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► **To cite this version:**

| Frédéric Lavancier, Paul Rochet. A general procedure to combine estimators. 2014. hal-00936024v3

HAL Id: hal-00936024

<https://hal.science/hal-00936024v3>

Preprint submitted on 3 Oct 2014 (v3), last revised 12 Mar 2015 (v4)

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A general procedure to combine estimators

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Abstract

We propose a general method to combine several estimators of the same quantity in order to produce a better estimate. In the spirit of model and forecast averaging, the final estimator is computed as a weighted average of the initial ones, where the weights are constrained to sum to one. In this framework, the optimal weights, minimizing the quadratic loss, are entirely determined by the mean square error matrix of the vector of initial estimators. The averaging estimator is derived using an estimation of this matrix, which can be computed from the same dataset. We prove a non-asymptotic error bound on the averaging estimator and we show that it is asymptotically optimal under mild conditions on the estimated mean square error matrix. This method is illustrated on standard statistical problems in parametric and semi-parametric models where the averaging estimator outperforms the initial estimators in most cases.

Keywords. Averaging ; Parametric estimation ; Weibull model ; Boolean model

1 Introduction

We are interested in estimating a parameter θ in a statistical model, based on a collection of preliminary estimators T_1, \dots, T_k . In general, the relative performance of each estimator depends on the true value of the parameter, the sample size, or other unknown factors, in which case deciding in advance what method to favor can be difficult. This situation occurs in numerous problems of modern statistics like forecasting or non-parametric regression, but it remains a major concern even in simple parametric problems. In this paper, we study a general methodology to combine linearly several estimators in order to produce a final single better estimator. In particular, we address the problem of inference for a parametric distribution and for a parametric stochastic process, in presence of several competing estimators.

The issue of dealing with several possibly competing estimators of the same quantity has been extensively studied in the literature these past decades. One of the main solution retained is to consider a weighted average of the T_i 's. The idea of estimator averaging actually goes back to the early 19th century with Pierre Simon de Laplace [19], who was interested in finding the best combination between the mean and the median to estimate the location parameter of a

symmetric distribution, see the discussion in [29]. More generally, the solution can be expressed as a linear combination of the initial estimators

$$\hat{\theta}_\lambda = \lambda^\top \mathbf{T} = \sum_{i=1}^k \lambda_i T_i, \quad (1)$$

for λ a vector of weights lying in a subset Λ of \mathbb{R}^k and $\mathbf{T} = (T_1, \dots, T_k)^\top$. A large number of statistical frameworks fit with this description. For example, model selection can be viewed as a particular case for Λ the set of vertices. Similarly, convex combinations corresponds to the simplex $\Lambda = \{\lambda : \sum_{i=1}^k \lambda_i = 1, \lambda_i \geq 0\}$ while linear combinations to $\Lambda = \mathbb{R}^k$. Another well-used framework consists in relaxing the positivity condition of convex combination, corresponding to the set $\Lambda = \{\lambda : \sum_{i=1}^k \lambda_i = 1\}$.

Estimator averaging has received a particular attention for prediction purposes. Ever since the paper of Bates and Granger [2], dealing with forecast averaging for time series, the literature on this subject has greatly developed, see for instance [10, 30] in econometrics and [7] in machine learning. In this framework, the parameter θ represents the future observation of a series to be predicted and $\mathbf{T} = (T_1, \dots, T_k)$ a collection of predictors. Averaging methods have also been widely used for prediction in a regression framework. In this case, θ is the response variable to predict given some regressors, and \mathbf{T} is a collection of models output. These so-called model averaging procedures are shown to provide good alternatives to model selection for parametric regression, see [23] for a survey. Model averaging has been studied in both Bayesian [26, 31] and frequentist contexts [3, 15, 13]. In closed relation, functional aggregation deals with the same problem in non-parametric regression [24, 34, 9, 4]. Aggregation methods have also been extensively studied for density estimation, as an alternative to classical bandwidth selection methods [27, 5, 33, 6].

In [13], Hansen introduced a least squares model average estimator, in the same spirit as the forecast average estimator proposed in [2]. Loosely speaking, this estimator aims to mimic the oracle, defined as the linear combination $\hat{\theta}_\lambda$ that minimizes the quadratic loss $\mathbb{E}(\hat{\theta}_\lambda - \theta)^2$, under the constraint on the weights $\sum_{i=1}^k \lambda_i = 1$. Under this constraint, the oracle expresses in terms of the mean squared error matrix Σ of \mathbf{T} . The averaging estimator is then defined by replacing Σ by an estimator $\hat{\Sigma}$.

The main objective of this paper is to apply the latter idea to classical estimation problems, not restricted to prediction. Although it can be applied to non-parametric models, our procedure is essentially designed for parametric or semi-parametric models, where the number k of available estimators is small compared to the sample size n and does not vary with n . The procedure works well in these situations because the estimation of Σ can be carried out efficiently by standard methods (e.g. plug-in or Monte-Carlo), and does not require the tuning of extra parameters. While it recovers some results of [11, 12, 20] and more recently [18] on estimator averaging for the mean in a Gaussian model, the method applies to a wide range of statistical models. It is implemented in Section 4 on three other examples. In the first one, θ represents the position of an unknown distribution, which can be estimated by both the empirical mean and median, as initially addressed by P. S. de Laplace in [19]. In the second example, θ is the

two-dimensional parameter of a Weibull distribution, for which several competing estimators exist. In the third one, we consider a stochastic process, namely the Boolean model, that also depends on a two-dimensional parameter and we apply averaging to get a better estimate.

An important contribution of our approach is to include the case where several parameters $\theta_1, \dots, \theta_d$ have to be estimated, and a collection of estimators is available for each of them. In order to fully exploit the available information to estimate say θ_1 , it may be profitable to average all estimators, including those designed for θ_j , $j \neq 1$. We show that a minimal requirement is that the weights associated to the latter estimators sum to 0, while the weights associated to the estimators of θ_1 sum to one (additional constraints on the weights can also be added as discussed in Section 2.3). To our knowledge, estimator averaging including estimators of other parameters is a new idea. Our simulations study shows that it can produce spectacular results in some specific situations such as the Boolean model treated in the third example of Section 4.

From a theoretical point of view, we provide an upper bound on the deviation of the averaging estimator to the oracle. Our result is non-asymptotic and involves the error to the oracle for the actual event, in contrast with usual criteria based on expected loss functions. In particular, our result strongly differs from classical oracle inequalities derived in the literature on aggregation for non-parametric regression [4, 9, 17, 34] or density estimation [5, 6, 27, 33]. Moreover, we deduce that under mild assumptions, our averaging estimator behaves asymptotically as the oracle, generalizing the asymptotic optimality result proved by Hansen and Racine [14] in the frame of model averaging where Σ is estimated by jackknife. Our result applies in particular if $\sqrt{n}(\mathbf{T} - \theta)$ converges in mean square to a gaussian law and a consistent estimator of the asymptotic covariance matrix is available, though these conditions are far from being necessary. This situation makes it possible to construct an asymptotic confidence interval based on the averaging estimator, the length of which is necessarily smaller than all confidence intervals based on the initial estimators.

The averaging procedure is detailed in Section 2, where we give some examples for the choice of the set of weights Λ , or equivalently of the constraints followed by the weights, and we detail some natural methods for the estimation of Σ . In Section 3 we prove a non-asymptotic bound on the error to the oracle and discuss the asymptotic optimality of the averaging estimator. Section 4 is devoted to some examples of problems, where we show that the method performs almost always better than the best estimator in the initial collection \mathbf{T} . These examples deal with the estimation of the position of a symmetric distribution, estimation in a Weibull model and a Boolean model. Proofs of our results are postponed to the Appendix.

2 The averaging procedure

The method is different whether it is applied to one parameter or several. For ease of comprehension, we first present the averaging procedure for one parameter, which follows the idea introduced in [2] for forecast averaging, though our choice of the set of weights Λ may be different. We then introduce a generalization of the procedure for averaging several parameters

simultaneously. Finally, we discuss in Sections 2.3 and 2.4 the choice Λ and the construction of $\hat{\Sigma}$.

2.1 Averaging for one parameter

Let $\mathbf{T} = (T_1, \dots, T_k)^\top$ be a collection of estimators of a real parameter θ . We search for a decision rule that combines suitably the T_i 's to provide a unique estimate of θ . Aiming for the best unconstrained transformation $f(\mathbf{T})$ is pointless since the solution is the trivial $f(\mathbf{T}) = \theta$. A reasonable alternative is to settle for linear transformations

$$\hat{\theta}_\lambda = \lambda^\top \mathbf{T}, \quad \lambda \in \Lambda,$$

where λ^\top denotes the transpose of λ and Λ is a given subset of \mathbb{R}^k . In this linear setting, a convenient way to measure the performance of $\hat{\theta}_\lambda$ is to compare it to the *oracle* $\hat{\theta}^*$, defined as the best linear combination $\hat{\theta}_\lambda$ obtained for a non-random vector $\lambda \in \Lambda$. Specifically, the oracle is the linear combination $\hat{\theta}^* = \lambda^{*\top} \mathbf{T}$ minimizing the mean square error (MSE), i.e.

$$\lambda^* = \arg \min_{\lambda \in \Lambda} \mathbb{E}(\lambda^\top \mathbf{T} - \theta)^2.$$

Of course, λ^* is unknown in practice and needs to be approximated by an estimator, say $\hat{\lambda}$.

Clearly, the larger the set Λ , the better the oracle. However, choosing the whole space $\Lambda = \mathbb{R}^k$ is generally not exploitable. Indeed, assuming that the Gram matrix $\mathbb{E}(\mathbf{T}\mathbf{T}^\top)$ exists and is non-singular, the oracle over $\Lambda = \mathbb{R}^k$ is given by

$$\lambda_{\text{lin}}^* = \arg \min_{\lambda \in \mathbb{R}^k} \mathbb{E}(\lambda^\top \mathbf{T} - \theta)^2 = \theta [\mathbb{E}(\mathbf{T}\mathbf{T}^\top)]^{-1} \mathbb{E}(\mathbf{T}).$$

For the solution $\hat{\theta} = \hat{\lambda}^\top \mathbf{T}$ to be comparable to the oracle, we need to be able to approach the optimal weights better than we can estimate θ . The presence of θ in the above expression shows that λ_{lin}^* should be at least as difficult to estimate as θ , rendering linear averaging inefficient. In fact, the performance of the average highly relies on the choice of the set Λ . Indeed, choosing a too large set Λ might increase the accuracy of the oracle but make it difficult to estimate λ^* . On the contrary, a too small set Λ might lead to a poorly efficient oracle but easy to approximate. Therefore, a good balance must be found for the oracle to be both accurate and reachable.

Writing the estimation error as

$$\hat{\theta} - \theta = \hat{\theta}^* - \theta + (\hat{\lambda} - \lambda^*)^\top \mathbf{T}, \quad (2)$$

a good rule of thumb is to consider a set Λ for which the residual term $(\hat{\lambda} - \lambda^*)^\top \mathbf{T}$ can be made negligible compared to the error of the oracle $\hat{\theta}^* - \theta$. In this purpose, a solution proposed in [2] and widely used in the averaging literature is to consider the affine constraint set

$$\Lambda_{\text{max}} = \{\lambda \in \mathbb{R}^k : \lambda^\top \mathbf{1} = 1\},$$

where $\mathbf{1}$ denotes the unit vector $\mathbf{1} = (1, \dots, 1)^\top$. If λ^* and $\hat{\lambda}$ are in Λ_{max} , the error term can be written as

$$(\hat{\lambda} - \lambda^*)^\top \mathbf{T} = (\hat{\lambda} - \lambda^*)^\top (\mathbf{T} - \theta \mathbf{1}),$$

where both $(\hat{\lambda} - \lambda^*)$ and $(\mathbf{T} - \theta \mathbf{1})$ can contribute to make the error term negligible. This is confirmed by the theoretical result stated later in Corollary 3.2.

