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THE DENSITY OF SETS AVOIDING DISTANCE 1 IN EUCLIDEAN SPACE

CHRISTINE BACHOC, ALBERTO PASSUELLO, AND ALAIN THIERY

ABSTRACT. We improve by an exponential factor the best known asymptotic upper bound for the density of sets avoiding 1 in Euclidean space. This result is obtained by a combination of an analytic bound that is an analogue of Lovász theta number and of a combinatorial argument involving finite subgraphs of the unit distance graph. In turn, we straightforwardly obtain an asymptotic improvement for the measurable chromatic number of Euclidean space. We also tighten previous results for the dimensions between 4 and 24.

1. INTRODUCTION

In the Euclidean space \mathbb{R}^n , a subset S is said to *avoid* 1 if $\|x - y\| \neq 1$ for all x, y in S . For example, one can take the union of open balls of radius $1/2$ with centers in $(2\mathbb{Z})^n$. It is natural to wonder how large S can be, given that it avoids 1. To be more precise, if S is a Lebesgue measurable set, its *density* $\delta(S)$ is defined by

$$\delta(S) = \limsup_{R \rightarrow \infty} \frac{\text{vol}([-R, R]^n \cap S)}{\text{vol}([-R, R]^n)},$$

where $\text{vol}(S)$ is the Lebesgue measure of S . We are interested in the supreme density $m_1(\mathbb{R}^n)$ of the Lebesgue measurable sets avoiding 1.

In terms of graphs, a set S avoiding 1 is an independent set of the *unit distance graph*, the graph drawn on \mathbb{R}^n that connects by an edge every pair of points at distance 1, and $m_1(\mathbb{R}^n)$ is a substitute for the *independence number* of this graph.

Larman and Rogers introduced in [14] the number $m_1(\mathbb{R}^n)$ in order to allow for analytic tools in the study of the *chromatic number* $\chi(\mathbb{R}^n)$ of the unit distance graph, i.e. the minimal number of colors needed to color \mathbb{R}^n so that points at distance 1 receive different colors. Indeed, the inequality

$$(1) \quad \chi_m(\mathbb{R}^n) \geq \frac{1}{m_1(\mathbb{R}^n)}.$$

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holds, where $\chi_m(\mathbb{R}^n)$ denotes the *measurable chromatic number* of \mathbb{R}^n . In the definition of $\chi_m(\mathbb{R}^n)$, the measurability of the color classes is required, so $\chi_m(\mathbb{R}^n) \geq \chi(\mathbb{R}^n)$. We note that (1) is the exact analogue of the well known relation between the chromatic number $\chi(G)$ and the independence number $\alpha(G)$ of a finite graph $G = (V, E)$:

$$\chi(G) \geq \frac{|V|}{\alpha(G)}.$$

Following (1), in order to lower bound $\chi_m(\mathbb{R}^n)$, it is enough to upper bound $m_1(\mathbb{R}^n)$. As shown in [14], finite configurations of points in \mathbb{R}^n can be used for this purpose. Indeed, if $G = (V, E)$ is a finite induced subgraph of the unit distance graph of \mathbb{R}^n , meaning that $V = \{v_1, \dots, v_M\} \subset \mathbb{R}^n$ and $E = \{\{i, j\} : \|v_i - v_j\| = 1\}$, then

$$(2) \quad m_1(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|}.$$

Combined with the celebrated Frankl and Wilson intersection theorem [22], this inequality has led to the asymptotic upper bound of 1.207^{-n} , proving the exponential decrease of $m_1(\mathbb{R}^n)$. This result was later improved to 1.239^{-n} in [23] following similar ideas. However, (2) can by no means result in a lower estimate for $\chi_m(\mathbb{R}^n)$ that would be tighter than that of $\chi(\mathbb{R}^n)$ since the inequalities $\chi(\mathbb{R}^n) \geq \chi(G) \geq \frac{|V|}{\alpha(G)}$ obviously hold. In [31], a more sophisticated configuration principle was introduced that improved the upper bounds of $m_1(\mathbb{R}^n)$ for dimensions $2 \leq n \leq 25$, but didn't move forward to an asymptotic improvement.

A completely different approach is taken in [20], where an analogue of Lovász theta number is defined and computed for the unit distance graph (see also [6] for an earlier approach dealing with the unit sphere of Euclidean space). This number, denoted here $\vartheta(\mathbb{R}^n)$, has an explicit expression in terms of Bessel functions, which will be recalled in section 2. However, the resulting upper bound of $m_1(\mathbb{R}^n)$ is asymptotically not as good as Frankl and Wilson. We introduce the following notations, which will be used throughout this paper:

Notations: Let u_n and $v_n \neq 0$ be two sequences. We denote $u_n \sim v_n$ if $\lim u_n/v_n = 1$, $u_n \approx v_n$ if there exists $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, such that $u_n/v_n \sim \beta n^\alpha$ and, for positive sequences, $u_n \lesssim v_n$ if there exists $\alpha, \beta \in \mathbb{R}$, $\beta > 0$ such that $u_n/v_n \leq \beta n^\alpha$.

Then, the asymptotic behavior of $\vartheta(\mathbb{R}^n)$ is

$$\vartheta(\mathbb{R}^n) \approx (\sqrt{e/2})^{-n} \lesssim (1.165)^{-n}.$$

Nevertheless, for small dimensions, $\vartheta(\mathbb{R}^n)$ does improve the previously known upper bounds of $m_1(\mathbb{R}^n)$. Moreover, this bound is further strengthened in [20] by adding extra inequalities arising from simplicial configurations of points, leading to the up to now tightest known upper bounds of $m_1(\mathbb{R}^n)$ for $2 \leq n \leq 24$ [20, Table 3.1].

In this paper, we build upon the results in [20], by considering more general configurations of points. More precisely, a linear program is associated to any

finite induced subgraph of the unit distance graph $G = (V, E)$, whose optimal value $\vartheta_G(\mathbb{R}^n)$ satisfies

$$m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n).$$

We prove that $\vartheta_G(\mathbb{R}^n)$ decreases exponentially faster than both $\vartheta(\mathbb{R}^n)$ and the ratio $\alpha(G)/|V|$, when G is taken in the family of graphs considered by Frankl and Wilson, or in the family of graphs defined by Raigorodskii. We obtain the improved estimate

Theorem 1.1.

$$(3) \quad m_1(\mathbb{R}^n) \lesssim (1.268)^{-n}.$$

We also present numerical results in the range of dimensions $4 \leq n \leq 24$ (see Table 2), where careful choices of graphs allow us to tighten the previously known upper estimates of $m_1(\mathbb{R}^n)$.

For the first time, tightening a theta-like upper bound using a subgraph constraint is applied in a systematic way. Both our numerical results in small dimensions and the asymptotic improvement that we have obtained (even if further improvement with this method is intrinsically limited, see Remark 3.5) show the relevance of this method. We believe this is a promising method that is worth to consider in other situations, in particular because it is much simpler than others such as further steps in hierarchies of semidefinite programs (see [15], [16], and for packing graphs in topological spaces, [13]). In section 5, we show how it can be applied in the framework of compact homogeneous graphs.

This paper is organized as follows: in section 2, we introduce $\vartheta_G(\mathbb{R}^n)$ and prove the announced inequality $m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n)$. In section 3, we prove Theorem 1.1. Section 4 is devoted to the numerical results in small dimension. Section 5 develops the method in the context of compact homogeneous graphs. The last section presents some open problems.

2. A LINEAR PROGRAMMING UPPER BOUND FOR $m_1(\mathbb{R}^n)$

Our goal in this section is to generalize to arbitrary induced subgraphs of the unit distance graph the linear programming upper bound of $m_1(\mathbb{R}^n)$ introduced in [20]. In order to formulate the result we need some preparation.

Let ω denote the surface measure of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$ and let $\omega_n = \omega(S^{n-1})$. We will need the Fourier transform of ω , which is by definition the function defined for $u \in \mathbb{R}^n$ by $\widehat{\omega}(u) = \int_{S^{n-1}} e^{-iu \cdot \xi} d\omega(\xi)$. This function is clearly invariant under the orthogonal group $O(\mathbb{R}^n)$, so let $\Omega_n(t)$ be the function of $t \geq 0$ such that, for all $u \in \mathbb{R}^n$,

$$\Omega_n(\|u\|) = \frac{1}{\omega_n} \int_{S^{n-1}} e^{-iu \cdot \xi} d\omega(\xi).$$

We note that $\Omega_n(0) = 1$. The function Ω_n expresses in terms of the Bessel function of the first kind [3, Chap.4 and Lemma 9.6.2], :

$$\Omega_n(t) = \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{t}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(t).$$

In [20, section 3.1], the following theorem is proved:

Theorem 2.1. [20]

$$m_1(\mathbb{R}^n) \leq \inf \left\{ \begin{array}{l} z_0 + z_c : z_c \geq 0, \\ z_0 + z_1 + z_c(n+1) \geq 1 \\ z_0 + z_1 \Omega_n(t) + z_c(n+1) \Omega_n\left(t \sqrt{\frac{1}{2} - \frac{1}{2n+2}}\right) \geq 0 \\ \text{for all } t \geq 0 \end{array} \right\}$$

Let $G = (V, E)$ a (not necessarily finite) induced subgraph of the unit distance graph. So, $V \subset \mathbb{R}^n$ and $E = \{\{x, y\} : x \in V, y \in V \text{ and } \|x - y\| = 1\}$. We assume that V is a Borel measurable set, endowed with a positive Borel measure λ , such that $0 < \lambda(V) < +\infty$. We introduce the λ -independence number of G :

$$\alpha_\lambda(G) := \sup\{\lambda(A) : A \subset V, A \text{ a Borel measurable independent set of } G\}.$$

If G is a finite graph, and λ is the counting measure, we recover the usual notion of the independence number $\alpha(G)$ of G .

