A motivic formula for the L-function of an abelian variety over a function field

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A MOTIVIC FORMULA FOR THE L-FUNCTION OF AN ABELIAN VARIETY OVER A FUNCTION FIELD

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Abstract. Let A be an abelian variety over the function field of a smooth projective curve C over an algebraically closed field k. We compute the l-adic cohomology groups

$$H^i(C, j_*H^1(\bar{A}, \mathbf{Q}_l)), \quad j : \eta \hookrightarrow C$$

in terms of arithmetico-geometric invariants of A. We apply this, when k is the algebraic closure of a finite field, to a motivic computation of the L-function of A.

Contents

Introduction

Let K be a global field, and let A be an abelian variety over K. Its L-function is classically defined as

$$L(A, s) = \prod_{v \in \Sigma_K^f} \det(1 - \pi_v N(v)^{-s} \mid H^1_{l_v}(A)^{I_v})^{-1}$$

where Σ_K^f is the set of non-archimedean places of K, $H^1_{l_v}(A) = H^1_{\text{ét}}(\bar{A}, \mathbf{Q}_{l_v})$ is geometric l_v-adic cohomology of A (alternately, the dual of the Tate module $V_{l_v}(A)$) for some prime l_v different from the residue characteristic at finite v, I_v is the absolute inertia group at v and π_v is the geometric Frobenius at v, well-defined modulo I_v as a conjugacy class.

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This function is independent of the choice of the l_v's, as a consequence of Weil's Riemann hypothesis for curves and the "weight-monodromy conjecture", which is known in this case by [SGA7, exp. IX, Th. 4.3 and Cor. 4.4]. In positive characteristic we have the more precise

Theorem 1. *Suppose* $\operatorname{char} K > 0$*; let* $k = \mathbf{F}_q$ *be the field of constants of* K*. Then*
a) (Grothendieck [GS]*, see also* [Dc, §10]*.) One has the formula*

$$L(A,s) = \frac{P_1(q^{-s})}{P_0(q^{-s})P_2(q^{-s})}$$

where $P_i \in \mathbf{Q}[t]$ *with* $P_i(0) = 1$*. Moreover,* $L(A,s)$ *has a functional equation of the form*

$$L(A, 2-s) = ab^s L(A,s)$$

for suitable integers a, b*.*
b) (Deligne [W.II]*.) The polynomials* P_i *have integer coefficients; the inverse roots of* P_i *are Weil* q*-numbers of weight* $i+1$*.*

In this note, we give a formula for the polynomial P_i in terms of *pure motives over* k. To express the result, let us take some notation:

- $B = \operatorname{Tr}_{K/k} A$ is the K/k-trace of A.
- $\operatorname{LN}(A, K\bar{k}/\bar{k}) = A(K\bar{k})/B(\bar{k})$ is the *geometric Lang-Néron group* of A, where $\bar{k}$ is an algebraic closure of k: it is finitely generated by the Lang-Néron theorem (e.g. [K1, App. B] or [C]). We view it as a Galois representation of k.

Theorem 2. *Let* $\mathcal{M} = \mathcal{M}_{\mathrm{rat}}(k, \mathbf{Q})$ *be the category of pure motives over* k *with rational coefficients, modulo rational equivalence.*[1]
a) We have

$$P_0(t) = Z(h^1(B), t), \quad P_2(t) = Z(h^1(B), qt)$$

where $h^1(B) \in \mathcal{M}$ *is the degree* 1 *part of the Künneth decomposition of the motive of* B*.*
b) We have

$$P_1(t) = Z(\ln(A, K/k), qt)^{-1} \cdot Z(\text{ш}(A, K/k), t)^{-1}$$

(a product of two polynomials), where $\ln(A, K/k)$ *is the Artin motive associated to* $\operatorname{LN}(A, K\bar{k}/\bar{k})$ *and* $\text{ш}(A, K/k) \in \mathcal{M}$ *is an effective Chow motive of weight* 2 *whose* l*-adic realization is* $V_l(\text{Ш}(A, K\bar{k}))$*, where* $\text{Ш}(A, K\bar{k})$ *is the geometric Tate-Šafarevič group of* A*.*

[1]Throughout this paper we use the contravariant convention for pure motives, e.g. as in [Kl].

In Theorem 2, we used the *Z-function* of a motive $M \in \mathcal{M}$ [Kl]. It is known to be a rational function of t, with a functional equation; more precisely, if M is homogeneous of weight w, then $Z(M,t)$ is a polynomial or the inverse of a polynomial according as w is odd or even. That its inverse roots are Weil q-numbers of weight w depends on [W.I] rather than [W.II]. Theorem 2 also provides a proof that $L(A,s)$ is independent of the l_v's avoiding [SGA7, Exp. IX].

The motive $\text{ш}(A,K/k)$ is really the new character in this story. We construct it "by hand" in Proposition 4.3; however, we will show in [K3] that it is actually canonical and functorial in A (for homomorphisms of abelian varieties).

Theorem 2 "reduces" the Birch and Swinnerton-Dyer conjecture for A to the non-vanishing of $Z(\text{ш}(A,K/k),t)$ at $t=q^{-1}$. The existence of $\text{ш}(A,K/k)$ actually yields a simple proof of the following theorem of Kato and Trihan by basically quoting the relevant literature [I, M, IR]:

Corollary 1 ([KT]). *The following conditions are equivalent:*

(i) $\operatorname{ord}_{s=1}L(A,s) = \operatorname{rk}A(K)$.
(ii) $\text{Ш}(A,K)\{l\}$ *is finite for some prime* l.
(iii) $\text{Ш}(A,K)\{l\}$ *is finite for all primes* l.
(iv) $\text{Ш}(A,K)$ *is finite.*

(We almost don't touch the special value at $s=1$, see however §6.3.)

To prove Theorem 2, we start from Grothendieck's formula for $P_i(t)$ (here we take $l_v = l \nmid q$ for all v):

$$P_i(t) = \det(1-\pi_k t \mid H^i(\bar{C}, j_* H^1_l(A))) \tag{0.1}$$

where π_k is the geometric Frobenius of k, C is the smooth projective k-curve with function field K and $j : \operatorname{Spec} K \hookrightarrow C$ is the inclusion of the generic point. The issue is then to give an expression of the cohomology groups $H^i(\bar{C}, j_* H^1_l(A)))$: this is done in Theorem 1.1 below when A is the Jacobian of a curve, and in Corollary 1.2 in general.