In the sequel, we let Λ be a non-empty closed subset of Λ_{\max} . Some examples for the choice of Λ are discussed in Section 2.3. We assume that the initial estimators have finite order-two moments and $\mathbf{1}, T_1, \dots, T_k$ are linearly independent so that the Gram matrix

$$\Sigma = \mathbb{E}[(\mathbf{T} - \theta \mathbf{1})(\mathbf{T} - \theta \mathbf{1})^\top]$$

is well defined and non-singular. From the identity $\lambda^\top \mathbf{1} = 1$, we see that the optimal weight λ^* defining the oracle $\hat{\theta}^* = \lambda^{*\top} \mathbf{T}$ writes

$$\lambda^* = \arg \min_{\lambda \in \Lambda} \mathbb{E}(\lambda^\top \mathbf{T} - \theta)^2 = \arg \min_{\lambda \in \Lambda} \lambda^\top \Sigma \lambda.$$

Remark that the assumptions made on Λ ensure the existence of a minimizer. If Λ is convex, the solution is unique, otherwise we agree that λ^* refers to one of the minimizers. In the particular important example where $\Lambda = \Lambda_{\max}$, we get the well known explicit solution

$$\lambda_{\max}^* = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}},$$

considered for instance in [2], [10] or [14]. Of course, in practice, the MSE matrix Σ is unknown and has to be approximated by some estimator $\hat{\Sigma}$ to yield the averaging estimator $\hat{\theta} = \hat{\lambda}^\top \mathbf{T}$, where

$$\hat{\lambda} = \arg \min_{\lambda \in \Lambda} \lambda^\top \hat{\Sigma} \lambda.$$

Natural methods to construct $\hat{\Sigma}$ are discussed in Section 2.4. While it may seem paradoxical to shift our attention from θ to the less accessible Σ , the effectiveness of the averaging process can be explained by a lesser sensibility to the errors on $\hat{\Sigma}$. As a result, the averaging estimator improves on the original collection as soon as we are able to build $\hat{\Sigma}$ sufficiently close from the true value, without stronger requirement such as consistency. On the contrary, the chances of considerably deteriorating the estimation of θ are expected to be small due to the smoothing effect of averaging.

2.2 Averaging for several parameters

We now discuss a generalization of the method that deals with several parameters simultaneously. Let $\theta = (\theta_1, \dots, \theta_d)^\top \in \mathbb{R}^d$ and assume we have access to a collection of estimators \mathbf{T}_j for each component θ_j . For sake of generality we allow the collections $\mathbf{T}_1, \dots, \mathbf{T}_d$ to have different sizes k_1, \dots, k_d with $k_j \geq 1$. So, let $\mathbf{T}_1 \in \mathbb{R}^{k_1}, \dots, \mathbf{T}_d \in \mathbb{R}^{k_d}$ and note $\mathbf{T} = (\mathbf{T}_1^\top, \dots, \mathbf{T}_d^\top)^\top \in \mathbb{R}^k$, with $k = \sum_{j=1}^d k_j$. We consider averaging estimators of θ of the form

$$\hat{\theta}_\lambda = \lambda^\top \mathbf{T} \in \mathbb{R}^d,$$

where here, λ is a $k \times d$ matrix. For similar reasons as previously, we choose to make some restrictions on the set of authorized values for λ . In this purpose, define the matrix

$$\mathbf{J} = \begin{pmatrix} \mathbf{1}_{k_1} & 0 & \dots & 0 \\ 0 & \mathbf{1}_{k_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{1}_{k_d} \end{pmatrix} \in \mathbb{R}^{k \times d},$$

where $\mathbf{1}_{k_j}$ is the vector composed of k_j ones (we simply note it $\mathbf{1}$ in the sequel to ease notation). We consider the maximal restriction set

$$\Lambda_{\max} = \{\lambda \in \mathbb{R}^{k \times d} : \lambda^\top \mathbf{J} = \mathbf{I}\}, \quad (3)$$

with \mathbf{I} the identity matrix. Let $\underline{\lambda}_j \in \mathbb{R}^k$ denote the j -th column of $\lambda \in \mathbb{R}^{k \times d}$. For each component θ_j , the average is given by

$$\hat{\theta}_{\lambda,j} = \underline{\lambda}_j^\top \mathbf{T} = \underline{\lambda}_{j,1}^\top \mathbf{T}_1 + \dots + \underline{\lambda}_{j,d}^\top \mathbf{T}_d,$$

where $\underline{\lambda}_j = (\underline{\lambda}_{j,1}^\top, \dots, \underline{\lambda}_{j,d}^\top)^\top$ with $\underline{\lambda}_{j,\ell} \in \mathbb{R}^{k_\ell}$, $\ell = 1, \dots, d$. Imposing that $\lambda \in \Lambda_{\max}$ means that for any $j = 1, \dots, d$

$$\underline{\lambda}_{j,\ell}^\top \mathbf{1} = \begin{cases} 0 & \text{if } \ell \neq j \\ 1 & \text{if } \ell = j. \end{cases} \quad (4)$$

In particular, this condition does not rule out using the entire collection \mathbf{T} to estimate each component θ_j , although the weights $\underline{\lambda}_{j,\ell}$ do not satisfy the same constraints depending on the relevance of \mathbf{T}_ℓ . While it may seem more natural to impose that only \mathbf{T}_j is involved in the estimation of θ_j (and this can be made easily through an appropriate choice of $\Lambda \subset \Lambda_{\max}$, letting $\underline{\lambda}_{j,\ell} = 0$ for $\ell \neq j$), allowing one to use the whole set \mathbf{T} to estimate each component enables to take into account possible dependencies, which may improve the results. Nevertheless, the condition $\lambda^\top \mathbf{J} = \mathbf{I}$ is imposed to have the equality $\lambda^\top \mathbf{T} - \theta = \lambda^\top (\mathbf{T} - \mathbf{J}\theta)$, which is used to derive the optimality result of Theorem 3.1 in Section 3.1. Letting $\|\cdot\|$ denote the usual Euclidean norm on \mathbb{R}^d , the expression of the mean square error becomes, using the classical trick of switching trace and expectation,

$$\mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2 = \mathbb{E} [\text{tr} [(\mathbf{T} - \mathbf{J}\theta)^\top \lambda \lambda^\top (\mathbf{T} - \mathbf{J}\theta)]] = \text{tr}(\lambda^\top \Sigma \lambda),$$

where $\Sigma = \mathbb{E}[(\mathbf{T} - \mathbf{J}\theta)(\mathbf{T} - \mathbf{J}\theta)^\top] \in \mathbb{R}^{k \times k}$. Here again, we assume that Σ exists and is non-singular. The simultaneous averaging process for several parameters generalizes the procedure presented in Section 2.1. In fact, averaging for one parameter just becomes the particular case with $d = 1$. Given a subset $\Lambda \subseteq \Lambda_{\max}$, we define the oracle as the linear transformation $\hat{\theta}^* = \lambda^{*\top} \mathbf{T}$ with

$$\lambda^* = \arg \min_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2 = \arg \min_{\lambda \in \Lambda} \text{tr}(\lambda^\top \Sigma \lambda). \quad (5)$$

Finally, assuming we have access to an estimator $\hat{\Sigma}$ of Σ , see Section 2.4, we define the averaging estimator as $\hat{\theta} = \hat{\lambda}^\top \mathbf{T}$ where

$$\hat{\lambda} = \arg \min_{\lambda \in \Lambda} \text{tr}(\lambda^\top \hat{\Sigma} \lambda). \quad (6)$$

If $\lambda^\top \Sigma \lambda$ is well approximated by $\lambda^\top \hat{\Sigma} \lambda$ for $\lambda \in \Lambda$, we can reasonably expect the average $\hat{\theta}$ to be close to the oracle $\hat{\theta}^*$, regardless of the possible dependency between $\hat{\Sigma}$ and \mathbf{T} .

2.3 Examples of constraint sets

As explained before, the set Λ must be included in Λ_{\max} defined in (3). This constraint is confirmed by the theoretical study of Section 3, where we further assume that Λ is convex. The latter condition is required for technical reasons and excludes the case where Λ is the set of vertices, corresponding to estimator selection. Applying the method in this case would simply consist in comparing the estimated mean square error of each estimator and this natural approach is already used in numerous situations. Other interesting situations are discussed below. They all fit the setting of our theoretical results and provide a better oracle than for estimator selection.

- When a good estimation of Σ can be provided, it is natural to consider the maximal constraint set $\Lambda = \Lambda_{\max}$, thus aiming for the best possible oracle. This set is actually an affine subspace of $\mathbb{R}^{k \times d}$ and in particular, it is convex. The oracle, obtained by minimizing the convex map $\lambda \mapsto \text{tr}(\lambda^\top \Sigma \lambda)$ subject to the constraint $\lambda^\top \mathbf{J} = \mathbf{I}$ is given by $\hat{\theta}_{\max}^* = \lambda_{\max}^{*\top} \mathbf{T}$ where

$$\lambda_{\max}^* = \Sigma^{-1} \mathbf{J} (\mathbf{J}^\top \Sigma^{-1} \mathbf{J})^{-1}, \quad (7)$$

generalizing the formula given in Section 2.1. Its mean-square error can be calculated directly

$$\mathbb{E}[(\hat{\theta}_{\max}^* - \theta)(\hat{\theta}_{\max}^* - \theta)^\top] = (\mathbf{J}^\top \Sigma^{-1} \mathbf{J})^{-1}.$$

This solution is a direct consequence of the equality

$$\lambda^\top \Sigma \lambda - (\mathbf{J}^\top \Sigma^{-1} \mathbf{J})^{-1} = \lambda^\top \Sigma \lambda - \lambda_{\max}^{*\top} \Sigma \lambda_{\max}^* = (\lambda - \lambda_{\max}^*)^\top \Sigma (\lambda - \lambda_{\max}^*) \quad (8)$$

that holds for all $\lambda \in \Lambda_{\max}$ due to the condition $\lambda^\top \mathbf{J} = \mathbf{I}$, and where the last matrix is positive definite.

Moreover, (8) shows that the oracle is not only the solution of our optimization problem over Λ_{\max} , but it is optimal to estimate any linear transformation of θ . In particular each component $\hat{\theta}_{\max, j}^*$ of the oracle is the best linear transformation $\lambda^\top \mathbf{T}$, $\lambda \in \Lambda_{\max}$, that one can get to estimate θ_j . Another desirable property of the choice $\Lambda = \Lambda_{\max}$ is that due to the closed expression (7), the averaging estimator $\hat{\theta}_{\max}$ obtained by replacing Σ by its estimation $\hat{\Sigma}$ has also a closed expression which makes it easily computable, namely

$$\hat{\theta}_{\max} = (\mathbf{J}^\top \hat{\Sigma}^{-1} \mathbf{J})^{-1} \mathbf{J}^\top \hat{\Sigma}^{-1} \mathbf{T}. \quad (9)$$

As mentioned earlier, the maximal constraint set allows one to use the information contained in external collections to estimate each parameter. This requires to estimate the whole MSE matrix, including the cross correlations between different collections \mathbf{T}_i . While this can produce surprisingly good results in some cases (see Section 4), it may deteriorate the estimator if the external collections do not contain significant additional information on the parameter.

