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Some finiteness results in the category \mathcal{U}

Nguyen The Cuong and Lionel Schwartz

UMR 7539 CNRS, Université Paris 13
LIA CNRS Formath Vietnam

Abstract

This note investigate some finiteness properties of the category \mathcal{U} of unstable modules. One shows finiteness properties for the injective resolution of finitely generated unstable modules. One also shows a stabilization result under Frobenius twist for Ext -groups.

1 Introduction

This note investigates some finiteness properties of the category \mathcal{U} of unstable modules. For simplicity the case of the prime 2 will only be considered. The main finiteness property of \mathcal{U} is to be locally noetherian [MP67], see also [LZ86]. This means that if M is a finitely generated unstable module any sub-module is also finitely generated. Injective objects of the category are described in [LS89]. It is natural and useful to ask for some control on injective resolutions. Some of the results of the note could have been written earlier. They are implicit in some places, however it is worth to present them all at the same time and explicitly. The first result of the paper is:

Theorem 3.1. *Let M be an unstable finitely generated module. It has an injective resolution \mathcal{I}^\bullet such that any \mathcal{I}^k is a finite direct sum of indecomposable injective unstable modules.*

This is to be compared with an analogous theorem in the category \mathcal{F} of functors from finite dimensional \mathbb{F}_2 -vector spaces to \mathbb{F}_2 -vector spaces [S94, FLS94, ER03]:

Theorem 1.1. *Let F be a polynomial functor taking finite dimensional values. There exist an injective resolution \mathcal{I}^\bullet of M such that any \mathcal{I}^k is a finite direct sum of indecomposable injective functors.*

In the sequel such injective resolutions (for both categories) will be called of finite type. The proof of 3.1 uses 1.1 but is not a direct consequence.

The link between 3.1 and 1.1 is done by the functor $f: \mathcal{U} \rightarrow \mathcal{F}$ [HLS93]. This functor induces an equivalence $\mathcal{U} \rightarrow \mathcal{U}/\mathcal{Nil} \cong \mathcal{F}_\omega$ of the quotient category of \mathcal{U} by the full subcategory \mathcal{Nil} of nilpotent modules to the category of analytic functors. It has a right adjoint m , the composition $\ell = m \circ f$ is the localization functor away from \mathcal{Nil} . The natural map $M \rightarrow \ell(M)$ is initial for morphisms $M \rightarrow L$, L being \mathcal{Nil} -closed *i.e.* such that $Ext_{\mathcal{U}}^i(N, L) = \{0\}$ for $i = 0, 1$, any $N \in \mathcal{Nil}$ [Gab62].

The functor ℓ is left exact and admits right derivatives ℓ^i . In some interesting cases described later the ℓ^i are computed by Mac-Lane homology. Here is a general result:

Theorem 3.2. *If M is a finitely generated unstable module the unstable modules $\ell^i(M)$ are finitely generated unstable modules.*

This is related to 3.1 but not (at least directly) equivalent.

Recall the Frobenius twist functor Φ on the category \mathcal{U} [S94]. The stabilization result which follows is a direct corollary of 3.1:

Theorem 3.3. *Let M and N be two finitely generated unstable modules. Consider the direct system induced by the maps $\Phi^{k+1}M \rightarrow \Phi^kM$:*

$$\dots \rightarrow Ext_{\mathcal{U}}^i(\Phi^k M, N) \rightarrow Ext_{\mathcal{U}}^i(\Phi^{k+1} M, N) \rightarrow \dots$$

For k large enough this map is an isomorphism, and the terms are isomorphic to $Ext_{\mathcal{F}}^i(f(M), f(N))$.

A similar stabilisation result holds in the category of strict polynomial functors for the map induced by the Frobenius twist [FFSS99], F and G denoting strict polynomial functors of the same degree, and $F^{(k)}$ being the k -th Frobenius twist: the system

$$\dots \rightarrow \mathrm{Ext}_{\mathcal{P}}^i(F^{(k)}, G^{(k)}) \rightarrow \mathrm{Ext}_{\mathcal{P}}^i(F^{(k+1)}, G^{(k+1)}) \rightarrow \dots$$

stabilizes for k large. But in general the colimit is not isomorphic to $\mathrm{Ext}_{\mathcal{F}}^i(\mathcal{O}(F), \mathcal{O}(G))$, $\mathcal{O}: \mathcal{P} \rightarrow \mathcal{F}$ is the forgetful functor. However, in interesting cases it is isomorphic, it is true in particular when F and G are the canonical liftings of the simple objects in \mathcal{F} in the category \mathcal{P} . Another difference is that on the right hand side of the Ext -group the module does come with a Frobenius twist: one has just N instead of $\Phi^k(N)$. Again in some interesting cases it is possible to replace N by $\Phi^k(N)$. The main reason to keep N is that the colimit considered in the corollary is more natural for topological applications.

