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# Fractional heat equations involving initial measure data and subcritical absorption

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## Abstract

We study the existence and uniqueness of weak solution to (F)  $\partial_t u + (-\Delta)^\alpha u + h(t, u) = 0$  in  $(0, \infty) \times \mathbb{R}^N$ , is given by the initial condition  $u(0, \cdot) = \nu$  in  $\mathbb{R}^N$ , where  $N \geq 1$ , the nonlocal operator  $(-\Delta)^\alpha$  denotes the fractional Laplacian with  $\alpha \in (0, 1)$ ,  $\nu$  is a bounded Radon measure and  $h : (0, \infty) \times \mathbb{R} \mapsto \mathbb{R}$  is a continuous function satisfying a subcritical integrability condition.

In particular of  $h(t, u) = t^\beta u^p$  with  $\beta > -1$ , there exists a unique solution  $u_k$  to (F) with  $\nu = k\delta_0$ , where  $\delta_0$  is the Dirac mass at the origin. We obtain that  $u_k \rightarrow \infty$  in  $(0, \infty) \times \mathbb{R}^N$  as  $k \rightarrow \infty$  for  $p \in (0, 1]$  and the limit of  $u_k$  exists as  $k \rightarrow \infty$  when  $p \in (1, 1 + \frac{2\alpha(1+\beta)}{N})$ , in this case denoting it by  $u_\infty$ . When  $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, 1 + \frac{2\alpha(1+\beta)}{N})$ ,  $u_\infty$  is the unique self-similar solution of  $(F)_\infty$   $\partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0$  in  $(0, \infty) \times \mathbb{R}^N$  with the initial condition  $u(0, \cdot) = 0$  in  $\mathbb{R}^N \setminus \{0\}$ . When  $p \in (1, 1 + \frac{2\alpha(1+\beta)}{N+2\alpha})$ ,  $u_\infty$  is a trivial self-similar solution to  $(F)_\infty$ , precisely,

$$u_\infty(t, x) = \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}, \quad (t, x) \in (0, \infty) \times \mathbb{R}^N.$$

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**Key words:** Fractional heat equation, Radon measure, Dirac mass, Self-similar solution, Very singular solution

**MSC2010:** 35R06, 35K05, 35R11

## 1 Introduction

Let  $h : (0, \infty) \times \mathbb{R} \mapsto \mathbb{R}$  be a continuous function and  $Q_\infty = (0, \infty) \times \mathbb{R}^N$  with  $N \geq 1$ . The first object of this paper is to consider the existence and uniqueness of weak solution to semilinear fractional heat equation involving initial measure data

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h(t, u) &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= \nu & \text{in } \mathbb{R}^N, \end{aligned} \tag{1.1}$$

where  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$  the bounded Radon measure space and the fractional Laplacian  $(-\Delta)^\alpha$  with  $\alpha \in (0, 1)$  is defined by

$$(-\Delta)^\alpha u(x) = \lim_{\epsilon \rightarrow 0^+} (-\Delta)_\epsilon^\alpha u(x),$$

here for  $\epsilon > 0$ ,

$$(-\Delta)_\epsilon^\alpha u(x) = - \int_{\mathbb{R}^N} \frac{u(z) - u(x)}{|z - x|^{N+2\alpha}} \chi_\epsilon(|x - z|) dz$$

and

$$\chi_\epsilon(r) = \begin{cases} 0, & \text{if } r \in [0, \epsilon], \\ 1, & \text{if } r > \epsilon. \end{cases}$$

In the pioneering work, Brezis and Friedman [4] have studied parabolic equation involving initial measure data

$$\begin{aligned} \partial_t u - \Delta u + u^p &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= k\delta_0 & \text{in } \mathbb{R}^N, \end{aligned} \tag{1.2}$$

where  $k > 0$  is a constant and  $\delta_0$  denotes the Dirac mass at the origin. They asserted that if  $1 < p < (N + 2)/N$ , then for every  $k > 0$  there exists a unique solution  $u_k$  to (1.2). When  $p \geq (N + 2)/N$ , problem (1.2) has no solution. Later on, Brezis, Peletier and Terman in [5] made use of dynamic method to obtain that there exists a unique very singular solution  $u_s$  to

$$\begin{aligned} \partial_t u - \Delta u + u^p &= 0 & \text{in } \mathbb{Q}_\infty, \\ u(0, \cdot) &= 0 & \text{in } \mathbb{R}^N \setminus \{0\} \end{aligned} \quad (1.3)$$

with  $1 < p < (N + 2)/N$ . In addition the asymptotic behavior of  $u_s(1, \cdot)$  as  $|x| \rightarrow \infty$  is given by

$$u_s(1, x) = c_1 e^{-\frac{1}{4}|x|^2} |x|^{\frac{2}{p-1}-N} \{1 - O(|x|^{-2})\},$$

where  $c_1$  is a certain positive constant. From then on, Kamin and Peletier in [18] built the connection that the weak solution  $u_k$  convergence to the very singular solution  $u_s$  as  $k \rightarrow \infty$ . Marcus and Véron [23], Al Sayed and Véron in [2] obtained the very singular solution of (1.3) replaced  $u^p$  by  $t^\beta u^p$  with  $\beta > -1$  and  $p \in (1, 1 + \frac{2(1+\beta)}{N})$ . The initial trace of nonlinear parabolic equations with general Radon measure and general nonlinearity

$$\begin{aligned} \partial_t u - \Delta u + h(t, u) &= 0 & \text{in } (0, T) \times \Omega, \\ u &= 0 & \text{in } (0, T) \times \partial\Omega, \\ u(0, \cdot) &= \nu & \text{in } \Omega, \end{aligned} \quad (1.4)$$

where  $\Omega$  is a domain in  $\mathbb{R}^N$  and  $T > 0$ , has been well studied by Marcus and Véron. In [21, 22], they proved the existence and uniqueness of solutions to (1.4). Therein, the very singular solution plays a fundamental role in dealing with the initial trace and they used dichotomy argument to make classification to the initial trace of the solutions to (1.4).

Motivated by great applications in physics and by important links on the theory of Lévy process, semilinear fractional equations has been attracted much interest in last few years, and see the references [6, 7, 8, 9, 11, 14, 15, 16]. Recently, in [12] we obtained existence and uniqueness of weak solution to semilinear fractional elliptic equation

$$\begin{aligned} (-\Delta)^\alpha u + f(u) &= \nu & \text{in } \Omega, \\ u &= 0 & \text{in } \Omega^c, \end{aligned} \quad (1.5)$$

when  $\nu$  is Radon measure and  $f$  satisfies a subcritical integrability condition.

Our purpose of this paper is to consider the existence and uniqueness of solutions to semilinear fractional heat equation (1.1) in the measure framework. We observe that most of the techniques used in the case of the Laplacian are not available. First, we make precise the notion of weak solution of (1.1) that we will use in this note.

**Definition 1.1** We say that  $u$  is a weak solution of (1.1), if for any  $T > 0$ ,  $u \in L^1(Q_T)$ ,  $h(t, u) \in L^1(Q_T)$  and

$$\begin{aligned} & \int_{Q_T} \{u(t, x)[- \partial_t \xi(t, x) + (-\Delta)^\alpha \xi(t, x)] + h(t, u)\xi(t, x)\} dxdt \\ & = \int_{\mathbb{R}^N} \xi(0, x)d\nu - \int_{\mathbb{R}^N} \xi(T, x)u(T, x)dx, \quad \forall \xi \in \mathbb{Y}_{\alpha, T}, \end{aligned} \quad (1.6)$$

where  $Q_T = (0, T) \times \mathbb{R}^N$  and  $\mathbb{Y}_{\alpha, T}$  is a space of functions  $\varsigma : [0, T] \times \mathbb{R}^N \mapsto \mathbb{R}$  satisfying

- (i)  $\|\varsigma\|_{L^1(Q_T)} + \|\varsigma\|_{L^\infty(Q_T)} + \|\partial_t \varsigma\|_{L^\infty(Q_T)} + \|(-\Delta)^\alpha \varsigma\|_{L^\infty(Q_T)} < +\infty$ ;
- (ii) for  $t \in (0, T)$ , there exist  $M > 0$  and  $\epsilon_0 > 0$  such that for all  $\epsilon \in (0, \epsilon_0]$ ,  $\|(-\Delta)_\epsilon^\alpha \varsigma(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} < M$ .

Before stating our first main theorem, we introduce the subcritical integrability condition for the nonlinearity  $h$ , that is,

- (H) (i) The function  $h : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and for any  $t \in (0, \infty)$ ,  $h(t, 0) = 0$  and  $h(t, r_1) \geq h(t, r_2)$  if  $r_1 \geq r_2$ .
- (ii) There exist  $\beta > -1$  and a continuous, nondecreasing function  $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$  such that

$$|h(t, r)| \leq t^\beta g(|r|), \quad \forall (t, r) \in (0, \infty) \times \mathbb{R}$$

and

$$\int_1^{+\infty} g(s)s^{-1-p_\beta^*} ds < +\infty, \quad (1.7)$$

where

$$p_\beta^* = 1 + \frac{2\alpha(1+\beta)}{N}. \quad (1.8)$$

Denote by  $H_\alpha : (0, \infty) \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}_+$  the heat kernel in  $(0, \infty) \times \mathbb{R}^N$  with homogeneous boundary conditions and by  $\mathbb{H}_\alpha[\nu]$  the heat potential of  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ , defined by

$$\mathbb{H}_\alpha[\nu](t, x) = \int_{\mathbb{R}^N} H_\alpha(t, x, y)d\nu(y).$$

We state the first theorem as follows.

**Theorem 1.1** Assume that  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$  and the function  $h$  satisfies (H). Then problem (1.1) admits a unique weak solution  $u_\nu$  such that

$$\mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(t, \mathbb{H}_\alpha[\nu_+])] \leq u_\nu \leq \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(t, -\mathbb{H}_\alpha[\nu_-])] \quad \text{in } Q_\infty, \quad (1.9)$$

where  $\nu_+$  and  $\nu_-$  are respectively the positive and negative part in the Jordan decomposition of  $\nu$ . Furthermore,

- (i) if  $\nu$  is nonnegative, so is  $u_\nu$ ;
- (ii) the mapping:  $\nu \mapsto u_\nu$  is increasing.

According to Theorem 1.1, there exists a unique positive weak solution  $u_k$  to fractional problem

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= k\delta_0 & \text{in } \mathbb{R}^N \end{aligned} \quad (1.10)$$

where  $\beta > -1$ ,  $k > 0$  and  $p \in (0, p_\beta^*)$ . It is of interest to investigate the limit of  $u_k$  as  $k \rightarrow \infty$  for  $p \in (1, p_\beta^*)$ , which exists since  $\{u_k\}$  are an increasing sequence functions and bounded by  $(\frac{1+\beta}{p-1})^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}$ . We denote the limit  $u_\infty$  by

$$u_\infty = \lim_{k \rightarrow \infty} u_k \quad \text{in } Q_\infty. \quad (1.11)$$

In fact,  $u_\infty$  and  $\{u_k\}_{k>0}$  are classical solutions to equation

$$\partial_t u + (-\Delta)^\alpha u + t^\beta u^p = 0 \quad \text{in } Q_\infty. \quad (1.12)$$

See Proposition 4.3.

**Definition 1.2** (i) A solution  $u$  of (1.12) is called a self-similar solution if

$$u(t, x) = t^{-\frac{1+\beta}{p-1}} u(1, t^{-\frac{1}{2\alpha}} x), \quad (t, x) \in Q_\infty.$$

(ii) A solution  $u$  of (1.12) is called a very singular solution if

$$\lim_{t \rightarrow 0^+} \frac{u(t, 0)}{\Gamma_\alpha(t, 0)} = +\infty,$$

where  $\Gamma_\alpha$  is the fundamental solution of

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= \delta_0 & \text{in } \mathbb{R}^N. \end{aligned} \quad (1.13)$$

We remark that for  $p \in (1, p_\beta^*)$ , a self-similar solution  $u$  of (1.12) is also a very singular solution, since

$$\lim_{t \rightarrow 0^+} \Gamma_\alpha(t, 0) t^{\frac{N}{2\alpha}} = c_2, \quad (1.14)$$

for some  $c_2 > 0$ . For any self-similar solution  $u$  of (1.12),  $v(\tilde{x}) := u(1, t^{-\frac{1}{2\alpha}} x)$  with  $\tilde{x} = t^{-\frac{1}{2\alpha}} x$  is a solution of

$$(-\Delta)^\alpha v - \frac{1}{2\alpha} \nabla v \cdot \tilde{x} - \frac{1+\beta}{p-1} v + v^p = 0 \quad \text{in } \mathbb{R}^N. \quad (1.15)$$

It is obvious that  $(\frac{1+\beta}{p-1})^{\frac{1}{p-1}}$  is a trivial solution of (1.15) and then the function

$$U_p(t) := \left(\frac{1+\beta}{p-1}\right)^{\frac{1}{p-1}} t^{-\frac{1+\beta}{p-1}}, \quad t > 0 \quad (1.16)$$

is the flat self-similar solution of (1.12). From Proposition 4.1,  $u_k$  is not self-similar solution of (1.12). Therefore, our second goal in this paper is to search for non-flat self-similar solution of (1.12).

**Theorem 1.2** *Assume that  $\beta > -1$ ,  $u_\infty$  is defined by (1.11) and*

$$p \in (p_\beta^{**}, p_\beta^*).$$

where  $p_\beta^{**} = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$  and  $p_\beta^* = 1 + \frac{2\alpha(1+\beta)}{N}$ . Then  $u_\infty$  is a positive self-similar solution of (1.12) with initial data  $u(0, \cdot) = 0$  in  $\mathbb{R}^N \setminus \{0\}$ .

Moreover, there exists  $c_3 > 1$  such that

$$\frac{c_3^{-1}}{1 + |x|^{N+2\alpha}} \leq u_\infty(1, x) \leq \frac{c_3 \log(2 + |x|)}{1 + |x|^{N+2\alpha}}, \quad x \in \mathbb{R}^N. \quad (1.17)$$

When  $p \in (p_\beta^{**}, p_\beta^*)$  with  $\beta > -1$ , we observe that  $u_\infty$  and  $U_p$  are self-similar solutions of (1.12) and  $u_\infty$  is not flat. An interest is to obtain the uniqueness of non flat self-similar solution of (1.12). We may propose our uniqueness for self-similar solution to

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + t^\beta u^p &= 0 \quad \text{in } Q_\infty, \\ \lim_{|x| \rightarrow \infty} u(1, x) &= 0. \end{aligned} \quad (1.18)$$

**Theorem 1.3** *Assume that  $p \in (p_\beta^{**}, p_\beta^*)$  and  $u_\infty$  is defined by (1.11), then  $u_\infty$  is the unique positive self-similar solution of (1.18).*

We note when  $u$  is self-similar,  $\lim_{|x| \rightarrow \infty} u(1, x) = 0$  is equivalent to  $\lim_{|x| \rightarrow \infty} u(t, x) = 0$  for any  $t > 0$ . Finally, we state the properties of  $u_\infty$  when  $p \in (1, p_\beta^{**}]$  as follows.

**Theorem 1.4** (i) *Assume that  $p \in (1, p_\beta^{**})$  and  $u_\infty$  is defined by (1.11). Then  $u_\infty$  is the flat self-similar solution of (1.12), that is,*

$$u_\infty = U_p,$$

where  $U_p$  is given by (1.16).

(ii) *Assume that  $p = p_\beta^{**}$  and  $u_\infty$  is defined by (1.11). Then  $u_\infty$  is a self-similar solution of (1.12) such that*

$$u_\infty(t, x) \geq \frac{c_4 t^{-\frac{N+2\alpha}{2\alpha}}}{1 + |t^{-\frac{1}{2\alpha}} x|^{N+2\alpha}}, \quad (t, x) \in (0, 1) \times \mathbb{R}^N, \quad (1.19)$$

for some  $c_4 > 0$ .

We note that Theorem 1.4 indicates that there is no self-similar solution of (1.12) with initial data  $u(0, \cdot) = 0$  in  $\mathbb{R}^N \setminus \{0\}$ , since  $u_\infty$  is the least self-similar solution. In Theorem 1.4 part (ii), we don't know if the self-similar solution is flat or not.

