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Castelnuovo-Mumford regularity and Segre-Veronese transform

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Abstract¹ Motivated by the paper of David A. Cox, and Evgeny Materov (2009), where is computed the Castelnuovo-Mumford regularity of the Segre Veronese embedding, we extend their result and compute the Castelnuovo-Mumford regularity of the Segre product of modules. Our proof is completely algebraic

Key words and phrases: Segre-Veronese, Castelnuovo-Mumford regularity, Cohen-Macaulay, local cohomology.

MSC 2000: Primary: 13D40, Secondary 14M25, 13C14, 14M05.

1 Introduction

Segre embeddings of projective variety plays a key role in Algebraic Geometry. From the algebraic point of view K -algebras and their free resolutions are important fields of research. One of the important numerical invariants of S -modules is the Castelnuovo-Mumford regularity. Several approaches to study the Castelnuovo-Mumford regularity of Segre embeddings was given. The case bi-graded was introduced in [H-W], and more generally the multigraded case, where the graduation is given by a finitely generated group was done in [M-S]. S.Goto and K. Watanabe have studied the local cohomology modules of Veronese and Segre transform of graded modules. Motivated by the paper of David A. Cox, and Evgeny Materov (2009), where is computed the Castelnuovo-Mumford regularity of the Segre Veronese embedding, we extend their result and compute the Castelnuovo-Mumford regularity of the Segre product of modules. In a previous paper [MD] we study the Castelnuovo-Mumford regularity by using the postulation number and avoiding to use local cohomology. In this paper we use the formula given by [G-W] for the local cohomology of a Segre product of modules with depth ≥ 2 , and with an hypothesis on the persistence of the local cohomology modules; this hypothesis is true for Cohen-Macaulay modules. By definition the Castelnuovo-Mumford regularity is the maximum of the "real" regularities of some Segre product of local cohomology modules, in this paper we define the "virtual" regularities and we prove that even if the

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”real” regularities and the ”virtual” regularities are different, taking the maximum over the set of all ”virtual” regularities gives the Castelnuovo-Mumford regularity.

2 Local cohomology and Segre transform

Let S be a polynomial ring over a field K , in a finite number of variables. We suppose that S is graded by the standard graduation $S = \bigoplus_{i \geq 0} S_i$. Let $\mathfrak{m} = \bigoplus_{i \geq 1} S_i$ be the maximal irrelevant ideal. Let M be a finitely generated graded S -module, $M = \bigoplus_{l \geq \sigma} M_l$, with $\sigma \in \mathbb{Z}$ and $M_\sigma \neq 0$.

Remark 2.1. *We choose to take as base ring polynomials rings, because of the paper [G-W], but in fact by using [B-S], all our results will be true over standard Noetherian graded rings $R = R_0[R_1]$, where R_0 is a local ring with infinite residue field.*

We say that a graded S -module $N = \bigoplus_{l \in \mathbb{Z}} N_l$, not necessarily finitely generated has no gaps if there is no integers $i < j < k$ such that

$$N_i \neq 0; N_k \neq 0; N_j = 0.$$

Example 2.2. *Let M be a finitely generated graded S -module with $\text{depth } M \geq 1$, $M = \bigoplus_{l \geq \sigma} M_l$, where $\sigma \in \mathbb{Z}$ and $M_\sigma \neq 0$. Assuming that the field K is infinite, there exists $x \in S_1$, a nonzero divisor of M . The multiplication by x defines an injective map $M_i \rightarrow M_{i+1}$, hence M has no gaps, and for all $l \geq \sigma$, we have $M_l \neq 0$.*

Let M be a finitely generated graded S -module, the local cohomology modules are graded, so we can define $\text{end}(H_{\mathfrak{m}}^i(M)) = \max\{\beta \in \mathbb{Z} \mid (H_{\mathfrak{m}}^i(M))_\beta \neq 0\}$. We recall the local duality’s theorem (see [S]) :

We have an isomorphism :

$$H_{\mathfrak{m}}^i(M) \simeq \text{Hom}_S(\text{Ext}_S^{n-i}(M, S), E(S/\mathfrak{m})).$$

We denote by $D^i(M)$ the finitely generated graded S -module $\text{Ext}_S^{n-i}(M, S)$. The following Lemma follows immediately from the local duality’s theorem and the Example 2.2.

