

# ON $(K_q, k)$ STABLE GRAPHS WITH SMALL $k$

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ABSTRACT. A graph  $G$  is  $(K_q, k)$  (vertex) stable if it contains a copy of  $K_q$  after deleting any subset of  $k$  vertices. We show that for  $q \geq 6$  and  $k \leq \frac{q}{2} + 1$  the only  $(K_q, k)$  stable graph with minimum size is isomorphic to  $K_{q+k}$ .

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## 1. INTRODUCTION

For terms not defined here we refer to [1]. As usually, the *order* of a graph  $G$  is the number of its vertices (it is denoted by  $|G|$ ) and the *size* of  $G$  is the number of its edges (it is denoted by  $e(G)$ ). The degree of a vertex  $v$  in a graph  $G$  is denoted by  $d_G(v)$ , or simply by  $d(v)$  if no confusion is possible. For any set  $S$  of vertices, we denote by  $G - S$  the subgraph induced by  $V(G) - S$ . If  $S = \{v\}$  we write  $G - v$  for  $G - \{v\}$ . When  $e$  is an edge of  $G$  we denote by  $G - e$  the spanning subgraph  $(V(G), E - \{e\})$ . The disjoint union of two graphs  $G_1$  and  $G_2$  is denoted by  $G_1 + G_2$ . The union of  $p$  mutually disjoint copies of a graph  $G$  is denoted by  $pG$ . A complete subgraph of order  $q$  of  $G$  is called a  $q$ -*clique* of  $G$ . The complete graph of order  $q$  is denoted by  $K_q$ . When a graph  $G$  contains a  $q$ -clique as subgraph, we say “ $G$  contains a  $K_q$ ”.

The following notion was introduced by Dudek et al. in [2].

**Definition 1.1.** Let  $H$  be a graph and  $k$  be a natural number. A graph  $G$  of order at least  $k$  is said to be a  $(H, k)$  *stable* graph if for any set  $S$  of  $k$  vertices the subgraph  $G - S$  contains a graph isomorphic to  $H$ .

By  $Q(H, k)$  we denote the size of a minimum  $(H, k)$  stable graph. It is clear that if  $G$  is an  $(H, k)$  stable graph with minimum size then the graph obtained from  $G$  by addition or deletion of some isolated vertices is also minimum  $(H, k)$  stable. Hence we shall assume that all the graphs considered in the paper have no isolated vertices. A  $(H, k)$  stable graph with minimum size shall be called a *minimum  $(H, k)$  stable graph*.

**Lemma 1.2.** [2] *Let  $q$  and  $k$  be integers,  $q \geq 2, k \geq 1$ . If  $G$  is  $(H, k)$  stable then, for every vertex  $v$  of  $G$ , the graph  $G - v$  is  $(H, k - 1)$  stable.*

**Proposition 1.3.** [2] *If  $G$  is a  $(H, k)$  stable graph with minimum size then every vertex as well as every edge is contained in a subgraph isomorphic to  $H$ .*

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**Proof.** Let  $e$  be an edge of  $G$  which is not contained in any subgraph of  $G$  isomorphic to  $H$ , then  $G - e$  would be a  $(H, k)$  stable graph with less edges than  $G$ , a contradiction. Let  $x$  be a vertex of  $G$  and  $e$  be an edge of  $G$  incident with  $x$ , since  $e$  is an edge of some subgraph isomorphic to  $H$ , say  $H_0$ , the vertex  $x$  is a vertex of  $H_0$ .  $\square$

## 2. PRELIMINARY RESULTS

We are interested by minimum  $(K_q, k)$  stable graphs (where  $q$  and  $k$  are integers such that  $q \geq 2$  and  $k \geq 0$ ). As a corollary to Proposition 1.3, every edge and every vertex of a minimum  $(K_q, k)$  stable graph is contained in a  $K_q$  (thus the minimum degree is at least  $q - 1$ ). Note that, for  $q \geq 2$  and  $k \geq 0$ , the graph  $K_{q+k}$  is  $(K_q, k)$  stable, hence  $Q(K_q, k) \leq \binom{q+k}{2}$ .

**Definition 2.1.** Let  $H$  be a non complete graph on  $q + t$  vertices ( $t \geq 1$ ). We shall say that  $H$  is a *near complete graph* when it has a vertex  $v$  such that

- $H - v$  is complete.
- $d_H(v) = q + r$  with  $-1 \leq r \leq t - 2$ .

