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A general procedure to combine estimators

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Abstract

We propose a general method to combine several estimators of the same quantity in order to produce a better estimate. In the spirit of model and forecast averaging, the final estimator is computed as a weighted average of the initial ones, where the weights are constrained to sum to one. In this framework, the optimal weights, minimizing the quadratic loss, are entirely determined by the mean square error matrix of the vector of initial estimators. The averaging estimator is derived using an estimation of this matrix, which can be computed from the same dataset. We show that the solution satisfies a non-asymptotic oracle inequality and is asymptotically optimal, provided the mean square error matrix is suitably estimated. This method is illustrated on standard statistical problems: estimation of the position of a symmetric distribution, estimation in a parametric model, density estimation, where the averaging estimator outperforms the initial estimators in most cases.

Keywords. Averaging ; Aggregation ; Oracle inequality ; Parametric estimation ; Weibull model ; Kernel density estimation

1 Introduction

We are interested in estimating a parameter θ in a statistical model, based on a collection of preliminary estimators $\mathbf{T} = (T_1, \dots, T_k)$. The issue of dealing with several possibly competing estimators of the same quantity arises in numerous situations for which methods have been developed to produce a single final estimator that hopefully performs as well as possible. A simple solution is to consider a weighted average of the T_i 's. The idea of estimator averaging goes back to the early 19th century with Pierre Simon de Laplace [18], who was interested in finding the best combination between the mean and the median to estimate the location parameter of a symmetric distribution, see the discussion in [28]. More generally, the solution can be expressed as a linear

combination of the initial estimators

$$\hat{\theta}_\lambda = \lambda^\top \mathbf{T} = \sum_{i=1}^k \lambda_i T_i, \quad (1)$$

for λ a vector of weights lying in a subset Λ of \mathbb{R}^k . A large number of statistical frameworks fit with this description. For example, model selection can be viewed as a particular case of estimator averaging for Λ the set of vertices. Similarly, convex combinations corresponds to $\Lambda = \{\lambda : \sum_{i=1}^k \lambda_i = 1, \lambda_i \geq 0\}$ while linear combinations to $\Lambda = \mathbb{R}^k$. Another well-used framework consists in relaxing the positivity condition of convex combination, corresponding to the set $\Lambda = \{\lambda : \sum_{i=1}^k \lambda_i = 1\}$.

Estimator averaging has received a particular attention for prediction purposes. Ever since the paper of Bates and Granger [2], dealing with forecast averaging for time series, the literature on this subject has greatly developed, see the surveys [8, 30]. In this framework, the parameter θ represents the future observation of a series to be predicted and \mathbf{T} a collection of predictors. However, as pointed out in [27], forecast averaging procedures remain often outperformed by the simple average in practice. Averaging methods have also been widely used for prediction in a regression framework, with somewhat more promising results. Then θ is the dependent variable given some regressors, and \mathbf{T} is a collection of models output. These so-called model averaging procedures are shown to provide good alternatives to model selection for parametric regression, see e.g. [20] for a survey and [11, 12] for asymptotic properties. In these studies, the averaging estimator aims to mimic the oracle, defined as the linear combination $\hat{\theta}_\lambda$ minimizing the quadratic loss, under the affine constraint on the weights $\sum_{i=1}^k \lambda_i = 1$. The main reason for using this particular constraint is that the oracle simply expresses in terms of the mean squared error matrix Σ of \mathbf{T} , as recalled in Section 2. The averaging estimator is then defined by replacing Σ by an estimator $\hat{\Sigma}$.

In this paper we aim to extend this approach to a more general setting, not restricted to prediction. Typically, the procedure we propose applies to classical estimation problems for a wide range of statistical models, when several estimators T_i of the same quantity θ are available. For instance θ may represent the position of an unknown distribution, which can be estimated by both the empirical mean and median, as initially addressed by P. S. de Laplace in [18]. Or θ may be the unknown parameter of a parametric model, say the 2-dimensional parameter of a Weibull distribution for which several competing estimators exist, as exemplified in Section 5. This method also covers averaging for the mean in a Gaussian model as discussed in [9, 10, 19] and more recently [17]. As a last example, θ can stand for the unknown probability density of a distribution, in which case θ belongs to a functional space and \mathbf{T} a collection of estimates obtained for instance from different choices of bandwidths. As in the above mentioned works, the averaging estimator is entirely determined by an estimate $\hat{\Sigma}$ of Σ . In most cases, this estimation

can be carried out by standard methods (e.g. plug-in or bootstrap), and does not require the tuning of any extra parameter. As revealed by our computational study of Section 5, the procedure appears to be quite robust even in semi-parametric and non-parametric models.

We extend the above averaging procedure towards two directions. First, apart from the condition $\sum_{i=1}^k \lambda_i = 1$, the weights λ may satisfy some additional constraints, leading for instance to convex averaging in the case of positive weights. Second, we include the case where several parameters $\theta_1, \dots, \theta_d$ have to be estimated (as for the Weibull model discussed earlier), and a collection of estimators is available for each of them. In order to fully exploit the available information to estimate say θ_1 , it may be profitable to average all estimators, including those designed for θ_j , $j \neq 1$. We show that a minimal requirement is that the weights associated to the latter estimators sum to 0, while the weights associated to the estimators of θ_1 sum to one.

The performance of the averaging procedure clearly depends on the accuracy of $\hat{\Sigma}$ to estimate Σ . In the frame of model averaging, Hansen and Racine [12] proved the asymptotic optimality of the method for a particular form of $\hat{\Sigma}$ obtained by jackknife. We establish a non asymptotic oracle inequality, under the general setting explained above, which implies in particular the asymptotic result of [12]. Note that some oracle inequalities have been proved in the closely related domain of functional aggregation, designed for non-parametric regression [16, 32, 7, 4] or density estimation [22, 5, 31, 6]. However, these theoretical results are derived in assuming that the averaging process is conducted independently from \mathbf{T} , which would correspond in our case to assume that $\hat{\Sigma}$ and \mathbf{T} are independent. This condition is generally achieved via sample splitting to separate training and validation, which can be a high price to pay especially when few data are available. On the contrary, it seems preferable to take profit of all the available information to produce both \mathbf{T} and $\hat{\Sigma}$, meaning that \mathbf{T} and $\hat{\Sigma}$ are built from the same data. Therefore we do not assume sample splitting and our oracle inequality is proved whatever the dependence between \mathbf{T} and $\hat{\Sigma}$.

The averaging procedure is detailed in Section 2, both for the estimation of one parameter, that we assume to belong to a Hilbert space, and for the estimation of several parameters. In Section 3, we discuss some examples of natural frameworks, i.e. choices of the set of weights Λ , or equivalently of the constraints followed by the weights. In Section 4 we derive a non asymptotic oracle inequality for the averaging estimator and we discuss its asymptotic optimality. Section 5 is devoted to some examples of problems, where we show that the method performs almost always better than the best estimator in the initial collection \mathbf{T} . These examples deal with the estimation of the position of a symmetric distribution, estimation in a parametric model, and kernel density estimators. Proofs of our results are postponed to the Appendix.

2 The averaging procedure

The method discussed in this section is slightly different whether it is applied to one parameter or several. For ease of comprehension, we first present the averaging procedure for one parameter, which follows the idea introduced in [2] for forecast averaging, though our choice of the set of weights Λ may be different. We then discuss a generalization of the procedure for averaging several parameters simultaneously.

2.1 Averaging for one parameter

Let $\mathbf{T} = (T_1, \dots, T_k)$ be a collection of estimators of a parameter θ lying in some Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. We search for a decision rule that combines suitably the T_i 's to provide a unique estimate of θ . The best unconstrained transformation $f(\mathbf{T})$ is trivially $f(\mathbf{T}) = \theta$ and this choice is pointless. A reasonable alternative is to settle for linear transformations

$$\hat{\theta}_\lambda = \lambda^\top \mathbf{T}, \quad \lambda \in \Lambda,$$

where λ^\top denotes the transpose of λ and Λ is a given subset of \mathbb{R}^k . In this linear setting, a convenient way to measure the performance of $\hat{\theta}_\lambda$ is to compare it to the best non random combination $\hat{\theta}^*$ in the class $\{\hat{\theta}_\lambda, \lambda \in \Lambda\}$, called *oracle*. Specifically, we define the oracle as the linear combination $\hat{\theta}^* = \lambda^{*\top} \mathbf{T}$ minimizing the mean square error (MSE), i.e.

$$\lambda^* = \arg \min_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2$$

where $\|\cdot\|$ denotes the norm on \mathcal{H} , i.e. for any $x \in \mathcal{H}$, $\|x\|^2 = \langle x, x \rangle$. Of course in practice λ^* is unknown and needs to be approximated by an estimator, say $\hat{\lambda}$.

Clearly, the larger the set Λ , the better the oracle. However, choosing the whole space $\Lambda = \mathbb{R}^k$ is generally not exploitable. Indeed, assuming that the Gram matrix $\mathbb{E}\langle \mathbf{T}, \mathbf{T}^\top \rangle$ (with entries $\mathbb{E}\langle T_i, T_j \rangle$) exists and is non-singular, the oracle over $\Lambda = \mathbb{R}^k$ is given by

$$\lambda_{\text{lin}}^* = \arg \min_{\lambda \in \mathbb{R}^k} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2 = \theta [\mathbb{E}\langle \mathbf{T}, \mathbf{T}^\top \rangle]^{-1} \mathbb{E}(\mathbf{T}).$$

For the solution $\hat{\theta} = \hat{\lambda}^\top \mathbf{T}$ to be comparable to the oracle, we need to be able to approach the optimal weights at least strictly better than we can estimate θ . The presence of θ in the above expression shows that λ_{lin}^* should be at least as difficult to estimate as θ , rendering linear averaging inefficient. In fact, the performance of the average highly relies on the choice of the set Λ . Indeed, choosing a too large set Λ might increase the accuracy of the oracle but make it difficult to estimate λ^* . On the contrary, a too small set Λ might lead to a poorly efficient oracle but easy to approximate. Therefore, a good balance must be found for the oracle to be both accurate and reachable.

Writing the estimation error as

$$\hat{\theta} - \theta = \hat{\theta}^* - \theta + (\hat{\lambda} - \lambda^*)^\top \mathbf{T}, \quad (2)$$

a good rule of thumb is to consider a set Λ for which the residual term $(\hat{\lambda} - \lambda^*)^\top \mathbf{T}$ can be made negligible compared to the error of the oracle $\hat{\theta}^* - \theta$. As seen above, this is not the case if $\Lambda = \mathbb{R}^k$. In this purpose, a solution proposed in [2] and widely used in the averaging literature is to consider the affine constraint set

$$\Lambda_{\max} = \{\lambda \in \mathbb{R}^k : \lambda^\top \mathbf{1} = 1\},$$

where $\mathbf{1}$ denotes the unit vector $\mathbf{1} = (1, \dots, 1)^\top$. If λ^* and $\hat{\lambda}$ are in Λ_{\max} , the error term can be written as

$$(\hat{\lambda} - \lambda^*)^\top \mathbf{T} = (\hat{\lambda} - \lambda^*)^\top (\mathbf{T} - \theta \mathbf{1}),$$

where both $(\hat{\lambda} - \lambda^*)$ and $(\mathbf{T} - \theta \mathbf{1})$ can contribute to make the error term negligible.