Theorem 2.2. *With the notations above,*

$$m_1(\mathbb{R}^n) \leq \inf \left\{ \begin{array}{l} z_0 + z_2 \frac{\alpha_\lambda(G)}{\lambda(V)} : z_2 \geq 0, \\ z_0 + z_1 + z_2 \geq 1 \\ z_0 + z_1 \Omega_n(t) + z_2 \frac{1}{\lambda(V)} \int_V \Omega_n(t\|v\|) d\lambda(v) \geq 0 \\ \text{for all } t \geq 0 \end{array} \right\}$$

The optimal value of this linear program will be denoted $\vartheta_G(\mathbb{R}^n)$.

Before we proceed with the proof, we would like to make a few comments on the choice of subgraph G .

In the next sections, we will apply Theorem 2.2 in the special case of a subgraph $G = (V, E)$ such that V is finite and lies on the sphere of radius r centered at 0^n , and the measure λ is the counting measure. Then, the linear program takes the simpler form

$$(4) \quad \inf \left\{ \begin{array}{l} z_0 + z_2 \frac{\alpha(G)}{|V|} : z_2 \geq 0, \\ z_0 + z_1 + z_2 \geq 1 \\ z_0 + z_1 \Omega_n(t) + z_2 \Omega_n(rt) \geq 0 \text{ for all } t \geq 0 \end{array} \right\}.$$

In particular, if V is a regular simplex with edges of length 1, centered at 0^n , the graph G is the complete graph of order $(n+1)$; the change of variable $z_2 = z_c(n+1)$ in (4), combined with Theorem 2.2, gives back Theorem 2.1.

Another natural choice is to take for V a sphere centered at 0^n , endowed with its surface measure. Of course, the exact value of the ratio $\frac{\alpha_\lambda(G)}{\lambda(V)}$ is not known in general, but one can upper bound it with a similar linear program. This will be explained in section 5, where the more general case of compact homogeneous graphs is discussed.

It is also possible to take account of several graphs at the same time; each graph would give rise to an additional variable z_i .

Theorem 2.2 can be obtained from a minor modification of the argument in [20]. In order to keep this paper self contained, we include a proof here.

Proof. One can come arbitrary close to $m_1(\mathbb{R}^n)$, with a Borel measurable subset of \mathbb{R}^n avoiding distance 1, which is moreover a periodic set. We refer to [20, proof of Theorem 1.1] for a proof. So, let S be a periodic Borel measurable subset of \mathbb{R}^n avoiding distance 1, having positive density, and let L denote its periodicity lattice. We consider the function:

$$f_S(x) := \frac{1}{\text{vol}(L)} \int_{\mathbb{R}^n/L} 1_S(x+y)1_S(y)dy$$

where $1_S(x)$ denotes the characteristic function of S and integration is with respect to the Lebesgue measure. One can verify that $f_S(0^n) = \delta(S)$, that f_S is L -periodic and that $\frac{1}{\text{vol}(L)} \int_{\mathbb{R}^n/L} f_S(x)dx = \delta(S)^2$. Moreover, f_S is a *positive definite function*, meaning that, for all choice of k points in \mathbb{R}^n , say x_1, \dots, x_k , the matrix with coefficients $f_S(x_i - x_j)$ is positive semidefinite (see [25, 1.4.1]).

δ_{0^n} denotes the Dirac measure at 0^n , and $\tilde{\omega}, \tilde{\lambda}$ stand for the natural extensions of ω and λ to \mathbb{R}^n (i.e. for $E \subset \mathbb{R}^n$ a Borel set, $\tilde{\omega}(E) := \omega(E \cap S^{n-1})$ and $\tilde{\lambda}(E) := \lambda(E \cap V)$). Let $(z_0, z_1, z_2) \in \mathbb{R}^3$ and let the Borel measure

$$\mu := z_0\delta_{0^n} + z_1\tilde{\omega}/\omega_n + z_2\tilde{\lambda}/\lambda(V).$$

We assume that $z_2 \geq 0$, that $\hat{\mu}(u) \geq 0$ for all $u \in \mathbb{R}^n$, and that $\hat{\mu}(0^n) \geq 1$. Then, we claim that the following inequalities hold:

$$(5) \quad \delta(S)^2 \leq \int f_S(x)d\mu(x) \leq \left(z_0 + z_2 \frac{\alpha_\lambda(G)}{\lambda(V)}\right) \delta(S).$$

To show the right hand side inequality, we observe that $\int f_S d\delta_{0^n} = f_S(0^n) = \delta(S)$, and, because S avoids 1, that $\int f_S d\tilde{\omega} = 0$. Less obvious is the inequality

$$\int f_S(x)d\tilde{\lambda}(x) \leq \alpha_\lambda(G)\delta(S).$$

It is easily obtained from a swap of integrals following Fubini's theorem and from the inequality

$$\int 1_S(x+y)d\tilde{\lambda}(x) = \lambda((S-y) \cap V) \leq \alpha_\lambda(G).$$

The left hand side inequality in (5) follows from basic results in Fourier analysis for which we refer to [25, Chapter 1]. Because f_S is continuous, positive definite on \mathbb{R}^n and L -periodic,

$$f_S(x) = \sum_{\gamma \in 2\pi L^*} \widehat{f_S}(\gamma) e^{i(\gamma \cdot x)}$$

where $L^* = \{x \in \mathbb{R}^n : x \cdot y \in \mathbb{Z} \text{ for all } y \in L\}$ is the dual lattice of L , and its Fourier coefficients

$$\widehat{f_S}(\gamma) = \frac{1}{\text{vol}(L)} \int_{\mathbb{R}^n/L} f_S(y) e^{-i(\gamma \cdot y)} dy$$

are non negative numbers (see Bochner's theorem [25, 1.4.3] and the inversion theorem [25, 1.5.1]). Then, applying Fubini, and the assumptions on $\widehat{\mu}$,

$$\int f_S(x)d\mu(x) = \sum_{\gamma \in 2\pi L^*} \widehat{f_S}(\gamma)\widehat{\mu}(-\gamma) \geq \widehat{f_S}(0^n) = \delta(S)^2.$$

So, from (5),

$$(6) \quad \delta(S) \leq z_0 + z_2 \frac{\alpha_\lambda(G)}{\lambda(V)}.$$

Now, we introduce the measure λ_0 , which is the average of λ with respect to the normalized Haar measure dg on the orthogonal group $O(\mathbb{R}^n)$, and we will apply (6) to λ_0 . The measure λ_0 is defined by: $\lambda_0(E) = \int_{O(\mathbb{R}^n)} \tilde{\lambda}(g^{-1}(E))dg$ for any Borel set $E \subset \mathbb{R}^n$ and we note that its support may be different than that of λ , but is anyway contained in the union V_0 of the images of V under elements of $O(\mathbb{R}^n)$. We have $\lambda_0(V_0) = \lambda(V)$ and, with obvious notations, $\alpha_{\lambda_0}(G_0) \leq \alpha_\lambda(G)$. The Fourier transform of λ_0 can be written:

$$\widehat{\lambda_0}(u) = \int \Omega_n(\|u\| \|x\|)d\tilde{\lambda}(x) = \int_V \Omega_n(\|u\| \|v\|)d\lambda(v)$$

so, if $\mu_0 := z_0\delta_{0^n} + z_1\tilde{\omega}/\omega_n + z_2\lambda_0/\lambda_0(V_0)$, the conditions that $\widehat{\mu_0}(u) \geq 0$ for all $u \in \mathbb{R}^n$ and $\widehat{\mu_0}(0^n) \geq 1$ translate to:

$$\begin{cases} z_0 + z_1\Omega_n(\|u\|) + z_2\frac{1}{\lambda(V)} \int_V \Omega_n(\|u\| \|v\|)d\lambda(v) \geq 0 \text{ for all } u \in \mathbb{R}^n \\ z_0 + z_1 + z_2 \geq 1 \end{cases}$$

which amounts, together with $z_2 \geq 0$, to (z_0, z_1, z_2) being feasible for the linear program in the theorem. Under these conditions, (6) holds for λ_0 and concludes the proof. \square

