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Aurélien Serge Tchakoutio Nguetcho, Jibin Li, Jean-Marie Bilbault. Bifurcations of Phase Portraits of a Singular Nonlinear Equation of the Second Class. *Communications in Nonlinear Science and Numerical Simulation*, 2014, <http://dx.doi.org/10.1016/j.cnsns.2013.12.022>. hal-00934971

HAL Id: hal-00934971

<https://hal.science/hal-00934971>

Submitted on 22 Jan 2014

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Bifurcations of Phase Portraits of a Singular Nonlinear Equation of the Second Class

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Abstract

The soliton dynamics is studied using the Frenkel Kontorova (FK) model with non-convex interparticle interactions immersed in a parameterized on-site substrate potential. The case of a deformable substrate potential allows theoretical adaptation of the model to various physical situations. Non-convex interactions in lattice systems lead to a number of interesting phenomena that cannot be produced with linear coupling alone. In the continuum limit for such a model, the particles are governed by a Singular Nonlinear Equation of the Second Class. The dynamical behavior of traveling wave solutions is studied by using the theory of bifurcations of dynamical systems. Under different parametric situations, we give various sufficient conditions leading to the existence of propagating wave solutions or dislocation threshold, highlighting namely that the deformability of the substrate potential plays only a minor role.

Key words: Hamiltonian system, Nonlinear wave equation, non-convex interparticle interactions, deformability of the substrate potential, Solitary wave solution, Kink wave solution, Periodic wave solution, Breaking wave solution.

1 Introduction

Simplicity and universality are two key requirements of a good physical model. Universal models, which can be used to describe a variety of different phenomena, are very rare and yet are of major importance. They have been proved useful at several scales and moreover possess an educational value. A simple example of such models is the Frenkel Kontorova (FK) model [1]. The FK model has become very popular in many niches of solid state and nonlinear physics. They invented their model in order to describe the motion of a dislocation in crystals. Meanwhile, the FK model has also become a model for an adsorbate layer on the surface of crystal, for ionic conductors, for glassy materials, for charge-density-wave (CDW) transport, and for chains of coupled Josephson junctions. Especially, a number of current researches, such as sliding friction [2,3], heat conduction [4–6], chaos control and nonlinear coupled pendulum [7] are also carried out using the FK model.

The standard FK model describes a chain of coupled atoms subjected to an external periodic potential and its Hamiltonian is defined as

$$H = T + U. \quad (1)$$

T and U are the kinetic and potential energies respectively. U consists of interparticle interactions V_c that take into account the linear and nonlinear couplings between the nearest neighbors of the chain and the substrate external potential V_e along the chain with spatial period a . b stands for the natural spacing of the unperturbed chain. The contribution of the particle i to potential energy U is then defined as follows:

$$U(x_i) = V_c(r_i) + V_e(x_i), \quad 1 \leq i \leq n \quad (2)$$

where $r_i = x_{i+1} - x_i - b + a$ represents the distance between the nearest neighbors.

One of the basic restrictions of this traditional model adopted originally by the authors, which makes it applicable for small lattice misfits only, is the purely elastic interaction between neighboring atoms as a substitute for the real inter-atomic forces. In addition, in real physical systems, the shape of the

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¹ This author is supported by the National Natural Science Foundation of China (10831003).

² This author is supported by the Regional Council of Burgundy.

21 substrate potential can deviate from the standard (sinusoidal or rigid) one and
 22 this may affect strongly the transport properties of the system.

23 As a better approach for modeling real systems, many researchers have con-
 24 sidered the FK model while either modifying the external periodic potential
 25 [1] or modifying interactions between atoms [8–12]. For realistic anharmonic
 26 interactions, such as those of Toda, Morse, Markov and Trayanov, besides the
 27 position of the minimum there exists a second characteristic length at which
 28 the tensile strength of the bond reaches its maximum, that is the inflection
 29 point r_{inf} . This makes the behavior of such systems more complex. The
 30 mathematical challenges of solving the problem analytically increase since the
 31 equations are no more single valued. It has been reported in a numerical study
 32 of the discrete FK model with anharmonic interactions (Toda, Morse) [13] that
 33 beyond some critical values of the independent parameters (the natural lattice
 34 mismatch and/or amplitude of the external periodic potential), these equa-
 35 tions have no solution which was interpreted as a disintegration of the system
 36 .

This work aims to consider a non-convex pair potential V_c [13–15] defined as
 :

$$V_c(r) = \frac{V_0 B \left(\frac{r}{r_0}\right)^2}{1 + B \left(\frac{r}{r_0}\right)^2} \quad (3)$$

37 with $r_0 = \frac{a}{2\pi}$. B changes the width of the potential well with depth V_0 .
 38 It is a single non-convex even potential with a single minimum and infinitely
 39 differentiable. The curvature (elastic constant) of $V_c(r)$ is $2BV_0$ at $r = 0$,
 and $V_c(r)$ has inflection points at $\pm r_{inf} = \frac{r_0}{\sqrt{3B}}$. This potential, plotted

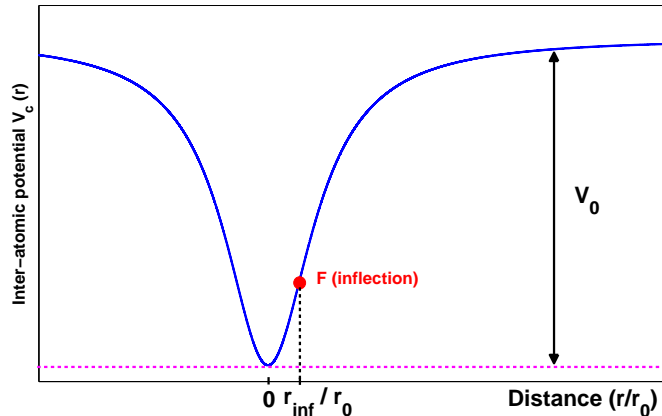


Fig. 1. Plot of the coupling potential $V_c(r)$

40 versus the nearest neighbors distance $\frac{r}{r_0}$ in Fig.1, is infinitely differentiable
 41

42 and supposed to bear all qualitatively important features of the problem under
 43 the influence of a nonlinear periodic deformable substrate V_e .

For this purpose, we used an important type of on-site potential that describes many realistic situations (e.g., the dynamics of atoms adsorbed on crystal surfaces, Josephson junctions, charge density wave condensate, \dots) [1], that is:

$$V_e(x_i) = \varepsilon_s \frac{\sin^2\left(\frac{\pi x_i}{a}\right)}{\sigma^2 + (1 - \sigma^2) \sin^2\left(\frac{\pi x_i}{a}\right)}, \quad (4)$$

44 where σ is related to the coefficient s introduced by Peyrard and Remoissenet
 45 [1] by $\sigma = \frac{1+s}{1-s}$. The parameter ε_s is the parameter that controls or adjusts
 46 the strength of the energy barriers of the substrate potential. In adsystem,
 47 ε_s is the activation energy for diffusion of an isolated adatom. At $\sigma = 1$
 48 the potential $V_e(x_i, \sigma)$ has a sinusoidal shape, while at $\sigma < 1$, broad wells
 49 separate narrow barriers, and at $\sigma > 1$, deep narrow wells separate broad
 gently sloping barriers (see Fig. 2).