According to theorems above, we make a summary for problem (1.15) as follows:

**Remark 1.1** (i) When  $p \in (p_\beta^{**}, p_\beta^*)$ , problem (1.15) under the condition  $\lim_{|x| \rightarrow \infty} v(x) = 0$  admits a unique positive solution.

(ii) When  $p \in (1, p_\beta^{**})$ , problem (1.15) under the condition  $\lim_{|x| \rightarrow \infty} v(x) = 0$  has no any positive solution.

This paper is organized as follows. In Section 2 we obtain that

$$\|\mathbb{H}_\alpha^\Omega[|\nu|]\|_{M^{p_\beta^*}(Q_T^\Omega, t^\beta dx dt)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\Omega)}$$

and the Kato's type inequality for the non-homogeneous problem. In Section 3 we prove Theorem 1.1. Section 4 is devoted to investigate the properties of solutions to (1.10). In Section 5 we give the proof of Theorem 1.2 and Theorem 1.4. Finally, we put the uniqueness of self-similar solution when  $p \in (p_\beta^{**}, p_\beta^*)$  in the section 6.

## 2 Linear estimates

### 2.1 The Marcinkiewicz spaces

We recall the definition and basic properties of the Marcinkiewicz spaces.

**Definition 2.1** Let  $\Theta \subset \mathbb{R}^{N+1}$  be an open domain and  $\mu$  be a positive Borel measure in  $\Theta$ . For  $\kappa > 1$ ,  $\kappa' = \kappa/(\kappa - 1)$  and  $u \in L_{loc}^1(\Theta, d\mu)$ , we set

$$\|u\|_{M^\kappa(\Theta, d\mu)} = \inf\{c \in [0, \infty] : \int_E |u| d\mu \leq c \left( \int_E d\mu \right)^{\frac{1}{\kappa'}}, \forall E \subset \Theta \text{ Borel set}\} \quad (2.1)$$

and

$$M^\kappa(\Theta, d\mu) = \{u \in L_{loc}^1(\Theta, d\mu) : \|u\|_{M^\kappa(\Theta, d\mu)} < \infty\}. \quad (2.2)$$

$M^\kappa(\Theta, d\mu)$  is called the Marcinkiewicz space of exponent  $\kappa$  or weak  $L^\kappa$  space and  $\|\cdot\|_{M^\kappa(\Theta, d\mu)}$  is a quasi-norm. The following property holds.

**Proposition 2.1** [3, 12] Assume  $1 \leq q < \kappa < \infty$  and  $u \in L_{loc}^1(\Theta, d\mu)$ . Then there exists  $c_6 > 0$  dependent of  $q, \kappa$  such that

$$\int_E |u|^q d\mu \leq c_6 \|u\|_{M^\kappa(\Theta, d\mu)} \left( \int_E d\mu \right)^{1-q/\kappa},$$

for any Borel set  $E$  of  $\Theta$ .



For an open regular domain  $\Omega$  in  $\mathbb{R}^N$ , we denote  $H_\alpha^\Omega : (0, \infty) \times \Omega \times \Omega \rightarrow \mathbb{R}_+$  the heat kernel in  $(0, \infty) \times \Omega$  with homogeneous boundary conditions and by  $\mathbb{H}_\alpha^\Omega[\nu]$  the heat potential of  $\nu \in \mathfrak{M}^b(\Omega)$ , defined by

$$\mathbb{H}_\alpha^\Omega[\nu](t, x) = \int_\Omega H_\alpha^\Omega(t, x, y) d\nu(y).$$

When  $\Omega = \mathbb{R}^N$ , by Fourier transform, it is obvious that

$$H_\alpha(t, x, y) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{i(x-y)\cdot\zeta - t|\zeta|^{2\alpha}} d\zeta.$$

**Proposition 2.2** *For any  $\beta > -1$  and  $T > 0$ , there exists a positive constant  $c_5 > 0$  dependent of  $N, \alpha, \beta$  such that for  $\nu \in \mathfrak{M}^b(\Omega)$ ,*

$$\|\mathbb{H}_\alpha^\Omega[|\nu|]\|_{M^{p_\beta^*}(Q_T^\Omega, t^\beta dxdt)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\Omega)}, \quad (2.3)$$

where  $p_\beta^*$  is defined by (1.8) and  $Q_T^\Omega = (0, T) \times \Omega$ .

In order to prove this proposition, we introduce some normal notations. For  $\lambda > 0$  and  $y \in \Omega$ , let us denote

$$A_\lambda(y) = \{(t, x) \in Q_T^\Omega : H_\alpha(t, x, y) > \lambda\} \quad \text{and} \quad m_\lambda(y) = \int_{A_\lambda(y)} t^\beta dxdt.$$

**Lemma 2.1** *There exists  $c_7 > 0$  such that for any  $\lambda > 1$ ,*

$$A_\lambda(y) \subset (0, c_7 \lambda^{-\frac{2\alpha}{N}}] \times B_{c_7 \lambda^{-\frac{1}{N}}}(y), \quad (2.4)$$

where  $B_r(y)$  is the ball with radius  $r$  and center  $y$  in  $\mathbb{R}^N$ .

**Proof.** We observe that  $H_\alpha(t, x, y) = t^{-\frac{N}{2\alpha}} \Gamma_\alpha(1, (x-y)t^{-\frac{1}{2\alpha}})$ , where  $\Gamma_\alpha$  is the fundamental solution of (1.13). From (1.1) in [10], there exists  $c_8 > 0$  such that

$$\Gamma_\alpha(1, z) \leq \frac{c_8}{1 + |z|^{N+2\alpha}}.$$

On the one hand, for  $(t, x) \in A_\lambda(y)$ , we have that

$$t^{-\frac{N}{2\alpha}} \Gamma_\alpha(1, 0) \geq t^{-\frac{N}{2\alpha}} \Gamma_\alpha(1, (x-y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

which implies that

$$t < \Gamma_\alpha^{\frac{2\alpha}{N}}(1, 0) \lambda^{-\frac{2\alpha}{N}}. \quad (2.5)$$

On the other hand, letting  $r = |x-y|$ ,

$$\frac{c_8 t}{t^{1+\frac{N}{2\alpha}} + r^{N+2\alpha}} \geq t^{-\frac{N}{2\alpha}} \Gamma_\alpha(1, (x-y)t^{-\frac{1}{2\alpha}}) > \lambda,$$

then

$$r \leq (c_8 t \lambda^{-1})^{\frac{1}{N+2\alpha}}, \quad (2.6)$$

which, together with (2.5), implies that

$$r \leq c_9 \lambda^{-\frac{1}{N}},$$

for some  $c_9 > 0$ .  $\square$

**Proof of Proposition 2.2.** By Lemma 2.1, there exists  $c_{10} > 0$  such that

$$m_\lambda(y) \leq c_{10} \lambda^{-1 - \frac{2\alpha(1+\beta)}{N}}.$$

We observe that

$$H_\alpha^\Omega(t, x, y) \leq H_\alpha(t, x, y). \quad (2.7)$$

Then for any Borel set  $E \subset Q_T^\Omega$  and  $y \in \Omega$ , we have that

$$\int_E H_\alpha^\Omega(t, x, y) t^\beta dx dt \leq \lambda \int_E t^\beta dx dt + \int_{A_\lambda(y)} H_\alpha(t, x, y) t^\beta dx dt$$

and

$$\begin{aligned} \int_{A_\lambda(y)} H_\alpha(t, x, y) t^\beta dx dt &= - \int_\lambda^{+\infty} s dm_s(y) \\ &= \lambda m_\lambda(y) + \int_\lambda^{+\infty} m_s(y) ds \\ &\leq c_{10} \lambda^{-\frac{2\alpha(1+\beta)}{N}} + c_{10} \int_\lambda^{+\infty} s^{-1 - \frac{2\alpha(1+\beta)}{N}} ds \\ &\leq c_{11} \lambda^{-\frac{2\alpha(1+\beta)}{N}}, \end{aligned}$$

where  $c_{11} = c_{10} [1 + \frac{N}{2\alpha(1+\beta)}]$ . As a consequence, we have that

$$\int_E H_\alpha^\Omega(t, x, y) t^\beta dx dt \leq \lambda \int_E t^\beta dx dt + c_{11} \lambda^{-\frac{2\alpha(1+\beta)}{N}}.$$

Taking  $\lambda = (\int_E t^\beta dx dt)^{-\frac{N}{N+2\alpha(1+\beta)}}$ , we obtain that

$$\int_E H_\alpha^\Omega(t, x, y) t^\beta dx dt \leq (c_{11} + 1) (\int_E t^\beta dx dt)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}. \quad (2.8)$$

By Fubini's theorem, we have that

$$\begin{aligned} \int_E \mathbb{H}_\alpha^\Omega[|\nu|](t, x) t^\beta dx dt &= \int_E \int_\Omega H_\alpha^\Omega(t, x, y) d|\nu(y)| t^\beta dx dt \\ &= \int_\Omega \int_E H_\alpha^\Omega(t, x, y) t^\beta dx dt d|\nu(y)|, \end{aligned}$$

which, together with (2.8), implies that

$$\int_E \mathbb{H}_\alpha^\Omega[|\nu|](t, x) t^\beta dx dt \leq (c_{11} + 1) \|\nu\|_{\mathfrak{M}^b(\Omega)} (\int_E t^\beta dx dt)^{\frac{2\alpha(1+\beta)}{N+2\alpha(1+\beta)}}.$$

Thus,

$$\|\mathbb{H}_\alpha^\Omega[\|\nu\|]\|_{M^{1+\frac{2\alpha(1+\beta)}{N}}(Q_T^\Omega, t^\beta dxdt)} \leq (c_{11} + 1)\|\nu\|_{\mathfrak{M}^b(\Omega)},$$

which ends the proof.  $\square$

## 2.2 Non-homogeneous problem

The following proposition is the Kato's type estimate which is essential tool to prove the uniqueness of solution to (1.1). For  $T > 0$ , we denote  $Q_T = (0, T) \times \mathbb{R}^N$ .

**Proposition 2.3** *Assume that  $\mu \in L^1(Q_T)$  and  $\nu \in L^1(\mathbb{R}^N)$ . Then there exists a unique weak solution  $u$  of the problem*

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u &= \mu & \text{in } Q_T, \\ u(0, \cdot) &= \nu & \text{in } \mathbb{R}^N \end{aligned} \quad (2.9)$$

and there exists  $c_{12} > 0$  such that

$$\int_{Q_T} |u| dxdt \leq c_{12} \int_{Q_T} |\mu| dxdt + c_{12} \int_{\mathbb{R}^N} |\nu| dx. \quad (2.10)$$

Moreover, for any  $\xi \in \mathbb{Y}_{\alpha, T}$ ,  $\xi \geq 0$ , we have

$$\begin{aligned} \int_{Q_T} |u| \{-\partial_t \xi + (-\Delta)^\alpha \xi\} dxdt + \int_{\mathbb{R}^N} |u(T, x)| \xi(T, x) dx \\ \leq \int_{Q_T} \xi \text{sign}(u) \mu dxdt + \int_{\mathbb{R}^N} \xi(0, x) |\nu| dx. \end{aligned} \quad (2.11)$$

To prove Proposition 2.3, we introduce following notations. We say that  $u : Q_T \mapsto \mathbb{R}$  is in  $C_{t,x}^{\sigma, \sigma'}(Q_T)$  for  $\sigma, \sigma' \in (0, 1)$  if

$$\|u\|_{C_{t,x}^{\sigma, \sigma'}(Q_T)} := \|u\|_{L^\infty(Q_T)} + \sup_{Q_T} \frac{|u(t, x) - u(s, y)|}{|t - s|^\sigma + |x - y|^{\sigma'}} < +\infty$$

and  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma'}(Q_T)$  if

$$\|u\|_{C_{t,x}^{1+\sigma, 2\alpha+\sigma'}(Q_T)} := \|u\|_{L^\infty(Q_T)} + \|\partial_t u\|_{C_{t,x}^{\sigma, \sigma'}(Q_T)} + \|(-\Delta)^\alpha u\|_{C_{t,x}^{\sigma, \sigma'}(Q_T)} < +\infty.$$

**Lemma 2.2** *Let  $\mu \in C^1(Q_T) \cap L^\infty(Q_T)$ ,  $\nu \in L^\infty(\mathbb{R}^N)$  and  $u$  is a solution of problem (2.9), then there exists  $\sigma \in (0, 1)$  such that  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$ .*

*In particular, if  $\|D^2 \nu\|_{L^\infty(\mathbb{R}^N)} + \|(-\Delta)^\alpha \nu\|_{C_x^{1-\alpha}(\mathbb{R}^N)} < \infty$ , we have  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ .*

**Proof.** *Step 1.* When  $\|D^2\nu\|_{L^\infty(\mathbb{R}^N)} + \|(-\Delta)^\alpha\nu\|_{C_x^{1-\alpha}(\mathbb{R}^N)} < \infty$ , it follows directly by [7, (A.1)] that  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ .

*Step 2.* When  $\nu \in L^\infty(\mathbb{R}^N)$ , we use [8, Theorem 6.1] to obtain that  $u \in C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}(Q_T)$  for some  $\sigma > 0$ . For any  $T_0 \in (0, T)$ , let  $\eta : [0, T] \rightarrow [0, 1]$  be a  $C^2$  functions such that  $\eta = 0$  in  $[0, \frac{T_0}{4}]$  and  $\eta = 1$  in  $[T_0, T]$  and  $v = \eta u$  in  $Q_T$ . Then we obtain that for  $t \in [\frac{T_0}{4}, T]$  and  $x \in \mathbb{R}^N$ ,

$$\partial_t v + (-\Delta)^\alpha v = \eta\mu + \eta'(t)u,$$

where  $\eta\mu + \eta'(t)u \in C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}(Q_T)$  and  $v(0, \cdot) = 0$  in  $\mathbb{R}^N$ , Then we apply argument in Step 1 to obtain that  $v \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ . Therefore,  $u$  is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$ . The proof is complete.  $\square$

**Lemma 2.3** (i) Let  $\mu \in C^1(Q_T) \cap L^\infty(Q_T)$  and  $\nu \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , then problem (2.9) admits a unique solution  $u$  and for some  $\sigma \in (0, 1)$ ,  $u$  is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$ .

(ii) Assume that  $\mu \in C^1(Q_T) \cap L^\infty(Q_T) \cap L^1(Q_T)$ ,  $\nu \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  and  $u$  be the solution of (2.9), then  $u \in L^1(Q_T)$ , is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$  and for any  $\xi \in \mathbb{Y}_{\alpha, T}$ ,

$$\begin{aligned} & \int_{Q_T} u(t, x) [-\partial_t \xi(t, x) + (-\Delta)^\alpha \xi(t, x)] dx dt \\ &= \int_{Q_T} \mu(t, x) \xi(t, x) dx dt + \int_{\mathbb{R}^N} \xi(0, x) \nu dx - \int_{\mathbb{R}^N} \xi(T, x) u(T, x) dx. \end{aligned} \quad (2.12)$$

(iii) Let  $\mu \in C^1(Q_T) \cap L^\infty(Q_T)$  and  $\nu \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ , then problem

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u &= \mu \quad \text{in } Q_T, \\ u(T, \cdot) &= \nu \quad \text{in } \mathbb{R}^N \end{aligned} \quad (2.13)$$

admits a unique solution  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$  for some  $\sigma \in (0, 1)$ .

Moreover,  $u \in \mathbb{Y}_{\alpha, T}$  if  $\mu \in C^1(Q_T) \cap L^\infty(Q_T) \cap L^1(Q_T)$  and  $\nu \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ .

**Proof.** (i) By [8, Theorem 2.6, Theorem 6.1], there exists a unique viscosity solution  $u \in C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}(Q_T)$  with  $\sigma > 0$  of (2.9), and then it follows by Lemma 2.2 that  $u$  is  $C_{t,x}^{1+\sigma', 2\alpha+\sigma'}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$  and some  $\sigma' \in (0, \min\{\frac{\sigma}{2\alpha}, \sigma\})$ . Then  $u$  is a classical solution of (2.9).