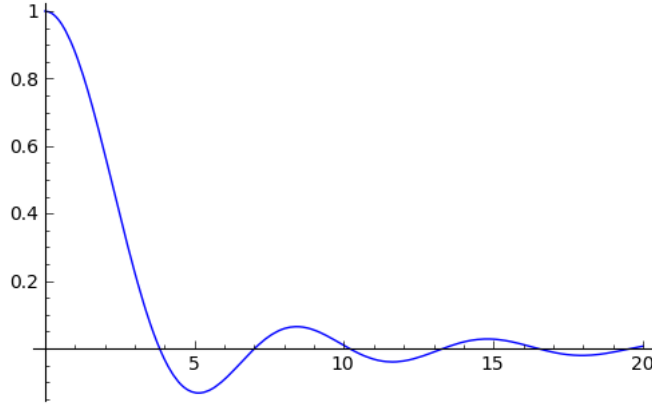
We recall from [20, section 3] and [19] that the linear program obtained from $\vartheta_G(\mathbb{R}^n)$ when the variable z_2 is set to 0, can be solved in full generality and that its optimal value, denoted here $\vartheta(\mathbb{R}^n)$, has the explicit expression:

$$(7) \quad \vartheta(\mathbb{R}^n) = \frac{-\Omega_n(j_{n/2,1})}{1 - \Omega_n(j_{n/2,1})}$$

where $j_{n/2,1}$ is the first positive zero of $J_{n/2}$ and is the value at which the function Ω_n reaches its absolute minimum (see Figure 1 for a plot of $\Omega_4(t)$). In particular, we have the inequalities:

$$m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n).$$

Unfortunately, the program $\vartheta_G(\mathbb{R}^n)$ cannot be solved explicitly in a similar fashion. Instead, we will content ourselves with the construction of explicit feasible solutions in section 3 and with numerical solutions in section 4.

FIGURE 1. $\Omega_4(t)$

3. THE PROOF OF THEOREM 1

In this section we will show that from Theorem 2.2 an asymptotic improvement of the known upper bounds for $m_1(\mathbb{R}^n)$ can be obtained. For this, we assume that $G_n = (V_n, E_n)$ is a sequence of induced subgraphs of the unit distance graph of dimension n , such that $|V_n| = M_n$ and V_n lies on the sphere of radius $r < 1$ centered at 0^n , where r is independent of n . We recall from Theorem 2.2 and (4) that $m_1(\mathbb{R}^n) \leq \vartheta_{G_n}(\mathbb{R}^n)$ where $\vartheta_{G_n}(\mathbb{R}^n)$ is the optimal value of:

$$(8) \quad \inf \left\{ z_0 + z_2 \frac{\alpha(G_n)}{M_n} : \begin{array}{l} z_2 \geq 0 \\ z_0 + z_1 + z_2 \geq 1 \\ z_0 + z_1 \Omega_n(t) + z_2 \Omega_n(rt) \geq 0 \quad (t > 0) \end{array} \right\}.$$

So, in order to upper bound $m_1(\mathbb{R}^n)$, it is enough to construct a feasible solution of (8). One that is suitable for our purpose is given in the following lemma:

Lemma 3.1. *For $0 < r < 1$, let $c(r)$ be defined by:*

$$c(r) := (1 + \sqrt{1 - r^2})e^{-\sqrt{1 - r^2}}$$

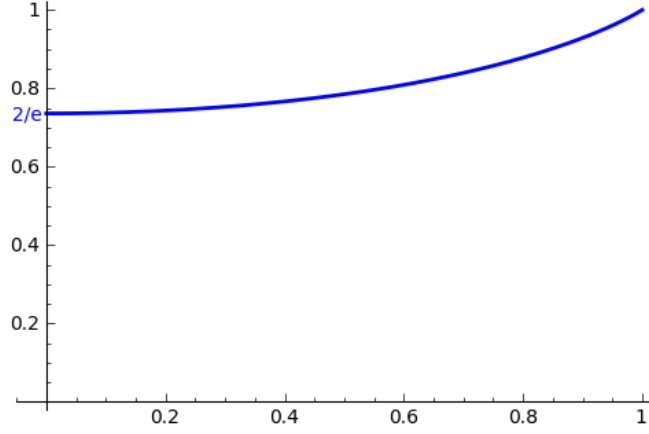
(see the plot of this function in Figure 2).

Let $\gamma > \sqrt{c(r)}$ and $m > \gamma\sqrt{2/e}$; there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$(9) \quad m^n + \Omega_n(t) + \gamma^n \Omega_n(rt) \geq 0, \text{ for all } t \geq 0.$$

Proof. After having established some preliminary inequalities, we will proceed in three steps. First, we will prove that the inequality (9) holds for “small” t , say $0 \leq t \leq \nu := \frac{n}{2} - 1$, then that it holds for “large” t , say $t \geq \alpha_0 \nu$ where α_0 is an explicit constant, and, at last, we will construct a decreasing sequence $\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_k \dots$ such that the inequality holds for $t \geq \alpha_k \nu$, and prove that $\lim_{k \rightarrow \infty} \alpha_k < 1$.

Let $j_{n/2,1}$ be the first zero of $J_{\nu+1}$, then Ω_n is a decreasing function on $[0, j_{n/2,1}]$ and Ω_n has a global minimum at $j_{n/2,1}$ (it follows from [3, (4.6.2), (4.14.1)] and

FIGURE 2. $c(r)$

[33, 15.31], a detailed proof is given in [19, sec. 4.3, (4.17)]). So, $\Omega_n(t) \geq \Omega_n(j_{n/2,1})$. Furthermore, $|J_\nu(t)| \leq 1$ for all $t \in \mathbb{R}$ (see [2, formula 9.1.60]), hence

$$|\Omega_n(j_{n/2,1})| \leq \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{j_{n/2,1}}\right)^{\frac{n}{2}-1}.$$

We apply the inequality $j_{n/2,1} > n/2$ ([33, 15.3 (1)]) and Stirling formula, which with our notation reads:

$$(10) \quad \Gamma(n) \approx \left(\frac{n}{e}\right)^n,$$

and we get

$$(11) \quad |\Omega_n(j_{n/2,1})| \lesssim \left(\sqrt{\frac{2}{e}}\right)^n.$$

Let $x \in]0, 1[$ and let us recall [2, formula 9.3.2] (see also [33, 8.4 (3)]):

$$(12) \quad J_\nu(\nu \operatorname{sech} \alpha) \sim \frac{e^{\nu(\tanh \alpha - \alpha)}}{\sqrt{2\pi\nu \tanh \alpha}} \quad (\alpha > 0, \nu \rightarrow +\infty).$$

Setting $x = \operatorname{sech} \alpha$, (12) with our notation, leads to:

$$(13) \quad J_\nu(x\nu) \approx \left(\sqrt{\frac{x}{c(x)}}\right)^n.$$

We note that (12) shows that, for n sufficiently large (possibly depending on x), $J_\nu(x\nu)$, and thus $\Omega_n(x\nu)$ is positive.

Combining (13) and (10), we get

$$(14) \quad \Omega_n(x\nu) = \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{x\nu}\right)^\nu J_\nu(x\nu) \approx \left(\sqrt{\frac{2}{ec(x)}}\right)^n.$$

First step. Suppose that $0 \leq t \leq \nu$. Since $\nu \leq j_{n/2,1}$ and $r < 1$, and since Ω_n is decreasing on $[0, j_{n/2,1}]$, it follows that $\Omega_n(rt) \geq \Omega_n(r\nu)$, so we have the inequality:

$$\Omega_n(t) + \gamma^n \Omega_n(rt) \geq -|\Omega_n(j_{n/2,1})| + \gamma^n \Omega_n(r\nu).$$

We assume that $\Omega_n(r\nu) \geq 0$, which holds for n sufficiently large as a consequence of (12). From (11), (14), and the assumption $\gamma > \sqrt{c(r)}$, the second term of the right hand side has asymptotically the largest absolute value, so, for n greater than some value m_0 , the sign of the right hand side is that of $\Omega_n(r\nu)$, hence is positive. So, for $n \geq m_0$, and for all $t \in [0, \nu]$, we have

$$\Omega_n(t) + \gamma^n \Omega_n(rt) \geq 0.$$

Second step. Let $\alpha_0 = \frac{1}{\gamma^2}$. For $t \geq \alpha_0\nu$, because $|J_\nu(t)| \leq 1$,

$$|\Omega_n(t)| \leq \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{\alpha_0\nu}\right)^{\frac{n}{2}-1} \approx \left(\gamma\sqrt{\frac{2}{e}}\right)^n.$$