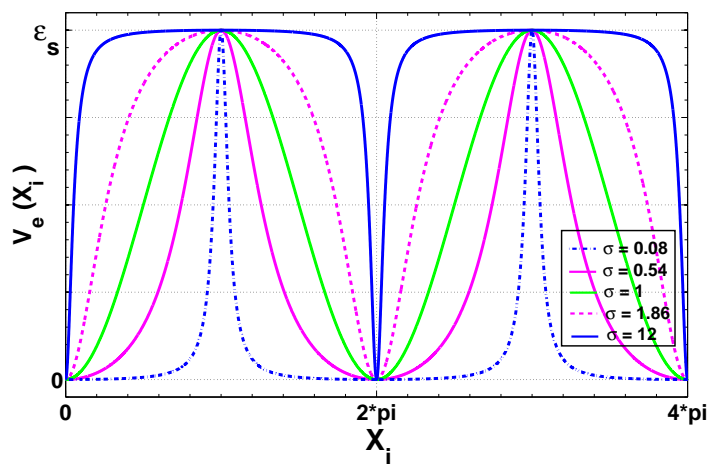


Fig. 2. Substrate potential $V_e(x_i)$ at different values of parameter σ

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51 The next section will be devoted to the presentation of the equations and
 52 integrals of motion. The phase portraits will be presented in section 3, while
 53 the main behaviors of the wave solutions will be studied in (§4), followed by
 54 a summary and some conclusion remarks.

55 2 The equations of motion

The Hamiltonian for such a system is defined as:

$$H = A \sum_i^n \left\{ \frac{1}{2} \left(\frac{d\phi_i}{dt} \right)^2 + \frac{C_0^2}{2b^2} \frac{(\phi_{i+1} - \phi_i - P)^2}{1 + B(\phi_{i+1} - \phi_i - P)^2} + \frac{\sin^2 \left(\frac{\phi_i}{2} \right)}{\sigma^2 + (1 - \sigma^2) \sin^2 \left(\frac{\phi_i}{2} \right)} \right\}, \quad (5)$$

56 where $\phi_i = \frac{2\pi x_i}{a}$ is the scalar dimensionless displacement of particles,
 57 $P = 2\pi \left(\frac{b}{a} - 1 \right)$ stands for the natural lattice incompatibility between sub-
 58 strate and overlayer, while the factor $A = m \left(\frac{a}{2\pi} \right)^2$ sets the energy scale.
 59 The parameters C_0 and ω_0 are the characteristic velocity and frequency
 60 respectively, and are related to the non-convex dimensionless constant B (the
 61 parameter that controls the strength of the nonlinear coupling) and the ampli-
 62 tude of the substrate potential ε_s through $C_0^2 = \frac{2BV_0b^2}{A}$ and $\omega_0^2 = \left(\frac{2\pi}{a} \right)^2 \frac{\varepsilon_s}{m}$.
 63 Finally, m represents the mass of atoms.

In the continuum limit, with dimensionless variables $X = \frac{x}{b}$ and $T = \omega_0 t$, when $\phi_i(t)$ is replaced by $\phi(X, T)$, the particles are governed by the following Euler-Lagrange equation:

$$\frac{\partial^2 \phi}{\partial T^2} - \frac{C_0^2}{b^2 \omega_0^2} \left[\frac{1 - 3B \left(\frac{\partial \phi}{\partial X} - P \right)^2}{\left[1 + B \left(\frac{\partial \phi}{\partial X} - P \right)^2 \right]^3} \right] \frac{\partial^2 \phi}{\partial X^2} + \frac{1}{2} \frac{\sigma^2 \sin(\phi)}{\left[(\sigma^2 - 1) \sin^2 \left(\frac{\phi}{2} \right) - \sigma^2 \right]^2} = 0. \quad (6)$$

In extreme cases $B \rightarrow 0$ and $\sigma \rightarrow 1$, equation (6) leads to the well-known Sine Gordon equation. Using the independent propagating variable $u = X - vT$, where v is a constant velocity, (6) yields:

$$\left(\frac{1 - 3B \left(\frac{d\phi}{du} - P \right)^2}{\left[1 + B \left(\frac{d\phi}{du} - P \right)^2 \right]^3} - \mathbb{V}^2 \right) \frac{d^2 \phi}{du^2} = \frac{1}{2} \frac{\lambda \sigma^2 \sin(\phi)}{\left[(\sigma^2 - 1) \sin^2 \left(\frac{\phi}{2} \right) - \sigma^2 \right]^2}, \quad (7)$$

where $\mathbb{V} = \frac{vb\omega_0}{C_0}$ is a more suitable dimensionless expression to determine the velocity of the travelling waves. $d_0 = \frac{C_0}{\omega_0}$ is the characteristic length of the system, while $\lambda = \left(\frac{b}{d_0}\right)^2$ denotes the scaled amplitude of the periodic potential and measures the effective depth of the substrate potential of the system. The study of Eq. (7) can be best driven in the phase plane $\left(\phi, y = \frac{d\phi}{du}\right)$. Eq(7) can then be rewritten as:

$$\frac{d\phi}{du} = y, \quad \frac{dy}{du} = \frac{1}{2f(y)} \frac{\lambda\sigma^2 \sin(\phi)}{\left[(\sigma^2 - 1) \sin^2\left(\frac{\phi}{2}\right) - \sigma^2\right]^2}, \quad (8)$$

with

$$f(y) = \frac{1 - 3B(y - P)^2}{[1 + B(y - P)^2]^3} - \mathbb{V}^2. \quad (9)$$

If the function $f(y)$ has at least a real zero at $y = y_0$, (8) is a singular nonlinear system of the second class (see [17–19]). By using the transformation $du = f(y)d\xi$ when $y \neq y_0$, system (8) becomes the associated regular system as follows:

$$\frac{d\phi}{d\xi} = yf(y), \quad \frac{dy}{d\xi} = \frac{1}{2} \frac{\lambda\sigma^2 \sin(\phi)}{\left[(\sigma^2 - 1) \sin^2\left(\frac{\phi}{2}\right) - \sigma^2\right]^2} \equiv Q(\phi). \quad (10)$$

Systems (8) and (10) have the same first integral defined as:

$$\begin{aligned} H(\phi, y) &= \frac{(1 - 3BP^2)y^2 + 4BP^3y - By^4}{(1 + BP^2)(1 + B(y - P)^2)^2} - \mathbb{V}^2 y^2 - \frac{2\lambda \sin^2\left(\frac{\phi}{2}\right)}{\left[\sigma^2 + (1 - \sigma^2) \sin^2\left(\frac{\phi}{2}\right)\right]} \\ &= h, \end{aligned} \quad (11)$$

64 where h is a constant.

65 System (10) is 2π -periodic in ϕ . Hence, the state (ϕ, y) can be viewed, for the
66 main properties, on a phase cylinder $S^1 \times R$, where $S^1 =] - \pi, \pi]$ and $-\pi$ is
67 identified with π (see Fig. 3).

68 Without losing generality, we will assume in the following that σ is fixed,
69 which leads to consider the systems (8) and (10) as four-parameter systems
70 depending on $(B, \mathbb{V}, P, \lambda)$. The role of σ will be discussed later.

