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# ON SIMULTANEOUS DIOPHANTINE APPROXIMATIONS TO $\zeta(2)$ AND $\zeta(3)$

SIMON DAUGUET AND WADIM ZUDILIN

## 1. INTRODUCTION

It is known that the Riemann zeta function  $\zeta(s)$  takes irrational values at positive even integers. This follows from Euler's evaluation  $\zeta(s)/\pi^s \in \mathbb{Q}$  for  $s = 2, 4, 6, \dots$  and from the transcendence of  $\pi$ . Less is known about the values of  $\zeta(s)$  at odd integers  $s > 1$ . Apéry was the first to establish the irrationality of such  $\zeta(s)$ : he proved [Apé79] in 1978 that  $\zeta(3)$  is irrational. The next major step in the direction was made by Ball and Rivoal [BR01] in 2000: they showed that there are infinitely many odd integers at which Riemann zeta function is irrational. Shortly after, Rivoal demonstrates [Riv02] in 2001 that one of the nine numbers  $\zeta(5), \zeta(7), \dots, \zeta(21)$  is irrational, while the second author [Zud01] reduces the nine to four: he proves that at least one of the four numbers  $\zeta(5), \zeta(7), \zeta(9)$  and  $\zeta(11)$  is irrational.

Already in 1978, Apéry constructs linear forms in 1 and  $\zeta(2)$ , as well as in 1 and  $\zeta(3)$ , with integer coefficients that produce the irrationality of the two zeta values in a quantitative form: the constructions imply upper bounds  $\mu(\zeta(2)) < 11.850878\dots$  and  $\mu(\zeta(3)) < 13.41782\dots$  for the irrationality measures. Recall that the irrationality exponent  $\mu(\alpha)$  of a real irrational  $\alpha$  is the supremum of the set of exponents  $\mu$  for which the inequality  $|\alpha - p/q| < q^{-\mu}$  has infinitely many solutions in rationals  $p/q$ . Hata improves the above mentioned results to  $\mu(\zeta(2)) < 5.687$  in [Hat95, Addendum] and to  $\mu(\zeta(3)) < 7.377956\dots$  in [Hat00]. Then Rhin and Viola study a permutation group related to  $\zeta(2)$  in [RV96] and show that  $\mu(\zeta(2)) < 5.441243$ . They later apply their new permutation group arithmetic method to  $\zeta(3)$  as well and prove the upper bound  $\mu(\zeta(3)) < 5.513891$ . In an attempt to unify the achievements of Ball–Rivoal and of Rhin–Viola, the second author re-interpreted the constructions using the classical theory of hypergeometric functions and integrals [Zud04]. In his recent work [Zud13], he uses the permutation group arithmetic method and a hypergeometric construction, closely related to the one in this paper, to sharpen the earlier irrationality exponent of  $\zeta(2)$  by proving that  $\mu(\zeta(2)) \leq 5.09541178\dots$

In this paper, we construct simultaneous rational approximations to both  $\zeta(2)$  and  $\zeta(3)$  using hypergeometric tools, and establish from them a lower bound for  $\mathbb{Z}$ -linear combinations of 1,  $\zeta(2)$  and  $\zeta(3)$  under some strong divisibility conditions on the coefficients. Namely, we prove

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**Theorem 1.** *Let  $\eta$  and  $\varepsilon$  be positive real numbers. For  $m$  sufficiently large with respect to  $\varepsilon$  and  $\eta$ , let  $(a_0, a_1, a_2) \in \mathbb{Q}^3 \setminus \{\mathbf{0}\}$  be such that*

- (i)  $D_m^2 D_{2m} a_0 \in \mathbb{Z}$ ,  $D_m a_1 \in \mathbb{Z}$  and  $\frac{D_{2m}}{D_m} a_2 \in \mathbb{Z}$ , where  $D_m$  denotes the least common multiple of  $1, 2, \dots, m$ ; and
- (ii)  $|a_0|, |a_1|, |a_2| \leq e^{-(\tau_0 + \varepsilon)m}$  hold with  $\tau_0 = 0.899668635\dots$

Then  $|a_0 + a_1\zeta(2) + a_2\zeta(3)| > e^{-(S+\eta)m}$  with  $S = 6.770732145\dots$

Theorem 1 contains the irrationality of both  $\zeta(2)$  and  $\zeta(3)$ , because  $\tau_0 < 1$ . Namely, taking

$$a_0 = \frac{-p}{D_m}, \quad a_1 = \frac{q}{D_m} \quad \text{and} \quad a_2 = 0$$

shows that  $\zeta(2) \neq p/q$ , while the choice

$$a_0 = \frac{-D_m p}{D_{2m}}, \quad a_1 = 0 \quad \text{and} \quad a_2 = \frac{D_m q}{D_{2m}}$$

implies that  $\zeta(3) \neq p/q$ . The theorem does not give however the expected linear independence of  $1, \zeta(2)$  and  $\zeta(3)$ : it remains an open problem.

Our proof of Theorem 1 heavily rests upon a general version of hypergeometric construction of linear forms in  $1$  and  $\zeta(2)$  on one hand, and in  $1$  and  $\zeta(3)$  on the other hand; some particular instances of this construction were previously outlined in [Zud11]. More precisely, the linear forms  $r_n = q_n\zeta(2) - p_n$  and  $\hat{r}_n = \hat{q}_n\zeta(3) - \hat{p}_n$  we construct in the proof are hypergeometric-type series that depend on certain sets of auxiliary integer parameters. Permuting parameters in the sets allows us to gain  $p$ -adic information about the coefficients  $q_n, \hat{q}_n, p_n$  and  $\hat{p}_n$ . In addition, a classical transformation from the theory of hypergeometric functions implies that  $q_n = \hat{q}_n$ . The latter fact leads us to *simultaneous* rational approximations  $r_n = q_n\zeta(2) - p_n$  and  $\hat{r}_n = q_n\zeta(3) - \hat{p}_n$  to  $\zeta(2)$  and  $\zeta(3)$ , with the following arithmetical and asymptotic properties:

$$\begin{aligned} \hat{\Phi}_n^{-1} q_n, \hat{\Phi}_n^{-1} D_{8n} D_{16n} p_n, \hat{\Phi}_n^{-1} D_{8n}^3 \hat{p}_n &\in \mathbb{Z}, \\ \lim_{n \rightarrow \infty} \frac{\log \hat{\Phi}_n}{n} &= \varphi = 5.70169601\dots, \end{aligned}$$

where  $\hat{\Phi}_n$  is an explicit product over primes, and

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\log |r_n|}{n} &= \limsup_{n \rightarrow \infty} \frac{\log |\hat{r}_n|}{n} = -\rho = -19.10095491\dots, \\ \lim_{n \rightarrow \infty} \frac{\log |q_n|}{n} &= \kappa = 27.86755317\dots \end{aligned}$$

Finally, executing the Gosper–Zeilberger algorithm of creative telescoping we find out a recurrence relation satisfied by the linear forms  $r_n$  and  $\hat{r}_n$ . Together with a standard argument using the nonvanishing determinants formed from the coefficients of the forms, we then deduce Theorem 1 (some further computational details can

be found in [Dau14]). Note that

$$\tau_0 = \frac{1}{8}(32 - \varphi - \rho) = 0.899668635\dots \quad \text{and} \quad S = \frac{1}{8}(32 - \varphi + \kappa) = 6.770732145\dots, \quad (1)$$

and the integer  $m$  from Theorem 1 is essentially  $8n$ .

In order to accommodate the atypical simultaneous approximations in Theorem 1 as well as to relate them to the context of previous results listed in the beginning of the section, we define a new diophantine exponent  $s_\tau(\xi_1, \xi_2)$  of two real numbers  $\xi_1$  and  $\xi_2$ , a characteristic of simultaneous irrationality of the numbers which depends on an additional parameter  $\tau$ . With this notion in mind, we restate Theorem 1 as  $s_{\tau_0}(\zeta(2), \zeta(3)) \leq S$ . Exploiting further the properties of the exponent, we demonstrate in Proposition 7 the unlikeness of linear dependence of  $1$ ,  $\zeta(2)$  and  $\zeta(3)$  over  $\mathbb{Q}$ : the latter would imply  $S = 6 - \tau_0$  or the belonging of both  $\zeta(2)$  and  $\zeta(3)$  to a certain set of Lebesgue measure 0.

In § 2 we introduce hypergeometric tools which depend on some parameters that lead to  $\mathbb{Q}$ -linear forms in  $1$  and  $\zeta(2)$  on one hand, and in  $1$  and  $\zeta(3)$  on the other, the forms having some common asymptotic properties.

In § 3 we specialise the parameters of the previous part to have the coefficients of  $\zeta(2)$  and  $\zeta(3)$  coincide. From this specialisation we derive the main theorem using recurrence relations satisfied by the linear forms and their coefficients.

In the final part, § 4, we introduce a new diophantine exponent. Some basic properties of this exponent are given, and it is compared to the irrationality exponents previously known. Then the main result is restated in terms of this diophantine exponent in Theorem 2, for consistency with previous results in the subject.

## 2. HYPERGEOMETRIC SERIES

In what follows, we always assume standard hypergeometric notation [Sla66]. For  $n \in \mathbb{N}$ , the Pochhammer symbol is given by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{k=0}^{n-1} (a+k),$$

with the convention  $(a)_0 = 1$ , while the generalized hypergeometric function is defined by the series

$${}_{p+1}F_p \left( \begin{matrix} a_0, a_1, \dots, a_p \\ b_1, \dots, b_p \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_p)_n}{n! (b_1)_n \cdots (b_p)_n} z^n.$$

**2.1. Integer-valued polynomials.** We reproduce here some auxiliary results about integer-valued polynomials; the proofs can be found in [Zud13].

**Lemma 1.** For  $\ell = 0, 1, 2, \dots$ ,

$$\frac{1}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \left( \frac{\pi}{\sin \pi t} \right)^2 \frac{(t-1)(t-2)\cdots(t-\ell)}{\ell!} dt = \frac{(-1)^\ell}{\ell+1}. \quad (2)$$

**Lemma 2.** *Given  $b < a$  integers, set*

$$R(t) = R(a, b; t) = \frac{(t+b)(t+b+1)\cdots(t+a-1)}{(a-b)!}.$$

Then

$$R(k) \in \mathbb{Z}, \quad D_{a-b} \cdot \frac{dR(t)}{dt} \Big|_{t=k} \in \mathbb{Z} \quad \text{and} \quad D_{a-b} \cdot \frac{R(k) - R(\ell)}{k - \ell} \in \mathbb{Z}$$

for any  $k, \ell \in \mathbb{Z}$ ,  $\ell \neq k$ .