Since $\Omega_n(rt) \geq -|\Omega_n(j_{n/2,1})|$ and $|\Omega_n(j_{n/2,1})| \lesssim \left(\sqrt{\frac{2}{e}}\right)^n$, it follows from the assumption $m > \gamma\sqrt{2/e}$ that

$$m^n + \Omega_n(t) + \gamma^n \Omega_n(rt) \geq m^n - |\Omega_n(t)| - \gamma^n |\Omega_n(j_{n/2,1})| \sim m^n$$

and hence that, for n greater than some m_1 , and for all $t \geq \alpha_0\nu$,

$$m^n + \Omega_n(t) + \gamma^n \Omega_n(rt) \geq 0.$$

Third step. Let us first study the function c . An elementary computation gives $c'(x) = xe^{-\sqrt{1-x^2}}$ for $x \in [0, 1]$. It implies that $0 \leq c'(x) \leq 1$, hence c is an increasing function and $c(x) \geq x$ with equality only for $x = 1$. Now, let us define ϕ by

$$\begin{aligned} \phi : [0, \frac{1}{r}] &\longrightarrow \mathbb{R} \\ x &\longmapsto \frac{1}{2}\left(\frac{c(rx)}{\gamma^2} + x\right) \end{aligned}$$

Since c is increasing, ϕ is also increasing. Furthermore, $\phi(0) = \frac{1}{e\gamma^2} > 0$ and $\phi(\frac{1}{r}) = \frac{1}{2}(\frac{1}{\gamma^2} + \frac{1}{r}) < \frac{1}{r}$ since $\gamma^2 > c(r) > r$. It follows that the interval $[0, \frac{1}{r}]$ is mapped into itself. One also gets immediately $\phi'(x) = \frac{1}{2}(\frac{rc'(rx)}{\gamma^2} + 1)$. Since $c'(rx) \leq 1$ and $\gamma^2 > r$, we have $\phi'(x) < 1$. Hence by Banach fixed point theorem, ϕ has only one fixed point, denoted by l and, for any $x_0 \geq l$, the sequence $\{x_k, k \geq 0\}$ defined by $x_{k+1} = \phi(x_k)$ is a decreasing sequence with limit l . Moreover, $\phi(1) < 1$, so $l < 1$.

We now return to the proof of the lemma. We have set $\alpha_0 = \frac{1}{\gamma^2}$. If $\alpha_0 \leq 1$, taking account of the previous steps, we are done, so, we assume $\alpha_0 > 1$. Let $\alpha_1 < \alpha_0$ and $t \in [\alpha_1\nu, \alpha_0\nu]$. By construction, $r\alpha_0 = \frac{r}{\gamma^2} < \frac{r}{c(r)} < 1$, hence $rt \leq r\alpha_0\nu < \nu \leq j_{n/2,1}$. Since Ω_n is decreasing on $[0, j_{n/2,1}]$, (14) gives

$$\Omega_n(rt) \geq \Omega_n(r\alpha_0\nu) \approx \left(\sqrt{\frac{2}{ec(r\alpha_0)}}\right)^n.$$

Now $|\Omega_n(t)| \leq \Gamma\left(\frac{n}{2}\right) \left(\frac{2}{\alpha_1 \nu}\right)^{\frac{n}{2}-1} \approx \left(\sqrt{\frac{2}{e\alpha_1}}\right)^n$. Hence, with the same reasoning as in step 1, we will have $\Omega_n(t) + \gamma^n \Omega_n(rt) \geq 0$, for n greater than some value m_2 , as soon as $\alpha_1 > \frac{c(r\alpha_0)}{\gamma^2}$ (here we need strict inequality). In order to achieve this constraint, we can take $\alpha_1 = \phi(\alpha_0)$. Defining the sequence $\{\alpha_k, k \geq 0\}$ by the recursive formula $\alpha_{k+1} = \phi(\alpha_k)$, we get, using the same method, that for every $k \geq 1$, there exists m_{k+1} , such that for all $n > m_{k+1}$, $\Omega_n(t) + \gamma^n \Omega_n(rt) \geq 0$ for $t \geq \alpha_k \nu$. Since $\lim \alpha_k = l < 1$, there exists an integer k_0 such that $\alpha_{k_0} < 1$.

Conclusion. With the three steps above, we have covered the whole range $t \geq 0$ by a finite number of intervals, and proved the wanted result on each of them. All together, we have that, for $n \geq n_0 := \max(m_0, m_1, \dots, m_{k_0+1})$, $m^n + \Omega_n(t) + \gamma^n \omega_n(rt) \geq 0$ for all $t \geq 0$. \square

Theorem 3.2. *We assume that, for some $b < \sqrt{2/e}$,*

$$(15) \quad \frac{\alpha(G_n)}{M_n} \lesssim b^n.$$

Let

$$c(r) = (1 + \sqrt{1-r^2})e^{-\sqrt{1-r^2}} \quad \text{and} \quad f(r) = \sqrt{2c(r)/e}.$$

Then, for every $\epsilon > 0$,

$$(16) \quad \vartheta_{G_n}(\mathbb{R}^n) \lesssim (f(r) + \epsilon)^n.$$

Proof. Let $\epsilon > 0$; let $\gamma = \sqrt{c(r)} + \epsilon$ and $m = \sqrt{2c(r)/e} + \epsilon$. Lemma 3.1 shows that for n sufficiently large, $(z_0, z_1, z_2) = (m^n, 1, \gamma^n)$ is a feasible solution of (8). So, for these values of n , the optimal value of (8) is upper bounded by $m^n + \gamma^n \alpha(G_n)/M_n$, leading to

$$\vartheta_{G_n}(\mathbb{R}^n) \lesssim (\sqrt{2c(r)/e} + \epsilon)^n + (b\sqrt{c(r)} + \epsilon)^n \lesssim (f(r) + \epsilon)^n. \quad \square$$

In order to complete the proof of Theorem 1.1, we will apply Theorem 3.2 to a certain sequence of graphs introduced by Raigorodskii in [23]. Before that, we introduce the family of *generalized Johnson graphs* and recall Frankl and Wilson upper bound on their independence number. We will explain with full details how this bound, combined with Theorem 3.2, reaches the inequality

$$m_1(\mathbb{R}^n) \lesssim (1.262)^{-n}.$$

Then, we will proceed to the graphs considered in [23], which will allow us to obtain the slightly better bound announced in Theorem 1.1 with similar techniques.

Definition 3.3. *We denote $J(n, w, i)$ and call generalized Johnson graph the graph with vertices the set of n -tuples of 0's and 1's, with w coordinates equal to 1, and with edges connecting pairs of n -tuples having exactly i coordinates in common equal to 1.*

An upper bound of $\alpha(J(n, w, i))$ is provided by Frankl and Wilson intersection theorem [22] and applies for certain values of the parameters w and i :

Theorem 3.4. [22] *If q is a power of a prime number,*

$$\alpha(J(n, 2q - 1, q - 1)) \leq \binom{n}{q - 1}.$$

Standard results on the density of prime numbers ensure that, for all $a > 0$, there exists a sequence of primes that grow like an . Indeed, the prime number theorem states that the m -th prime number $\pi(m)$ satisfies $\pi(m) \sim m \ln(m)$ where $\ln(t)$ denotes the natural base- e logarithm function (see [4] for a historical survey). Since $m \ln(m)$ is strictly increasing to infinity, for any $n \in \mathbb{N}$ there exists $m_n \geq 1$ such that

$$m_n \ln(m_n) \leq an < (m_n + 1) \ln(m_n + 1).$$

In particular $an \sim m_n \ln(m_n)$. We set $p_n = \pi(m_n)$, and then $p_n \sim m_n \ln(m_n) \sim an$.

Taking $q = p_n \sim an$, with $a < 1/4$ and $H(a) := -a \ln(a) - (1 - a) \ln(1 - a)$, Theorem 3.4 leads to

$$(17) \quad \frac{\alpha(J(n, 2q - 1, q - 1))}{|J(n, 2q - 1, q - 1)|} \leq \frac{\binom{n}{q-1}}{\binom{n}{2q-1}} \approx e^{-(H(2a) - H(a))n}.$$

Moreover, this upper bound is optimal in the sense that it cannot be tightened by an exponential factor (see [1]). The optimal choice of a , i.e. the value of a that maximizes $H(2a) - H(a)$ is $a = (2 - \sqrt{2})/4$, from which one obtains the upper estimate $(1.207)^{-n}$. Let us recall that this result gave the first lower estimate of exponential growth for the chromatic number of \mathbb{R}^n [22].