The previous definition generalizes Definition 1.5 in [3] initially given for  $r \in \{-1, 0, 1\}$  and the following lemma generalizes Proposition 2.1 in [3].

**Lemma 2.2.** *Every minimum  $(K_q, k)$  stable graph  $G$ , where  $q \geq 3$  and  $k \geq 1$ , has no component  $H$  isomorphic to a near complete graph.*

**Proof.** Suppose, contrary to our claim, that  $G$  has such a component  $H$  and let  $v$  be the vertex of  $H$  such that  $H - v$  is a clique of  $G$ . Then,  $H$  has  $q + t$  vertices with  $q - 1 \leq d(v) \leq q + t - 2$  and  $d(v) = q + r$ . Since  $G$  is minimum  $(K_q, k)$  stable,  $G - v$  is  $(K_q, k - 1)$  stable and is not  $(K_q, k)$  stable. Then,  $G - v$  contains a set  $S$  with at most  $k$  vertices intersecting every subgraph of  $G - v$  isomorphic to a  $K_q$ . The graph  $G - S$  contains some  $K_q$  (at least one) and clearly every subgraph of  $G - S$  isomorphic to a  $K_q$  contains  $v$ . Since  $N(v)$  is a  $K_{q+r}$  and  $N(v) - S$  contains no  $K_q$ ,  $|N(v) - S| \leq q - 1$ . Since there exists a  $K_q$  containing  $v$  in  $H - S$ ,  $|N(v) - S| = q - 1$  (and hence  $|S \cap N(v)| = r + 1$ ). Since  $H - v - S$  contains no  $K_q$ ,  $H - v - S = N(v) - S$ . Let  $a$  be a vertex of  $H - v$  not adjacent to  $v$  and let  $b$  be a vertex in  $N(v) - S$ , and consider  $S' = S - \{a\} + \{b\}$ . We have  $|S'| \leq k$  and  $G - S'$  contains no  $K_q$ , a contradiction.  $\square$

It is clear that  $Q(K_q, 0) = \binom{q}{2}$  and the only minimum  $(K_q, 0)$  stable graph is  $K_q$ . It is an easy exercise to see that  $Q(K_2, k) = k + 1$  and that the matching  $(k + 1)K_2$  is the unique minimum  $(K_2, k)$  stable graph.

**Theorem 2.3.** [3] *Let  $G$  be a minimum  $(K_3, k)$  stable graph, with  $k \geq 0$ . Then  $G$  is isomorphic to  $sK_4 + tK_3$ , for any choice of  $s$  and  $t$  such that  $2s + t = k + 1$ .*

In [3] it was proved that if  $q \geq 4$  and  $k \in \{1, 2\}$  then  $Q(K_q, k) = \binom{q+k}{2}$  and the only minimum  $(K_q, k)$  stable graph is  $K_{q+k}$ . We have proved also that if  $q \geq 5$  then  $Q(K_q, 3) = \binom{q+3}{2}$  and the only minimum  $(K_q, 3)$  stable graph is  $K_{q+3}$ . Dudek, Szymański and Zwonek proved the following result.

**Theorem 2.4.** [2] *For every  $q \geq 4$ , there exists an integer  $k(q)$  such that  $Q(K_q, k) \leq (2q - 3)(k + 1)$  for  $k \geq k(q)$ .*

As a consequence of this last result, they have deduced that for every  $k \geq k(q)$   $K_{q+k}$  is not minimum  $(K_q, k)$  stable.

*Remark 2.5.* From now on, throughout this section we assume that  $q$  and  $k$  are integers such that  $q \geq 4$ ,  $k \geq 1$  and for every  $r$  such that  $0 \leq r < k$  we have  $Q(K_q, r) = \binom{q+r}{2}$  and the only minimum  $(K_q, r)$  stable graph is  $K_{q+r}$ .

In view of Theorem 2.4,  $k$  is bounded from above and we are interested to obtain the greatest possible value of  $k$ .