In the sequel, we let Λ be a non-empty closed convex subset of Λ_{\max} . We assume that the initial estimators have finite order-two moments and $1, T_1, \dots, T_k$ are linearly independent so that the Gram matrix

$$\Sigma = \mathbb{E} \langle \mathbf{T} - \theta \mathbf{1}, (\mathbf{T} - \theta \mathbf{1})^\top \rangle = (\mathbb{E} \langle T_i - \theta, T_j - \theta \rangle)_{i,j=1,\dots,k}$$

is well defined and non-singular. From the identity $\lambda^\top \mathbf{1} = 1$, we see that the optimal weight λ^* defining the oracle $\hat{\theta}^* = \lambda^{*\top} \mathbf{T}$ writes

$$\lambda^* = \arg \min_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2 = \arg \min_{\lambda \in \Lambda} \lambda^\top \Sigma \lambda.$$

Remark that the assumptions made on Λ ensure both existence and unicity of the minimizer. In the particular important example where $\Lambda = \Lambda_{\max}$, we get the well known explicit solution

$$\lambda_{\max}^* = \frac{\Sigma^{-1} \mathbf{1}}{\mathbf{1}^\top \Sigma^{-1} \mathbf{1}},$$

considered for instance in [2], [8] or [12]. Of course, in practice, the MSE matrix Σ is unknown and has to be approximated by some estimator $\hat{\Sigma}$ to yield the averaging estimator $\hat{\theta} = \hat{\lambda}^\top \mathbf{T}$, where

$$\hat{\lambda} = \arg \min_{\lambda \in \Lambda} \lambda^\top \hat{\Sigma} \lambda.$$

There are natural methods to construct $\hat{\Sigma}$ that essentially differ whether the model is parametric or not. In a fully specified parametric model in which Σ is known up to θ , the MSE can be estimated by plugging in an initial estimate of θ . Precisely, assuming that the MSE can be expressed as the image of θ through a known map $\Sigma(\cdot) : \mathcal{H} \rightarrow \mathbb{R}^{k \times k}$, one can choose $\hat{\Sigma} = \Sigma(\hat{\theta}_0)$, where $\hat{\theta}_0$ is an initial estimate of θ . A natural choice for $\hat{\theta}_0$ is to

take one of the initial estimator or the average $\bar{\mathbf{T}} = \frac{1}{k} \sum_{i=1}^k T_i$. In this case, the averaging procedure does not require any information other than the initial collection of estimator. Remark that even if the map $\Sigma(\cdot)$ is not explicitly known, $\Sigma(\hat{\theta}_0)$ may be approximated by parametric bootstrap. On the other hand, in a non-parametric setting, an estimation of Σ may be achieved by standard (non-parametric) bootstrap. Alternatively, a parametric closed-form expression for Σ may be available asymptotically, i.e. when the sample size on which \mathbf{T} is built tends to infinity, and the plugging method explained above then becomes possible. Some of these methods are illustrated in our examples in Section 5.

2.2 Averaging for several parameters

We now discuss a generalization of the method that deals with several parameters simultaneously. Let $\theta = (\theta_1, \dots, \theta_d)^\top \in \mathcal{H}^d$ and assume we have access to several collections of estimators, $\mathbf{T}_1, \dots, \mathbf{T}_d$, one for each component θ_j . For sake of generality we allow the collections \mathbf{T}_j 's to have different sizes denoted k_1, \dots, k_d respectively. So, let $\mathbf{T}_1 \in \mathcal{H}^{k_1}, \dots, \mathbf{T}_d \in \mathcal{H}^{k_d}$ and denote $\mathbf{T} = (\mathbf{T}_1^\top, \dots, \mathbf{T}_d^\top)^\top \in \mathcal{H}^k$, with $k = \sum_{j=1}^d k_j$. We consider averaging estimators of θ of the form

$$\hat{\theta}_\lambda = \lambda^\top \mathbf{T} \in \mathcal{H}^d,$$

where here, λ is a $k \times d$ matrix. For similar reasons as previously, we choose to make some restrictions on the set of authorized values for λ . In this purpose, let

$$\mathbf{J} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \in \mathbb{R}^{k \times d},$$

where the j -th column of \mathbf{J} contains exactly k_j ones, and define the maximal restriction set

$$\Lambda_{\max} = \{\lambda \in \mathbb{R}^{k \times d} : \lambda^\top \mathbf{J} = \mathbf{I}\}, \quad (3)$$

with \mathbf{I} the identity matrix. Let $\Pi_j(\lambda)$ denote the j -th column of $\lambda \in \mathbb{R}^{k \times d}$. For each component θ_j , the average is given by

$$\hat{\theta}_{\lambda,j} = \Pi_j(\lambda)^\top \mathbf{T} = \underline{\lambda}_{j,1}^\top \mathbf{T}_1 + \dots + \underline{\lambda}_{j,d}^\top \mathbf{T}_d,$$

where $\Pi_j(\lambda) = (\underline{\lambda}_{j,1}^\top, \dots, \underline{\lambda}_{j,d}^\top)^\top$ with $\underline{\lambda}_{j,\ell} \in \mathbb{R}^{k_\ell}$, $\ell = 1, \dots, d$. Imposing that $\lambda \in \Lambda_{\max}$ means that for any $j = 1, \dots, d$

$$\underline{\lambda}_{j,\ell}^\top \mathbf{1} = \begin{cases} 0 & \text{if } \ell \neq j \\ 1 & \text{if } \ell = j. \end{cases} \quad (4)$$

In particular, this condition does not rule out using the entire collection \mathbf{T} to estimate each component θ_j , although the weights $\underline{\lambda}_{j,\ell}$ do not satisfy the same constraints depending on the relevance of \mathbf{T}_ℓ . While it may seem more natural to impose that only \mathbf{T}_j is involved in the estimation of θ_j (and this can be made easily through an appropriate choice of $\Lambda \subset \Lambda_{\max}$, letting $\underline{\lambda}_{j,\ell} = 0$ for $\ell \neq j$), allowing one to use the whole set \mathbf{T} to estimate each component enables to take into account possible dependencies, which may improve the results. Moreover, the condition $\lambda^\top \mathbf{J} = \mathbf{I}$ appears as a minimal requirement to obtain an oracle inequality, as shown further in Theorem 4.1.

Since there is no ambiguity, we shall use abusively the same notation $\|\cdot\|$ to refer to the norm in \mathcal{H}^d , i.e. for any $a = (a_1, \dots, a_d)^\top \in \mathcal{H}^d$

$$\|a\| = \|a\|_{\mathcal{H}^d} = \sqrt{\|a_1\|_{\mathcal{H}}^2 + \dots + \|a_d\|_{\mathcal{H}}^2}.$$

Similarly, for a and b in \mathcal{H}^d , $\langle a^\top, b \rangle$ stands for $\sum \langle a_i, b_i \rangle$, while $\langle a, b^\top \rangle$ denotes as before the Gram matrix with entries $\langle a_i, b_j \rangle$.

Notice that the condition $\lambda^\top \mathbf{J} = \mathbf{I}$ implies that

$$\lambda^\top \mathbf{T} - \theta = \lambda^\top (\mathbf{T} - \mathbf{J}\theta),$$

and the expression of the mean square error can be rewritten

$$\begin{aligned} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2 &= \mathbb{E} [\langle (\mathbf{T} - \mathbf{J}\theta)^\top \lambda, \lambda^\top (\mathbf{T} - \mathbf{J}\theta) \rangle] \\ &= \mathbb{E} [\text{tr} (\langle (\mathbf{T} - \mathbf{J}\theta)^\top \lambda, \lambda^\top (\mathbf{T} - \mathbf{J}\theta) \rangle)] \\ &= \mathbb{E} [\text{tr} (\langle \lambda^\top (\mathbf{T} - \mathbf{J}\theta), (\mathbf{T} - \mathbf{J}\theta)^\top \lambda \rangle)] \\ &= \text{tr}(\lambda^\top \Sigma \lambda), \end{aligned}$$

where $\Sigma = \mathbb{E} \langle \mathbf{T} - \mathbf{J}\theta, (\mathbf{T} - \mathbf{J}\theta)^\top \rangle \in \mathbb{R}^{k \times k}$ and $\text{tr}(\cdot)$ denotes the trace operator. Here again, we assume that Σ exists and is non-singular.

The simultaneous averaging process for several parameters generalizes the procedure presented in Section 2.1. In fact, averaging for one parameter just becomes the particular case with $d = 1$. Given a subset $\Lambda \subseteq \Lambda_{\max}$, we define the oracle as the linear transformation $\hat{\theta}^* = \lambda^{*\top} \mathbf{T}$ with

$$\lambda^* = \arg \min_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2 = \arg \min_{\lambda \in \Lambda} \text{tr}(\lambda^\top \Sigma \lambda). \quad (5)$$

Finally, assuming we have access to an estimator $\hat{\Sigma}$ of Σ , we define the averaging estimator as $\hat{\theta} = \hat{\lambda}^\top \mathbf{T}$ where

$$\hat{\lambda} = \arg \min_{\lambda \in \Lambda} \text{tr}(\lambda^\top \hat{\Sigma} \lambda). \quad (6)$$

If $\lambda^\top \Sigma \lambda$ is well approximated by $\lambda^\top \hat{\Sigma} \lambda$ for $\lambda \in \Lambda$, we may reasonably think that the average $\hat{\theta}$ will be close to the oracle $\hat{\theta}^*$, regardless of the possible dependency between $\hat{\Sigma}$ and \mathbf{T} .

3 Examples of constraint sets

3.1 Maximal constraint set

When a good estimation of Σ can be provided, it is natural to consider the maximal constraint set $\Lambda = \Lambda_{\max}$ defined in (3), thus aiming for the best possible oracle. This set is actually an affine subspace of $\mathbb{R}^{k \times d}$ and in particular, it is convex. The oracle, obtained by minimizing the convex map $\lambda \mapsto \text{tr}(\lambda^\top \Sigma \lambda)$ subject to the constraint $\lambda^\top \mathbf{J} = \mathbf{I}$ is given by $\hat{\theta}_{\max}^* = \lambda_{\max}^{*\top} \mathbf{T}$ where

$$\lambda_{\max}^* = \Sigma^{-1} \mathbf{J} (\mathbf{J}^\top \Sigma^{-1} \mathbf{J})^{-1}, \quad (7)$$

generalizing the formula given in Section 2.1, and its mean-square error can be calculated directly

$$\mathbb{E} \langle \hat{\theta}_{\max}^* - \theta, (\hat{\theta}_{\max}^* - \theta)^\top \rangle = (\mathbf{J}^\top \Sigma^{-1} \mathbf{J})^{-1}.$$

This solution is a direct consequence of the equality

$$\lambda^\top \Sigma \lambda - (\mathbf{J}^\top \Sigma^{-1} \mathbf{J})^{-1} = \lambda^\top \Sigma \lambda - \lambda_{\max}^{*\top} \Sigma \lambda_{\max}^* = (\lambda - \lambda_{\max}^*)^\top \Sigma (\lambda - \lambda_{\max}^*) \quad (8)$$

that holds for all $\lambda \in \Lambda_{\max}$ due to the condition $\lambda^\top \mathbf{J} = \mathbf{I}$, and where the last matrix is positive definite.