Lemma 2.3. • *If $\text{depth}(D^i(M)) \geq 1$ then $H_{\mathfrak{m}}^i(M)$ has no gaps and for all $l \leq \text{end}(H_{\mathfrak{m}}^i(M))$, we have $(H_{\mathfrak{m}}^i(M))_l \neq 0$.*

- *Let M be a finitely generated graded S -module with $\text{depth } M \geq 1$ and $\dim M = d$. It is known that $\text{depth } D^d(M) \geq \min\{d, 2\}$. Hence the top local cohomology $H_{\mathfrak{m}}^d(M)$ has no gaps and for all $l \leq \text{end}(H_{\mathfrak{m}}^d(M))$, we have $(H_{\mathfrak{m}}^d(M))_l \neq 0$.*
- *It follows from [M1] that if A is a standard graded simplicial toric ring of dimension d , and $\text{depth } A = d - 1$, then $D^{d-1}(A)$ is a Cohen-Macaulay Module of dimension $d - 2$, so if $d \geq 3$, the module $H_{\mathfrak{m}}^{d-1}(A)$ has no gaps, and for all $l \leq \text{end}(H_{\mathfrak{m}}^{d-1}(M))$, we have $(H_{\mathfrak{m}}^{d-1}(M))_l \neq 0$.*

We recall the following theorem [G-W, Theorem 4.1.5].

Theorem 2.4. *Let S_1, S_2 be the two polynomial rings on two disjoint sets of variables, for $i = 1, 2$, M_i be finitely generated graded S_i -module. Let \mathfrak{m} be the maximal irrelevant ideal of $S_1 \otimes S_2$. Assume that $\text{depth } M_i \geq 2$, for all i . Then for all $j \in \mathbb{Z}$*

$$H_{\mathfrak{m}}^j(M_1 \otimes M_2) = (M_1 \otimes H_{\mathfrak{m}_2}^j(M_2)) \oplus (H_{\mathfrak{m}_1}^j(M_1) \otimes M_2) \oplus_{k,l \mid j=k+l-1} (H_{\mathfrak{m}_1}^k(M_1) \otimes H_{\mathfrak{m}_2}^l(M_2)).$$

Remark 2.5. *Let M_1 (respectively M_2) be a finitely generated graded S_1 -module (respectively a finitely generated graded S_2 -module), where S_1, S_2 are the polynomial rings in disjoint sets of variables. For $i = 1, 2$, let denote*

$$\text{end}(H_{\mathfrak{m}_i}^j(M_i)) = \max\{\beta \in \mathbb{Z} \mid (H_{\mathfrak{m}_i}^j(M_i))_{\beta} \neq 0\};$$

$$r_j(M_i) = \text{end}(H_{\mathfrak{m}_i}^j(M_i)) + j.$$

If $(H_{\mathfrak{m}_1}^j(M_1)), (H_{\mathfrak{m}_2}^k(M_2))$ have no gaps, and the Segre product $(H_{\mathfrak{m}_1}^j(M_1)) \otimes (H_{\mathfrak{m}_2}^k(M_2))$ is non null, then $(H_{\mathfrak{m}_1}^j(M_1)) \otimes (H_{\mathfrak{m}_2}^k(M_2))$ has no gaps and

$$\begin{aligned} \text{end}(H_{\mathfrak{m}_1}^j(M_1) \otimes H_{\mathfrak{m}_2}^k(M_2)) &= \min(\text{end}(H_{\mathfrak{m}_1}^j(M_1)), \text{end}(H_{\mathfrak{m}_2}^k(M_2))) \\ &= \min(r_j(M_1) - j, r_k(M_2) - k). \end{aligned}$$

Definition 2.6. *From now on, we consider s -polynomial rings (with disjoint set of variables) S_1, \dots, S_s , graded, with irrelevant ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$. For $i = 1, \dots, s$, let M_i be a finitely generated graded S_i -module such that $\text{depth}(M_i) \geq 2$. We set the following notations:*