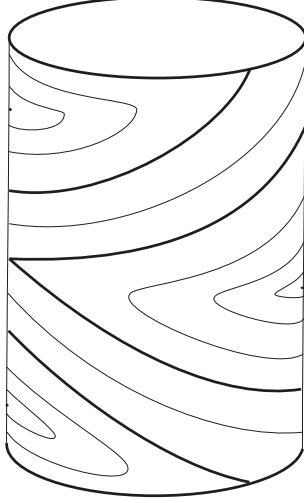


Fig. 3. The phase cylinder $S^1 \times R$ of system (10).

71 Obviously, $y = G(\phi)$ can not be derived explicitly from (11). Therefore, it is
 72 not possible to obtain explicit solutions by solving equation (7). However with
 73 the qualitative theory of differential equations, Eq. (7) can be solved and all
 74 related informations to its solutions can be derived, as well as phase portraits
 75 [20,21].

Let $Y = (y - P)^2$. For $\mathbb{V} \neq 0$, to find the zeros of function $f(y)$ is equivalent
 to find the roots of the cubic polynomial

$$F(Y) = Y^3 + \frac{3}{B}Y^2 + \frac{3}{B^2} \left(1 + \frac{1}{\mathbb{V}^2}\right)Y + \frac{1}{B^3} \left(1 - \frac{1}{\mathbb{V}^2}\right) \quad (12)$$

$$\equiv Y^3 + a_2Y^2 + a_1Y + a_0.$$

76 Let $q = \frac{1}{3}a_1 - \frac{1}{9}a_2^2$, $r = \frac{1}{6}(a_1a_2 - 3a_0) - \frac{1}{27}a_2^3$. Then, the discriminant $S =$
 77 $q^3 + r^2$ of cubic equation $F(Y) = 0$ is $S = \frac{1 + 4\mathbb{V}^2}{\mathbb{V}^6 B^6} > 0$. It follows that $F(Y)$
 78 has only one real root $Y = Y_0$.

79 If and only if $a_0 < 0$, i.e., $0 < \mathbb{V} < 1$, the only real root Y_0 of cubic polynomial
 80 $F(Y)$ is positive. Then $f(y)$ has two real roots y_1 and y_2 .

81 Therefore, for any $P \neq 0$, looking for the equilibrium points of system (10),
 82 we have the following conclusions:

- 83 (1) **For the static ($\mathbb{V} = 0$) or subsonic velocity ($0 < \mathbb{V} < 1$) cases:**
 84 For $\mathbb{V} = 0$, $f(y)$ becomes zero for $y_1 = P + r_{inf}$, and $y_2 = P - r_{inf}$,
 85 while if $0 < \mathbb{V} < 1$, $y_1 = P - \sqrt{Y_0}$ and $y_2 = P + \sqrt{Y_0}$.
 86 Then, for ($0 \leq \mathbb{V} < 1$), there exist six equilibrium points of system (10)
 87 in the phase cylinder, at $O(0,0)$, $A(\pi,0)$, $B_1(0,y_1)$, $B_2(\pi,y_1)$, $C_1(0,y_2)$

88 and $C_2(\pi, y_2)$. On the straight lines $y = y_1$ and $y = y_2$, $\frac{dy}{du}$, defined
 89 by system (8), becomes infinite. These two straight lines are then two
 90 singular straight lines for system (8).

91 (2) **For the case of sonic velocities** ($\mathbb{V} = 1$).

92 There exist four equilibrium points of system (10) at $O(0, 0)$, $A(\pi, 0)$,
 93 $B_1(0, P)$ and $B_2(\pi, P)$. On the straight lines $y = P$, $\frac{dy}{du}$, defined by
 94 system (8), becomes infinite. Thus, the straight line $y = P$ is a singular
 95 straight line for system (8).

96 (3) **For the case of supersonic velocities** ($1 < \mathbb{V} < \infty$).

97 There exist two equilibrium points of system (10) at $O(0, 0)$ and $A(\pi, 0)$.
 98 In this case, because $f(y)$ has no real zero, system (8) is a regular dy-
 99 namical system.

100 We notice that $f(0) = -\frac{\mathbb{V}^2 F(P^2) B^3}{(1 + BP^2)^3}$.

101 (a) When $1 < \mathbb{V} < \infty$, $F(P^2) > 0$ and $f(0) < 0$.

102 (b) When $0 < \mathbb{V} < 1$, if $P^2 < Y_0$, then $f(0) > 0$, while if $P^2 > Y_0$, then
 103 $f(0) < 0$.

Let $M(\phi_i, y_j)$ be the matrix of the linearized approximation of system (10) at the equilibrium point (ϕ_i, y_j) . Then, we have

$$\det M(\phi_i, y_j) = -Q'(\phi_i)[y_j f'(y_j) + f(y_j)],$$

where

$$f'(y) = \frac{12B(y - P) [B(y - P)^2 - 1]}{[B(y - P)^2 + 1]^4},$$

104 and $\text{Trace } M(\phi_i, y_j) = 0$.

Thus,

$$J(0, 0) = \det M(0, 0) = -\frac{\lambda}{2\sigma^2}f(0),$$

$$J(\pi, 0) = \det M(\pi, 0) = \frac{\lambda\sigma^2}{2}f(0),$$

$$J(0, y_j) = \det M(0, y_j) = -\frac{\lambda}{2\sigma^2}y_j f'(y_j),$$

$$J(\pi, y_j) = \det M(\pi, y_j) = \frac{\lambda\sigma^2}{2}y_j f'(y_j), \quad j = 1, 2.$$

105 The theory of planar dynamical systems gives the following results for an
106 equilibrium point of a planar integrable system:

- 107 • if $J < 0$, then the equilibrium point is a saddle point;
- 108 • If $J > 0$ then it is a center point;
- 109 • if $J = 0$ and the Poincaré index of the equilibrium point is 0, then this
110 equilibrium point is a cusp [19,20].

For the Hamiltonian values defined by (11), we have

$$h_0 = H(0, 0) = 0,$$

$$h_1 = H(\pi, 0) = -2\lambda,$$

$$h_{2,3} = H(0, y_{1,2}) = \frac{(1 - 3BP^2)y_{1,2}^2 + 4BP y_{1,2}^3 - B y_{1,2}^4}{(1 + BP^2)(1 + B(y_{1,2} - P)^2)^2} - \mathbb{V}^2 y_{1,2}^2,$$

$$h_{4,5} = H(\pi, y_{1,2}) = \frac{(1 - 3BP^2)y_{1,2}^2 + 4BP y_{1,2}^3 - B y_{1,2}^4}{(1 + BP^2)(1 + B(y_{1,2} - P)^2)^2} - \mathbb{V}^2 y_{1,2}^2 - 2\lambda.$$

111 3 The phase portraits of the model

112 Note first that the case P and $-P$ are symmetric with respect to the ϕ -axe,
113 which allows us to restrict our study to $P \geq 0$.

114 In order to solve the four-parameter systems (8) or (10), and to draw the
115 portrait phases, we will discuss according the different cases $\mathbb{V} = 0$, $0 <$
116 $\mathbb{V} < 1$, $\mathbb{V} = 1$ and $V > 1$. Using the above informations to perform
117 qualitative analysis, we also represent in phase portraits the corresponding
118 singular straight lines. They intersect transversally some families of orbits.