**Lemma 3.** *Let  $R(t)$  be a product of several integer-valued polynomials*

$$R_j(t) = R(a_j, b_j; t) = \frac{(t+b_j)(t+b_j+1)\cdots(t+a_j-1)}{(a_j-b_j)!}, \quad \text{where } b_j < a_j,$$

and  $m = \max_j \{a_j - b_j\}$ . Then

$$R(k) \in \mathbb{Z}, \quad D_m \cdot \frac{dR(t)}{dt} \Big|_{t=k} \in \mathbb{Z} \quad \text{and} \quad D_m \cdot \frac{R(k) - R(\ell)}{k - \ell} \in \mathbb{Z} \quad (3)$$

for any  $k, \ell \in \mathbb{Z}$ ,  $\ell \neq k$ .

**2.2. Construction of linear forms in 1 and  $\zeta(2)$ .** The construction in this subsection is a general case of the one considered in [Zud07, Section 2].

For a set of parameters

$$(\mathbf{a}, \mathbf{b}) = \begin{pmatrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3, b_4 \end{pmatrix}$$

subject to the conditions

$$\begin{aligned} b_1, b_2, b_3 &\leq a_1, a_2, a_3, a_4 < b_4, \\ d = (a_1 + a_2 + a_3 + a_4) - (b_1 + b_2 + b_3 + b_4) &\geq 0, \end{aligned} \quad (4)$$

define the rational function

$$\begin{aligned} R(t) = R(\mathbf{a}, \mathbf{b}; t) &= \frac{(t+b_1)\cdots(t+a_1-1)}{(a_1-b_1)!} \cdot \frac{(t+b_2)\cdots(t+a_2-1)}{(a_2-b_2)!} \\ &\times \frac{(t+b_3)\cdots(t+a_3-1)}{(a_3-b_3)!} \cdot \frac{(b_4-a_4-1)!}{(t+a_4)\cdots(t+b_4-1)} \end{aligned} \quad (5)$$

$$= \Pi(\mathbf{a}, \mathbf{b}) \cdot \frac{\Gamma(t+a_1)\Gamma(t+a_2)\Gamma(t+a_3)\Gamma(t+a_4)}{\Gamma(t+b_1)\Gamma(t+b_2)\Gamma(t+b_3)\Gamma(t+b_4)}, \quad (6)$$

where

$$\Pi(\mathbf{a}, \mathbf{b}) = \frac{(b_4 - a_4 - 1)!}{(a_1 - b_1)! (a_2 - b_2)! (a_3 - b_3)!}.$$

We also introduce the ordered versions  $a_1^* \leq a_2^* \leq a_3^* \leq a_4^*$  of the parameters  $a_1, a_2, a_3, a_4$  and  $b_1^* \leq b_2^* \leq b_3^*$  of  $b_1, b_2, b_3$ , so that  $\{a_1^*, a_2^*, a_3^*, a_4^*\}$  coincides with  $\{a_1, a_2, a_3, a_4\}$  and  $\{b_1^*, b_2^*, b_3^*\}$  coincides with  $\{b_1, b_2, b_3\}$  as multi-sets (that is, sets with possible repetition of elements). Then  $R(t)$  has poles at  $t = -k$  where  $k = a_4^*, a_4^* + 1, \dots, b_4 - 1$ , zeroes at  $t = -\ell$  where  $\ell = b_1^*, b_1^* + 1, \dots, a_3^* - 1$ , and double zeroes at  $t = -\ell$  where  $\ell = b_2^*, b_2^* + 1, \dots, a_2^* - 1$ .

Decomposing  $R(t)$  into the sum of partial fractions, we get

$$R(t) = \sum_{k=a_4^*}^{b_4-1} \frac{C_k}{t+k} + P(t), \quad (7)$$

where  $P(t)$  is a polynomial of which the degree  $d$  is defined in (4) and

$$\begin{aligned} C_k &= (R(t)(t+k))|_{t=-k} \\ &= (-1)^{d+b_4+k} \binom{k-b_1}{k-a_1} \binom{k-b_2}{k-a_2} \binom{k-b_3}{k-a_3} \binom{b_4-a_4-1}{k-a_4} \in \mathbb{Z} \end{aligned} \quad (8)$$

for  $k = a_4^*, a_4^* + 1, \dots, b_4 - 1$ .

**Lemma 4.** *Set  $c = \max\{a_1 - b_1, a_2 - b_2, a_3 - b_3\}$ . Then  $D_c P(t)$  is an integer-valued polynomial of degree  $d$ .*

*Proof.* Write  $R(t) = R_1(t)R_2(t)$ , where

$$R_1(t) = \frac{\prod_{j=b_1}^{a_1-1} (t+j)}{(a_1-b_1)!} \cdot \frac{\prod_{j=b_2}^{a_2-1} (t+j)}{(a_2-b_2)!} \cdot \frac{\prod_{j=b_3}^{a_3-1} (t+j)}{(a_3-b_3)!}$$

is the product of three integer-valued polynomials and

$$R_2(t) = \frac{(b_4-a_4-1)!}{\prod_{j=a_4}^{b_4-1} (t+j)} = \sum_{k=a_4}^{b_4-1} \frac{(-1)^{k-a_4} \binom{b_4-a_4-1}{k-a_4}}{t+k}.$$

It follows from Lemma 3 that

$$\begin{aligned} D_c \cdot \frac{dR_1(t)}{dt} \Big|_{t=j} &\in \mathbb{Z} \quad \text{for } j \in \mathbb{Z} \quad \text{and} \\ D_c \cdot \frac{R_1(j) - R_1(m)}{j-m} &\in \mathbb{Z} \quad \text{for } j, m \in \mathbb{Z}, j \neq m. \end{aligned} \quad (9)$$

Furthermore, note that

$$\begin{aligned} C_k &= R_1(-k) \cdot (R_2(t)(t+k))|_{t=-k} \\ &= R_1(-k) \cdot (-1)^{k-a_4} \binom{b_4-a_4-1}{k-a_4} \quad \text{for } k \in \mathbb{Z}, \end{aligned}$$

and the expression in fact vanishes if  $k$  is outside the range  $a_4^* \leq k \leq b_4 - 1$ .

For  $\ell \in \mathbb{Z}$  we have

$$\begin{aligned} \frac{d}{dt} (R(t)(t+\ell)) \Big|_{t=-\ell} &= \frac{d}{dt} (R_1(t) \cdot R_2(t)(t+\ell)) \Big|_{t=-\ell} \\ &= \frac{dR_1(t)}{dt} \Big|_{t=-\ell} \cdot (R_2(t)(t+\ell))|_{t=-\ell} + R_1(-\ell) \cdot \frac{d}{dt} (R_2(t)(t+\ell)) \Big|_{t=-\ell} \\ &= \frac{dR_1(t)}{dt} \Big|_{t=-\ell} \cdot (-1)^{\ell-a_4} \binom{b_4-a_4-1}{\ell-a_4} \\ &\quad + R_1(-\ell) \cdot \frac{d}{dt} \sum_{k=a_4}^{b_4-1} (-1)^{k-a_4} \binom{b_4-a_4-1}{k-a_4} \left(1 - \frac{-\ell+k}{t+k}\right) \Big|_{t=-\ell} \end{aligned}$$

$$= \frac{dR_1(t)}{dt} \Big|_{t=-\ell} \cdot (-1)^{\ell-a_4} \binom{b_4 - a_4 - 1}{\ell - a_4} + R_1(-\ell) \sum_{\substack{k=a_4 \\ k \neq \ell}}^{b_4-1} \frac{(-1)^{k-a_4} \binom{b_4-a_4-1}{k-a_4}}{-\ell + k}$$

and

$$\begin{aligned} \frac{d}{dt} \left( \sum_{k=a_4^*}^{b_4-1} \frac{C_k}{t+k} \cdot (t+\ell) \right) \Big|_{t=-\ell} &= \frac{d}{dt} \left( \sum_{k=a_4}^{b_4-1} \frac{C_k}{t+k} \cdot (t+\ell) \right) \Big|_{t=-\ell} \\ &= \frac{d}{dt} \sum_{k=a_4}^{b_4-1} C_k \left( 1 - \frac{-\ell+k}{t+k} \right) \Big|_{t=-\ell} = \sum_{\substack{k=a_4 \\ k \neq \ell}}^{b_4-1} \frac{C_k}{-\ell+k} \\ &= \sum_{\substack{k=a_4 \\ k \neq \ell}}^{b_4-1} \frac{R_1(-k) \cdot (-1)^{k-a_4} \binom{b_4-a_4-1}{k-a_4}}{-\ell+k}. \end{aligned}$$

Therefore,

$$\begin{aligned} P(-\ell) &= \frac{d}{dt} (P(t)(t+\ell)) \Big|_{t=-\ell} = \frac{d}{dt} \left( R(t)(t+\ell) - \sum_{k=a_4^*}^{b_4-1} \frac{C_k}{t+k} \cdot (t+\ell) \right) \Big|_{t=-\ell} \\ &= \frac{dR_1(t)}{dt} \Big|_{t=-\ell} \cdot (-1)^{\ell-a_4} \binom{b_4 - a_4 - 1}{\ell - a_4} \\ &\quad + \sum_{\substack{k=a_4 \\ k \neq \ell}}^{b_4-1} (-1)^{k-a_4} \binom{b_4 - a_4 - 1}{k - a_4} \frac{R_1(-\ell) - R_1(-k)}{-\ell + k}, \end{aligned}$$

and this implies, on the basis of the inclusions (9) above, that  $D_c P(-\ell) \in \mathbb{Z}$  for all  $\ell \in \mathbb{Z}$ .  $\square$

Finally, define the quantity

$$r(\mathbf{a}, \mathbf{b}) = \frac{(-1)^d}{2\pi i} \int_{C-i\infty}^{C+i\infty} \left( \frac{\pi}{\sin \pi t} \right)^2 R(\mathbf{a}, \mathbf{b}; t) dt, \quad (10)$$

where  $C$  is arbitrary from the interval  $-a_2^* < C < 1 - b_2^*$ . The definition does not depend on the choice of  $C$ , as the integrand does not have singularities in the strip  $-a_2^* < \operatorname{Re} t < 1 - b_2^*$ .