Moreover, (8) shows that the oracle is not only the solution of our optimization problem over Λ_{\max} , but it is optimal to estimate any linear transformation of θ . In particular each component $\hat{\theta}_{\max,j}^*$ of the oracle is the best linear transformation $\lambda^\top \mathbf{T}$, $\lambda \in \Lambda_{\max}$, that one can get to estimate θ_j . Another desirable property of the choice $\Lambda = \Lambda_{\max}$ is that due to the closed expression (7), the averaging estimator $\hat{\theta}_{\max}$ obtained by replacing Σ by its estimation $\hat{\Sigma}$ has also a closed expression which makes it easily computable, namely

$$\hat{\theta}_{\max} = (\mathbf{J}^\top \hat{\Sigma}^{-1} \mathbf{J})^{-1} \mathbf{J}^\top \hat{\Sigma}^{-1} \mathbf{T}. \quad (9)$$

3.2 Component-wise averaging

A natural and simpler framework is to consider component-wise averaging, for which only the collection \mathbf{T}_j is involved in the estimation of θ_j . The associated set of weights is the set of matrices λ whose support is included in the support of \mathbf{J} , that is

$$\Lambda = \{\lambda \in \Lambda_{\max} : \text{supp}(\lambda) \subseteq \text{supp}(\mathbf{J})\},$$

where for a matrix $A = (A_{i,j}) \in \mathbb{R}^{k \times d}$, $\text{supp}(A) := \{(i, j), A_{i,j} \neq 0\}$. In this particular framework, the covariance of two initial estimators in different collection $\mathbf{T}_i, \mathbf{T}_j$, $i \neq j$ is not involved in the computation of the oracle, so that the corresponding entries of Σ need not be estimated. Consequently, each component of θ is combined regardless of the others and as a result, the oracle is given by

$$\hat{\theta}_j^* = \frac{\mathbf{1}^\top \Sigma_j^{-1} \mathbf{T}_j}{\mathbf{1}^\top \Sigma_j^{-1} \mathbf{1}}, \quad j = 1, \dots, d.$$

where

$$\Sigma_j = \mathbb{E}\langle \mathbf{T}_j - \theta_j \mathbf{1}, (\mathbf{T}_j - \theta_j \mathbf{1})^\top \rangle \in \mathbb{R}^{k_j \times k_j}, \quad j = 1, \dots, d.$$

In order to build the averaging estimator, it is sufficient to plug an estimate of Σ_j , $j = 1, \dots, d$, in the above expression, which makes it easily computable. See Section 5.2 for further discussion.

3.3 Convex averaging

Convex averaging corresponds to the choice

$$\Lambda = \{\lambda \in \Lambda_{\max} : \lambda_{i,j} \geq 0, \quad i = 1, \dots, k, \quad j = 1, \dots, d\}.$$

Observe that the positivity restriction combined with the condition $\lambda^\top \mathbf{J}$ results in λ having its support included in that of \mathbf{J} , making convex averaging a particular case of component-wise averaging. This means in particular that each component of θ can be dealt with separately. So, for sake of simplicity in this example, we only consider the case $d = 1$.

Convex combination of estimators is a natural choice that has been widely studied in the literature. An advantage lies in the increased stability of the solution, due to the restriction of λ to a compact set, though the oracle may of course be less efficient than in the case $\Lambda = \Lambda_{\max}$. The use of convex combinations is also particularly convenient to preserve some properties of the initial estimators, such as positivity or boundedness. Moreover, imposing non-negativity often leads to sparse solutions.

In this convex constrained optimization problem, the minimizer $\hat{\lambda} = \arg \min_{\lambda \in \Lambda} \lambda^\top \hat{\Sigma} \lambda$ can either lie in the interior of the domain, in which case $\hat{\lambda} = \hat{\Sigma}^{-1} \mathbf{1} / \mathbf{1}^\top \hat{\Sigma}^{-1} \mathbf{1}$ corresponds

to the global minimizer over Λ_{\max} , or on the edge, meaning that it has at least one zero coordinate. Letting $\hat{m} \subseteq \{1, \dots, k\}$ denote the support of $\hat{\lambda}$, it follows that the averaging procedure obtained with the estimators $\mathbf{T}_{\hat{m}} := (T_i)_{i \in \hat{m}}$ leads to a solution $\hat{\lambda}_{\hat{m}}$ with full support. As a result, it can be expressed as the global minimizer for the collection $\mathbf{T}_{\hat{m}}$,

$$\hat{\lambda}_{\hat{m}} = \frac{\hat{\Sigma}_{\hat{m}}^{-1} \mathbf{1}}{\mathbf{1}^\top \hat{\Sigma}_{\hat{m}}^{-1} \mathbf{1}},$$

where $\hat{\Sigma}_{\hat{m}}$ is the submatrix composed of the entries $\hat{\Sigma}_{i,j}$ for $(i, j) \in \hat{m}^2$. Since we have by construction $\hat{\lambda}_{\hat{m}}^\top \mathbf{T}_{\hat{m}} = \hat{\lambda}^\top \mathbf{T} = \hat{\theta}$, we deduce the following characterization of the convex averaging solution:

$$\hat{\theta} = \frac{\mathbf{1}^\top \hat{\Sigma}_{\hat{m}}^{-1} \mathbf{T}_{\hat{m}}}{\mathbf{1}^\top \hat{\Sigma}_{\hat{m}}^{-1} \mathbf{1}},$$

where \hat{m} must be the support which is both admissible and provides the minimal mean-square error, i.e. $\hat{m} = \arg \max_{m \subseteq \{1, \dots, k\}} \mathbf{1}^\top \hat{\Sigma}_m^{-1} \mathbf{1}$ subject to the constraint that $\hat{\Sigma}_m^{-1} \mathbf{1}$ has all its coordinates positive. This provides an easy method to implement convex averaging in practice. Remark that this method is only efficient if k is not too large, otherwise we recommend to use a standard quadratic programming solver to get $\hat{\lambda}$, see for instance [21].

4 Theoretical results

4.1 Oracle inequality

The performance of the averaging estimator relies on the accuracy of $\hat{\Sigma}$, but more specifically, on the ability to evaluate the error $\text{tr}(\lambda^\top \Sigma \lambda)$ as λ ranges over Λ . As a result, it is not crucial that $\hat{\Sigma}$ be a perfect estimate of Σ as long as $\text{tr}(\lambda^\top \hat{\Sigma} \lambda)$ is close to its true value for $\lambda \in \Lambda$. In order to measure the accuracy of $\hat{\Sigma}$ for this particular purpose, we introduce the following criterion. For two symmetric positive definite matrices A and B and for any non-empty set Λ that does not contain 0, let $\delta_\Lambda(A|B)$ denote the maximal divergence of the ratio $\text{tr}(\lambda^\top A \lambda) / \text{tr}(\lambda^\top B \lambda)$ over Λ ,

$$\delta_\Lambda(A|B) = \sup_{\lambda \in \Lambda} \left| 1 - \frac{\text{tr}(\lambda^\top A \lambda)}{\text{tr}(\lambda^\top B \lambda)} \right|,$$

and $\delta_\Lambda(A, B) = \max\{\delta_\Lambda(A|B), \delta_\Lambda(B|A)\}$. We are now in position to state our main result.

Theorem 4.1 *Let Λ be a non-empty closed convex subset of Λ_{\max} and $\hat{\Sigma}$ a symmetric positive definite $k \times k$ matrix. The averaging estimator $\hat{\theta} = \hat{\lambda}^\top \mathbf{T}$ defined through (6) satisfies*

$$\|\hat{\theta} - \hat{\theta}^*\|^2 \leq \left[\inf_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2 \right] \left(2\delta_\Lambda(\hat{\Sigma}, \Sigma) + \delta_\Lambda(\hat{\Sigma}, \Sigma)^2 \right) \|\Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)\|^2, \quad (10)$$

where $\hat{\theta}^*$ is the oracle given by (5).

In this theorem, we provide an upper bound on the distance of the averaging estimator to the oracle. We emphasize that this result holds without requiring any condition on the joint behavior of \mathbf{T} and $\hat{\Sigma}$ (in particular, they may be strongly dependent). The influence of the constraint set Λ in the process becomes apparent through both the minimal error $\inf_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2$ and the maximal divergence $\delta_\Lambda(\hat{\Sigma}, \Sigma)$. This result conveys that while the efficiency of the oracle is increased for large sets Λ , one must settle for combinations λ for which $\text{tr}(\lambda^\top \Sigma \lambda)$ can be well evaluated, thus yielding a small value of $\delta_\Lambda(\hat{\Sigma}, \Sigma)$. Remark moreover that the last term $\|\Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)\|^2$ influences the efficiency of the averaging estimator essentially through the size of the initial collection \mathbf{T} , in view of the equality

$$\mathbb{E} \|\Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)\|^2 = k.$$

4.2 Asymptotic study

The properties of the averaging estimator established in Theorem 4.1 do not rely on any assumption on the construction of \mathbf{T} or $\hat{\Sigma}$. In this section, we investigate the asymptotic properties of the averaging estimator in a situation where both \mathbf{T} and $\hat{\Sigma}$ are computed from a set of observations X_1, \dots, X_n of size n growing to infinity. From now on, we modify our notations to $\mathbf{T}_n, \hat{\Sigma}_n, \Sigma_n, \lambda_n^*, \hat{\lambda}_n, \hat{\theta}_n$ and $\hat{\theta}_n^*$ to emphasize the dependency on n .

In practice, we expect the oracle $\hat{\theta}_n^*$ to satisfy good properties such as consistency and asymptotic normality. Theorem 4.1 suggests that $\hat{\theta}_n$ should inherit these asymptotic properties if Σ_n can be sufficiently well estimated. Remark that if the initial estimators T_i are consistent in quadratic mean, Σ_n converges to the null matrix as $n \rightarrow \infty$. In this case, providing an estimator $\hat{\Sigma}_n$ such that $\hat{\Sigma}_n - \Sigma_n \xrightarrow{p} 0$ is clearly not sufficient for $\hat{\theta}_n$ to achieve the asymptotic performance of the oracle (here p stands for the convergence in probability while d is used for distribution). On the contrary, requiring that $\hat{\Sigma}_n^{-1} - \Sigma_n^{-1} \xrightarrow{p} 0$ is unnecessarily too strong and would be nearly impossible to achieve. In fact, we show in Proposition 4.2 below that the condition

$$\hat{\Sigma}_n \Sigma_n^{-1} \xrightarrow{p} \mathbf{I} \tag{11}$$

appears as a simple compromise, both sufficient for asymptotic optimality and reasonable enough to be verified in numerous situations with regular estimators of Σ_n . General sufficient conditions for (11) to hold are discussed in Lemma 4.4.