119 We should emphasize that when these singular straight lines of system (8)

120 exist, the vector fields defined by system (8) and system (10) are different. If
 121 an orbit of system (8) intersect a singular straight line $y = y_j$, that is $f(y_j) = 0$
 122 in a particular point (ϕ_0, y_j) , where ϕ_0 is not a multiple of π , then at
 123 this point, the second derivative $\frac{d^2\phi}{du^2}$ of $\phi(u)$ becomes infinite as shown
 124 by equation (7). The direction of vector field defined by system (8) changes
 125 rapidly to the inverse direction of the vector field defined by system (10). This
 126 lets the wave solution $\phi(u)$ of (7) associated with system (8) degenerate into
 127 a breaking solution, meaning physically a dislocation of the particle lattice, as
 128 we will show now.

Considering first the static case $\mathbb{V} = 0$, the coupling potential V_c can be rewritten as

$$\sum_i V_c(r_i) = \sum_i \frac{V_0 B (\phi_{i+1} - \phi_i - P)^2}{1 + B(\phi_{i+1} - \phi_i - P)^2}. \quad (13)$$

Deriving V_c versus ϕ_i gives the resulting force from the lattice on particle number i , namely

$$F_i = -2V_0 B \left[\frac{\phi_{i+1} - \phi_i - P}{[1 + B(\phi_{i+1} - \phi_i - P)^2]^2} - \frac{\phi_i - \phi_{i-1} - P}{[1 + B(\phi_i - \phi_{i-1} - P)^2]^2} \right] \quad (14)$$

Using the continuous media approximation $(\phi_{i+1} - \phi_i \rightarrow \frac{\partial\phi}{\partial X})$ gives

$$F_i = -2V_0 B \frac{\partial^2\phi}{\partial X^2} \left[\frac{1 - 3B \left(\frac{\partial\phi}{\partial X} - P \right)^2}{\left[1 + B \left(\frac{\partial\phi}{\partial X} - P \right)^2 \right]^3} \right] \quad (15)$$

129 This result shows that when $\frac{\partial\phi}{\partial X} = P \pm \frac{1}{\sqrt{3B}}$, the resulting force on particle i
 130 from the lattice vanishes. The attraction on particle i from the left neighbors is
 131 exactly balanced by the one from the right neighbors, resulting on a breaking
 132 of the lattice cohesion.

133 Note that the particle remains submitted to the substrate potential $V_e(x_i)$ (see
 134 Eq. (4)) but the links to their neighbors are broken.

135 Finally, with $y = \frac{\partial\phi}{\partial X} = \frac{d\phi}{du}$, (we recall that $u = X - vT = X$ for $\mathbb{V} = 0$),
 136 the lattice submits a dislocation when $f(y) = 0$, that is on the singular
 137 straight lines.

138 When $\mathbb{V} \neq 0$, the coupling potential must be replaced by an effective one,

139 taking into consideration the propagative variable $u = X - vT$. Then, dislo-
 140 cations occur when the slope $y = \frac{d\phi}{du}$ reaches one of the roots of $f(y) = 0$,
 141 when they exist. In the phase portraits, this corresponds to any intersection
 142 of the orbits with the singular straight lines, except at the minima of the
 143 potential $V_e(\phi_i)$.

144 (i) **Phase portraits in the static case ($\mathbb{V} = 0$) with $P = 0$**

145 For a given B , we have $y_1 = -y_2 = -\frac{1}{\sqrt{3B}}$, $h_2 = h_3 = \frac{1}{8B}$, $h_4 =$
 146 $h_5 = \frac{1}{8B} - 2\lambda$. Hence, when $\lambda = \frac{1}{16B}$, it follows that $h_4 = h_5 = h_0 = 0$.
 147 When λ increases, the cases $h_1 < 0 < h_4 < h_2$, $h_1 < h_4 = 0 < h_2$ and
 148 $h_1 < h_4 < 0 < h_2$ are successively obtained, as shown in Fig.4.

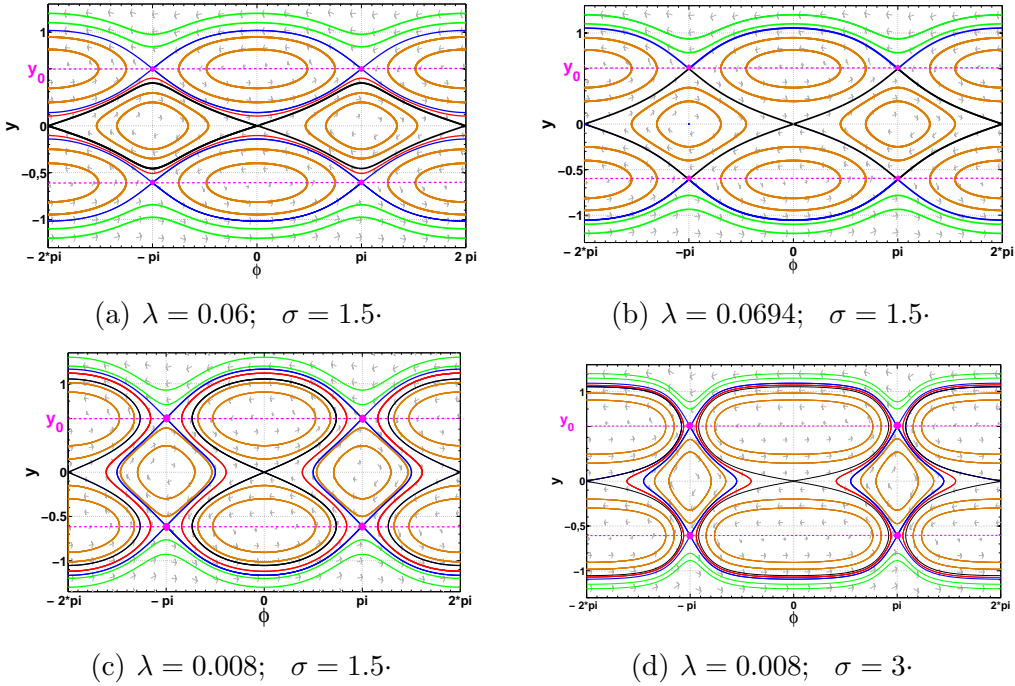


Fig. 4. The bifurcation of phase portraits of system (10) for $\mathbb{V} = 0$, $P = 0$, $B = 0.9$,
 (a) $\lambda < \frac{1}{16B}$, (b) $\lambda = \frac{1}{16B}$, (c) $\lambda > \frac{1}{16B}$ and (d) $\lambda > \frac{1}{16B}$ but σ is now 3.

149 (ii) **Phase portraits of the subsonic velocity case ($0 < \mathbb{V} < 1$)**

150 In this case, there exist a fixed positive root Y_0 of $F(Y)$.

151 If $P = 0$, $y_{1,2} = \pm\sqrt{Y_0}$ and $h_2 = h_3 = \frac{Y_0(1 - BY_0)}{4(1 + BY_0)^2} - \mathbb{V}^2 Y_0$, $h_4 =$
 152 $h_5 = \frac{Y_0(1 - BY_0)}{4(1 + BY_0)^2} - \mathbb{V}^2 Y_0 - 2\lambda$. Let $\lambda_b = \frac{1}{2} \left(\frac{Y_0(1 - BY_0)}{4(1 + BY_0)^2} - \mathbb{V}^2 Y_0 \right)$. It
 153 is easy to see that respectively the case $\lambda < \lambda_b$ ($\lambda = \lambda_b$, then $\lambda > \lambda_b$)
 154 leads to $h_1 < 0 < h_4 < h_2$ (respectively $h_1 < h_4 = 0 < h_2$, then

155
156

$h_1 < h_4 < 0 < h_2$), while these different phase portraits are shown in Fig.5.