(ii) To prove  $u \in L^1(Q_T)$  and  $u(t, \cdot) \in L^1(\mathbb{R}^N)$  for  $t \in (0, T)$ . By Duhamel formula, we have

$$\begin{aligned} \|u(t, \cdot)\|_{L^1(\mathbb{R}^N)} &\leq \int_{\mathbb{R}^N} \int_{Q_T} H_\alpha(t, s, x, y) |\mu(s, y)| dy ds dx \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} H_\alpha(t, x, y) |\nu(y)| dy dx \\ &\leq \|\mu\|_{L^1(Q_T)} + \|\nu\|_{L^1(\mathbb{R}^N)} \end{aligned}$$

and

$$\|u\|_{L^1(Q_T)} = \int_0^T \|u(t, \cdot)\|_{L^1(\mathbb{R}^N)} dt \leq T(\|\mu\|_{L^1(Q_T)} + \|\nu\|_{L^1(\mathbb{R}^N)}).$$

To prove that  $\|(-\Delta)_\epsilon^\alpha u(t, \cdot)\|_{L^\infty(\mathbb{R}^N)}$  is uniformly bounded with to  $\epsilon \in (0, \epsilon_0)$ . Since  $u(t, \cdot) \in C_x^{2\alpha+\sigma}(\mathbb{R}^N)$  for some  $\sigma \in (0, \min\{2-2\alpha, 1\})$ , then for  $x \in \mathbb{R}^N$  and  $y \in B_1(0)$ ,  $|u(x+y) + u(x-y) - 2u(x)| \leq \|u(t, \cdot)\|_{C_x^{2\alpha+\sigma}(\mathbb{R}^N)} |y|^{2\alpha+\sigma}$ . Thus

$$\begin{aligned} \|(-\Delta)_\epsilon^\alpha u(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} &\leq \sup_{x \in \mathbb{R}^N} \left[ \int_{\mathbb{R}^N \setminus B_1(0)} \frac{|u(x+y) - u(x)|}{|y|^{N+2\alpha}} dy \right. \\ &\quad \left. + \frac{1}{2} \int_{B_1(0) \setminus B_\epsilon(0)} \frac{|u(x+y) + u(x-y) - 2u(x)|}{|y|^{N+2\alpha}} dy \right] \\ &\leq 2\|u\|_{L^1(\mathbb{R}^N)} + \int_{B_1(0)} |y|^{\sigma-N} dy \|u(t, \cdot)\|_{C_x^{2\alpha+\sigma}(\mathbb{R}^N)}. \end{aligned}$$

To prove (2.12). We claim that

$$\int_{Q_T} \xi(-\Delta)_\epsilon^\alpha u dx dt = \int_{Q_T} u(-\Delta)_\epsilon^\alpha \xi dx dt, \quad \forall \xi \in \mathbb{Y}_{\alpha, T}. \quad (2.14)$$

Indeed, by using the fact that for any  $t > 0$ , we have

$$\begin{aligned} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(t, z) - u(t, x)] \xi(t, x)}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|) dz dx \\ = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(t, x) - u(t, z)] \xi(t, z)}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|) dz dx, \end{aligned}$$

then we have

$$\begin{aligned} \int_{\mathbb{R}^N} \xi(t, x) (-\Delta)_\epsilon^\alpha u(t, x) dx \\ = -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left[ \frac{(u(t, z) - u(t, x)) \xi(t, x)}{|z-x|^{N+2\alpha}} + \frac{(u(t, x) - u(t, z)) \xi(t, z)}{|z-x|^{N+2\alpha}} \right] \chi_\epsilon(|x-z|) dz dx \\ = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(t, z) - u(t, x)] [\xi(t, z) - \xi(t, x)]}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|) dz dx. \end{aligned}$$

Similarly,

$$\int_{\mathbb{R}^N} u(t, x) (-\Delta)_\epsilon^\alpha \xi(t, x) dx = \frac{1}{2} \int_{\mathbb{R}^{2N}} \frac{[u(t, z) - u(t, x)] [\xi(t, z) - \xi(t, x)]}{|z-x|^{N+2\alpha}} \chi_\epsilon(|x-z|) dz dx.$$

Then (2.14) holds. Since  $u$  is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}$  in  $(T_0, T) \times \mathbb{R}^N$  for any  $T_0 \in (0, T)$  and  $\xi$  belongs to  $\mathbb{Y}_{\alpha, T}$ ,  $(-\Delta)_\epsilon^\alpha \xi(t, \cdot) \rightarrow (-\Delta)^\alpha \xi(t, \cdot)$  and  $(-\Delta)_\epsilon^\alpha u(t, \cdot) \rightarrow (-\Delta)^\alpha u(t, \cdot)$  as  $\epsilon \rightarrow 0$  in  $\mathbb{R}^N$  and  $(-\Delta)_\epsilon^\alpha \xi(t, \cdot)$ ,  $(-\Delta)_\epsilon^\alpha u(t, \cdot) \in L^\infty(\mathbb{R}^N)$  and  $\xi(t, \cdot), u(t, \cdot) \in L^1(\mathbb{R}^N)$ , then it follows by the Dominated Convergence Theorem

$$\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N} \xi(t, x) (-\Delta)_\epsilon^\alpha u(t, x) dx = \int_{\mathbb{R}^N} \xi(t, x) (-\Delta)^\alpha u(t, x) dx$$

and

$$\lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N} (-\Delta)_\epsilon^\alpha \xi(t, x) u(t, x) dx = \int_{\mathbb{R}^N} (-\Delta)^\alpha \xi(t, x) u(t, x) dx.$$

Combining with (2.14), taking  $\epsilon \rightarrow 0^+$ , we have that

$$\int_{\mathbb{R}^N} \xi(t, x) (-\Delta)^\alpha u(t, x) dx = \int_{\mathbb{R}^N} (-\Delta)^\alpha \xi(t, x) u(t, x) dx,$$

integrating over  $[0, T]$  and by (2.9), we conclude that (2.12) holds.

(iii) Let  $u$  be the solution of problem (2.9) and

$$w(t, x) = u(T - t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^N.$$

Then  $w$  is a solution of (2.13) and for some  $\sigma \in (0, 1)$ ,  $w$  is  $C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ . On the contrary, if  $w$  is a solution of (2.13), then  $u(t, x) = w(T - t, x)$  for  $(t, x) \in [0, T] \times \mathbb{R}^N$  is a solution of (2.9), then the uniqueness holds since the solution of (2.9) is unique. Since  $u \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T)$ , then  $(-\Delta)^\alpha u(t, \cdot) \in C_x^\sigma$  and then  $(-\Delta)_\epsilon^\alpha u(t, \cdot)$  is bounded, which implies  $u \in \mathbb{Y}_{\alpha, T}$ .  $\square$

**Proof of Proposition 2.3.** *Uniqueness.* Let  $v$  be a weak solution of

$$\begin{aligned} \partial_t v + (-\Delta)^\alpha v &= 0 & \text{in } Q_T, \\ v(0, \cdot) &= 0 & \text{in } \mathbb{R}^N. \end{aligned} \tag{2.15}$$

We claim that  $v = 0$  a.e. in  $Q_T$ .

In fact, let  $\omega$  be a Borel subset of  $Q_T$  and  $\eta_{\omega, n}$  be the solution of

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u &= \zeta_n & \text{in } Q_T, \\ u(T, \cdot) &= 0 & \text{in } \mathbb{R}^N, \end{aligned} \tag{2.16}$$

where  $\zeta_n : \bar{Q}_T \mapsto [0, 1]$  is a function  $C_0^1(Q_T)$  such that

$$\zeta_n \rightarrow \chi_\omega \quad \text{in } L^\infty(\bar{Q}_T) \quad \text{as } n \rightarrow \infty.$$

Then by Lemma 2.3, we have that  $\eta_{\omega, n} \in \mathbb{Y}_{\alpha, T}$  and

$$\int_{Q_T} v \zeta_n dx dt = 0.$$

Thus passing the limit of  $n \rightarrow \infty$ , we have

$$\int_\omega v dx dt = 0.$$

This implies  $v = 0$  a.e. in  $Q_T$ .

*Existence and estimate (2.11).* For  $\delta > 0$ , we define an even convex function  $\phi_\delta$  by

$$\phi_\delta(t) = \begin{cases} |t| - \frac{\delta}{2}, & \text{if } |t| \geq \delta, \\ \frac{t^2}{2\delta}, & \text{if } |t| < \delta/2. \end{cases} \quad (2.17)$$

Then for any  $t, s \in \mathbb{R}$ ,  $|\phi'_\delta(t)| \leq 1$ ,  $\phi_\delta(t) \rightarrow |t|$  and  $\phi'_\delta(t) \rightarrow \text{sign}(t)$  when  $\delta \rightarrow 0^+$ . Moreover

$$\phi_\delta(s) - \phi_\delta(t) \geq \phi'_\delta(t)(s - t). \quad (2.18)$$

Let  $\{\mu_n\}, \{\nu_n\}$  be two sequences functions in  $C_0^2(Q_T), C_0^2(\mathbb{R}^N)$  respectively such that

$$\lim_{n \rightarrow \infty} \int_{Q_T} |\mu_n - \mu| dx dt = 0, \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nu_n - \nu| dx = 0.$$

We denote  $u_n$  be the corresponding solution to (2.9) replaced  $\mu, \nu$  by  $\mu_n, \nu_n$ , respectively. By Lemma 2.2 and Lemma 2.3(ii),  $u_n \in C_{t,x}^{1+\sigma, 2\alpha+\sigma}(Q_T) \cap L^1(Q_T)$  and then we use Lemma 2.3 in [12] and Lemma 2.3 (ii) to obtain that for any  $\delta > 0$  and  $\xi \in \mathbb{Y}_{\alpha, T}$ ,  $\xi \geq 0$ ,

$$\begin{aligned} & \int_{Q_T} \phi_\delta(u_n) [-\partial_t \xi + (-\Delta)^\alpha \xi] dx dt + \int_{\mathbb{R}^N} \xi(T, x) \phi_\delta(u_n(T, x)) dx \\ &= \int_{Q_T} \xi [\partial_t \phi_\delta(u_n) + (-\Delta)^\alpha \phi_\delta(u_n)] dx dt + \int_{\mathbb{R}^N} \xi(0, x) \phi_\delta(\nu_n) dx \\ &\leq \int_{Q_T} \xi \phi'_\delta(u_n) [\partial_t u_n + (-\Delta)^\alpha u_n] dx dt + \int_{\mathbb{R}^N} \xi(0, x) \phi_\delta(\nu_n) dx \\ &= \int_{Q_T} \xi \phi'_\delta(u_n) \mu_n dx dt + \int_{\mathbb{R}^N} \xi(0, x) \phi_\delta(\nu_n) dx. \end{aligned}$$

Taking  $\delta \rightarrow 0^+$ , we obtain

$$\begin{aligned} & \int_{Q_T} |u_n| [-\partial_t \xi + (-\Delta)^\alpha \xi] dx dt + \int_{\mathbb{R}^N} \xi(T, x) |u_n(T, x)| dx \\ &\leq \int_{Q_T} \xi \text{sign}(u_n) \mu_n dx dt + \int_{\mathbb{R}^N} \xi(0, x) |\nu_n| dx. \end{aligned} \quad (2.19)$$

Denote by  $\eta_k$  the solution of

$$\begin{aligned} -\partial_t u + (-\Delta)^\alpha u &= \varsigma_k & \text{in } Q_T, \\ u(T, \cdot) &= 0 & \text{in } \mathbb{R}^N, \end{aligned} \quad (2.20)$$

where  $\varsigma_k : Q_T \mapsto [0, 1]$  is a  $C_0^2$  function such that  $\varsigma_k = 1$  in  $(0, T) \times B_k(0)$ . From the proof of Lemma 2.3,  $\tilde{\eta}_k(t, x) := \eta_k(T - t, x)$  solves

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u &= \varsigma_k & \text{in } Q_T, \\ u(0, \cdot) &= 0 & \text{in } \mathbb{R}^N. \end{aligned}$$

By Lemma 2.2, then  $\tilde{\eta}_k \in C_{t,x}^{1+\sigma,2\alpha+\sigma}(Q_T)$  with some  $\sigma \in (0,1)$  and

$$\begin{aligned} 0 \leq \tilde{\eta}_k(t,x) &\leq c_8 \int_t^T \int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}}}{1 + |(s-t)^{-\frac{1}{2\alpha}}(y-x)|^{N+2\alpha}} dy ds \\ &\leq c_8 \int_t^T \int_{\mathbb{R}^N} \frac{dz}{1 + |z|^{N+2\alpha}} ds \\ &= c_{13}(T-t). \end{aligned}$$

Taking  $\xi = \eta_k$  in (2.19), we derive

$$\int_{Q_T} |u_n| \chi_{(0,T) \times B_k(0)} dx dt \leq c_{13}T \int_{Q_T} |\mu_n| dx dt + c_{13}T \int_{\mathbb{R}^N} |\nu_n| dx.$$

Then taking  $k \rightarrow \infty$ ,

$$\int_{Q_T} |u_n| dx dt \leq c_{13}T \int_{Q_T} |\mu_n| dx dt + c_{13}T \int_{\mathbb{R}^N} |\nu_n| dx. \quad (2.21)$$

Similarly,

$$\int_{Q_T} |u_n - u_m| dx \leq c_{13}T \int_{Q_T} |\mu_n - \mu_m| dx dt + c_{13}T \int_{\mathbb{R}^N} |\nu_n - \nu_m| dx. \quad (2.22)$$

Therefore,  $\{u_n\}$  is a Cauchy sequence in  $L^1(Q_T)$  and its limit  $u$  is a weak solution of (2.9). Letting  $n \rightarrow \infty$ , (2.11) and (2.10) follow by (2.19) and (2.21), respectively.  $\square$

### 3 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first consider the classical solution of (1.1) in domain  $Q_T$  with regular initial data for  $T > 0$ . Assume that

$(\widetilde{H})$  The function  $h$  satisfies (H) part (i) and  $h \in C^\gamma([0,T] \times \mathbb{R})$  for some  $\gamma \in (0,1)$ .

**Lemma 3.1** *Let  $h$  satisfy  $(\widetilde{H})$ ,  $\phi \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  and  $T > 0$ . Then there exists a unique solution  $u_\phi$  of (1.1) in  $Q_T$  with  $\nu = \phi$  such that*

$$\mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, \mathbb{H}_\alpha[\phi_+])] \leq u_\phi \leq \mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, -\mathbb{H}_\alpha[\phi_-])] \quad \text{in } Q_T, \quad (3.1)$$

where  $\phi_\pm = \max\{0, \pm\phi\}$ . Moreover,

- (i)  $u_\phi \geq 0$  if  $\phi \geq 0$  in  $\Omega$ ;
- (ii) the mapping  $\phi \mapsto u_\phi$  is increasing.



**Proof.** To prove that there exists  $T' > 0$  such that problem

$$\begin{aligned} \partial_t u + (-\Delta)^\alpha u + h(t, u)\chi_{(0, T')}(t) &= 0 & \text{in } Q_\infty, \\ u(0, \cdot) &= \phi & \text{in } \mathbb{R}^N \end{aligned} \quad (3.2)$$

admits a solution  $u_{0, \phi}$ .

For parameter  $T' > 0$ , we denote the operator  $\Upsilon$  by

$$\Upsilon(u) = \mathbb{H}_\alpha[\phi - h(t, u)\chi_{[0, T']}], \quad u \in L^\infty(Q_\infty).$$

We observe that  $\mathbb{H}_\alpha[\phi](0, \cdot) = \phi$  in  $\mathbb{R}^N$ , then

$$\begin{aligned} \|\phi\|_{L^\infty(\mathbb{R}^N)} &\leq \|\mathbb{H}_\alpha[\phi]\|_{L^\infty(Q_\infty)} \\ &\leq c_{14}\|\phi\|_{L^\infty(\mathbb{R}^N)} \left\| \int_{\mathbb{R}^N} \frac{t^{-\frac{N}{2\alpha}}}{1+|t^{-\frac{1}{2\alpha}}(x-y)|^{N+2\alpha}} dy \right\|_{L^\infty(Q_\infty)} \\ &\leq c_{15}\|\phi\|_{L^\infty(\mathbb{R}^N)}, \end{aligned} \quad (3.3)$$

where  $c_{14} > 0$  and  $c_{15} = c_{14} \int_{\mathbb{R}^N} \frac{1}{1+|z|^{N+2\alpha}} dz$ . Let  $\mathcal{B}_r = \{u \in L^\infty(Q_\infty) : \|u\|_{L^\infty(Q_\infty)} \leq r\}$  with  $r = 2c_{15}\|\phi\|_{L^\infty(\mathbb{R}^N)}$  and  $A = [0, T'] \times [-r, r]$ . For  $T' > 0$  small enough and  $u \in \mathcal{B}_r$ , we derive that

$$\begin{aligned} &\|\mathbb{H}_\alpha[h(\cdot, u)\chi_{(0, T')}(0, \cdot)]\|_{L^\infty(Q_\infty)} \\ &\leq c_{14} \max_{(t, s) \in A} |h(t, s)| \left\| \int_t^{T'} \int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}}}{1+|(s-t)^{-\frac{1}{2\alpha}}(x-y)|^{N+2\alpha}} dy ds \right\|_{L^\infty(Q_{T'})} \\ &\leq c_{15} T' \max_{(t, s) \in A} |h(t, s)|. \end{aligned}$$

Therefore, choosing  $T' > 0$  such that  $T' \max_{(t, s) \in A} |h(t, s)| \leq \frac{r}{2}$ , we derive that

$$\begin{aligned} \|\Upsilon(u)\|_{L^\infty(Q_\infty)} &\leq c_{15} \left( \|\phi\|_{L^\infty(\mathbb{R}^N)} + T' \max_{(t, s) \in A} |h(t, s)| \right) \\ &\leq r. \end{aligned} \quad (3.4)$$

For our choosing  $r$  and  $T'$ , we have that  $\Upsilon(\mathcal{B}_r) \subset \mathcal{B}_r$ .