**Lemma 2.6.** *Let  $G$  be a  $(K_q, k)$  stable graph such that  $e(G) \leq \binom{q+k}{2}$ . Then either for every vertex  $v$  we have  $d(v) \leq q+k-2$  or  $G$  is isomorphic to  $K_{q+k}$ .*

**Proof.** Suppose that some vertex  $v$  has degree at least  $q+k-1$ . By Lemma 1.2 the graph  $G-v$  is  $(K_q, k-1)$  stable, hence  $Q(K_q, k-1) \leq e(G-v) = e(G) - d(v)$ . Since  $Q(K_q, k-1) = \binom{q+k-1}{2}$ , we have

$$\binom{q+k-1}{2} \leq e(G) - d(v) \leq \binom{q+k}{2} - (q+k-1) = \binom{q+k-1}{2}.$$

It follows that  $e(G-v) = \binom{q+k-1}{2}$ ,  $G-v$  is isomorphic to  $K_{q+k-1}$  and  $d(v) = q+k-1$ . Hence,  $G$  is isomorphic to  $K_{q+k}$ .  $\square$

**Lemma 2.7.** *Let  $G$  be a minimum  $(K_q, k)$  stable graph. Then one of the following statements is true*

- $G$  has no component isomorphic to  $K_q$
- $Q(K_q, k-1) + \binom{q}{2} \leq Q(K_q, k)$

**Proof.** Suppose that some component  $H$  of  $G$  is isomorphic to a  $K_q$ . If  $G-H$  is not  $(K_q, k-1)$  stable, then there is a set  $S$  with at most  $k-1$  vertices intersecting each  $K_q$  of  $G-H$ . Then, for any vertex  $a$  of  $H$ ,  $S+a$  intersects each  $K_q$  of  $G$  while  $S$  has at most  $k-1$  vertices, a contradiction. Hence  $G-H$  is  $(K_q, k-1)$  stable and we have  $Q(K_q, k-1) \leq e(G-H) = Q(K_q, k) - \binom{q}{2}$ .  $\square$

**Lemma 2.8.** [3] *Let  $G$  be a minimum  $(K_q, k)$  stable graph and let  $u$  be a vertex of degree  $q-1$ . Then one of the following statements is true*

- $\forall v \in N(u) \quad d(v) \geq q+1$
- $Q(K_q, k-1) + 3(q-2) \leq Q(K_q, k)$

**Proof.** By Proposition 1.3, since  $d(u) = q-1$ ,  $\{u\} + N(u)$  induces a complete graph on  $q$  vertices. Assume that some vertex  $w \in N(u)$  has degree  $q+r$  where  $r = -1$  or  $r = 0$ , and let  $v$  be a neighbour of  $u$  distinct from  $w$ . Since the degree of  $u$  in  $G-v$  is  $q-2$ , no edge incident with  $u$  can be contained in a  $K_q$  of  $G-v$ . Since  $G-v$  is  $(K_q, k-1)$  stable, we can delete the  $q-2$  edges incident with  $u$  in  $G-v$  and the resulting graph  $G'$  is still  $(K_q, k-1)$  stable. By deleting  $v$ , we have  $e(G-v) \leq e(G) - (q-1)$  and hence

$$e(G') \leq e(G) - (q-1) - (q-2).$$

In  $G'$ , the degree of  $w$  is now  $q+r-2$ . Hence, no edge incident with  $w$  in  $G'$  can be contained in a  $K_q$ . Deleting these  $q+r-2$  edges from  $G'$  leads to a graph  $G''$  which remains to be  $(K_q, k-1)$  stable. We get thus

$$Q(K_q, k-1) \leq e(G'') \leq e(G) - (q-1) - (q-2) - (q+r-2).$$

Since  $e(G) \leq Q(K_q, k)$ , the result follows.  $\square$

**Lemma 2.9.** *Let  $G$  be a minimum  $(K_q, k)$  stable graph, where  $1 \leq k \leq 2q - 6$ , and let  $v$  be a vertex of degree  $q - 1$ . Then for every vertex  $w \in N(v)$  we have  $d(w) \geq q + 1$ .*

**Proof.** Suppose, contrary to the assertion of the lemma, that  $d(w) \leq q$  for some vertex  $w \in N(v)$ . By Lemma 2.8, we have  $Q(K_q, k - 1) + 3(q - 2) \leq Q(K_q, k)$ . Since  $Q(K_q, k - 1) = \binom{q+k-1}{2}$  and  $Q(K_q, k) \leq \binom{q+k}{2}$  we have  $\binom{q+k-1}{2} + 3q - 6 \leq \binom{q+k}{2}$ . Then we obtain  $k \geq 2q - 5$ , a contradiction.  $\square$