When A is the Jacobian J of a curve Γ, we also get a precise relationship between $L(J,s)$ and the zeta function of a smooth projective k-surface spreading Γ, which was my original motivation for this work. More precisely, let Γ be a regular, projective, geometrically irreducible curve over K and S a smooth projective surface over k, fibred over C by a flat morphism f, with generic fibre Γ:

$$\begin{array}{ccc} \Gamma & \longrightarrow & S \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow \\ \operatorname{Spec} K & \xrightarrow{j} & C \\ & & {\scriptstyle p}\downarrow \\ & & \operatorname{Spec} k. \end{array} \tag{0.2}$$

Define (cf. [S])

$$L(h^i(\Gamma),s) = \prod_{v\in\Sigma_K} \det(1-\pi_v N(v)^{-s} \mid H^i_l(\Gamma)^{I_v})^{(-1)^{i+1}}$$

$$L(h(\Gamma),s) = \prod_{i=0}^{2} L(h^i(\Gamma),s)$$

so that

$$L(h^0(\Gamma),s) = \zeta(C,s),\quad L(h^2(\Gamma),s) = \zeta(C,s-1),$$
$$L(h^1(\Gamma),s) = L(J,s)^{-1}$$

(beware the exponent change!).

Theorem 3. *We have*

$$\frac{\zeta(S,s)}{L(K,h(\Gamma),s)} = Z(a(D),q^{1-s})$$

where $a(D)$ is the Artin motive associated to the "divisor of multiple fibres"

$$D = \bigoplus_{c\in C_{(0)}} \bar D_c,\quad \bar D_c = \operatorname{Coker}(\mathbf{Z} \xrightarrow{f^*} \bigoplus_{x\in Supp(f^{-1}(c))} \mathbf{Z}).$$

Theorem 2, Corollary 1 and Theorem 3 are results on abelian varieties over a global field of positive characteristic. More intriguing for me is that Theorem 2 leads to a definition of the L-function of an abelian variety over a *finitely generated field of Kronecker dimension* 2, see Definition 7.1. This might be viewed a step towards answering the awkwardness of [T2, §4]: meanwhile, it raises more questions than it answers.

The main technical part of this work is to prove Theorem 1.1 below. The method is to "l-adify" Grothendieck's computations with $\mathbb{G}_m$ coefficients in [Br.III, §4][2]. In a forthcoming work with Amílcar Pacheco [KP], we shall extend these results to a general fibration of smooth projective k-varieties, with a different and (hopefully) less unpleasant proof.

Contents of this paper. Theorem 1.1 and Corollary 1.2 are stated in Section 1. The first is proven in Section 2 and the second in Section 3: as explained above, Corollary 1.2 implies Theorem 2. In section 4, we show how Theorem 1.1 yields an identity in K_0 of a category of l-adic representations or pure motives, see Theorem 4.1: it implies

[2]These computations also appear with less generality in two other exposés of the volume *Dix exposés sur la cohomologie des schémas*: [Ra, §3] and [T1, Th. 3.1].

Theorem 3. In Section 5, we recall well-known facts on the crystalline realisation and present them in a convenient way. In Section 6, we examine what Theorem 2 teaches us on the functional equation and special values of $L(A,s)$; in particular, we prove Corollary 1 in §6.2. Finally, in Section 7, we get a formula for the total L-function of a surface S over a global field k in terms of L-functions of motives over k associated to a fibration of S over a curve (Theorem 7.4).

Acknowledgements. This work was partly inspired by the papers of Hindry-Pacheco [HP] and Hindry-Pacheco-Wazir [HPW]; I would also like to acknowledge several discussions with Amílcar Pacheco around it, which eventually led to [KP]. For this, I thank the Réseau franco-brésilien de mathématiques (RFBM) for its support for two visits to Rio de Janeiro in 2008 and 2010.

Theorems 2 (for the Jacobian of a curve), 3, 1.1 and 4.1 were obtained in the fall 2008 at the Tata Institute of Fundamental Research of Mumbai during its p-adic semester; I thank this institution for its hospitality and R. Sujatha for having invited me. These results were initially part of a more ambitious project on adjunctions in categories of motives [K3], from which I extracted them. The rest of the present article was obtained more recently.

1. Cohomological results

Consider the situation of (0.2), with k separably closed. Take a prime number l invertible in k. We write

$$H^2_{\rm tr}(S,\mathbf{Q}_l(1)) = \operatorname{Coker}\left(\operatorname{NS}(S)\otimes\mathbf{Q}_l \to H^2(S,\mathbf{Q}_l(1))\right)$$

where $\operatorname{NS}(S)$ is the Néron-Severi group of S. Here are two other descriptions of this group:

(1.1) $H^2_{\rm tr}(S,\mathbf{Q}_l(1)) \simeq V_l(\operatorname{Br}(S))$ (Kummer exact sequence).

(1.2) $H^2_{\rm tr}(S,\mathbf{Q}_l(1)) \simeq V_l(Ш(J,K))$ [Br.III, pp. 120/121]

where J is the Jacobian variety of Γ and $Ш(J,K)$ denotes its Tate-Šafarevič group.

Theorem 1.1. *Suppose k separably closed. There are isomorphisms*

$$H^0(C,j_*R^1f'_*\mathbf{Q}_l(1)) \simeq H^1(\operatorname{Tr}_{K/k} J,\mathbf{Q}_l(1))$$
$$H^2(C,j_*R^1f'_*\mathbf{Q}_l(1)) \simeq H^1(\operatorname{Tr}_{K/k} J,\mathbf{Q}_l)$$

and an exact sequence

$$0\to \operatorname{LN}(J,K/k)\otimes\mathbf{Q}_l \to H^1(C,j_*R^1f'_*\mathbf{Q}_l(1)) \to H^2_{\rm tr}(S,\mathbf{Q}_l(1))\to 0.$$

Corollary 1.2. *Let $f' : A \to \operatorname{Spec} K$ be an abelian variety over K. Suppose k separably closed. There are isomorphisms*

$$H^0(C, j_*R^1f'_*\mathbf{Q}_l(1)) \simeq H^1(\operatorname{Tr}_{K/k} A, \mathbf{Q}_l(1))$$
$$H^2(C, j_*R^1f'_*\mathbf{Q}_l(1)) \simeq H^1(\operatorname{Tr}_{K/k} A, \mathbf{Q}_l)$$

and an exact sequence

$$0 \to \operatorname{LN}(A, K/k) \otimes \mathbf{Q}_l \to H^1(C, j_*R^1f'_*\mathbf{Q}_l(1)) \to V_l(\text{Ш}(A, K)) \to 0.$$

If $k = \mathbf{C}$, it seems likely that similar results hold for the analytic cohomology of $R^1f_*\mathbf{Q}(1)$ with similar techniques as those used in the next section (replacing Kummer sequences by exponential sequences), but I haven't tried to prove them.