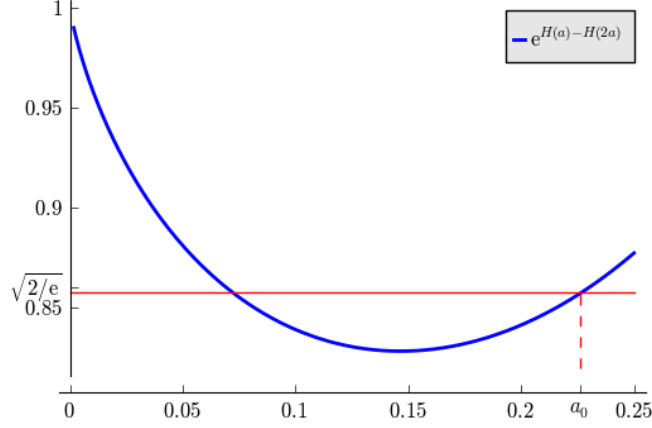
Let $G_n = J(n, 2p_n - 1, p_n - 1)$ where p_n is, as above, a sequence of prime numbers such that $p_n \sim an$. The value of $a < 1/4$ will be chosen later in order to optimize the resulting bound (16) (interestingly, it will turn to be different than the one that optimizes (17)). So, we have $M_n = \binom{n}{2p_n - 1}$ and the constant b in (15) is $b(a) = e^{-(H(2a) - H(a))n}$.

From G_n , we construct unit distance graphs in \mathbb{R}^n by assigning the real value t_0 to the 0-coordinates and the real value t_1 to the 1-coordinates. The squared Euclidean distance between two vertices is equal to $2(t_0 - t_1)^2 p_n$ so, assuming $t_0 \geq t_1$, we must have $t_0 = t_1 + 1/\sqrt{2p_n}$. The vertices then belong to the sphere centered at 0^n and of radius r with $r^2 = t_0^2(n - 2p_n + 1) + t_1^2 p_n = t_1^2 n + 2t_1(n - 2p_n + 1)/\sqrt{2p_n} + (n - 2p_n + 1)/2p_n$. So, we obtain infinitely many induced subgraphs of the unit distance graph, all of them being combinatorially equivalent to G_n , and realizing every radius r such that

$$r \geq r_{\min}(n, p_n) := \sqrt{\frac{(n - 2p_n + 1)(2p_n - 1)}{2np_n}} \sim r(a) := \sqrt{1 - 2a}.$$

The function $f(r(a)) = \sqrt{2c(r(a))}/e$ is decreasing with a , so we will take the largest possible value for a , under the constraint $b(a) \leq \sqrt{2}/e$. Let this value be denoted a_0 ; then $b(a_0) = \sqrt{2}/e$ and (see Figure 3)

$$0.2268 \leq a_0 \leq 0.2269.$$

FIGURE 3. $e^{H(a)-H(2a)}$

We fix now $a = a_0$. For a given $\epsilon > 0$, because the function $f(r) = \sqrt{2c(r)}/e$ is continuous, there is a $r > r(a_0)$ such that $f(r) = f(r(a_0)) + \epsilon$, and such that, for n sufficiently large, r is a valid radius for all the graphs G_n . Applying Theorem 3.2 to this value of r and to ϵ , we obtain

$$\vartheta_{G_n}(\mathbb{R}^n) \lesssim (f(r(a_0)) + \epsilon)^n$$

with

$$f(r(a_0)) = \sqrt{2(1 + \sqrt{2a_0})e^{-(1+\sqrt{2a_0})}} < (1.262)^{-1}.$$

In [23], Raigorodskii considers graphs with vertices in $\{-1, 0, 1\}^n$, where the number of -1 , respectively of 1 , grows linearly with n . If the number of 1 , respectively of -1 , is of the order of x_1n , respectively x_2n , with $x_2 \leq x_1$, if $z = (x_1 + 3x_2)/2$, and $y_1 = (-1 + \sqrt{-3z^2 + 6z + 1})/3$, he shows that:

$$(18) \quad \frac{\alpha(G_n)}{M_n} \lesssim b(x_1, x_2)^n \text{ where } b(x_1, x_2) = e^{-(H_2(x_1, x_2) - H_2(y_1, (z - y_1)/2))}$$

where $H_2(u, v) = -u \ln(u) - v \ln(v) - (1 - u - v) \ln(1 - u - v)$. The proof of (18) relies on a similar argument as in Frankl-Wilson intersection theorem. These graphs can be realized as subgraphs of the unit distance graph in \mathbb{R}^n with minimal radius

$$r(x_1, x_2) = \sqrt{\frac{(x_1 + x_2) - (x_1 - x_2)^2}{(x_1 + 3x_2)}}.$$

For $x_1 = 0.22$ and $x_2 = 0.20$, the inequality $b(x_1, x_2) < \sqrt{2}/e$ holds and $f(r(x_1, x_2)) < 1.268^{-1}$, leading to the announced inequality (3).

Remark 3.5. *The possibility to further improve the basis of exponential growth using Theorem 3.2 is rather limited. Indeed, $f(r) \geq \sqrt{2c(1/2)}/e > (1.316)^{-1}$. So, with this method, we cannot reach a better basis than 1.316.*

4. NUMERICAL RESULTS FOR DIMENSIONS UP TO 24

In this range of dimensions, we have tried many graphs in order to improve the known upper estimates of $m_1(\mathbb{R}^n)$ (and, in turn, the lower estimates of the measurable chromatic number). We didn't improve upon the results obtained in [20] for dimension 2 and 3. We report here the best we could achieve for $4 \leq n \leq 24$. Table 2 displays a feasible solution (z_0, z_1, z_2) of (4) where the notations are those of section 2: G is an induced subgraph of the unit distance graph in dimension n , and it has M vertices at distance r from 0^n . The number given in the third column is the exact value, or an upper bound, of its independence number $\alpha(G)$, and replaces $\alpha(G)$ in (4). The last column contains the objective value of (4), thus an upper bound for $m_1(\mathbb{R}^n)$. Table 3 gives the corresponding lower bounds for $\chi_m(\mathbb{R}^n)$, compared to the previous best known ones.

The computation of (z_0, z_1, z_2) was performed in a similar way as in [20]: the interval $[0, 50]$ is sampled in order to replace the condition $z_0 + z_1\Omega_n(t) + z_2\Omega_n(rt) \geq 0$ for all $t > 0$ by a finite number of inequalities; the resulting linear program is solved leading to a solution (z_0^*, z_1^*, z_2^*) . The function $z_0^* + z_1^*\Omega_n(t) + z_2^*\Omega_n(rt)$ is then almost feasible for (4), in the sense that its absolute minimum is reached in the range $[0, 50]$ and is a (small) negative number. Then, we need only slightly increase z_0^* in order to turn it to a true feasible solution. The computations were performed with the help of the softwares SAGE [29] and Ipsolve [8].

In the next two subsections, we give more details on the graphs involved in the computations and on how we dealt with their independence number.

4.1. Johnson graphs. The generalized Johnson graphs were introduced in section 3. They give the best upper bound for dimensions between 12 and 23.

According to the definition 3.3, the coordinates of any vertex of $J(n, w, i)$ sum to w , and the squared Euclidean distance between two vertices connected by an edge is equal to $2(w - i)$, so, after rescaling by $1/\sqrt{2(w - i)}$, $J(n, w, i)$ is an induced subgraph of the unit distance graph of dimension $n - 1$. A straightforward calculation shows that it lies on a sphere of radius equal to $\sqrt{w(1 - w/n)/(2(w - i))}$.

For these graphs, several strategies are available to deal with their independence number, that we will discuss now.

A direct computation of $\alpha(J(n, w, i))$ for all w, i turns successful only up to $n = 10$ (we have performed the computations using the package GRAPE of the computational system GAP, that deals with graphs with symmetries [28]; indeed, the graphs $J(n, w, i)$ are invariant under the group of permutations of the n coordinates). On the other hand, for the graphs $J(n, 3, 1)$, there is an explicit formula due to Erdős and Sós (see [14, Lemma 18]), but the number of vertices in this case, which is roughly equal to n^3 , is too small to lead to a good bound.

If, being less demanding, we seek only for an upper estimate of $\alpha(J(n, w, i))$, we have two possibilities. One of them is offered by Frankl and Wilson bound recalled in Theorem 3.4, if the parameters (n, w, i) are of the specific form $(n, 2q - 1, q - 1)$ where q is the power of a prime number.

