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QUASIHOMOGENEOUS THREE-DIMENSIONAL REAL ANALYTIC LORENTZ METRICS

SORIN DUMITRESCU

ABSTRACT. We classify germs at the origin of real analytic Lorentz metrics on \mathbf{R}^3 which are *quasihomogeneous*, in the sense that they are *locally homogeneous* on an open set containing the origin in its closure, but not locally homogeneous in the neighborhood of the origin.

1. INTRODUCTION

The most symmetric Riemannian and pseudo-Riemannian metrics are those for which the *Killing Lie algebra* is transitive. They are called *locally homogeneous* and their study is a traditional field in differential and Riemannian geometry. In dimension two, this means exactly that the (pseudo)-Riemannian metric is of constant sectional curvature. Locally homogeneous Riemannian metrics of dimension three, are the context of Thurston's 3-dimensional geometrization program [Thu97]. The classification of compact locally homogeneous Lorentz 3-manifolds was given in [DZ10].

This article deals with the classification of germs at the origin of three-dimensional real analytic Lorentz metrics which are *quasihomogeneous*, in the sense that they are *locally homogeneous* on an open set containing the origin in its closure, but not locally homogeneous in the neighborhood of the origin. The quasihomogeneous Lorentz metrics are the most symmetric ones, after those which are locally homogeneous. In particular, all their scalar invariants are constant. Recall that, in the Riemannian setting, that implies local homogeneity (see [PTV96] for an effective result).

The main theorem in this article is the following:

Theorem 1. *Let g be a real-analytic Lorentz metric in a neighborhood of the origin in \mathbf{R}^3 . Suppose that the maximal open set on which the Killing Lie algebra \mathcal{G} of g is transitive contains the origin in its closure, but does not contain the origin. Then, in adapted analytic coordinates in the neighborhood of the origin, the germ of g at 0 is*

$$g = dx^2 + dh dz + Cz^2 dh^2$$

with $C \in \mathbb{R} \setminus \{0\}$.

In these coordinates $\mathcal{G} = \langle \partial/\partial x, \partial/\partial h, h\partial/\partial h - z\partial/\partial z \rangle$. In particular, the Killing Lie algebra is isomorphic to $\mathbb{R} \oplus \mathfrak{aff}(\mathbf{R})$, where $\mathfrak{aff}(\mathbf{R})$ is the Lie algebra of the affine group of the real line.

Remark 2. *The closed set on which g is not locally homogeneous is the totally geodesic surface S given by $z = 0$. The curvature tensor of g vanishes exactly on S . The Killing Lie algebra \mathcal{G} also preserves the flat Lorentz metric $g_0 = dx^2 + dh dz$.*

The quasihomogeneous germs of Lorentz metrics constructed in theorem 1 never extend to a compact manifold. Indeed, the main theorem in [Dum08] asserts that *a real analytic Lorentz metric on a compact 3-manifold which is locally homogeneous on a nontrivial open set is locally homogeneous on all of the manifold.*

Key words and phrases. real analytic Lorentz metrics, transitive Killing Lie algebras, local differential invariants.

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The same is known to be true for real analytic Lorentz metrics in higher dimension, under the assumptions that the Killing algebra is semi-simple, the metric is geodesically complete and the universal cover is acyclic, as a consequence of a more general result of K. Melnick [Mel09].

In the smooth category, A. Zeghib proved in [Zeg96] that compact Lorentz 3-manifolds which admit essential Killing fields are necessarily locally homogeneous.

These results are motivated by Gromov's *open-dense orbit theorem* [DG91, Gro88] (see also [Ben97, Fer02]). Gromov's result asserts that, if the pseudogroup of local automorphisms of a *rigid geometric structure* (a Lorentz metric or an analytic connection, for example) acts with a dense orbit, then this orbit is open. In this case, the rigid geometric structure is locally homogeneous on an open dense set. Gromov's theorem says nothing about this maximal open and dense set of local homogeneity which appears to be mysterious. In many interesting geometric situations, it may be all of the (connected) manifold. This was proved, for instance, for Anosov flows with differentiable stable and unstable foliations and transverse contact structure [BFL92]. In [BF05], the authors deal with this question and their results indicate ways in which some rigid geometric structures cannot degenerate off the open dense set.

In a recent common work with A. Guillot we obtained, with different methods, the classification of germs of quasihomogeneous real analytic torsion free affine connections on surfaces [DG13].

The composition of this article is the following. In Section 2 we use the geometry of Killing fields and geometric invariant theory to prove that the Killing Lie algebra of a three-dimensional quasihomogeneous Lorentz metric g is a three-dimensional solvable Lie algebra. We also show that g is locally homogeneous away of a (totally geodesic) surface S on which the isotropy is an one parameter semi-simple group. Theorem 1 is proved in Section 3. As a by-product of the proof of Theorem 1, we get the following more technical result.

Proposition 3. *Let g be a real-analytic Lorentz metric in a neighborhood of the origin in \mathbf{R}^3 . Suppose that there exists a three-dimensional solvable subalgebra of the Killing Lie algebra acting transitively on an open set admitting the origin in its closure, but not in the neighborhood of the origin. If the isotropy at the origin is an one parameter semi-simple subgroup in $O(2,1)$, then there exists local analytic coordinates (x,h,z) in the neighborhood of the origin, and real constants C,D such that the germ of g at 0 is*

$$dx^2 + dh dz + Cz^2 dh^2 + Dz dx dh.$$

In these coordinates, the three-dimensional solvable subalgebra is $\mathcal{G} = \langle \partial/\partial x, \partial/\partial h, h\partial/\partial h - z\partial/\partial z \rangle$. In particular, it is isomorphic to $\mathbb{R} \oplus \mathfrak{aff}(\mathbf{R})$, where $\mathfrak{aff}(\mathbf{R})$ is the Lie algebra of the affine group of the real line.