**Lemma 2.10.** *Let  $G$  be a minimum  $(K_q, k)$  stable graph, where  $q \geq 5$  and  $1 \leq k \leq q - 1$ . Then the minimum degree of  $G$  is at least  $q$ .*

**Proof.** Suppose that there is a vertex  $v$  of degree  $q - 1$  and let  $w$  be a neighbour of  $v$ . Since  $q - 1 \leq 2q - 6$ , by Lemma 2.9,  $w$  has degree at least  $q + 1$ . By Lemma 1.2 the graph  $G - w$  is  $(K_q, k - 1)$  stable. In that graph  $v$  is not contained in any  $K_q$  since its degree is  $q - 2$ . Hence  $G - \{w, v\}$  is still  $(K_q, k - 1)$  stable. We have  $e(G - \{w, v\}) = e(G) - (d(v) + d(w) - 1) \leq e(G) - 2q + 1$ . Since  $Q(K_q, k - 1) = \binom{q+k-1}{2}$  and  $Q(K_q, k) \leq \binom{q+k}{2}$  we have  $\binom{q+k-1}{2} \leq e(G) - 2q + 1 \leq \binom{q+k}{2} - 2q + 1$ . It follows that  $k \geq q$ , a contradiction.  $\square$

**Lemma 2.11.** *Let  $G$  be a minimum  $(K_q, k)$  stable graph, where  $q \geq 5$  and  $1 \leq k \leq q - 1$ , and let  $v$  be a vertex of degree  $q$ . Then  $N(v)$  is complete.*

**Proof.** Assume, by contradiction, that  $v$  is a vertex of degree  $q$  and  $N(v)$  contains two non adjacent vertices  $a$  and  $b$ . Let  $w \in N(v)$  distinct from  $a$  and  $b$  ( $w$  must exist since  $q \geq 4$ ). By Lemma 1.2 the graph  $G - w$  is  $(K_q, k - 1)$  stable. In that graph  $v$  is not contained in a  $K_q$  since its two neighbours  $a$  and  $b$  are not adjacent. Hence  $G - \{w, v\}$  is still  $(K_q, k - 1)$  stable. By Lemma 2.10,  $d(w) \geq q$  and hence  $e(G - \{w, v\}) = e(G) - (d(v) + d(w) - 1) \leq e(G) - 2q + 1$ . We have, as in the proof of Lemma 2.10,  $\binom{q+k-1}{2} \leq e(G) - 2q + 1 \leq \binom{q+k}{2} - 2q + 1$ , and we obtain  $k \geq q$ , a contradiction.  $\square$

**Lemma 2.12.** *Let  $G$  be a minimum  $(K_q, k)$  stable graph, where  $q \geq 4$  and  $2 \leq k \leq \frac{q}{2} + 1$ , and let  $v$  be a vertex of degree at least  $q + 1$ . Then either  $N(v)$  induces a complete graph or there exists an ordering  $v_1, \dots, v_{d(v)}$  of the vertices of  $N(v)$  such that  $\{v_1, \dots, v_{q-1}\}$  induces a complete graph and  $v_{d(v)-1}v_{d(v)}$  is not in  $E(G)$ . Moreover, there exists a vertex  $w$  in  $\{v_1, \dots, v_{q-1}\}$  adjacent to  $v_{d(v)-1}$  and  $v_{d(v)}$ .*

**Proof.** Suppose that the subgraph induced by  $N(v)$  is not complete and let  $a$  and  $b$  be two non adjacent neighbours of  $v$ . Assume that every complete graph on  $q - 1$  vertices contained in  $N(v)$  intersects  $\{a, b\}$ . The graph  $G - \{a, b\}$  is  $(K_q, k - 2)$  stable. In that graph,  $v$  is not contained in a  $K_q$ , hence  $G - \{a, b, v\}$  is still  $(K_q, k - 2)$  stable. We have  $e(G - \{a, b, v\}) = e(G) - (d(v) + d(a) + d(b) - 2) \leq e(G) - 3q - 1$ . Since  $Q(K_q, k - 2) = \binom{q+k-2}{2}$  and  $Q(K_q, k) \leq \binom{q+k}{2}$ , we have  $\binom{q+k-2}{2} \leq e(G - \{a, b, v\}) \leq e(G) - 3q - 1 \leq \binom{q+k}{2} - 3q - 1$  and hence  $\frac{q}{2} + 2 \leq k$ , a contradiction with  $k \leq \frac{q}{2} + 1$ .