Let us introduce some additional definitions and notation. For each component θ_j , $j = 1, \dots, d$, we define

$$\alpha_{n,j} := \mathbb{E} \|\hat{\theta}_{n,j}^* - \theta_j\|^2 = \Pi_j(\lambda_n^*)^\top \Sigma_n \Pi_j(\lambda_n^*),$$

where we recall that $\Pi_j(\lambda_n^*)$ is the j -th column of λ_n^* . Similarly, let $\hat{\alpha}_{n,j} = \Pi_j(\hat{\lambda}_n)^\top \hat{\Sigma}_n \Pi_j(\hat{\lambda}_n)$. We assume that the quadratic error of the oracle, given by

$$\alpha_n := \mathbb{E} \|\hat{\theta}_n^* - \theta\|^2 = \text{tr}(\lambda_n^{*\top} \Sigma_n \lambda_n^*) = \sum_{j=1}^d \alpha_{n,j},$$

converges to zero as $n \rightarrow \infty$. For a given constraint set $\Lambda \subset \mathbb{R}^{k \times d}$, we note $\Lambda_j = \{\Pi_j(\lambda) : \lambda \in \Lambda\} \subset \mathbb{R}^k$. We say that Λ is a *cylinder* if $\Lambda = \{\lambda : \Pi_1(\lambda) \in \Lambda_1, \dots, \Pi_d(\lambda) \in \Lambda_d\}$, i.e., if Λ is the Cartesian product of its marginal sets Λ_j . We point out that choosing a constraint set Λ that satisfies this property is very natural, as it simply states that each vector of weights $\Pi_j(\lambda_n^*)$ used to produce $\hat{\theta}_{n,j}^*$ can be computed independently of the others. In particular, all the constraint sets discussed in Section 3 are cylinders.

Proposition 4.2 *If (11) holds, then*

$$\|\hat{\theta}_n - \theta\|^2 = \|\hat{\theta}_n^* - \theta\|^2 + o_p(\alpha_n). \quad (12)$$

Moreover, if Λ is a cylinder and $\alpha_{n,j}^{-\frac{1}{2}}(\hat{\theta}_{n,j}^* - \theta_j) \xrightarrow{d} \mathcal{Z}$ for some $j = 1, \dots, d$, then

$$\hat{\alpha}_{n,j}^{-\frac{1}{2}}(\hat{\theta}_{n,j} - \theta_j) \xrightarrow{d} \mathcal{Z}. \quad (13)$$

This proposition establishes that building an estimate $\hat{\Sigma}_n$ for which (11) holds ensures that the error of the average $\hat{\theta}_n$ is asymptotically comparable to that of the oracle, up to $o_p(\alpha_n)$. If in addition Λ is a cylinder, it is possible to provide asymptotic confidence regions for θ_j when the limit distribution \mathcal{Z} is known. If $\mathcal{H} = \mathbb{R}$, this situation occurs for instance when \mathbf{T}_n is asymptotically unbiased and asymptotically Gaussian. In this case, the normalization $\alpha_{n,j} = \mathbb{E}(\hat{\theta}_{n,j}^* - \theta_j)^2$ guarantees that $\mathcal{Z} \stackrel{d}{=} \mathcal{N}(0, 1)$ and (13) enables to build an asymptotic confidence interval for θ_j . From (12), this confidence interval is of minimal length amongst all possible confidence intervals based on a linear combination of \mathbf{T}_n . Note that no extra estimation is needed to approach the asymptotic variance, as $\hat{\alpha}_{n,j}$ is entirely determined by $\hat{\lambda}_n$ and $\hat{\Sigma}_n$, which are already used to compute $\hat{\theta}_n$.

Remark 4.3 *The convergence result (13) is in contrast with Theorem 4.1 in [13] where the limit distribution of the averaging estimator differs from that of the oracle. However the authors deal with model averaging, where some particular local misspecification is assumed, implying a specific construction of the averaging weights. We consider here a more regular framework, which explains why we are able to achieve the same limit distribution as the oracle.*

These properties rely on the assumption (11) that can be difficult to check in practice. In the following lemma, we discuss a particular situation where this condition is verified.

Lemma 4.4 *Assume there exist an orthogonal matrix P (i.e. with $P^\top P = \mathbf{I}$) and a known deterministic sequence $(A_n)_{n \in \mathbb{N}}$ of diagonal invertible matrices such that*

$$\lim_{n \rightarrow \infty} A_n P \Sigma_n P^\top = D,$$

for some non-singular diagonal matrix D . If \hat{P}_n and \hat{D}_n are consistent estimators of P and D respectively, then $\hat{\Sigma}_n = \hat{P}_n^\top A_n^{-1} \hat{D}_n \hat{P}_n$ satisfies (11).

This lemma concerns the situation in which the normed eigenvectors of Σ_n converge as $n \rightarrow \infty$, their limits being the rows of P . In this case, a natural estimator $\hat{\Sigma}_n$ is provided by the asymptotic expansion of Σ_n . This result covers the particular case where all constant combinations $\lambda^\top \mathbf{T}_n$ converge to θ at the same rate, as illustrated by the following result.

Corollary 4.5 *Assume there exists a sequence $(a_n)_{n \in \mathbb{N}}$ tending to zero such that*

$$\Sigma_n = a_n W + o(a_n), \quad (14)$$

for some non-singular matrix W . Then, if \hat{W}_n is a consistent estimate of W , the averaging estimator $\hat{\theta}_n$ obtained by minimizing $\lambda \mapsto \text{tr}(\lambda^\top \hat{W}_n \lambda)$ satisfies (12). Moreover, if Λ is a cylinder and $\alpha_{n,j}^{-\frac{1}{2}}(\hat{\theta}_{n,j}^ - \theta_j) \xrightarrow{d} \mathcal{Z}$, then (13) holds.*

The proof follows directly from Proposition 4.2 and Lemma 4.4 with $A_n = a_n^{-1} \mathbf{I}$, in which case the scaling a_n has no influence on the value of $\hat{\theta}_n$, as shown by (6), and needs not be known. If (14) holds, the situation becomes particularly convenient if the limit matrix W follows a known parametric expression $W = W(\eta, \theta)$, with η a nuisance parameter. If the map $W(\cdot, \cdot)$ is continuous, plugging consistent estimators $\hat{\eta}_0, \hat{\theta}_0$ yields an estimator $\hat{W}_n = W(\hat{\eta}_0, \hat{\theta}_0)$ that fulfills the sufficient conditions for $\hat{\theta}_n$ to satisfy (12).

In Proposition 4.2, the asymptotic optimality of $\hat{\theta}_n$ is stated in probability. Remark however that asymptotic optimality in quadratic loss can be obtained easily under additional assumptions. If, for instance, $\hat{\Sigma}_n$ and \mathbf{T} are computed from independent samples (which may be achieved by sample splitting), $\hat{\theta}_n$ is asymptotically optimal in quadratic loss as soon as $\mathbb{E}[\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n)^2]$ tends to 0. Indeed, we have in this case

$$\mathbb{E}\|\hat{\theta}_n - \theta\|^2 = \mathbb{E}\|\hat{\theta}_n^* - \theta\|^2 + o(\alpha_n), \quad (15)$$

which follows directly from (23) in the proof of Proposition 4.2, taking the expectation on both sides. We emphasize however that the use of sample splitting may considerably deteriorate the oracle, as it would be computed from fewer data. One can argue that this is a high price to pay to obtain asymptotic optimality in \mathbb{L}^2 and is not to be recommended in this framework.

Asymptotic optimality in \mathbb{L}^2 can also be achieved if one can show there exists $p > 1$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E} \|\Sigma_n^{-\frac{1}{2}}(\mathbf{T}_n - \mathbf{J}\theta)\|_{\frac{2p}{p-1}} < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{E}[\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n)^{2p}] = 0.$$

In this case, Equation (15) follows directly by applying Hölder's inequality in (23). These conditions ensure the asymptotic optimality in \mathbb{L}^2 of the averaging estimator without sample splitting, but they remain nonetheless extremely difficult to check in practice.

5 Applications

5.1 Estimating the position of a symmetric distribution

Let us consider a continuous real distribution with density f , symmetric around some parameter θ . To estimate θ from a sample of n realisations x_1, \dots, x_n , a natural choice is to use the mean \bar{x}_n or the median $x_{(n/2)}$. Both estimators are consistent whenever $\sigma^2 = \int (x - \theta)^2 f(x) dx$ is finite.

As remarked in introduction, the idea of combining the mean and the median to construct a better estimator goes back to Pierre Simon de Laplace [18]. P. S. de Laplace obtains the expression of the weights in Λ_{\max} that ensure a minimal asymptotic variance for the averaging estimator. In particular, he deduced that for a Gaussian distribution, the better combination is to take the mean only, showing for the first time the efficiency of the latter. For other distributions, he noticed that the best combination is not available in practice because it depends on the unknown distribution.

Similarly, we consider the averaging of the mean and the median over Λ_{\max} . In the setting of the previous sections, we have two initial estimators $T_1 = \bar{x}_n$, $T_2 = x_{(n/2)}$, the space \mathcal{H} is simply \mathbb{R} , and the averaging estimator is given by (9) where \mathbf{J} is just in this case the vector $(1, 1)^\top$. The MSE matrix between the two estimators is denoted by Σ_n . We assume that the n realisations are independent and we propose two ways to estimate Σ_n :

1. *Based on the asymptotic equivalent of Σ_n .* The latter, obtained in P. S. de Laplace's work and recalled in [28], is $n^{-1}W$ where

$$W = \begin{pmatrix} \sigma^2 & \frac{\mathbb{E}|X-\theta|}{2f(\theta)} \\ \frac{\mathbb{E}|X-\theta|}{2f(\theta)} & \frac{1}{4f(\theta)^2} \end{pmatrix}.$$

Each entry of W may be naturally estimated from an initial estimate $\hat{\theta}_0$ of θ as follows: σ^2 by the empirical variance s_n^2 ; $\mathbb{E}|X - \theta|$ by $\hat{m} = 1/n \sum_{i=1}^n |x_i - \hat{\theta}_0|$; and $f(\theta)$ by the kernel estimator $\hat{f}(\hat{\theta}_0) = 1/(nh) \sum_{i=1}^n \exp(-(x_i - \hat{\theta}_0)^2/(2h^2))$, where

h is chosen, e.g., by the so-called Silverman's rule of thumb (see [26]). With this estimation of Σ_n , we get the following averaging estimator:

$$\hat{\theta}_{AV} = \frac{p_1}{p_1 + p_2} \bar{x}_n + \frac{p_2}{p_1 + p_2} x_{(n/2)} \quad (16)$$

where $p_1 = 1/(4\hat{f}(\hat{\theta}_0)) - \hat{m}/2$ and $p_2 = s_n^2 \hat{f}(\hat{\theta}_0) - \hat{m}/2$. This estimator corresponds to an empirical version of the best combination obtained by P. S. de Laplace.