- A natural and simpler framework is to consider component-wise averaging, for which only the collection \mathbf{T}_j is involved in the estimation of θ_j . The associated set of weights is the set of matrices λ whose support is included in the support of \mathbf{J} , that is

$$\Lambda = \{\lambda \in \Lambda_{\max} : \text{supp}(\lambda) \subseteq \text{supp}(\mathbf{J})\},$$

where for a matrix $A = (A_{i,j}) \in \mathbb{R}^{k \times d}$, $\text{supp}(A) := \{(i,j), A_{i,j} \neq 0\}$. In this particular framework, the covariance of two initial estimators in different collection $\mathbf{T}_i, \mathbf{T}_j$, $i \neq j$ is not involved in the computation of the oracle, so that the corresponding entries of Σ need not be estimated. Consequently, each component of θ is combined regardless of the others and as a result, the oracle is given by

$$\hat{\theta}_j^* = \frac{\mathbf{1}^\top \Sigma_j^{-1} \mathbf{T}_j}{\mathbf{1}^\top \Sigma_j^{-1} \mathbf{1}}, \quad j = 1, \dots, d.$$

where

$$\Sigma_j = \mathbb{E}[(\mathbf{T}_j - \theta_j \mathbf{1})(\mathbf{T}_j - \theta_j \mathbf{1})^\top] \in \mathbb{R}^{k_j \times k_j}, \quad j = 1, \dots, d.$$

In order to build the averaging estimator, it is sufficient to plug an estimate of Σ_j , $j = 1, \dots, d$, in the above expression, which makes it easily computable. See Section 4.2 for further discussion.

- Convex averaging corresponds to the choice

$$\Lambda = \{\lambda \in \Lambda_{\max} : \lambda_{i,j} \geq 0, \quad i = 1, \dots, k, \quad j = 1, \dots, d\}.$$

Observe that the positivity restriction combined with the condition $\lambda^\top \mathbf{J}$ results in λ having its support included in that of \mathbf{J} , making convex averaging a particular case of component-wise averaging. This means that each component of θ can be dealt with separately. So, for sake of simplicity in this example, we only consider the case $d = 1$.

Convex combination of estimators is a natural choice that has been widely studied in the literature. An advantage lies in the increased stability of the solution, due to the restriction of λ to a compact set, though the oracle may of course be less efficient than in the maximal case $\Lambda = \Lambda_{\max}$. The use of convex combinations is also particularly convenient to preserve some properties of the initial estimators, such as positivity or boundedness. Moreover, imposing non-negativity often leads to sparse solutions.

In this convex constrained optimization problem, the minimizer $\hat{\lambda} = \arg \min_{\lambda \in \Lambda} \lambda^\top \hat{\Sigma} \lambda$ can either lie in the interior of the domain, in which case $\hat{\lambda} = \hat{\Sigma}^{-1} \mathbf{1} / \mathbf{1}^\top \hat{\Sigma}^{-1} \mathbf{1}$ corresponds to the global minimizer over Λ_{\max} , or on the edge, meaning that it has at least one zero coordinate. Letting $\hat{m} \subseteq \{1, \dots, k\}$ denote the support of $\hat{\lambda}$, it follows that the averaging procedure obtained with the estimators $\mathbf{T}_{\hat{m}} := (\mathbf{T}_i)_{i \in \hat{m}}$ leads to a solution $\hat{\lambda}_{\hat{m}}$ with full support. As a result, it can be expressed as the global minimizer for the collection $\mathbf{T}_{\hat{m}}$,

$$\hat{\lambda}_{\hat{m}} = \frac{\hat{\Sigma}_{\hat{m}}^{-1} \mathbf{1}}{\mathbf{1}^\top \hat{\Sigma}_{\hat{m}}^{-1} \mathbf{1}},$$

where $\hat{\Sigma}_{\hat{m}}$ is the submatrix composed of the entries $\hat{\Sigma}_{i,j}$ for $(i,j) \in \hat{m}^2$. Since we have by construction $\hat{\lambda}_{\hat{m}}^\top \mathbf{T}_{\hat{m}} = \hat{\lambda}^\top \mathbf{T} = \hat{\theta}$, we deduce the following characterization of the convex averaging solution:

$$\hat{\theta} = \frac{\mathbf{1}^\top \hat{\Sigma}_{\hat{m}}^{-1} \mathbf{T}_{\hat{m}}}{\mathbf{1}^\top \hat{\Sigma}_{\hat{m}}^{-1} \mathbf{1}},$$

where \hat{m} must be the support which is both admissible and provides the minimal mean-square error, i.e. $\hat{m} = \arg \max_{m \subseteq \{1, \dots, k\}} \mathbf{1}^\top \hat{\Sigma}_m^{-1} \mathbf{1}$ subject to the constraint that $\hat{\Sigma}_m^{-1} \mathbf{1}$ has all its coordinates positive. This provides an easy method to implement convex averaging in practice. Remark that this method is only efficient if k is not too large, otherwise we recommend to use a standard quadratic programming solver to get $\hat{\lambda}$, see for instance [25].

2.4 Estimation of the MSE matrix

The accuracy of $\hat{\Sigma}$ is clearly a main factor to the performance of the averaging method. There are natural methods to construct $\hat{\Sigma}$ that essentially differ whether the model is parametric or not. In all cases, the estimation of Σ can be carried out from the same data as those used to produce the initial estimators T_i and no sample splitting is needed.

In a fully specified parametric model, the MSE matrix Σ can be estimated by plugging an initial estimate of θ . Precisely, assuming that the MSE matrix can be expressed as the image of θ through a known map $\Sigma(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^{k \times k}$, one can choose $\hat{\Sigma} = \Sigma(\hat{\theta}_0)$, where $\hat{\theta}_0$ is a consistent estimate of θ . A natural choice for $\hat{\theta}_0$ is to take one of the initial estimators if it is known to be consistent, or the average $\frac{1}{k} \sum_{i=1}^k T_i$ provided all initial estimators are consistent. If the map $\Sigma(\cdot)$ is not explicitly known, $\Sigma(\hat{\theta}_0)$ may be approximated by Monte-Carlo simulations of the model using the estimated parameter $\hat{\theta}_0$, a procedure sometimes called parametric bootstrap. This method is illustrated in our examples in Sections 4.2 and 4.3. Remark that in this parametric situation, the averaging procedure does not require any information other than the initial collection \mathbf{T} .

In some cases, Σ may also depend on a nuisance parameter η . In this situation, $\hat{\Sigma}$ can be built similarly by plugging or Monte-Carlo, provided η can be estimated from the observations. This situation requires that the sample X_1, \dots, X_n used to built the initial estimators T_i is available to the user.

In a semi and non-parametric setting, a parametric closed-form expression for Σ may be available asymptotically, i.e. when the sample size on which \mathbf{T} is built tends to infinity, and the above plugging method then becomes possible, see also (i) in Section 3.2. Alternatively, Σ can be estimated by standard bootstrap if no extra information is available. These two methods are implemented in the first example of Section 4.

3 Theoretical results

3.1 Non-asymptotic error bound

The performance of the averaging estimator relies on the accuracy of $\hat{\Sigma}$, but more specifically, on the ability to evaluate the error $\text{tr}(\lambda^\top \Sigma \lambda)$ as λ ranges over Λ . As a result, it is not crucial that $\hat{\Sigma}$ be a perfect estimate of Σ as long as $\text{tr}(\lambda^\top \hat{\Sigma} \lambda)$ is close to its true value for $\lambda \in \Lambda$. In order to measure the accuracy of $\hat{\Sigma}$ for this particular purpose, we introduce the following criterion. For two symmetric positive definite matrices A and B and for any non-empty set Λ that does not contain 0, let $\delta_\Lambda(A|B)$ denote the maximal divergence of the ratio $\text{tr}(\lambda^\top A \lambda) / \text{tr}(\lambda^\top B \lambda)$ over

Λ ,

$$\delta_\Lambda(A|B) = \sup_{\lambda \in \Lambda} \left| 1 - \frac{\text{tr}(\lambda^\top A \lambda)}{\text{tr}(\lambda^\top B \lambda)} \right|,$$

and $\delta_\Lambda(A, B) = \max\{\delta_\Lambda(A|B), \delta_\Lambda(B|A)\}$. We are now in position to state our main result.

Theorem 3.1 *Let Λ be a non-empty closed convex subset of Λ_{\max} with associated oracle $\hat{\theta}^*$ defined through (5), and $\hat{\Sigma}$ a symmetric positive definite $k \times k$ matrix. The averaging estimator $\hat{\theta} = \hat{\lambda}^\top \mathbf{T}$ defined through (6) satisfies*

$$\|\hat{\theta} - \hat{\theta}^*\|^2 \leq \left[\inf_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2 \right] \left(2\delta_\Lambda(\hat{\Sigma}, \Sigma) + \delta_\Lambda(\hat{\Sigma}, \Sigma)^2 \right) \|\Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)\|^2. \quad (10)$$

In this theorem, we provide an upper bound on the distance of the averaging estimator to the oracle. We emphasize that this result holds without requiring any condition on the joint behavior of \mathbf{T} and $\hat{\Sigma}$ (in particular, they may be strongly dependent). Moreover, we point out that the upper bound applies to the actual error to the oracle (for the current event ω), contrary to classical oracle inequalities which generally involve an expected loss of some kind.

The influence of the constraint set Λ in the process becomes apparent through both the minimal error $\inf_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2$ and the maximal divergence $\delta_\Lambda(\hat{\Sigma}, \Sigma)$. It appears that while the efficiency of the oracle is increased for large sets Λ , one must settle for combinations λ for which $\text{tr}(\lambda^\top \Sigma \lambda)$ can be well evaluated, thus yielding a small value of $\delta_\Lambda(\hat{\Sigma}, \Sigma)$. The presence of strong correlations among the initial estimators T_i can in turn influence negatively the value of $\delta_\Lambda(\hat{\Sigma}, \Sigma)$. This is because a near singular matrix Σ will result in amplified errors when computing $\hat{\Sigma}^{-1}$. For this reason, one must be careful when including too many estimators to the initial collection as it increases the possibilities of strong correlations. If this problem does not occur, adding a new estimator generally improves the efficiency of the oracle, but the final term $\|\Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)\|^2$ in the right-hand side shows the price to pay in view of the equality

$$\mathbb{E} \|\Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)\|^2 = k.$$

In Lemma 5.1 in the Appendix, we establish an upper bound of the divergence $\delta_\Lambda(\hat{\Sigma}, \Sigma)$ by

$$\delta_\Lambda(\hat{\Sigma}, \Sigma) \leq \|\hat{\Sigma}\Sigma^{-1} - \Sigma\hat{\Sigma}^{-1}\|, \quad (11)$$

where $\|\cdot\|$ denotes the operator norm. The right-hand side measures how well $\hat{\Sigma}\Sigma^{-1}$ approximates the identity matrix. Thus, the quantity

$$L := \|\hat{\Sigma}\Sigma^{-1} - \Sigma\hat{\Sigma}^{-1}\|$$

provides a good indicator of efficiency for the averaging method, as illustrated by the next result.