It is also worth to note that it is not clear to the authors how to prove the result using projective resolutions. The category \mathcal{U} is locally noetherian. Thus, given a finitely generated module M , it is possible to construct a projective resolution of M such that each term of the resolution is a finite direct sum of indecomposable projective modules. However when replacing M by $\Phi^k(M)$, if the finite generation is preserved, one has no control on the size of a given term in the resolution.

In particular if $\mathrm{Ext}_{\mathcal{F}}^i(f(M), f(N)) \cong \{0\}$ for k large enough $\mathrm{Ext}_{\mathcal{U}}^i(\Phi^{k+1}M, N) \cong \{0\}$. This case applies for $i = 1$ in to the following case. Let \mathcal{U}_n , $n \in \mathbb{N}$, the Krull filtration on \mathcal{U} (see the next section). Let M be a finitely generated unstable module, assume $M \in \mathcal{U}_n/\mathcal{U}_{n-1}$. Let R be the smallest sub-module in \mathcal{U}_n such that $M/R \in \mathcal{U}_{n-1}$. The following is a corollary of 3.3:

Proposition 1.2. *For k large enough the \mathbb{F}_2 -vector space $\mathrm{Ext}_{\mathcal{U}}^1(\Phi^k(R \otimes R), M)$ is trivial.*

This is applied in [CGS14]. This statement can be proved directly using only Steenrod operations, however it is much more tedious.

2 Recollections: the nilpotent filtration, the Krull filtration and functors

This section recalls briefly facts about the category \mathcal{U} . One refers mostly to [S94] and [K95] for all of this material. The subcategory $\mathcal{N}il_s$, $s \geq 0$, of \mathcal{U} is the smallest thick subcategory stable under colimits and containing all s -suspensions of unstable modules.

$$\mathcal{U} = \mathcal{N}il_0 \supset \mathcal{N}il_1 \supset \mathcal{N}il_2 \supset \dots \supset \mathcal{N}il_s \supset \dots$$

Proposition 2.1. *Any unstable module M has a convergent decreasing filtration $\{M_s\}_{s \geq 0}$ with $M_s/M_{s+1} \cong \Sigma^s R_s(M)$ where $R_s(M)$ is a reduced unstable module, i.e. which does not contain a non trivial. suspension.*

An unstable module is nilpotent if it belongs to $\mathcal{N}il_1$. The following is proved in [S94] Lemma 6.1.4, see also [K95]:

Proposition 2.2. *Let M be a finitely generated unstable module. Then the $R_i(M)$ are finitely generated and trivial if i is large enough.*

The category of unstable modules \mathcal{U} , as any abelian category, has a Krull filtration. It is by thick subcategories which are stable under colimits

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \dots \subset \mathcal{U}$$

The category \mathcal{U}_0 is the largest thick sub-category generated by simple objects and stable under colimits. It is the subcategory of locally finite modules, $M \in \mathcal{U}$ is locally finite if the span over \mathcal{A}_p of any $x \in M$ is finite. Having defined by induction \mathcal{U}_n one defines \mathcal{U}_{n+1} as follows. One introduces the quotient category $\mathcal{U}/\mathcal{U}_n$ whose objects are the same of those of \mathcal{U} but where morphisms in \mathcal{U} that have kernel and cokernel in \mathcal{U}_n are formally inverted. Then $(\mathcal{U}/\mathcal{U}_n)_0$ is defined as above and \mathcal{U}_{n+1} is the pre-image of this subcategory in \mathcal{U} via the canonical projection functor, see [Gab62] for details. One has

Theorem 2.3. *Let $M \in \mathcal{U}$ and $K_n(M)$ be the largest sub-object of M that is in \mathcal{U}_n , then*

$$M = \cup_n K_n(M)$$

Let us give some examples:

- $\Sigma^k F(n) \in \mathcal{U}_n \setminus \mathcal{U}_{n-1}$, the unstable modules $F(n)$ are the canonical generators of \mathcal{U} , generated in degree n by ι_n and \mathbb{F}_p -basis $P^I \iota_n$, I an admissible multi-index of excess less than n ;
- $\Lambda^n(F(1)), F(1)^{\otimes n} \in \mathcal{U}_n \setminus \mathcal{U}_{n-1} \dots$

There is a characterization of the Krull filtration in terms of the functor \bar{T} introduced by Jean Lannes. The functor T_V , V being an elementary abelian p -group, is left adjoint to $M \mapsto H^*BV \otimes M$. If $V = \mathbb{F}_2$ denote it by T . As $H^*B\mathbb{Z}/2$ splits up, in \mathcal{U} , as $\mathbb{F}_2 \oplus \tilde{H}^*B\mathbb{Z}/2$ the functor T is naturally equivalent to $Id \oplus \bar{T}$. Below are the main properties of T_V :

Theorem 2.4 ([La92, S94]). *The functor T_V commutes with colimits (as a left adjoint). It is exact. Moreover there is a canonical isomorphism*

$$T_V(M \otimes N) \cong T_V(M) \otimes T_V(N)$$

If $M_2 = \Sigma\mathbb{F}_2$ it writes as $T_V(\Sigma M) \cong \Sigma T_V(M)$.

Below is the characterisation of the Krull filtration alluded to above:

Theorem 2.5. *The following two conditions are equivalent:*

- $M \in \mathcal{U}_n$,
- $\bar{T}^{n+1}(M) = \{0\}$.

A very nice proof of this result is in [K13], as well as in Nguyen The Cuong's thesis [NTC14].

Corollary 2.6. *If $M \in \mathcal{U}_m$ and $N \in \mathcal{U}_n$ then $M \otimes N \in \mathcal{U}_{m+n}$*

Let \mathcal{F} be the category of functors from finite dimensional \mathbb{F}_2 -vector spaces to all vector spaces. Define a functor, [HLS93], $f: \mathcal{U} \rightarrow \mathcal{F}$ by

$$f(M)(V) = \text{Hom}_{\mathcal{U}}(M, H^*(BV))^* = T_V(M)^0$$

Let \mathcal{F}_n be the sub-category of polynomial functors of degree less than n . It is defined as follows. Let $F \in \mathcal{F}$, let $\Delta(F) \in \mathcal{F}$ defined by

$$\Delta(F)(V) = \text{Ker}(F(V \oplus \mathbb{F}_p) \rightarrow F(V))$$

Then by definition $F \in \mathcal{F}_n$ if and only if $\Delta^{n+1}(F) = 0$. As an example $V \mapsto V^{\otimes n}$ is in \mathcal{F}_n . The following holds for any $M \in \mathcal{U}$

$$\Delta(f(M)) \cong f(\bar{T}(M)) \cong T_V^0(M)$$

Thus, the diagram below commutes:

$$\begin{array}{ccccccc} \mathcal{U}_0 & \hookrightarrow & \dots & \hookrightarrow & \mathcal{U}_{n-1} & \hookrightarrow & \mathcal{U}_n & \hookrightarrow & \mathcal{U} \\ \downarrow f & & & & \downarrow f & & \downarrow f & & \downarrow f \\ \mathcal{F}_0 & \hookrightarrow & \dots & \hookrightarrow & \mathcal{F}_{n-1} & \hookrightarrow & \mathcal{F}_n & \hookrightarrow & \mathcal{F} \end{array}$$

An injective unstable module always splits up as the direct sum of a reduced module and of a nilpotent one, moreover (by definition) there are no non trivial maps from a nilpotent module to a reduced one ([S94] chapter 2 and 3). Thus, any injective resolution \mathcal{I}^\bullet in the \mathcal{U} of an unstable module M has the following properties.