Another upper bound of $\alpha(J(n, w, i))$ is given by the Lovász theta number $\vartheta(J(n, w, i))$ of the graph $J(n, w, i)$. The theta number of a graph was introduced

by Lovász in [17]. It is a semidefinite programming relaxation of the independence number (its definition and properties will be recalled in section 5). The group of permutations of the n coordinates acts transitively on the vertices as well as on the edges of the graph $J(n, w, i)$ so from [17, Theorem 9], its theta number expresses in terms of the largest and smallest eigenvalues of the graph; taking into account that these eigenvalues, being the eigenvalues of the Johnson scheme, are computed in [11] in terms of Hahn polynomials, we have, if $Z_k(i) := Q_k(w-i)/Q_k(0)$ with the notations of [11]:

$$(19) \quad \frac{\vartheta(J(n, w, i))}{|J(n, w, i)|} = \frac{-\min_{k \in [w]} Z_k(i)}{1 - \min_{k \in [w]} Z_k(i)}.$$

Remark 4.1. *We note that this expression is completely analogous to (7); indeed, both graphs afford an automorphism group that is edge transitive. We refer to [5] for an interpretation of (7) in terms of eigenvalues of operators.*

The bound on $\alpha(J(n, w, i))$ given by (19) unfortunately turns to be poor. Computing $\vartheta'(J(n, w, i))$ instead of $\vartheta(J(n, w, i))$ (ϑ' is a standard strengthening of ϑ obtained by adding a non negativity constraint on the matrix variables, see Remark 5.1) represents an easy way to tighten it. Indeed, one can see that $\vartheta(J(n, w, i)) = \vartheta'(J(n, w, i))$ only if

$$(20) \quad Z_{k_0}(i) = \min_{j \in [w]} Z_{k_0}(j)$$

where k_0 satisfies $Z_{k_0}(i) = \min_{k \in [w]} Z_k(i)$. It turns out that (20) is not always fulfilled and in these cases $\vartheta'(J(n, w, i)) < \vartheta(J(n, w, i))$.

A more serious improvement is provided by semidefinite programming following [27, (67)] where constant weight codes with given minimal distance are considered. In order to apply this framework to our setting, we only need to change the range of avoided Hamming distances in [27, (65-iv)].

Table 1 displays the numerical values of the three bounds for certain parameters (n, w, i) , selected either because they allow for Frankl and Wilson bound, or because they give the best upper bound for $m_1(\mathbb{R}^n)$ that we could achieve. For most of these parameters, the semidefinite programming bound turns to be the best one and is significantly better than the theta number. It would be of course very interesting to understand the asymptotic behavior of this bound when n grows to $+\infty$, unfortunately this problem seems to be out of reach to date.

The computation of the semidefinite programming bound was performed on the NEOS website (<http://www.neos-server.org/neos/>) with the solver SDPA [34] and double checked with SDPT3 version 4.0-beta [32].

4.2. Other graphs. The 600-cell is a regular polytope of dimension 4 with 120 vertices: the sixteen points $(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$, the eight permutations of $(\pm 1, 0, 0, 0)$ and the 96 points that are even permutations of $(0, \pm 1/(2\phi), \pm 1/2, \pm \phi/2)$, where $\phi = (1 + \sqrt{5})/2$. If d is the distance between two non antipodal vertices, we have $d^2 \in \{(5 \pm \sqrt{5})/2, 3, (3 \pm \sqrt{5})/2, 2, 1\}$. Each value of d gives raise to a graph connecting the vertices that are at distance d apart; these graphs, after rescaling so that the edges have length 1, lie on the sphere of radius $r = 1/d$. Their

independence numbers are respectively equal to: 39, 26, 24, 26, 20. We note that applying the conjugation $\sqrt{5} \rightarrow -\sqrt{5}$ will obviously not change the independence number. Among these graphs, the best result in dimension 4, recorded in Table 2, was obtained with $d = \sqrt{3}$. It turned out that the same graph gave the best result we could achieve in dimensions 5 and 6.

The root system E_8 is the following set of 240 points in \mathbb{R}^8 : the points $((\pm 1)^2, 0^6)$ and all their permutations, and the points $((\pm 1/2)^8)$ with an even number of minus signs. The distances between two non antipodal points take three different values: $d = \sqrt{2}, 2, \sqrt{6}$. The unit distance subgraph associated to a value of d lies on a sphere of radius $r = \sqrt{2}/d$ and has an independence number equal respectively to 16, 16, 36. The one with the smallest radius $r = \sqrt{1/3}$ gives the best bound in dimension 8 as well as in dimensions 9, 10, and 11 (in these dimensions we have compared with Johnson graphs).

The configuration in dimension 7 is derived from E_8 : given $p \in E_8$, we take the set of points in E_8 closest to p . Independently of p , this construction leads to 56 points that lie on a hyperplane. The graph defined by the distance $\sqrt{6}$ after suitable rescaling corresponds to $r = \sqrt{6}/4$ and has independence number 7.

In dimension 24, we obtained the best result with the so-called orthogonality graph $\Omega(24)$. For $n = 0 \pmod{4}$, $\Omega(n)$ denotes the graph with vertices in $\{0, 1\}^n$, where the edges connect the points at Hamming distance $n/2$. Using semidefinite programming, an upper bound of its independence number is computed for $n = 16, 20, 24$ in [12].

5. TIGHTENING THE THETA NUMBER OF COMPACT GRAPHS WITH SUBGRAPHS

In this section, we would like to show that the method presented in section 2 to upper bound $m_1(\mathbb{R}^n)$ is flexible enough to be adapted to a broad variety of situations, in order to design tight upper bounds for the independence number of a graph. In fact, this method represents an interesting way to strengthen the upper bound given by Lovász theta number of a graph, by exploiting an additional constraint arising from a subgraph.

The framework in which we will develop the method is that of a graph $\mathcal{G} = (X, E)$ where X is a compact topological space, endowed with the continuous and transitive action of a compact topological group Γ (X is called a homogeneous space). This framework includes the case of finite graphs where the vertex set is given the discrete topology. There are two reasons why we do not limit ourselves to the finite case: one is that going from finite graphs to compact graphs does not raise essential difficulties; the other reason is that the compact case includes spaces of special interest to us, in particular that of the unit sphere S^{n-1} (see Remark 5.3).

Before we dive into this rather general framework, we review the theta number of a finite graph.

(n, w, i)	$\alpha(J(n, w, i))$	FW bound [22]	$\vartheta'(J(n, w, i))$	SDP bound
(6, 3, 1)	4	6	4	4
(7, 3, 1)	5	7	5	5
(8, 3, 1)	8	8	8	8
(9, 3, 1)	8	9	11	8
(10, 5, 2)	27	45	30	27
(11, 5, 2)	37	55	42	37
(12, 5, 2)	57	66	72	57
(12, 6, 2)			130	112
(13, 5, 2)		78	109	72
(13, 6, 2)			191	148
(14, 7, 3)		364	290	184
(15, 7, 3)		455	429	261
(16, 7, 3)		560	762	464
(16, 8, 3)			1315	850
(17, 7, 3)		680	1215	570
(17, 8, 3)			2002	1090
(18, 9, 4)		3060	3146	1460
(19, 9, 4)		3876	4862	2127
(20, 9, 3)			13765	6708
(20, 9, 4)		4845	8840	3625
(21, 9, 4)		5985	14578	4875
(21, 10, 4)			22794	8639
(22, 9, 4)		7315	22333	6480
(22, 11, 5)			36791	11360
(23, 9, 4)		8855	32112	8465
(23, 11, 5)			58786	17055
(24, 9, 4)		10626	38561	10796
(24, 12, 5)			172159	53945
(25, 9, 4)		12650	46099	13720
(26, 13, 6)		230230	453169	101494
(27, 13, 6)		296010	742900	163216

TABLE 1. Bounds for the independence number of $J(n, w, i)$

5.1. The theta number of a finite graph. This number, denoted $\vartheta(\mathcal{G})$ and introduced in [17], is the optimal value of a semidefinite program that satisfies

$$\alpha(\mathcal{G}) \leq \vartheta(\mathcal{G}) \leq \chi(\overline{\mathcal{G}})$$

where $\alpha(\mathcal{G})$ denotes as before the independence number of \mathcal{G} , $\overline{\mathcal{G}}$ is the complementary graph, and $\chi(\overline{\mathcal{G}})$ is its chromatic number, the least number of colors needed to color all vertices so that adjacent vertices receive different colors.