- $d_i = \dim M_i$; $\sigma_i = \min\{l \in \mathbb{Z} : (M_i)_l \neq 0\}$; $r_{i,j} = \text{end}(H_{\mathfrak{m}_i}^j(M_i)) + j$,
- $\tilde{C} := [0, \dots, d_1] \times [0, \dots, d_2] \times \dots \times [0, \dots, d_s]$,
- for $u \in \tilde{C}$, $\text{supp } u = \{j \in \{1, \dots, s\} \mid u_j \neq 0\}$,
- $C := \{u \in \tilde{C} \mid \text{supp } u \neq \emptyset \text{ and } \forall i \in \text{supp } u, H_{\mathfrak{m}_i}^{u_i}(M_i) \neq 0\}$
- $E_{i,j} = \begin{cases} H_{\mathfrak{m}_i}^j(M_i) & \text{if } j > 0 \\ M_i & \text{if } j = 0 \end{cases}$; for $u \in C$, $E_u = \bigotimes_{i=1}^s E_{i,u_i}$.
- For $u \in C$, $\Gamma_u = \text{end}(E_u) + (1 + \sum_{l \in \text{supp } u} (u_l - 1))$.
- For $u \in C$, $\gamma_u = \min_{i \in \text{supp } u} (r_{i,u_i} + \sum_{l \in \text{supp } u, l \neq i} (u_l - 1))$,
- For $j \geq 2$, $C_j := \{u \in C; 1 + \sum_{l \in \text{supp } u} (u_l - 1) = j\}$.

We can state the following corollary of Theorem 2.4.

Corollary 2.7. *With the notations introduced in 2.6, for $j \geq 2$, we have*

$$H_{\mathfrak{m}}^j(M_1 \otimes \dots \otimes M_s) = \bigoplus_{u \in C_j} E_u,$$

where \mathfrak{m} is the irrelevant maximal ideal of $S_1 \otimes \dots \otimes S_s$.

Hence we have that

$$\text{end}(H_{\mathfrak{m}}^j(M_1 \otimes \dots \otimes M_s)) + j = \max_{u \in C_j} \Gamma_u,$$

and

$$\text{reg}(M_1 \otimes \dots \otimes M_s) = \max_{u \in C} \Gamma_u.$$

Assumption 2.8. **In all this paper we assume :** for any $i = 1, \dots, s; j$; $\text{depth } M_i \geq 2$, $H_{\mathfrak{m}_i}^j(M_i)$ has no gaps, and $(H_{\mathfrak{m}_i}^j(M_i))_k \neq 0$ for infinitely many k .

In order to state our main theorem we need some preparatory Lemmas.

Lemma 2.9. *With the notations introduced in 2.6, and the Assumption 2.8. Let $\epsilon_1, \dots, \epsilon_s$ be the canonical basis of \mathbb{Z}^s . For any $u \in C$:*

1. *If $E_u \neq 0$ (i.e. $\Gamma_u \neq -\infty$) then $\Gamma_u = \gamma_u$.*
2. *For any $k \notin \text{supp } u$, $\lambda_k \in \mathbb{N}^*$, $\lambda_k \leq d_k$, we have*

$$\gamma_{u+\lambda_k \epsilon_k} = \min(\gamma_u + \lambda_k - 1, r_{k, \lambda_k} + \sum_{l \in \text{supp } u} (u_l - 1)).$$

Proof. (1) Suppose $E_u \neq 0$ (i.e. $\Gamma_u \neq -\infty$). By the Remark 2.5, we have

$$\text{end}\left(\bigotimes_{i \in \text{supp } u} H_{\mathfrak{m}_i}^{u_i}(M_i)\right) = \min_{i \in \text{supp } u} (r_{i, u_i} - u_i).$$

since for any $i = 1, \dots, s$, M_i and $H_{\mathfrak{m}_i}^{u_i}(M_i)$ has no gaps then $\text{end}\left(\bigotimes_{i \in \text{supp } u} H_{\mathfrak{m}_i}^{u_i}(M_i)\right) = \text{end}(E_u)$.