**Proposition 1.** *We have*

$$r(\mathbf{a}, \mathbf{b}) = q(\mathbf{a}, \mathbf{b})\zeta(2) - p(\mathbf{a}, \mathbf{b}), \quad \text{with } q(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}, \quad D_{c_1} D_{c_2} p(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}, \quad (11)$$

where

$$c_1 = \max\{a_1 - b_1, a_2 - b_2, a_3 - b_3, b_4 - a_2^* - 1\} \quad \text{and} \quad c_2 = \max\{d + 1, b_4 - a_2^* - 1\}.$$

In addition,

$$q(\mathbf{a}, \mathbf{b}) = (-1)^{b_4 - a_4^* - 1} \binom{a_4^* - b_1}{a_4^* - a_1} \binom{a_4^* - b_2}{a_4^* - a_2} \binom{a_4^* - b_3}{a_4^* - a_3} \binom{b_4 - a_4 - 1}{a_4^* - a_4} \\ \times {}_4F_3 \left( \begin{matrix} -(b_4 - a_4^* - 1), a_4^* - b_1 + 1, a_4^* - b_2 + 1, a_4^* - b_3 + 1 \\ a_4^* - a_1^* + 1, a_4^* - a_2^* + 1, a_4^* - a_3^* + 1 \end{matrix} \middle| 1 \right), \quad (12)$$

and the quantity  $r(\mathbf{a}, \mathbf{b})/\Pi(\mathbf{a}, \mathbf{b})$  is invariant under any permutation of the parameters  $a_1, a_2, a_3, a_4$ .

*Proof.* We choose  $C = 1/2 - a_2^*$  in (10) and write (7) as

$$R(t) = \sum_{k=a_4^*}^{b_4-1} \frac{C_k}{t+k} + \sum_{\ell=0}^d A_\ell P_\ell(t + a_2^*),$$

where

$$P_\ell(t) = \frac{(t-1)(t-2)\cdots(t-\ell)}{\ell!}$$

and  $D_c A_\ell \in \mathbb{Z}$  in accordance with Lemma 4. Applying Lemma 1 we obtain

$$r(\mathbf{a}, \mathbf{b}) = \frac{(-1)^d}{2\pi i} \int_{1/2-i\infty}^{1/2+i\infty} \left( \frac{\pi}{\sin \pi t} \right)^2 R(t - a_2^*) dt \\ = (-1)^d \sum_{m=1-a_2^*}^{\infty} \sum_{k=a_4^*}^{b_4-1} \frac{C_k}{(m+k)^2} + \sum_{\ell=0}^d \frac{(-1)^{d+\ell} A_\ell}{\ell+1} \\ = \zeta(2) \cdot (-1)^d \sum_{k=a_4^*}^{b_4-1} C_k - (-1)^d \sum_{k=a_4^*}^{b_4-1} C_k \sum_{\ell=1}^{k-a_2^*} \frac{1}{\ell^2} + \sum_{\ell=0}^d \frac{(-1)^{d+\ell} A_\ell}{\ell+1}.$$

This representation clearly implies that  $r(\mathbf{a}, \mathbf{b})$  has the desired form (11), while the hypergeometric form (12) follows from

$$q(\mathbf{a}, \mathbf{b}) = (-1)^d \sum_{k=a_4^*}^{b_4-1} C_k$$

and the explicit formula (8) for  $C_k$ . Finally, the invariance of  $r(\mathbf{a}, \mathbf{b})/\Pi(\mathbf{a}, \mathbf{b})$  under permutations of  $a_1, a_2, a_3, a_4$  follows from (6) and definition (10) of  $r(\mathbf{a}, \mathbf{b})$ .  $\square$

Assume that the parameters  $(\mathbf{a}, \mathbf{b})$  are chosen in the following way:

$$\begin{aligned} a_1 &= \alpha_1 n + 1, & a_2 &= \alpha_2 n + 1, & a_3 &= \alpha_3 n + 1, & a_4 &= \alpha_4 n + 1, \\ b_1 &= \beta_1 n + 1, & b_2 &= \beta_2 n + 1, & b_3 &= \beta_3 n + 1, & b_4 &= \beta_4 n + 2, \end{aligned} \quad (13)$$

where the *fixed* integers  $\alpha_j$  and  $\beta_j$ ,  $j = 1, \dots, 4$ , satisfy

$$\begin{aligned} \beta_1, \beta_2, \beta_3 &< \alpha_1, \alpha_2, \alpha_3, \alpha_4 < \beta_4, \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &> \beta_1 + \beta_2 + \beta_3 + \beta_4. \end{aligned}$$

The quantities (11) in these settings become dependent on a single parameter  $n = 0, 1, 2, \dots$ , so we let  $r_n = r(\mathbf{a}, \mathbf{b})$ ,  $q_n = q(\mathbf{a}, \mathbf{b})$ ,  $p_n = p(\mathbf{a}, \mathbf{b})$  and identify the characteristics  $c_1 = \gamma_1 n$  and  $c_2 = \gamma_2 n$  of Proposition 1, where  $\gamma_1$  and  $\gamma_2$  are



completely determined by  $\alpha_j$  and  $\beta_j$ ,  $j = 1, \dots, 4$ . The statement below is proven by standard techniques and is very similar to [Zud04, Lemmas 10–12].

**Proposition 2.** *In the above notation, let  $\tau_0, \bar{\tau}_0 \in \mathbb{C} \setminus \mathbb{R}$  and  $\tau_1 \in \mathbb{R}$  be the zeroes of the cubic polynomial  $\prod_{j=1}^4(\tau - \alpha_j) - \prod_{j=1}^4(\tau - \beta_j)$ . Define*

$$f_0(\tau) = \sum_{j=1}^4 (\alpha_j \log(\tau - \alpha_j) - \beta_j \log(\tau - \beta_j)) \\ - \sum_{j=1}^3 (\alpha_j - \beta_j) \log(\alpha_j - \beta_j) + (\beta_4 - \alpha_4) \log(\beta_4 - \alpha_4).$$

Then

$$\limsup_{n \rightarrow \infty} \frac{\log |r_n|}{n} = \operatorname{Re} f_0(\tau_0) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log |q_n|}{n} = \operatorname{Re} f_0(\tau_1).$$

Furthermore,

$$\Phi_n^{-1} q_n, \Phi_n^{-1} D_{\gamma_1 n} D_{\gamma_2 n} p_n \in \mathbb{Z}$$

with

$$\Phi_n = \prod_{\substack{p \text{ prime} \\ p \leq \min\{\gamma_1, \gamma_2\}n}} p^{\varphi(n/p)},$$

where

$$\varphi(x) = \max_{\alpha' = \sigma \alpha: \sigma \in \mathfrak{S}_4} \left( \lfloor (\beta_4 - \alpha_4)x \rfloor - \lfloor (\beta_4 - \alpha'_4)x \rfloor \right. \\ \left. - \sum_{j=1}^3 (\lfloor (\alpha_j - \beta_j)x \rfloor - \lfloor (\alpha'_j - \beta_j)x \rfloor) \right),$$

so that the maximum is taken over all permutations  $(\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4)$  of  $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ , and we have

$$\lim_{n \rightarrow \infty} \frac{\log \Phi_n}{n} = \int_0^1 \varphi(x) d\psi(x) - \int_0^{1/\min\{\gamma_1, \gamma_2\}} \varphi(x) \frac{dx}{x^2},$$

where  $\psi(x)$  is the logarithmic derivative of the gamma function.

Here and in what follows, the notation  $\lfloor \cdot \rfloor$  and  $\lceil \cdot \rceil$  is used for the floor and ceiling integer-part functions.

**2.3. Construction of linear forms in 1 and  $\zeta(3)$ .** The construction in this subsection depends on another set of integral parameters

$$(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \begin{pmatrix} \hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3 \\ \hat{b}_0, \hat{b}_1, \hat{b}_2, \hat{b}_3 \end{pmatrix}$$

which satisfies the conditions

$$\frac{1}{2}\hat{b}_0, \hat{b}_1 \leq \frac{1}{2}\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3 < \hat{b}_2, \hat{b}_3, \\ \hat{a}_0 + \hat{a}_1 + \hat{a}_2 + \hat{a}_3 \leq \hat{b}_0 + \hat{b}_1 + \hat{b}_2 + \hat{b}_3 - 2. \tag{14}$$

To this set we assign the rational function

$$\begin{aligned} \hat{R}(t) = \hat{R}(\hat{\mathbf{a}}, \hat{\mathbf{b}}; t) &= \frac{(2t + \hat{b}_0)(2t + \hat{b}_0 + 1) \cdots (2t + \hat{a}_0 - 1)}{(\hat{a}_0 - \hat{b}_0)!} \cdot \frac{(t + \hat{b}_1) \cdots (t + \hat{a}_1 - 1)}{(\hat{a}_1 - \hat{b}_1)!} \\ &\quad \times \frac{(\hat{b}_2 - \hat{a}_2 - 1)!}{(t + \hat{a}_2) \cdots (t + \hat{b}_2 - 1)} \cdot \frac{(\hat{b}_3 - \hat{a}_3 - 1)!}{(t + \hat{a}_3) \cdots (t + \hat{b}_3 - 1)} \end{aligned} \quad (15)$$

$$= \hat{\Pi}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) \cdot \frac{\Gamma(2t + \hat{a}_0) \Gamma(t + \hat{a}_1) \Gamma(t + \hat{a}_2) \Gamma(t + \hat{a}_3)}{\Gamma(2t + \hat{b}_0) \Gamma(t + \hat{b}_1) \Gamma(t + \hat{b}_2) \Gamma(t + \hat{b}_3)}, \quad (16)$$

where

$$\hat{\Pi}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \frac{(\hat{b}_2 - \hat{a}_2 - 1)! (\hat{b}_3 - \hat{a}_3 - 1)!}{(\hat{a}_0 - \hat{b}_0)! (\hat{a}_1 - \hat{b}_1)!}.$$

As in Section 2.2 we introduce the ordered versions  $\hat{a}_1^* \leq \hat{a}_2^* \leq \hat{a}_3^*$  of the parameters  $\hat{a}_1, \hat{a}_2, \hat{a}_3$  and  $\hat{b}_2^* \leq \hat{b}_3^*$  of  $\hat{b}_2, \hat{b}_3$ . Then this ordering and conditions (14) imply that  $\hat{R}(t) = O(1/t^2)$  as  $t \rightarrow \infty$ , the rational function has poles at  $t = -k$  for  $\hat{a}_2^* \leq k \leq \hat{b}_3^* - 1$ , double poles at  $t = -k$  for  $\hat{a}_3^* \leq k \leq \hat{b}_2^* - 1$ , and double zeroes at  $t = -\ell$  for  $\max\{\lceil \hat{b}_0/2 \rceil, \hat{b}_1\} \leq \ell \leq \min\{\lfloor (\hat{a}_0 - 1)/2 \rfloor, \hat{a}_1^* - 1\}$ .