Proposition 2.7. *For any k \mathcal{I}^k writes as a direct sum $\mathcal{R}^k \oplus \mathcal{N}^k$, the first module being reduced and the second one nilpotent. The differential $\partial_k: \mathcal{I}^k \rightarrow \mathcal{I}^{k+1}$ writes as $\begin{pmatrix} \partial_r^k & 0 \\ \rho^k & \partial_n^k \end{pmatrix}: \mathcal{R}^k \oplus \mathcal{N}^k \rightarrow \mathcal{R}^{k+1} \oplus \mathcal{N}^{k+1}$. The k -th cohomology module of the quotient complex \mathcal{R}^\bullet is (by definition) the k -th derived functor $\ell^k(M)$ of the localization functor away from $\mathcal{N}il$ applied to M .*

The functor f has a right adjoint m , the composition $\ell = m \circ f$ is the localization functor away from $\mathcal{N}il$. The natural map $M \rightarrow \ell(M)$ is initial for $M \rightarrow L$, L being $\mathcal{N}il$ -closed i.e. such that $\text{Ext}_{\mathcal{U}}^i(N, L) = \{0\}$ for $i = 0, 1$, any $N \in \mathcal{N}il$ [Gab62]. In particular the localization of a nilpotent unstable module is trivial. It follows that:

$$H^{k+1}(\mathcal{N}^\bullet) \cong \ell^k(M)$$

and the modules $\ell^k(M)$, $k \geq 1$, are nilpotent (this can be seen directly). Moreover as the functor T preserves reduced modules and nilpotent modules it commutes with ℓ one gets $T(\ell(M)) \cong \ell(T(M))$. Thus because of 2.5

Proposition 2.8. *If $M \in \mathcal{U}_n$, then $\ell^i(M) \in \mathcal{U}_{n-1}$, if $i > 0$.*

As the tensor product of reduced injective unstable modules is still injective on has:

Corollary 2.9 (Kunneth formula). *Let M and N be two unstable modules then:*

$$\ell^k(M \otimes N) \cong \bigoplus_{i+j=k} \ell^i(M) \otimes \ell^j(N)$$

One keeps the notation introduced above. Assume the resolution to be minimal, \mathcal{I}^{k+1} is the injective hull of $\text{coker}(\partial^{k-1})$. On the other hand \mathcal{R}^{k+1} is the injective hull of the quotient R^k of $\text{coker}(\partial_r^{k-1})$ by its largest nilpotent submodule $N(\text{coker}(\partial_r^{k-1}))$. The results which follow are standard homological algebra.

Proposition 2.10. *Let N^{rk} be the largest nilpotent submodule of $\text{coker}(\partial^{k-1})$. One has the following exact sequence:*

$$\{0\} \rightarrow \ell^k(M) \rightarrow \text{coker}(\partial_n^{k-1}) \rightarrow N^{rk} \rightarrow \ell^{k+1}(M) \rightarrow \{0\}$$

If the resolution is minimal the unstable module R^{k+1} is the injective hull of R^k , N^{k+1} is the injective hull of N^{rk} .

If one is given an injective resolution \mathcal{I}^\bullet of an unstable module M , $f(\mathcal{I}^\bullet)$ is an injective resolution of $f(M)$. If moreover one assumes M to be finitely generated then $f(M)$ is a finite functor, using Kuhn's terminology: it is polynomial and takes finite dimensional values [K94-1]. Then 1.1, implies that a minimal resolution of $f(M)$ is of finite type. This implies easily that the reduced part of a minimal resolution of M is of finite type and in fact:

Proposition 2.11. *Let M be a finitely generated unstable module, and \mathcal{I}^\bullet a minimal injective resolution. Then $f(\mathcal{I}^\bullet)$ is a minimal injective resolution of $f(M)$.*

As said it follows from what precedes and in particular if R is a reduced injective unstable module the localization $R \rightarrow \ell(R)$ is an isomorphism, also if I is injective analytic functor [HLS93] then $f \circ m(I) \rightarrow I$ is an equivalence.

3 Proofs

One formulates 3.1 in a slightly different way than in the introduction, but equivalent.

Theorem 3.1. *Let M be a finitely generated unstable module, there is an injective resolution \mathcal{I}^\bullet of M in \mathcal{U} such that any \mathcal{I}^k is a finite direct sum of modules of the type $J(n) \otimes H^*V$.*

In the introduction it is formulated for a minimal injective resolution. In this setting the result will be that \mathcal{I}^k is a finite direct sum of indecomposable injective unstable \mathcal{A}_p -modules. Such modules are known to be of the form $J(n) \otimes L_\lambda$, L_λ being an indecomposable factor of some H^*V , [LS89] and see below. The two formulations are equivalent because H^*V is a finite direct sum of indecomposable modules. The gain with the first one is that allows to use the functor T_V , instead of the division functor by indecomposable factors L_λ (see below). For what is necessary to the proofs all of these functors share the same essential properties, however T_V is much more manageable.