Next we prove that the operator  $\Upsilon$  is compact in  $\mathcal{B}_r$ . We observe that for  $u \in \mathcal{B}_r$ ,

$$\|u(t, \cdot)\|_{L^1(\mathbb{R}^N, \tilde{\omega} dx)} \leq c_{16}\|u\|_{L^\infty(Q_\infty)} \leq c_{16}r, \quad (3.5)$$

where  $c_{16} > 0$  and  $\tilde{\omega}(x) = (1 + |x|^{N+2\alpha})^{-1}$ . By [8, Theorem 5.1], there exist  $\sigma \in (0, 1)$  and  $c_{17}, c_{18} > 0$  only dependent of  $\alpha, N$  such that

$$\begin{aligned} \|\Upsilon(u)\|_{C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}(Q_\infty)} &\leq c_{17} \left\{ \|u\|_{L^\infty(Q_\infty)} + \sup_{\tau > 0} \|u(\tau, \cdot)\|_{L^1(\mathbb{R}^N, \tilde{\omega} dx)} \right. \\ &\quad \left. + \|h(\cdot, u)\|_{L^\infty(Q_{T'})} \right\} \\ &\leq c_{18} \left\{ \|\phi\|_{L^\infty(\mathbb{R}^N)} + \max_{(t, s) \in A} |h(t, s)| \right\}. \end{aligned} \quad (3.6)$$

Therefore,  $\Upsilon$  is compact. It follows by Schauder's fixed point theorem, that there exists  $u_{0,\phi} \in \mathcal{B}_r$  such that  $\Upsilon(u_{0,\phi}) = u_{0,\phi}$ . It infers by (3.6) that  $u_{0,\phi} \in C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}(Q_\infty)$ . By Lemma 2.2,  $u_{0,\phi}$  is in  $C_{t,x}^{1+\sigma', 2\alpha+\sigma'}(Q_\infty)$  for some  $\sigma' > 0$  and  $u_{0,\phi}$  is a classical solution of (3.2).

Moreover, since  $\mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, -\mathbb{H}_\alpha[\phi_-])]$  and  $\mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, \mathbb{H}_\alpha[\phi_+])]$  are super and sub solution of (3.2) respectively, then by Comparison Principle, we have that

$$\mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, \mathbb{H}_\alpha[\phi_+])] \leq u_{0,\phi} \leq \mathbb{H}_\alpha[\phi] - \mathbb{H}_\alpha[h(t, -\mathbb{H}_\alpha[\phi_-])] \quad \text{in } Q_\infty. \quad (3.7)$$

*The mapping  $\phi \mapsto u_{0,\phi}$  is increasing.* Let  $v_1, v_2$  be solutions of (3.2) with initial data  $\phi_1$  and  $\phi_2$  respectively, where  $\phi_1 \geq \phi_2$  and denote  $w = v_1 - v_2$ . We prove this argument by contradiction. Assume that there exists  $(t_0, x_0) \in Q_\infty$  satisfying  $w(t_0, x_0) < 0$ . Since

$$w(0, x) = \phi_1(x) - \phi_2(x) \geq 0, \quad x \in \mathbb{R}^N,$$

then there exist  $t_1 \in (0, t_0]$  and  $x_1 \in \mathbb{R}^N$  such that

$$w(t_1, x_1) = \min_{(t,x) \in [0, t_1] \times \mathbb{R}^N} w(t, x) < 0,$$

thus,  $\partial_t w(t_1, x_1) \leq 0$  and  $(-\Delta)^\alpha w(t_1, x_1) < 0$  and by monotonicity of  $h$ , we have  $h(t_1, v_1(t_1, x_1)) \leq h(t_1, v_2(t_1, x_1))$ . Therefore,

$$\partial_t w(t_1, x_1) + (-\Delta)^\alpha w(t_1, x_1) + h(t_1, v_1(t_1, x_1)) - h(t_1, v_2(t_1, x_1)) < 0.$$

which contradicts that  $v_1$  and  $v_2$  are solutions of (3.2).

Moreover, the uniqueness of solution to (3.2) follows by the fact that the mapping  $\phi \mapsto u_{0,\phi}$  is increasing. Then 0 is the unique solution of (3.2) with 0 initial data. Therefore,  $u_{0,\phi} \geq 0$  if  $\phi \geq 0$ .

*To prove that (1.1) in  $Q_T$  with  $T > 0$  admits a solution  $u_\phi$ .* Since  $\int_{\mathbb{R}^N} H_\alpha(t, x - y) dy = 1$  for any  $t > 0$  and  $x \in \mathbb{R}^N$ , then

$$\begin{aligned} \|\phi\|_{L^\infty(\mathbb{R}^N)} &\leq \|\mathbb{H}_\alpha[\phi]\|_{L^\infty(Q_\infty)} \\ &\leq \|\phi\|_{L^\infty(\mathbb{R}^N)} \left\| \int_{\mathbb{R}^N} H_\alpha(t, x - y) dy \right\|_{L^\infty(Q_\infty)} \\ &= \|\phi\|_{L^\infty(\mathbb{R}^N)}, \end{aligned}$$

which, combining (3.7) with  $|\phi|$ , infers that

$$\|u_{0,\phi}(T', \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \|\mathbb{H}_\alpha[|\phi|](T', \cdot)\|_{L^\infty(\mathbb{R}^N)} \leq \|\phi\|_{L^\infty(\mathbb{R}^N)}. \quad (3.8)$$

Thus, there exists a unique solution  $u_{1,\phi}$  to (3.2) with initial data  $u_{1,\phi}(0, \cdot) = u_{0,\phi}(T', \cdot)$  and we note that it is also able to choose the same  $T'$  by (3.8) in this step and it could iterate this process by times  $k$  and we obtain a unique

solution  $u_{k,\phi}$  of (3.2) with  $u_{k,\phi}(0, \cdot) = u_{k-1,\phi}(T', \cdot)$  initial data for the same  $T' > 0$ . Since for any  $T > 0$ , there exists  $m \in \mathbb{N}$  such that  $mT' \geq T$  and  $(m-1)T' < T$ , letting

$$u_\phi(t, x) = \sum_{i=0}^{m-1} u_{i,\phi}(t - iT', x) \chi_{(iT', (i+1)T']}(t), \quad (t, x) \in (0, T) \times \mathbb{R}^N,$$

then  $u_\phi$  is the unique solution of (1.1) in  $Q_T$  with  $\nu = \phi$  and (3.1) follows by (3.7).  $\square$

**Proof of Theorem 1.1.** *Existence for  $\nu \geq 0$ .* We consider a sequence nonnegative functions  $\{\nu_n\} \subset C_0^2(\mathbb{R}^N)$  such that  $\nu_n \rightarrow \nu$  in the duality sense of

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \zeta \nu_n dx = \int_{\mathbb{R}^N} \zeta d\nu, \quad \forall \zeta \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \quad (3.9)$$

It follows from the Banach-Steinhaus theorem that  $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)}$  is bounded independently of  $n$  and we assume that  $\|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)} \leq 2\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}$ . For any  $T > 0$ , we consider a sequence functions  $\{h_n\}$  satisfying  $(\widetilde{H})$ ,

$$|h_n| \leq |h| \quad \text{and} \quad \lim_{n \rightarrow \infty} \|h_n - h\|_{L^\infty([0, T] \times \mathbb{R})} = 0.$$

By Lemma 3.1, we denote by  $u_n$  the corresponding solution of (1.1) with nonlinearity  $h_n$  and initial  $\nu_n$ , then  $u_n$  is nonnegative and satisfies that

$$0 \leq u_n = \mathbb{H}_\alpha[\nu_n - h_n(t, u_n)] \leq \mathbb{H}_\alpha[\nu_n] \quad \text{in } Q_T. \quad (3.10)$$

By (2.3) it follows that

$$\|u_n\|_{M^{p\beta}_*(Q_T, t^\beta dx dt)} \leq c_5 \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}. \quad (3.11)$$

For  $\epsilon > 0$  and  $\sigma \in (0, \alpha)$ , set  $\xi_\epsilon = (\eta_k + \epsilon)^{\frac{\sigma}{\alpha}} - \epsilon^{\frac{\sigma}{\alpha}}$ , where  $\eta_k$  is given by (2.20), then by [12, Lemma 2.3 part (ii)],

$$\begin{aligned} -\partial_t \xi_\epsilon + (-\Delta)^\alpha \xi_\epsilon &= -\frac{\sigma}{\alpha} (\eta_k + \epsilon)^{\frac{\sigma-\alpha}{\alpha}} \partial_t \eta_k + \frac{\sigma}{\alpha} (\eta_k + \epsilon)^{\frac{\sigma-\alpha}{\alpha}} (-\Delta)^\alpha \eta_k \\ &\quad - \frac{\sigma(\sigma-\alpha)}{\alpha^2} (\eta_k + \epsilon)^{\frac{\sigma-2\alpha}{\alpha}} \int_{\mathbb{R}^N} \frac{(\eta_k(y) - \eta_k(x))^2}{|y-x|^{N+2\alpha}} dy \\ &\geq \frac{\sigma}{\alpha} (\eta_k + \epsilon)^{\frac{\sigma-\alpha}{\alpha}}, \end{aligned}$$

and  $\xi_\epsilon \in \mathbb{Y}_{\alpha, T}$ . By (2.11)

$$\begin{aligned} \int_{Q_T} (|u_n| [-\partial_t \xi_\epsilon + (-\Delta)^\alpha \xi_\epsilon] + |h_n(t, u_n)| \xi_\epsilon) dx dt + \int_{\mathbb{R}^N} |u(T, x)| \xi_\epsilon(T, x) dx \\ \leq \int_{\mathbb{R}^N} \xi_\epsilon(0, x) |\nu_n(x)| dx, \end{aligned}$$

then we obtain

$$\int_{Q_T} |h_n(t, u_n)| \xi_\epsilon(t, x) dx dt \leq \int_{\mathbb{R}^N} \xi_\epsilon(0, x) |\nu_n(x)| dx.$$

If we let  $\epsilon \rightarrow 0$ , we obtain

$$\int_{Q_T} |h_n(t, u_n)| \eta_k^\sigma(t, x) dx dt \leq \int_{\mathbb{R}^N} \eta_k^\sigma(0, x) |\nu_n(x)| dx.$$

Taking  $k \rightarrow \infty$  and then  $\sigma \rightarrow 0^+$ , we obtain that

$$\|h_n(t, u_n)\|_{L^1(Q_T)} \leq \|\nu_n\|_{\mathfrak{M}^b(\mathbb{R}^N)} \leq 2\|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}. \quad (3.12)$$

Since

$$H_\alpha(t, x, y) = t^{-\frac{N}{2\alpha}} \Gamma_\alpha(1, t^{-\frac{1}{2\alpha}}(x - y)),$$

then combining with (3.10), we obtain that for  $t > 0$

$$\begin{aligned} \|u_n(t, \cdot)\|_{L^\infty(\mathbb{R}^N)} &\leq c_8 t^{-\frac{N}{2\alpha}} \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\nu_n(y)}{1 + |t^{-\frac{1}{2\alpha}}(x - y)|^{N+2\alpha}} dy \\ &\leq c_8 t^{-\frac{N}{2\alpha}} \|\nu_n\|_{L^1(\mathbb{R}^N)} \\ &\leq 2c_8 t^{-\frac{N}{2\alpha}} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)} \end{aligned}$$

and

$$\begin{aligned} \|\mathbb{H}_\alpha[\nu_n](t, \cdot)\|_{L^1(\mathbb{R}^N, \tilde{\omega} dx)} &\leq c_8 t^{-\frac{N}{2\alpha}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\nu_n(y)}{1 + |t^{-\frac{1}{2\alpha}}(x - y)|^{N+2\alpha}} \frac{dy dx}{1 + |x|^{N+2\alpha}} \\ &\leq c_{15} t^{-\frac{N}{2\alpha}} \|\nu_n\|_{L^1(\mathbb{R}^N)} \\ &\leq 2c_{15} t^{-\frac{N}{2\alpha}} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}, \end{aligned}$$

then

$$\begin{aligned} \|\mathbb{H}_\alpha[h_n(t, u_n)]\|_{L^1(\mathbb{R}^N, \tilde{\omega} dx)} &\leq c_8 t^{-\frac{N}{2\alpha}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|h_n(t, u_n)|}{1 + |t^{-\frac{1}{2\alpha}}(x - y)|^{N+2\alpha}} \frac{dy dx}{1 + |x|^{N+2\alpha}} \\ &\leq c_{15} \|h_n(t, u_n)\|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{1 + |x|^{N+2\alpha}} \frac{1}{1 + |y|^{N+2\alpha}} dy dx \\ &\leq c_{20} t^\beta g(2c_8 t^{-N/2\alpha} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}) \end{aligned}$$

and then

$$\begin{aligned} \|u_n(t, \cdot)\|_{L^1(\mathbb{R}^N, \tilde{\omega} dx)} &\leq \|\mathbb{H}_\alpha[\nu_n]\|_{L^1(\mathbb{R}^N, \tilde{\omega} dx)} + \|\mathbb{H}_\alpha[h_n(t, u_n)]\|_{L^1(\mathbb{R}^N, \tilde{\omega} dx)} \\ &\leq 2c_{15} t^{-\frac{N}{2\alpha}} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)} + c_{20} t^\beta g(2c_8 t^{-N/2\alpha} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}). \end{aligned}$$

Then by [8, Theorem 5.1], there exists  $\sigma \in (0, 1)$  such that for any  $T_0 \in (0, T)$ ,

$$\begin{aligned} \|u_n\|_{C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}((T_0, T) \times \mathbb{R}^N)} &\leq c_{18} \{ \|u_n\|_{L^\infty((T_0, T) \times \mathbb{R}^N)} + \sup_{\tau \in (T_0, T)} \|u_n(\tau, \cdot)\|_{L^1(\mathbb{R}^N, \bar{\omega} dx)} \\ &\quad + \|h_n(\cdot, u_n)\|_{L^\infty((T_0, T) \times \mathbb{R}^N)} \} \\ &\leq c_{21} T_0^{-\frac{N}{2\alpha}} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)} + c_{22} (T^\beta + T_0^\beta) g(2c_8 T_0^{-N/2\alpha} \|\nu\|_{\mathfrak{M}^b(\mathbb{R}^N)}). \end{aligned} \quad (3.13)$$

Therefore, there exist a sub-sequence  $\{u_{n_k}\}$  and some  $u \in C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}$  locally in  $Q_T$  such that  $u_{n_k} \rightarrow u$  in  $Q_T$  and then  $h_{n_k}(\cdot, u_{n_k}) \rightarrow h(\cdot, u)$  in  $Q_T$ .