$$\begin{aligned} \Gamma_u &= \min_{i \in \text{supp } u} (r_{i, u_i} - u_i) + \sum_{l \in \text{supp } u} (u_l - 1) + 1 \\ &= \min_{i \in \text{supp } u} (r_{i, u_i} + \sum_{l \in \text{supp } u; l \neq i} (u_l - 1)) = \gamma_u. \end{aligned}$$

Now we prove (2). By definition

$$\begin{aligned} \gamma_{u+\lambda_k \epsilon_k} &= \min\left(\min_{i \in \text{supp } u} (r_{i, u_i} + \sum_{l \in \text{supp } u; l \neq i} (u_l - 1) + (\lambda_k - 1)), r_{k, \lambda_k} + \sum_{l \in \text{supp } u} (u_l - 1)\right) \\ &= \min(\gamma_u + (\lambda_k - 1), r_{k, \lambda_k} + \sum_{l \in \text{supp } u} (u_l - 1)). \end{aligned}$$

□

Lemma 2.10. *With the notations introduced in 2.6, and the Assumption 2.8. For any $i = 1, \dots, s$ let δ_i be an integer such that $\text{reg}(M_i) = \text{end}(H_{\mathfrak{m}_i}^{\delta_i}(M_i)) + \delta_i$. For any $u \in C$, we set*

$$\Sigma_u = \{\Gamma_v \mid v = u + \sum_{l \in L} \delta_l \epsilon_l, L \subset \{1, \dots, s\}, L \cap \text{supp } u = \emptyset\};$$

$$S_u = \{\gamma_v \mid v = u + \sum_{l \in L} \delta_l \epsilon_l, L \subset \{1, \dots, s\}, L \cap \text{supp } u = \emptyset\}.$$

Then for all $u \in C$, we have $\max \Sigma_u = \max S_u$.

Proof. For any $k = 0, \dots, s - |\text{supp } u|$, we set

$$\Sigma_{u,k} = \{\Gamma_v \mid v = u + \sum_{l \in L} \delta_l \epsilon_l, L \cap \text{supp } u = \emptyset, |L| \geq k\};$$

$$S_{u,k} = \{\gamma_v \mid v = u + \sum_{l \in L} \delta_l \epsilon_l, L \cap \text{supp } u = \emptyset, |L| \geq k\}.$$

Let remark that by the Assumption 2.8 if $\text{supp } u = \{1, \dots, s\}$ then $E_u \neq 0$, hence by Lemma 2.9 $\Sigma_u = S_u$.

We will prove by induction on k , that for any $k = 0, \dots, s - |\text{supp } u|$, $\max \Sigma_{u,k} = \max S_{u,k}$.

The Lemma will be proved since our claim is the case $k = 0$. For $k = s - |\text{supp } u|$, there is only one set L with $|L| = s - |\text{supp } u|$, it is the complementary of $\text{supp } u$, the local cohomology module $E_{u + \sum_{l \notin \text{supp } u} \delta_l \epsilon_l}$ is non zero by the Assumption 2.8. So by Lemma 2.9,

$$\Gamma_{u + \sum_{l \notin \text{supp } u} \delta_l \epsilon_l} = \gamma_{u + \sum_{l \notin \text{supp } u} \delta_l \epsilon_l}$$

which implies $\Sigma_{u, s - |\text{supp } u|} = S_{u, s - |\text{supp } u|}$, and $\max(\Sigma_{u, s - |\text{supp } u|}) = \max(S_{u, s - |\text{supp } u|})$.