The partial-fraction decomposition of  $\hat{R}(t)$  assumes the form

$$\hat{R}(t) = \sum_{k=\hat{a}_3^*}^{\hat{b}_2^*-1} \frac{A_k}{(t+k)^2} + \sum_{k=\hat{a}_2^*}^{\hat{b}_3^*-1} \frac{B_k}{t+k}, \quad (17)$$

where

$$\begin{aligned} A_k &= (\hat{R}(t)(t+k)^2)|_{t=-k} \\ &= (-1)^{\hat{d}} \binom{2k - \hat{b}_0}{2k - \hat{a}_0} \binom{k - \hat{b}_1}{k - \hat{a}_1} \binom{\hat{b}_2 - \hat{a}_2 - 1}{k - \hat{a}_2} \binom{\hat{b}_3 - \hat{a}_3 - 1}{k - \hat{a}_3} \in \mathbb{Z} \end{aligned} \quad (18)$$

with  $\hat{d} = \hat{a}_0 + \hat{a}_1 + \hat{a}_2 + \hat{a}_3 - \hat{b}_0 - \hat{b}_1$ , for  $k = \hat{a}_3^*, \hat{a}_3^* + 1, \dots, \hat{b}_2^* - 1$  and, similarly,

$$B_k = \frac{d}{dt} (\hat{R}(t)(t+k)^2)|_{t=-k}$$

for  $k = \hat{a}_2^*, \hat{a}_2^* + 1, \dots, \hat{b}_3^* - 1$ . The inclusions

$$D_{\max\{\hat{a}_0 - \hat{b}_0, \hat{a}_1 - \hat{b}_1, \hat{b}_3^* - \hat{a}_2 - 1, \hat{b}_3^* - \hat{a}_3 - 1\}} \cdot B_k \in \mathbb{Z} \quad (19)$$

follow then from standard consideration; see, for example, Lemma 3 and the proof of Lemma 4 in [Zud04]. In addition,

$$\sum_{k=\hat{a}_2^*}^{\hat{b}_3^*-1} B_k = -\operatorname{Res}_{t=\infty} \hat{R}(t) = 0 \quad (20)$$

by the residue sum theorem.

The quantity of our interest in this section is

$$\hat{r}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \frac{(-1)^{\hat{d}}}{4\pi i} \int_{C-i\infty}^{C+i\infty} \left( \frac{\pi}{\sin \pi t} \right)^2 \hat{R}(\hat{\mathbf{a}}, \hat{\mathbf{b}}; t) dt, \quad (21)$$

where  $C$  is arbitrary from the interval  $-\min\{\hat{a}_0/2, \hat{a}_1^*\} < C < 1 - \max\{\hat{b}_0/2, \hat{b}_1\}$ .

**Proposition 3.** *We have*

$$\hat{r}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \hat{q}(\hat{\mathbf{a}}, \hat{\mathbf{b}})\zeta(3) - \hat{p}(\hat{\mathbf{a}}, \hat{\mathbf{b}}), \quad \text{with } \hat{q}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) \in \mathbb{Z}, \quad 2D_{\hat{c}_1}D_{\hat{c}_2}^2\hat{p}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) \in \mathbb{Z}, \quad (22)$$

where

$$\begin{aligned} \hat{c}_1 &= \max\{\hat{a}_0 - \hat{b}_0, \hat{a}_1 - \hat{b}_1, \hat{b}_3^* - \hat{a}_2 - 1, \hat{b}_3^* - \hat{a}_3 - 1, \hat{b}_2^* - \lceil \hat{a}_0/2 \rceil - 1, \hat{b}_2^* - \hat{a}_1^* - 1\}, \\ \hat{c}_2 &= \max\{\hat{b}_3^* - \lceil \hat{a}_0/2 \rceil - 1, \hat{b}_3^* - \hat{a}_1^* - 1\}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \hat{q}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) &= \binom{2\hat{a}_3^* - \hat{b}_0}{2\hat{a}_3^* - \hat{a}_0} \binom{\hat{a}_3^* - \hat{b}_1}{\hat{a}_3^* - \hat{a}_1} \binom{\hat{b}_2 - \hat{a}_2 - 1}{\hat{a}_3^* - \hat{a}_2} \binom{\hat{b}_3 - \hat{a}_3 - 1}{\hat{a}_3^* - \hat{a}_3} \\ &\times {}_5F_4 \left( \begin{matrix} -(\hat{b}_2 - \hat{a}_3^* - 1), -(\hat{b}_3 - \hat{a}_3^* - 1), \hat{a}_3^* - \hat{b}_1 + 1, \hat{a}_3^* - \frac{1}{2}\hat{b}_0 + \frac{1}{2}, \hat{a}_3^* - \frac{1}{2}\hat{b}_1 + 1 \\ \hat{a}_3^* - \hat{a}_1^* + 1, \hat{a}_3^* - \hat{a}_2^* + 1, \hat{a}_3^* - \frac{1}{2}\hat{a}_0 + \frac{1}{2}, \hat{a}_3^* - \frac{1}{2}\hat{a}_0 + 1 \end{matrix} \middle| 1 \right), \end{aligned} \quad (23)$$

and the quantity  $\hat{r}(\hat{\mathbf{a}}, \hat{\mathbf{b}})/\hat{\Pi}(\hat{\mathbf{a}}, \hat{\mathbf{b}})$  is invariant under any permutation of the parameters  $\hat{a}_1, \hat{a}_2, \hat{a}_3$ .

*Proof.* Denote  $\hat{a}^* = \min\{\lceil \hat{a}_0/2 \rceil, \hat{a}_1^*\}$  and choose  $C = 1/2 - \hat{a}^*$  in (21) to write

$$\begin{aligned} \hat{r}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) &= -\frac{(-1)^{\hat{d}}}{2} \sum_{m=1-\hat{a}^*}^{\infty} \frac{d\hat{R}(t)}{dt} \Big|_{t=m} \\ &= (-1)^{\hat{d}} \sum_{m=1-\hat{a}^*}^{\infty} \sum_{k=\hat{a}_3^*}^{\hat{b}_2^*-1} \frac{A_k}{(m+k)^3} + \frac{(-1)^{\hat{d}}}{2} \sum_{m=1-\hat{a}^*}^{\infty} \sum_{k=\hat{a}_2^*}^{\hat{b}_3^*-1} \frac{B_k}{(m+k)^2} \\ &= \zeta(3) \cdot (-1)^{\hat{d}} \sum_{k=\hat{a}_3^*}^{\hat{b}_2^*-1} A_k \\ &\quad - (-1)^{\hat{d}} \sum_{k=\hat{a}_3^*}^{\hat{b}_2^*-1} A_k \sum_{\ell=1}^{k-\hat{a}^*} \frac{1}{\ell^3} - \frac{(-1)^{\hat{d}}}{2} \sum_{k=\hat{a}_2^*}^{\hat{b}_3^*-1} B_k \sum_{\ell=1}^{k-\hat{a}^*} \frac{1}{\ell^2}, \end{aligned}$$

where equality (20) was used. In view of the inclusions (18), (19) the found representation of  $\hat{r}(\hat{\mathbf{a}}, \hat{\mathbf{b}})$  implies the form (22). The hypergeometric form (23) follows from

$$\hat{q}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = (-1)^{\hat{d}} \sum_{k=\hat{a}_3^*}^{\hat{b}_2^*-1} A_k$$

and the explicit formula (18) for  $A_k$ . Finally, the invariance of  $\hat{r}(\hat{\mathbf{a}}, \hat{\mathbf{b}})/\hat{\Pi}(\hat{\mathbf{a}}, \hat{\mathbf{b}})$  under permutations of  $\hat{a}_1, \hat{a}_2, \hat{a}_3$  follows from (16) and definition (21) of  $\hat{r}(\hat{\mathbf{a}}, \hat{\mathbf{b}})$ .  $\square$

Similar to our choice in Section 2.2, we take the parameters  $(\hat{\mathbf{a}}, \hat{\mathbf{b}})$  as follows:

$$\begin{aligned} \hat{a}_0 &= \hat{\alpha}_0 n + 2, & \hat{a}_1 &= \hat{\alpha}_1 n + 1, & \hat{a}_2 &= \hat{\alpha}_2 n + 1, & \hat{a}_3 &= \hat{\alpha}_3 n + 1, \\ \hat{b}_0 &= \hat{\beta}_0 n + 2, & \hat{b}_1 &= \hat{\beta}_1 n + 1, & \hat{b}_2 &= \hat{\beta}_2 n + 2, & \hat{b}_3 &= \hat{\beta}_3 n + 2, \end{aligned} \quad (24)$$

where the fixed integers  $\hat{\alpha}_j$  and  $\hat{\beta}_j$ ,  $j = 0, \dots, 3$ , satisfy

$$\begin{aligned} \frac{1}{2}\hat{\beta}_0, \hat{\beta}_1 &< \frac{1}{2}\hat{\alpha}_0\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3 < \hat{\beta}_2, \hat{\beta}_3, \\ \hat{\alpha}_0 + \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\alpha}_3 &= \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3; \end{aligned}$$

note that the equality is assumed in the latter relation (compare to (14)) to simplify the asymptotic consideration in Proposition 4. The quantities (22) then depend on  $n = 0, 1, 2, \dots$ ; we write  $\hat{r}_n = \hat{r}(\hat{\mathbf{a}}, \hat{\mathbf{b}})$ ,  $\hat{q}_n = \hat{q}(\hat{\mathbf{a}}, \hat{\mathbf{b}})$ ,  $\hat{p}_n = \hat{p}(\hat{\mathbf{a}}, \hat{\mathbf{b}})$  and identify the characteristics  $\hat{c}_1 = \hat{\gamma}_1 n$  and  $\hat{c}_2 = \hat{\gamma}_2 n$  of Proposition 3. Proving the analytical part of the following statement is again similar to what is done in [Zud04, Lemma 12 or Lemma 20], while the arithmetic part follows from the results in [Zud04, Section 7] (cf. [Zud04, Lemma 19]).