The indecomposable injective unstable modules are, as is said above, of the form $J(n) \otimes L_\lambda$, where L_λ is a direct factor in some H^*V . The isomorphism classes of indecomposable reduced injective unstable modules are indexed by the simple representations over \mathbb{F}_2 of the groups $\mathbb{GL}_n(\mathbb{F}_2)$. These representations are themselves indexed by 2-regular partitions λ , 2-regular partitions are the strictly decreasing ones. The unstable module $J(n) \otimes L_\lambda$ is the injective hull in \mathcal{U} of of the unstable module $\Sigma^n S_\lambda(F(1))$; where S_λ is the simple functor in the category \mathcal{F} associated to λ , this functor is of degree $|\lambda|$ ([PS98, Dj07, K94-2]).

Theorem 3.2. *Let $M \in \mathcal{U}_n$ be finitely generated, the derivatives of ℓ , for $i > 0$, $\ell^i(M)$ belong to \mathcal{U}_{n-1} . Moreover the unstable modules $\ell^i(M)$ are finitely generated unstable modules.*

The first part of the theorem is classical and follows from 2.5 and the commutation of T with ℓ .

The next result is the stabilization theorem:

Theorem 3.3. *Let M and N be two finitely generated unstable modules. Then for any i the \mathbb{F}_2 -vector space $\text{Ext}_{\mathcal{F}}^i(f(M), f(N))$ is the colimit over k of the \mathbb{F}_2 -vector spaces $\text{Ext}_{\mathcal{U}}^i(\Phi^k M, N)$. Moreover for k large enough the colimit, along the maps $\Phi^{k+1}(M) \rightarrow \Phi^k(M)$, of the system:*

$$\dots \rightarrow \dots \text{Ext}_{\mathcal{U}}^i(\Phi^k M, N) \rightarrow \text{Ext}_{\mathcal{U}}^i(\Phi^{k+1} M, N) \rightarrow \dots$$

moreover for k large enough:

$$\text{Ext}_{\mathcal{U}}^i(\Phi^k M, N) \cong \text{Ext}_{\mathcal{F}}^i(f(M), f(N))$$

The proofs of 3.1 and 3.2 will be done at the same time by induction over the Krull filtration. To prove 3.1 it is enough to prove:

Proposition 3.4. *Let M be a finitely generated unstable module and k an integer. There exists only a finite number of unstable modules $\Sigma^n S_\lambda(F(1))$ such that $\text{Ext}_{\mathcal{U}}^k(\Sigma^n S_\lambda(F(1)), M)$ is non trivial, and if it is non-trivial it is finite dimensional.*

Indeed, $J(n) \otimes L_\lambda$ is the injective hull of $\Sigma^n S_\lambda(F(1))$, and the dimension of the \mathbb{F}_2 -vector space $\text{Ext}_{\mathcal{U}}^k(\Sigma^n S_\lambda(F(1)), M)$ is greater or equal to the number of occurrences of $J(n) \otimes L_\lambda$ in the term \mathcal{I}^k of a minimal resolution of M .

First step of the proofs of 3.1 and 3.2: \mathcal{U}_0 .

Lemma 3.5. *Both 3.1 and 3.2 are true for finitely generated locally finite unstable modules.*

That the lemma is true for finitely generated objects in \mathcal{U}_0 is easy and very classical. The derived functors of the localization are trivial and one can construct a finite resolution of a finitely generated object M in \mathcal{U}_0 because the injective hull E_M of M is also finite and thus finitely generated. Moreover, denote by $v(M)$ the largest integer such that M is trivial in degrees strictly larger than $v(M)$, then $v(E_M/M) < v(M)$. This allows to show that minimal resolution are finite.

Second step of the proofs of 3.1 and 3.2: $F(1)$.

Lemma 3.6. *Both 3.1 and 3.2 is true for the unstable module $F(1)$.*

This relies on computations in Mac Lane homology and explained in great details in [NTC14]. We give some informations below that are enough for our purpose.