Corollary 3.2 *Under the assumptions of Theorem 3.1, the averaging estimator satisfies, for all $\epsilon \in (0, 1]$,*

$$\mathbb{P} \left(\frac{\|\hat{\theta} - \hat{\theta}^*\|^2}{\mathbb{E} \|\hat{\theta}^* - \theta\|^2} < \frac{k}{\epsilon} (2L + L^2) \right) \geq 1 - \epsilon. \quad (12)$$

Remark that, contrary to $\delta_\Lambda(\hat{\Sigma}, \Sigma)$, L no longer depends on the constraint set Λ . Hence, the sharpest bounds in (11) and (12) are obtained for $\Lambda = \Lambda_{\max}$, although they trivially apply to smaller constraint sets.

3.2 Asymptotic study

The properties of the averaging estimator established in Theorem 3.1 do not rely on any assumption on the construction of \mathbf{T} or $\hat{\Sigma}$. In this section, we investigate the asymptotic properties of the averaging estimator in a situation where both \mathbf{T} and $\hat{\Sigma}$ are computed from a set of observations X_1, \dots, X_n of size n growing to infinity. From now on, we modify our notations to $\mathbf{T}_n, \hat{\Sigma}_n, \Sigma_n, \lambda_n^*, \hat{\lambda}_n, \hat{\theta}_n$ and $\hat{\theta}_n^*$ to emphasize the dependency on n .

In practice, we expect the oracle $\hat{\theta}_n^*$ to satisfy good properties such as consistency or asymptotic normality. Theorem 3.1 suggests that $\hat{\theta}_n$ should inherit these asymptotic properties if Σ_n can be sufficiently well estimated. Remark that if the initial estimators T_i are consistent in quadratic mean, Σ_n converges to the null matrix as $n \rightarrow \infty$. In this case, providing an estimator $\hat{\Sigma}_n$ such that $\hat{\Sigma}_n - \Sigma_n \xrightarrow{p} 0$ is clearly not sufficient for $\hat{\theta}_n$ to achieve the asymptotic performance of the oracle (here p stands for the convergence in probability while d is used for distribution). On the contrary, requiring that $\hat{\Sigma}_n^{-1} - \Sigma_n^{-1} \xrightarrow{p} 0$ is unnecessarily too strong and would be nearly impossible to achieve. In fact, we show in Proposition 3.3 below that the condition

$$\hat{\Sigma}_n \Sigma_n^{-1} \xrightarrow{p} \mathbf{I} \quad (13)$$

appears as a simple compromise, both sufficient for asymptotic optimality and reasonable enough to be verified in numerous situations with regular estimators of Σ_n . We briefly discuss a few examples.

- (i) If $\sqrt{n}(\mathbf{T}_n - \mathbf{J}\theta)$ converges in \mathbb{L}^2 to a Gaussian vector $\mathcal{N}(0, W)$ with W a non-singular matrix, providing a consistent estimator, say \hat{W}_n , of W is sufficient to verify (13), taking $\hat{\Sigma}_n = \hat{W}_n$. The situation becomes particularly convenient if the limit matrix W follows a known parametric expression $W = W(\eta, \theta)$, with η a nuisance parameter (see the first example in Section 4). If the map $W(\cdot, \cdot)$ is continuous, plugging consistent estimators $\hat{\eta}_0, \hat{\theta}_0$ yields an estimator $\hat{W}_n = W(\hat{\eta}_0, \hat{\theta}_0)$ that fulfills (13). Observe that knowing the rate \sqrt{n} in this example is not necessary as it simplifies in the expression of $\hat{\theta}_n$. In fact, a different rate of convergence, even unknown, would lead to the exact same result. In this case, the asymptotic normality can make it possible to construct asymptotic confidence intervals of minimal length for the parameter, as shown in Proposition 3.3 below.

- (ii) More generally, if Σ_n satisfies

$$\Sigma_n = a_n W + o(a_n), \quad (14)$$

for some vanishing sequence a_n , building a consistent estimator of W is sufficient to achieve (13). Here again, the rate of convergence must not be known.

- (iii) If we have different rates of convergence within the collection \mathbf{T}_n , the condition (13) can be verified if the normed eigenvectors of Σ_n converge as $n \rightarrow \infty$. Precisely, if there exist an orthogonal matrix P (i.e. with $P^\top P = \mathbf{I}$) and a known deterministic sequence $(A_n)_{n \in \mathbb{N}}$ of diagonal invertible matrices such that

$$\lim_{n \rightarrow \infty} A_n P \Sigma_n P^\top = D,$$

for some non-singular diagonal matrix D , producing consistent estimators \hat{P}_n and \hat{D}_n of P and D respectively enables to verify (13) by $\hat{\Sigma}_n = \hat{P}_n^\top A_n^{-1} \hat{D}_n \hat{P}_n$. Here, the limit of the normed eigenvectors of Σ_n are given by the rows of P and the estimator $\hat{\Sigma}_n$ is constructed from the asymptotic expansion of Σ_n . This example allows to have different rates of convergence within the collection \mathbf{T}_n but also covers the previously mentioned examples where all constant combinations $\lambda^\top \mathbf{T}_n$ converge to θ at the same rate.

Let us introduce some additional definitions and notation. For each component θ_j , $j = 1, \dots, d$, we define

$$\alpha_{n,j} := \mathbb{E} \|\hat{\theta}_{n,j}^* - \theta_j\|^2 = \underline{\lambda}_{n,j}^{*\top} \Sigma_n \underline{\lambda}_{n,j}^*,$$

where we recall that $\underline{\lambda}_{n,j}^*$ is the j -th column of λ_n^* . Similarly, let $\hat{\alpha}_{n,j} = \hat{\lambda}_{n,j}^\top \hat{\Sigma}_n \hat{\lambda}_{n,j}$. We assume that the quadratic error of the oracle, given by

$$\alpha_n := \mathbb{E} \|\hat{\theta}_n^* - \theta\|^2 = \text{tr}(\lambda_n^{*\top} \Sigma_n \lambda_n^*) = \sum_{j=1}^d \alpha_{n,j},$$

converges to zero as $n \rightarrow \infty$. For a given constraint set $\Lambda \subset \mathbb{R}^{k \times d}$, we note $\Lambda_j = \{\underline{\lambda}_j : \lambda \in \Lambda\} \subset \mathbb{R}^k$ its marginal set. We say that Λ is a *cylinder* if $\Lambda = \{\lambda : \underline{\lambda}_1 \in \Lambda_1, \dots, \underline{\lambda}_d \in \Lambda_d\}$, i.e., if Λ is the Cartesian product of its marginal sets Λ_j . We point out that choosing a constraint set Λ that satisfies this property is very natural, as it simply states that each vector of weights $\underline{\lambda}_j$ used to estimate θ_j can be computed independently of the others. In particular, all the constraint sets discussed in Section 2.3 are cylinders.

Proposition 3.3 *If (13) holds, then*

$$\|\hat{\theta}_n - \theta\|^2 = \|\hat{\theta}_n^* - \theta\|^2 + o_p(\alpha_n). \quad (15)$$

Moreover, if Λ is a cylinder and $\alpha_{n,j}^{-\frac{1}{2}}(\hat{\theta}_{n,j}^* - \theta_j) \xrightarrow{d} \mathcal{Z}$ for some $j = 1, \dots, d$, where \mathcal{Z} is a random variable, then

$$\hat{\alpha}_{n,j}^{-\frac{1}{2}}(\hat{\theta}_{n,j} - \theta_j) \xrightarrow{d} \mathcal{Z}. \quad (16)$$

This proposition establishes that building an estimate $\hat{\Sigma}_n$ for which (13) holds ensures that the error of the average $\hat{\theta}_n$ is asymptotically comparable to that of the oracle, up to $o_p(\alpha_n)$. If in addition Λ is a cylinder, it is possible to provide asymptotic confidence regions for θ_j when the limit distribution \mathcal{Z} is known. From (6), we know this confidence interval is of minimal length amongst all possible confidence intervals based on a linear combination of \mathbf{T}_n . Note that no extra estimation is needed to approach the asymptotic variance, as $\hat{\alpha}_{n,j}$ is entirely determined by $\hat{\lambda}_n$ and $\hat{\Sigma}_n$, which are already used to compute $\hat{\theta}_n$.

Remark 3.4 *The convergence result (16) is in contrast with Theorem 4.1 in [15] where the limit distribution of the averaging estimator differs from that of the oracle. However the authors deal with model averaging, where some particular local misspecification is assumed, implying a specific construction of the averaging weights. We consider here a more regular framework, which explains why we are able to achieve the same limit distribution as the oracle.*

In Proposition 3.3, the asymptotic optimality of $\hat{\theta}_n$ is stated in probability. Remark however that asymptotic optimality in quadratic loss can be obtained easily under additional assumptions. If, for instance, $\hat{\Sigma}_n$ and \mathbf{T}_n are computed from independent samples (which may be achieved by sample splitting), $\hat{\theta}_n$ is asymptotically optimal in quadratic loss as soon as $\mathbb{E}[\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n)^2]$ tends to 0. Indeed, we have in this case

$$\mathbb{E}\|\hat{\theta}_n - \theta\|^2 = \mathbb{E}\|\hat{\theta}_n^* - \theta\|^2 + o(\alpha_n), \quad (17)$$

which follows directly from (24) in the proof of Proposition 3.3, taking the expectation on both sides. We emphasize however that the use of sample splitting may reduce the performance of the oracle, as it would be computed from fewer data. One can argue that this is a high price to pay to obtain asymptotic optimality in \mathbb{L}^2 and is not to be recommended in this framework. Asymptotic optimality in \mathbb{L}^2 can also be achieved if one can show there exists $p > 1$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E}\|\Sigma_n^{-\frac{1}{2}}(\mathbf{T}_n - \mathbf{J}\theta)\|_{\frac{2p}{p-1}} < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{E}[\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n)^{2p}] = 0.$$

In this case, Equation (17) follows directly by applying Hölder's inequality in (24). These conditions ensure the asymptotic optimality in \mathbb{L}^2 of the averaging estimator without sample splitting, but they remain nonetheless extremely difficult to check in practice.

4 Applications

4.1 Estimating the position of a symmetric distribution

Let us consider a continuous real distribution with density f , symmetric around some parameter θ . To estimate θ from a sample of n realisations x_1, \dots, x_n , a natural choice is to use the mean \bar{x}_n or the median $x_{(n/2)}$. Both estimators are consistent whenever $\sigma^2 = \int (x - \theta)^2 f(x) dx$ is finite.

As remarked in introduction, the idea of combining the mean and the median to construct a better estimator goes back to Pierre Simon de Laplace [19]. P. S. de Laplace obtains the expression of the weights in Λ_{\max} that ensure a minimal asymptotic variance for the averaging estimator. In particular, he deduced that for a Gaussian distribution, the better combination is to take the mean only, showing for the first time the efficiency of the latter. For other distributions, he noticed that the best combination is not available in practice because it depends on the unknown distribution.