For  $\lambda > 0$ , we denote  $S_\lambda = \{(t, x) \in Q_T : |u_{n_k}(t, x)| > \lambda\}$  and  $\omega(\lambda) = \int_{S_\lambda} t^\beta dx dt$ . Then for any Borel set  $E \subset Q_T$ , we use (H) to obtain

$$\begin{aligned} \int_E |h_{n_k}(t, u_{n_k})| dx dt &\leq \int_{E \cap S_\lambda^c} g(|u_{n_k}|) t^\beta dx dt + \int_{E \cap S_\lambda} g(|u_{n_k}|) t^\beta dx dt \\ &\leq g(\lambda) \int_E t^\beta dx dt + \int_{S_\lambda} g(u_{n_k}) t^\beta dx dt \\ &\leq g(\lambda) \int_E t^\beta dx dt - \int_\lambda^\infty g(s) d\omega(s), \end{aligned}$$

where

$$\int_\lambda^\infty g(s) d\omega(s) = \lim_{M \rightarrow \infty} \int_\lambda^M g(s) d\omega(s).$$

By (3.11), we have  $\omega(s) \leq c_{23} s^{-p_\beta^*}$  and then

$$\begin{aligned} - \int_\lambda^M g(s) d\omega(s) &= - \left[ g(s) \omega(s) \right]_{s=\lambda}^{s=M} + \int_\lambda^M \omega(s) dg(s) \\ &\leq g(\lambda) \omega(\lambda) - g(M) \omega(M) + c_{23} \int_\lambda^M s^{-p_\beta^*} dg(s) \\ &\leq g(\lambda) \omega(\lambda) - g(M) \omega(M) + c_{23} \left( M^{-p_\beta^*} g(M) - \lambda^{-p_\beta^*} g(\lambda) \right) \\ &\quad + \frac{c_{23}}{p_\beta^* + 1} \int_\lambda^M s^{-1-p_\beta^*} g(s) ds. \end{aligned}$$

By (1.7) and Lemma 4.1 in [12], we have  $\lim_{M \rightarrow \infty} M^{-p_\beta^*} g(M) = 0$ . Since  $\omega(s) \leq c_{23} s^{-p_\beta^*}$ , we obtain  $g(\lambda) \omega(\lambda) \leq c_{23} \lambda^{-p_\beta^*} g(\lambda)$  and then

$$- \int_\lambda^\infty g(s) d\omega(s) \leq \frac{c_{23}}{p_\beta^* + 1} \int_\lambda^\infty s^{-1-p_\beta^*} g(s) ds.$$

Notice that the above quantity on the right-hand side tends to 0 when  $\lambda \rightarrow \infty$ . The conclusion follows: for any  $\epsilon > 0$  there exists  $\lambda > 0$  such that

$$\frac{c_{23}}{p_\beta^* + 1} \int_\lambda^\infty s^{-1-p_\beta^*} g(s) ds \leq \frac{\epsilon}{2}$$

and there exists  $\delta > 0$  such that

$$\int_E t^\beta dxdt \leq \delta \implies g(\lambda) \int_E t^\beta dxdt \leq \frac{\epsilon}{2}.$$

This means that  $\{h_{n_k}(\cdot, u_{n_k})\}$  is uniformly integrable in  $L^1(Q_T)$ . Combining (3.12), we have  $h_{n_k}(\cdot, u_{n_k}) \rightarrow h(\cdot, u)$  in  $L^1(Q_T)$  by Vitali convergence theorem. Letting  $n_k \rightarrow \infty$  in the identity

$$\begin{aligned} & \int_{Q_T} [-u_{n_k} \partial_t \xi + u_{n_k} (-\Delta)^\alpha \xi + \xi h_{n_k}(t, u_{n_k})] dxdt \\ &= \int_{\mathbb{R}^N} \xi(0, x) \nu_{n_k} dx - \int_{\mathbb{R}^N} u_{n_k}(T, x) \xi(T, x) dx, \quad \forall \xi \in \mathbb{Y}_{\alpha, T}, \end{aligned}$$

which infers that  $u$  is a weak solution of (1.1).

*Existence for general  $\nu$ .* For  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$ , a sequence  $\{\nu_n\}$  in  $C_0^2(\mathbb{R}^N)$  converge to  $\nu$  in the dual sense of (3.9). From the monotonicity of  $h_n(t, \cdot)$

$$-\mathbb{H}_\alpha[|\nu_n|] \leq u_{-|\nu_n|} \leq u_{\nu_n} \leq u_{|\nu_n|} \leq \mathbb{H}_\alpha[|\nu_n|].$$

Then by above analysis, the sequence  $\{h_n(\cdot, u_{-|\nu_n|})\}$  and  $\{h_n(\cdot, u_{|\nu_n|})\}$  are compact in  $L^1(Q_T)$  for any  $T > 0$  and (3.11) holds for  $\{u_{\nu_n}\}$ . Therefore  $\{u_{\nu_n}\}$  is compact in  $L^1(Q_T^\Omega)$  and there exist some subsequence  $\{u_{\nu_{n_k}}\}$  and  $u_\nu \in L^1(Q_T)$  such that

$$u_{\nu_{n_k}} \rightarrow u_\nu \quad \text{as } k \rightarrow \infty \quad \text{in } Q_T$$

and

$$\begin{aligned} & \int_{Q_T} [-u_\nu \partial_t \xi + u_\nu (-\Delta)^\alpha \xi + \xi h(t, u_\nu)] dxdt \\ &= \int_{\mathbb{R}^N} \xi(0, x) d\nu - \int_{\mathbb{R}^N} u_\nu(T, x) \xi(T, x) dx, \quad \forall \xi \in \mathbb{Y}_{\alpha, T}. \end{aligned}$$

*Uniqueness.* Let  $u_1, u_2$  be two weak solutions of (1.1) with the same initial  $\nu$  and  $w = u_1 - u_2$ . Then

$$\partial_t w + (-\Delta)^\alpha w = h(t, u_2) - h(t, u_1) \quad \text{in } Q_T.$$

Since  $h(t, u_2) - h(t, u_1) \in L^1(Q_T)$ , then by (2.11), for  $\xi \in \mathbb{Y}_{\alpha, T}$ ,  $\xi \geq 0$ , we have

$$\begin{aligned} & \int_{Q_T} |w| [-\partial_t \xi + (-\Delta)^\alpha \xi] dxdt + \int_{\mathbb{R}^N} |w(T, x)| \xi(T, x) dxdt \\ &+ \int_{Q_T} [h(t, u_1) - h(t, u_2)] \text{sign}(w) \xi dxdt \leq 0. \end{aligned}$$

Together with  $\int_{Q_T} [h(t, u_1) - h(t, u_2)] \text{sign}(w) \xi dx dt \geq 0$ , then

$$w = 0 \quad \text{a.e. in } Q_T.$$

To prove that the mapping  $\nu \mapsto u_\nu$  is increasing. Let  $\nu_1, \nu_2 \in \mathfrak{M}^b(\mathbb{R}^N)$  and  $\nu_1 \geq \nu_2$ , then there exist two sequences  $\{\nu_{1,n}\}$  and  $\{\nu_{2,n}\}$  in  $C_0^2(\mathbb{R}^N)$  such that  $\nu_{1,n} \geq \nu_{2,n}$ , converge to  $\nu_1$  and  $\nu_2$  respectively in the sense of (3.9). Let  $u_{i,n}$  be the corresponding unique solution of (1.1) with initial  $\nu_{i,n}$  and  $u_{\nu_i}$  be the unique solution of (1.1) with initial  $\nu_i$  for  $i = 1, 2$ . Then  $u_{1,n} \geq u_{2,n}$ . Moreover, by uniqueness,  $u_{i,n}$  convergence to  $u_{\nu_i}$  in  $L^1(Q_T)$  for  $i = 1, 2$ . Then we have  $u_{\nu_1} \geq u_{\nu_2}$ .

To prove (1.9). We make Jordan decomposition of  $\nu \in \mathfrak{M}^b(\mathbb{R}^N)$  by the positive part  $\nu_+$  and the negative part  $\nu_-$ . For  $\nu_\pm$ , it is able to find two sequences of nonnegative functions  $\{\nu_{n\pm}\}$  in  $C_0^2(\mathbb{R}^N)$  which converge to  $\nu_\pm$  in the dual sense of (3.9) respectively. Then the sequence functions  $\nu_n = \nu_{n+} - \nu_{n-}$  converge to  $\nu$  in the dual sense of (3.9). From the monotonicity of  $h(t, \cdot)$ ,

$$-\mathbb{H}_\alpha[\nu_{n-}] \leq u_{-\nu_{n-}} \leq u_{\nu_n} \leq u_{\nu_{n+}} \leq \mathbb{H}_\alpha[\nu_{n+}],$$

which, together with the uniqueness of  $u_\nu$ , implies that

$$-\mathbb{H}_\alpha[\nu_-] \leq u_\nu \leq \mathbb{H}_\alpha[\nu_+]. \quad (3.14)$$

Therefore, by monotonicity of  $h(t, \cdot)$ ,

$$\begin{aligned} u_\nu &= \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(\cdot, u_\nu)] \\ &\leq \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(\cdot, -\mathbb{H}_\alpha[\nu_-])] \end{aligned}$$

and

$$\begin{aligned} u_\nu &= \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(\cdot, u_\nu)] \\ &\geq \mathbb{H}_\alpha[\nu] - \mathbb{H}_\alpha[h(\cdot, \mathbb{H}_\alpha[\nu_+])], \end{aligned}$$

which imply (1.9). The proof is complete.  $\square$

## 4 Initial Dirac mass

In this section, we will consider the properties of solutions to (1.1) in the particular case of  $h(t, r) = t^\beta r^p$  and the initial data  $\nu = k\delta_0$ , where  $\beta > -1$  and  $p \in (0, p_\beta^*)$ .

**Proposition 4.1** *Let  $p \in (0, p_\beta^*)$  and  $u_k$  be the solution of (1.10), then there exists  $c_{24} > 0$  such that*

$$\lim_{t \rightarrow 0^+} u_k(t, 0) t^{\frac{N}{2\alpha}} = c_{24} k. \quad (4.1)$$

**Proof.** By (1.9) it follows that

$$u_k(t, 0) \leq k\mathbb{H}_\alpha[\delta_0](t, 0) = k\Gamma_\alpha(t, 0), \quad t > 0. \quad (4.2)$$

We claim that there exists  $c_{25} > 0$  independent of  $k$  such that

$$u_k(t, 0) \geq k\Gamma_\alpha(t, 0) - c_{25}k^p t^{-\frac{N}{2\alpha}p+1+\beta}, \quad t \in (0, 1/2). \quad (4.3)$$

Indeed, from (1.9), it infers that

$$u_k(t, 0) \geq k\Gamma_\alpha(t, 0) - k^p W(t, 0), \quad t \in (0, 1/2),$$

where

$$W(t, x) = \mathbb{H}_\alpha[t^\beta(\mathbb{H}_\alpha[\delta_0])^p](t, x), \quad (t, x) \in Q_\infty.$$

For  $t \in (0, 1/4)$ , there exists  $c_{26}, c_{27} > 0$  such that

$$\begin{aligned} W(t, 0) &\leq c_{26} \int_t^1 \int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}} s^\beta}{1 + |(s-t)^{-\frac{1}{2\alpha}} y|^{N+2\alpha}} \left( \frac{s^{-\frac{N}{2\alpha}}}{1 + |s^{-\frac{1}{2\alpha}} y|^{N+2\alpha}} \right)^p dy ds + c_{27} \\ &\leq c_{26} \int_t^1 \int_{\mathbb{R}^N} \frac{(s-t)^{-\frac{N}{2\alpha}} s^{-\frac{N}{2\alpha}p+\beta}}{1 + |(\frac{s-t}{s})^{-\frac{1}{2\alpha}} y|^{N+2\alpha}} dy ds + c_{27} \\ &\leq c_{26} \int_t^1 (s-t)^{-\frac{N}{2\alpha}} s^{-\frac{N}{2\alpha}p+\beta} \left(\frac{s-t}{s}\right)^{\frac{N}{2\alpha}} ds + c_{27} \\ &\leq c_{26} t^{-\frac{N}{2\alpha}p+1+\beta} + c_{27}. \end{aligned}$$

Combining (1.14) and  $-\frac{N}{2\alpha}p + 1 + \beta > -\frac{N}{2\alpha}$ , we obtain that

$$\lim_{t \rightarrow 0^+} t^{\frac{N}{2\alpha}} W(t, 0) = 0.$$

Therefore, (4.1) holds.  $\square$

In what follows we consider the limit of the solution  $\{u_k\}$  of (1.10) as  $k \rightarrow \infty$  for  $p \in (0, 1]$ .

**Proposition 4.2** *Assume that  $p \in (0, 1]$  and  $u_k$  is the solution of (1.10), then*

$$\lim_{k \rightarrow \infty} u_k = \infty \quad \text{in } Q_\infty.$$

**Proof.** We observe that  $\mathbb{H}_\alpha[\delta_0]$  and  $\mathbb{H}_\alpha[t^\beta(\mathbb{H}_\alpha[\delta_0])^p]$  are positive in  $(0, \infty) \times \mathbb{R}^N$ . By (1.9), for  $p \in (0, 1)$  and  $(t, x) \in (0, \infty) \times \mathbb{R}^N$ , we have that

$$\begin{aligned} u_k &\geq k\mathbb{H}_\alpha[\delta_0] - k^p \mathbb{H}_\alpha[t^\beta(\mathbb{H}_\alpha[\delta_0])^p] \\ &\rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

For  $p = 1$ , it is obvious that  $u_k = ku_1$  and  $u_1 > 0$  in  $(0, \infty) \times \mathbb{R}^N$ , then

$$\lim_{k \rightarrow \infty} u_k = \infty \quad \text{in } Q_\infty.$$

The proof is complete.  $\square$

Now we deal with the range  $p \in (1, p_\beta^*)$ .



**Lemma 4.1** Let  $p \in (1, p_\beta^*)$  and  $u_k$  be the solution of (1.10), then for any  $k > 0$ ,

$$0 \leq u_k \leq U_p \quad \text{in } Q_\infty, \quad (4.4)$$

where  $U_p$  is given by (1.16).

**Proof.** Let  $\{f_{n,k}\}$  be a sequence of nonnegative functions in  $C_0^1(\mathbb{R}^N)$  that converge to  $k\delta_0$  in the distribution sense as  $n \rightarrow \infty$ , denote by  $u_{n,k}$  the corresponding solution of (1.12) with initial data by  $f_{n,k}$ .

We claim that

$$u_{n,k} \leq U_p \quad \text{in } Q_\infty. \quad (4.5)$$

Now we assume this claim holds at this moment, then it follows that

$$u_k \leq U_p \quad \text{in } Q_\infty,$$

where  $u_k$  is the solution of (1.10), since  $\lim_{n \rightarrow \infty} u_{n,k} = u_k$  in  $(0, \infty) \times \mathbb{R}^N$ .

*Step 1.* To prove  $\lim_{|x| \rightarrow \infty} u_{n,k}(t, x) = 0$  for any  $t > 0$ . From [10, 14], there exists  $c_8 > 0$  such that for any  $x, y \in \mathbb{R}^N$  and  $t \in (0, \infty)$ ,

$$0 < \Gamma_\alpha(t, x - y) \leq \frac{c_8 t^{-\frac{N}{2\alpha}}}{1 + (|x - y| t^{-\frac{1}{2\alpha}})^{N+2\alpha}}.$$

Then for  $|x| > 1$ ,

$$\begin{aligned} 0 \leq \mathbb{H}_\alpha[f_{n,k}](t, x) &\leq c_8 t^{-\frac{N}{2\alpha}} \int_{\mathbb{R}^N} \frac{f_{n,k}(y)}{1 + (|x - y| t^{-\frac{1}{2\alpha}})^{N+2\alpha}} dy \\ &= c_8 \int_{\mathbb{R}^N} \frac{f_{n,k}(x - z t^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \\ &= c_8 \left( \int_{\mathbb{R}^N \setminus B_R} \frac{f_{n,k}(x - z t^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz + \int_{B_R} \frac{f_{n,k}(x - z t^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \right), \end{aligned}$$

where  $R = \frac{1}{2}|x|t^{-\frac{1}{2\alpha}}$  and  $B_R = \{z \in \mathbb{R}^N : |z| < R\}$ . It is obvious that

$$|x - z t^{\frac{1}{2\alpha}}| \geq |x| - |z| t^{\frac{1}{2\alpha}} \geq |x|/2 \quad \text{for all } z \in B_R.$$

Then

$$\begin{aligned} \int_{B_R} \frac{f_{n,k}(x - z t^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz &\leq \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) \int_{B_R} \frac{1}{1 + |z|^{N+2\alpha}} dz \\ &\leq \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) \int_{\mathbb{R}^N} \frac{1}{1 + |z|^{N+2\alpha}} dz \\ &= c_{25} \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) \end{aligned}$$

and

$$\int_{\mathbb{R}^N \setminus B_R} \frac{f_{n,k}(x - zt^{\frac{1}{2\alpha}})}{1 + |z|^{N+2\alpha}} dz \leq \int_{\mathbb{R}^N \setminus B_R} \frac{\|f_{n,k}\|_{L^\infty(\mathbb{R}^N)}}{1 + |z|^{N+2\alpha}} dz \leq c_{26} R^{-2\alpha} = \frac{c_{27}t}{|x|^{2\alpha}},$$

for some  $c_{27} > 0$  independent of  $x, t$  and  $R$ . Since  $f_{n,k} \in C_0^1(\mathbb{R}^N)$ , we have  $\lim_{|x| \rightarrow \infty} \sup_{|y| \geq \frac{|x|}{2}} f_{n,k}(y) = 0$  and then  $0 \leq u_{n,k}(t, x) \leq \mathbb{H}_\alpha[f_{n,k}](t, x) \rightarrow 0$  as  $|x| \rightarrow \infty$  for  $t > 0$ .