2. Proof of theorem 1.1

2.1. Reduction to the cohomology of the Néron model.

Lemma 2.1. *Let $\mathcal{J} = j_*J$ be the Néron model of J over C. There are short exact sequences*

$$0 \to (\varprojlim H^{p-1}(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} \to H^p(C, j_*R^1f'_*\mathbf{Q}_l(1)) \to V_l(H^p(C, \mathcal{J})) \to 0.$$

Proof. Given $c \in C$, write $i_c : c \hookrightarrow C$ for the corresponding closed immersion and let Φ_c be the group of connected components of the special fibre of $\mathcal{J}$ at c. Then Φ_c is finite for any c and is 0 except for a finite number of c's. Write $\mathcal{J}^0 = \operatorname{Ker}(\mathcal{J} \to \bigoplus_{c \in C}(i_c)_*\Phi_c)$ for the connected component of $\mathcal{J}$. Since k is separably closed, we have isomorphisms $H^p(C, \mathcal{J}^0) \xrightarrow{\sim} H^p(C, \mathcal{J})$ for $p > 0$ and an injection with finite cokernel $H^0(C, \mathcal{J}^0) \hookrightarrow H^0(C, \mathcal{J})$. So,

$$(\varprojlim H^*(C, \mathcal{J}^0)/l^\nu) \otimes \mathbf{Q} \xrightarrow{\sim} (\varprojlim H^*(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q},$$
$$V_l(H^*(C, \mathcal{J}_0)) \xrightarrow{\sim} V_l(H^*(C, \mathcal{J})).$$

To handle the cohomology of $\mathcal{J}^0$, we may use the Kummer exact sequences

$$0 \to {}_{l^\nu}\mathcal{J}^0 \to \mathcal{J}^0 \xrightarrow{l^n} \mathcal{J}^0 \to 0$$

which yield exact sequences with finite central terms

$$0 \to H^{p-1}(C, \mathcal{J}^0)/l^\nu \to H^p(C, {}_{l^\nu}\mathcal{J}^0) \to {}_{l^\nu}H^p(C, \mathcal{J}^0) \to 0$$

hence other exact sequences

$$0 \to \varprojlim H^{p-1}(C, \mathcal{J}^0)/l^\nu) \otimes \mathbf{Q} \to H^p(C, V_l(\mathcal{J}^0)) \to V_l(H^p(C, \mathcal{J}^0)) \to 0.$$

But $V_l(\mathcal{J}^0) \xrightarrow{\sim} V_l(\mathcal{J})$; as $R^1f'_*\mu_{l^\nu} \xrightarrow{\sim} {}_{l^\nu}J$ and $j_*{}_{l^\nu}J = {}_{l^\nu}\mathcal{J}$, the lemma follows. □

2.2. **Cohomology of $B := j_* \operatorname{Pic}_{\Gamma/K}$.** For $c \in C$, view $f^{-1}(c)$ as a divisor on S. Let $D_c = \bigoplus_{x\in \operatorname{Supp}(f^{-1}(c))} \mathbf{Z}$ and $\bar{D}_c = D_c/\langle f^{-1}(c)\rangle$: thus $\bar{D}_c = 0$ whenever f is smooth over c. Write

$$D = \bigoplus_{c\in C} \bar{D}_c \tag{2.1}$$

(a finite sum).

Lemma 2.2. *There is an isogeny*

$$\operatorname{Pic}^0_{S/k} / \operatorname{Pic}^0_{C/k} \to \operatorname{Tr}_{K/k} J$$

and a complex

$$0 \to \operatorname{NS}(C) \to \operatorname{NS}(S) \to \operatorname{Pic}(\Gamma)/ \operatorname{Tr}_{K/k} J(k) \to 0$$

which, modulo finite groups, is acyclic except at $\operatorname{NS}(S)$*, where its homology is* D*.*

Proof. This follows from [HP, prop. 3.3 et 3.8] or [K2, 3.2 a)]. □

Lemma 2.3. *a) There is an exact sequence*

$$0 \to D \to \operatorname{Pic}(S/C) \to H^0(C, B) \to 0 \tag{2.2}$$

where $\operatorname{Pic}(S/C) = H^0(C, \operatorname{Pic}_{S/C})$ *and* $B = j_* \operatorname{Pic}_{\Gamma/K}$.
b) There is an exact sequence

$$0 \to \operatorname{Pic}(C) \to \operatorname{Pic}(S) \to \operatorname{Pic}(S/C) \to 0 \tag{2.3}$$

and isomorphisms

$$H^n(S, \mathbb{G}_m) \xrightarrow{\sim} H^{n-1}(C, B) \text{ for } n > 1. \tag{2.4}$$

In particular,

$$\operatorname{Br}(S) \xrightarrow{\sim} H^1(C, B) \tag{2.5}$$

and $H^n(C, B) = 0$ *for* $n > 3$.

Proof. This follows from the computations in [Br.III, §4]. After [Br.III, (4.1)], we have a long cohomology exact sequence (a consequence of *op. cit.*, (3.2)):

$$\dots \to H^n(C, \mathbb{G}_m) \to H^n(S, \mathbb{G}_m) \to H^{n-1}(C, P) \to \dots$$

where $P = \operatorname{Pic}_{S/C}$. Moreover, the homomorphism $P \to B$ is epi and its kernel is a skyscraper sheaf whose global sections are D [Br.III, p. 114]: (2.2) follows, as well as isomorphisms $H^n(C, P) \xrightarrow{\sim} H^n(C, B)$ for $n > 0$. As $H^n(C, \mathbb{G}_m) = 0$ for $n > 1$ and $H^n(S, \mathbb{G}_m) = 0$ for $n > 4$, one gets (2.3) and (2.4). □