(i) *The Killing algebra is of dimension three (and g is quasihomogeneous) if and only if $C \neq 0$ and $D = 0$.*

(ii) *If $C = 0$ and $D \neq 0$, the Killing algebra is solvable of dimension 4 and g is locally homogeneous. An extra Killing field is $e^{-Dx}\partial/\partial z$.*

(iii) *If $C \neq 0$ and $D \neq 0$, then g is isomorphic to a left invariant metric on $SL(2, \mathbb{R})$ and the Killing algebra is $\mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R})$. An extra Killing field is $T = ah\partial/\partial x + \frac{1}{2}bh^2\partial/\partial h + (-bzh - \frac{a}{D})\partial/\partial z$, with a, b real constants such that $a(D - \frac{C}{D}) = b$.*

(iv) *If $C = 0$ and $D = 0$, g is flat and the Killing algebra is of dimension six.*

2. KILLING LIE ALGEBRA. INVARIANT THEORY

Let g be a real analytic Lorentz metric defined in an open neighborhood U of the origin in \mathbb{R}^3 .

Classically (see, for instance [Gro88, DG91]) one considers the k -jet of g , by taking at each point $u \in U$ the expression of g in exponential coordinates, up to order k . In these coordinates, the 1-jet of g is the standard flat Lorentz metric $dx^2 + dy^2 - dz^2$. At each point $u \in U$, the space of exponential coordinates is acted on simply transitively by $O(2, 1)$ (which is isomorphic to $PSL(2, \mathbb{R})$). The space of all exponential coordinates in U is a principal $PSL(2, \mathbb{R})$ -bundle over U , also called the orthonormal frame bundle and denoted by $R(U)$.

Geometrically, the k -jet of g is an (analytic) $PSL(2, \mathbb{R})$ -equivariant map $g^{(k)} : R(U) \rightarrow V^{(k)}$, where $V^{(k)}$ is the finite dimensional affine space of k -jets of Lorentz metrics with origin at the 1-jet $dx^2 + dy^2 - dz^2$, endowed with the linear action of $O(2, 1) \simeq PSL(2, \mathbb{R})$ (notice that this action preserves the origin). More precisely, consider a system of local exponential coordinates at u with respect to g and, for all $k \in \mathbb{N}$ take the k -jet of g in these coordinates. Any linear isomorphism of $(T_u U, g(u))$ gives another system of local exponential coordinates at u , with respect to which we consider the k -jet of g . This gives a linear *algebraic* $PSL(2, \mathbb{R})$ -action on the vector space $V^{(k)}$ of k -jets of Lorentz metrics in exponential coordinates. One can find the details of this classical construction in [DG91].

Recall also that a (local) vector field is a *Killing field* for a Lorentz metric g if its (local) flow preserves g . The collection of all local Killing fields in the neighborhood of a point has the structure of a finite dimensional Lie algebra called the *Killing algebra* of g . We will denote it by \mathcal{G} .

At a given point $u \in U$, the subalgebra \mathcal{I} of \mathcal{G} consisting on those vector fields $X \in \mathcal{G}$ such that $X(u) = 0$, is called the *isotropy algebra* at u .

Definition 4. The Lorentz metric g is said to be locally homogeneous on an open subset $W \subset U$, if for any $w \in W$ and any tangent vector $V \in T_w W$, there exists a local Killing field X of g such that $X(w) = V$. In this case, we will say that the Killing algebra \mathcal{G} is transitive on W .

Our proof of theorem 1 will need analyticity in an essential way. We will make use of an extendability result for local Killing fields proved first by Nomizu in the Riemannian setting [Nom60] and generalized then for any rigid geometric structures by Amores and Gromov [Amo79, Gro88, DG91]. This phenomenon states that a local Killing field of g can be extended by monodromy along any curve γ in U and the resulting Killing field only depends on the homotopy type of γ .

Here we will assume that U is connected and simply connected and hence, *local Killing fields extend on all of U* . In particular, the Killing algebra in the neighborhood of any point is the same.

Notice that Nomizu's extension phenomenon doesn't imply that the extension of a family of pointwise linearly independent Killing fields, stays linearly independent.

Assume for now on that g is a quasihomogeneous Lorentz metric in the neighborhood of the origin in \mathbb{R}^3 .

The set of points s in U at which the Killing algebra \mathcal{G} of g does not span the tangent space $T_s U$ is a nontrivial locally closed analytic subset S in U passing through origin. In this case, g is locally homogeneous on each connected component of $U \setminus S$, but not in the neighborhood of points in S .

Moreover, at each point s of S one will get a *nontrivial isotropy algebra* given by the kernel of the canonical evaluation morphism $ev : \mathcal{G} \rightarrow T_s U$ (which is, by definition, of nonmaximal rank at points of S).

Let us prove the following

Lemma 5. *The Killing algebra \mathcal{G} cannot be both 3-dimensional and unimodular.*

Proof. Let K_1, K_2 and K_3 be a basis of the Killing algebra. Consider the analytic function $v = \text{vol}(K_1, K_2, K_3)$, where vol is the volume of the Lorentz metric. Since \mathcal{G} is unimodular, the function v is $\neq 0$ and constant on each open set where \mathcal{G} is transitive. On the other hand, v vanishes on S : a contradiction. \square

We prove now that:

- Lemma 6.** (i) *The dimension of the isotropy at a point $u \in U$ is $\neq 2$.*
(ii) *The Killing algebra \mathcal{G} is of dimension 3.*
(iii) *The Killing algebra \mathcal{G} is solvable.*

Proof. (i) Assume by contradiction that the isotropy algebra \mathcal{I} at a point $u \in U$ has dimension two.

Elements of \mathcal{I} linearize in exponential coordinates at u .

Since elements of \mathcal{I} preserve g , they preserve, in particular, the k -jet of g at u , for all $k \in \mathbb{N}$. This gives an embedding of \mathcal{I} in the Lie algebra of $PSL(2, \mathbb{R})$ such that the corresponding (two dimensional) connected subgroup of $PSL(2, \mathbb{R})$ preserves the k -jet of g at u , for all $k \in \mathbb{N}$.