2. *Based on non-parametric bootstrap.* We draw with replacement B samples of size n from the original dataset. We compute the mean and the median of each sample, respectively denoted $\bar{x}_n^{(b)}$ and $x_{(n/2)}^{(b)}$ for $b = 1, \dots, B$. The MSE matrix Σ_n is then estimated by

$$\frac{1}{B} \begin{pmatrix} \sum_{b=1}^B (\bar{x}_n^{(b)} - \bar{x}_n)^2 & \sum_{b=1}^B (\bar{x}_n^{(b)} - \bar{x}_n)(x_{(n/2)}^{(b)} - x_{(n/2)}) \\ \sum_{b=1}^B (\bar{x}_n^{(b)} - \bar{x}_n)(x_{(n/2)}^{(b)} - x_{(n/2)}) & \sum_{b=1}^B (x_{(n/2)}^{(b)} - x_{(n/2)})^2 \end{pmatrix}.$$

This leads to another averaging estimator, denoted by $\hat{\theta}_{AVB}$.

Let us note that the first procedure above fits the asymptotic justification presented in Section 4.2, as (14) holds. For this reason, $\hat{\theta}_{AV}$ is asymptotically as efficient as the oracle, provided $\hat{\theta}_0$ is consistent. Moreover, since the initial estimators are asymptotically Gaussian and unbiased, an asymptotic confidence interval for θ can be provided without further estimation, see Section 4.2. For the second procedure, theory is lacking to study the behaviour of δ in (10) when $\hat{\Sigma}$ is estimated by non-parametric bootstrap, so no consistency can be claimed at this point. However the latter is a very natural procedure, easy to implement in practice, so it is natural to assess its performances in our simulation study.

Table 1 summarizes the estimated MSE of \bar{x}_n , $x_{(n/2)}$, $\hat{\theta}_{AV}$ and $\hat{\theta}_{AVB}$, for $n = 30, 50, 100$, and for different distributions, namely: Cauchy, Student with 5 degrees of freedom, Student with 7 degrees of freedom, Logistic, standard Gaussian, and an equal mixture distribution of a $\mathcal{N}(-2, 1)$ and a $\mathcal{N}(2, 1)$. For all distributions, $\theta = 0$. For the initial estimate $\hat{\theta}_0$ in (16), we take the median $x_{(n/2)}$, because it is well defined and consistent for any continuous distribution. The number of bootstrap samples taken for $\hat{\theta}_{AVB}$ is $B = 1000$.

While the best estimator between \bar{x}_n and $x_{(n/2)}$ depends on the underlying distribution, the averaging estimators $\hat{\theta}_{AV}$ and $\hat{\theta}_{AVB}$ perform better than both \bar{x}_n and $x_{(n/2)}$, for all distributions considered in Table 1 except the Gaussian law. For the latter distribution, we know that the oracle is the mean, so the averaging estimator cannot improve on \bar{x}_n . However the MSE of $\hat{\theta}_{AV}$ and $\hat{\theta}_{AVB}$ are very close to that of \bar{x}_n in this case, proving that the optimal weights $(1, 0)$ are fairly well estimated. Moreover, note that the Cauchy distribution does not belong to our theoretical setting because it has no finite moments and \bar{x}_n should not be used. But it turns out that the averaging estimators are very robust

in this case, as they manage to select $x_{(n/2)}$. Choosing the median $x_{(n/2)}$ as the initial estimator $\hat{\theta}_0$ is of course crucial in this case.

Finally, while $\hat{\theta}_{AVB}$ suffers from a lack of theoretical justification, it behaves pretty much like $\hat{\theta}_{AV}$, except for the mixture distribution where it performs slightly better than $\hat{\theta}_{AV}$. This may be explained by the fact that $\hat{\theta}_{AV}$ is more sensitive than $\hat{\theta}_{AVB}$ to the initial estimate $\hat{\theta}_0$, the variance of which is large for the mixture distribution because $f(0)$ is close to 0. Nevertheless, $\hat{\theta}_{AV}$ demonstrates very good performance in this case, for the sample sizes considered in Table 1.

	n=30				n=50				n=100			
	MEAN	MED	AV	AVB	MEAN	MED	AV	AVB	MEAN	MED	AV	AVB
Cauchy	2.10 ⁶ (1.10 ⁶)	9 (0.14)	8.95 (0.15)	8.99 (0.15)	4.10 ⁷ (4.10 ⁷)	5.07 (0.08)	4.92 (0.08)	4.9 (0.08)	2.10 ⁷ (2.10 ⁷)	2.56 (0.04)	2.49 (0.04)	2.49 (0.04)
St(4)	6.68 (0.1)	5.71 (0.08)	5.4 (0.08)	5.43 (0.08)	4.12 (0.06)	3.53 (0.05)	3.33 (0.05)	3.34 (0.05)	1.99 (0.03)	1.74 (0.02)	1.61 (0.02)	1.62 (0.02)
St(7)	4.8 (0.07)	5.51 (0.08)	4.6 (0.07)	4.64 (0.07)	2.82 (0.04)	3.32 (0.05)	2.74 (0.04)	2.8 (0.04)	1.42 (0.02)	1.67 (0.02)	1.37 (0.02)	1.38 (0.02)
Logistic	10.89 (0.16)	12.7 (0.18)	10.76 (0.16)	10.87 (0.16)	6.64 (0.09)	7.93 (0.11)	6.52 (0.09)	6.6 (0.09)	3.3 (0.05)	4 (0.06)	3.2 (0.05)	3.26 (0.05)
Gauss	3.39 (0.05)	5.11 (0.07)	3.53 (0.05)	3.61 (0.05)	2.04 (0.03)	3.1 (0.04)	2.1 (0.03)	2.15 (0.03)	1 (0.01)	1.51 (0.02)	1.02 (0.01)	1.06 (0.01)
Mix	16.79 (0.23)	87 (0.82)	15.03 (0.29)	13.41 (0.3)	10.08 (0.14)	66.53 (0.64)	7.57 (0.15)	6.68 (0.18)	5.05 (0.07)	42.35 (0.43)	3.09 (0.06)	2.36 (0.07)

Table 1: Monte Carlo estimation of the MSE of \bar{x}_n (MEAN), $x_{(n/2)}$ (MED), $\hat{\theta}_{AV}$ (AV) and $\hat{\theta}_{AVB}$ (AVB) in the estimation of the position of a symmetric distribution, depending on the distribution and the sample size. The number of replications is 10^4 and the standard deviation of the MSE estimations is given in parenthesis. Each entry has been multiplied by 100 for ease of presentation.

5.2 Estimating the parameters of a Weibull distribution

We consider estimators averaging in a parametric setting, namely the Weibull distribution with shape parameter $\beta > 0$ and scale parameter $\eta > 0$, the density function of which is

$$f(x) = \frac{\beta}{\eta} \left(\frac{x}{\eta}\right)^{\beta-1} e^{-(x/\eta)^\beta}, \quad x > 0.$$

Based on a sample of n independent realisations, many estimators of β and η are available (see [14]). We consider the following three standard methods:

- the maximum likelihood estimator (ML) is the solution of the system

$$\frac{n}{\beta} + \sum_{i=1}^n \log(x_i) - n \frac{\sum_{i=1}^n x_i^\beta \log(x_i)}{\sum_{i=1}^n x_i^\beta} = 0, \quad \eta = \left(\frac{1}{n} \sum_{i=1}^n x_i^\beta\right)^{1/\beta}.$$

- the method of moments (MM), based on the two first moments, reduces to solve:

$$\frac{s_n^2}{\bar{x}_n^2} = \frac{\Gamma(1 + 2/\beta)}{\Gamma(1 + 1/\beta)^2} - 1, \quad \eta = \frac{\bar{x}_n}{\Gamma(1 + 1/\beta)},$$

where \bar{x}_n and s_n denote the empirical sample mean and the unbiased sample variance.

- the ordinary least squares method (OLS) is based on the fact that for any $x > 0$, $\log(-\log(1 - F(x))) = \beta \log(x) - \beta \log \eta$, where F denotes the cumulative distribution function of the Weibull distribution. More precisely, denoting $x_{(1)}, \dots, x_{(n)}$ the ordered sample, an estimation of β and η is deduced from the simple linear regression of $(\log(-\log(1 - F(x_{(i)}))))_{i=1\dots n}$ on $(\log x_{(i)})_{i=1\dots n}$, where according to the "mean rank" method $F(x_{(i)})$ may be estimated by $i/(n + 1)$. This fitting method is very popular in the engineer community (see [1]): the estimation of β simply corresponds to the slope in a "Weibull plot".

The performances of these three estimators are variable, depending on the value of the parameters and the sample size. In particular, no one is uniformly better than the others, see Figure 1 for an illustration.

Let us now consider the averaging of these estimators. In the setting of the previous sections, we have $d = 2$ parameters in $\mathcal{H} = \mathbb{R}$ to estimate and $k_1 = 3$, $k_2 = 3$ initial estimators of each are available. The averaging over the maximal constraint set Λ_{\max} demands to estimate the 6×6 MSE matrix Σ , that involves 21 unknown values. The Weibull distribution is often used to model lifetimes, and typically only a low number of observations are available to estimate the parameters. As a consequence averaging over Λ_{\max} of the 6 initial estimators above could be too demanding. Moreover, between the two parameters β and η , the shape parameter β is often the most important to identify, as it characterizes for instance the type of failure rate in reliability engineering. For these reasons, we choose to average the three estimators of β presented above, $\hat{\beta}_{ML}$, $\hat{\beta}_{MM}$ and $\hat{\beta}_{OLS}$, and to consider only one estimator of η : $\hat{\eta}_{ML}$ (where $\hat{\beta}_{ML}$ is used for its computation). The averaging over Λ_{\max} of these 4 estimators has three consequences: First, the number of unknown values in the MSE matrix is reduced to 10. Second, the averaging estimator of β depends only on $\hat{\beta}_{ML}$, $\hat{\beta}_{MM}$ and $\hat{\beta}_{OLS}$, because $\hat{\eta}_{ML}$ has a zero weight from (4). This means that we actually implement a component-wise averaging for β . Third, the averaging estimator of η equals $\hat{\eta}_{ML}$ plus some linear combination of $\hat{\beta}_{ML}$, $\hat{\beta}_{MM}$ and $\hat{\beta}_{OLS}$ where the weights sum to zero. This particular situation will allow us to see if $\hat{\eta}_{ML}$ can be improved by exploiting the correlation with the estimators of β , or if it is deteriorated.