n	G	α	M	r	z_0	z_1	z_2	$z_0 + z_2\alpha/M$
4	600-cell	26	120	$\sqrt{3}/3$	0.0421343	0.690511	0.267355	0.100062
5	600-cell	26	120	$\sqrt{3}/3$	0.023477	0.772059	0.204465	0.0677778
6	600-cell	26	120	$\sqrt{3}/3$	0.0141514	0.830343	0.155506	0.0478444
7	E_8 kissing	7	56	$\sqrt{6}/4$	0.007948	0.834435	0.157617	0.0276502
8	E_8	36	240	$\sqrt{3}/3$	0.0053364	0.899613	0.0950508	0.0195941
9	E_8	36	240	$\sqrt{3}/3$	0.0033303	0.921154	0.0755157	0.0146577
10	E_8	36	240	$\sqrt{3}/3$	0.00209416	0.937453	0.0604529	0.0111621
11	E_8	36	240	$\sqrt{3}/3$	0.00132364	0.949973	0.0487036	0.00862918
12	$J(13, 6, 2)$	148	1716	$\sqrt{21/52}$	9.002e-04	0.938681	0.0604188	0.00611112
13	$J(14, 7, 3)$	184	3432	$\sqrt{7/16}$	5.933e-04	0.936921	0.0624857	0.00394335
14	$J(15, 7, 3)$	261	6435	$\sqrt{7/15}$	3.9393e-04	0.935283	0.0643239	0.00300288
15	$J(16, 8, 3)$	850	12870	$\sqrt{2/5}$	2.7212e-04	0.967168	0.0325604	0.00242258
16	$J(17, 8, 3)$	1090	24310	$\sqrt{36/85}$	1.9080e-04	0.968014	0.0317961	0.00161646
17	$J(18, 9, 4)$	1460	48620	$\sqrt{9/20}$	1.34658e-04	0.967557	0.0323093	0.00110487
18	$J(19, 9, 4)$	2127	92378	$\sqrt{9/19}$	9.50746e-05	0.96714	0.032765	8.49488e-04
19	$J(20, 9, 3)$	6708	167960	$\sqrt{33/80}$	5.944e-05	0.98275	0.0171908	7.46008e-04
20	$J(21, 10, 4)$	8639	352716	$\sqrt{55/126}$	4.44363e-05	0.982618	0.0173381	4.69095e-04
21	$J(22, 11, 5)$	11360	705432	$\sqrt{11/24}$	3.2936e-05	0.982495	0.0174727	3.1431e-04
22	$J(23, 11, 5)$	17055	1352078	$\sqrt{11/23}$	2.4315e-05	0.982385	0.0175913	2.46211e-04
23	$J(24, 12, 5)$	53945	2704156	$\sqrt{3/7}$	1.40898e-05	0.990052	0.00993429	2.12269e-04
24	$\Omega(m)$	183373	2^{24}	$\sqrt{1/2}$	1.30001e-05	0.984309	0.0156786	1.84366e-04

TABLE 2. Feasible solutions of (4) and corresponding upper bounds for $m_1(\mathbb{R}^n)$

n	previous best lower bound for $\chi_m(\mathbb{R}^n)$	new lower bound for $\chi_m(\mathbb{R}^n)$
4	9 [20]	10
5	14[20]	15
6	20 [20]	21
7	28 [20]	37
8	39[20]	52
9	54[20]	69
10	73 [20]	90
11	97 [20]	116
12	129 [20]	164
13	168[20]	254
14	217[20]	334
15	279[20]	413
16	355 [20]	619
17	448[20]	906
18	563[20]	1178
19	705 [20]	1341
20	879[20]	2132
21	1093[20]	3182
22	1359[20]	4062
23	1690 [20]	4712
24	2106[20]	5424

TABLE 3. Lower bounds for the measurable chromatic number

One of the many equivalent definitions of $\vartheta(\mathcal{G})$ involves matrices S whose rows and columns are indexed by X ; for such a matrix, whose coefficients will be denoted $S(x, y)$, we write $S \succeq 0$ if S is symmetric and positive semidefinite. Then, following [17, Theorem 4]:

$$(21) \quad \vartheta(\mathcal{G}) = \sup \left\{ \sum_{(x,y) \in X^2} S(x, y) : \begin{array}{l} S \in \mathbb{R}^{X \times X}, S \succeq 0, \\ \sum_{x \in X} S(x, x) = 1, \\ S(x, y) = 0 \text{ } (\{x, y\} \in E) \end{array} \right\}.$$

The inequality:

$$\alpha(\mathcal{G}) \leq \vartheta(\mathcal{G})$$

follows from the properties of a certain matrix naturally associated to a subset set $A \subset X$:

$$S_A(x, y) := 1_A(x)1_A(y)/|A|.$$

This matrix satisfies a number of linear conditions:

$$\sum_{x \in X} S_A(x, x) = 1, \quad \sum_{(x,y) \in X^2} S_A(x, y) = |A|,$$

and, if A is an independent set, $S_A(x, y) = 0$ for all $\{x, y\} \in E$. Moreover, S_A is positive semidefinite, so it defines a feasible solution of the semidefinite program in (21), with objective value equal to $|A|$. The inequality $\alpha(\mathcal{G}) \leq \vartheta(\mathcal{G})$ follows immediately.

Remark 5.1. *In order to tighten the inequality $\alpha(G) \leq \vartheta(G)$ for finite graphs, it is customary to add the condition $S \geq 0$ (meaning all coefficients of S are non negative) to the constraints in (21); the new optimal value is denoted $\vartheta'(G)$ and coincides with the linear programming bound introduced earlier by P. Delsarte for the cardinality of codes in polynomial association schemes (see [10] and [26]).*

5.2. Compact homogeneous graphs. From now on, we assume that X is a compact space, acted upon by a compact topological group Γ which is a subgroup of the automorphism group of the graph \mathcal{G} . We assume that the application $(\gamma, x) \mapsto \gamma x$ defining the action of Γ on X is continuous, and that this action is transitive. We choose a base point $p \in X$, and let Γ_p denote the stabilizer of p in Γ , so that X can be identified with the quotient space Γ/Γ_p (see [21, section 2.6]).

The group Γ is equipped with its Haar measure (see [21, section 2.2]), normalized so that its total volume equals 1, which induces a Borel regular measure on X , such that for any measurable function φ on X ,

$$\int_X \varphi(x) dx = \int_\Gamma \varphi(\gamma p) d\gamma.$$

(see [21, section 2.6]). Volumes for this measure will be denoted vol_X . The *independence volume* $\alpha_X(\mathcal{G})$ of \mathcal{G} is by definition the supremum of the volume of a measurable independent set of X :

$$\alpha_X(\mathcal{G}) := \sup\{\text{vol}_X(A) : A \subset X, A \text{ is Borel measurable and independent}\}.$$

We will assume from now on that $\alpha_X(\mathcal{G}) > 0$. We note that, if X is finite, the measure induced on X is simply

$$\int_X \varphi(x) dx = \frac{1}{|X|} \sum_{x \in X} \varphi(x).$$

In particular, if $A \subset X$, $\text{vol}_X(A) = |A|/|X|$ so $\alpha_X(\mathcal{G}) = \alpha(\mathcal{G})/|X|$.

Now let $V \subset X$ be a Borel measurable subset of X together with a finite positive Borel measure λ on V , such that $0 < \lambda(V) < +\infty$. We introduce as in section 2 the λ -independence number of the subgraph G induced on V by \mathcal{G} :

$$\alpha_\lambda(G) := \sup\{\lambda(A) : A \subset V, A \text{ a Borel measurable independent set}\}.$$

In order to define $\vartheta_G(\mathcal{G})$, we need to introduce positive definite functions on X . If f belongs to the space $\mathcal{C}(X)$ of real valued continuous functions on X , we say that f is *positive definite* and denote $f \succeq 0$ if, for all $k \geq 1$, for any choice of $\gamma_1, \dots, \gamma_k \in \Gamma$, the matrix with coefficients $f(\gamma_j^{-1} \gamma_i p)$ is symmetric positive semidefinite. Because Γ is compact and f is continuous, this condition is equivalent to the property that the function $\gamma \rightarrow f(\gamma p)$ is a function of positive type on Γ in the sense of [21, section 3.3] (see also [9]).

We note that this notion coincides with the notion of positive definite functions that came into play in section 2. Indeed, the situation of section 2 corresponds to the case of the additive group $\Gamma = \mathbb{R}^n/L$, acting on itself by translations, with $p = 0^n$.

Now we can state the main result of this section:

Theorem 5.2. *With the notations introduced above, let*

$$(22) \quad \vartheta_G(\mathcal{G}) = \sup \left\{ \int_X f(x) dx : \begin{array}{l} f \in \mathcal{C}(X), f(\gamma x) = f(x) (\gamma \in \Gamma_p), \\ f \succeq 0, \\ f(p) = 1, \\ f(x) = 0 (\{x, p\} \in E) \\ \int_V f(v) d\lambda(v) \leq \alpha_\lambda(G) \end{array} \right\}.$$

Assuming $\alpha_X(\mathcal{G}) > 0$, we have

$$\alpha_X(\mathcal{G}) \leq \vartheta_G(\mathcal{G}).$$

Proof. Let $A \subset X$ be a Borel measurable independent set of positive measure. We introduce

$$f_A(x) := \frac{1}{\text{vol}_X(A)} \int_\Gamma 1_A(\gamma x) 1_A(\gamma p) d\gamma.$$

This function $f_A \in \mathbb{R}^X$ will play the role of the matrix S_A that occurred in the proof of the inequality $\alpha(\mathcal{G}) \leq \vartheta(\mathcal{G})$ for finite graphs. We claim that f_A satisfies the constraints required by the program defining $\vartheta_G(\mathcal{G})$. Indeed, being the convolution over Γ of two bounded functions, f_A is continuous (see [21, (2.39)]). The other conditions (numbered (i) to (v) in order of appearance in (22)) are easily obtained, applying Fubini's theorem to swap integrals and the invariance by left and right multiplication of the Haar measure on Γ . We skip the details for (i) and (iii). Condition (iv) holds because on one hand, if $\{x, p\} \in E$, also $\{\gamma x, \gamma p\} \in E$, and on the other hand, A is an independent set of \mathcal{G} , so $1_A(\gamma x) 1_A(\gamma p) = 0$.