Now we use the fact that *the stabilizers of a finite dimensional linear algebraic $PSL(2, \mathbb{R})$ -action are of dimension $\neq 2$* . Indeed, it suffices to check this statement for irreducible linear representations of $PSL(2, \mathbb{R})$, for which it is well-known that the stabilizer in $PSL(2, \mathbb{R})$ of a nonzero element is one dimensional [Kir74].

It follows that the stabilizer in $PSL(2, \mathbb{R})$ of the k -jet of g at u is of dimension three and hence contains the connected component of identity in $PSL(2, \mathbb{R})$. Consequently, in exponential coordinates at u , each element of the connected component of the identity in $PSL(2, \mathbb{R})$ gives rise to a local linear vector field which preserves g (for it preserves all k -jets of g). The isotropy algebra \mathcal{I} contains a copy of the Lie algebra of $PSL(2, \mathbb{R})$: a contradiction, since \mathcal{I} is of dimension two.

(ii) Since g is quasihomogeneous, the Killing algebra is of dimension at least 3.

For a three-dimensional Lorentz metric, the maximal dimension of the Killing algebra is 6. This characterizes Lorentz metrics of constant sectional curvature. Indeed, in this case, the isotropy is, at each point, of dimension three and acts transitively on the nondegenerate 2-planes (see, for instance, [Wol67]). These Lorentz metrics are locally homogeneous (and not quasihomogeneous).

Assume, by contradiction, that the Killing algebra of g is of dimension 5. Then, on any open set of local homogeneity the isotropy is two-dimensional. This is in contradiction with point (i).

Assume by contradiction that the Killing algebra of g is of dimension 4. Then, at a point $s \in S$, the isotropy has dimension ≥ 2 . Hence, point (i) implies that the isotropy at s has dimension three (isomorphic to $PSL(2, \mathbb{R})$). Moreover, the standard linear action of the isotropy on $T_s U$ preserves the image of the evaluation morphism $ev(s) : \mathcal{G} \rightarrow T_s U$, which is a line. But the standard 3-dimensional $PSL(2, \mathbb{R})$ -representation does not admit invariant lines: a contradiction. Therefore, the Killing algebra is three-dimensional.

(iii) A Lie algebra of dimension three is semi-simple or solvable [Kir74]. Since semi-simple Lie algebras are unimodular, Lemma 5 implies that \mathcal{G} is solvable. \square

Lemma 7. *S is a connected real analytic submanifold of codimension one.*

Proof. If needed, one can shrink the open set U in order to get S connected. Let (K_1, K_2, K_3) a basis of the Killing algebra \mathcal{G} . Then S is defined by the equation $v = \text{vol}(K_1, K_2, K_3) = 0$, which is an analytic subset in U . By point (i) in Lemma 6, the isotropy algebra at points in S has dimension one or three. We prove that this dimension must be equal to one.

Assume, by contradiction, that there exists $s \in S$ such that the isotropy at s has dimension three. Then, the isotropy algebra at s is isomorphic to the Lie algebra of (the full) linear group $PSL(2, \mathbb{R})$. On the other hand,

since both are 3-dimensional, the isotropy algebra at s is isomorphic to \mathcal{G} . Hence, \mathcal{G} is isomorphic to the Lie algebra of $PSL(2, \mathbb{R})$ which is semi-simple. This is in contradiction with Lemma 6 (point (iii)).

It follows that the isotropy algebra at each point $s \in S$ is of dimension one. Equivalently, the evaluation morphism $ev(s) : \mathcal{G} \rightarrow T_s U$ has rank two. Since the \mathcal{G} -action preserves S , this implies that S is a smooth submanifold of codimension one in U and $T_s S$ coincides with the image of $ev(s)$. Moreover, \mathcal{G} acts transitively on S (which locally coincides with the \mathcal{G} -orbit of s in U).

□

Let us recall Singer's result [Sin60, DG91, Gro88] which asserts that g is locally homogeneous if and only if the image of $g^{(k)}$ is exactly one $PSL(2, \mathbb{R})$ -orbit in $V^{(k)}$, for a certain k (big enough).

As a consequence of Singer's theorem we get:

Proposition 8. *If g is quasihomogeneous, then the Killing algebra \mathcal{G} does not preserve any vector field of constant norm ≤ 0 .*

Proof. Let $k \in \mathbb{N}$ be given by Singer's theorem.

First suppose, by contradiction, that there exists a isotropic vector field X in U , preserved by \mathcal{G} . Then the \mathcal{G} -action on $R(U)$ (lifted from the action on U) preserves the subbundle $R'(U)$, which is a reduction of the structural group $PSL(2, \mathbb{R})$ to the stabilizer $H = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$ (with $T \in \mathbb{R}$) of an isotropic vector in the standard linear representation of $PSL(2, \mathbb{R}) \simeq O(2, 1)$ on \mathbb{R}^3 . One consider now only exponential coordinates with respect to frames preserving X and get a H -equivariant k -jet map $g^{(k)} : R'(U) \rightarrow V^{(k)}$.

On each open set W on which g is locally homogeneous, the image $g^{(k)}(R'(W))$ is exactly one H -orbit \mathcal{O} in $V^{(k)}$. Let $s \in S$ being a point in the closure of W . Then the image through $g^{(k)}$ of the fiber of $R'(W)_s$ above s lies in the closure of \mathcal{O} . But, here H is unipotent and a classical result due to Konstant and Rosenlicht [Ros61] asserts that *for algebraic representations of unipotent groups, the orbits are closed*. This implies that the image $g^{(k)}(R'(W)_s)$ is also \mathcal{O} . Since \mathcal{G} acts transitively on S , this holds for all $s \in S$.