Thus,  $N(v) - \{a, b\}$  contains a  $K_{q-1}$  and we can order the vertices of  $N(v)$  in such a way that the  $q-1$  first ones  $v_1, \dots, v_{q-1}$  induce a complete graph and the two last vertices  $v_{d(v)-1}$  and  $v_{d(v)}$  are not adjacent, as claimed.

Set  $d(v) = q + r$  with  $r \geq 1$ . By Proposition 1.3, the edges  $vv_{q+r-1}$  and  $vv_{q+r}$  are contained in two distinct  $q$ -cliques, say  $Q_1$  and  $Q_2$ . Since  $v_{q+r-1}$  and  $v_{q+r}$  are not adjacent, each  $Q_i$  contains at most  $r$  vertices in  $N(v) - \{v_1, \dots, v_{q-1}\}$ . Hence, each  $Q_i$  must have at least  $q-r-1$  vertices in  $\{v_1 \dots v_{q-1}\}$ . Since  $N(v)$  is not complete and  $e(G) \leq \binom{q+k}{2}$ , by Lemma 2.6 we have  $r \leq k-2$ . Since  $k \leq \frac{q}{2} + 1$ ,  $Q_1$  (as well as  $Q_2$ ) has at least  $q-r-1 \geq q-k+1 \geq \frac{q}{2}$  vertices in  $\{v_1 \dots v_{q-1}\}$ . Hence  $Q_1$  and  $Q_2$  have at least one common vertex  $w$  in  $N(v)$ , and the lemma follows.  $\square$

**Lemma 2.13.** *Let  $G$  be a minimum  $(K_q, k)$  stable graph, where  $q \geq 5$  and  $2 \leq k \leq \frac{q}{2} + 1$ , and let  $H$  be a component of  $G$ . Assume that  $v$  is a vertex of maximum degree in  $H$ . Then either  $H$  is complete or the subgraph induced by  $N(v)$  contains no complete subgraph on  $d(v) - 1$  vertices.*

**Proof.** Assume that  $H$  is not complete. First, we prove that the maximum degree in  $H$  is at least  $q+1$ . If the minimum degree in  $H$  is at least  $q+1$ , we are done. If there exists a vertex  $u$  of degree  $q-1$  in  $H$  then, by Lemma 2.9, the degree of any vertex of  $N(u)$  is at least  $q+1$ . If there exists a vertex  $u$  of degree  $q$  then, by Lemma 2.11,  $N(u) \cup \{u\}$  induces a  $K_{q+1}$ , and hence, since  $H$  is connected, there exists a vertex in  $H - (N(u) \cup \{u\})$  having at least one neighbour  $w$  in  $N(u)$  and  $d(w) \geq q+1$ . Thus, the maximum degree in  $H$  is  $q+r$  with  $r \geq 1$ .

Since  $H$  is not complete and  $v$  is a vertex of maximum degree in  $H$ , the subgraph induced on  $N(v)$  is not complete. By Lemma 2.12, there exists an ordering  $\{v_1 \dots v_{q+r}\}$  of the vertices of  $N(v)$  such that  $\{v_1 \dots v_{q-1}\}$  induces a complete graph and  $v_{q+r-1}v_{q+r}$  is not an edge of  $G$ . If the subgraph induced by  $N(v)$  contains a complete subgraph on  $q+r-1$  vertices then without loss of generality we may suppose that it contains the vertex  $v_{q+r-1}$ .