2.3. **l-adic conversion.** Using (2.3), (2.4) and the structure of $\operatorname{Pic}(C)$ and $\operatorname{Pic}(S)$, we find exact sequences

$$0 \to V_l(\operatorname{Pic}^0(C)) \to V_l(\operatorname{Pic}^0(S)) \to V_l(\operatorname{Pic}(S/C))$$
$$\to \mathbf{Q}_l \to \operatorname{NS}(S) \otimes \mathbf{Q}_l \to (\varprojlim \operatorname{Pic}(S/C)/l^\nu) \otimes \mathbf{Q} \to 0$$

$$0 \to V_l(\operatorname{Pic}(S/C)) \to V_l(H^0(C,B)) \to D \otimes \mathbf{Q}_l$$
$$\to (\varprojlim \operatorname{Pic}(S/C)/l^\nu) \otimes \mathbf{Q} \to (\varprojlim H^0(C,B)/l^\nu) \otimes \mathbf{Q} \to 0$$

$$H^2_{\mathrm{tr}}(S,\mathbf{Q}_l(1)) \xrightarrow{\sim} V_l(H^1(C,B)), \quad (\varprojlim H^1(C,B)/l^\nu) \otimes \mathbf{Q} = 0.$$

Using Lemma 2.2, we derive new exact sequences

$$0 \to V_l(\operatorname{Pic}^0(C)) \to V_l(\operatorname{Pic}^0(S) \to V_l(\operatorname{Tr}_{K/k} J) \to 0$$

$$0 \to D \otimes \mathbf{Q}_l \to (\operatorname{NS}(S)/\mathbf{Z}) \otimes \mathbf{Q}_l \to (\operatorname{Pic}(\Gamma)/(\operatorname{Tr}_{K/k} J)(k)) \otimes \mathbf{Q}_l \to 0.$$

Hence

$$V_l(\operatorname{Tr}_{K/k} J) \xrightarrow{\sim} V_l(\operatorname{Pic}(S/C))$$
$$(\operatorname{NS}(S)/\mathbf{Z}) \otimes \mathbf{Q}_l \xrightarrow{\sim} (\varprojlim \operatorname{Pic}(S/C)/l^\nu) \otimes \mathbf{Q}$$

then

$$V_l(\operatorname{Tr}_{K/k} J) \xrightarrow{\sim} V_l(H^0(C,B))$$
$$(\operatorname{Pic}(\Gamma)/(\operatorname{Tr}_{K/k} J)(k)) \otimes \mathbf{Q}_l \xrightarrow{\sim} (\varprojlim H^0(C,B)/l^\nu) \otimes \mathbf{Q}.$$

2.4. **From B to $\mathcal{J}$.** To pass from B to $\mathcal{J}$, we work in the category $\mathcal{C}$ of abelian groups modulo the thick sub-category of finite groups, which does not affect the functor V_l.

Lemma 2.4. *In $\mathcal{C}$, we have*

1. *A split exact sequence* $0 \to H^0(C,\mathcal{J}) \to H^0(C,B) \to \mathbf{Z} \to 0$.
2. *An isomorphism* $H^1(C,\mathcal{J}) \simeq \operatorname{Br}(S)$.
3. *An isomorphism* $H^2(C,\mathcal{J})\{l\} \simeq \operatorname{Im}_{K/k} J\{l\}(-1)$; $H^2(C,\mathcal{J})$ *is torsion.*
4. $H^p(C,\mathcal{J}) = 0$ *for* $p \geq 3$.

Proof. We have an exact sequence, split in $\mathcal{C}$

$$0 \to \mathcal{J} \to B \to \mathbf{Z} \to 0 \tag{2.6}$$

which gives (still in $\mathcal{C}$) split exact sequences

$$0 \to H^p(C,\mathcal{J}) \to H^p(C,B) \to H^p(C,\mathbf{Z}) \to 0.$$

For $p=0$, we get (1). For $p=1$, we get (2) in view of $H^1(C,\mathbf{Z}) = 0$ and (2.5).

For $p>1$, this gives in view of (2.4)

$$0 \to H^p(C,\mathcal{J}) \to H^{p+1}(S,\mathbb{G}_m) \xrightarrow{\tau} H^p(C,\mathbf{Z}) \to 0$$

Still for $p > 1$, we have isomorphisms

$$H^{p-1}(C, \mathbf{Q}_l/\mathbf{Z}_l) \xrightarrow{\sim} H^p(C, \mathbf{Z})\{l\}, \quad H^{p+1}(S, \mathbf{Q}_l/\mathbf{Z}_l(1)) \xrightarrow{\sim} H^{p+1}(S, \mathbb{G}_m)\{l\}.$$

The morphism τ then gets identified to the trace morphism, which is an isomorphism for $p \geq 3$ (hence (4)), while for $p = 2$ it is the dual of

$$f^* : H^1(C, \mathbf{Z}_l(1)) \to H^1(S, \mathbf{Z}_l(1)).$$

We can then identify $H^2(C, \mathcal{J})$ with $\mathrm{Ker}(\mathrm{Alb}(S)\{l\}(-1) \to \mathrm{Alb}(C)\{l\}(-1)$, *i.e.*, to $\mathrm{Im}_{K/k} J\{l\}(-1)$. □

2.5. **Conclusion.** From Lemma 2.4 and the computations in §2.3, we derive

$$V_l(\mathrm{Tr}_{K/k} J) \xrightarrow{\sim} V_l(H^0(C, \mathcal{J}))$$
$$V_l(H^1(C, \mathcal{J})) \simeq V_l(\mathrm{Br}(S)) \simeq H^2_{\mathrm{tr}}(S, \mathbf{Q}_l(1)).$$
$$\mathrm{LN}(J, K/k) \otimes \mathbf{Q}_l \xrightarrow{\sim} (\varprojlim H^0(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q}$$
$$H^2_{\mathrm{tr}}(S, \mathbf{Q}_l(1)) \xrightarrow{\sim} V_l(H^1(C, \mathcal{J})), \quad (\varprojlim H^1(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} = 0$$
$$V_l(H^2(C, \mathcal{J})) \simeq V_l(\mathrm{Im}_{K/k} J)(-1) \simeq V_l(\mathrm{Tr}_{K/k} J)(-1)$$
$$(\varprojlim H^2(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} = 0$$
$$V_l(H^p(C, \mathcal{J})) = (\varprojlim H^p(C, \mathcal{J})/l^\nu) \otimes \mathbf{Q} = 0 \text{ for } p > 2$$

and finally the isomorphisms and the exact sequence of Theorem 1.1, using Lemma 2.1 and the isomorphisms

$$V_l(A)(-1) \simeq V_l(A)^* \simeq H^1(A, \mathbf{Q}_l)$$

valid for any abelian variety A over a separably closed field. □

3. Proofs of Corollary 1.2 and Theorem 2

If X is a smooth projective variety of dimension d over a field F, we write $CH^d_{\equiv}(X \times_F X)$ for the quotient of the ring of Chow correspondences on X by the ideal generated by those $Z \subset X \times X$ such that $p_1(Z) \neq X$ or $p_2(Z) \neq X$, where p_1, p_2 are the two projections $X \times X \to X$ (*cf.* [F, ex. 16.1.2 (b)].)