But any open set of local homogeneity in U admits points of S in its closure. It follows that the image of $R'(U)$ through $g^{(k)}$ is exactly the orbit \mathcal{O} and Singer's theorem implies that g is locally homogeneous (and not quasihomogeneous): a contradiction.

If, also by contradiction, there exists a \mathcal{G} -invariant vector field X in U , of constant strictly negative g -norm, then the \mathcal{G} -action on $R(U)$ also preserves a subbundle $R'(U)$, with structural group H' . Here H' is the stabilizer of a strictly negative vector in the standard linear representation of $PSL(2, \mathbb{R})$ on \mathbb{R}^3 . Moreover, H' is a compact one parameter (elliptic) group in $PSL(2, \mathbb{R})$. The previous argument works, replacing Konstant-Rosenlicht theorem, by the obvious fact that orbits of compact group (smooth) actions are closed.

□

Lemma 9. *At each point s in S , the isotropy is an one parameter semi-simple subgroup in $PSL(2, \mathbb{R})$.*

Proof. Pick up a point $s \in S$ and consider a vector $X(s) \in T_s U$ which is fixed by the isotropy at s . Three distinct possibilities might appear: either $X(s)$ is isotropic (the isotropy corresponds to an one parameter unipotent subgroup in $PSL(2, \mathbb{R})$), or $X(s)$ is of strictly negative norm (the isotropy corresponds to an one parameter elliptic subgroup in $PSL(2, \mathbb{R})$), or $X(s)$ is of strictly positive norm (the isotropy corresponds to an one parameter semi-simple subgroup in $PSL(2, \mathbb{R})$).

Consider a real analytic arc $c(t)$ transverse to S in s and extend $X(s)$ to a real analytic vector field \bar{X} of constant g -norm defined along the curve $c(t)$ (this extension is not unique). Then extend \bar{X} in a unique way, by \mathcal{G} -invariance, to a real analytic vector field X , defined in a neighborhood of s in U . The vector field X

is well defined even on S (on which the \mathcal{G} -action is not simply transitive), because $X(s)$ is invariant by the isotropy.

By Proposition 8, the \mathcal{G} -invariant vector field X should be of constant strictly positive norm. Hence, the isotropy at s is an one parameter semi-simple subgroup in $PSL(2, \mathbb{R})$. \square

3. QUASIHOMOGENEOUS LORENTZ METRICS WITH SEMI-SIMPLE ISOTROPY

In this section we settle the remaining case, where the isotropy at S is semi-simple. Lemma 9 constructed a \mathcal{G} -invariant vector field X , in U , of constant strictly positive g -norm. We normalize X and suppose that X is of constant g -norm equal to 1. This vector field is not unique (only its restriction to S is) and the following Lemma 10 shows that among all \mathcal{G} -invariant vector fields of norm 1, there exists exactly one (denoted by X') which is Killing.

In the sequel, we will work only with the restriction of X to S , which we will still denote by X .

Recall that the affine group of the real line Aff is the group of transformations of \mathbb{R} , given by $x \rightarrow ax + b$, with $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$. If Y is the infinitesimal generator of the one parameter group of homotheties and H the infinitesimal generator of the one parameter group of translations, then $[Y, H] = H$.

Lemma 10. (i) *The Killing algebra \mathcal{G} is isomorphic to that of $\mathbb{R} \times Aff$. The isotropy corresponds to the one parameter group of homotheties in Aff .*

(ii) *The vector field X is the restriction to S of a central element X' in \mathcal{G} .*

(iii) *The restriction of the Killing algebra to S has, in adapted analytic coordinates (x, h) , the following basis $(-h \frac{\partial}{\partial h}, \frac{\partial}{\partial h}, \frac{\partial}{\partial x})$.*

(iv) *In the previous coordinates, the restriction of g to S is dx^2 .*

Proof. (i) We show first that the derivative Lie algebra $[\mathcal{G}, \mathcal{G}]$ is 1-dimensional.

It is a general fact that the derivative algebra of a solvable Lie algebra is nilpotent [Kir74]. Remark first that $[\mathcal{G}, \mathcal{G}] \neq 0$. Indeed, if not \mathcal{G} is abelian and the action of the isotropy $\mathcal{I} \subset \mathcal{G}$ at a point $s \in S$ would be trivial on \mathcal{G} and hence on $T_s S$, which is identified to \mathcal{G}/\mathcal{I} . The isotropy action on the tangent space $T_s S$ being trivial, this implies that the isotropy action is trivial on $T_s U$ (an element of $O(2, 1)$ which acts trivially on a plane in \mathbb{R}^3 is trivial). This implies that the isotropy is trivial at $s \in S$: a contradiction.

As \mathcal{G} is 3-dimensional, its derivative algebra $[\mathcal{G}, \mathcal{G}]$ is a nilpotent Lie algebra of dimension 1 or 2, hence $[\mathcal{G}, \mathcal{G}] \simeq \mathbb{R}$, or $[\mathcal{G}, \mathcal{G}] \simeq \mathbb{R}^2$.

Assume, by contradiction, that $[\mathcal{G}, \mathcal{G}] \simeq \mathbb{R}^2$.

We first prove that the isotropy \mathcal{I} lies in $[\mathcal{G}, \mathcal{G}]$. Assume, by contradiction, that this is not the case. Then, $[\mathcal{G}, \mathcal{G}] \simeq \mathbb{R}^2$ will act freely and so transitively on S , preserving the vector field X . In particular, X is the restriction to S of a Killing vector field $X' \in [\mathcal{G}, \mathcal{G}]$.

Let Y be a generator of the isotropy at $s \in S$. Since X is fixed by the isotropy, one gets, in restriction to S , the following Lie bracket relation: $[Y, X'] = [Y, X] = aY$, for some $a \in \mathbb{R}$. On the other hand, Y is supposed not to belong to $[\mathcal{G}, \mathcal{G}]$, meaning that $a = 0$.