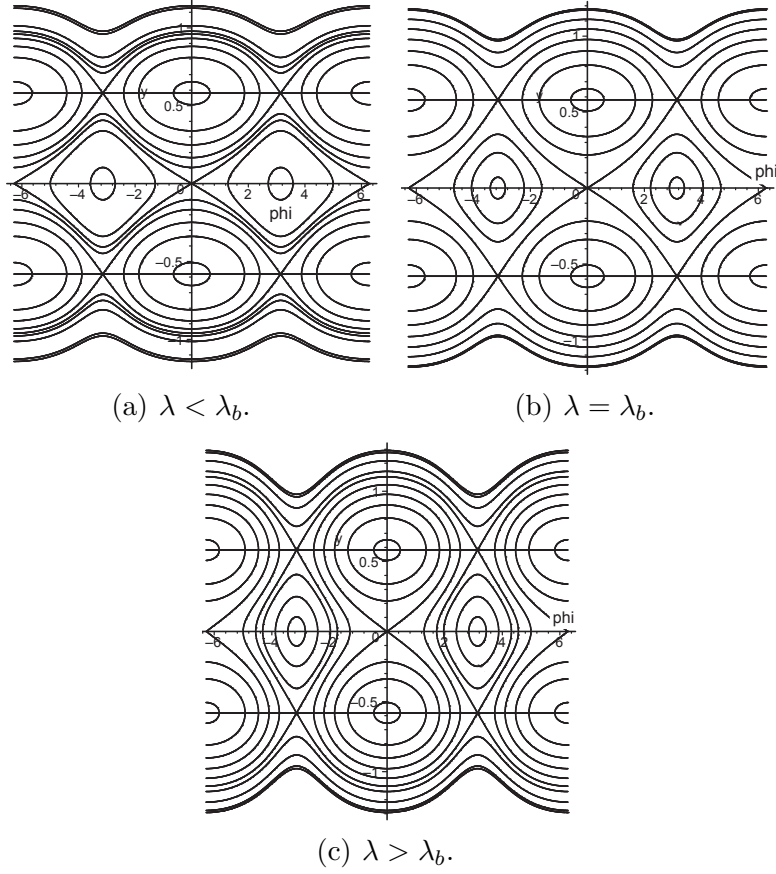


Fig. 5. The bifurcation of phase portraits of system (10) for $P = 0$, $0 < \mathbb{V} < 1$. (Parameters: $B = 0.9$, $\mathbb{V} = 0.2$, $\sigma = 1.5$, (a) $\lambda = 0.025$, (b) $\lambda = \lambda_b = 0.03118632263$, (c) $\lambda = 0.05$.)

157 If $P = \sqrt{Y_0}$, then $y_1 = 0$, $y_2 = 2Y_0$, $F(P^2) = 0$. Consequently $f(0) = 0$
158 and the two equilibrium points $O(0, 0)$ and $A(\pi, 0)$ are cusps.

159 When P varies from 0 to ∞ , there exists a value $P = P_b$ such
160 that $h_2 = h_5$. On Fig. 6, we present the phase portraits highlighting
161 the bifurcations obtained when the parameter P increases.

162 (iii) **Phase portraits in the sonic velocity case ($\mathbb{V} = 1$)**

163 We have now $Y_0 = 0$, and $y_1 = y_2 = P$. The two equilibrium points
164 $B_1(0, P)$ and $B_2(\pi, P)$ are cusps, and $h_{2,3} = -\frac{BP^4}{1 + BP^2}$. Thus, for a
165 given parameter pair (B, λ) , $h_1 = h_{2,3}$ when $P = P_a \equiv \lambda + \sqrt{\lambda^2 + \frac{2\lambda}{B}}$;
166 we see in Fig. 7 the phase portraits for $P < P_a$, $P = P_a$ and $P > P_a$.

167 (iv) **Phase portraits in the supersonic velocity case ($V > 1$)**

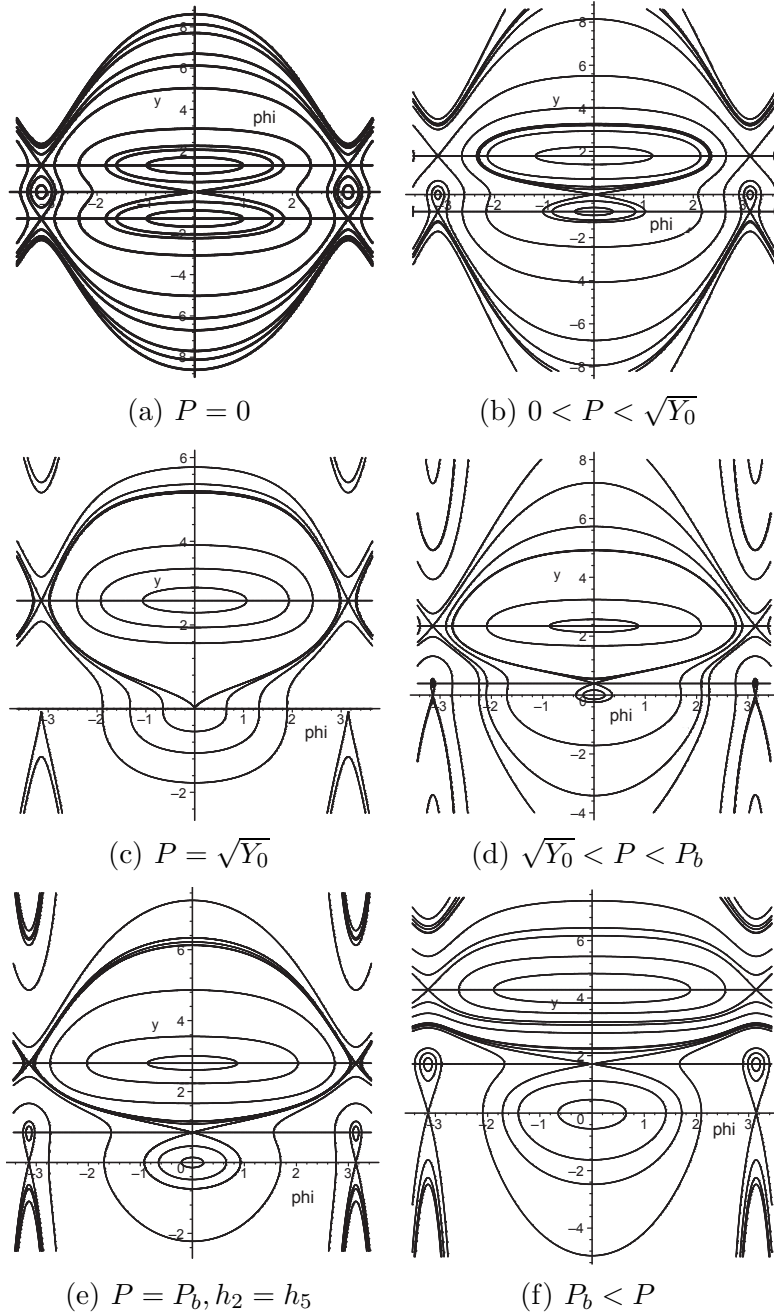


Fig. 6. The bifurcation of phase portraits of system (10) for $0 < V < 1$ and for $B = 0.9$, $\sigma = 1.5$, $\lambda = 0.03$.