**Proposition 4.** *In the above notation, let  $\hat{\tau}_0, \overline{\hat{\tau}_0} \in \mathbb{C} \setminus \mathbb{R}$  and  $\hat{\tau}_1 \in \mathbb{R}$  be the zeroes of the cubic polynomial  $(\tau - \hat{\alpha}_0/2)^2 \prod_{j=1}^3 (\tau - \hat{\alpha}_j) - (\tau - \hat{\beta}_0/2)^2 \prod_{j=1}^3 (\tau - \hat{\beta}_j)$ . Define*

$$\begin{aligned} \hat{f}_0(\tau) &= \hat{\alpha}_0 \log(\tau - \hat{\alpha}_0/2) - \hat{\beta}_0 \log(\tau - \hat{\beta}_0/2) + \sum_{j=1}^3 (\hat{\alpha}_j \log(\tau - \hat{\alpha}_j) - \hat{\beta}_j \log(\tau - \hat{\beta}_j)) \\ &\quad - (\hat{\alpha}_0 - \hat{\beta}_0) \log(\hat{\alpha}_0/2 - \hat{\beta}_0/2) - (\hat{\alpha}_1 - \hat{\beta}_1) \log(\hat{\alpha}_1 - \hat{\beta}_1) \\ &\quad + (\hat{\beta}_2 - \hat{\alpha}_2) \log(\hat{\beta}_2 - \hat{\alpha}_2) + (\hat{\beta}_3 - \hat{\alpha}_3) \log(\hat{\beta}_3 - \hat{\alpha}_3). \end{aligned}$$

Then

$$\limsup_{n \rightarrow \infty} \frac{\log |\hat{r}_n|}{n} = \operatorname{Re} \hat{f}_0(\hat{\tau}_0) \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log |\hat{q}_n|}{n} = \operatorname{Re} \hat{f}_0(\hat{\tau}_1).$$

Furthermore,

$$\hat{\Phi}_n^{-1} \hat{q}_n, 2\hat{\Phi}_n^{-1} D_{\hat{\gamma}_1 n} D_{\hat{\gamma}_2 n}^2 \hat{p}_n \in \mathbb{Z}$$

with

$$\hat{\Phi}_n = \prod_{\substack{p \text{ prime} \\ p \leq \min\{\hat{\gamma}_1, \hat{\gamma}_2\}n}} p^{\hat{\varphi}(n/p)},$$

where

$$\begin{aligned} \hat{\varphi}(x) &= \min_{0 \leq y < 1} \left( [2y - \hat{\beta}_0 x] - [2y - \hat{\alpha}_0 x] - [(\hat{\alpha}_0 - \hat{\beta}_0)x] \right. \\ &\quad + [y - \hat{\beta}_1 x] - [y - \hat{\alpha}_1 x] - [(\hat{\alpha}_1 - \hat{\beta}_1)x] \\ &\quad + [(\hat{\beta}_2 - \hat{\alpha}_2)x] - [\hat{\beta}_2 x - y] - [y - \hat{\alpha}_2 x] \\ &\quad \left. + [(\hat{\beta}_3 - \hat{\alpha}_3)x] - [\hat{\beta}_3 x - y] - [y - \hat{\alpha}_3 x] \right), \end{aligned}$$

so that we have

$$\lim_{n \rightarrow \infty} \frac{\log \hat{\Phi}_n}{n} = \int_0^1 \hat{\varphi}(x) \, d\psi(x) - \int_0^{1/\min\{\hat{\gamma}_1, \hat{\gamma}_2\}} \hat{\varphi}(x) \frac{dx}{x^2}.$$

### 3. SIMULTANEOUS DIOPHANTINE PROPERTIES OF $\zeta(2)$ AND $\zeta(3)$

In this section we prove Theorem 1 stated in the introduction by combining the constructions of Subsect. 2.2 and 2.3.

**Construction 1.** If we specialize the set of parameters  $(\mathbf{a}, \mathbf{b})$  of Subsect. 2.2 to be

$$\begin{aligned} a_1 &= 8n + 1, & a_2 &= 7n + 1, & a_3 &= 10n + 1, & a_4 &= 9n + 1, \\ b_1 &= 1, & b_2 &= n + 1, & b_3 &= 2n + 1, & b_4 &= 15n + 2, \end{aligned} \quad (25)$$

then Propositions 1 and 2 imply that

$$r_n = r(\mathbf{a}, \mathbf{b}) = q_n \zeta(2) - p_n, \quad \text{where } \Phi_n^{-1} q_n, \Phi_n^{-1} D_{8n} D_{16n} p_n \in \mathbb{Z}, \quad (26)$$

and

$$q_n = \frac{(-1)^n (9n)! (10n)!}{n! (2n)! (3n)! (5n)! (8n)!} {}_4F_3 \left( \begin{matrix} -5n, 10n + 1, 9n + 1, 8n + 1 \\ 3n + 1, 2n + 1, n + 1 \end{matrix} \middle| 1 \right). \quad (27)$$

The corresponding function  $\varphi(x)$  which defines  $\Phi_n$  is

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in \left[ \frac{1}{10}, \frac{1}{9} \right) \cup \left[ \frac{1}{7}, \frac{2}{9} \right) \cup \left[ \frac{2}{7}, \frac{1}{3} \right) \cup \left[ \frac{2}{5}, \frac{1}{2} \right) \cup \left[ \frac{5}{9}, \frac{4}{7} \right) \cup \left[ \frac{2}{3}, \frac{5}{7} \right) \cup \left[ \frac{4}{5}, \frac{6}{7} \right), \\ 2 & \text{if } x \in \left[ \frac{1}{9}, \frac{1}{8} \right) \cup \left[ \frac{2}{9}, \frac{1}{4} \right) \cup \left[ \frac{1}{3}, \frac{3}{8} \right) \cup \left[ \frac{4}{7}, \frac{5}{8} \right) \cup \left[ \frac{5}{7}, \frac{3}{4} \right) \cup \left[ \frac{6}{7}, \frac{7}{8} \right), \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\lim_{n \rightarrow \infty} \frac{\log \Phi_n}{n} = 6.61268356 \dots,$$

and the growth of  $r_n$  and  $q_n$  as  $n \rightarrow \infty$  is determined by

$$\limsup_{n \rightarrow \infty} \frac{\log |r_n|}{n} = -19.10095491 \dots \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log |q_n|}{n} = 27.86755317 \dots$$

**Construction 2.** If we specialize the set of parameters  $(\hat{\mathbf{a}}, \hat{\mathbf{b}})$  of Subsect. 2.3 to be

$$\begin{aligned} \hat{a}_0 &= 16n + 2, & \hat{a}_1 &= 8n + 1, & \hat{a}_2 &= 9n + 1, & \hat{a}_3 &= 10n + 1, \\ \hat{b}_0 &= 11n + 2, & \hat{b}_1 &= 1, & \hat{b}_2 &= 16n + 2, & \hat{b}_3 &= 16n + 2, \end{aligned} \quad (28)$$

we obtain from Propositions 3 and 4 that

$$\hat{r}_n = \hat{r}(\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \hat{q}_n \zeta(3) - \hat{p}_n, \quad \text{where } \hat{\Phi}_n^{-1} \hat{q}_n, 2\hat{\Phi}_n^{-1} D_{8n}^3 \hat{p}_n \in \mathbb{Z}, \quad (29)$$

and

$$\hat{q}_n = \frac{(7n)! (9n)! (10n)!}{n! (2n)! (4n)! (5n)! (6n)! (8n)!} {}_5F_4 \left( \begin{matrix} -6n, -6n, 10n + 1, \frac{9}{2}n + \frac{1}{2}, \frac{9}{2}n + 1 \\ 2n + 1, n + 1, 2n + \frac{1}{2}, 2n + 1 \end{matrix} \middle| 1 \right). \quad (30)$$

The corresponding function  $\hat{\varphi}(x)$  assumes the form

$$\hat{\varphi}(x) = \begin{cases} 1 & \text{if } x \in \left[\frac{1}{10}, \frac{1}{8}\right) \cup \left[\frac{1}{7}, \frac{1}{4}\right) \cup \left[\frac{2}{7}, \frac{1}{3}\right) \cup \left[\frac{3}{7}, \frac{1}{2}\right) \cup \left[\frac{5}{9}, \frac{4}{7}\right) \cup \left[\frac{3}{5}, \frac{5}{8}\right) \cup \left[\frac{2}{3}, \frac{5}{7}\right) \cup \left[\frac{5}{6}, \frac{6}{7}\right), \\ 2 & \text{if } x \in \left[\frac{1}{3}, \frac{3}{8}\right) \cup \left[\frac{4}{7}, \frac{3}{5}\right) \cup \left[\frac{5}{7}, \frac{3}{4}\right) \cup \left[\frac{6}{7}, \frac{7}{8}\right), \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\lim_{n \rightarrow \infty} \frac{\log \hat{\Phi}_n}{n} = \varphi = 5.70169601\dots,$$

and the growth of  $\hat{r}_n$  and  $\hat{q}_n$  as  $n \rightarrow \infty$  is determined by

$$\limsup_{n \rightarrow \infty} \frac{\log |\hat{r}_n|}{n} = -\rho = -19.10095491\dots \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log |\hat{q}_n|}{n} = \kappa = 27.86755317\dots$$

with the same letters  $\varphi$ ,  $\kappa$  and  $\rho$  as in the introduction.

**Connection between the constructions.** Surprisingly — and this could be guessed from the asymptotics above, the coefficients in (26) of  $\zeta(2)$  and in (29) of  $\zeta(3)$  coincide:  $q_n = \hat{q}_n$ . This follows from the following classical identity — Whipple's transformation [Sla66, p. 65, eq. (2.4.2.3)], in which we assume that  $b = -N$  is a negative integer:

$$\begin{aligned} {}_4F_3 \left( \begin{matrix} f, 1+f-h, h-a, b \\ h, 1+f+a-h, g \end{matrix} \middle| 1 \right) &= \frac{(g-f)_N}{(g)_N} \\ &\times {}_5F_4 \left( \begin{matrix} a, b, 1+f-g, \frac{1}{2}f, \frac{1}{2}f + \frac{1}{2} \\ h, 1+f+a-h, \frac{1}{2}(1+f+b-g), \frac{1}{2}(1+f+b-g) + \frac{1}{2} \end{matrix} \middle| 1 \right). \end{aligned} \quad (31)$$

The particular choices (25) and (28) correspond to taking  $a = b = -6n$ ,  $f = 9n + 1$ ,  $h = n + 1$  and  $g \rightarrow -n + 1$  in (31). The equality  $q_n = \hat{q}_n$  can be alternatively established by examining the recurrence equation satisfied by both  $q_n$  and  $\hat{q}_n$ ; we outline the equation in our proof of Theorem 1 below.