Now suppose that for some $1 \leq k \leq s - |\text{supp } u|$, we have $\max(\Sigma_{u,k}) = \max(S_{u,k})$. We will prove that $\max(\Sigma_{u, k-1}) = \max(S_{u, k-1})$. We have two cases:

(i) For all L such that $L \subset \{1, \dots, s\}, L \cap \text{supp } u = \emptyset, |L| = k - 1$, we have $E_v \neq 0$, where $v = u + \sum_{l \in L} \delta_l \epsilon_l$, then $\Gamma_v = \gamma_v$, hence

$$\begin{aligned} \max(\Sigma_{u, k-1}) &= \max(\Gamma_v \mid v = u + \sum_{l \in L} \delta_l \epsilon_l, |L| = k - 1, \max \Sigma_{u,k}) \\ &= \max\{\gamma_v \mid v = u + \sum_{l \in L} \delta_l \epsilon_l, |L| = k - 1, \max S_{u,k}\} \\ &= \max\{S_{u, k-1}\}. \end{aligned}$$

(ii) There exists $L \subset \{1, \dots, s\}, L \cap \text{supp } u = \emptyset, |L| = k - 1$, such that $E_v = 0$, for $v = u + \sum_{l \in L} \delta_l \epsilon_l$. We have

$$E_v = 0 \Leftrightarrow \text{end}\left(\bigotimes_{i \in \text{supp } v} H_{\mathfrak{m}_i}^{v_i}(M_i)\right) < \max_{j \notin \text{supp } v} \sigma_j.$$

Let $n \notin \text{supp } v$ such that $\max_{j \in \text{supp } v} \sigma_j = \sigma_n$. Thus the condition $E_v = 0$ is equivalent to $\min_{i \in \text{supp } v} (r_{i,v_i} - v_i) < \sigma_n$. But $\sigma_n \leq \text{reg}(M_n) = r_{n,\delta_n}$, by [B-S, Theorem 15.3.1]. So

$$\min_{i \in \text{supp } v} (r_{i,v_i} - v_i) \leq r_{n,\delta_n} - 1.$$

It implies that

$$\min_{i \in \text{supp } v} (r_{i,v_i} + \sum_{l \in \text{supp } v; l \neq i} (v_l - 1)) \leq r_{n,\delta_n} + \sum_{l \in \text{supp } v} (v_l - 1).$$

That is $\gamma_v \leq r_{n,\delta_n} + \sum_{l \in \text{supp } v} (v_l - 1)$. Since $\text{depth}(M_n) \geq 2$, $\delta_n \geq 2$, we obtain $\gamma_v \leq \gamma_v + (\delta_n - 1)$ which implies that

$$\gamma_v \leq \min(\gamma_v + \delta_n - 1, r_{n,\delta_n} + \sum_{l \in \text{supp } v} (v_l - 1)) = \gamma_{v+\delta_n \epsilon_n} \in S_{u,k}.$$

The last equality follows from Lemma 2.9. In conclusion $\max \Sigma_{u,k-1} = \max S_{u,k-1}$, the proof of the Lemma is over. \square

Now we can state our main theorem.

Theorem 2.11. *With the notations introduced in 2.6, and the Assumption 2.8. We have:*

- $\max\{\Gamma_u \mid u \in C\} = \max\{\gamma_u \mid u \in C\}$.
- $\text{reg}(M_1 \underline{\otimes} \dots \underline{\otimes} M_s) = \max_{u \in C} \{1 + \sum_{l \in \text{supp } u} u_l + \min_{i \in \text{supp } u} (\text{end}(H_{\mathfrak{m}_i}^{u_i}(M_i)))\}$.

Proof. The first claim from the Lemma 2.10, since

$$\max(\Gamma_u \mid u \in C) = \max(\max \Sigma_u) = \max(\max S_u) = \max(\gamma_u \mid u \in C).$$

The second claim follows from the definition of Castelnuovo-Mumford regularity, the first claim, the Corollary 2.7 and the definition of γ_u . \square

We recall the following Proposition from [MD]:

Proposition 2.12. *If M is a Cohen-Macaulay module of dimension d , then $\text{reg } M[\tau]^{<n>} = d - \lceil \frac{d - \text{reg } M + \tau}{n} \rceil$.*

Hence we have the following consequence:

Theorem 2.13. *Let S_1, \dots, S_s be graded polynomial rings on disjoint sets of variables. For all $i = 1, \dots, s$, let M_i be a graded finitely generated S_i -Cohen-Macaulay module with $\text{depth } M_i \geq 2$. Let $d_i = \dim M_i, b_i = d_i - 1 \geq 1, \alpha_i = d_i - \text{reg}(M_i)$, where $\text{reg}(M_i)$ is the Castelnuovo-Mumford regularity of M_i . Then*

$$(1) \operatorname{reg}(M_1 \underline{\otimes} \dots \underline{\otimes} M_s) = \max_{u \in C} \left\{ 1 + \sum_{l \in \operatorname{supp} u} b_l - \max_{l \in \operatorname{supp} u} \{\alpha_l\} \right\}.$$

(2) For $n_i \in \mathbb{N}$, let $M_i[\tau_i]^{<n_i>}$ be the shifted n_i -Veronese transform of M_i , then

$$\operatorname{reg}(M_1[\tau_1]^{<n_1>} \underline{\otimes} \dots \underline{\otimes} M_s[\tau_s]^{<n_s>}) = \max_{u \in C} \left\{ 1 + \sum_{l \in \operatorname{supp} u} b_l - \max_{l \in \operatorname{supp} u} \left\{ \left\lceil \frac{\alpha_l + \tau_l}{n_l} \right\rceil \right\} \right\}.$$

As a Corollary we generalize one of the main results of [C-M][Theorem 1.4]

Corollary 2.14. (*[C-M][Theorem 1.4]*) For $i = 1, \dots, s$, let S_i be graded polynomial rings on disjoint sets of variables, with $\dim S_i \geq 2$, let $m_i, n_i \in \mathbb{Z}$, and $S_i[m_i]^{<n_i>}$ be the n_i -Veronese transform of $S_i[m_i]$, then

$$\operatorname{reg}(S_1[m_1]^{<n_1>} \underline{\otimes} \dots \underline{\otimes} S_s[m_s]^{<n_s>}) = \max_{u \in C} \left\{ \sum_{l \in \operatorname{supp} u} b_l - \max_{l \in \operatorname{supp} u} \left\{ \left\lceil \frac{b_l + m_l}{n_l} \right\rceil \right\} \right\}.$$

Remark 2.15. Let S_1, \dots, S_s be graded polynomial rings on disjoint sets of variables. For all $i = 1, \dots, s$, let M_i be a graded finitely generated S_i -module with $\operatorname{depth} M_i \geq 2$. If we don't have the assumption: for any $i = 1, \dots, s; j; H_{\mathfrak{m}_i}^j(M_i)$ has no gaps. and $(H_{\mathfrak{m}_i}^j(M_i))_k \neq 0$ for infinitely many k , we get trivially the inequality:

$$\operatorname{reg}(M_1 \underline{\otimes} \dots \underline{\otimes} M_s) \leq \max_{u \in C} \left\{ 1 + \sum_{l \in \operatorname{supp} u} u_l + \min_{i \in \operatorname{supp} u} (\operatorname{end}(H_{\mathfrak{m}_i}^{u_i}(M_i))) \right\}.$$

3 Square free monomial ideals

Let Δ be a simplicial complex with support on n vertices, labeled by the set $[n] = \{1, \dots, n\}$, $S := K[x_1, \dots, x_n]$ be a polynomial ring, $I_\Delta \subset S$ be the Stanley Reisner ideal associated to Δ , that is $I_\Delta = \{x^F / F \notin \Delta\}$. We quote the following theorem from [Sb][Proposition 3.8]:

Proposition 3.1. Let $K[\Delta] := S/I_\Delta$ be the Stanley-Reisner ring associated to Δ , $a = (a_1, \dots, a_n) \in \mathbb{Z}$, where for all i , $a_i \leq 0$. Let $F = \operatorname{supp}(a) \subset [n]$ and $|F| := \operatorname{card}(F)$ then:

$$\dim_K(H_{\mathfrak{m}}^i(K[\Delta]))_a = \beta_{i+1-|F|, [n] \setminus F}(K[\Delta^*]),$$

where Δ^* is the Alexander dual of Δ .