Assume that  $\{v_1, \dots, v_{q+r-2}, v_{q+r-1}\}$  induces a complete graph. Since  $G$  is a minimum  $(K_q, k)$  stable graph, by Proposition 1.3, the edge  $vv_{q+r}$  must be contained in a  $K_q$ . Hence  $v_{q+r}$  has at least  $q-2$  neighbours in  $\{v_1, \dots, v_{q+r-2}\}$ . Let us denote by  $A$  this set of neighbours. Since the subgraph induced by  $(N(v) - \{v_{q+r}\})$  is complete, every vertex in  $A$  has degree  $q+r$  in  $G$  (let us say that these vertices are *saturated*). Henceforth, every vertex in  $A$  has no neighbour outside  $N(v) \cup \{v\}$ . By Lemma 2.2, the  $(q+r)$ -clique  $(N(v) - \{v_{q+r}\}) \cup \{v\}$  is a proper subgraph of  $H - \{v_{q+r}\}$ . Since  $H$  is connected, there exists a vertex  $w$  outside  $N(v) \cup \{v\}$  adjacent to a vertex  $u$  in  $N(v)$ . Clearly, the vertex  $u$  has also degree  $q+r$  and it has no other neighbour outside  $N(v) \cup \{v\}$  than  $w$ . The edge  $uw$  being contained in a  $K_q$  by Proposition 1.3,  $w$  must have at least  $q-2$  common neighbours with  $u$  in  $N(v)$ . Let us denote by  $B$  the set of neighbours of  $w$  in  $N(v)$ . It is easy to see that every vertex in  $B$  is saturated. Since  $A$  and  $B$  are disjoint, we have  $2q-3 \leq |A \cup B| \leq |N(v)| = q+r$  and hence  $q \leq r+3$ . Since  $r \leq k-2$  by Lemma 2.6, we have  $q \leq k+1$ . Thus, we obtain  $q \leq \frac{q}{2} + 2$ , which implies  $q \leq 4$ , a contradiction.

Hence the subgraph induced by the vertices  $\{v_1, \dots, v_{q+r-2}, v_{q+r-1}\}$  is not complete, and the lemma follows.  $\square$

**Proposition 2.14.** *Let  $G$  be a minimum  $(K_q, k)$  stable graph, where  $q \geq 6$  and  $2 \leq k \leq \frac{q}{2} + 1$ . Then every component of  $G$  is a complete graph.*

**Proof.** Let  $H$  be a component of  $G$  and  $v$  be a vertex of maximum degree in  $H$ . If the subgraph induced on  $N(v)$  is complete then  $H$  is obviously complete. We can thus suppose that  $N(v)$  is not a clique. By Lemmas 2.10 and 2.11, the minimum degree is at least  $q + 1$ , and hence  $d(v) = q + r$  with  $r \geq 1$ .

**Claim 2.14.1.** *The graph  $G - (N(v) \cup \{v\})$  is  $(K_q, k - r)$  stable.*

**Proof** By Lemma 2.12, we can consider an ordering  $v_1, \dots, v_{q+r}$  of  $N(v)$  such that the set  $\{v_1, \dots, v_{q-1}\}$  induces a  $K_{q-1}$ ,  $v_{q+r-1}v_{q+r} \notin E(G)$  and there is a vertex  $w \in \{v_1, \dots, v_{q-1}\}$  adjacent to  $v_{q+r-1}$  and  $v_{q+r}$ . By Lemma 2.13, we can find two non adjacent vertices  $a$  and  $b$  in  $N(v) - \{v_{q+r}\}$  and two non adjacent vertices  $c$  and  $d$  in  $N(v) - \{v_{q+r-1}\}$ . Let us note that since the set  $\{v_1, \dots, v_{q-1}\}$  induces a complete graph, it contains at most one vertex of the set  $\{a, b\}$  and at most one vertex of  $\{c, d\}$ . Then,  $|\{v_1, \dots, v_{q-1}\} \cap \{w, a, b, c, d\}| \leq 3$ .

Since  $H$  is not complete, the graph  $G$  is not complete and by Lemma 2.6 we have  $r \leq k - 2$ . Since  $k \leq \frac{q}{2} + 1$  and  $q \geq 6$ , there exists a subset  $A \subseteq \{v_1 \dots v_{q-1}\}$  such that

- $|A| = r$
- $w \notin A$
- $A \cap \{a, b, c, d\} = \emptyset$

By repeated applications of Lemma 1.2, the graph  $G_1$  obtained from  $G$  by deleting  $A$  is  $(K_q, k - r)$  stable. In  $G_1$ , the degree of  $v$  is equal to  $q$ .

Without loss of generality, suppose that  $a$  is distinct from  $v_{q+r-1}$  and  $c$  is distinct from  $v_{q+r}$ . If there exists a  $q$ -clique in  $G_1$  containing the edge  $vv_{q+r-1}$  then  $\{v_1, \dots, v_{q+r-2}\} - A$  is a  $(q - 2)$ -clique containing  $a$ . Since  $ab$  is not an edge, we must have  $b = v_{q+r-1}$ , a contradiction to the fact that  $av_{q+r-1}$  is an edge. Thus, there is no  $q$ -clique in  $G_1$  containing  $vv_{q+r-1}$ . Analogously, we prove that there is no  $q$ -clique in  $G_1$  containing  $vv_{q+r}$ .