Note that we also have  $\Phi_n$  divisible by  $\hat{\Phi}_n$  in the construction above, so that we can 'merge' the corresponding arithmetic properties (26) and (29) as follows:

$$\hat{\Phi}_n^{-1} q_n, \hat{\Phi}_n^{-1} D_{8n} D_{16n} p_n, 2\hat{\Phi}_n^{-1} D_{8n}^3 \hat{p}_n \in \mathbb{Z}. \quad (32)$$

In both situations we get

$$\lim_{n \rightarrow \infty} \frac{\log(\hat{\Phi}_n^{-1} D_{8n} D_{16n})}{n} = \lim_{n \rightarrow \infty} \frac{\log(2\hat{\Phi}_n^{-1} D_{8n}^3)}{n} = 24 - \varphi = 18.29830398\dots$$

and

$$\lim_{n \rightarrow \infty} \frac{\log |\hat{q}_n|}{n} = \kappa = 27.86755317\dots,$$

so that both families of rational approximations to  $\zeta(2)$  and  $\zeta(3)$  are diophantine:

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\log |\hat{\Phi}_n^{-1} D_{8n} D_{16n} \hat{r}_n|}{n} &= \limsup_{n \rightarrow \infty} \frac{\log |2\hat{\Phi}_n^{-1} D_{8n}^3 \hat{r}_n|}{n} \\ &= 24 - \varphi - \rho = -0.80265093\dots < 0. \end{aligned}$$

*Proof of Theorem 1.* Using the notation above we define  $\tau_0$  and  $S$  in accordance with (1).

To prove the theorem, we use a recurrence relation satisfied by  $q_n$ ,  $p_n$  and  $\hat{p}_n$ . We execute the Gosper–Zeilberger algorithm of creative telescoping separately for the rational function  $R_n(t) = R(t)$  defined in (5) and specialised by (25), and for  $\hat{R}_n(t) = \hat{R}(t)$  defined in (15) with the choice of parameters (28). The results in both cases are polynomials  $P_0(n), \dots, P_3(n) \in \mathbb{Z}[n]$  and rational functions  $S_n(t), \hat{S}_n(t)$  such that

$$\begin{aligned} P_3(n)R_{n+3}(t) + P_2(n)R_{n+2}(t) + P_1(n)R_{n+1}(t) + P_0(n)R_n(t) &= S_n(t+1) - S_n(t), \\ P_3(n)\hat{R}_{n+3}(t) + P_2(n)\hat{R}_{n+2}(t) + P_1(n)\hat{R}_{n+1}(t) + P_0(n)\hat{R}_n(t) &= \hat{S}_n(t+1) - \hat{S}_n(t). \end{aligned}$$

Applying then the argument as in the proof of Theorem 5.4 in [BBBC07] we find out that both the hypergeometric integrals

$$r_n = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left( \frac{\pi}{\sin \pi t} \right)^2 R_n(t) dt \quad \text{and} \quad \hat{r}_n = \frac{1}{4\pi i} \int_{-i\infty}^{i\infty} \left( \frac{\pi}{\sin \pi t} \right)^2 \hat{R}_n(t) dt$$

satisfy the *same* recurrence equation

$$P_3(n)y_{n+3} + P_2(n)y_{n+2} + P_1(n)y_{n+1} + P_0(n)y_n = 0.$$

Since  $r_n = q_n\zeta(2) - p_n$ ,  $\hat{r}_n = q_n\zeta(3) - \hat{p}_n$  and both  $\zeta(2)$  and  $\zeta(3)$  are irrational, we deduce that the coefficients  $q_n$ ,  $p_n$  and  $\hat{p}_n$  satisfy the same equation. Using this fact we obtain that the sequence of determinants

$$\Delta_n = \begin{vmatrix} q_n & q_{n+1} & q_{n+2} \\ p_n & p_{n+1} & p_{n+2} \\ \hat{p}_n & \hat{p}_{n+1} & \hat{p}_{n+2} \end{vmatrix}$$

satisfies the recurrence equation  $P_3(n)\Delta_{n+1} + P_0(n)\Delta_n = 0$ . The coefficients of  $P_3(n)$  are all positive, while the coefficients of  $P_0(n)$  are all negative; the details of this computation can be found on the webpage [Dau14] of the first author. This implies that the nonvanishing of  $\Delta_n$  for some  $n$  is equivalent to the nonvanishing of  $\Delta_0$ . We have explicitly

$$\begin{aligned} q_0 &= 1, & q_1 &= 12307565655, & q_2 &= 5669931265166541788415, \\ p_0 &= 0, & p_1 &= \frac{199536684432021}{9856}, & p_2 &= \frac{6500408024275547867356589727409007}{696970391040}, \\ \hat{p}_0 &= 0, & \hat{p}_1 &= \frac{7953492001094261}{537600}, & \hat{p}_2 &= \frac{37762843816152998347068580008855083}{5540664729600}, \end{aligned}$$

so that

$$\Delta_0 = \begin{vmatrix} q_0 & q_1 & q_2 \\ p_0 & p_1 & p_2 \\ \hat{p}_0 & \hat{p}_1 & \hat{p}_2 \end{vmatrix} = \frac{288666665737256181552839214834819523}{107268868422523551744000} \neq 0.$$

Thus,  $\Delta_n \neq 0$  for any  $n \geq 0$ .

Now let  $\varepsilon, \eta > 0$ ; for simplicity we may assume  $\eta \leq \varepsilon$ . Let  $m$  be a sufficiently large integer as in the statement of Theorem 1. Let  $a_0, a_1, a_2$  satisfy the hypotheses

in Theorem 1. We take  $n = \lceil m/8 \rceil$ , so that  $8n - 7 \leq m \leq 8n$ . Since the determinant  $\Delta_n$  does not vanish, there exists an  $\ell \in \{n, n+1, n+2\}$  such that

$$a_0q_\ell + a_1p_\ell + a_2\hat{p}_\ell \neq 0.$$

Now we have  $m \leq 8n \leq 8\ell$ , so that  $D_{8\ell}^2 D_{16\ell} a_0 \in \mathbb{Z}$  and  $D_{8\ell} a_1 \in \mathbb{Z}$ . Letting  $e_{m,\ell}$  denote the least positive integer such that  $2D_{8\ell} \mid e_{m,\ell} D_m$ , we get the property

$$\frac{D_{2m}}{D_m} \mid e_{m,\ell} \frac{D_{16\ell}}{2D_{8\ell}},$$

so that  $e_{m,\ell} \frac{D_{16\ell}}{2D_{8\ell}} a_2 \in \mathbb{Z}$ . Therefore, using the arithmetic properties of  $q_\ell$ ,  $p_\ell$  and  $\hat{p}_\ell$  we conclude that

$$e_{m,\ell} (D_{8\ell}^2 D_{16\ell} a_0) (\Phi_\ell^{-1} q_\ell) + e_{m,\ell} (D_{8\ell} a_1) (\Phi_\ell^{-1} D_{8\ell} D_{16\ell} p_\ell) + e_{m,\ell} \left( \frac{D_{16\ell}}{2D_{8\ell}} a_2 \right) (2\Phi_\ell^{-1} D_{8\ell}^3 \hat{p}_\ell) \quad (33)$$

is a nonzero integer. Note that  $\ell \leq \frac{m}{8} + 3$ , so that the asymptotic contribution of  $e_{m,\ell}$  is almost invisible:  $e_{m,\ell} \leq 2(8\ell)^{24} = e^{o(\ell)}$ .

Let us bound the integer (33) from above. Writing hypothesis (ii) as

$$|a_0 + a_1\zeta(2) + a_2\zeta(3)| \leq e^{-(S+\eta)m} \leq e^{-(32-\varphi+\kappa+8\eta)(n-1)},$$

we obtain

$$\begin{aligned} & |a_0q_\ell + a_1p_\ell + a_2\hat{p}_\ell| \\ & \leq |q_\ell| |a_0 + a_1\zeta(2) + a_2\zeta(3)| + |a_1| |q_\ell\zeta(2) - p_\ell| + |a_2| |q_\ell\zeta(3) - \hat{p}_\ell| \\ & \leq e^{-(32-\varphi+8\varepsilon)n+o(n)}, \end{aligned}$$

since  $\varepsilon \leq \eta$ . On the other hand, the common denominator of the coefficients used above is

$$e_{m,\ell} D_{8\ell}^2 D_{16\ell} \Phi_\ell^{-1} \leq e^{(2 \cdot 8 + 16 - \varphi)\ell + o(\ell)} = e^{(32 - \varphi)n + o(n)}.$$

This means that the non-zero integer (33) has absolute value at most  $e^{-8\varepsilon n + o(n)}$ , which is not possible for a sufficiently large  $n$ , thus implying the truth of Theorem 1.  $\square$

#### 4. A NEW DIOPHANTINE EXPONENT

**4.1. Definition and basic properties.** We now introduce a new exponent that depends on some  $\tau \in \mathbb{R}$  and is related to Theorem 1.

**Definition 1.** Let  $\xi_1, \xi_2 \in \mathbb{R}$  and  $\tau \in \mathbb{R}$ . We denote by  $s_\tau(\xi_1, \xi_2)$  the infimum of the set  $E_\tau(\xi_1, \xi_2)$  of all  $s \in \mathbb{R}$  with the following property. Let  $\varepsilon > 0$  and  $n$  be sufficiently large in terms of  $\varepsilon$ . Let  $(a_0, a_1, a_2) \in \mathbb{Q}^3 \setminus \{\mathbf{0}\}$  be such that:

- (i)  $D_n^2 D_{2n} a_0 \in \mathbb{Z}$ ,  $D_n a_1 \in \mathbb{Z}$  and  $\frac{D_{2n}}{D_n} a_2 \in \mathbb{Z}$ ; and
- (ii)  $|a_0|, |a_1|, |a_2|$  are bounded from above by  $e^{-(\tau+\varepsilon)n}$ .



Then  $|a_0 + a_1\xi_1 + a_2\xi_2| > e^{-sn}$ .