Proposition 3.1. *a) In the situation of* (0.2)*, there is a ring isomorphism* $CH^1_{\equiv}(\Gamma \times_K \Gamma) \xrightarrow{\sim} \mathrm{End}_K(J)$*, and a ring homomorphism*

$$r : CH^1_{\equiv}(\Gamma \times_K \Gamma) \to CH^2_{\equiv}(S \times_k S).$$

b) The rings $\mathrm{End}_K(J) \otimes \mathbf{Q}$ *and* $CH^2_{\equiv}(S \times_k S) \otimes \mathbf{Q}$ *act compatibly on the isomorphisms and the exact sequence of Theorem 1.1, as well as on* (1.1) *and* (1.2).

Proof. a) The first isomorphism is due to Weil [W, ch. 6, th. 22][3]. We have a homomorphism

$$R : Z^1(C \times_K C) \to Z^2(S \times_k S)$$

defined as follows: let $Z \subset \Gamma \times_K \Gamma$ be an irreducible cycle of codimension 1. Write $\mathcal{Z}$ for its closure in $S \times_C S$. We set $R(Z) =$ image of $\mathcal{Z}$ in $Z^2(S \times_k S)$. One checks that R passes to rational equivalence and to the equivalences $\equiv$, and that the induced map r is compatible with composition of correspondances.

A more functorial construction of r will be given in [K3].

b) This is a long but eventless verification. □

In view of Theorem 1, Theorem 2 immediately follows from (0.1) and Corollary 1.2.

4. Comparing classes in K_0; proof of Theorem 3

In (0.2), let us come back to the case of an arbitrary base field k. Let k_s be a separable closure of k and $G = Gal(k_s/k)$. Write $C_s, \Gamma_s \dots$ for the objects of (0.2) after base change to k_s. Then Theorem 1.1 "over k_s" is G-equivariant; moreover, the Kk_s/k_s-trace of J_s is $(\operatorname{Tr}_{K/k} J)_s$ ([K1, Prop. 6] or [C, Th. 6.8]). One might want to compare

$$R(pf)_*\mathbf{Q}_l$$

and

$$Rp_*j_*Rf'_*\mathbf{Q}_l$$

in the derived category of $\mathbf{Q}_l[[G]]$-modules. Unfortunately this has no meaning, because j_* has no meaning in the derived category.

On the other hand, let $\mathbf{K}_l$ be the Grothendieck group of continuous, finite dimensional $\mathbf{Q}_l$-representations G. We may consider in $\mathbf{K}_l$:

(1) $H_l = [H_l^*(S_s)]$, the alternating sum of the l-adic cohomology groups of S;
(2) $H'_l = [R^*p_*j_*R^*f'_*\mathbf{Q}_l]$ (9 terms).

We may also consider D as a discrete topological G-module.

Theorem 4.1. $H_l - H'_l = [D \otimes \mathbf{Q}_l(-1)]$.

Proof. For simplicity, put $B := \operatorname{Tr}_{K/k} J$. Let $\mathbf{K}$ be the Grothendieck group of the category $\mathbf{Mot}_{\mathrm{rat}}(k, \mathbf{Q})$ of Chow motives over k, and let $R_l : \mathbf{K} \to \mathbf{K}_l$ be the homomorphism given by l-adic realisation. (To avoid confusion, we adopt cohomological notation as in [Kl] but contrary to [KMP], also for Tate twists.) Then $H_l = R_l(h)$, with $h = h(S)$.

[3]The homomorphism is constructed in [F, ex. 16.1.2 (c)], but its bijectivity is not mentioned…

Similarly, there exists a canonical $h' \in \mathbf{K}$ such that $H'_l = R_l(h')$. Indeed, Theorem 1.1 shows that

$$[p_* j_* R^1 f'_* \mathbf{Q}_l] = R_l([h^1(B]), [R^2 p_* j_* R^1 f'_* \mathbf{Q}_l] = R_l([h^1(B(-1)]),$$
$$[R^1 p_* j_* R^1 f'_* \mathbf{Q}_l] = R_l([t^2(S)] + [\ln(J, K/k)(-1)])$$

where $\ln(J, K/k)$ is the Artin motive associated to the Galois module $\mathrm{LN}(J, Kk_s/k_s)$ (see [KMP] for $t^2(S)$).

Using $f'_* \mathbf{Q}_l = \mathbf{Q}_l$ and $R^2 f'_* \mathbf{Q}_l = \mathbf{Q}_l(-1)$, we similarly get

$$[R^q p_* j_* f'_* \mathbf{Q}_l] = R_l([h^q(C)]), \quad [R^q p_* j_* R^2 f'_* \mathbf{Q}_l] = R_l([h^q(C)(-1)]).$$

Set

$$h' = \sum_{q=0}^{2} (-1)^q [h^q(C)] + \sum_{q=0}^{2} (-1)^q [h^q(C)(-1)]$$
$$- \big([h^1(B] + [h^1(B(-1)] - ([t^2(S)] + [\ln(J, K/k)(-1)])\big) \in \mathbf{K}.$$

To prove Theorem 4.1, it therefore suffices to show:

$$h - h' = [D(-1)]. \tag{4.1}$$

From Lemma 2.2, one gets identities in $\mathbf{K}$:

$$[h^1(S)] = [h^1(C)] + [h^1(B], [h^3(S)] = [h^1(C)(-1)] + [h^1(B(-1)],$$

$[\mathrm{NS}(S)] = [\mathrm{NS}(C)] + [\mathbf{1}] + [\ln(J, K/k)] + [D] = 2[\mathbf{1}] + [\ln(J, K/k)] + [D]$ from which (4.1) easily follows. □

Remark 4.2. The same proof gives a more precise identity in $\mathbf{K}$:

$$[h^2(S)] - [Rp_* j_* h^1(J)] = [D(-1)] + 2[\mathbb{L}] - [h^1(B)] - [h^1(B)(-1)]$$

where $\mathbb{L}$ is the Lefschetz motive and $[Rp_* j_* h^1(J)]$ stands for the canonical element of $\mathbf{K}$ whose l-adic realisation is $\sum_{i=0}^{2} (-1)^i [R^i p_* j_* f'_* \mathbf{Q}_l]$.

Let us pass to the case of an abelian variety A over K. We would like to interpret the terms in Corollary 1.2 as realisations of pure motives over k. This is clearly possible, except perhaps for the term $V_l(Ш(A, Kk_s))$.