So we have $d = 2$, $k_1 = 3$, $k_2 = 1$, $\mathbf{T}_1 = (\hat{\beta}_{ML}, \hat{\beta}_{MM}, \hat{\beta}_{OLS})^\top$, $\mathbf{T}_2 = \hat{\eta}_{ML}$ and the averaging estimator over Λ_{\max} is given by (9), denoted by $(\hat{\beta}_{AV}, \hat{\eta}_{AV})^\top$. The matrix Σ

is estimated by parametric bootstrap: Starting from initial estimates $\hat{\beta}_0, \hat{\eta}_0$, we simulate B samples of size n of a Weibull distribution with parameters $\hat{\beta}_0, \hat{\eta}_0$. Then the four estimators are computed, which gives $\hat{\beta}_{ML}^{(b)}, \hat{\beta}_{MM}^{(b)}, \hat{\beta}_{OLS}^{(b)}$ and $\hat{\eta}_{ML}^{(b)}$, for $b = 1, \dots, B$, and each entry of Σ is estimated by its bootstrap counterpart. For instance the estimation of $\mathbb{E}(\hat{\beta}_{ML} - \beta)(\hat{\beta}_{MM} - \beta)$ is $(1/B) \sum_{b=1}^B (\hat{\beta}_{ML}^{(b)} - \hat{\beta}_0)(\hat{\beta}_{MM}^{(b)} - \hat{\beta}_0)$. In our simulations, we chose $\hat{\beta}_0$ as the mean of \mathbf{T}_1 , and $\hat{\eta}_0 = \hat{\eta}_{ML}$. Note that Σ having a parametric form ensures that $(\hat{\beta}_{AV}, \hat{\eta}_{AV})^\top$ is asymptotically as efficient as the oracle, see Corollary 4.5.

Table 2 gives the MSE, estimated from 10^4 replications, of each estimator of β , for $n = 10, 20, 50$, and for $\beta = 0.5, 1, 2, 3, \eta = 10$. The averaging estimator has by far the lowest MSE, even for small samples. As an illustration, the repartition of each estimator, for $n = 20$ and $\beta = 0.5, 3$, is represented in Figure 1.

Table 3 shows the MSE for $\hat{\eta}_{ML}$ and $\hat{\eta}_{AV}$ where only estimators of β were used in attempt to improve $\hat{\eta}_{ML}$ by averaging. The performances of both estimators are similar, showing that the information coming from \mathbf{T}_1 did not help significantly improving $\hat{\eta}_{ML}$. On the other hand, the estimation of these (almost zero) weights might have deteriorated $\hat{\eta}_{ML}$, especially for small sample sizes. This did not happen.

	n=10				n=20				n=50			
	ML	MM	OLS	AV	ML	MM	OLS	AV	ML	MM	OLS	AV
$\beta = 0.5$	35.53 (0.91)	76.95 (1.27)	24.41 (0.40)	25.27 (0.64)	12.06 (0.26)	35.57 (0.52)	13.74 (0.19)	10.5 (0.19)	3.7 (0.07)	14.19 (0.20)	6.04 (0.08)	3.52 (0.06)
$\beta = 1$	152.4 (3.8)	131.6 (3.1)	98.1 (1.5)	85.5 (1.7)	49.2 (1.1)	53.6 (1.1)	54.2 (0.7)	36.9 (0.7)	14.4 (0.2)	19.3 (0.3)	23.9 (0.3)	12.8 (0.2)
$\beta = 2$	596.4 (14.4)	444.6 (11.9)	399.4 (6.3)	355.5 (6.7)	194.5 (3.8)	164.5 (3.3)	218 (2.8)	163.3 (2.7)	57.9 (1.0)	53.9 (0.9)	94.8 (1.3)	54.3 (0.9)
$\beta = 3$	1369 (34.6)	1080 (29.7)	905 (14.6)	770 (18.1)	452 (9.8)	394 (8.9)	486 (6.7)	343 (6.2)	128 (2.2)	122 (2.0)	211 (2.7)	120 (1.9)

Table 2: Monte Carlo estimation of the MSE of $\hat{\beta}_{ML}, \hat{\beta}_{MM}, \hat{\beta}_{OLS}$ and $\hat{\beta}_{AV}$, based on 10^4 replications of a sample of size $n = 10, 20, 50$ from a Weibull distribution with parameters $\beta = 0.5, 1, 2, 3$ and $\eta = 10$. The standard deviation of the MSE estimations are given in parenthesis. Each entry has been multiplied by 100 for ease of presentation.

5.3 Aggregating kernel density estimators

Let x_1, \dots, x_n be a sample from a real random variable with density f . The kernel density estimator of f at $x \in \mathbb{R}$ is

$$\hat{f}_{n,h}(x) = \frac{1}{nh} \sum_{i=1}^n K\left(\frac{x - x_i}{h}\right),$$

	n=10		n=20		n=50	
	ML	AV	ML	AV	ML	AV
$\beta = 0.5$	60.59 (1.60)	55.61 (1.48)	25.96 (0.53)	24.56 (0.5)	9.57 (0.17)	9.38 (0.17)
$\beta = 1$	11.15 (0.18)	10.88 (0.17)	5.53 (0.08)	5.43 (0.08)	2.23 (0.03)	2.22 (0.03)
$\beta = 2$	2.71 (0.04)	2.74 (0.04)	1.36 (0.02)	1.37 (0.02)	0.55 (0.01)	0.56 (0.01)
$\beta = 3$	1.21 (0.02)	1.23 (0.02)	0.61 (0.01)	0.61 (0.01)	0.247 (0.003)	0.248 (0.004)

Table 3: Monte Carlo estimation of the MSE of $\hat{\eta}_{ML}$ and $\hat{\eta}_{AV}$, based on 10^4 replications of a sample of size $n = 10, 20, 50$ from a Weibull distribution with parameters $\beta = 0.5, 1, 2, 3$ and $\eta = 10$. The standard deviation of the MSE estimations are given in parenthesis. Each entry has been multiplied by 100 for ease of presentation.

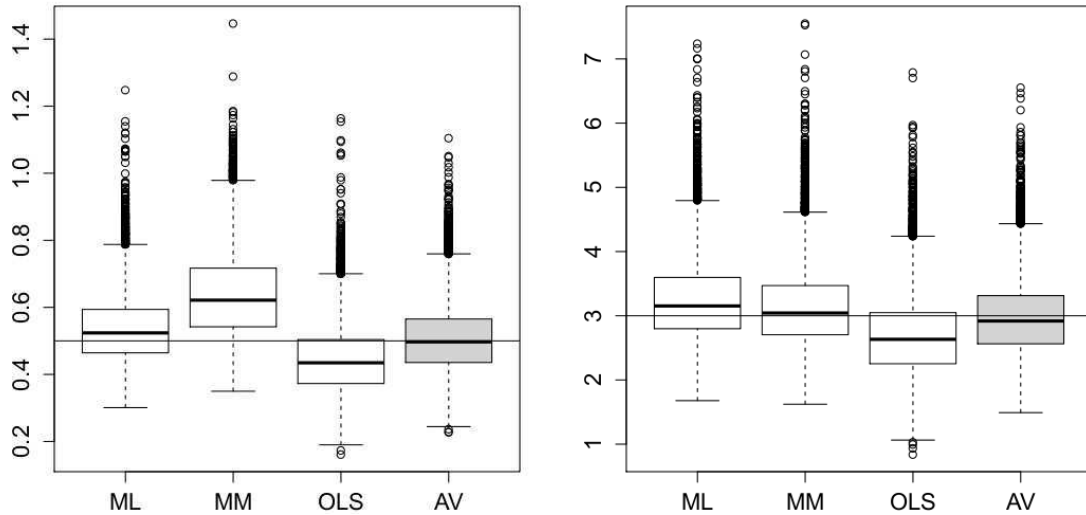


Figure 1: Repartition of $\hat{\beta}_{ML}$, $\hat{\beta}_{MM}$, $\hat{\beta}_{OLS}$ and $\hat{\beta}_{AV}$ (from left to right) based on 10^4 replications of a sample of size $n = 20$ from a Weibull distribution with $\beta = 0.5$ (left), $\beta = 3$ (right) and $\eta = 10$.

where the function K is the kernel and h is a smoothing parameter called bandwidth. It is well known that the choice of K has only a small impact on the performances of $\hat{f}_{n,h}$, while the choice of h is crucial. Many works are dedicated to propose some data-based bandwidth selection. We refer to [24] for a review.

For our simulations, we choose the standard Gaussian kernel $K(x) = e^{-x^2/2}/\sqrt{2\pi}$ and we consider four choices of bandwidth, see [24] for more details and references:

- The Silverman's rule-of-thumb, namely $h_1 = 0.9 \min(s_n, IQR/1.34) n^{-1/5}$, where s_n denotes the standard deviation and IQR the interquartile range.
- The variation proposed in [23], where the constant 0.9 above is replaced by 1.06.
- The unbiased (or least squares) cross-validation method.
- The plug-in method of Sheather and Jones [25].

These four possible choices, called h_1, \dots, h_4 in the following, correspond to the bandwidths proposed in the function `density` implemented in the freeware R: `nrd0`, `nrd`, `ucv` and `SJ` respectively.

Our goal is to combine \hat{f}_{n,h_i} , $i = 1, \dots, 4$ to obtain a better estimate of f . In most existing methods of density aggregation (see the references cited in introduction), the observations are assumed independent and the procedure mainly relies on a sample splitting, where a training sample is used to compute the initial estimators \hat{f}_{n,h_i} and the weights are estimated from the validation sample. In contrast, we propose to construct an averaging estimator that minimizes the mean integrated square error (MISE), defined for an estimator \hat{f} of f by $\int \mathbb{E}(\hat{f}(x) - f(x))^2 dx$. In the setting of the previous sections, $\mathcal{H} = L^2(\mathbb{R})$ and the oracle given by (5) involves the MISE matrix Σ with entries $\int \mathbb{E}(\hat{f}_{n,h_i}(x) - f(x))(\hat{f}_{n,h_j}(x) - f(x)) dx$. The averaging estimator over Λ_{\max} is then given by (9), where $d = 1$ and $k_1 = 4$. In particular, our procedure does not require sample splitting, and we do not assume independence of the observations.

The main difficulty is the estimation of the MISE matrix Σ which is at the heart of most methods of bandwidth selection. A standard procedure consists in estimating the asymptotic form AMISE of the MISE. If the bandwidths h_i are deterministic, assuming that the observations are independent or weakly dependent (and under further mild assumptions, see for instance [3]), the asymptotic equivalent of each entry of Σ is given by:

$$AMISE(h_i, h_j) = \frac{1}{n\sqrt{h_i h_j}} I_K \left(\sqrt{\frac{h_j}{h_i}} \right) + \frac{h_i^2 h_j^2}{4} \mu_2^2(K) R(f'') \quad (17)$$

where for $\alpha > 0$, $I_K(\alpha) = \int K(\alpha u) K(\alpha^{-1} u) du$, $\mu_2(K) = \int x^2 K(x) dx$ and $R(f'') = \int (f''(x))^2 dx$. When K is the Gaussian kernel, we have $I_K(\alpha) = (1/\sqrt{2\pi}) \alpha/\sqrt{\alpha^4 + 1}$ and

$\mu_2(K) = 1$. If the bandwidths h_i depend on the observations, as for the four above choices, then (17) is the conditional AMISE given h_1, \dots, h_4 . We choose to estimate Σ by (17) where $R(f'')$ is estimated by the standard plug-in method proposed in [15], which turns out to be also used for the computation of h_4 , see [25]. From a theoretical point of view, if all h_i 's are deterministic and of the form $h_i = c_i n^{-1/5}$, $c_i > 0$, then (14) holds and our procedure provides a consistent averaging estimator. In presence of random c_i 's, some further investigations are necessary to prove consistency.