Similarly, we consider the averaging of the mean and the median over Λ_{\max} . We have two initial estimators $T_1 = \bar{x}_n$, $T_2 = x_{(n/2)}$ and the averaging estimator is given by (9) where \mathbf{J} is just in this case the vector $(1, 1)^\top$. The MSE matrix between the two estimators is denoted by Σ_n . We assume that the n realisations are independent and we propose two ways to estimate Σ_n :

1. *Based on the asymptotic equivalent of Σ_n .* The latter, obtained in P. S. de Laplace's work

and recalled in [29], is $n^{-1}W$ where

$$W = \begin{pmatrix} \sigma^2 & \frac{\mathbb{E}|X-\theta|}{2f(\theta)} \\ \frac{\mathbb{E}|X-\theta|}{2f(\theta)} & \frac{1}{4f(\theta)^2} \end{pmatrix}.$$

Each entry of W may be naturally estimated from an initial consistent estimate $\hat{\theta}_0$ of θ as follows: σ^2 by the empirical variance s_n^2 ; $\mathbb{E}|X - \theta|$ by $\hat{m} = 1/n \sum_{i=1}^n |x_i - \hat{\theta}_0|$; and $f(\theta)$ by the kernel estimator $\hat{f}(\hat{\theta}_0) = 1/(nh) \sum_{i=1}^n \exp(-(x_i - \hat{\theta}_0)^2/(2h^2))$, where h is chosen, e.g., by the so-called Silverman's rule of thumb (see [28]). With this estimation of Σ_n , we get the following averaging estimator:

$$\hat{\theta}_{AV} = \frac{p_1}{p_1 + p_2} \bar{x}_n + \frac{p_2}{p_1 + p_2} x_{(n/2)} \quad (18)$$

where $p_1 = 1/(4\hat{f}(\hat{\theta}_0)) - \hat{m}/2$ and $p_2 = s_n^2 \hat{f}(\hat{\theta}_0) - \hat{m}/2$. This estimator corresponds to an empirical version of the best combination obtained by P. S. de Laplace.

2. *Based on bootstrap.* We draw with replacement B samples of size n from the original dataset. We compute the mean and the median of each sample, respectively denoted $\bar{x}_n^{(b)}$ and $x_{(n/2)}^{(b)}$ for $b = 1, \dots, B$. The MSE matrix Σ_n is then estimated by

$$\frac{1}{B} \begin{pmatrix} \sum_{b=1}^B (\bar{x}_n^{(b)} - \bar{x}_n)^2 & \sum_{b=1}^B (\bar{x}_n^{(b)} - \bar{x}_n)(x_{(n/2)}^{(b)} - x_{(n/2)}) \\ \sum_{b=1}^B (\bar{x}_n^{(b)} - \bar{x}_n)(x_{(n/2)}^{(b)} - x_{(n/2)}) & \sum_{b=1}^B (x_{(n/2)}^{(b)} - x_{(n/2)})^2 \end{pmatrix}.$$

This leads to another averaging estimator, denoted by $\hat{\theta}_{AVB}$.

Let us note that the first procedure above fits the asymptotic justification presented in example (i) of Section 3.2. For this reason, $\hat{\theta}_{AV}$ is asymptotically as efficient as the oracle, provided $\hat{\theta}_0$ is consistent. Moreover, since the initial estimators are asymptotically Gaussian and unbiased, an asymptotic confidence interval for θ can be provided without further estimation, see Section 3.2. For the second procedure, theory is lacking to study the behaviour of δ in (10) when $\hat{\Sigma}_n$ is estimated by bootstrap, so no consistency can be claimed at this point. However the latter is a very natural procedure, easy to implement in practice, so it is natural to assess its performances in our simulation study.

Table 1 summarizes the estimated MSE of \bar{x}_n , $x_{(n/2)}$, $\hat{\theta}_{AV}$ and $\hat{\theta}_{AVB}$, for $n = 30, 50, 100$, and for different distributions, namely: Cauchy, Student with 5 degrees of freedom, Student with 7 degrees of freedom, Logistic, standard Gaussian, and an equal mixture distribution of a $\mathcal{N}(-2, 1)$ and a $\mathcal{N}(2, 1)$. For all distributions, $\theta = 0$. For the initial estimate $\hat{\theta}_0$ in (18), we take the median $x_{(n/2)}$, because it is well defined and consistent for any continuous distribution. The number of bootstrap samples taken for $\hat{\theta}_{AVB}$ is $B = 1000$.

While the best estimator between \bar{x}_n and $x_{(n/2)}$ depends on the underlying distribution, the averaging estimators $\hat{\theta}_{AV}$ and $\hat{\theta}_{AVB}$ perform better than both \bar{x}_n and $x_{(n/2)}$, for all distributions considered in Table 1 except the Gaussian law. For the latter distribution, we know that the

| | n=30 | | | | n=50 | | | | n=100 | | | |
|----------|-------------------------------------------|----------------|-----------------|-----------------|-------------------------------------------|-----------------|----------------|----------------|-------------------------------------------|-----------------|----------------|----------------|
| | MEAN | MED | AV | AVB | MEAN | MED | AV | AVB | MEAN | MED | AV | AVB |
| Cauchy | 2.10 ⁶ (1.10 ⁶) | 9 (0.14) | 8.95 (0.15) | 8.99 (0.15) | 4.10 ⁷ (4.10 ⁷) | 5.07 (0.08) | 4.92 (0.08) | 4.9 (0.08) | 2.10 ⁷ (2.10 ⁷) | 2.56 (0.04) | 2.49 (0.04) | 2.49 (0.04) |
| St(4) | 6.68 (0.1) | 5.71 (0.08) | 5.4 (0.08) | 5.43 (0.08) | 4.12 (0.06) | 3.53 (0.05) | 3.33 (0.05) | 3.34 (0.05) | 1.99 (0.03) | 1.74 (0.02) | 1.61 (0.02) | 1.62 (0.02) |
| St(7) | 4.8 (0.07) | 5.51 (0.08) | 4.6 (0.07) | 4.64 (0.07) | 2.82 (0.04) | 3.32 (0.05) | 2.74 (0.04) | 2.8 (0.04) | 1.42 (0.02) | 1.67 (0.02) | 1.37 (0.02) | 1.38 (0.02) |
| Logistic | 10.89 (0.16) | 12.7 (0.18) | 10.76 (0.16) | 10.87 (0.16) | 6.64 (0.09) | 7.93 (0.11) | 6.52 (0.09) | 6.6 (0.09) | 3.3 (0.05) | 4 (0.06) | 3.2 (0.05) | 3.26 (0.05) |
| Gauss | 3.39 (0.05) | 5.11 (0.07) | 3.53 (0.05) | 3.61 (0.05) | 2.04 (0.03) | 3.1 (0.04) | 2.1 (0.03) | 2.15 (0.03) | 1 (0.01) | 1.51 (0.02) | 1.02 (0.01) | 1.06 (0.01) |
| Mix | 16.79 (0.23) | 87 (0.82) | 15.03 (0.29) | 13.41 (0.3) | 10.08 (0.14) | 66.53 (0.64) | 7.57 (0.15) | 6.68 (0.18) | 5.05 (0.07) | 42.35 (0.43) | 3.09 (0.06) | 2.36 (0.07) |

Table 1: Monte Carlo estimation of the MSE of \bar{x}_n (MEAN), $x_{(n/2)}$ (MED), $\hat{\theta}_{AV}$ (AV) and $\hat{\theta}_{AVB}$ (AVB) in the estimation of the position of a symmetric distribution, depending on the distribution and the sample size. The number of replications is 10^4 and the standard deviation of the MSE estimations is given in parenthesis. Each entry has been multiplied by 100 for ease of presentation.

oracle is the mean, so the averaging estimator cannot improve on \bar{x}_n . However the MSE of $\hat{\theta}_{AV}$ and $\hat{\theta}_{AVB}$ are very close to that of \bar{x}_n in this case, proving that the optimal weights $(1, 0)$ are fairly well estimated. Moreover, note that the Cauchy distribution does not belong to our theoretical setting because it has no finite moments and \bar{x}_n should not be used. But it turns out that the averaging estimators are very robust in this case, as they manage to highly favor $x_{(n/2)}$. Choosing the median $x_{(n/2)}$ as the initial estimator $\hat{\theta}_0$ is of course crucial in this case.

As explained above, the averaging procedure allows us to deduce an asymptotic confidence interval without any further estimation. By construction, the length of these intervals is smaller than the length of any similar confidence interval based on \bar{x}_n or $x_{(n/2)}$. Further, the empirical rate of coverage of these intervals is reported in Table 2 for the previous simulations, and turns out to be close to the nominal level 95%.

| | n=30 | | n=50 | | n=100 | |
|----------|-------|-------|-------|-------|-------|-------|
| | AV | AVB | AV | AVB | AV | AVB |
| Cauchy | 98.08 | 96.18 | 98.21 | 95.55 | 97.75 | 95.22 |
| St(4) | 93.59 | 91.45 | 94.38 | 92.71 | 94.71 | 92.55 |
| St(7) | 93.34 | 91.25 | 93.93 | 91.77 | 94.27 | 92.73 |
| Logistic | 92.48 | 90.33 | 93.96 | 92.05 | 93.91 | 92.21 |
| Gauss | 92.97 | 91.13 | 93.54 | 91.94 | 94.09 | 92.59 |
| Mix | 93.19 | 93.83 | 94.77 | 95.97 | 94.94 | 97.91 |

Table 2: Empirical rate of coverage (in %) of the asymptotic 95% confidence intervals based on $\hat{\theta}_{AV}$ and $\hat{\theta}_{AVB}$ in the estimation of the position of a symmetric distribution, deduced from the same simulations as in Table 1.

Finally, while $\hat{\theta}_{AVB}$ suffers from a lack of theoretical justification, it behaves pretty much like $\hat{\theta}_{AV}$, except for the mixture distribution where it performs slightly better than $\hat{\theta}_{AV}$. This may be explained by the fact that $\hat{\theta}_{AV}$ is more sensitive than $\hat{\theta}_{AVB}$ to the initial estimate $\hat{\theta}_0$, the variance of which is large for the mixture distribution because $f(0)$ is close to 0. Nevertheless, $\hat{\theta}_{AV}$ demonstrates very good performance in this case, for the sample sizes considered in Tables 1 and 2.

4.2 Estimating the parameters of a Weibull distribution

We consider estimators averaging in a parametric setting, namely the Weibull distribution with shape parameter $\beta > 0$ and scale parameter $\eta > 0$, the density function of which is

$$f(x) = \frac{\beta}{\eta^\beta} x^{\beta-1} e^{-(x/\eta)^\beta}, \quad x > 0.$$

Based on a sample of n independent realisations, many estimators of β and η are available (see [16]). We consider the following three standard methods:

- the maximum likelihood estimator (ML) is the solution of the system

$$\frac{n}{\beta} + \sum_{i=1}^n \log(x_i) - n \frac{\sum_{i=1}^n x_i^\beta \log(x_i)}{\sum_{i=1}^n x_i^\beta} = 0, \quad \eta = \left(\frac{1}{n} \sum_{i=1}^n x_i^\beta \right)^{1/\beta}.$$

- the method of moments (MM), based on the two first moments, reduces to solve:

$$\frac{s_n^2}{\bar{x}_n^2} = \frac{\Gamma(1 + 2/\beta)}{\Gamma(1 + 1/\beta)^2} - 1, \quad \eta = \frac{\bar{x}_n}{\Gamma(1 + 1/\beta)},$$

where \bar{x}_n and s_n denote the empirical sample mean and the unbiased sample variance.