By convention, we set  $s_\tau(\xi_1, \xi_2) = +\infty$  if  $E_\tau(\xi_1, \xi_2) = \emptyset$ , and  $s_\tau(\xi_1, \xi_2) = -\infty$  if  $E_\tau(\xi_1, \xi_2) = \mathbb{R}$ .

This definition allows us to restate Theorem 1 as follows.

**Theorem 2.** *With  $\tau_0 = 0.899668635\dots$  and  $S = 6.770732145\dots$  as in (1), we have  $s_{\tau_0}(\zeta(2), \zeta(3)) \leq S$ .*

To begin with, let us state and prove general results on this diophantine exponent  $s_\tau(\xi_1, \xi_2)$  depending on the range when  $\tau$  varies; it turns out that it carries diophantine information on  $\xi_1$  and  $\xi_2$  only if  $\tau < 1$ .

**Proposition 5.** (1) *If  $\tau > 4$ , then  $s_\tau(\xi_1, \xi_2) = -\infty$ .*

(2) *If  $1 \leq \tau \leq 4$ , then  $s_\tau(\xi_1, \xi_2) = 4$ .*

(3) *If  $\tau < 1$ , then  $s_\tau(\xi_1, \xi_2) \geq 6 - 2\tau$ .*

(4) *If  $\tau < 1$  and at least one of  $\xi_1$  or  $\xi_2$  is rational, then  $s_\tau(\xi_1, \xi_2) = +\infty$ .*

(5) *If  $\tau < 0$  and the numbers  $1, \xi_1$  and  $\xi_2$  are linearly dependent over  $\mathbb{Q}$ , then  $s_\tau(\xi_1, \xi_2) = +\infty$ .*

(6) *If  $\tau \leq \tau'$ , then  $s_\tau(\xi_1, \xi_2) \geq s_{\tau'}(\xi_1, \xi_2)$ .*

*Proof.* (1) We see that whenever the coefficient  $a_i$  is not zero, we must have  $|a_i| \geq 1/(D_n^2 D_{2n}) = e^{-4n+o(n)}$  if  $i = 0$ , and an even larger estimate from below (namely,  $e^{-n+o(n)}$ ) if  $i = 1$  or  $2$ . Therefore, having at least one triple  $(a_0, a_1, a_2) \in \mathbb{Q}^3 \setminus \{\mathbf{0}\}$  that satisfies both (i) and (ii) of Definition 1 means  $\tau \leq 4$ ; having no such triple implies  $E_\tau(\xi_1, \xi_2) = \mathbb{R}$ .

(2) Assuming now  $1 \leq \tau \leq 4$  in Definition 1 and choose  $n$  sufficiently large to accommodate  $D_n < e^{(1+\varepsilon)n}$  and  $D_{2n}/D_n < e^{(1+\varepsilon)n}$ . Condition (ii) implies that  $|a_1| \leq e^{-(\tau+\varepsilon)n} \leq e^{-n-\varepsilon n}$ , so that the integer  $|D_n a_1| \leq D_n e^{-n-\varepsilon n} < 1$  must be zero,  $a_1 = 0$ . Similar consideration shows that  $a_2 = 0$ , hence the only nonzero element in the triple  $(a_0, a_1, a_2) \in \mathbb{Q}^3 \setminus \{\mathbf{0}\}$  is  $a_0$ . Then condition (i) implies that  $|a_0| \geq 1/(D_n^2 D_{2n}) = e^{-4n+o(n)}$  with the equality possible by simply taking  $a_0 = 1/(D_n^2 D_{2n})$ . Thus,  $s_\tau(\xi_1, \xi_2) = 4$  for all  $\xi_1, \xi_2$  whenever  $4 \geq \tau \geq 1$ .

(3) Take  $s < 6 - 2\tau$  and define  $\varepsilon = \frac{1}{3}(6 - 2\tau - s)$ , so that  $s = 6 - 2\tau - 3\varepsilon > \tau + \varepsilon/2$  because of  $\tau < 1$ . Let  $n$  be sufficiently large to have  $(D_n D_{2n})^2 > e^{(6-\varepsilon)n} = e^{-(s+2\tau+2\varepsilon)n}$  satisfied. Define the set

$$K = \{(x_0, x_1, x_2) \in \mathbb{R}^3 : |x_1|, |x_2| \leq e^{-(\tau+\varepsilon)n}, |x_0 + x_1\xi_1 + x_2\xi_2| \leq e^{-sn}\} \subset \mathbb{R}^3,$$

which is compact, convex, symmetric with respect to  $\mathbf{0}$  and has volume  $8e^{-(s+2\tau+2\varepsilon)n}$ . Consider the lattice

$$\Gamma = \frac{1}{D_n^2 D_{2n}} \mathbb{Z} \oplus \frac{1}{D_n} \mathbb{Z} \oplus \frac{D_n}{D_{2n}} \mathbb{Z},$$

whose fundamental domain has volume

$$\frac{1}{D_n^2 D_{2n}} \cdot \frac{1}{D_n} \cdot \frac{D_n}{D_{2n}} < e^{-(s+2\tau+2\varepsilon)n}.$$

By Minkowski's theorem,  $K$  contains a nonzero point  $(a_0, a_1, a_2)$  of the lattice  $\Gamma$ , for which we have

$$\begin{aligned} |a_0| &\leq |a_1| |\xi_1| + |a_2| |\xi_2| + |a_0 + a_1\xi_1 + a_2\xi_2| \\ &\leq (|\xi_1| + |\xi_2|)e^{-(\tau+\varepsilon)n} + e^{-sn} \leq e^{-(\tau+\varepsilon/2)n}. \end{aligned}$$

The estimate means that  $s \notin E_\tau(\xi_1, \xi_2)$ ; as  $s_\tau(\xi_1, \xi_2)$  is the infimum of the set  $E_\tau(\xi_1, \xi_2)$ , we get  $s_\tau(\xi_1, \xi_2) \geq 6 - 2\tau$ .

(4) Assume  $\xi_1 = p/q \in \mathbb{Q}$ , take  $\varepsilon \in (0, 1 - \tau)$ . By choosing  $a_0 = q\xi_1/D_n$ ,  $a_1 = -q/D_n$  and  $a_2 = 0$  we see that properties (i) and (ii) in the definition of  $E_\tau(\xi_1, \xi_2)$  are satisfied for *any*  $n$  sufficiently large. In addition,  $|a_0 + a_1\xi_1 + a_2\xi_2| = 0 < e^{-sn}$  for any  $s \in \mathbb{R}$ , which means that  $E_\tau(\xi_1, \xi_2) = \emptyset$ , hence  $s_\tau(\xi_1, \xi_2) = +\infty$ .

If  $\xi_2 = p/q \in \mathbb{Q}$ , then the choice  $a_0 = q\xi_2 D_n/D_{2n}$ ,  $a_1 = 0$  and  $a_2 = -qD_n/D_{2n}$  does the job.

(5) Assume now that there exist integers  $q_0, q_1$  and  $q_2$ , not all zero, such that  $p_0 + p_1\xi_1 + p_2\xi_2 = 0$ . Setting  $a_0 = p_0$ ,  $a_1 = p_1$  and  $a_2 = p_2$  we see that properties (i) and (ii) are satisfied with any choice of  $\tau < 0$  and  $\varepsilon$ , for all  $n$  sufficiently large in terms of  $\varepsilon$ . At the same time  $|a_0 + a_1\xi_1 + a_2\xi_2| = 0 < e^{-sn}$  for any  $s \in \mathbb{R}$ , meaning that  $E_\tau(\xi_1, \xi_2) = \emptyset$ , hence  $s_\tau(\xi_1, \xi_2) = +\infty$ .

(6) Using (1), (2) and (3), we may assume that  $\tau' < 1$ . Let  $s \in E_\tau(\xi_1, \xi_2)$  meaning that for all  $\varepsilon > 0$  and for all triples  $(a_0, a_1, a_2) \in \mathbb{Q}^3 \setminus \{\mathbf{0}\}$  which satisfy  $D_n^2 D_{2n} a_0, D_n a_1, \frac{D_{2n}}{D_n} a_2 \in \mathbb{Z}$  and  $|a_0|, |a_1|, |a_2| \leq e^{-(\tau+\varepsilon)n}$ , we have  $|a_0 + a_1\xi_1 + a_2\xi_2| > e^{-sn}$ . For  $n$  be sufficiently large and  $(a_0, a_1, a_2) \in \mathbb{Q}^3 \setminus \{\mathbf{0}\}$  such that  $D_n^2 D_{2n} a_0, D_n a_1, \frac{D_{2n}}{D_n} a_2 \in \mathbb{Z}$  and  $|a_i| \leq e^{-(\tau'+\varepsilon)n}$ , we also have  $|a_i| \leq e^{-(\tau+\varepsilon)n}$ . This means that  $|a_0 + a_1\xi_1 + a_2\xi_2| > e^{-sn}$  and  $s \in E_{\tau'}(\xi_1, \xi_2)$ , so that  $E_\tau(\xi_1, \xi_2) \subset E_{\tau'}(\xi_1, \xi_2)$ , which leads to claim (6) by taking the infimum of both sets.  $\square$

From now on we assume  $\tau$  to be real  $< 1$ .

**Remarks.** Theorem 2 is nontrivial since  $\tau_0 < 1$ . However, it does not imply that 1,  $\zeta(2)$  and  $\zeta(3)$  are  $\mathbb{Q}$ -linearly independent since  $\tau_0 > 0$ .

Part (3) of Proposition 5 yields  $s_{\tau_0}(\zeta(2), \zeta(3)) \geq 4.20$ , so that the statement of Theorem 2 is far from being best possible.

The fact that  $s_{\tau_0}(\zeta(2), \zeta(3)) < +\infty$  in Theorem 2 is already new.

**4.2. Omitting one number.** Recall the definition of the usual exponent of irrationality of  $\mu(\xi)$  of a number  $\xi \in \mathbb{R}$  from the introductory part. Here comes its generalisation, the  $\psi$ -exponent of irrationality, given by S. Fischler in [Fis09].