*Step 2. To prove (4.5).* If (4.5) fails, there exists  $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^N$  such that

$$(U_p - u_{n,k})(t_0, x_0) = \min_{(t,x) \in (0,\infty) \times \mathbb{R}^N} (U_p - u_{n,k})(t, x) < 0,$$

since  $U_p(t) > 0 = \lim_{|x| \rightarrow \infty} u_{n,k}(t, x)$  for any  $t \in (0, \infty)$ ,  $U_p(0) = \infty > f_{n,k}(x) = u_{n,k}(0, x)$  for  $x \in \mathbb{R}^N$  and  $\lim_{t \rightarrow \infty} U_p(t) = \lim_{t \rightarrow \infty} u_{n,k}(t, x) = 0$  for  $x \in \mathbb{R}^N$ . Then  $\partial_t(U_p - u_{n,k})(t_0, x_0) = 0$ ,  $(-\Delta)^\alpha(U_p - u_{n,k})(t_0, x_0) \leq 0$  and

$$\partial_t(U_p - u_{n,k})(t_0, x_0) + (-\Delta)^\alpha(U_p - u_{n,k})(t_0, x_0) + t_0^\beta U_p^p(t_0) - t_0^\beta u_{n,k}^p(t_0, x_0) < 0,$$

which is impossible with

$$\begin{aligned} & \partial_t(U_p - u_{n,k})(t_0, x_0) + (-\Delta)^\alpha(U_p - u_{n,k})(t_0, x_0) + t_0^\beta U_p^p(t_0) - t_0^\beta u_{n,k}^p(t_0, x_0) \\ &= \partial_t U_p(t_0) + t_0^\beta U_p^p(t_0) - \{\partial_t u_{n,k}(t_0, x_0) + (-\Delta)^\alpha u_{n,k}(t_0, x_0) + t_0^\beta u_{n,k}^p(t_0, x_0)\} \\ &= 0. \end{aligned}$$

The proof is complete.  $\square$

**Proposition 4.3** (i) Assume that  $p \in (0, p_\beta^*)$  and  $u_k$  is the solution of (1.10). Then  $u_k$  is a classical solution of (1.12).

(ii) Assume that  $p \in (1, p_\beta^*)$  and  $u_\infty$  is defined by (1.11). Then  $u_\infty$  is a classical solution of (1.12).

**Proof.** (i) Since  $u_k \leq k\mathbb{H}_\alpha[\delta_0]$ , then it infers that  $u_k$  is bounded in  $(T_0, \infty) \times \mathbb{R}^N$  with  $T_0 > 0$ . Let  $\{g_{n,k}\}$  be a sequence of nonnegative functions in  $C_0^1(\mathbb{R}^N)$  that converge to  $k\delta_0$  in the distribution sense as  $n \rightarrow \infty$  and  $u_{n,k}$  the corresponding solution of (1.12) with initial data  $g_{n,k}$ . Then  $\mathbb{H}_\alpha[g_{n,k}] \rightarrow k\mathbb{H}_\alpha[\delta_0]$  as  $n \rightarrow \infty$  uniformly in  $[T_0, \infty) \times \mathbb{R}^N$  for any  $T_0 > 0$  and by Comparison Principle, there exists  $c_{28} > 1$  such that

$$0 \leq u_{n,k}(t, x) \leq k\mathbb{H}_\alpha[g_{n,k}] \leq c_{28}k\mathbb{H}_\alpha[\delta_0] \quad \text{in } [T_0, \infty) \times \mathbb{R}^N$$

and there exists  $\sigma \in (0, 1)$  such that  $\{u_{n,k}\}$  are uniformly bounded with respect to  $n$  in  $C_{t,x}^{\frac{\sigma}{2\alpha}, \sigma}((T_0, \infty) \times \mathbb{R}^N)$  with  $T_0 > 0$ . Therefore, by the Arzela-Ascoli theorem,  $u_{n,k}$  converges to  $u_k$  in  $C_{t,x}^{\frac{\sigma'}{2\alpha}, \sigma'}((T_0, \infty) \times \mathbb{R}^N)$  with  $\sigma' \in (0, \sigma)$

and then  $u_k$  is a viscosity solution of (1.12) in  $(T_0, \infty) \times \mathbb{R}^N$ . By estimate (A.1) in [7],  $u_k$  is in  $C_{t,x}^{1+\sigma', 2\alpha+\sigma'}((T_0, \infty) \times \mathbb{R}^N)$  and  $u_k$  is a classical solution of (1.12) in  $(T_0, \infty) \times \mathbb{R}^N$ .

(ii) The proof is the same as part (i) just replacing  $u_k \leq k\mathbb{H}_\alpha[\delta_0]$  by  $u_\infty \leq U_p$ .  $\square$

## 5 Self-similar solution when $p \in (1, 1 + \frac{2\alpha(1+\beta)}{N})$

### 5.1 Non-flat self-similar solution for $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, 1 + \frac{2\alpha(1+\beta)}{N})$

By Theorem 1.1 and (4.4), we see that  $\{u_k\}$  are a sequence of nonnegative and increasing functions and controlled by function  $U_p$  defined in (1.16), then for  $p \in (1, p_\beta^*)$ , the limit of  $u_k$  as  $k \rightarrow \infty$  exists and unique, then we denote it by  $u_\infty$ , see (1.11). By Proposition 4.3 (ii) and (4.4),  $u_\infty$  is a classical solution of (1.12) and

$$u_\infty \leq U_p \quad \text{in } Q_\infty. \quad (5.1)$$

**Proposition 5.1** *Assume that  $p \in (1, p_\beta^*)$  and  $u_\infty$  is defined in (1.16). Then  $u_\infty$  is a self-similar solution of (1.12).*

**Proof.** For  $\lambda > 0$ , we denote

$$\tilde{u}_\lambda(t, x) = \lambda^{\frac{2\alpha(1+\beta)}{p-1}} u_k(\lambda^{2\alpha}t, \lambda x), \quad (t, x) \in Q_\infty.$$

By direct computation, we have

$$\begin{aligned} & \partial_t \tilde{u}_\lambda(t, x) + (-\Delta)^\alpha \tilde{u}_\lambda(t, x) + t^\beta \tilde{u}_\lambda^p(t, x) \\ &= \lambda^{\frac{2\alpha p(1+\beta)}{p-1}} [\partial_t u_k(\lambda^{2\alpha}t, \lambda x) + (-\Delta)^\alpha u_k(\lambda^{2\alpha}t, \lambda x) + t^\beta u_k^p(\lambda^{2\alpha}t, \lambda x)] \\ &= 0. \end{aligned} \quad (5.2)$$

Moreover, for  $f \in C_0(\mathbb{R}^N)$

$$\begin{aligned} \langle \tilde{u}_\lambda(0, \cdot), f \rangle &= \lambda^{\frac{2\alpha(1+\beta)}{p-1}} \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^N} u_k(\lambda^{2\alpha}t, \lambda x) f(x) dx \\ &= \lambda^{\frac{2\alpha(1+\beta)}{p-1} - N} \lim_{t \rightarrow 0^+} \int_{\mathbb{R}^N} u_k(\lambda^{2\alpha}t, z) f\left(\frac{z}{\lambda}\right) dz \\ &= \lambda^{\frac{2\alpha(1+\beta)}{p-1} - N} k f(0), \end{aligned} \quad (5.3)$$

where  $\frac{2\alpha(1+\beta)}{p-1} - N > 0$  by the fact of  $p \in (1, p_\beta^*)$ . Thus,

$$\tilde{u}_\lambda(0, \cdot) = \lambda^{\frac{2\alpha(1+\beta)}{p-1} - N} k \delta_0 \quad \text{in } \mathbb{R}^N.$$

By (5.3),  $u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}$  is a unique solution of (1.12) with initial data  $\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}k\delta_0$ , then by (5.2) it infers that for  $(t, x) \in Q_\infty$ ,

$$\lambda^{\frac{2\alpha(1+\beta)}{p-1}}u_k(\lambda^{2\alpha}t, \lambda x) = \tilde{u}_\lambda(t, x) = u_{k\lambda^{\frac{2\alpha(1+\beta)}{p-1}-N}}(t, x) \quad (5.4)$$

and letting  $k \rightarrow \infty$  we have that

$$u_\infty(t, x) = \lambda^{\frac{2\alpha(1+\beta)}{p-1}}u_\infty(\lambda^{2\alpha}t, \lambda x), \quad (t, x) \in Q_\infty,$$

which implies that  $u_\infty$  is a self-similar solution (1.12).  $\square$

Let us denote

$$U_\infty(z) = u_\infty(1, z), \quad z \in \mathbb{R}^N$$

and we observe that  $U_\infty$  is a classical solution of (1.15). It is obvious that the constant  $(\frac{1+\beta}{p-1})^{\frac{1}{p-1}}$  is a trivial positive classical solution of (1.15). We observe that  $N < \frac{2\alpha(1+\beta)}{p-1} < N + 2\alpha$  for  $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, p_\beta^*)$ . To be convenient, we introduce the auxiliary function

$$w_\lambda(t, x) = \lambda t^{-\frac{1+\beta}{p-1}}w(t^{-\frac{1}{2\alpha}}|x|), \quad (t, x) \in Q_\infty, \quad (5.5)$$

where  $w(s) = \frac{\log(e+s^2)}{1+s^{N+2\alpha}}$  with  $e$  is the natural number.

**Lemma 5.1** *Assume that  $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, p_\beta^*)$  and  $w_\lambda$  is defined by (5.5). Then there exists  $\Lambda_0 > 0$  such that for  $\lambda \geq \Lambda_0$ ,*

$$\partial_t w_\lambda(t, x) + (-\Delta)^\alpha w_\lambda(t, x) + t^\beta w_\lambda^p(t, x) \geq 0, \quad \forall (t, x) \in Q_\infty. \quad (5.6)$$

**Proof.** By direct computation,

$$\partial_t w_\lambda(t, x) = -\frac{\lambda(1+\beta)}{p-1}t^{-\frac{1+\beta}{p-1}-1}w(t^{-\frac{1}{2\alpha}}|x|) - \frac{\lambda}{2\alpha}t^{-\frac{1+\beta}{p-1}-1}w'(t^{-\frac{1}{2\alpha}}|x|)|t^{-\frac{1}{2\alpha}}x|$$

and

$$(-\Delta)^\alpha w_\lambda(t, x) = \lambda t^{-\frac{1+\beta}{p-1}-1}(-\Delta)^\alpha w(t^{-\frac{1}{2\alpha}}|x|),$$

which implies that

$$\begin{aligned} & \partial_t w_\lambda(t, x) + (-\Delta)^\alpha w_\lambda(t, x) + t^\beta w_\lambda^p(t, x) \\ &= \lambda t^{-\frac{1+\beta}{p-1}-1} \left[ (-\Delta)^\alpha w(s) - \frac{1}{2\alpha}w'(s)s - \frac{1+\beta}{p-1}w(s) + \lambda^{p-1}w^p(s) \right], \end{aligned} \quad (5.7)$$

where  $s = |z|$  with  $z = t^{-\frac{1}{2\alpha}}x$ .

For  $s > 0$ , by the direct computation, we obtain that

$$-\frac{1}{2\alpha}w'(s)s - \frac{1+\beta}{p-1}w(s) = \left[ \frac{N+2\alpha}{2\alpha} \frac{s^{N+2\alpha}}{1+s^{N+2\alpha}} - \frac{1+\beta}{p-1} - \frac{s^2(e+s^2)^{-1}}{\alpha \log(e+s^2)} \right] w(s).$$

Since  $\frac{N+2\alpha}{2\alpha} > \frac{1+\beta}{p-1}$ ,  $\lim_{s \rightarrow \infty} \frac{s^{N+2\alpha}}{1+s^{N+2\alpha}} = 1$  and  $\lim_{s \rightarrow \infty} \frac{1}{\log(e+s^2)} = 0$ , then there exists  $R_0 > 0$  and  $\sigma_0 > 0$  such that

$$-\frac{1}{2\alpha}w'(s)s - \frac{1+\beta}{p-1}w(s) \geq \sigma_0w(s), \quad s \geq R_0. \quad (5.8)$$

For  $|z| > 2$ , by the definition of fractional Laplacian, we have that

$$\begin{aligned} -(-\Delta)^\alpha w(|z|) &= \frac{1}{2} \int_{\mathbb{R}^N} \frac{\frac{\log(e+|z+\tilde{y}|^2)}{1+|z+\tilde{y}|^{N+2\alpha}} + \frac{\log(e+|z-\tilde{y}|^2)}{1+|z-\tilde{y}|^{N+2\alpha}} - \frac{2\log(e+|z|^2)}{1+|z|^{N+2\alpha}}}{|\tilde{y}|^{N+2\alpha}} d\tilde{y} \\ &= \frac{w(|z|)}{2|z|^{2\alpha}} \int_{\mathbb{R}^N} \frac{I_z(y)}{|y|^{N+2\alpha}} dy, \end{aligned} \quad (5.9)$$

where

$$\begin{aligned} I_z(y) &= \frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z+y|^{N+2\alpha}} \frac{\log(e+|z|^2|e_z+y|^2)}{\log(e+|z|^2)} \\ &\quad + \frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z-y|^{N+2\alpha}} \frac{\log(e+|z|^2|e_z-y|^2)}{\log(e+|z|^2)} - 2 \end{aligned}$$

and  $e_z = \frac{z}{|z|}$ .