- the ordinary least squares method (OLS) is based on the fact that for any $x > 0$, $\log(-\log(1 - F(x))) = \beta \log(x) - \beta \log \eta$, where F denotes the cumulative distribution function of the Weibull distribution. More precisely, denoting $x_{(1)}, \dots, x_{(n)}$ the ordered sample, an estimation of β and η is deduced from the simple linear regression of $(\log(-\log(1 - F(x_{(i)}))))_{i=1 \dots n}$ on $(\log x_{(i)})_{i=1 \dots n}$, where according to the "mean rank" method $F(x_{(i)})$ may be estimated by $i/(n+1)$. This fitting method is very popular in the engineer community (see [1]): the estimation of β simply corresponds to the slope in a "Weibull plot".

The performances of these three estimators are variable, depending on the value of the parameters and the sample size. In particular, no one is uniformly better than the others, see Figure 1 for an illustration.

Let us now consider the averaging of these estimators. In the setting of the previous sections, we have $d = 2$ parameters to estimate and $k_1 = 3$, $k_2 = 3$ initial estimators of each are available. The averaging over the maximal constraint set Λ_{\max} demands to estimate the 6×6 MSE matrix Σ , that involves 21 unknown values. The Weibull distribution is often used to

model lifetimes, and typically only a low number of observations are available to estimate the parameters. As a consequence averaging over Λ_{\max} of the 6 initial estimators above could be too demanding. Moreover, between the two parameters β and η , the shape parameter β is often the most important to identify, as it characterizes for instance the type of failure rate in reliability engineering. For these reasons, we choose to average the three estimators of β presented above, $\hat{\beta}_{ML}$, $\hat{\beta}_{MM}$ and $\hat{\beta}_{OLS}$, and to consider only one estimator of η : $\hat{\eta}_{ML}$ (where $\hat{\beta}_{ML}$ is used for its computation). The averaging over Λ_{\max} of these 4 estimators has three consequences: First, the number of unknown values in the MSE matrix is reduced to 10. Second, the averaging estimator of β depends only on $\hat{\beta}_{ML}$, $\hat{\beta}_{MM}$ and $\hat{\beta}_{OLS}$, because $\hat{\eta}_{ML}$ has a zero weight from (4). This means that we actually implement a component-wise averaging for β . Third, the averaging estimator of η equals $\hat{\eta}_{ML}$ plus some linear combination of $\hat{\beta}_{ML}$, $\hat{\beta}_{MM}$ and $\hat{\beta}_{OLS}$ where the weights sum to zero. This particular situation will allow us to see if $\hat{\eta}_{ML}$ can be improved by exploiting the correlation with the estimators of β , or if it is deteriorated.

So we have $d = 2$, $k_1 = 3$, $k_2 = 1$, $\mathbf{T}_1 = (\hat{\beta}_{ML}, \hat{\beta}_{MM}, \hat{\beta}_{OLS})^\top$, $\mathbf{T}_2 = \hat{\eta}_{ML}$ and the averaging estimator over Λ_{\max} is given by (9), denoted by $(\hat{\beta}_{AV}, \hat{\eta}_{AV})^\top$. The matrix Σ is estimated by Monte Carlo simulations: Starting from initial estimates $\hat{\beta}_0$, $\hat{\eta}_0$, we simulate B samples of size n of a Weibull distribution with parameters $\hat{\beta}_0$, $\hat{\eta}_0$. Then the four estimators are computed, which gives $\hat{\beta}_{ML}^{(b)}$, $\hat{\beta}_{MM}^{(b)}$, $\hat{\beta}_{OLS}^{(b)}$ and $\hat{\eta}_{ML}^{(b)}$, for $b = 1, \dots, B$, and each entry of Σ is estimated by its empirical counterpart. For instance the estimation of $\mathbb{E}(\hat{\beta}_{ML} - \beta)(\hat{\beta}_{MM} - \beta)$ is $(1/B) \sum_{b=1}^B (\hat{\beta}_{ML}^{(b)} - \hat{\beta}_0)(\hat{\beta}_{MM}^{(b)} - \hat{\beta}_0)$. In our simulations, we chose $\hat{\beta}_0$ as the mean of \mathbf{T}_1 and $\hat{\eta}_0 = \hat{\eta}_{ML}$. Note that Σ having a parametric form ensures that $(\hat{\beta}_{AV}, \hat{\eta}_{AV})^\top$ is asymptotically as efficient as the oracle, as explained in Section 3.2.

Table 3 gives the MSE, estimated from 10^4 replications, of each estimator of β , for $n = 10, 20, 50$, and for $\beta = 0.5, 1, 2, 3$, $\eta = 10$, where for each replication $B = 1000$. The averaging estimator has by far the lowest MSE, even for small samples. As an illustration, the repartition of each estimator, for $n = 20$ and $\beta = 0.5, 3$, is represented in Figure 1.

Table 4 shows the MSE for $\hat{\eta}_{ML}$ and $\hat{\eta}_{AV}$ where only estimators of β were used in attempt to improve $\hat{\eta}_{ML}$ by averaging. The performances of both estimators are similar, showing that the information coming from \mathbf{T}_1 did not help significantly improving $\hat{\eta}_{ML}$. On the other hand, the estimation of these (almost zero) weights might have deteriorated $\hat{\eta}_{ML}$, especially for small sample sizes. This did not happen.

Finally, the empirical rate of coverage of the asymptotic confidence intervals based on $\hat{\beta}_{AV}$ and $\hat{\eta}_{AV}$ is given in Table 5, showing that it is not far from the nominal level 95%, even for the small sample sizes considered in this simulation. On the other hand, the length of these intervals are by construction smaller than the length of similar confidence intervals based on the initial estimators.

| | n=10 | | | | n=20 | | | | n=50 | | | |
|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|----------------|---------------|-----------------|----------------|----------------|
| | ML | MM | OLS | AV | ML | MM | OLS | AV | ML | MM | OLS | AV |
| $\beta = 0.5$ | 35.53 (0.91) | 76.95 (1.27) | 24.41 (0.40) | 25.27 (0.64) | 12.06 (0.26) | 35.57 (0.52) | 13.74 (0.19) | 10.5 (0.19) | 3.7 (0.07) | 14.19 (0.20) | 6.04 (0.08) | 3.52 (0.06) |
| $\beta = 1$ | 152.4 (3.8) | 131.6 (3.1) | 98.1 (1.5) | 85.5 (1.7) | 49.2 (1.1) | 53.6 (1.1) | 54.2 (0.7) | 36.9 (0.7) | 14.4 (0.2) | 19.3 (0.3) | 23.9 (0.3) | 12.8 (0.2) |
| $\beta = 2$ | 596.4 (14.4) | 444.6 (11.9) | 399.4 (6.3) | 355.5 (6.7) | 194.5 (3.8) | 164.5 (3.3) | 218 (2.8) | 163.3 (2.7) | 57.9 (1.0) | 53.9 (0.9) | 94.8 (1.3) | 54.3 (0.9) |
| $\beta = 3$ | 1369 (34.6) | 1080 (29.7) | 905 (14.6) | 770 (18.1) | 452 (9.8) | 394 (8.9) | 486 (6.7) | 343 (6.2) | 128 (2.2) | 122 (2.0) | 211 (2.7) | 120 (1.9) |

Table 3: Monte Carlo estimation of the MSE of $\hat{\beta}_{ML}$, $\hat{\beta}_{MM}$, $\hat{\beta}_{OLS}$ and $\hat{\beta}_{AV}$, based on 10^4 replications of a sample of size $n = 10, 20, 50$ from a Weibull distribution with parameters $\beta = 0.5, 1, 2, 3$ and $\eta = 10$. The standard deviation of the MSE estimations are given in parenthesis. Each entry has been multiplied by 100 for ease of presentation.

| | n=10 | | n=20 | | n=50 | |
|---------------|-----------------|-----------------|-----------------|----------------|------------------|------------------|
| | ML | AV | ML | AV | ML | AV |
| $\beta = 0.5$ | 60.59 (1.60) | 55.61 (1.48) | 25.96 (0.53) | 24.56 (0.5) | 9.57 (0.17) | 9.38 (0.17) |
| $\beta = 1$ | 11.15 (0.18) | 10.88 (0.17) | 5.53 (0.08) | 5.43 (0.08) | 2.23 (0.03) | 2.22 (0.03) |
| $\beta = 2$ | 2.71 (0.04) | 2.74 (0.04) | 1.36 (0.02) | 1.37 (0.02) | 0.55 (0.01) | 0.56 (0.01) |
| $\beta = 3$ | 1.21 (0.02) | 1.23 (0.02) | 0.61 (0.01) | 0.61 (0.01) | 0.247 (0.003) | 0.248 (0.004) |

Table 4: Monte Carlo estimation of the MSE of $\hat{\eta}_{ML}$ and $\hat{\eta}_{AV}$, based on 10^4 replications of a sample of size $n = 10, 20, 50$ from a Weibull distribution with parameters $\beta = 0.5, 1, 2, 3$ and $\eta = 10$. The standard deviation of the MSE estimations are given in parenthesis. Each entry has been multiplied by 100 for ease of presentation.

| | n=10 | | n=20 | | n=50 | |
|---------------|--------------------|-------------------|--------------------|-------------------|--------------------|-------------------|
| | $\hat{\beta}_{AV}$ | $\hat{\eta}_{AV}$ | $\hat{\beta}_{AV}$ | $\hat{\eta}_{AV}$ | $\hat{\beta}_{AV}$ | $\hat{\eta}_{AV}$ |
| $\beta = 0.5$ | 89.84 | 87.48 | 93.43 | 90.01 | 95.41 | 93.07 |
| $\beta = 1$ | 87.25 | 89.24 | 90.98 | 91.61 | 93.81 | 93.62 |
| $\beta = 2$ | 89.96 | 91.36 | 91.77 | 93.39 | 93.09 | 94.20 |
| $\beta = 3$ | 92.19 | 92.38 | 92.86 | 93.83 | 94.25 | 94.77 |

Table 5: Empirical rate of coverage (in %) of the asymptotic 95% confidence intervals based on $\hat{\beta}_{AV}$ and $\hat{\eta}_{AV}$ for the parameters of a Weibull distribution, deduced from the same simulations as in Tables 3 and 4.