Proposition 4.3. *There exists an effective k-Chow motive* $ш(A, K/k)$ *such that* $R_l(ш(A, K/k)) = V_l(Ш(A, Kk_s))(-1)$.

Proof. Write A as a direct summand of a Jacobian J, up to isogeny, where J comes from a situation (0.2). *Via* Proposition 3.1, the corresponding projector $\pi \in \mathrm{End}(A) \otimes \mathbf{Q}$ defines a projector $r(\pi) \in CH^2_{\equiv}(S \times_k S) \otimes \mathbf{Q} = \mathrm{End}(t^2(S))$ [KMP]. Define $ш(A, K/k)$ as the image of $r(\pi)$. □

Remark 4.4. We shall see in [K3] that the motive $Ш(A, K/k)$ is independent of the choice of J, and is *functorial in A.*

Theorem 3 immediately follows from Theorem 4.1.

5. The crystalline realisation

In this section, k is any perfect field of characteristic $p > 0$.

5.1. **Isocrystals.** We rely here on the crystal-clear exposition of Saavedra [Saa, Ch. VI, §3].

Let $W(k)$ be the ring of Witt vectors on k and $K(k)$ be the field of fractions of $W(k)$. The Frobenius automorphism $x \mapsto x^p$ of k lifts to an endomorphism on $W(k)$ and an automorphism of $K(k)$, written σ: we have $K(k)^\sigma = \mathbf{Q}_p$. A *$k$-isocrystal* is a finite-dimensional $K(k)$-vector space M provided with a σ-linear automorphism F_M. k-isocrystals form a $\mathbf{Q}_p$-linear tannakian category $\mathbf{Fcriso}(k)$, provided with a canonical $K(k)$-valued fibre functor (forgetting F_M) [Saa, VI.3.2.1]. We have

$$\mathbf{Fcriso}(k)(\mathbf{1}, M) = M^{F_M} = \{m \in M \mid F_M m = m\}$$

for $M \in \mathbf{Fcriso}(k)$, where $\mathbf{1} = (K(k), \sigma)$ is the unit object. For $n \in \mathbf{Z}$, we write more generally

$$M^{(n)} = M^{F_M = p^n} = \mathbf{Fcriso}(k)(\mathbb{L}^n_{\text{crys}}, M) = \mathbf{Fcriso}(k)(\mathbf{1}, M(n)) \tag{5.1}$$

where $M(n) = M \otimes \mathbb{L}^{-n}_{\text{crys}}$ with $\mathbb{L}_{\text{crys}} := (K(k), p\sigma)$.

5.2. **The realisation.** By [Saa, VI.4.1.4.3], the formal properties of crystalline cohomology yield a $\otimes$-functor

$$R_p : \mathbf{Mot}_{\text{rat}}(k, \mathbf{Q}) \to \mathbf{Fcriso}(k).$$

This functor sends the motive of a smooth projective variety X to $H^*_{\text{crys}}(X/W(k)) \otimes_{W(k)} K(k)$ and the Lefschetz motive $\mathbb{L}$ to $\mathbb{L}_{\text{crys}}$.

5.3. **The case of a finite field.** Suppose that $k = \mathbf{F}_q$, with $q = p^m$. Then any object $M \in Mot_{\text{rat}}(k, \mathbf{Q})$ has its *Frobenius endomorphism* π_M: if $M = h(X)$ for a smooth projective variety X, $\pi_M = \pi_X$ is the graph of the Frobenius endomorphism F^m on X. This implies:

Lemma 5.1. *The action of π_M on $R_p(M)$ equals that of F^m.* □

Let $\bar{k}$ be an algebraic closure of k. There is an obvious functor

$$\mathbf{Fcriso}(k) \to \mathbf{Fcriso}(\bar{k}), \quad M \mapsto \bar{M} := M \otimes_{K(k)} K(\bar{k}) \tag{5.2}$$

which is compatible with the extension of scalars $\mathbf{Mot}_{\text{rat}}(k, \mathbf{Q}) \to \mathbf{Mot}_{\text{rat}}(\bar{k}, \mathbf{Q})$ via the realisation functors R_p for k and $\bar{k}$. Moreover F^m is $K(k)$-linear, therefore one can talk of its eigenvalues. We have the following result of Milne [M, Lemma 5.1]:

Lemma 5.2. *One has an equality*

$$\det(1-\gamma t \mid \bar{M}^{(n)}) = \prod_{v(a)=v(q^n)} (1-(q^n/at))$$

where γ is the arithmetic Frobenius and a runs through the eigenvalues of F^m having same valuation as q^n.

5.4. **Logarithmic de Rham-Witt cohomology.**

Proposition 5.3. *Let X/k be smooth projective. Then, for any $i,n \in$ $\mathbf{Z}$, there is a canonical isomorphism*

$$H^i(X,\mathbf{Q}_p(n)) \xrightarrow{\sim} (H^i_{\text{crys}}(X/W(k)) \otimes_{W(k)} K(k))^{(n)}$$

where the left hand side is logarithmic Hodge-Witt cohomology as in Milne [M, p. 309].

Proof. This is [M, Prop. 1.15], but unfortunately its proof is garbled (the last line of loc. cit., p. 310 is wrong). Let us recapitulate it. For simplicity, let $W = W(k)$ and $K = K(k)$.

1) The slope spectral sequence

$$E_1^{i,j} = H^j(X, W\Omega^i) \Rightarrow H^{i+j}(X, W\Omega^\cdot) \simeq H^{i+j}_{\text{crys}}(X/W)$$

degenerates up to torsion, yielding canonical isomorphisms of k-isocrystals

$$H^{i-n}(X, W\Omega^n) \otimes_W K \xrightarrow{\sim} (H^i(X/W) \otimes_W K)_{[n,n+1[}$$

where the index $[n, n+1[$ means the sum of summands of slope λ for $n \le \lambda < n+1$ [I, Th. 3.2 p. 615 and (3.5.4) p. 616].

2) If k is algebraically closed, the homomorphism

$$H^i(X, \mathbf{Z}_p(n)) := H^{i-n}(X, W\Omega^n_{\log}) \to H^{i-n}(X, W\Omega^n)^F$$

is bijective [IR, Cor. 3.5 p. 194].