This implies that X' is a central element in \mathcal{G} . In particular, $[\mathcal{G}, \mathcal{G}]$ is one-dimensional: a contradiction.

Hence $\mathcal{I} \subset [\mathcal{G}, \mathcal{G}]$.

Let Y be a generator of \mathcal{I} , $\{Y, X'\}$ be generators of $[\mathcal{G}, \mathcal{G}]$ and $\{Y, X', Z\}$ be a basis of \mathcal{G} . The tangent space of S , at some point $s \in S$, is identified with \mathcal{G}/\mathcal{I} and the infinitesimal (isotropic) action of Y on this

tangent space is given in the basis $\{X', Z\}$ by the matrix $ad(Y) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. This is because $[\mathcal{G}, \mathcal{G}] \simeq \mathbb{R}^2$ and $ad(Y)(\mathcal{G}) \subset [\mathcal{G}, \mathcal{G}]$. Moreover, $ad(Y) \neq 0$, since the restriction to the isotropy action to the tangent space of S is injective.

From this form of $ad(Y)$, we see that the isotropy is unipotent with fixed direction $\mathbb{R}X'$: a contradiction.

We proved that $[\mathcal{G}, \mathcal{G}]$ is 1-dimensional. Notice that $\mathcal{I} \neq [\mathcal{G}, \mathcal{G}]$. Indeed, if one assume the contrary, then the action of the isotropy on the tangent space $T_s U$ at $s \in S$ is trivial: a contradiction.

Let H be a generator of $[\mathcal{G}, \mathcal{G}]$. If Y is (still) the generator of \mathcal{I} , it follows that $[Y, H] = aH$, with $a \in \mathbb{R}$.

Assume, by contradiction, that $a = 0$. Then the image of $ad(Y)$ (which lies in $[\mathcal{G}, \mathcal{G}]$) belongs to the kernel of $ad(Y)$: a contradiction (since the isotropy is semi-simple).

Therefore $a \neq 0$ and we can assume, by changing the generator Y of the isotropy, that $a = 1$. We have $[Y, H] = H$.

Let $X' \in \mathcal{G}$ such that $\{X', H\}$ span the kernel of $ad(H)$. Then $\{Y, X', H\}$ is a basis of \mathcal{G} . We also have $[X', Y] = bH$, with $b \in \mathbb{R}$. After replacing X' by $X' + bH$, we can assume $[X', Y] = 0$.

It follows that \mathcal{G} is the Lie algebra $\mathbb{R} \oplus \mathfrak{aff}(\mathbf{R})$, where $\mathfrak{aff}(\mathbf{R})$ is the Lie algebra of the affine group of the real line. The Killing field X' span the center, the isotropy Y span the one parameter group of the homotheties and H span the one parameter group of translations in the affine group.

(ii) This comes from the fact that X is the unique vector field tangent to S invariant by the isotropy.

(iii) The commuting Killing vector fields X' and H are nonsingular on S . This implies that, in adapted coordinates (x, h) on S , we get $H = \frac{\partial}{\partial h}$ and $X = \frac{\partial}{\partial x}$. The isotropy preserves X . It follows that, in restriction to S , the expression of Y is $f(h)\frac{\partial}{\partial h}$, with f an analytic function vanishing at the origin. The Lie bracket relation $[Y, H] = H$ reads

$$[f(h)\frac{\partial}{\partial h}, \frac{\partial}{\partial h}] = \frac{\partial}{\partial h},$$

and leads to $f(h) = -h$. Therefore, the isotropy Y is linear: $-h\frac{\partial}{\partial h}$.

(iv) Since $H = \frac{\partial}{\partial h}$ and $X = \frac{\partial}{\partial x}$ are Killing fields, the restriction of g to S admits constant coefficients with respect to the coordinates (x, h) . Since H is expanded by the isotropy, it follows that H is of constant g -norm equal to 0. On the other hand, X is of constant g -norm equal to one. It follows that the expression of g on S is dx^2 . □

Lemma 11. *In adapted analytic coordinates (x, h, z) , in the neighborhood of the origin,*

$$g = dx^2 + dh dz + Cz^2 dh^2 + Dz dx dh,$$

with C and D real numbers.

Moreover, in these coordinates, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial h}$ and $-h\frac{\partial}{\partial h} + z\frac{\partial}{\partial z}$ are Killing fields.

Proof. Let us consider the commuting Killing vector fields X' and H , constructed in Lemma 10. Their restrictions to S have the expressions $H = \frac{\partial}{\partial h}$ and $X = \frac{\partial}{\partial x}$. Recall that H is of constant g -norm equal to 0 and X is of constant g -norm equal to one. Point (iv) in Lemma 10 also shows that $g(X, H) = 0$ on S . Moreover, being central, X' is of constant g -norm equal to one in all of U .

We define the geodesic vector field Z as follows. At each point in S , there exists a unique vector Z (transverse to S) such that $g(Z, Z) = 0, g(X, Z) = 0, g(H, Z) = 1$. In fact, Z spans the second isotropic line

(other than that generated by H) in X^\perp . On this line Z is uniquely determined by the relation $g(H, Z) = 1$. Now Z uniquely extends in the neighborhood of the origin to a geodesic vector field.

The image of S through the geodesic flow of Z defines a foliation by surfaces. Each leaf is given by $\exp_S(zZ)$, for some z , small enough. The leaf S corresponds to $z = 0$.

Since X' and H are Killing, then Z commutes with both X' and H . Let (x, h, z) be analytic coordinates in the neighborhood of the origin such that $X' = \frac{\partial}{\partial x}, H = \frac{\partial}{\partial h}, Z = \frac{\partial}{\partial z}$.

As the scalar product between the geodesic vector field Z and the Killing vector field X' is constant along the orbits of Z , the invariance by the commutative Killing algebra generated by X' and H , implies that $dx dz = 0$ and $dhdz = 1$. Also the coefficients of dh^2 and $dx dh$ depend only on z (since H and X' are Killing fields).