To estimate that there exists  $c_{29} > 0$  such that

$$\int_{B_{\frac{1}{2}}(-e_z) \cup B_{\frac{1}{2}}(e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq \frac{c_{29}}{w(|z|)|z|^N}. \quad (5.10)$$

In fact, for  $y \in B_{\frac{1}{2}}(-e_z)$ , there exists  $c_{30} > 0$  such that

$$\frac{1+|z|^{N+2\alpha}}{1+|z|^{N+2\alpha}|e_z-y|^{N+2\alpha}} \frac{\log(e+|z|^2|e_z-y|^2)}{\log(e+|z|^2)} \leq c_{30}$$

and then

$$\begin{aligned} \int_{B_{\frac{1}{2}}(-e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy &\leq \omega_N \int_0^{\frac{1}{2}} \frac{1+|z|^{N+2\alpha}}{1+(|z|r)^{N+2\alpha}} \frac{\log(e+|z|^2r^2)}{\log(e+|z|^2)} r^{N-1} dr + c_{31} \\ &\leq \frac{\omega_N}{w(|z|)|z|^N} \int_0^\infty \frac{t^{N-1} \log(e+t^2)}{1+t^{N+2\alpha}} dt + c_{31} \\ &\leq \frac{c_{32}}{w(|z|)|z|^N}, \end{aligned}$$

where  $c_{31}, c_{32} > 0$  and the last inequality holds since  $w(|z|)|z|^N \rightarrow 0$  as  $|z| \rightarrow \infty$ .

$$\int_{B_{\frac{1}{2}}(e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy = \int_{B_{\frac{1}{2}}(-e_z)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq \frac{c_{32}}{w(|z|)|z|^N}.$$

To estimate that there exists  $c_{33} > 0$  such that

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq c_{33}. \quad (5.11)$$

Indeed, since function  $I_z$  is  $C^2$  in  $\bar{B}_{\frac{1}{2}}(0)$  such that

$$I_z(0) = 0, \quad I_z(y) = I_z(-y),$$

then  $\nabla I_z(0) = 0$  and there exists  $c_{34} > 0$  such that

$$|D^2 I_z(y)| \leq c_{34}, \quad y \in B_{\frac{1}{2}}(0).$$

Then we have

$$I_z(y) \leq c_{34}|y|^2, \quad y \in B_{\frac{1}{2}}(0),$$

which implies that

$$\int_{B_{\frac{1}{2}}(0)} \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq c_{34} \int_{B_{\frac{1}{2}}(0)} \frac{|y|^2}{|y|^{N+2\alpha}} dy \leq c_{33}.$$

To estimate that there exists  $c > 0$  such that

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq c, \quad (5.12)$$

where  $A = \mathbb{R}^N \setminus (B_{\frac{1}{2}}(0) \cup B_{\frac{1}{2}}(e_z) \cup B_{\frac{1}{2}}(-e_z))$ . In fact, for  $y \in A$ , we observe that there exists  $c_{35} > 0$  such that  $I_z(y) \leq c_{35}$  and

$$\int_A \frac{I_z(y)}{|y|^{N+2\alpha}} dy \leq \int_{\mathbb{R}^N \setminus B_{\frac{1}{2}}(0)} \frac{c_{35}}{|y|^{N+2\alpha}} \leq c_{36},$$

for some  $c_{36} > 0$ . Therefore, by (5.7)-(5.12), there exists  $c_{37} > 0$  such that

$$(-\Delta)^\alpha w(|z|) \geq -\frac{c_{37}}{1 + |z|^{N+2\alpha}}, \quad |z| \geq 2. \quad (5.13)$$

By (5.8) and (5.13), there exists  $R_1 \geq R_0 + 2$  such that for  $|z| > R_1$ ,

$$\begin{aligned} (-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) &\geq \sigma_0 w(|z|) - \frac{c_{37}}{1 + |z|^{N+2\alpha}} \\ &= w(|z|) \left( \sigma_0 - \frac{c_{37}}{\log(e + |z|^2)} \right) \\ &\geq 0. \end{aligned}$$

For  $|z| \leq R_1$ , it is obvious that there exists  $c_{38} > 0$  such that

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) \geq -c_{38}.$$

Then there exists  $\Lambda_0 > 0$  such that for  $\lambda \geq \Lambda_0$ ,

$$(-\Delta)^\alpha w(|z|) - \frac{1}{2\alpha} w'(|z|)|z| - \frac{1+\beta}{p-1} w(|z|) + \lambda^{p-1} w^p(|z|) \geq 0, \quad z \in \mathbb{R}^N, \quad (5.14)$$

which together with (5.7), implies that (5.6) holds.  $\square$

**Lemma 5.2** *Assume that  $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, p_\beta^*)$ ,  $w_{\Lambda_0}$  is given in (5.5) and  $u_\infty$  is given in (1.11). Then*

$$u_\infty(t, x) \leq w_{\Lambda_0}(t, x), \quad (t, x) \in Q_\infty. \quad (5.15)$$

In particular,

$$u_\infty(0, x) = 0, \quad x \in \mathbb{R}^N \setminus \{0\}. \quad (5.16)$$

**Proof.** Let us denote

$$f_0(r) = \frac{k_0 \log(e + r^2)}{1 + r^{N+2\alpha}}, \quad r \geq 0 \quad \text{and} \quad f_{n,k}(x) = kn^N f_0(n|x|), \quad x \in \mathbb{R}^N,$$

where

$$k_0 = \left[ \omega_N \int_0^\infty \frac{\log(e + r^2)}{1 + r^{N+2\alpha}} r^{N-1} dr \right]^{-1}.$$

Then for any  $\eta \in C_0(\mathbb{R}^N)$ , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle f_{n,k}, \eta \rangle &= k \lim_{n \rightarrow \infty} n^N \int_{\mathbb{R}^N} f_0(n|x|) \eta(x) dx \\ &= k \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f_0(|x|) \eta\left(\frac{x}{n}\right) dx \\ &= k \eta(0). \end{aligned}$$

Let  $t_n = n^{-2\alpha}$  and then

$$\begin{aligned} w_{\Lambda_0}(t_n, x) &= \Lambda_0 t_n^{-\frac{1+\beta}{p-1}} \frac{\log(e + (t_n^{-\frac{1}{2\alpha}} |x|)^2)}{1 + (t_n^{-\frac{1}{2\alpha}} |x|)^{N+2\alpha}} = \Lambda_0 n^{\frac{2\alpha(1+\beta)}{p-1}} \frac{\log(e + (n|x|)^2)}{1 + (n|x|)^{N+2\alpha}} \\ &= \frac{\Lambda_0}{k_0} n^{\frac{2\alpha(1+\beta)}{p-1} - N} n^N f_0(n|x|) \\ &\geq \frac{\Lambda_0}{k_0} \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N} n^N f_0(n|x|) = f_{n, k_{\tilde{n}}}(x), \end{aligned}$$

where  $\tilde{n} \leq n$  and  $k_{\tilde{n}} = \Lambda_0 \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N}$ . We see that  $k_{\tilde{n}} = \Lambda_0 \tilde{n}^{\frac{2\alpha(1+\beta)}{p-1} - N} \rightarrow \infty$  as  $\tilde{n} \rightarrow \infty$ , since  $\frac{2\alpha(1+\beta)}{p-1} - N > 0$ . Let  $u_{n, k_{\tilde{n}}}$  be the solution of (1.12) with the initial data  $f_{n, k_{\tilde{n}}}$ . By Lemma 5.1,  $w_{\Lambda_0}(\cdot + t_n, \cdot)$  is a super solution of (1.12) with the initial data  $w_{\Lambda_0}(t_n, \cdot)$ , that is, for  $(t, x) \in Q_\infty$ ,

$$\partial_t w_\lambda(t + t_n, x) + (-\Delta)^\alpha w_\lambda(t + t_n, x) + (t + t_n)^\beta w_\lambda^p(t + t_n, x) \geq 0.$$

By Comparison Principle,

$$u_{n,k_{\tilde{n}}}(t,x) \leq w_{\Lambda_0}(t+t_n,x), \quad (t,x) \in Q_\infty,$$

for any  $\tilde{n} \leq n$ . Then taking  $n \rightarrow \infty$ , implies that

$$u_{k_{\tilde{n}}}(t,x) \leq w_{\Lambda_0}(t,x), \quad (t,x) \in Q_\infty, \quad (5.17)$$

where  $u_{k_{\tilde{n}}}$  is the solution of (1.12) with  $k_{\tilde{n}}\delta_0$  initial data. We derive (5.15) by taking  $\tilde{n} \rightarrow \infty$ .

In particular, the argument (5.16) follows by the fact of

$$\lim_{t \rightarrow 0^+} w_{\Lambda_0}(t,x) = 0, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

The proof is complete.  $\square$

**Lemma 5.3** *Assume that  $p \in (1, p_\beta^*)$ , then there exists  $c_{39} > 0$  such that*

$$u_\infty(t,x) \geq \frac{c_{39}t^{-\frac{1+\beta}{p-1}}}{1 + |t^{-\frac{1}{2\alpha}}x|^{N+2\alpha}}, \quad \forall (t,x) \in (0,1) \times \mathbb{R}^N. \quad (5.18)$$

**Proof.** We divide the proof into several steps.

*Step 1.* Let  $\sigma_0 = 1 + \beta - \frac{N}{2\alpha}(p-1) > 0$ ,

$$\eta(t) = 2 - t^{\sigma_0}, \quad t > 0$$

and denote

$$v_\epsilon(t,x) = \epsilon\eta(t)\Gamma_\alpha(t,x),$$

where  $\Gamma_\alpha$  is the solution of (1.13).

In this step we prove that there exists  $\epsilon_0 > 0$  such that

$$u_{k_0} \geq v_{\epsilon_0} \quad \text{in } (0,1) \times \mathbb{R}^N, \quad (5.19)$$

where  $k_0 = 2\epsilon_0$  and  $u_{k_0}$  is the solution of (1.12) with initial data  $k_0\delta_0$ . Indeed,

$$\partial_t v_\epsilon(t,x) = \epsilon\eta'(t)\Gamma_\alpha(t,x) + \epsilon\eta(t)\partial_t\Gamma_\alpha(t,x)$$

and

$$(-\Delta)^\alpha v_\epsilon(t,x) = \epsilon\eta(t)(-\Delta)^\alpha\Gamma_\alpha(t,x),$$

Let  $\Gamma_1(t^{-\frac{1}{2\alpha}}x) = \Gamma_\alpha(1, t^{-\frac{1}{2\alpha}}x)$ , then there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \leq \epsilon_0$  and  $(t,x) \in (0,1) \times \mathbb{R}^N$

$$\begin{aligned} & \partial_t v_\epsilon(t,x) + (-\Delta)^\alpha v_\epsilon(t,x) + t^\beta v_\epsilon^p(t,x) \\ &= \epsilon\eta'(t)t^{-\frac{N}{2\alpha}}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + \epsilon^p\eta^p(t)t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x) \\ &\leq -\epsilon\sigma_0 t^{-\frac{N}{2\alpha}-1+\sigma_0}\Gamma_1(t^{-\frac{1}{2\alpha}}x) + 2^p\epsilon^p t^{-\frac{N}{2\alpha}p+\beta}\Gamma_1^p(t^{-\frac{1}{2\alpha}}x) \\ &\leq 0, \end{aligned}$$



the last inequality holds since  $-\frac{N}{2\alpha} - 1 + \sigma_0 = -\frac{N}{2\alpha}p + \beta$  and  $\Gamma_1$  is bounded. In particular, we have that

$$\partial_t v_{\epsilon_0}(t, x) + (-\Delta)^\alpha v_{\epsilon_0}(t, x) + t^\beta v_{\epsilon_0}^p(t, x) \leq 0, \quad \forall (t, x) \in (0, 1) \times \mathbb{R}^N. \quad (5.20)$$

Let  $f_n(x) = v_{\epsilon_0}(t_n, x)$  with  $t_n = n^{-2\alpha}$ . Since  $\lim_{t \rightarrow 0^+} \eta(t) = 2$ , then we have that  $f_n \rightarrow 2\epsilon_0\delta_0$  as  $n \rightarrow \infty$  in the distribution sense. There exists  $N_0 > 0$  such that  $t_n \in (0, \frac{1}{8})$  for  $n \geq N_0$ . Let  $w_n$  be the solution of (1.12) with initial data  $f_n$ , then it infers that

$$w_n(t, x) \geq v_{\epsilon_0}(t + t_n, x), \quad (t, x) \in (0, 1 - t_n) \times \mathbb{R}^N$$

By the uniqueness of  $u_{k_0}$ , we have

$$w_n \rightarrow u_{k_0} \quad \text{as } n \rightarrow \infty \quad \text{in } (0, 1) \times \mathbb{R}^N$$

and

$$\lim_{n \rightarrow \infty} v_{\epsilon_0}(t + t_n, x) = v_{\epsilon_0}(t, x), \quad (t, x) \in (0, 1) \times \mathbb{R}^N,$$

which imply (5.19).

*Step 2. To prove (5.18).* Since

$$v_{\epsilon_0}(t, x) \geq \epsilon_0 t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x), \quad (t, x) \in (0, 1) \times \mathbb{R}^N,$$

then, along with (5.4), we observe that for any  $\lambda > 0$ ,

$$\begin{aligned} u_{k_0 \lambda^{\frac{2\alpha(1+\beta)}{p-1}} - N}(t, x) &= \lambda^{\frac{2\alpha(1+\beta)}{p-1}} u_{k_0}(\lambda^{2\alpha} t, \lambda x) \\ &\geq \lambda^{\frac{2\alpha(1+\beta)}{p-1}} v_{\epsilon_0}(\lambda^{2\alpha} t, \lambda x) \\ &\geq \epsilon_0 \lambda^{\frac{2\alpha(1+\beta)}{p-1} - N} t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x). \end{aligned}$$

Let  $\varrho = \lambda^{\frac{2\alpha(1+\beta)}{p-1} - N}$ ,  $t_\varrho = (2\varrho)^{\frac{1}{\frac{N}{2\alpha} - \frac{1+\beta}{p-1}}}$  and  $T_\varrho = \varrho^{\frac{1}{\frac{N}{2\alpha} - \frac{1+\beta}{p-1}}}$ , then

$$0 < t_\varrho < T_\varrho \rightarrow 0 \quad \text{as } \varrho \rightarrow \infty.$$

For  $(t, x) \in (t_\varrho, T_\varrho) \times \mathbb{R}^N$ ,

$$u_{k_0 \varrho}(t, x) \geq \epsilon_0 \varrho t^{-\frac{N}{2\alpha}} \Gamma_1(t^{-\frac{1}{2\alpha}} x) \geq \frac{\epsilon_0}{2} t^{-\frac{1+\beta}{p-1}} \Gamma_1(t^{-\frac{1}{2\alpha}} x),$$

then

$$u_\infty(t, x) \geq \frac{\epsilon_0}{2} t^{-\frac{1+\beta}{p-1}} \Gamma_1(t^{-\frac{1}{2\alpha}} x), \quad (t, x) \in (t_\varrho, T_\varrho) \times \mathbb{R}^N.$$

which, choosing a sequence  $\{\varrho_n\}$  such that  $(0, 1) \subset \cup(t_{\varrho_n}, T_{\varrho_n})$ , infers (5.18). The proof is complete.  $\square$

**Proof of Theorem 1.2.** By Proposition 5.1 and Lemma 5.2, we have that  $u_\infty$  is a nontrivial self-similar solution of (1.12) and (1.17) follows by (5.15), (5.18) and  $\log(e + |t^{-\frac{1}{2\alpha}} x|^2) \leq 2 \log(2 + |t^{-\frac{1}{2\alpha}} x|)$ . The proof is complete.  $\square$

## 5.2 Trivial self-similar solution for $p \in (1, 1 + \frac{2\alpha(1+\beta)}{N+2\alpha})$

For  $p \in (1, 1 + \frac{2\alpha(1+\beta)}{N+2\alpha})$ , it derive from Lemma 5.3 that

$$\lim_{t \rightarrow 0^+} u_\infty(t, x) = \infty, \quad \forall x \in \mathbb{R}^N. \quad (5.21)$$

**Proof of Theorem 1.4 (i).** Let  $f_0 : \mathbb{R}^N \rightarrow \mathbb{R}$  be a nonnegative continuous function such that

$$\text{supp} f_0 \subset B_1(0) \quad \text{and} \quad \max_{x \in B_1(0)} f_0 = 1.$$

Denote

$$f_{n,k}(x) = kn^{\theta N} f_0(n^\theta(x - x_0)),$$

where  $k \leq n^\tau$  with  $\tau = \frac{1}{2}(\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha) > 0$ ,  $\theta = \frac{\tau}{N}$  and  $x_0 \in \mathbb{R}^N$ . Since  $f_{n,k}(x) \leq n^\tau$  for  $x \in B_1(x_0)$ ,  $f_n(x) = 0$  for  $x \in B_1^c(x_0)$  and

$$v_{\varepsilon_0}(t_n, x) \geq \frac{c_{38} n^{\frac{2\alpha(1+\beta)}{p-1} - N - 2\alpha}}{(2 + |x_0|)^{N+2\alpha}}, \quad x \in B_1(x_0),$$

where  $t_n = n^{-2\alpha}$ . Then there exists  $N_0 > 0$  such that for any  $n \geq N_0$

$$f_{n,k}(x) \leq v_{\varepsilon_0}(t_n, x), \quad x \in B_1(x_0).$$

Since  $n^{\theta N} f_0(n^\theta(x - x_0)) \rightarrow c_{41} \delta_{x_0}$ , as  $n \rightarrow \infty$  in the distribution sense, for some  $c_{41} > 0$ .