168 The phase portraits are simple without singular lines. As shown on Fig.8,
 169 no bifurcation occurs when P or λ are varied.

170 In this section, we have studied the main characteristics of the phase portraits
 171 for the systems (8) and (10). We notice at this stage that the deformability
 172 parameter σ acting on substrate potential (4) does not play an important role.
 173 In fact, it only slightly and qualitatively changes the shapes of the trajectories

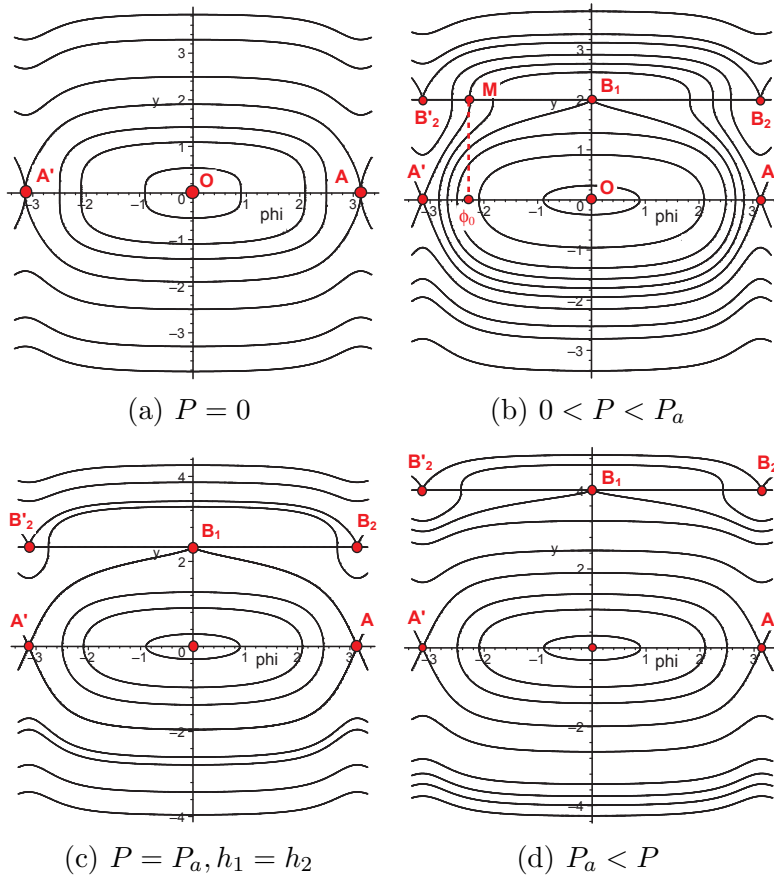


Fig. 7. The bifurcation of phase portraits of system (10) for $\mathbb{V} = 1$ (sketch for any given parameter set (B, λ)).

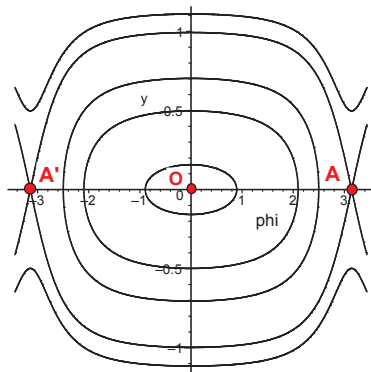


Fig. 8. Example of phase portraits of system (10) for $\mathbb{V} > 1$ (sketch for any given parameter set (B, P, λ)).

174 in the different phase portraits (compare Fig. 4(c) and Fig. 4(d)) where only
 175 the parameter σ is varied.

176 **4 Dynamics of the solutions of equation (7)**

177 In this section, we consider the solutions of equation (7) giving the wave
 178 profile $\phi(u)$ and corresponding to some trajectories of the phase portraits of
 179 section 3. For a sake of simplicity, we will consider the velocity as a decreasing
 180 parameter.

181 **(I) Case of supersonic velocities: $V > 1$. (see Fig. 8) .**

182 In this case, the function $f(y)$ has no real zero. Thus, systems (8) and
 183 (10) lead both to the phase portraits of Fig. 8 for any kind of parameter
 184 set (B, P, λ) .

185 (i) Corresponding to the family of closed orbits defined by $H(\phi, y) =$
 186 $h, h \in] - 2\lambda, 0[$, there exists a family of oscillating periodic solu-
 187 tions of equation (7), as depicted in Fig.9(a).

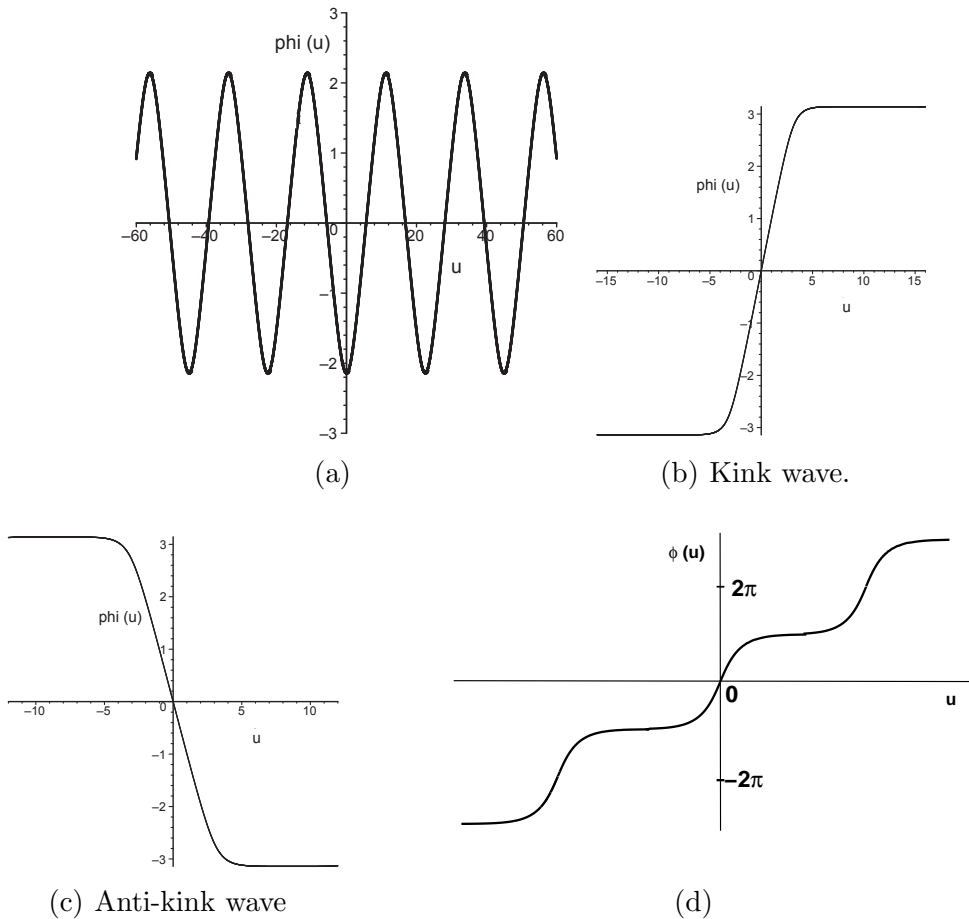


Fig. 9. Solution wave $\phi(u)$ of equation (7) corresponding to the case $V > 1$. The abscisse scale is arbitrary. [The initial values for the pair (ϕ, y) are respectively: a) $(-2.4, 0)$. b) $(-\pi, 0)$. c) $(\pi, 0)$. d) $(-3\pi, 0.1)$.]

