,k} = u_k$, where u_k is the solution of (1.1) with initial data $k\delta_0$. Then for any k, we have $u_k \le \tilde{u}$ in Q_∞, which implies that

$$\tilde{u} \ge u_\infty \quad \text{in} \quad Q_\infty.$$

Now we assume there exists $(t_0, x_0) \in Q_\infty$ such that

$$\tilde{u}(t_0, x_0) = u_\infty(t_0, x_0).$$

Since $\tilde{u}$ and u_∞ are self-similar, then

$$\tilde{u}(t, x_0) = u_\infty(t, x_0), \quad t > 0.$$

Now for any $t > 0$, $w(t,\cdot) = \tilde{u}(t,\cdot) - u_\infty(t,\cdot)$ achieves the minimum at x_0. Combining with $\partial_t w(t, x_0) = 0$ and $t^\beta \tilde{u}^p(t, x_0) = t^\beta u_\infty^p(t, x_0)$, we derive that $(-\Delta)^\alpha w(t, x_0) = 0$, which implies that

$$\tilde{u}(t,\cdot) = u_\infty(t,\cdot) \quad \text{in} \quad \mathbb{R}^N.$$

Then $\tilde{u} \equiv u_\infty$ in Q_∞. The proof is complete. □

Lemma 6.2 *Assume that $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, 1 + \frac{2\alpha(1+\beta)}{N})$ and $\tilde{u}$ is a positive self-similar solution of (1.18). Then for any $\epsilon > 0$, there exists $\lambda_\epsilon \ge 1$ such that for any $\lambda \ge \lambda_\epsilon$,*

$$\tilde{u} \le \lambda u_\infty + \epsilon \quad \text{in} \quad [1,\infty) \times \mathbb{R}^N =: Q_1. \tag{6.3}$$

Proof. For $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\tilde{u}(1,x) \le \epsilon, \quad x \in B^c_{R_\epsilon}(0).$$

Then there exists $\sigma_0 > 0$ such that $u_\infty(1,x) \ge \sigma_0$ for $x \in B_{R_\epsilon}(0)$. By continuity of $\tilde{u}(1,\cdot)$, there exists $\lambda_\epsilon \ge 1$ such that $\tilde{u}(1,x) \le \lambda_\epsilon u_\infty(1,x)$ for

$x \in B_{R_\epsilon}(0)$. Then $\tilde{u}(1,x) \le \lambda_\epsilon u_\infty(1,x) + \epsilon$ for $x \in \mathbb{R}^N$. By the definition of self-similar solution,

$$\begin{aligned} \tilde{u}(t,x) &\le \lambda_\epsilon u_\infty(t,x) + \epsilon t^{-\frac{1+\beta}{p-1}} \\ &\le \lambda_\epsilon u_\infty(t,x) + \epsilon, \quad (t,x) \in Q_1, \end{aligned}$$

which ends the proof. □

Proof of Theorem 1.3. Let $\tilde{u}$ be a positive self-similar solution of (1.18). By Lemma 6.1, $\tilde{u} > u_\infty$ in Q_∞ or $\tilde{u} = u_\infty$ in Q_∞. Our aim is to rule out the case $\tilde{u} > u_\infty$ in Q_∞. To this end, for $\epsilon \in (0,1]$ we denote

$$\underline{\lambda}_\epsilon = \min\{\lambda \in [1,\infty) : \ \tilde{u} \le \lambda u_\infty + \epsilon \text{ in } Q_1\}.$$

It follows by Lemma 6.2 that $\underline{\lambda}_\epsilon$ is well-defined and $\underline{\lambda}_\epsilon \le \lambda_\epsilon$.

We first claim that

$$\underline{\lambda}_\epsilon = 1 \quad \text{for any } \epsilon \in (0,1]. \tag{6.4}$$

In fact, if (6.4) fails, then there exists $\epsilon_0 \in (0,1]$ such that $\underline{\lambda}_{\epsilon_0} > 1$.
Step 1. To prove $\underline{\lambda}_{\epsilon_0} u_\infty + \epsilon_0 > \tilde{u}$ *in* Q_1. By continuity, we see that $\underline{\lambda}_{\epsilon_0} u_\infty + \epsilon_0 \ge \tilde{u}$ in Q_1. If there exists $(t_0,x_0) \in Q_1$ such that

$$\underline{\lambda}_{\epsilon_0} u_\infty(t_0,x_0) + \epsilon_0 = \tilde{u}(t_0,x_0).$$

Let $w = \underline{\lambda}_{\epsilon_0} u_\infty + \epsilon_0 - \tilde{u}$. Since $\epsilon_0 > 0$, there exists R_0 such that $\tilde{u}(t_0,x) < \epsilon_0$ for $x \in B^c_{R_0}$, then w achieves the minimum at (t_0,x_0) then $\partial_t w(t_0,x_0) = 0$, $t_0^\beta(\underline{\lambda}_{\epsilon_0} u_\infty(t_0,x_0) + \epsilon_0)^p = t_0^\beta \tilde{u}^p(t_0,x_0)$ and

$$(-\Delta)^\alpha w(t_0,x_0) = -\int_{\mathbb{R}^N} \frac{w(t_0,x_0+y)}{|y|^{N+2\alpha}} dy < 0.$$

Thus, we obtain a contradiction, since $\underline{\lambda}_{\epsilon_0} u_\infty + \epsilon_0$ is a super solution of (1.12) and $\tilde{u}$ is a solution of (1.12).

Step 2. To prove that there exists $\sigma \in (0, \underline{\lambda}_{\epsilon_0} - 1)$ *such that* $(\underline{\lambda}_{\epsilon_0} - \sigma) u_\infty + \epsilon_0 \ge \tilde{u}$ *in* Q_1.

For $\epsilon_0 > 0$, there exists R_0 such that $\tilde{u}(1,x) < \epsilon_0$ for $x \in B^c_{R_0}(0)$. By Step 1, $\tilde{u}(1,x) < \underline{\lambda}_{\epsilon_0} u_\infty(1,x) + \epsilon_0$ for $x \in \mathbb{R}^N$, then there exists $\sigma \in (0, \underline{\lambda}_{\epsilon_0} - 1)$ such that

$$\tilde{u}(1,x) \le \underline{\lambda}_{\epsilon_0} u_\infty(1,x) + \epsilon_0 - \sigma u_\infty(1,0), \quad x \in \bar{B}_{R_0}(0).$$

Then we have

$$\tilde{u}(1,x) \le (\underline{\lambda}_{\epsilon_0} - \sigma) u_\infty(1,x) + \epsilon_0, \quad x \in \mathbb{R}^N.$$

By Comparison Principle, we have

$$\tilde{u}(t,x) \le (\underline{\lambda}_{\epsilon_0} - \sigma) u_\infty(t,x) + \epsilon_0, \quad (t,x) \in Q_1,$$

which contradicts the definition of $\underline{\lambda}_{\epsilon_0}$. Therefore, (6.4) holds.

By (6.4), we have $\tilde{u} \leq u_\infty + \epsilon$ in Q_1 for any $\epsilon > 0$. Taking $\epsilon \to 0$, we derive that

$$\tilde{u} \leq u_\infty \quad \text{in} \quad Q_1$$

and then it follows by Lemma 6.1 that $\tilde{u} = u_\infty$ in Q_∞. □

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