Let  $w_{n,k}$  be the solution of (1.12) with initial data  $f_n$  and then

$$w_{n,k}(0, x) = f_{n,k}(x) \leq v_{\varepsilon_0}(t_n, x) \leq u_\infty(t_n, x), \quad x \in \mathbb{R}^N.$$

Therefore, by comparison principle

$$w_{n,k}(t, x) \leq u_\infty(t + t_n, x), \quad (t, x) \in Q_\infty.$$

We observe that

$$\lim_{k \rightarrow \infty} [\lim_{n \rightarrow \infty} w_{n,k}(t, x)] = u_\infty(t, x - x_0), \quad (t, x) \in Q_\infty.$$

Thus, we derive that

$$u_\infty(t, x - x_0) \leq u_\infty(t, x), \quad (t, x) \in Q_\infty. \quad (5.22)$$

By changing the role of  $x_0$  and 0 in (5.22), we have that

$$u_\infty(t, x - x_0) = u_\infty(t, x), \quad (t, x) \in Q_\infty,$$

which implies that  $u_\infty$  is independent of  $x$ . Combining (5.1) and (5.18), implies that

$$u_\infty = \left(\frac{1+\beta}{p-1}\right)^{\frac{1+\beta}{p-1}} t^{-\frac{1+\beta}{p-1}}.$$

The proof is complete.  $\square$

In the case of  $p = 1 + \frac{2\alpha(1+\beta)}{N+2\alpha}$ , it derive from Lemma 5.3 that

$$\liminf_{t \rightarrow 0^+} u_\infty(t, x) \geq \lim_{t \rightarrow 0^+} \frac{c_{39} t^{-\frac{1+\beta}{p-1}}}{1 + |t^{-\frac{1}{2\alpha}} x|^{N+2\alpha}} = \frac{c_{39}}{|x|^{N+2\alpha}}, \quad \forall x \in \mathbb{R}^N.$$

**Proof of Theorem 1.4 (ii).** We note that  $u_\infty$  is a self-similar solution of (1.12). Moreover, we derive (1.19) by (5.18), ends the proof.  $\square$

## 6 Uniqueness of self-similar solution when $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, 1 + \frac{2\alpha(1+\beta)}{N})$

Theorem 1.3 will be proved by contradiction. We first introduce some auxiliary lemmas.

**Lemma 6.1** *Assume that  $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, 1 + \frac{2\alpha(1+\beta)}{N})$  and  $\tilde{u}$  is a positive self-similar solution of (1.18). Then either*

$$\tilde{u} > u_\infty \quad \text{in } Q_\infty \tag{6.1}$$

or

$$\tilde{u} \equiv u_\infty \quad \text{in } Q_\infty. \tag{6.2}$$

**Proof.** For any  $r > 0$ , we have that

$$\begin{aligned} \int_{B_r(0)} \tilde{u}(t, x) dx &= t^{-\frac{1+\beta}{p-1}} \int_{B_r(0)} \tilde{u}(1, t^{-\frac{1}{2\alpha}} x) dx \\ &= t^{-\frac{1+\beta}{p-1} + \frac{N}{2\alpha}} \int_{B_{t^{-\frac{1}{2\alpha}} r}(0)} \tilde{u}(1, z) dz \\ &\geq t^{-\frac{1+\beta}{p-1} + \frac{N}{2\alpha}} \int_{B_1(0)} \tilde{u}(1, z) dz \\ &\rightarrow +\infty \quad \text{as } t \rightarrow 0^+, \end{aligned}$$

where last inequality holds for  $t \in (0, r^{2\alpha}]$ . Let  $\{\epsilon_n\}$  be a sequence positive decreasing numbers converging to 0 as  $n \rightarrow \infty$ . For  $\epsilon_n$  and  $k > 0$ , there exists  $t_{n,k} > 0$  such that

$$\int_{B_{\epsilon_n}(0)} \tilde{u}(t_{n,k}, x) dx = k.$$

We observe that for any fixed  $k$ ,  $t_{n,k} \rightarrow 0$  as  $n \rightarrow \infty$  since  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ . Let  $\eta_0 : \mathbb{R}^N \rightarrow [0, 1]$  be a  $C^2$  function such that  $\text{supp} \eta_0 \subset \bar{B}_2(0)$ ,  $\eta_0 = 1$  in  $B_1(0)$  and  $\eta_n(x) = \eta_0(\epsilon_n^{-1}x)$  for  $x \in \mathbb{R}^N$ . Choosing  $\{f_{n,k}\}$  be a sequence of  $C^2$  functions such that

$$0 \leq f_{n,k}(x) \leq \eta_n(x)\tilde{u}(t_{n,k}, x), \quad x \in \mathbb{R}^N$$

and

$$f_{n,k} \rightarrow k\delta_0 \quad \text{as } n \rightarrow \infty.$$

Let  $u_{n,k}$  be the solution of (1.1) with initial data  $f_{n,k}$ , then

$$u_{n,k}(t, x) \leq u(t_{n,k} + t, x), \quad (t, x) \in Q_\infty$$

and by uniqueness of  $u_k$ ,  $\lim_{n \rightarrow \infty} u_{n,k} = u_k$ , where  $u_k$  is the solution of (1.1) with initial data  $k\delta_0$ . Then for any  $k$ , we have  $u_k \leq \tilde{u}$  in  $Q_\infty$ , which implies that

$$\tilde{u} \geq u_\infty \quad \text{in } Q_\infty.$$

Now we assume there exists  $(t_0, x_0) \in Q_\infty$  such that

$$\tilde{u}(t_0, x_0) = u_\infty(t_0, x_0).$$

Since  $\tilde{u}$  and  $u_\infty$  are self-similar, then

$$\tilde{u}(t, x_0) = u_\infty(t, x_0), \quad t > 0.$$

Now for any  $t > 0$ ,  $w(t, \cdot) = \tilde{u}(t, \cdot) - u_\infty(t, \cdot)$  achieves the minimum at  $x_0$ . Combining with  $\partial_t w(t, x_0) = 0$  and  $t^\beta \tilde{u}^p(t, x_0) = t^\beta u_\infty^p(t, x_0)$ , we derive that  $(-\Delta)^\alpha w(t, x_0) = 0$ , which implies that

$$\tilde{u}(t, \cdot) = u_\infty(t, \cdot) \quad \text{in } \mathbb{R}^N.$$

Then  $\tilde{u} \equiv u_\infty$  in  $Q_\infty$ . The proof is complete.  $\square$

**Lemma 6.2** *Assume that  $p \in (1 + \frac{2\alpha(1+\beta)}{N+2\alpha}, 1 + \frac{2\alpha(1+\beta)}{N})$  and  $\tilde{u}$  is a positive self-similar solution of (1.18). Then for any  $\epsilon > 0$ , there exists  $\lambda_\epsilon \geq 1$  such that for any  $\lambda \geq \lambda_\epsilon$ ,*

$$\tilde{u} \leq \lambda u_\infty + \epsilon \quad \text{in } [1, \infty) \times \mathbb{R}^N =: Q_1. \quad (6.3)$$

**Proof.** For  $\epsilon > 0$ , there exists  $R_\epsilon > 0$  such that

$$\tilde{u}(1, x) \leq \epsilon, \quad x \in B_{R_\epsilon}^c(0).$$

Then there exists  $\sigma_0 > 0$  such that  $u_\infty(1, x) \geq \sigma_0$  for  $x \in B_{R_\epsilon}(0)$ . By continuity of  $\tilde{u}(1, \cdot)$ , there exists  $\lambda_\epsilon \geq 1$  such that  $\tilde{u}(1, x) \leq \lambda_\epsilon u_\infty(1, x)$  for

$x \in B_{R_\epsilon}(0)$ . Then  $\tilde{u}(1, x) \leq \lambda_\epsilon u_\infty(1, x) + \epsilon$  for  $x \in \mathbb{R}^N$ . By the definition of self-similar solution,

$$\begin{aligned}\tilde{u}(t, x) &\leq \lambda_\epsilon u_\infty(t, x) + \epsilon t^{-\frac{1+\beta}{p-1}} \\ &\leq \lambda_\epsilon u_\infty(t, x) + \epsilon, \quad (t, x) \in Q_1,\end{aligned}$$

which ends the proof.  $\square$

**Proof of Theorem 1.3.** Let  $\tilde{u}$  be a positive self-similar solution of (1.18). By Lemma 6.1,  $\tilde{u} > u_\infty$  in  $Q_\infty$  or  $\tilde{u} = u_\infty$  in  $Q_\infty$ . Our aim is to rule out the case  $\tilde{u} > u_\infty$  in  $Q_\infty$ . To this end, for  $\epsilon \in (0, 1]$  we denote

$$\underline{\lambda}_\epsilon = \min\{\lambda \in [1, \infty) : \tilde{u} \leq \lambda u_\infty + \epsilon \text{ in } Q_1\}.$$

It follows by Lemma 6.2 that  $\underline{\lambda}_\epsilon$  is well-defined and  $\underline{\lambda}_\epsilon \leq \lambda_\epsilon$ .

We first claim that

$$\underline{\lambda}_\epsilon = 1 \quad \text{for any } \epsilon \in (0, 1]. \quad (6.4)$$

In fact, if (6.4) fails, then there exists  $\epsilon_0 \in (0, 1]$  such that  $\underline{\lambda}_{\epsilon_0} > 1$ .

*Step 1.* To prove  $\underline{\lambda}_{\epsilon_0} u_\infty + \epsilon_0 > \tilde{u}$  in  $Q_1$ . By continuity, we see that  $\underline{\lambda}_{\epsilon_0} u_\infty + \epsilon_0 \geq \tilde{u}$  in  $Q_1$ . If there exists  $(t_0, x_0) \in Q_1$  such that

$$\underline{\lambda}_{\epsilon_0} u_\infty(t_0, x_0) + \epsilon_0 = \tilde{u}(t_0, x_0).$$

Let  $w = \underline{\lambda}_{\epsilon_0} u_\infty + \epsilon_0 - \tilde{u}$ . Since  $\epsilon_0 > 0$ , there exists  $R_0$  such that  $\tilde{u}(t_0, x) < \epsilon_0$  for  $x \in B_{R_0}^c$ , then  $w$  achieves the minimum at  $(t_0, x_0)$  then  $\partial_t w(t_0, x_0) = 0$ ,  $t_0^\beta (\underline{\lambda}_{\epsilon_0} u_\infty(t_0, x_0) + \epsilon_0)^p = t_0^\beta \tilde{u}^p(t_0, x_0)$  and

$$(-\Delta)^\alpha w(t_0, x_0) = - \int_{\mathbb{R}^N} \frac{w(t_0, x_0 + y)}{|y|^{N+2\alpha}} dy < 0.$$

Thus, we obtain a contradiction, since  $\underline{\lambda}_{\epsilon_0} u_\infty + \epsilon_0$  is a super solution of (1.12) and  $\tilde{u}$  is a solution of (1.12).

*Step 2.* To prove that there exists  $\sigma \in (0, \underline{\lambda}_{\epsilon_0} - 1)$  such that  $(\underline{\lambda}_{\epsilon_0} - \sigma)u_\infty + \epsilon_0 \geq \tilde{u}$  in  $Q_1$ .

For  $\epsilon_0 > 0$ , there exists  $R_0$  such that  $\tilde{u}(1, x) < \epsilon_0$  for  $x \in B_{R_0}^c(0)$ . By Step 1,  $\tilde{u}(1, x) < \underline{\lambda}_{\epsilon_0} u_\infty(1, x) + \epsilon_0$  for  $x \in \mathbb{R}^N$ , then there exists  $\sigma \in (0, \underline{\lambda}_{\epsilon_0} - 1)$  such that

$$\tilde{u}(1, x) \leq \underline{\lambda}_{\epsilon_0} u_\infty(1, x) + \epsilon_0 - \sigma u_\infty(1, 0), \quad x \in \bar{B}_{R_0}(0).$$

Then we have

$$\tilde{u}(1, x) \leq (\underline{\lambda}_{\epsilon_0} - \sigma)u_\infty(1, x) + \epsilon_0, \quad x \in \mathbb{R}^N.$$

By Comparison Principle, we have

$$\tilde{u}(t, x) \leq (\underline{\lambda}_{\epsilon_0} - \sigma)u_\infty(t, x) + \epsilon_0, \quad (t, x) \in Q_1,$$

which contradicts the definition of  $\underline{\lambda}_{c_0}$ . Therefore, (6.4) holds.

By (6.4), we have  $\tilde{u} \leq u_\infty + \epsilon$  in  $Q_1$  for any  $\epsilon > 0$ . Taking  $\epsilon \rightarrow 0$ , we derive that

$$\tilde{u} \leq u_\infty \quad \text{in } Q_1$$

and then it follows by Lemma 6.1 that  $\tilde{u} = u_\infty$  in  $Q_\infty$ .  $\square$

## References

- [1] D. R. Adams and M. Pierre, Capacity strong type estimates in semi-linear problems, *Ann. Inst. Fourier Grenoble* 41, 117-135 (1991).
- [2] W. Al Sayed and L. Véron, Initial trace of solutions of semilinear heat equations with absorption, *Nonlinear Analysis* 93, 197-225 (2013).
- [3] Ph. Bénéilan, H. Brezis and M. Crandall, A semilinear elliptic equation in  $L^1(\mathbb{R}^N)$ , *Ann. Sc. Norm. Sup. Pisa Cl. Sci.* 2, 523-555 (1975).
- [4] H. Brezis and A. Friedman, Nonlinear parabolic equations involving measures as initial conditions, *J. Math. Pures Appl.* 62(9), 73-97 (1983).
- [5] H. Brezis, L. A. Peletier and D. Terman, A very singular solution of the heat equation with absorption, *Arch. Rational Mech. Anal.* 95, 185-209 (1986).
- [6] L. Caffarelli, C. Chan and A. Vasseur, Regularity theory for nonlinear integral operators, *Journal of the American Mathematical Society*, 24(3), 849-869 (2011).
- [7] L. Caffarelli and A. Figalli, Regularity of solutions to the parabolic fractional obstacle problem, *arXiv:1101.5170*, (2011).
- [8] H. Chang Lara and G. Dávila, Regularity for solutions of non local parabolic equations, *Calculus of Variations and Partial Differential Equations*, 1-34 (2011).
- [9] H. Chen, P. Felmer and A. Quaas, Large solution to elliptic equations involving fractional Laplacian, *arXiv:1311.6044*, (2013)
- [10] Z. Chen, P. Kim and R. Song, Heat kernel estimates for the Dirichlet fractional Laplacian. *J. Eur. Math. Soc.* 12, 1307-1329 (2010).
- [11] H. Chen and L. Véron, Semilinear fractional elliptic equations with gradient nonlinearity involving measures, *Journal of Functional Analysis* to appear, *arXiv:1308.6720*, (2013).
- [12] H. Chen and L. Véron, Semilinear fractional elliptic equations involving measures, *arXiv:1305.0945*, (2013).

- [13] H. Chen and L. Véron, Weak and strong singular solutions of semilinear fractional elliptic equations, *arXiv:1307.7023*, (2013).
- [14] Z. Chen and J. Tokle, Global heat kernel estimates for fractional laplacians in unbounded open sets, *Probab. Theory Related Field* 149, 373-395 (2011).
- [15] J. Dronion and C. Imbert, Fracta first-order partial differential equations, *Arch. Rational Mech. Anal.* 182, 299-331 (2006).
- [16] P. Felmer and Y. Wang, Radial symmetry of positive solutions to equations involving the fractional laplacian, *Commun. Contemp. Math.* to appear.
- [17] A. Fino and G. Karch, Decay of mass for nonlinear equation with fractional laplacian, *Monatsh. Math.* 160, 375-384 (2010).
- [18] S. Kamin and L.A. Peletier, Singular solutions of the heat equation with absorption, *Proc. Am. Math. Soc.* 95, 205-210 (1985).
- [19] Y. Naito and T. Suzuki, Radial symmetry of self-similar solutions for semilinear heat equations, *J. Differential Equations*, 163, 407-428 (2000).
- [20] T. Nguyen-Phuoc and L. Véron, Initial trace of positive solutions of a class of degenerate heat equation with absorption, *arXiv:1101.1576*, (2011)
- [21] M. Marcus and L. Véron, Initial trace of positive solutions of some nonlinear parabolic equations, *Comm. Partial Differential Equations* 24, 1445-1499 (1999).
- [22] M. Marcus and L. Véron, Semilinear parabolic equations with measure boundary data and isolated singularities, *J. Anal. Math.* 85, 245-290 (2001).
- [23] M. Marcus and L. Véron, Initial trace of positive solutions to semilinear parabolic inequalities, *Advanced Nonlinear Studies* 2, 395-436 (2002).
- [24] I. Moutoussamy and L. Véron, Isolated singularities and asymptotic behaviour of the solutions of a semi-linear heat equation, *Asymptotic Anal.* 9, 259-289 (1994).
- [25] L. Oswald, Isolated positive singularities for a nonlinear heat equation, *Houston J. Math.* 14, 543-572 (1988).