- 188 (ii) Corresponding to heteroclinic orbits joining points A to A' in
 189 Fig.8, such as $H(\phi, y) = -2\lambda$, there exist two kink solutions as
 190 shown in Fig. 9(b) and 9(c).
 191 (iii) Corresponding to open orbits defined by $H(\phi, y) = h$, $h \in] -$
 192 $\infty, -2\lambda[$, there exist rotating solutions of equation (7) with only
 193 periodic changes of positive slope y . Although these solutions may
 194 not correspond to realistic behaviors in the context of lattice systems,
 195 they are represented on figure 9(d).

196 (II) **Case of sonic velocities** ($\mathbb{V} = 1$, see Fig. 7).

- 197 (II-1) If $P = 0$ (see Fig. 7(a)), there exist only the equilibrium points
 198 O , A and A' , and the families of solutions for Eq. (7) are the
 199 same as the ones reported in § 4(I).
 200 (II-2) If $0 < P < P_a$ (see Fig. 7(b)) or $P = P_a$ (Fig. 7(c)) we have
 201 $h_4 < h_1 < h_2 < 0$ or $h_4 < h_1 = h_2 < 0$ respectively.

- 202 • Corresponding to the family of closed orbits defined by $H(\phi, y) =$
 203 h , $h \in]h_2, 0[$, there exists a family of oscillating periodic
 204 solutions of Eq. (7) similar to those shown on Fig. 9(a).
 205 • Corresponding to the family of closed orbits defined by
 206 $H(\phi, y) = h$, $h \in]h_1, h_2]$, there exists a family of oscillating
 207 periodic solutions of Eq. (7). But these orbits intersect the
 208 singular straight line $y = P$ first for $-\pi < \phi < 0$. For
 209 example, the orbit starting from the point $A'(\phi = -\pi, y = 0)$
 210 in Fig. 7(b) intersects the singular straight line at point M .
 211 The wave profile, as shown on Fig. 10 leads to the lattice
 212 dislocation when $\phi = \phi_0$ and $y = P$. At this point, this
 213 gives rise to a breaking solutions $\phi(u)$.

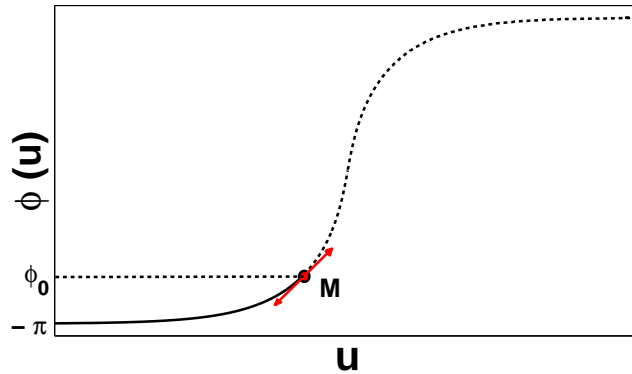


Fig. 10. Sketch of a dislocation. Starting from the point A' in Fig. 7(b), the orbit intersects the singular straight line at point M , when the slope $\frac{d\phi(u)}{du} = P$. (Same parameter set as for Fig. 7).

- 214 • Considering the two heteroclinic orbits linking the equi-

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librium points A and A' [$H(\phi, y) = h_1$], the upper one intersects also the singular straight line $y = P$ first for $-\pi < \phi < 0$ (behavior similar to the one depicted in Fig. 10) while the lower one is a standard anti-kink wave (similar to Fig. 9(c)).

- Corresponding to $H(\phi, y) < h_1$, there exist different families of trajectories. For the lower ones, the profiles $\phi(u)$ resemble to the curves on Fig. 9(d) while the upper ones intersect the singular straight line $y = P$ and correspond to breaking wave solution as on Fig. 10.

(II-3) $P > P_a$. In this case $h_4 < h_2 < h_1 < 0$. In addition to the previous case, we notice the presence of a heteroclinic orbit joining the points A' and A , corresponding to a kink wave solution $\phi(u)$ similar to the curve on Fig. 9(b).

(III) **Case of subsonic velocities** ($0 < \mathbb{V} < 1$).

(III-1) Let us consider first the case $P = 0$ (see Fig. 5 (a, b, c)).

Some families of orbits do not intersect the singular straight lines $y = y_1 = \sqrt{Y_0}$ and $y = y_2 = -\sqrt{Y_0}$. There are limit cycles around the equilibrium points A' or A . The corresponding wave solution $\phi(u)$ of Eq. (7) is similar to those depicted on Fig. 9(a), with only a difference concerning the ordinate values.

We notice even, when $\lambda < \lambda_b$ (Fig. 5(a)), that a kink (antikink) solution links the point O to the point $O'(2\pi, 0)$, with the corresponding wave solutions $\phi(u)$ sketched in Fig. 11. Note here the influence of parameter σ on the shape of the wave $\phi(u)$: it only concerns the width of the kink (compare Fig. 11 (a), (b) and (d)).

The other families of orbits in this case all intersect one of the singular straight lines. They correspond thus to breaking waves solution $\phi(u)$, with the dislocation happening precisely when their slope $y = \frac{d\phi}{du}$ reaches either $\sqrt{Y_0}$ or $-\sqrt{Y_0}$.

(III-2) $0 < \mathbb{V} < 1$ and $P \neq 0$ (see Fig. 6(a)→ 6(f)).

In this case, almost all orbits of systems (8) and (10) intersect the singular straight lines $y = P + \sqrt{Y_0}$ and $y = P - \sqrt{Y_0}$, giving rise to breaking solutions $\phi(u)$. The only difference between these waves stands on the position of the initial condition.

A few orbits do not intersect the singular straight lines. They correspond to periodic oscillations around the equilibrium points A' and A (if $P \leq \sqrt{Y_0}$) and around the equilibrium point O (if $P > P_b$). In this later case, an heteroclinic orbit linking A to A' exist, whose corresponding $\phi(u)$ is an anti-kink wave.

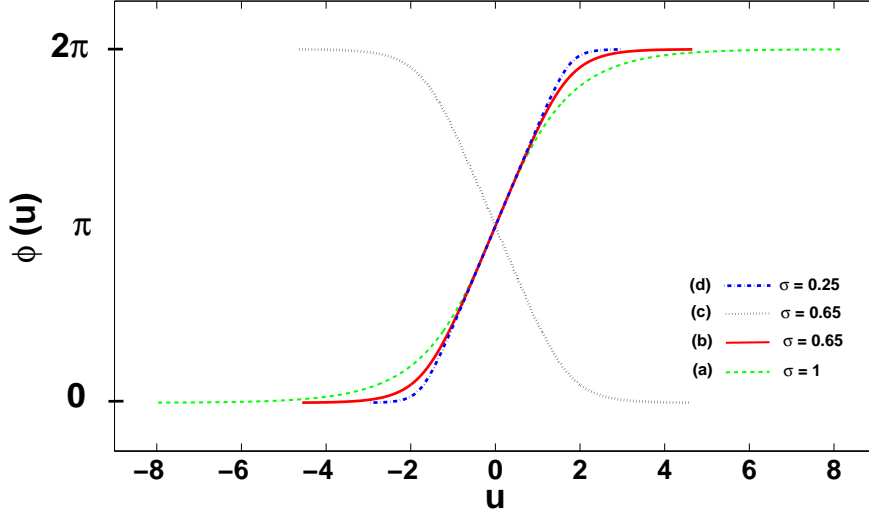


Fig. 11. The kink wave and anti-kink wave of Eq (7) for $0 < \mathbb{V} < 1$ (Parameters: $P = 0$, $\mathbb{V} = 0.2$, $B = 0.9$, $\lambda = 0.025$. For (a), (b) and (d) $\phi(-\infty) = 0$, $y(-\infty) = 0$ but σ takes different values. For (c) $\phi(-\infty) = 2\pi$, $y(-\infty) = 0$).