Hence, the graph  $G_2$  obtained from  $G_1$  by deletion of the edges  $vv_{q+r-1}$  and  $vv_{q+r}$  is still  $(K_q, k - r)$  stable. In  $G_2$ ,  $v$  has degree  $q - 2$ , so it is not contained in any  $K_q$ . We can thus delete  $v$  and we get a  $(K_q, k - r)$  stable graph  $G_3$ .

Since the maximum degree in  $G$  is  $q + r$ , the degree of  $w$  in  $G_3$  is at most  $q - 1$ . Recall that  $w$  is adjacent to the two non adjacent vertices  $v_{q+r-1}$  and  $v_{q+r}$ . Hence  $w$  is not contained in any  $K_q$  of  $G_3$ , which means that  $G_4 = G_3 - w$  is still  $(K_q, k - r)$  stable. Since the degree of each vertex in  $\{v_1, \dots, v_{q-1}\} - (A \cup \{w\})$  is at most  $q - 2$  in  $G_4$ , none of these vertices can be contained in any  $K_q$  of  $G_4$ . Hence by deletion of these vertices we get again a  $(K_q, k - r)$  stable graph  $G_5$ . We shall prove that none of the  $r + 1$  vertices  $v_q, \dots, v_{q+r}$  is contained in a  $K_q$  of  $G_5$ .

Note that  $G_5 = G - \{v, v_1, \dots, v_{q-1}\}$ . For  $q \leq j \leq q + r$ , denote by  $d_j$  the degree of the vertex  $v_j$  in the subgraph induced by  $\{v_q, \dots, v_{q+r}\}$ . Clearly we have  $0 \leq d_j \leq r$ . In  $G$ , by Proposition 1.3, the edge  $vv_j$  is contained in a  $K_q$ . Hence  $v_j$  is adjacent (in  $G$ ) to at least  $q - 2 - d_j$  vertices in  $\{v_1, \dots, v_{q-1}\}$ . Since we have deleted the vertex  $v$  and the vertices  $v_1, \dots, v_{q-1}$ , we have thus  $d_{G_5}(v_j) \leq q + r - (q - 2 - d_j) - 1 = r + 1 + d_j$ . If  $d_j \leq r - 1$  then  $d_{G_5}(v_j) \leq 2r \leq 2(k - 2) \leq q - 2$  and there is no  $K_q$  in  $G_5$  containing  $v_j$ . The equality  $d_{G_5}(v_j) = q - 1$  can only be obtained when  $d_j = r$ , that is  $v_j$  has  $r$  neighbours in  $v_q \dots v_{q+r}$ . Since  $v_{q+r-1}$  and  $v_{q+r}$  are not adjacent,  $v_j$  is not contained in any  $K_q$  of  $G_5$ .

Hence, the graph  $G_6 = G - (N(v) \cup \{v\})$  obtained from  $G_5$  by deletion of all the vertices  $v_q, v_{q+1}, \dots, v_{q+r}$  is still  $(K_q, k-r)$  stable, and the claim follows.  $\square$

**Claim 2.14.2.**

$$(2.1) \quad \binom{q+k-r}{2} + q+r + \binom{q-1}{2} + \frac{1}{2}(r+1)(2q-r-2) + 1 \leq \binom{q+k}{2}$$

**Proof** To get back  $G$  from  $G - (N(v) \cup \{v\})$  we add, at least

- the  $q+r$  edges incident with  $v$ ,
- the  $\binom{q-1}{2}$  edges of the  $(q-1)$ -clique induced by the set  $\{v_1 \dots v_{q-1}\}$ ,
- the edges incident with  $v_q, \dots, v_{q+r}$  and not incident with  $v$ .