3) In general, descend from $\bar{k}$ to k by taking Galois invariants. □

By [M, §2] and [G], Chow correspondences act on logarithmic Hodge-Witt cohomology by respecting the isomorphisms of Proposition 5.3: this yields functors

$$H^i(-, \mathbf{Q}_p(n)) : \mathbf{Mot}_{\text{rat}}(k, \mathbf{Q}) \to \mathbf{Vec}^*_{\mathbf{Q}_p}$$

and natural isomorphisms

$$(5.3) \quad H^i(M, \mathbf{Q}_p(n)) \xrightarrow{\sim} \mathbf{Fcriso}(k)(\mathbb{L}^n_{\text{crys}}, R_p(M)),\ M \in \mathbf{Mot}_{\text{rat}}(k, \mathbf{Q}).$$

5.5. **The Brauer group and the Tate-Šafarevič group.** We have

Proposition 5.4 ([I, (5.8.5) p. 629]). *Let k be algebraically closed and X/k be smooth projective. Then there is an exact sequence*

$$0 \to \mathrm{NS}(X) \otimes \mathbf{Z}_p \to H^2(X, \mathbf{Z}_p(1)) \to T_p(\mathrm{Br}(X)) \to 0.$$

As before, Chow correspondences act on this exact sequence. Therefore if $X = S$ is a surface, applying the projector π^2_{tr} defining $t^2(S)$, we get an isomorphism

$$H^2(t^2(X), \mathbf{Q}_p(1)) \simeq V_p(\mathrm{Br}(X))$$

hence, taking (5.3) into account:

$$\mathbf{Fcriso}(k)(\mathbb{L}_{\mathrm{crys}}, R_p(t^2(X))) \simeq V_p(\mathrm{Br}(X)).$$

If now K/k is a function field in one variable and A is an abelian variety over K, using the projector $r(\pi)$ from the proof of Proposition 4.3, we get an isomorphism

$$(5.4) \qquad \mathbf{Fcriso}(k)(\mathbb{L}_{\mathrm{crys}}, R_p(ш(A, K/k))) \simeq V_p(Ш(A, K)).$$

6. Functional equation, order of zero and special value

6.1. **Functional equation.** Recall the functional equation of the zeta function of a pure motive M of weight w over a finite field k with q elements:

$$\zeta(M^*, -s) = \det(M)(-q^{-s})^{\chi(M)}\zeta(M, s)$$

where M^* is the dual of M, $\chi(M)$ is the Euler characteristic of M (computed for example with the help of its l-adic realisation) and

$$\det(M) = \pm q^{w\chi(M)/2}$$

is the determinant of the Frobenius endomorphism of M. Applying this to Theorem 2, we get the following functional equation for $L(K, A, s)$:

$$L(K, A, 2 - s) = a(-q^{-s})^{\beta}L(K, A, s)$$

with

$$\begin{aligned}\beta &= -2\chi(h^1(B)) - \chi(ш(A, K/k)) - \chi(\mathrm{ln}(A, K/k)) \\ &= 4\dim B - \mathrm{cork}Ш(A, K\bar{k}) - \mathrm{rk}A(K\bar{k}) \\ a &= \left(\det h^1(B)\det h^1(B)(-1)\det ш(A, K/k)\det \mathrm{ln}(A, K/k)(-1)\right)^{-1} \\ &= \pm q^{\beta}.\end{aligned}$$

The exponent β compares mysteriously with the one appearing in the functional equation of Grothendieck:

$$\beta = \chi(j_*H^1(\bar{A}, \mathbf{Q}_l)) = 2 - 2g - \deg(\mathfrak{f})$$

where g is the genus of C and $\mathfrak{f}$ is the conductor of A (relative to K/k), and the second equality follows from [Ra, Th. 1].

6.2. **Order of zero.** From Theorem 2, one immediately gets the well-known equality and inequality

$$\begin{aligned}(6.1)\quad \operatorname{ord}_{s=1}L(A,s) &= \operatorname{rk}A(K) + \operatorname{cork}_l^1 l Ш(A, K\bar{k})\\ &\geq \operatorname{rk}A(K) + \operatorname{cork}_l l Ш(A,K)\end{aligned}$$

where $\operatorname{cork}_l^1 Ш(A, K\bar{k})$ is the corank of the generalised eigensubgroup for the eigenvalue 1 of the action of the arithmetic Frobenius γ on an arbitrary l-primary component of $Ш(A, K\bar{k})$ (cf. [Sch, Lemma 2 (i)] for $l \neq p$).

Indeed, let us show that $\operatorname{ord}_{s=1}Z(ш(A, K/k)) = \operatorname{cork}_l^1 Ш(A, K\bar{k})$ for any prime l. This order can be computed through the action of the Frobenius endomorphism $\pi_ш$ of $ш(A, K/k)(1)$ on $R(ш(A, K/k)(1)) = R_l(ш(A, K/k))(-1)$ for any realisation functor R on $\mathbf{Mot}_{\mathrm{rat}}(k, \mathbf{Q})$. If we use the l-adic realisation R_l for $l \neq p$, the claim is clear by Proposition 4.3 since $R_l(\pi_ш)$ acts like the inverse of γ. If we now take R_p, we find from (5.1), (5.2) and (5.4):

$$R_p(ш(A, K\bar{k}/\bar{k})(1))^{(0)} \simeq \overline{R_p(ш(A, K/k)(1))}^{(0)} \simeq V_p(Ш(A, K\bar{k})).$$

By Lemma 5.2, we have

$$\det(1 - \gamma t \mid \overline{R_p(ш(A, K/k)(1))}^{(0)}) = \prod_{v(a)=0} (1 - 1/at)$$

where a runs through the eigenvalues of F^m acting on $R_p(ш(A, K/k)(1))$, with valuation 0. By lemma 5.1, $F^m = R_p(\pi_ш)$, so we are done.

This argument does not show that $\operatorname{cork}_l Ш(A,K) = \operatorname{cork}_l Ш(A, K\bar{k})^\gamma$ is independent of l. However, it does yield:

Proof of Corollary 1. For any l, $\operatorname{cork}_l Ш(A,K) = 0 \iff Ш(A,K)\{l\}$ is finite. In view of (6.1), this shows (i) $\iff$ (ii) $\iff$ (iii). To get (iii) $\Rightarrow$ (iv), consider a surface S/k used to construct the motive $ш(A, K/k)$. The projector $r(\pi)$ in the proof of Proposition 4.3 is represented by an algebraic correspondence with $\mathbf{Q}$ coefficients, which have a common denominator D. For l prime to D, $Ш(A,K)\{l\}$ is then a direct summand of $\operatorname{Br}(S)\{l\}$, a group of cofinite type whose finite quotient is 0 for almost all l. □

6.3. **Special value.** It is less obvious to relate Theorem 2 to the value of the principal part in the Birch and Swinnerton-Dyer conjecture ([Sch, Theorem p. 509], [KT]):

$$\lim_{s\to 1}\frac{L(A,s)}{(s-1)^\rho}\sim \pm q^\rho \frac{|Ш(A,K)||\det\langle,\rangle_{A(K)}|}{|A(K)_{\rm tors}||A'(K)_{\rm tors}|}\prod_{c\in C}|\Phi_c(k(c))|, \tag{6.2}$$

where $\rho = \mathrm{rk} A(K)$, $\langle,\rangle_{A(K)}$ is the height pairing constructed in [Sch, p. 502] and the Φ_c are the groups of connected components of the Néron model of A over C, as in §2.1.