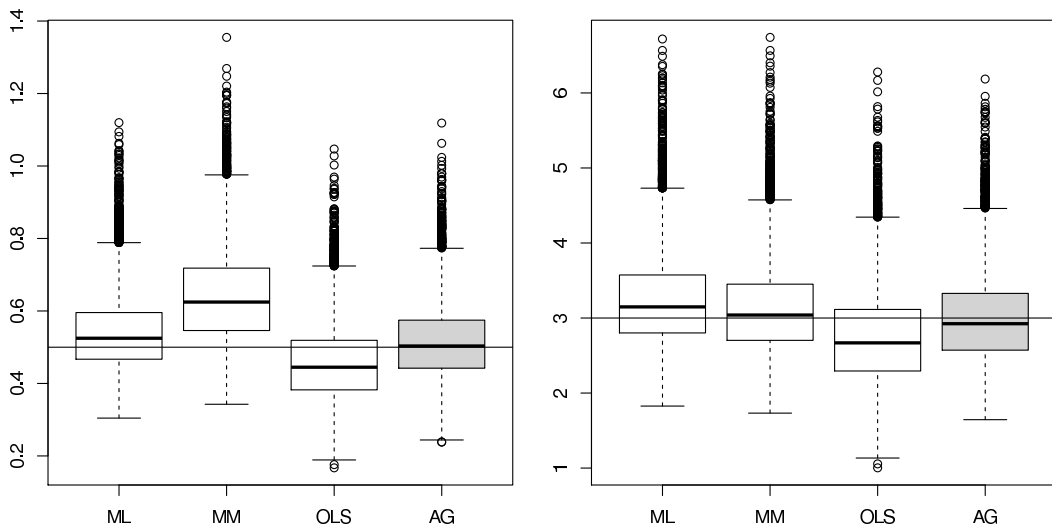


Figure 1: Repartition of $\hat{\beta}_{ML}$, $\hat{\beta}_{MM}$, $\hat{\beta}_{OLS}$ and $\hat{\beta}_{AV}$ (from left to right) based on 10^4 replications of a sample of size $n = 20$ from a Weibull distribution with $\beta = 0.5$ (left), $\beta = 3$ (right) and $\eta = 10$.

4.3 Estimation in a Boolean model

The Boolean model is the main model of random sets used in spatial statistics and random geometry, see [8]. It is a germ-grain model where, in the planar and stationary case, the germs come from a homogeneous Poisson point process on \mathbb{R}^2 with intensity ρ and the grains are independent random discs, the radii of which are distributed according to a probability law μ . Figure 2 contains four realisations of a Boolean model on $[0, 1]^2$ where $\rho = 25, 50, 100, 150$ respectively and the law of the radii μ is the uniform distribution over $[0, 0.1]$. We assume in the following that μ is the beta distribution over $[0, 0.1]$ with parameter $(1, \alpha)$, $\alpha > 0$, denoted by $B(1, \alpha)$, i.e. μ has density $10\alpha(1 - 10x)^{\alpha-1}$ on $[0, 0.1]$. The simulations of Figure 2 correspond to $\alpha = 1$.

The estimation of parameters ρ and α from the observation of random sets as in Figure 2 is challenging, since the individual grains cannot be identified and likelihood-based inference becomes impossible. The standard method of inference, see [22], is based on the following equations proved in [32]. They relate the expected area per unit area \mathcal{A} and the expected perimeter per unit area \mathcal{P} of the random set to the intensity ρ and the two first moments of μ , namely

$$\mathcal{A} = 1 - \exp(-\pi\rho E_\mu(R^2)), \quad \mathcal{P} = 2\pi\rho E_\mu(R) \exp(-\pi\rho E_\mu(R^2)),$$

where R denotes a random variable with distribution μ . Developing $E_\mu(R)$ and $E_\mu(R^2)$ in terms of α , we obtain the following estimates of α and ρ :

$$\hat{\alpha}_1 = \frac{\mathcal{P}_{obs}}{10(\mathcal{A}_{obs} - 1) \log(1 - \mathcal{A}_{obs})} - 2, \quad \hat{\rho}_1 = \frac{5(\hat{\alpha}_1 + 1)\mathcal{P}_{obs}}{\pi(1 - \mathcal{A}_{obs})},$$

where \mathcal{A}_{obs} and \mathcal{P}_{obs} denote the observed area and perimeter per unit area of the set.

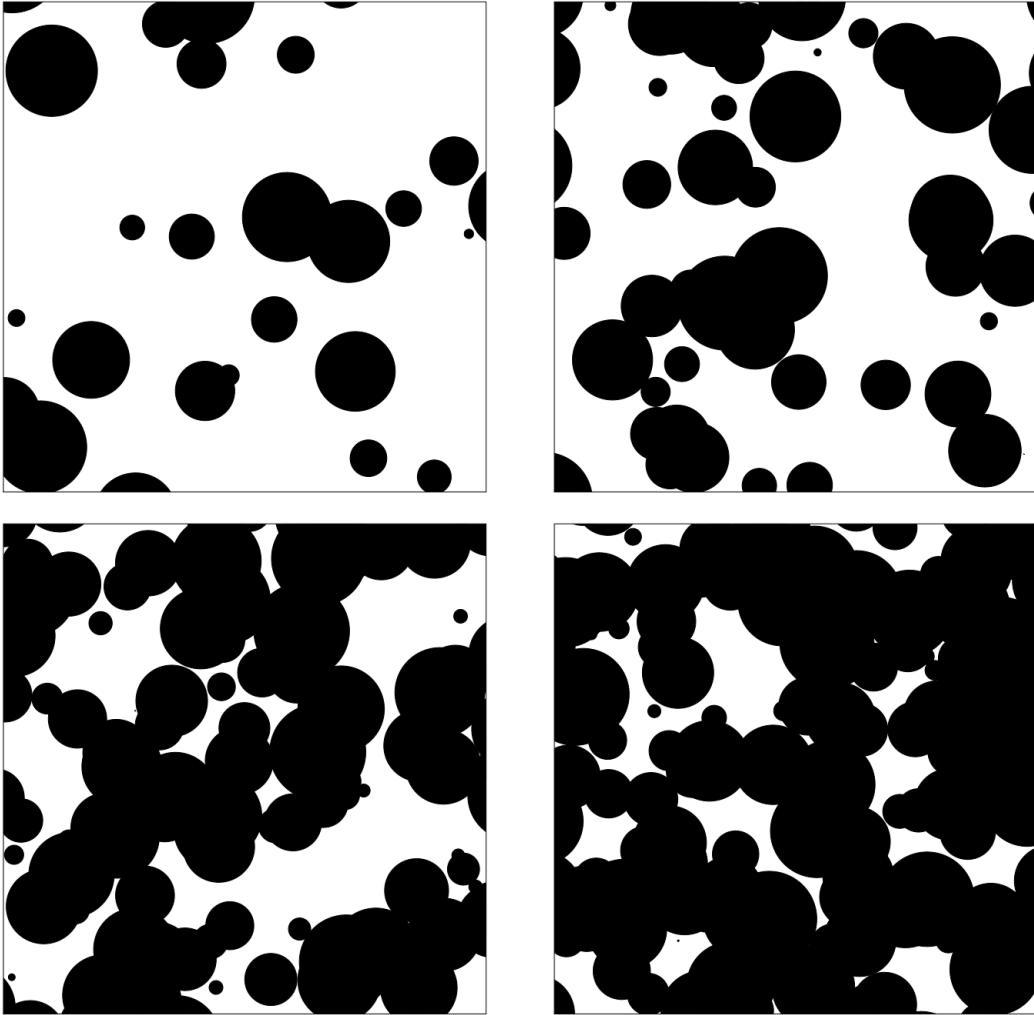


Figure 2: Samples from a Boolean model on $[0, 1]^2$ with intensity $\rho = 25, 50, 100, 150$, from top left to bottom right respectively, and law of radii $B(1, \alpha)$ where $\alpha = 1$.

An alternative procedure to estimate the intensity ρ is based on the number of tangent points to the random set in a given direction. Let u be a vector in \mathbb{R}^2 . We denote by $N(u)$ the number of tangent points to the random set such that the associated tangent line is orthogonal to u and the boundary of the set is convex in direction u . Considering k distinct vectors u_1, \dots, u_k , an estimator of ρ , studied in [21], is

$$\hat{\rho}_2 = \frac{\frac{1}{k} \sum_{i=1}^k N(u_i)}{|W|(1 - \mathcal{A}_{obs})},$$

where $|W|$ denotes the area of the observation window. Although this estimator is consistent and asymptotically normal for $k = 1$, it becomes more efficient as k increases, see [21]. In the following, we consider $k = 100$ and the directions of u_1, \dots, u_k are randomly drawn from a uniform distribution over $[0, 2\pi]$.

Let us now consider the combination of the above estimators. In connection with the previous sections, we have $d = 2$, $k_1 = 2$, $k_2 = 1$, $\mathbf{T}_1 = (\hat{\rho}_1, \hat{\rho}_2)$ and $\mathbf{T}_2 = \hat{\alpha}_1$. The averaging estimator over Λ_{\max} is denoted by $(\hat{\rho}_{AV}, \hat{\alpha}_{AV})$. In this setting, we recall that $\hat{\rho}_{AV}$ is a linear combination of $\hat{\rho}_1$ and $\hat{\rho}_2$ where the weights sum to one, whereas $\hat{\alpha}_{AV}$ equals $\hat{\alpha}_1$ plus a linear combination of $\hat{\rho}_1$ and $\hat{\rho}_2$ where the weights sum to zero. The weights are estimated according to (7), where Σ is obtained from Monte-Carlo simulations of the model with parameters $0.5(\hat{\rho}_1 + \hat{\rho}_2)$ and $\hat{\alpha}_1$ (see the previous section for more details).

Table 6 reports the MSE of each estimator, estimated from 10^4 replications from a Boolean model with parameters $\rho = 25, 50, 100, 150$ and $\alpha = 1$. For each replication, 100 Monte-Carlo samples were used. The averaging estimators have better performances than the initial estimators. It is worth noticing the improvement of $\hat{\alpha}_1$ when it is corrected by $\hat{\rho}_1$ and $\hat{\rho}_2$ through $\hat{\alpha}_{AV}$. Though this procedure might seem unnatural, the result is astonishing for this model. More simulations with other values of α (not reported in this paper) gave similar results.

| | $\hat{\rho}_1$ | $\hat{\rho}_2$ | $\hat{\rho}_{AV}$ | $\hat{\alpha}_1$ | $\hat{\alpha}_{AV}$ |
|--------------|------------------|-----------------|-------------------|------------------|---------------------|
| $\rho = 25$ | 34.15 (0.55) | 14.63 (0.22) | 14.60 (0.22) | 8.09 (0.15) | 6.70 (0.13) |
| $\rho = 50$ | 131.63 (2.26) | 47.41 (0.72) | 45.65 (0.67) | 4.69 (0.067) | 3.24 (0.048) |
| $\rho = 100$ | 949 (21.8) | 272 (4.9) | 223 (3.6) | 5.70 (0.086) | 2.29 (0.034) |
| $\rho = 150$ | 7606 (341) | 1656 (46.5) | 1005 (24.4) | 14.7 (0.34) | 4.1 (0.11) |

Table 6: Monte Carlo estimation of the MSE of $\hat{\rho}_1$, $\hat{\rho}_2$, $\hat{\rho}_{AV}$ and $\hat{\alpha}_1$, $\hat{\alpha}_{AV}$ based on 10^4 replications of a Boolean model with parameters $\rho = 25, 50, 100, 200$ and $\mu \sim B(1, \alpha)$ with $\alpha = 1$. The standard deviation of the MSE estimations are given in parenthesis. The two last columns have been multiplied by 100 for ease of presentation.

As explained before, we can deduce from $\hat{\rho}_{AV}$ and $\hat{\alpha}_{AV}$ an asymptotic confidence interval

without any further estimation. The length of these intervals is smaller than the length of any similar confidence interval based on the initial estimators and Table 7 reports the empirical rate of coverage of these intervals, showing that it is close to the nominal level 95%.

| | $\rho = 25$ | $\rho = 50$ | $\rho = 100$ | $\rho = 150$ |
|---------------------|-------------|-------------|--------------|--------------|
| $\hat{\rho}_{AV}$ | 98.3 % | 97.6 % | 96.5 % | 93.4 % |
| $\hat{\alpha}_{AV}$ | 95.9 % | 94.3 % | 93.9 % | 94.9 % |

Table 7: Empirical rate of coverage (in %) of the asymptotic 95% confidence intervals based on $\hat{\rho}_{AV}$ and $\hat{\alpha}_{AV}$ for the parameters of the Boolean model, deduced from the same simulations as in Table 6.