The unstable modules $\ell^i(F(1))$ are known to belong to \mathcal{U}_0 2.8, it remains to prove they are finitely generated. In fact, they are explicitly known, computations depending on Mac Lane homology show that they are finite and thus finitely generated. This follows from theorem 12.13 of "Algèbre de Steenrod, modules instables et foncteurs polynômiaux", [ER03]. This allows to have some control on the \mathcal{N}^k using 2.10 and to prove by induction they are finite. It follows directly from these results that the nilpotent part of the minimal resolution of $F(1)$ is in each degree a finite direct sum of Brown-Gitler modules.

It remains to show the result for the reduced part: that \mathcal{R}^k of the minimal resolution of $F(1)$ is a finite direct sum of indecomposable injective unstable modules. This is a consequence of the corresponding result in the category \mathcal{F} 1.1, 2.11.

Below are given some more informations on the reduced part of the resolution. One starts by a theorem which is proved in [NTC14]. It is not necessary here but worth to be mentioned.

Proposition 3.7 (A. Touzé [To13, 4.18]). *Let S be a simple functor, if $\text{Ext}_{\mathcal{F}}^*(S, Id)$ is non trivial then the degree of S must be a power of 2.*

One offers a proof different from the one of [To13], it depends on:

Lemma 3.8. *Let λ be a 2-regular partition, such that $|\lambda| = n$. The functor $\Lambda^\lambda \cong \Lambda^{\lambda_1} \otimes \dots \otimes \Lambda^{\lambda_t}$ has a filtration whose sub-quotients are either simple functors of degree m such that there exists h with $2^h m = n$, either tensor products $F \otimes G$ of functors with no constant part ($F(\{0\}) = G(\{0\}) = \{0\}$).*

To prove the proposition consider $\Lambda^\lambda \cong \Lambda^{\lambda_1} \otimes \dots \otimes \Lambda^{\lambda_t}$ as a strict polynomial functor and use the classification of simple objects in \mathcal{P} . It follows from a theorem of Steinberg they are of the form, [K02]

$$S_{\lambda_1}^{(i_1)} \otimes S_{\lambda_1}^{(i_2)} \otimes \dots \otimes S_{\lambda_t}^{(i_t)}$$

where $\lambda^1, \dots, \lambda^t$ are 2-regular partitions, $i_1 < i_2 < \dots < i_t$, and $S_{\lambda}^{(i)}$ is the i -th Frobenius twist of the canonical lifting of the simple functor $S_\lambda \in \mathcal{F}$ to \mathcal{P} .

The following more precise form of 3.8 is necessary to complete the proof and will be necessary later:

Lemma 3.9. *Let λ be a 2-regular partition of the integer n . The composition series of the strict polynomial functor Λ^λ has*

- one sub-quotient S_λ ,

- all other simple sub-quotients of degree n as polynomial functors are S_μ with $\mu > \lambda$, for the natural order on partitions;
- Frobenius twists of functors $S_\nu^{(r)}$, with $2^r|\nu| = n$;
- non-trivial tensor products.

This result can be deduced easily from various publications, in particular [PS98, Dj07] which do not claim for originality.

Proposition 3.7 follows by induction. As if F and G are functors with no constant part, $\text{Ext}_{\mathcal{F}}^*(F \otimes G, Id)$ is trivial, if $\text{Ext}_{\mathcal{F}}^*(\Lambda^n, Id)$ is trivial, $\text{Ext}_{\mathcal{F}}^*(S_\lambda, Id)$ will be for all simple functors of degree n by an increasing induction on the degree and a descending one on the 2-regular partitions of n . The result follows because $\text{Ext}_{\mathcal{F}}^*(\Lambda^n, Id)$ is trivial if (and only if) n is not a power of 2 [FLS94].

Third step of the proofs of 3.1 and 3.2: tensor products and \mathcal{U}_1 .

Proposition 3.10. *If 3.1 and 3.2 hold for M and N , they hold for the tensor product $M \otimes N$.*

Theorem 3.2 is true in this case, because of 2.9, it is true for the tensor product $M \otimes N$.

For theorem 3.1, let \mathcal{I}^\bullet and \mathcal{J}^\bullet be injective resolutions of M and N having the required property. The tensor product $\mathcal{H}^\bullet = \mathcal{I}^\bullet \otimes \mathcal{J}^\bullet$ is not an injective resolution of $M \otimes N$. Construct a Cartan-Eilenberg resolution $\mathcal{H}^{\bullet, \bullet}$ of \mathcal{H}^\bullet . Applying to this double complex the functor $\text{Hom}_{\mathcal{U}}(S, -)$ yields an hypercohomology spectral sequence converging to $\text{Ext}_{\mathcal{U}}^{p+q}(S, M \otimes N)$ with E_2 -term $\text{Ext}_{\mathcal{U}}^p(S, \mathcal{H}^q)$.