We get that $g = dx^2 + dhdz + g(z)dh^2 + f(z)dx dh$, with f, g analytic functions which are both constant equal to 0 on S ($f(0)=g(0)=0$).

We need also to write down the invariance of g by the isotropy $\mathbb{R}Y$. Remember that the Lie bracket relations in \mathcal{G} are $[Y, X'] = 0$ and $[Y, H] = H$. Since the isotropy preserves X , it must preserve also the two isotropic directions of X^\perp . Moreover, since $g(H, Z) = 1$, the isotropy must expand H and contract Z at the same rate. This implies the Lie bracket relation $ad(Y) \cdot Z = [Y, Z] = -Z$.

Now, since Y and X' commute, the general expression for Y is $u(h, z)\frac{\partial}{\partial h} + v(h, z)\frac{\partial}{\partial z} + t(h, z)\frac{\partial}{\partial x}$, with u, v and t analytic functions vanishing at the origin.

The other Lie bracket relations read $[u(h, z)\frac{\partial}{\partial h} + v(h, z)\frac{\partial}{\partial z} + t(h, z)\frac{\partial}{\partial x}, \frac{\partial}{\partial h}] = \frac{\partial}{\partial h}$ and $[u(h, z)\frac{\partial}{\partial h} + v(h, z)\frac{\partial}{\partial z} + t(h, z)\frac{\partial}{\partial x}, \frac{\partial}{\partial z}] = -\frac{\partial}{\partial z}$.

The first Lie bracket relation leads to the following equations: $\frac{\partial u}{\partial h} = -1, \frac{\partial v}{\partial h} = 0, \frac{\partial t}{\partial h} = 0$.

The second one leads to $\frac{\partial u}{\partial z} = 0, \frac{\partial v}{\partial z} = 1, \frac{\partial t}{\partial z} = 0$.

We get $u(h, z) = -h, v(h, z) = z, t(h, z) = 0$.

Hence, in our coordinates, $Y = -h\frac{\partial}{\partial h} + z\frac{\partial}{\partial z}$. The invariance of g under the action of this linear vector field implies $g(e^{-t}z)e^{2t} = g(z)$ and $f(e^{-t}z)e^t = f(z)$, for all $t \in \mathbb{R}$. This implies then that $f(z) = Cz$ and $g(z) = Dz^2$, with C, D real constants. \square

3.1. Computation of the Killing algebra. We need to understand now if all the metrics

$$g_{C,D} = dx^2 + dhdz + Cz^2dh^2 + Dzdx dh$$

constructed in Lemma 11 are quasihomogeneous. In other words, do the metrics in this family admit other Killing fields than $\frac{\partial}{\partial x}, \frac{\partial}{\partial h}$ and $h\frac{\partial}{\partial h} - z\frac{\partial}{\partial z}$?

In this section we compute the full Killing algebra of $g_{C,D}$ and prove Proposition 3. In particular, we obtain that the metrics $g_{C,D} = dx^2 + dhdz + Cz^2dh^2 + Dzdx dh$ admit extra Killing fields and are locally homogeneous, unless $C \neq 0$ and $D = 0$.

Using the formula [KN96]

$$(L_T g_{C,D})(\partial/\partial x_i, \partial/\partial x_j) = T \cdot g_{C,D}(\partial/\partial x_i, \partial/\partial x_j) + g_{C,D}([\partial/\partial x_i, T], \partial/\partial x_j) + g_{C,D}(\partial/\partial x_i, [\partial/\partial x_j, T])$$

one gets the following PDE system for the coefficients of a Killing field $T = \alpha\partial/\partial x + \beta\partial/\partial h + \gamma\partial/\partial z$.

$$\begin{aligned}
 (1) \quad & 0 = \beta_z, \\
 (2) \quad & 0 = \alpha_x + Dz\beta_x, \\
 (3) \quad & 0 = \beta_x + Dz\beta_z + \alpha_z, \\
 (4) \quad & 0 = \gamma D + Dz\alpha_x + Cz^2\beta_x + \gamma_x + \alpha_h + Dz\beta_h, \\
 (5) \quad & 0 = \beta_h + Cz^2\beta_z + Dz\alpha_z + \gamma_z, \\
 (6) \quad & 0 = zC\gamma + Cz^2\beta_h + Dz\alpha_h + \gamma_h.
 \end{aligned}$$

These correspond, in the equation satisfied by T , to the couples

$$(\partial/\partial x_i, \partial/\partial x_j) \in \{(\partial/\partial z, \partial/\partial z), (\partial/\partial x, \partial/\partial x), (\partial/\partial x, \partial/\partial z), (\partial/\partial x, \partial/\partial h), (\partial/\partial h, \partial/\partial z), (\partial/\partial h, \partial/\partial h)\}.$$

We prove now the final part of Proposition 3:

Proposition 12. (i) If $C = 0$ and $D \neq 0$, the Killing algebra of $g_{C,D}$ is solvable of dimension 4 and g is locally homogeneous.

(ii) If $C \neq 0$ and $D \neq 0$, then $g_{C,D}$ is a left invariant metric on $SL(2, \mathbb{R})$ and its Killing algebra is $\mathbb{R} \oplus sl(2, \mathbb{R})$.

(iii) The Killing algebra of $g_{C,D}$ is of dimension three (and $g_{C,D}$ is quasihomogeneous) if and only if $C \neq 0$ and $D = 0$.

(iv) If $C = 0$ and $D = 0$, $g_{C,D}$ is flat and its Killing Lie algebra is of dimension six.