256 5 Conclusion

257 In summary, we have studied the dynamic behaviors of traveling wave solu-
 258 tions of the generalized Frenkel-Kontorova model. Our model extends previ-
 259 ous lattice soliton-bearing one-dimensional models discussed in the literature,
 260 by considering the contribution of anharmonic couplings. Such couplings are
 261 achieved through a non-convex interaction potential and a deformable sub-
 262 strate potential. In the continuum limit, a Singular Nonlinear Equation of the
 263 Second Class governs the displacement of the particles. Under different para-
 264 metric conditions, we have shown that various sufficient conditions lead to the
 265 existence of a rich diversity of solitary and periodic wave solutions. Another
 266 effect is related to an opening of phase trajectories of the system taking place
 267 beyond some threshold values of the lattice parameters, and depending on
 268 the velocity of the soliton patterns. Analytical expression for the breakdown
 269 threshold has been derived although it is quite indifferent of the deformabil-
 270 ity of the substrate (σ being not a bifurcation parameter). The deformability
 271 of substrate in such model gradually has only an influence on the width of
 272 the wave solutions. The parameter P , as well as other defects, may locally
 273 influence the breakdown threshold and thus plays a major role with respect
 274 to nonlinear excitations in such systems.

275 The bifurcations and dynamic behaviors of traveling wave solutions can be
 276 suitable for another nonlinear wave equation. Although it is tremendously
 277 challenging to analytically determine the explicit travelling wave solutions,
 278 our preliminaries results suggest that they can be combined with a bifurca-
 279 tion method in order to provide an insight of at least implicit solutions. The

280 findings reported in this paper might be of critical impacts in several areas of
281 applied sciences. For instance, they can help to explain the formation of cracks
282 originating from dislocations that are observed in semiconductor heterostruc-
283 tures [1,14–16]. Moreover, they can help to better understand the break up
284 (rupture) of a stretched polymer chain by pulling and its relation to soliton
285 destruction [22].

286 References

- 287 [1] Braun OM, Kivshar YS. “*The Frenkel-Kontorova Model, Concepts, Methods,*
288 *and Applications*”. Springer-Verlag: Heidelberg New York; 2004.
- 289 [2] Yuan XP, Zheng ZG. “*Nonlinear Dynamics of a Sliding Chain in a Periodic*
290 *Potential*”. Chin Phys Lett (2007); **24**: 2513.
- 291 [3] Vanossi A, Braun OM. “*Driven dynamics of simplified tribological models*”, J
292 Phys: Condens Matter (2007); **19**: 305017.
- 293 [4] Li BW, Wang L, Casati G. “*Negative differential thermal resistance and thermal*
294 *transistor*”. Appl Phys Lett (2006); **88**: 143501.
- 295 [5] Li BW, Lan JH, Wang L. “*Interface Thermal Resistance between Dissimilar*
296 *Anharmonic Lattices*” Phys Rev Lett (2005); **95**: 104302.
- 297 [6] Hu B, Yang L, Zhang Y. “*Asymmetric Heat Conduction in Nonlinear Lattices*”.
298 Phys Rev Lett (2006); **97**: 124302.
- 299 [7] Martínez PJ, Chacón R. “*Taming Chaotic Solitons in Frenkel-Kontorova*
300 *Chains by Weak Periodic Excitations*”. Phys Rev Lett (2004); **93**: 237006.
- 301 [8] Chacón R, Martínez PJ. “*Controlling Chaotic Solitons in Frenkel-Kontorova*
302 *Chains by Disordered Driving Forces*”. Phys Rev Lett (2007); **98**: 224102.
- 303 [9] Martínez PJ, Chacón R. “*Taming Chaotic Solitons in Frenkel-Kontorova*
304 *Chains by Weak Periodic Excitations*”. Phys Rev Lett (2004); **93**: 237006.
- 305 [10] Xu H, Chen WZ, Zhu YF. “*Influence of the bond defect in driven Frenkel-*
306 *Kontorova chains*”. Phys Rev B (2007); **75**: 224303.
- 307 [11] Tchakoutio Nguetcho AS, Bogning JR, Yemele D, Kofane TC. “*Kink*
308 *compactons in models with parametrized periodic double-well and asymmetric*
309 *substrate potentials*”. Chaos, Solitons and Fractals (2004); **21**: 165.
- 310 [12] Tchakoutio Nguetcho AS, Kofane TC. “*Soliton patterns and breakup thresholds*
311 *in hydrogen-bonded chains*”. Eur Phys J B (2007); **57**: 411.
- 312 [13] Milchev A, Markov I. “*The effect of anharmonicity in epitaxial interfaces: I.*
313 *Substrate-induced dissociation of finite epitaxial islands*”. Surf Sci (1984); **136**:
314 503.

- 315 [14] Braun OM, Kivshar YS. “*Nonlinear dynamics of the Frenkel-Kontorova model*”.
316 Phys Rep (1998); **306**: 1.
- 317 [15] Milchev A. “*Solitary waves in a frenkel-kontorova model with non-convex*
318 *interactions*”. Physica D (1990); **41**: 262.
- 319 [16] Malomed BA, Milchev A. “*Interaction of dislocations with a local defect in an*
320 *atomic chain with a nonconvex interparticle potential*”. Phys Rev B (1990); **41**:
321 4240.
- 322 [17] Li Jibin. “*Singular Nonlinear Traveling Wave Equations: Bifurcations and*
323 *Exact Solutions*”. Beijing: Science Press; (2013).
- 324 [18] Li Jibin, Shenjianwei. “*Travelling wave solutions in a model of the helix*
325 *polypeptide chains*”. Chaos Solitons Fractals (2004); **20**: 827.
- 326 [19] Li Jibin, Zhao Xiaohua, Chen Guanrong. “*On the Breaking Property for the*
327 *Second Class of Singular Nonlinear Traveling Wave Equations*”. Int J of
328 Bifurcation Chaos (2009); **19**: 1289.
- 329 [20] Asit Saha. “*Bifurcation of travelling wave solutions for the generalized KP-*
330 *MEW equations*”. Commun Nonlinear Sci Numer Simul (2012); **17**: 3539.
- 331 [21] Fang Yana, Haihong Liub, Zengrong Liu. “*The bifurcation and exact travelling*
332 *wave solutions for the modified BenjaminBonaMahoney (mBBM) equation*”.
333 Commun Nonlinear Sci Numer Simul (2012); **17**: 2824.
- 334 [22] Paturej J, Milchev A, Rostiashvili VG, Vilgis TA. “*Thermal degradation of*
335 *unstrained single polymer chain: Non-linear effects at work*”. J Chem Phys
336 (2011); **134**: 224901.