Let us check (ii), i.e. $f_A \succeq 0$: thanks to the right invariance of the Haar measure, we have

$$\begin{aligned} f_A(\gamma_j^{-1} \gamma_i p) &= \frac{1}{\text{vol}_X(A)} \int_\Gamma 1_A(\gamma \gamma_j^{-1} \gamma_i p) 1_A(\gamma p) d\gamma \\ &= \frac{1}{\text{vol}_X(A)} \int_\Gamma 1_A(\gamma \gamma_i p) 1_A(\gamma \gamma_j p) d\gamma. \end{aligned}$$

So, the matrix with coefficients $f_A(\gamma_j^{-1} \gamma_i p)$ is symmetric. Moreover, for $(x_1, \dots, x_k) \in \mathbb{R}^k$,

$$\sum_{1 \leq i, j \leq k} x_i x_j f_A(\gamma_j^{-1} \gamma_i p) = \frac{1}{\text{vol}_X(A)} \int_\Gamma \left(\sum_{i=1}^k x_i 1_A(\gamma \gamma_i p) \right)^2 d\gamma \geq 0.$$

In order to verify (v), we remark that

$$(23) \quad \int_V 1_A(\gamma v) d\lambda(v) = \lambda((\gamma^{-1} A) \cap V) \leq \alpha_\lambda(G).$$

This inequality, combined with Fubini's theorem, leads to the result. Indeed,

$$\begin{aligned} \int_V f_A(v) d\lambda(v) &= \frac{1}{\text{vol}_X(A)} \int_V \int_\Gamma 1_A(\gamma v) 1_A(\gamma p) d\gamma d\lambda(v) \\ &= \frac{1}{\text{vol}_X(A)} \int_\Gamma \left(\int_V 1_A(\gamma v) d\lambda(v) \right) 1_A(\gamma p) d\gamma \\ &\leq \frac{1}{\text{vol}_X(A)} \int_\Gamma \alpha_\lambda(G) 1_A(\gamma p) d\gamma = \alpha_\lambda(G). \end{aligned}$$

It remains to compute the objective value of $f_A(x)$; for this, we apply Fubini's theorem once more:

$$\begin{aligned} \int_X f_A(x) dx &= \int_X \frac{1}{\text{vol}_X(A)} \int_\Gamma 1_A(\gamma x) 1_A(\gamma p) d\gamma dx \\ &= \frac{1}{\text{vol}_X(A)} \int_\Gamma \left(\int_X 1_A(\gamma x) dx \right) 1_A(\gamma p) d\gamma \\ &= \frac{1}{\text{vol}_X(A)} \int_\Gamma \text{vol}_X(\gamma^{-1}A) 1_A(\gamma p) d\gamma \\ &= \int_\Gamma 1_A(\gamma p) d\gamma = \text{vol}_X(A). \end{aligned}$$

□

Remark 5.3. Taking $X = S^{n-1}$, and $E = \{\{x, y\} : \|x - y\| = d\}$, defines a graph $G(S^{n-1}, d)$ homogeneous under the action of the orthogonal group that fits into our setting. Moreover, up to a suitable rescaling, this graph is an induced subgraph of the unit distance graph. So, Theorem 5.2 can be applied to compute tight bounds for $\alpha(G(S^{n-1}, d))$, which in turn may be used in Theorem 2.2, suggesting an inductive method to calculate better upper bounds for $m_1(\mathbb{R}^n)$.

In the remaining of this section, we discuss some connections between Theorem 5.2 and previous results.

5.3. The relationship between $\vartheta_G(\mathcal{G})$ and $\vartheta(\mathcal{G})$ for finite homogeneous graphs.

If \mathcal{G} is a finite homogeneous graph for the group Γ , its theta number can be rewritten as:

$$\vartheta(\mathcal{G}) = \sup \left\{ \sum_{x \in X} f(x) : \begin{array}{l} f \in \mathbb{R}^X, f(\gamma x) = f(x) (\gamma \in \Gamma_p), \\ f \succeq 0, \\ f(p) = 1, \\ f(x) = 0 (\{x, p\} \in E) \end{array} \right\}.$$

We refer to [9, Theorem 2] where this reformulation is given in the special case of Cayley graphs. The generalization to homogeneous graphs is straightforward. So, $\vartheta_G(\mathcal{G})$ is a tightening of $\vartheta(\mathcal{G})$ with an additional constraint relative to the subgraph G , and we have

$$\vartheta_G(\mathcal{G}) \leq \vartheta(\mathcal{G}).$$

5.4. The analogy between $\vartheta_G(\mathcal{G})$ and $\vartheta_G(\mathbb{R}^n)$. Our notations suggest an analogy between $\vartheta_G(\mathcal{G})$ as defined in (22) and $\vartheta_G(\mathbb{R}^n)$ as introduced in Theorem 2.2. This analogy will be more transparent from the expression of the dual program of $\vartheta_G(\mathcal{G})$. Here we apply the duality theory of conic linear programs in locally convex topological vector spaces for which we refer to [7, Chapter IV]. The dual space of the space $\mathcal{C}(X)$ of real valued continuous functions on X , i.e. the space of continuous linear forms on $\mathcal{C}(X)$ equipped with the topology defined by the supremum norm, is the space $\mathcal{M}(X)$ of signed regular measures on X (it follows from Riesz representation theorem, see [24, Theorem 6.19]). For $\mu \in \mathcal{M}(X)$, the notation $\mu \succeq 0$ (μ positive definite) stands for $\int f d\mu \geq 0$ for all $f \in \mathcal{C}(X)$, f being positive definite. The support of μ is denoted by $\text{supp}(\mu)$. The dual program of $\vartheta_G(\mathcal{G})$ in the sense of [7, Chapter 4, section 6] becomes:

$$\inf \left\{ z_0 + z_2 \frac{\alpha_\lambda(G)}{\lambda(V)} : \begin{array}{l} \mu \in \mathcal{M}(X), \text{supp}(\mu) \subset \{x \in X : \{x, p\} \in E\} \\ z_2 \geq 0 \\ z_0 \delta_p + \mu + z_2 \frac{\lambda}{\lambda(V)} - dx \succeq 0 \end{array} \right\}$$

We recall that weak duality holds, i.e. that $\vartheta_G(\mathcal{G})$ is upper bounded by the optimal value of its dual program (see [7, Theorem 6.2]).

5.5. An inequality relating $\alpha_X(\mathcal{G})$ and $\alpha_\lambda(G)$ and its connection to $\vartheta_G(\mathcal{G})$. Let us go back to the inequality (23). If we integrate it over Γ , and then take the supremum over the independent sets of \mathcal{G} , we obtain

$$\alpha_X(\mathcal{G}) \leq \frac{\alpha_\lambda(G)}{\lambda(V)}.$$

In particular, if \mathcal{G} is a finite graph and λ is the counting measure, the above inequality becomes

$$(24) \quad \frac{\alpha(\mathcal{G})}{|X|} \leq \frac{\alpha(G)}{|V|}.$$

We recover a standard inequality that has proved to be useful in several instances, in particular if one has a special hint on G . For example, in coding theory it is applied to relate the sizes of codes in Hamming and Johnson spaces respectively, following Elias and Bassalygo principle (see e.g. [18]). Also, Larman and Rogers inequality (2) can be seen as an analogue of (24) for the unit distance graph.

6. OPEN PROBLEMS

We present here a few questions that we believe would be worth to look at. Some of them have already been mentioned previously.

- (1) There are several possible variants in the way we apply Theorem 2.2 to find upper bounds for $m_1(\mathbb{R}^n)$. There is no reason to restrict to graphs that embed in a sphere centered at 0^n as we do, and also several subgraphs could be used simultaneously. Can the bounds of Tables 2 and 3 be improved this way ?

- (2) The subgraph method can be applied to strengthen the theta number of finite graphs, in particular it could be used to obtain better bounds for the independence number of the graphs $J(n, w, i)$. In turn, the resulting upper bounds may lead to further improvements on the bounds for $m_1(\mathbb{R}^n)$. More generally, can Theorem 5.2 applied to the unit sphere lead to improved bounds for $m_1(\mathbb{R}^n)$ (see Remark 5.3)?
- (3) In coding theory, the so-called MRRW-bound [18] is an asymptotic upper bound for the size of codes of given minimal Hamming distance, which derives from Delsarte linear programming bound (Lovász theta number provides essentially the same bound). For binary codes and for a certain range of minimal distances, it is superseded by the so-called second MRRW-bound, which is obtained from the inequality (24), where X is the Hamming space and V is a Johnson space with suitable weight (i.e. the set of binary words of fixed weight). Again, Delsarte linear programming bound is applied to V .

Following Theorem 5.2, it is possible to design a program that combines the two bounds in one. Is it possible to improve the MRRW bounds by analyzing this bound ?

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