We get the following Corollary, which statement in [Sb][Lemma 3.9] has minor mistakes:

Corollary 3.2. 1.

$$\dim_K(H_{\mathfrak{m}}^i(K[\Delta]))_0 = \beta_{i+1, n}(K[\Delta^*]),$$

2. For any integer $j > 0$:

$$\dim_K(H_{\mathfrak{m}}^i(K[\Delta]))_{-j} = \sum_{h=1}^{\min(j, n)} \binom{j-1}{h} \beta_{i+1-h, n-h}(K[\Delta^*]).$$

Note also that even if [H-S][Theorem 4.1] is true, its proof has minor mistakes. In that proof it was used the relation $H^t = AB$, where A, B, H are three matrices, by using the Corollary 3.2, we can give the correct expressions for the matrices A, B, H . Let

$$1 \leq i \leq n-1, 1 \leq j \leq n-1, 1 \leq h \leq n-1, h_{i,j} = \dim_K(H_m^i(K[\Delta]))_{-j}, H = (h_{i,j})$$

$$a_{j,h} = \begin{cases} \binom{j-1}{h-1} & \text{if } h \leq j \\ 0 & \text{else} \end{cases}, A = (a_{j,h}), b_{h,i} = \begin{cases} \beta_{i+1-h,n-h}(K[\Delta^*]) & \text{if } h \leq i+1 \\ 0 & \text{else} \end{cases}, B = (b_{h,i}).$$

Then

$$H^t = AB.$$

$$\begin{pmatrix} h_{1,1} & \dots & \dots & h_{n-1,1} \\ h_{1,2} & \dots & \dots & h_{n-1,2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ h_{1,n-1} & \dots & \dots & h_{n-1,n-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & \binom{2}{1} & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{n-2}{1} & \binom{n-2}{2} & \dots & \binom{n-2}{n-2} \end{pmatrix} \begin{pmatrix} \beta_{1,n-1} & \beta_{2,n-1} & \dots & \beta_{n-1,n-1} \\ 0 & \beta_{1,n-2} & \dots & \beta_{n-2,n-2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta_{1,1} \end{pmatrix}$$

Corollary 3.3. *Let i be an integer. Let k_i be the smallest h such that $\beta_{i+1-h,n-h}(K[\Delta^*]) \neq 0$. If $k_i = 1$ or $(H_m^i(S/I_\Delta))_0 = 0$, then $H_m^i(S/I_\Delta)$ has no gaps and $(H_m^i(S/I_\Delta))_k \neq 0$ for all $k \leq \text{end}(H_m^i(S/I_\Delta))$.*

Proof. We can assume that $(H_m^i(S/I_\Delta)) \neq 0$. The formula proved in 3.2 implies that $(H_m^i(S/I_\Delta))_k \neq 0$ for all $k \leq k_i$. The claim is over. \square

For $i = 1, \dots, s$ consider Δ_i be a simplicial complex with support on n_i vertices, labeled by the set $[n_i] = \{1, \dots, n_i\}$, S_i be a polynomial ring on n_i variables, $I_{\Delta_i} \subset S_i$ be the Stanley Reisner ideal associated to Δ_i . Let $\tilde{C} := [0, \dots, n_1] \times [0, \dots, n_2] \times \dots \times [0, \dots, n_s]$, and $C_1 := \{u \in \tilde{C} \mid H_{m_i}^{u_i}(S_i/I_{\Delta_i}) \neq 0 \forall i = 1, \dots, s\}$

Theorem 3.4. *Let consider $\Delta_1, \dots, \Delta_s$, s simplicial complexes such that for all $i = 1, \dots, s$, $\text{depth } S_i/I_{\Delta_i} \geq 2$, and $k_i = 1$ or $(H_{m_i}^i(S/I_{\Delta_i}))_0 = 0$. Then*

$$\text{reg}(S_1/I_{\Delta_1} \underline{\otimes} \dots \underline{\otimes} S_s/I_{\Delta_s}) = \max_{u \in C} \{1 + \sum_{l \in \text{supp } u} u_l + \min_{i \in \text{supp } u} (\text{end}(H_{m_i}^{u_i}(S_i/I_{\Delta_i})))\}.$$

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