It seems that the explicit expression of $L(A,s)$ could actually be used to provide an expression of the left hand side of (6.2) independently of the Birch and Swinnerton-Dyer conjecture, in the spirit of (6.1). This can presumably be done by the method of [Sch]: I did not succeed and leave it to better experts. Let me only note that in Theorem 2, the factors $Z(h^1(B),q^{-s})$ and $Z(h^1(B),q^{1-s})$ respectively contribute by $|B'(k)|$ and $|B(k)|$ (as usual, $B := \mathrm{Tr}_{K/k} A$), while $Z(\ln(A,K/k),q^{1-s})$ contributes by

$$\pm q^{-\mathrm{rk} A(K)}\frac{\det\langle,\rangle_{A(K\bar k)}}{\det\langle,\rangle_{A(K)}}\frac{|A(K)_{\rm tors}/B(k)|}{|(\mathrm{LN}(A,K\bar k/\bar k)_F)_{\rm tors}|}$$

Where $\langle,\rangle_{A(K)}$ and $\langle,\rangle_{A(K\bar k)}$ are the height pairings constructed in [Sch, p. 502]. This folllows from the elementary lemma, in the spirit of [T1, Lemma z.4]:

Lemma 6.1. *Let $\langle,\rangle : M\times M'\to \mathbf{Q}$ be a $\mathbf{Q}$-non-degenerate pairing between finitely generated abelian groups. Suppose M and M' are provided with operators F, F' which are adjoint for the pairing, and $\mathbf{Q}$-semi-simple (e.g., F is of finite order). Let $P = \det(1-FT)$ be the inverse characteristic polynomial of F acting on $M_{\mathbf{Q}}$. Then $\rho := \mathrm{ord}_{T=1} P = \mathrm{rk} M^F$ and, if $P' = P/(1-T)^\rho$,*

$$|P'(1)| = \frac{\det\langle,\rangle^F}{\det\langle,\rangle}\frac{|(M^F)_{\rm tors}|}{|(M_F)_{\rm tors}|}$$

where M^F (resp. M_F) denotes the F-invariants (resp. coinvariants) of F and $\langle,\rangle^F$ is the ($\mathbf{Q}$-non-degenerate) pairing induced by $\langle,\rangle$ on $M^F\times M'^{F'}$.

7. Surfaces over a global field

In §4, suppose k *global*: K is a function field in one variable over k.

Definition 7.1. If A is an abelian variety over K, we set

$$L(K, h^1(A), s) = L(k, h^1(\operatorname{Tr}_{K/k} A), s) L(k, h^1(\operatorname{Tr}_{K/k} A), s-1)$$
$$L(k, Ш(A, K/k), s) L(k, \ln(A, K/k), s-1)$$

where the right hand side is defined in terms of l-adic realisations.

In the right hand side, the motive $\ln(A, K/k)$ is of weight 0 (it is an Artin motive), $h^1(\operatorname{Tr}_{K/k} A)$ is of weight 1 and $Ш(A, K/k)$ is of weight 2, a direct summand of h^2 of a suitable surface. Definition 7.1 is independent of the choice of l (invertible in k) because this is so for each individual factor (for $Ш(A, K/k)$, it follows from [RZ, Satz 2.13] and [Sa, cor. 0.6]).

If k is a number field, it may always be chosen as the algebraic closure of $\mathbf{Q}$ in K, and this choice is unique. On the other hand, I don't know the answer to:

Question 7.2. If $\operatorname{char} k > 0$, is Definition 7.1 independent of the choice of k?

(Said differently: does Definition 7.1 only depend on K, a function field in 2 variables over a finite field, and on A?)

Question 7.3. Can one interpret $L(K, h^1(A), s)$, *via* a trace formula, as an "Euler" product of the form

$$L(C, j_* H^1_l(A), s) = \prod_{x \in C_{(0)}} L(k(x), i_x^* H^1_l(\mathcal{A}), s)$$

where $\mathcal{A}$ is the Néron model of A over C?

(It is not even clear that the right hand side converges!)

Let us now place ourselves in the situation of (0.2). Set $L(K, h^1(\Gamma), s) = L(K, h^1(J), s)$, and define similarly:

$$L(K, h^0(\Gamma), s) = L(k, h(C), s)$$
$$L(K, h^2(\Gamma), s) = L(k, h(C), s-1)$$
$$L(K, h(\Gamma), s) = \prod_{i=0}^{2} L(k, h^i(\Gamma), s).$$

Theorem 4.1 then gives the following analogue to Theorem 3:

Theorem 7.4. *One has*

$$\frac{L(k, h(S), s)}{L(K, h(\Gamma), s)} = L(k, D(-1), s) = L(k, D, s-1). \quad \square$$

Question 7.5. The height pairing defined by Schneider in [Sch, p. 507]:

$$\mathcal{A}^0(C_s) \times A'(Kk_s) \to \mathrm{Pic}(C_s)$$

induces a pairing

$$\mathcal{A}^0(C_s)/B(k_s) \times \mathrm{LN}(A', Kk_s/k_s) \to \mathbf{Z}$$

because $B(k_s)$ and $B'(k_s)$ are divisible; moreover it presumably restricts to a pairing

$$B(k_s) \times \mathrm{LN}(A', Kk_s/k_s) \to \mathrm{Pic}^0(C_s). \tag{7.1}$$

One way to justify (7.1) would be to show that the functor $S \mapsto \Gamma(S, \mathcal{A} \times_k S)$ on k-schemes of finite type is representable by a k-group scheme of finite type with connected component B, and that Schneider's pairing emanates from a pairing of k-group schemes. Then (7.1) would induce a Galois-equivariant homomorphism

$$\mathrm{LN}(A', Kk_s/k_s) \to \mathrm{Hom}_{k_s}(B, J).$$

Can one use these pairings to describe the special values of $L(K, A, s)$?

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