Proof. (i) If $C = 0$, one directly checks that $\alpha = \beta = 0$ and $\gamma = e^{-Dx}$ is a solution of the PDE system, meaning that $e^{-Dx}\partial/\partial z$ is an extra Killing field. In this case the Killing algebra is of dimension 4, generated by $\langle \partial/\partial x, \partial/\partial h, h\partial/\partial h - z\partial/\partial z, e^{-Dx}\partial/\partial z \rangle$. Indeed, Lemma 6 shows that the Killing Lie algebra cannot be bigger, since $g_{C,D}$ is not of constant sectional curvature, except for $C = D = 0$ (see computations at point (iii)). Observe that the Killing algebra is solvable and transitive. The Lorentz metrics $g_{0,D}$ are locally homogeneous.

(ii) Assume that there exists an extra Killing field T . Then by Lemma 6, $g_{C,D}$ is locally homogeneous and the full Killing Lie algebra \mathcal{G} is of dimension four (except for $C = D = 0$, for which $g_{C,D}$ is flat and the Killing algebra is of dimension 6), generated by (X', Y, H, T) .

Since the isotropy $\mathbb{R}Y$ fixes X and expands the direction $\mathbb{R}H$ (because of the relation $[Y, H] = H$), we can choose as fourth generator T of \mathcal{G} , at the origin, a generator of the second isotropic direction of the Lorentz plane X^\perp . Then we will have $[Y, T] = -T + aY$, for some constant $a \in \mathbb{R}$ and we can replace T with $T - aY$ in order to get $[Y, T] = -T$.

In the following, we assume that $[Y, T] = -T$.

We will first show that necessarily we have $[H, T] = aX' - bY$, for some $a, b \in \mathbb{R}$.

For this, we use the Jacobi relation $[Y, [T, H]] = [[Y, T], H] + [T, [Y, H]] = [-T, H] + [T, H] = 0$, to get that $[T, H]$ commutes with Y and, consequently, lies in $\mathbb{R}Y \oplus \mathbb{R}X'$.

Observe also that X' et Y commute, and thus T (which is an eigenvector of $ad(Y)$), is also an eigenvector of $ad(X')$. This gives $[T, X'] = cT$, for some $c \in \mathbb{R}$.

We will construct a Killing field $T = \alpha\partial/\partial x + \beta\partial/\partial h + \gamma\partial/\partial z$, such that $c = 0$ (meaning that X' is central in \mathcal{G}) and $b \neq 0$ (\mathcal{G} semi-simple).

It follows that $[H, T] = aX' - bY$, with $a, b \in \mathbb{R}$ and $b \neq 0$. The Lie algebra \mathcal{L} generated by $aX' - bY, H, T$ is $sl(2, \mathbb{R})$.

We also assume that $a \neq 0$, which implies that \mathcal{L} does not contain the isotropy and hence acts simply transitively.

Since T and X' commute, the coefficients α, β and γ do not depend on x .

The Lie bracket relation $[H, T] = aX' - bY$ reads $[\partial/\partial h, T] = a\partial/\partial x + b(h\partial/\partial h - z\partial/\partial z)$.

This leads to $\alpha_h = a, \beta_h = bh, \gamma_h = -bz$.

It follows that $\alpha = ah + t(z), \beta = \frac{1}{2}bh^2$ and $\gamma = -bzh + s(z)$, with t, s analytic functions of z .

We check equation 5 which gives $bh + Dz't'(z) + s'(z) - bh = 0$, or equivalently $Dz't'(z) + s'(z) = 0$.

Equation 4 leads to $[-bzh + s(z)]D + a + Dz'bh$, which simplifies in $Ds(z) + a = 0$.

Since here $D \neq 0$, we get $s(z) = -\frac{a}{D}$. It follows that $Dz't'(z) = 0$, and, since $D \neq 0, t'(z) = 0$, and t is a constant. We can assume that $t = 0$ (since $X' = \partial/\partial x$ is a Killing field).

Equation 6 leads to $-zC[bzh + \frac{a}{D}] + Cz^2bh + Dza - bz = 0$, which gives

$$a(D - \frac{C}{D}) = b.$$

The extra Killing field is $T = ah\partial/\partial x + \frac{1}{2}bh^2\partial/\partial h + (-bzh - \frac{a}{D})\partial/\partial z$, with a, b such that $a(D - \frac{C}{D}) = b$.

Indeed, since α and β depend only on h , then T also satisfies equations 1, 2 and 3. It is a Killing field of $g_{C,D}$.

Consequently, $g_{C,D}$ is locally isomorphic to a left invariant metric on $SL(2, \mathbb{R})$. The isotropy is diagonally embedded in $\mathbb{R} \oplus sl(2, \mathbb{R})$. This terminates the proof of point (ii).

(iii). We give here a description of the curvature of the Lorentz metrics $g_{C,D}$.

The formulas one needs to use for the computation are classical and can be found in [Wol67]. In the sequel, the local coordinates (x, h, z) are denoted by (x_1, x_2, x_3) .

First one computes Christoffel coefficients Γ_{ij}^m using the formula

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k} \right) g^{km}.$$

Here (g^{ij}) , the inverse of the matrix (g_{ij}) is the following: $\begin{pmatrix} 1 & 0 & -Dz \\ 0 & 0 & 1 \\ -Dz & 1 & (D^2 - C)z^2 \end{pmatrix}$.

For our metrics $g_{C,D}$, only two derivatives are nontrivial: $\frac{\partial g_{22}}{\partial x_3} = 2zC$ and $\frac{\partial g_{12}}{\partial x_3} = D$.