Our averaging estimator, say $\hat{f}_{n,AV}$, is thus given by (9) with $\hat{\Sigma}$ obtained as explained above. Table 4 summarises the estimated MISE of \hat{f}_{n,h_i} , $i = 1, \dots, 4$ and $\hat{f}_{n,AV}$, for some standard densities f and different sample sizes $n = 250, 500, 1000$, when the observations are independently drawn from f . Specifically, we consider the densities of the standard Gaussian law $\mathcal{N}(0, 1)$, of the equal mixture of a $\mathcal{N}(-1.5, 1)$ and a $\mathcal{N}(1.5, 1)$, of the Gamma distribution $\Gamma(2, 1)$ and of the Cauchy distribution. The MISE is estimated by averaging over 10^4 replications the integrated square error, obtained by the sum of the square error $(\hat{f}_{n,h_i}(x) - f(x))^2$ over 100 points x equally spaced on $[-2, 2]$, $[-3, 3]$, $[0.5, 4]$ and $[-4, 4]$, respectively. Note that Table 4 gives an estimation of the unconditional MISE, even if the estimation of the weights in $\hat{f}_{n,AV}$ is based on the conditional AMISE (17). Moreover, Figure 2 shows the MSE functions $(\hat{f}_{n,h_i}(x) - f(x))^2$ for each distribution f when $n = 500$.

	n=250					n=500					n=1000				
	h_1	h_2	h_3	h_4	AV	h_1	h_2	h_3	h_4	AV	h_1	h_2	h_3	h_4	AV
Gauss	29.9	27.2	26.8	29.9	24.9	17.7	16.2	16.2	17.3	14.4	10.5	9.7	9.8	10.1	8.4
Mix	24.0	27.5	27.1	25.2	26.7	14.8	17.6	15.3	14.9	14.2	9.1	11.1	8.9	8.8	7.4
Gamma	28.0	32.7	29.5	28.9	27.9	17.1	20.6	17.0	17.2	15.8	10.3	12.7	10.0	10.3	9.0
Cauchy	31.2	37.0	830	132	32.8	18.9	23.2	945	180	18.7	11.4	14.4	1068	226	10.6

Table 4: Monte Carlo estimation of the MISE of \hat{f}_{n,h_i} , for $i = 1, \dots, 4$ and of the averaging estimator $\hat{f}_{n,AV}$, for different f and $n = 250, 500, 1000$. With the notation of R: $h_1 = \text{nr}d0$, $h_2 = \text{nr}d$, $h_3 = \text{ucv}$ and $h_4 = \text{SJ}$. The MISE are estimated by the mean over 10^4 replications of the integrated square error, obtained by summing up the square error of 100 equally spaced points on the support of f . Each entry has been multiplied by $n \cdot 10^4$ for ease of presentation.

As a result, $\hat{f}_{n,AV}$ has a lower MISE than the experts, except for the mixture distribution when $n = 250$. In fact, our procedure works very well for large samples, but is less efficient for small samples. One obvious reason is that we do not actually estimate the MISE matrix Σ but its (conditional) asymptotic expression given by (17). Some simulations (not presented here) show that even the oracle estimator based on the asymptotic MISE matrix may have a larger MISE than the initial estimators for small values of n . This is for instance the case for the mixture distribution when $n = 250$. Moreover, the performances of $\hat{f}_{n,AV}$ for the Cauchy distribution are remarkably good, in spite of the

presence of two unadapted initial estimators (namely \hat{f}_{n,h_3} and \hat{f}_{n,h_4}).

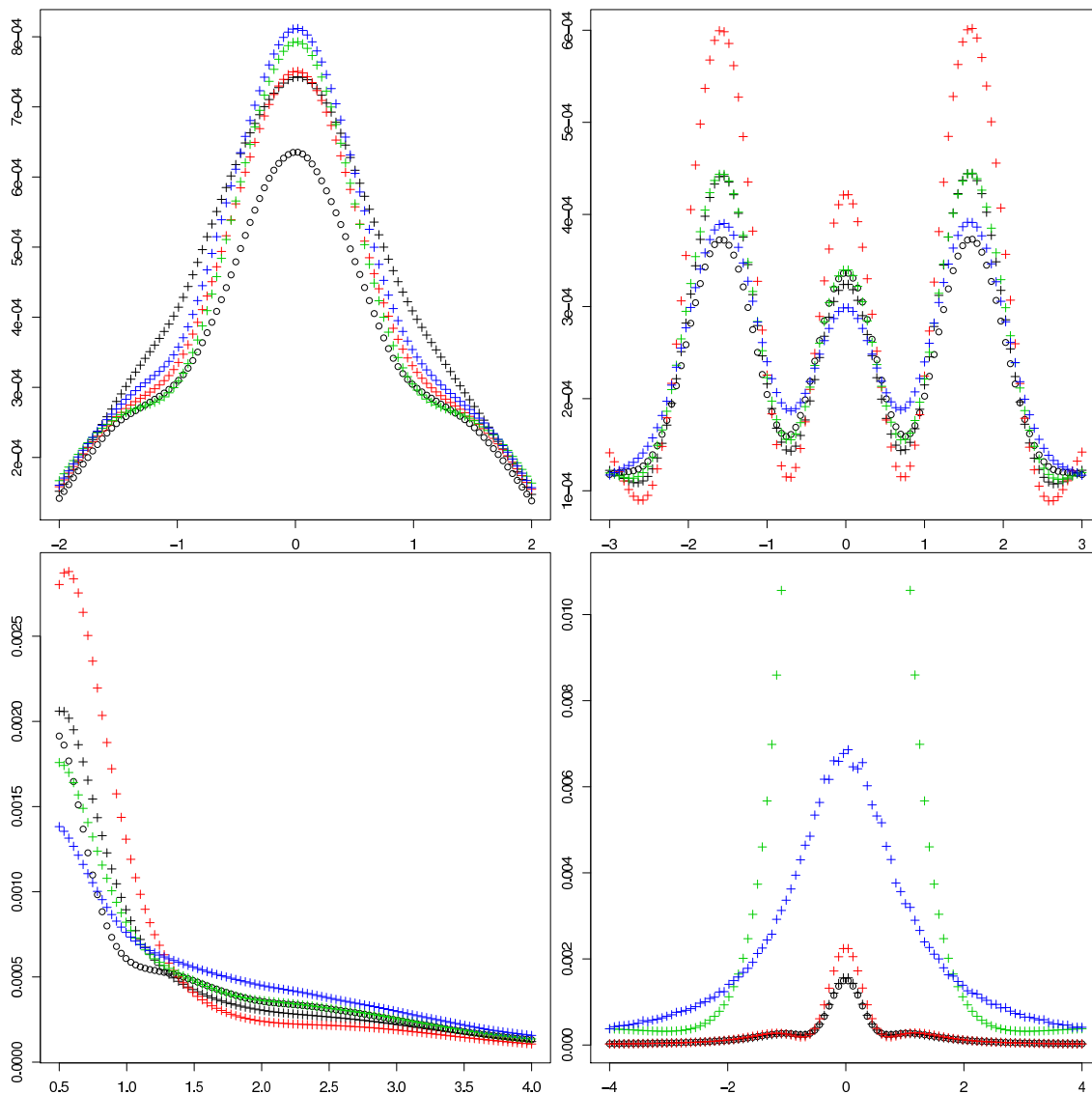


Figure 2: Monte Carlo estimation of the MSE of \hat{f}_{n,h_1} (black crosses), \hat{f}_{n,h_2} (red crosses), \hat{f}_{n,h_3} (green crosses), \hat{f}_{n,h_4} (blue crosses) and $\hat{f}_{n,AV}$ (black circles), based on 10^4 replications, for $n = 500$ and when f is the density of the Gaussian distribution (top left), the mixture distribution (top right), the Gamma distribution (bottom left) and the Cauchy distribution (bottom right). The estimated MISE in Table 4 (for $n = 500$) are the approximated integrals of these curves.

An alternative procedure to estimate the MISE matrix is smooth bootstrap [29]. Recall

that standard bootstrap fails to estimate the bias in a non-parametric setting. Smooth bootstrap amounts to resample according to a continuous density \hat{f} that estimates f . At least two appealing features arise: First, smooth bootstrap estimates the MISE and not the asymptotic MISE, which could improve our procedure for small samples. Second, when K is the Gaussian kernel, closed-form formulas are available for the smooth bootstrap estimate of the conditional MISE given h_1, \dots, h_4 , and no Monte-Carlo step is needed. The unconditional MISE can also be estimated by smooth bootstrap, but some Monte-Carlo approximations are then needed. Unfortunately, smooth bootstrap depends on a pilot bandwidth for which we are unable to propose a satisfactory data-driven choice. Therefore, although smooth bootstrap seems a promising perspective for averaging kernel density estimators based on small samples, its implementation deserves further analysis. Nevertheless, our simulation study shows that the averaging based on the asymptotic approximation of the MISE produces satisfactory results for moderate sample sizes (e.g. $n \geq 250$).