Let  $l$  be the number of edges incident with  $v_q, \dots, v_{q+r}$  and not incident with  $v$ . We have

$$(2.2) \quad e(G - (N(v) \cup \{v\})) + q+r + \binom{q-1}{2} + l \leq e(G)$$

In order to find a lower bound of the number of edges incident with the vertices  $v_q, \dots, v_{q+r}$ , for each  $i \in \{q, \dots, q+r\}$  let us denote by  $d_i$  the degree of the vertex  $v_i$  in the subgraph induced by the set  $\{v_q, \dots, v_{q+r}\}$ . Then,

$$l = \frac{1}{2} \sum_{i=q}^{q+r} d_i + \sum_{i=q}^{q+r} (d_G(v_i) - 1 - d_i) = \sum_{i=q}^{q+r} d_G(v_i) - (r+1) - \frac{1}{2} \sum_{i=q}^{q+r} d_i .$$

Since by Lemma 2.10 the minimum degree in  $G$  is at least  $q$ , we have

$$l \geq q(r+1) - (r+1) - \frac{1}{2} \sum_{i=q}^{q+r} d_i .$$

Since for every  $i$  in  $\{q, \dots, q+r-2\}$   $d_i \leq r$ ,  $d_{q+r-1} \leq r-1$  and  $d_{q+r} \leq r-1$ , we obtain

$$l \geq q(r+1) - (r+1) - \frac{1}{2}r(r-1) - (r-1) ,$$

and hence

$$l \geq \frac{1}{2}(r+1)(2q-r-2) + 1 .$$

By the assumption made at the beginning of the section (see Remark 2.5), a minimum  $(K_q, k-r)$  stable graph has  $\binom{q+k-r}{2}$  edges. Since  $e(G) \leq \binom{q+k}{2}$ , the inequality (2.1) follows from Claim 2.14.1 and the inequality (2.2).  $\square$

A simple calculation shows that the inequality

$$q^2 + q + 2 \leq 2kr$$

can be obtained by starting from the inequality (2.1).

Since  $r \leq k-2$  and  $k \leq \frac{q}{2} + 1$ , we have  $q^2 + q + 2 \leq 2k(k-2) \leq (q+2)(\frac{q}{2}-1)$ , hence  $\frac{q^2}{2} + q + 4 \leq 0$ , a contradiction. Thus,  $N(v)$  is a clique and the proposition follows.  $\square$

## 3. RESULT

In [3], it is shown that if  $G$  is minimum  $(K_q, k)$  stable and the numbers  $k$  and  $q$  verify one of the following conditions:

- $k = 1$  and  $q \geq 4$
- $k = 2$  and  $q \geq 4$
- $k = 3$  and  $q \geq 5$

then  $G$  is isomorphic to  $K_{q+k}$ .

**Theorem 3.1.** *Let  $G$  be a minimum  $(K_q, k)$  stable graph, where  $q \geq 6$  and  $k \leq \frac{q}{2} + 1$ . Then  $G$  is isomorphic to  $K_{q+k}$ .*

**Proof.** For  $0 \leq k \leq 3$  the graph  $G$  is isomorphic to  $K_{q+k}$ . Let  $k$  be such that  $4 \leq k \leq \frac{q}{2} + 1$  and suppose that for every  $r$  with  $0 \leq r < k$  the only minimum  $(K_q, r)$  stable graph is  $K_{q+r}$ . By Proposition 2.14, the graph  $G$  is the disjoint union of  $p$  complete graphs  $H_1 \equiv K_{q+k_1}$ ,  $H_2 \equiv K_{q+k_2}$ ,  $\dots$ ,  $H_p \equiv K_{q+k_p}$ . Suppose, without loss of generality, that  $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$  and that there exist two components  $H_i$  and  $H_j$  with  $i < j$  such that  $k_i - k_j \geq 2$ . By substituting  $H'_i \equiv K_{q+k_i-1}$  for  $H_i$  and  $H'_j \equiv K_{q+k_j+1}$  for  $H_j$ , we obtain a new  $(K_q, k)$  stable graph  $G'$  such that  $e(G') = e(G) - (k_i - k_j - 1) < e(G)$ , which is a contradiction. Thus, for any  $i$  and any  $j$ ,  $0 \leq |k_i - k_j| \leq 1$  (cf [2] Proposition 7).

To conclude that  $G$  has a unique component, observe the following facts.

- The graphs  $2K_{q+l}$  and  $K_{q+2l+1}$  are both  $(K_q, 2l+1)$  stable, but if  $2l+1 \leq \frac{q}{2} + 1$  then  $\binom{q+2l+1}{2} < 2\binom{q+l}{2}$
- The graphs  $K_{q+l} \cup K_{q+l+1}$  and  $K_{q+2l+2}$  are both  $(K_q, 2l+2)$  stable but if  $2l+2 \leq \frac{q}{2} + 1$  then  $\binom{q+2l+2}{2} < \binom{q+l+1}{2} + \binom{q+l}{2}$

□

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