We get $\Gamma_{12}^m = -\frac{1}{2}Dg^{3m}, \Gamma_{13}^m = \frac{1}{2}g^{2m}\frac{\partial g_{21}}{\partial x_3} = \frac{1}{2}Dg^{2m}, \Gamma_{22}^m = -\frac{1}{2}g^{3m}\frac{\partial g_{22}}{\partial x_3} = -Czg^{3m}$ and $\Gamma_{23}^m = \frac{1}{2}g^{1m}\frac{\partial g_{12}}{\partial x_3} + \frac{1}{2}g^{2m}\frac{\partial g_{22}}{\partial x_3} = \frac{1}{2}Dg^{1m} + zCg^{2m}$, for $m \in \{1, 2, 3\}$

A straightforward computation leads to $\Gamma_{12}^1 = D^2z, \Gamma_{12}^2 = -\frac{1}{2}D, \Gamma_{12}^3 = -\frac{1}{2}D(D^2 - C)z^2, \Gamma_{13}^1 = \Gamma_{13}^2 = 0, \Gamma_{13}^3 = \frac{1}{2}D, \Gamma_{23}^1 = \frac{1}{2}D, \Gamma_{23}^2 = 0, \Gamma_{23}^3 = (C - \frac{D^2}{2})z, \Gamma_{22}^1 = CDz^2, \Gamma_{22}^2 = -zC, \Gamma_{22}^3 = C(C - D^2)z^3, \Gamma_{11}^m = 0, \Gamma_{33}^m = 0$.

Starting with Christoffel symbols we compute the curvature components using the standard formula:

$$R_{ijk}^s = \sum_l \Gamma_{ik}^l \Gamma_{jl}^s - \sum_l \Gamma_{jk}^l \Gamma_{il}^s + \frac{\partial \Gamma_{ik}^s}{\partial x_j} - \frac{\partial \Gamma_{jk}^s}{\partial x_i}.$$

The straightforward computation of the components of the curvature of the metrics $g_{C,D}$ leads to the following:

$$R_{121}^2 = -\frac{1}{4}D^2, R_{121}^3 = 0, R_{131}^3 = -\frac{1}{4}D^2, R_{131}^2 = 0, R_{132}^3 = (\frac{3}{4}D^3 - 2CD)z, R_{232}^3 = (-\frac{5}{4}CD^2 + C^2)z^2, R_{122}^3 = 0.$$

This shows that for $C \neq 0$ and $D = 0$, the curvature tensor of $g_{C,D}$ vanishes exactly on the surface $z = 0$. This implies that the Lorentz metrics $g_{C,0}$, with $C \neq 0$ are not locally homogeneous. They are quasihomogeneous.

Notice also that all Christoffel symbols Γ_{ij}^3 , with $i, j \in \{1, 2\}$, vanish on $z = 0$. Consequently, S is totally geodesic.

(iv) The previous computations show that the Lorentz metric $g_{C,D}$ is flat if and only if $C = D = 0$. \square

REFERENCES

- [Amo79] A. M. Amores. Vector fields of a finite type G -structure. *J. Differential Geom.*, 14(1):1–6 (1980), 1979.
- [Ben97] Yves Benoist. Orbites des structures rigides (d’après M. Gromov). In *Integrable systems and foliations/Feuilletages et systèmes intégrables (Montpellier, 1995)*, volume 145 of *Progr. Math.*, pages 1–17. Birkhäuser Boston, Boston, MA, 1997.
- [BF05] E. Jerome Benveniste and David Fisher. Nonexistence of invariant rigid structures and invariant almost rigid structures. *Comm. Anal. Geom.*, 13(1):89–111, 2005.
- [BFL92] Yves Benoist, Patrick Foulon, and François Labourie. Flots d’Anosov à distributions stable et instable différentiables. *J. Amer. Math. Soc.*, 5(1):33–74, 1992.
- [DG91] G. D’Ambra and M. Gromov. Lectures on transformation groups: geometry and dynamics. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 19–111. Lehigh Univ., Bethlehem, PA, 1991.
- [DG13] Sorin Dumitrescu and Adolfo Guillot. Quasihomogeneous real analytic connections on surfaces. *J. Topol. Anal.*, 5(4):491–532, 2013.
- [Dum08] Sorin Dumitrescu. Dynamique du pseudo-groupe des isométries locales sur une variété lorentzienne analytique de dimension 3. *Ergodic Theory Dynam. Systems*, 28(4):1091–1116, 2008.
- [DZ10] Sorin Dumitrescu and Abdelghani Zeghib. Géométries lorentziennes de dimension trois : classification et complétude. *Geom. Dedicata*, 149:243–273, 2010.
- [Fer02] Renato Feres. Rigid geometric structures and actions of semisimple Lie groups. In *Rigidité, groupe fondamental et dynamique*, volume 13 of *Panor. Synthèses*, pages 121–167. Soc. Math. France, Paris, 2002.
- [Gro88] Michael Gromov. Rigid transformations groups. In *Géométrie différentielle (Paris, 1986)*, volume 33 of *Travaux en Cours*, pages 65–139. Hermann, Paris, 1988.
- [Kir74] A. Kirilov. *Eléments de la théorie des représentations*. M.I.R. Moscou, 1974.
- [KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996. Reprint of the 1963 original.
- [Mel09] Karin Melnick. Compact Lorentz manifolds with local symmetry. *J. Differential Geom.*, 81(2):355–390, 2009.
- [Nom60] Katsumi Nomizu. On local and global existence of Killing vector fields. *Ann. of Math. (2)*, 72:105–120, 1960.
- [PTV96] Friedbert Prüfer, Franco Tricerri, and Lieven Vanhecke. Curvature invariants, differential operators and local homogeneity. *Trans. Amer. Math. Soc.*, 348(11):4643–4652, 1996.
- [Ros61] M. Rosenlicht. On quotient varieties and the affine embedding of certain homogeneous spaces. *Trans. Amer. Math. Soc.*, 101:211–223, 1961.
- [Sin60] I. Singer. Infinitesimally homogeneous spaces. *Comm. Pure Appl. Math.*, 13:685–697, 1960.
- [Thu97] William P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.
- [Wol67] Joseph A. Wolf. *Spaces of constant curvature*. McGraw-Hill Book Co., New York, 1967.
- [Zeg96] Abdelghani Zeghib. Killing fields in compact Lorentz 3-manifolds. *J. Differential Geom.*, 43(4):859–894, 1996.

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