The unstable module \mathcal{H}^q is a finite direct sum of modules $J(k) \otimes J(\ell) \otimes H^*V$. Thus, the group $\text{Ext}_{\mathcal{U}}^p(S, \mathcal{H}^q)$ is isomorphic to a finite direct sum of groups $\text{Ext}_{\mathcal{U}}^p(T_V(S), J(k) \otimes J(\ell))$, the \mathbb{F}_2 -vector space V being of bounded dimension as there are only finitely many factors. For $i = p + q$ let d be an upper bound of the dimensions.

Suppose now that S is of the form $\Sigma^n S_\lambda(F(1))$. These groups are trivial as soon as $n > \text{sup}(k + \ell)$ or as soon as the connectivity of $T_V(S_\lambda(F(1)))$, $c(T_V(S_\lambda(F(1))))$ is greater than $\text{sup}(k + \ell)$.

Lemma 3.11. *The connectivity of $S_\lambda(F(1))$, $c(S_\lambda(F(1)))$, is $\lambda'_1 + 2\lambda'_2 + \dots + 2^{t-1}\lambda'_{t-1} - 1 \geq \lambda'_1 - 1$, λ' being the associated partition.*

Using the properties of $T_V \cong T^{\dim(V)}$, the decomposition $T \cong \mathbb{F}_2 \oplus \bar{T}$. and the fact that $S_\lambda(F(1)) \subset \tilde{H}^*(\mathbb{Z}/2)^{\lambda'1}$, one shows that $c(T_V(S_\lambda(F(1)))) \geq \lambda'_1 - d - 1$.

As a consequence, for a given V only a finite number of reduced simple unstable modules $T_V S_\lambda(F(1))$ are of connectivity less than a given constant.

The result follows.

This proves the theorems for \mathcal{U}_1 . Indeed, they are true for \mathcal{U}_0 , $F(1)$ and by tensor product for any $\Sigma^n F(1)$, and any finitely generated object $M \in \mathcal{U}_1$ enters in a short exact sequence [S98], see also [K13]:

$$\{0\} \rightarrow L' \rightarrow M \rightarrow L \otimes F(1) \rightarrow L'' \rightarrow \{0\}$$

where $L, L', L'' \in \mathcal{U}_0$ and are finitely generated (see [K13] for a generalization). The result follows.

From now on one assumes that the theorems has been proved for objects in \mathcal{U}_{n-1} .

Fourth step of the proofs of 3.1 and 3.2: The case of $\Lambda^n(F(1))$.

The following step is the case of exterior powers. One assume the theorems hold for $\Lambda^k(F(1))$, and it holds for $\Lambda^{k+1}(F(1))$.

If k is even $\Lambda^{k+1}(F(1))$ is a direct summand in $\Lambda^k(F(1)) \otimes F(1)$, and 3.10 implies that 3.1 and 3.2 hold for $\Lambda^{k+1}(F(1))$.

If k is odd the situation is more complicate, one has short exact sequences:

$$\{0\} \rightarrow W_{(k,1)}(F(1)) \rightarrow \Lambda^k(F(1)) \otimes F(1) \rightarrow \Lambda^{k+1}(F(1)) \rightarrow \{0\}$$

$$\{0\} \rightarrow \Lambda^{k+1}(F(1)) \rightarrow W_{(k,1)}(F(1)) \rightarrow S_{(k,1)}(F(1)) \rightarrow \{0\}$$

and

$$\{0\} \rightarrow \Lambda^{k+1}(F(1)) \rightarrow \Lambda^k(F(1)) \otimes F(1) \rightarrow DW_{(k,1)}(F(1)) \rightarrow \{0\}$$

$$\{0\} \rightarrow S_{(k,1)}(F(1)) \rightarrow DW_{(k,1)}(F(1)) \rightarrow \Lambda^{k+1}(F(1)) \rightarrow \{0\}$$

These exact sequences define $W_{(k,1)}(F(1))$ and $DW_{(k,1)}(F(1))$.