6 Appendix

Proof of Theorem 4.1

Since $\Lambda \subseteq \Lambda_{\max}$, we know that $\lambda^\top \mathbf{J} = \mathbf{I}$ for all $\lambda \in \Lambda$. Let $\mathbf{S} = \Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)$, we have

$$\|\hat{\theta} - \hat{\theta}^*\|^2 = \|(\hat{\lambda} - \lambda^*)^\top (\mathbf{T} - \mathbf{J}\theta)\|^2 = \|(\hat{\lambda} - \lambda^*)^\top \Sigma^{\frac{1}{2}} \mathbf{S}\|^2 \leq \|(\hat{\lambda} - \lambda^*)^\top \Sigma^{\frac{1}{2}}\|_F^2 \|\mathbf{S}\|^2, \quad (18)$$

where $\|A\|_F = \sqrt{\text{tr}(A^\top A)}$ denotes the Frobenius norm of A . The map $\phi : \lambda \mapsto \text{tr}(\lambda^\top \Sigma \lambda)$ is coercive, and strictly convex by assumption. So, since Λ is closed and convex, the minimum of ϕ on Λ is reached at a unique point $\lambda^* \in \Lambda$. Moreover, we know that for $\lambda \in \Lambda$, $\lambda^* + t(\lambda - \lambda^*)$ lies in Λ for all $t \in [0, 1]$, to which we deduce the optimality condition

$$\lim_{t \rightarrow 0^+} \frac{\phi(\lambda^* + t(\lambda - \lambda^*)) - \phi(\lambda^*)}{t} = \text{tr} [\nabla \phi(\lambda^*)^\top (\lambda - \lambda^*)] = 2\text{tr} [\lambda^{*\top} \Sigma (\lambda - \lambda^*)] \geq 0,$$

for all $\lambda \in \Lambda$. It follows that

$$\begin{aligned} \|(\hat{\lambda} - \lambda^*) \Sigma^{\frac{1}{2}}\|_F^2 &= \text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*) - 2\text{tr} [\lambda^{*\top} \Sigma (\hat{\lambda} - \lambda^*)] \\ &\leq \text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*). \end{aligned} \quad (19)$$

By construction of $\hat{\lambda}$, we know that $\text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) \leq \text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*)$, yielding

$$\begin{aligned} \text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*) &\leq \text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) + \text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \\ &\leq \text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) \delta_\Lambda(\Sigma | \hat{\Sigma}) + \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \delta_\Lambda(\hat{\Sigma} | \Sigma) \\ &\leq \left[\text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) + \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \right] \delta_\Lambda(\hat{\Sigma}, \Sigma) \end{aligned}$$

where we recall $\delta_\Lambda(A|B) = \sup_{\lambda \in \Lambda} \left| 1 - \frac{\text{tr}(\lambda^\top A \lambda)}{\text{tr}(\lambda^\top B \lambda)} \right|$ and $\delta_\Lambda(A, B) = \max\{\delta_\Lambda(A|B), \delta_\Lambda(B|A)\}$. Now using that $\text{tr}(\hat{\lambda}^\top \hat{\Sigma} \hat{\lambda}) \leq \text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*)$ and

$$\text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*) = \text{tr}(\lambda^{*\top} \Sigma \lambda^*) + \left[\text{tr}(\lambda^{*\top} \hat{\Sigma} \lambda^*) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \right] \leq \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \left[1 + \delta_\Lambda(\hat{\Sigma}, \Sigma) \right],$$

we obtain

$$\text{tr}(\hat{\lambda}^\top \Sigma \hat{\lambda}) - \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \leq \text{tr}(\lambda^{*\top} \Sigma \lambda^*) \left[2\delta_\Lambda(\hat{\Sigma}, \Sigma) + \delta_\Lambda(\hat{\Sigma}, \Sigma)^2 \right]. \quad (20)$$

Recall that $\text{tr}(\lambda^{*\top} \Sigma \lambda^*) = \inf_{\lambda \in \Lambda} \mathbb{E} \|\lambda^\top \mathbf{T} - \theta\|^2$, the result follows from (18), (19) and (20).

Proof of Proposition 4.2

We use the following preliminary result.

Lemma 6.1 *Let A, B be two positive definite matrices of order k . For any non-empty set Λ that does not contain 0 ,*

$$\delta_\Lambda(A, B) \leq \|\| AB^{-1} - BA^{-1} \|\|,$$

where $\|\|A\|\| = \sup_{\|x\|_F=1} \|Ax\|_F$ stands for the operator norm.

Proof. By symmetry, it is sufficient to show that the result holds for $\delta_\Lambda(A|B)$. We have

$$\delta_\Lambda(A|B) = \sup_{\lambda \in \Lambda} \frac{|\text{tr}[\lambda^\top (B - A)\lambda]|}{\text{tr}(\lambda^\top B \lambda)} \leq \sup_{\lambda \neq 0} \frac{|\text{tr}[\lambda^\top (B - A)\lambda]|}{\text{tr}(\lambda^\top B \lambda)}.$$

By Cauchy-Schwarz inequality,

$$\begin{aligned} |\text{tr}[\lambda^\top (B - A)\lambda]| &= \left| \text{tr} \left[\lambda^\top B^{\frac{1}{2}} (I - B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) B^{\frac{1}{2}} \lambda \right] \right| \\ &\leq \|B^{\frac{1}{2}} \lambda\|_F \|(I - B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) B^{\frac{1}{2}} \lambda\|_F \\ &\leq \|\|I - B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\|\| \|B^{\frac{1}{2}} \lambda\|_F^2. \end{aligned} \quad (21)$$

Recall that $\|B^{\frac{1}{2}} \lambda\|_F^2 = \text{tr}(\lambda^\top B \lambda)$, it follows

$$\delta_\Lambda(A|B) \leq \|\|I - B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\|\|.$$

Since the matrix $C = I - B^{-\frac{1}{2}} A B^{-\frac{1}{2}}$ is symmetric, it is diagonalizable in an orthogonal basis. In particular, denoting $\text{sp}(\cdot)$ the spectrum, $\|\|C\|\| = \sup_{t \in \text{sp}(C)} |t|$. Finally, observe that $\text{sp}(C) = 1 - \text{sp}(B^{-\frac{1}{2}} A B^{-\frac{1}{2}}) = 1 - \text{sp}(AB^{-1})$, so that AB^{-1} has positive eigenvalues and

$$\|\|I - B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\|\| = \sup_{t \in \text{sp}(AB^{-1})} |1 - t| \leq \sup_{t \in \text{sp}(AB^{-1})} \left| t - \frac{1}{t} \right| \leq \|\|AB^{-1} - BA^{-1}\|\|,$$

ending the proof. ■

By this lemma, we deduce that

$$\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n) \leq \|\hat{\Sigma}_n \Sigma_n^{-1} - \Sigma_n \hat{\Sigma}_n^{-1}\|. \quad (22)$$

In particular, $\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n) = o_p(1)$ by the assumption $\hat{\Sigma}_n \Sigma_n^{-1} \xrightarrow{p} \mathbf{I}$. Write for $c > 0$,

$$\|\hat{\theta}_n - \theta\|^2 \leq (1+c)\|\hat{\theta}_n^* - \theta\|^2 + (1+c^{-1})\|\hat{\theta}_n - \hat{\theta}_n^*\|^2.$$

Applying Theorem 4.1, we get

$$\|\hat{\theta}_n - \theta\|^2 \leq (1+c)\|\hat{\theta}_n^* - \theta\|^2 + (1+c^{-1})\alpha_n \left(2\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n) + \delta_\Lambda(\hat{\Sigma}_n, \Sigma_n)^2 \right) \|\mathbf{S}_n\|^2, \quad (23)$$

where $\mathbf{S}_n = \Sigma_n^{-\frac{1}{2}}(\mathbf{T}_n - \mathbf{J}\theta)$. Since $\mathbb{E}\|\mathbf{S}_n\|^2 = k$, we know that $\|\mathbf{S}_n\|^2 = O_p(1)$. Equation (23) holds for all $c > 0$ so we can take $c = c_n$ such that $c_n \rightarrow 0$ and $\delta_\Lambda(\hat{\Sigma}_n, \Sigma_n)/c_n \xrightarrow{p} 0$ as $n \rightarrow \infty$, yielding

$$\|\hat{\theta}_n - \theta\|^2 \leq \|\hat{\theta}_n^* - \theta\|^2 + c_n \|\hat{\theta}_n^* - \theta\|^2 + o_p(\alpha_n) = \|\hat{\theta}_n^* - \theta\|^2 + o_p(\alpha_n).$$

We shall now prove the second part of the proposition. Write,

$$\hat{\alpha}_{n,j}^{-\frac{1}{2}}(\hat{\theta}_{n,j} - \theta_j) = \sqrt{\frac{\alpha_{n,j}}{\hat{\alpha}_{n,j}}} \alpha_{n,j}^{-\frac{1}{2}} \left[(\hat{\theta}_{n,j}^* - \theta_j) + (\hat{\theta}_{n,j} - \hat{\theta}_{n,j}^*) \right].$$

To prove the result, it suffices to show that $\alpha_{n,j}^{-\frac{1}{2}}\|\hat{\theta}_{n,j} - \hat{\theta}_{n,j}^*\| = o_p(1)$ and $\alpha_{n,j}/\hat{\alpha}_{n,j} \xrightarrow{p} 1$. When Λ is a cylinder, it is easy to see that the following holds

$$\Pi_j(\hat{\lambda}_n) = \arg \min_{\lambda \in \Lambda_j} \lambda^\top \hat{\Sigma}_n \lambda \quad \text{and} \quad \Pi_j(\lambda_n^*) = \arg \min_{\lambda \in \Lambda_j} \lambda^\top \Sigma_n \lambda,$$

where we recall $\Lambda_j = \{\Pi_j(\lambda) : \lambda \in \Lambda\}$. Moreover, it is easy to adapt the proof of Theorem 4.1 to get

$$\|\hat{\theta}_{n,j} - \hat{\theta}_{n,j}^*\|^2 \leq \alpha_{n,j} \left(2\delta_{\Lambda_j}(\hat{\Sigma}, \Sigma) + \delta_{\Lambda_j}(\hat{\Sigma}, \Sigma)^2 \right) \|\Sigma^{-\frac{1}{2}}(\mathbf{T} - \mathbf{J}\theta)\|^2.$$

We deduce that $\alpha_{n,j}^{-\frac{1}{2}}(\hat{\theta}_{n,j} - \hat{\theta}_{n,j}^*) = o_p(1)$ in view of (11) and Lemma 6.1. Now, remark that

$$\frac{\alpha_{n,j}}{\hat{\alpha}_{n,j}} = \frac{\Pi_j(\lambda_n^*)^\top \Sigma_n \Pi_j(\lambda_n^*)}{\Pi_j(\hat{\lambda}_n)^\top \hat{\Sigma}_n \Pi_j(\hat{\lambda}_n)} \leq \frac{\Pi_j(\hat{\lambda}_n)^\top \Sigma_n \Pi_j(\hat{\lambda}_n)}{\Pi_j(\hat{\lambda}_n)^\top \hat{\Sigma}_n \Pi_j(\hat{\lambda}_n)} - 1 + 1 \leq \delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n) + 1.$$

Similarly,

$$\frac{\hat{\alpha}_{n,j}}{\alpha_{n,j}} \leq \delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n) + 1.$$

So, we get

$$\frac{1}{1 + \delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n)} \leq \frac{\alpha_{n,j}}{\hat{\alpha}_{n,j}} \leq 1 + \delta_{\Lambda_j}(\hat{\Sigma}_n, \Sigma_n),$$

proving that $\alpha_{n,j}/\hat{\alpha}_{n,j} \xrightarrow{p} 1$.

Proof of Lemma 4.4

Using that by assumption $P\Sigma_n^{-1}P^\top A_n^{-1}$ tends to D^{-1} , and that \hat{P}_n (resp. \hat{D}_n) converges in probability to P (resp. D), it follows that

$$\hat{\Sigma}_n \Sigma_n^{-1} = \hat{P}_n^\top A_n^{-1} \hat{D}_n \hat{P}_n \Sigma_n^{-1} = \hat{P}_n^\top A_n^{-1} \hat{D}_n \hat{P}_n P^\top P \Sigma_n^{-1} P^\top A_n^{-1} A_n P$$

converges in probability to I.

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