5 Appendix

Proof of Theorem 3.1

Since $\Lambda \subseteq \Lambda_{\max}$, we know that $\lambda^\top \mathbf{J} = \mathbf{I}$ for all $\lambda \in \Lambda$. Let $\mathbf{S} = \Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)$, we have

$$\|\hat{\theta} - \hat{\theta}^*\|^2 = \|(\hat{\lambda} - \lambda^*)^\top (\mathbf{T} - \mathbf{J}\theta)\|^2 = \|(\hat{\lambda} - \lambda^*)^\top \Sigma^{\frac{1}{2}} \mathbf{S}\|^2 \leq \|(\hat{\lambda} - \lambda^*)^\top \Sigma^{\frac{1}{2}}\|_F^2 \|\mathbf{S}\|^2, \quad (19)$$

where $\|A\|_F = \sqrt{\text{tr}(A^\top A)}$ denotes the Frobenius norm of A . The map $\phi : \lambda \mapsto \text{tr}(\lambda^\top \Sigma \lambda)$ is coercive, and strictly convex by assumption. So, since Λ is closed and convex, the minimum of ϕ on Λ is reached at a unique point $\lambda^* \in \Lambda$. Moreover, we know that for $\lambda \in \Lambda$, $\lambda^* + t(\lambda - \lambda^*)$ lies in Λ for all $t \in [0, 1]$, to which we deduce the optimality condition

$$\lim_{t \rightarrow 0^+} \frac{\phi(\lambda^* + t(\lambda - \lambda^*)) - \phi(\lambda^*)}{t} = \text{tr}[\nabla \phi(\lambda^*)^\top (\lambda - \lambda^*)] = 2 \text{tr}[\lambda^{*\top} \Sigma (\lambda - \lambda^*)] \geq 0,$$

for all $\lambda \in \Lambda$. It follows that

$$\begin{aligned} \|(\hat{\lambda} - \lambda^*)^\top \Sigma^{\frac{1}{2}}\|_F^2 &= \text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*) - 2 \text{tr}[\lambda^{*\top} \Sigma (\hat{\lambda} - \lambda^*)] \\ &\leq \text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*). \end{aligned} \quad (20)$$

By construction of $\hat{\lambda}$, we know that $\text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) \leq \text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*)$, yielding

$$\begin{aligned} \text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*) &\leq \text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) + \text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \\ &\leq \text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) \delta_\Lambda(\Sigma | \hat{\Sigma}) + \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \delta_\Lambda(\hat{\Sigma} | \Sigma) \\ &\leq [\text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) + \text{tr}(\lambda^{*\top} \Sigma \lambda^*)] \delta_\Lambda(\hat{\Sigma}, \Sigma) \end{aligned}$$

where we recall $\delta_\Lambda(A|B) = \sup_{\lambda \in \Lambda} \left| 1 - \frac{\text{tr}(\lambda^\top A \lambda)}{\text{tr}(\lambda^\top B \lambda)} \right|$ and $\delta_\Lambda(A, B) = \max\{\delta_\Lambda(A|B), \delta_\Lambda(B|A)\}$. Now using that $\text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) \leq \text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*)$ and

$$\text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*) = \text{tr}(\lambda^{*\top} \Sigma \lambda^*) + [\text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*)] \leq \text{tr}(\lambda^{*\top} \Sigma \lambda^*) [1 + \delta_\Lambda(\hat{\Sigma}, \Sigma)],$$

we obtain

$$\text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \leq \text{tr}(\lambda^{*\top} \Sigma \lambda^*) [2\delta_\Lambda(\hat{\Sigma}, \Sigma) + \delta_\Lambda(\hat{\Sigma}, \Sigma)^2]. \quad (21)$$

Recall that $\text{tr}(\lambda^{*\top} \Sigma \lambda^*) = \inf_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2$, the result follows from (19), (20) and (21).

Proof of Corollary 3.2

The proof follows directly from an application of Markov's inequality to the term $\|\Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)\|^2$ and the next lemma.

Lemma 5.1 *Let A, B be two positive definite matrices of order k . For any non-empty set Λ that does not contain 0,*

$$\delta_\Lambda(A, B) \leq \| \|AB^{-1} - BA^{-1}\| \|,$$

where $\| \|A\| \| = \sup_{\|x\|_F=1} \|Ax\|_F$ stands for the operator norm.

Proof. By symmetry, it is sufficient to show that the result holds for $\delta_\Lambda(A|B)$. We have

$$\delta_\Lambda(A|B) = \sup_{\lambda \in \Lambda} \frac{|\text{tr}[\lambda^\top (B - A)\lambda]|}{\text{tr}(\lambda^\top B\lambda)} \leq \sup_{\lambda \neq 0} \frac{|\text{tr}[\lambda^\top (B - A)\lambda]|}{\text{tr}(\lambda^\top B\lambda)}.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} |\text{tr}[\lambda^\top (B - A)\lambda]| &= |\text{tr}[\lambda^\top B^{\frac{1}{2}} (I - B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) B^{\frac{1}{2}}\lambda]| \\ &\leq \|B^{\frac{1}{2}}\lambda\|_F \|(I - B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) B^{\frac{1}{2}}\lambda\|_F \\ &\leq \|I - B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\| \|B^{\frac{1}{2}}\lambda\|_F^2. \end{aligned} \quad (22)$$

Recall that $\|B^{\frac{1}{2}}\lambda\|_F^2 = \text{tr}(\lambda^\top B\lambda)$, it follows

$$\delta_\Lambda(A|B) \leq \| \|I - B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\| \|.$$

Since the matrix $C = I - B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is symmetric, it is diagonalizable in an orthogonal basis. In particular, denoting $\text{sp}(\cdot)$ the spectrum, $\| \|C\| \| = \sup_{t \in \text{sp}(C)} |t|$. Finally, observe that $\text{sp}(C) = 1 - \text{sp}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}) = 1 - \text{sp}(AB^{-1})$, so that AB^{-1} has positive eigenvalues and

$$\| \|I - B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\| \| = \sup_{t \in \text{sp}(AB^{-1})} |1 - t| \leq \sup_{t \in \text{sp}(AB^{-1})} |t - \frac{1}{t}| \leq \| \|AB^{-1} - BA^{-1}\| \|,$$

ending the proof. ■

Proof of Proposition 3.3

By Lemma 5.1, we know that

$$\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n) \leq \| \|\hat{\Sigma}_n \Sigma_n^{-1} - \Sigma_n \hat{\Sigma}_n^{-1}\| \| . \quad (23)$$

In particular, $\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n) = o_p(1)$ by the assumption $\hat{\Sigma}_n \Sigma_n^{-1} \xrightarrow{p} I$. Write for $c > 0$,

$$\|\hat{\theta}_n - \theta\|^2 \leq (1 + c)\|\hat{\theta}_n^* - \theta\|^2 + (1 + c^{-1})\|\hat{\theta}_n - \hat{\theta}_n^*\|^2.$$

Applying Theorem 3.1, we get

$$\|\hat{\theta}_n - \theta\|^2 \leq (1 + c)\|\hat{\theta}_n^* - \theta\|^2 + (1 + c^{-1})\alpha_n \left(2\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n) + \delta_\Lambda(\hat{\Sigma}_n, \Sigma_n)^2 \right) \|\mathbf{S}_n\|^2, \quad (24)$$

where $\mathbf{S}_n = \Sigma_n^{-\frac{1}{2}}(\mathbf{T}_n - \mathbf{J}\theta)$. Since $\mathbb{E}\|\mathbf{S}_n\|^2 = k$, we know that $\|\mathbf{S}_n\|^2 = O_p(1)$. Equation (24) holds for all $c > 0$ so we can take $c = c_n$ such that $c_n \rightarrow 0$ and $\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n)/c_n \xrightarrow{p} 0$ as $n \rightarrow \infty$, yielding

$$\|\hat{\theta}_n - \theta\|^2 \leq \|\hat{\theta}_n^* - \theta\|^2 + c_n \|\hat{\theta}_n^* - \theta\|^2 + o_p(\alpha_n) = \|\hat{\theta}_n^* - \theta\|^2 + o_p(\alpha_n).$$

We shall now prove the second part of the proposition. Write,

$$\hat{\alpha}_{n,j}^{-\frac{1}{2}}(\hat{\theta}_{n,j} - \theta_j) = \sqrt{\frac{\alpha_{n,j}}{\hat{\alpha}_{n,j}}} \alpha_{n,j}^{-\frac{1}{2}} [(\hat{\theta}_{n,j}^* - \theta_j) + (\hat{\theta}_{n,j} - \hat{\theta}_{n,j}^*)].$$

To prove the result, it suffices to show that $\alpha_{n,j}^{-\frac{1}{2}} \|\hat{\theta}_{n,j} - \hat{\theta}_{n,j}^*\| = o_p(1)$ and $\alpha_{n,j}/\hat{\alpha}_{n,j} \xrightarrow{p} 1$. When Λ is a cylinder, it is easy to see that the following holds

$$\hat{\lambda}_{n,j} = \arg \min_{\lambda \in \Lambda_j} \lambda^\top \hat{\Sigma}_n \lambda \quad \text{and} \quad \lambda_{n,j}^* = \arg \min_{\lambda \in \Lambda_j} \lambda^\top \Sigma_n \lambda,$$

where we recall $\Lambda_j = \{\lambda_j : \lambda \in \Lambda\}$. Moreover, we easily adapt the proof of Theorem 3.1 to get

$$\|\hat{\theta}_{n,j} - \hat{\theta}_{n,j}^*\|^2 \leq \alpha_{n,j} (2\delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n) + \delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n)^2) \|\Sigma_n^{-\frac{1}{2}}(\mathbf{T}_n - \mathbf{J}\theta)\|^2.$$

We deduce that $\alpha_{n,j}^{-\frac{1}{2}}(\hat{\theta}_{n,j} - \hat{\theta}_{n,j}^*) = o_p(1)$ in view of (13) and Lemma 5.1. Now, remark that

$$\frac{\alpha_{n,j}}{\hat{\alpha}_{n,j}} = \frac{\lambda_{n,j}^{*\top} \Sigma_n \lambda_{n,j}^*}{\hat{\lambda}_{n,j}^\top \hat{\Sigma}_n \hat{\lambda}_{n,j}} \leq \frac{\hat{\lambda}_{n,j}^\top \Sigma_n \hat{\lambda}_{n,j}}{\hat{\lambda}_{n,j}^\top \hat{\Sigma}_n \hat{\lambda}_{n,j}} - 1 + 1 \leq \delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n) + 1.$$

Similarly,

$$\frac{\hat{\alpha}_{n,j}}{\alpha_{n,j}} \leq \delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n) + 1.$$

So, we get

$$\frac{1}{1 + \delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n)} \leq \frac{\alpha_{n,j}}{\hat{\alpha}_{n,j}} \leq 1 + \delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n),$$

proving that $\alpha_{n,j}/\hat{\alpha}_{n,j} \xrightarrow{p} 1$.

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