Assume (for the integer k) that $\ell^i(\Lambda^{k+1}(F(1)))$, $\ell^i(S_{(k,1)}(F(1)))$ are finitely generated and show the result for $k + 1$. The same is true for $\ell^i(W_{(k,1)}(F(1)))$ and $\ell^i(DW_{(k,1)}(F(1)))$.

Using the long exact sequences associated to the above exact sequences one first shows that $\ell^i(W_{(k+1,1)}(F(1)))$ is finitely generated. Then one gets the result for $\ell^i(\Lambda^{(k+2)}(F(1)))$, next for $\ell^i(S_{(k+1,1)}(F(1)))$ and finally for $\ell^i(DW_{(k+1,1)}(F(1)))$. This last case is not used in the induction.

The case of the exterior powers follows also from explicit computations [FFSS99].

Last step of the proofs of 3.1 and 3.2.

As the result holds for all exterior powers $\Lambda^k(F(1))$, it holds for tensor products of such modules. Then, 3.9 implies it holds for any module $S_\lambda(F(1))$ $|\lambda| = n$ and using 3.10 for all $\Sigma^n S_\lambda(F(1))$.

To finish the proof one uses the following two lemmas and the induction hypothesis on the Krull filtration.

Lemma 3.12. *If a reduced unstable module M belongs to \mathcal{U}_n , ΩM belongs to \mathcal{U}_{n-1} .*

The proof is left to the reader.

Lemma 3.13. *If the theorem holds for a reduced unstable module M of Krull filtration n , and for all unstable of Krull filtration $n - 1$ it holds for any submodule N of M so that there exist ℓ with $\Phi^\ell M \subset N \subset M$.*

This is because $M/\Phi^k M \in \mathcal{U}_{n-1}$.

The proof ends using 2.2 and the following:

Proposition 3.14. *A finitely generated reduced unstable R of Krull filtration n has a finite filtration whose quotients are suspensions of reduced modules of Krull filtration less than n , and whose associated functors are simple (of degree less than n).*

This means that the sub-quotients are or the form $\Sigma^n R$, for some R so that $\Phi^k S_\lambda \subset R \subset S_\lambda$ for some λ . The required filtration is obtained as follows. One considers the localization $R \rightarrow \ell(R)$, following Kuhn, [K94-2], one knows that $f(R) = f(\ell(R))$ has a finite composition series. The proof is done by induction on the length of the composition series of $f(R)$. Consider an epimorphism on a simple functor: $f(R) \rightarrow S_\lambda$, and the associated unstable modules map $R \rightarrow S_\lambda(F(1))$, and its kernel K which is reduced, and to which one can apply the induction hypothesis.

To finish the proof of the theorem one applies the preceding lemma.

Proof of 3.3 and of the corollary

The proof of 3.3 follows directly from 3.1 from the fact the properties of the adjoint $\tilde{\Phi}$ of Φ , and in particular from the computation $\tilde{\Phi}(L \otimes J(n)) \cong L \otimes J(n/2)$, (L a reduced injective, and $J(n/2)$ is trivial if n is not an integer). If \mathcal{N}^i is a finite direct sum it is clear that for k large enough $\tilde{\Phi}^k(\mathcal{N}^i)$ is trivial. The use of $\tilde{\Phi}$ can be replaced by:

Proposition 3.15. *Let M be a finitely generated unstable modules, assume N is nilpotent and has finite nilpotent filtration. If k is large enough $\text{Hom}_{\mathcal{U}}(\Phi^k(M), N) \cong \{0\}$.*

Note that in particular a finitely generated module has a finite nilpotent filtration [S94], [K95]. Suppose $N_t\{0\} \cong \{0\}$ for $t > t_0$. The unstable module $\Omega\Phi^k(M)$ is isomorphic to $\Sigma^{2^k-1}\Phi^k\Omega M$, and has trivial image in N if $2^k - 1 > t_0$.

For the corollary, one observes that 3.3 allows to reduce to a computation in the category \mathcal{F} . It is enough to show the following:

Proposition 3.16. *Let F be a polynomial functor of degree n , such that $F(\{0\}) = \{0\}$, R the smallest sub-functor such that F/R is of degree $n - 1$. Then $\text{Ext}_{\mathcal{F}}^1(R \otimes R, F) \cong \{0\}$.*

This last result is proved using the category $bi - \mathcal{F}$, [FFSS99]. This result is used in[CGS14].

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