**Definition 2.** Let  $\mathcal{E}$  be the set of all  $\psi: \mathbb{N}^* \rightarrow \mathbb{N}^*$  with the following properties: for any  $q \geq 1$ ,  $\psi(q+1)$  is a multiple of  $\psi(q)$ , and the limit

$$\gamma_\psi = \lim_{q \rightarrow \infty} \frac{\log \psi(q)}{\log q}$$

exists and belongs to the interval  $[0, 1)$ . For  $\psi \in \mathcal{E}$  and  $\xi \in \mathbb{R} \setminus \mathbb{Q}$ , denote by  $\mu_\psi(\xi)$  the supremum of the set  $M_\psi(\xi)$  of all  $\mu \in \mathbb{R}$  such that there are infinitely many

$q \geq 1$  which are divisible by  $\psi(q)$  and satisfy

$$\left| \xi - \frac{p}{q} \right| \leq \frac{1}{q^\mu} \quad \text{for some } p \in \mathbb{Z}.$$

If  $M_\psi(\xi)$  is not bounded from above, that is, if  $M_\psi(\xi) = \mathbb{R}$ , we get  $\mu_\psi(\xi) = +\infty$ .

An equivalent way of defining  $\mu_\psi(\xi)$ , is by letting  $\mu_\psi(\xi)$  be the infimum of the set of exponents  $\mu$  such that for all  $q$  large enough with  $\psi(q) \mid q$  one has  $|\xi - p/q| > 1/q^{-\mu}$ , and taking  $\mu_\psi(\xi) = +\infty$  if the set is empty.

When  $\psi(q) = 1$  for all  $q$ , the  $\psi$ -exponent  $\mu_\psi(\xi)$  coincides with the usual exponent of irrationality  $\mu(\xi)$ . It is known [Fis09, Corollary 3] that  $\mu_\psi(\xi) = +\infty$  if and only if  $\xi$  is a Liouville number, that is,  $\mu(\xi) = +\infty$ . If this is not the case, then

$$(1 - \gamma_\psi)\mu(\xi) \leq \mu_\psi(\xi) \leq \mu(\xi).$$

Fischler proves in [Fis09] that  $\mu_\psi(\xi) \geq 2 - \gamma_\psi$  for any  $\psi \in \mathcal{E}$  and any  $\xi \in \mathbb{R} \setminus \mathbb{Q}$ , with the equality holding for almost all  $\xi \in \mathbb{R}$  in the sense of Lebesgue measure. More precisely, he shows that, given an  $\eta > 2 - \gamma_\psi$ , the set of  $\xi$  such that  $\mu_\psi(\xi) > \eta$  has Hausdorff dimension  $(2 - \gamma_\psi)/\eta$ .

The usual construction of a function  $\psi \in \mathcal{E}$  is as follows. One takes  $\psi(q) = \delta_n$  with  $n = \lfloor (\log q)/(\delta - \alpha) \rfloor$ , where  $(\delta_n)_{n \geq 1}$  is a sequence of positive integers such that  $\delta_n$  divides  $\delta_{n+1}$  for each  $n \geq 1$  and  $\delta_n = e^{\delta n + o(n)}$  as  $n \rightarrow \infty$ , while  $\alpha \in \mathbb{R}$  is chosen to satisfy  $\alpha < \delta$ . In this construction, we have  $\gamma_\psi = \delta/(\delta - \alpha)$ .

Definition 2 allows us deducing diophantine results involving only quantity,  $\xi_1$  or  $\xi_2$ , from a nontrivial upper bound for the exponent  $s_\tau(\xi_1, \xi_2)$  from Definition 1.

**Proposition 6.** *Let  $\xi_1, \xi_2$  be real numbers and  $\tau < 1$ . Define  $\psi_1, \psi_2: \mathbb{N}^* \rightarrow \mathbb{N}^*$  by taking  $\psi_1(q) = D_n D_{2n}$  and  $\psi_2(q) = D_n^3$ , where  $n = \lfloor (\log q)/(4 - \tau) \rfloor$ . Then*

$$\mu_{\psi_i}(\xi_i) \leq \frac{s_\tau(\xi_1, \xi_2) - \tau}{4 - \tau} \quad \text{for } i = 1, 2.$$

*Proof.* Let  $\tau' \in \mathbb{R}$  satisfy  $\tau < \tau' < 1$ . Take  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}^*$  sufficiently large,  $\psi_1(q) \mid q$ , and  $m = \lfloor (\log q)/(4 - \tau') \rfloor$ , so that  $\psi_1(q) = D_m D_{2m}$ . We may assume that  $|p/q - \xi_1| < 1$ .

For an  $s > s_\tau(\xi_1, \xi_2)$ , choose  $\varepsilon > 0$  such that

$$\varepsilon < \frac{1}{2} \min \left\{ \tau' - \tau, \frac{(s-4)(\tau' - \tau)}{2s - \tau' - 4} \right\};$$

part (3) of Proposition 5 implies  $s > 4$ .

Take

$$n = \left\lfloor \frac{4 - \tau'}{4 - \tau - 2\varepsilon} (m + 1) \right\rfloor + 1 < m, \quad a_0 = \frac{p}{D_n^2 D_{2n}} \quad \text{and} \quad a_1 = \frac{-q}{D_n^2 D_{2n}}.$$

Then

$$D_n^2 D_{2n} a_0 \in \mathbb{Z} \quad \text{and} \quad D_n a_1 = \frac{-q}{D_m D_{2m}} \frac{D_m D_{2m}}{D_n D_{2n}} = \frac{-q}{\psi_1(q)} \frac{D_m D_{2m}}{D_n D_{2n}} \in \mathbb{Z},$$

and  $q < e^{(4-\tau')(m+1)}$  implying that  $|a_1| \leq e^{-(\tau+\varepsilon)n}$ ; for  $a_0$  we have  $|a_0| = |p|e^{-4n+o(n)}$ . Therefore,  $|p| \leq |p - q\xi_1| + |q\xi_1| \leq q(1 + |\xi_1|)$ , which leads to

$$|a_0| \leq q(1 + |\xi_1|)e^{-4n+o(n)} \leq e^{(4-\tau')(m+1)-(4-\varepsilon)n} \leq e^{-(\tau+\varepsilon)n}$$

for  $q$  sufficiently large. Letting  $a_2 = 0$  and using  $s > s_\tau(\xi_1, \xi_2)$  we deduce that  $|a_0 + a_1\xi_1| > e^{-sn}$ ; from the definition of  $a_0$  and  $a_1$  it follows that

$$\left| \frac{p}{q} - \xi_1 \right| > \frac{e^{(4-s-\varepsilon)n}}{q}$$

provided  $q$  is sufficiently large. The assumption on  $\varepsilon$  results in the estimate

$$\left| \frac{p}{q} - \xi_1 \right| > q^{-(s-\tau')/(4-\tau')},$$

which implies  $\mu_{\psi_1}(\xi_1) \leq (s - \tau')/(4 - \tau')$ . This upper bound holds for all  $s > s_\tau(\xi_1, \xi_2)$ ; taking the infimum over  $s$  and then choosing  $\tau' \in (\tau, 1)$  sufficiently close to  $\tau$ , completes the proof for  $i = 1$ . The proof for  $i = 2$  is similar.  $\square$

Since  $\mu_{\psi_i}(\xi_i) \geq 2 - \gamma_{\psi_i}$  with  $\gamma_{\psi_i} = 3/(4 - \tau) < 1$ , Proposition 6 implies the lower bound  $s_\tau(\xi_1, \xi_2) \geq 5 - \tau$ , which is however weaker than the one from statement (3) of Proposition 5.

**Corollary 1.** *For  $\xi_1$  and  $\xi_2$  real numbers and  $\tau < 1$ , the following inequalities hold for the ordinary irrationality exponent:*

$$\mu(\xi_i) \leq \frac{s_\tau(\xi_1, \xi_2) - \tau}{1 - \tau} \quad \text{for } i = 1, 2.$$

*Proof.* In the notation of Proposition 6, use  $(1 - \gamma_{\psi_i})\mu(\xi_i) \leq \mu_{\psi_i}(\xi_i)$ .  $\square$

**4.3. Case of linear dependence.** In this subsection, we prove a converse result to Proposition 6 above; namely, under the linear dependence of 1,  $\xi_1$  and  $\xi_2$  over  $\mathbb{Q}$ , we deduce an upper bound on  $s_\tau(\xi_1, \xi_2)$  from an upper bound on the irrationality exponent of either  $\xi_1$  or  $\xi_2$ .

**Proposition 7.** *For  $\xi_1, \xi_2 \notin \mathbb{Q}$  assume that 1,  $\xi_1$  and  $\xi_2$  are linearly dependent over  $\mathbb{Q}$ . Take  $0 \leq \tau < 1$  and define  $\psi \in \mathcal{E}$  by  $\psi(q) = D_n^2$  with  $n = \lfloor (\log q)/(4 - \tau) \rfloor$ . Then  $\mu_\psi(\xi_1) = \mu_\psi(\xi_2)$  and*

$$s_\tau(\xi_1, \xi_2) \leq 4 + (\mu_\psi(\xi_i) - 1)(4 - \tau) \quad \text{for } i = 1, 2.$$

*In addition,*

$$s_\tau(\xi_1, \xi_2) \leq 6 - \tau$$

*unless both  $\xi_1$  and  $\xi_2$  belong to a certain set of Lebesgue measure zero.*

Note that the inequalities of this proposition do not hold if  $\tau < 0$ , since  $s_\tau(\xi_1, \xi_2) = +\infty$  in this case (see Proposition 5).

*Proof.* The equality  $\mu_\psi(\xi_1) = \mu_\psi(\xi_2)$  is trivially true for any  $\psi \in \mathcal{E}$ .

Let  $\alpha_0, \alpha_1 \in \mathbb{Q}$  be such that  $\alpha_0 + \alpha_1\xi_1 = \xi_2$  and  $\psi$  the function defined in the statement of Proposition 7. Denote by  $A$  a common denominator of  $\alpha_0$  and  $\alpha_1$ . Take  $\varepsilon > 0$ ,  $\nu > 0$ ,  $\mu > \mu_\psi(\xi_1)$  and  $n$  be sufficiently large with respect to  $\varepsilon$ ,  $\nu$  and

$\mu$ . Let  $(a_0, a_1, a_2) \in \mathbb{Q}^3 \setminus \{\mathbf{0}\}$  satisfy  $|a_i| \leq e^{-(\tau+\varepsilon)n}$  and  $D_n^2 D_{2n} a_0, D_n a_1, \frac{D_{2n}}{D_n} a_2 \in \mathbb{Z}$ ; set  $\eta = |a_0 + a_1 \xi_1 + a_2 \xi_2|$ .

To begin with, we claim that  $\eta \neq 0$ . Indeed, if  $\eta = 0$ , then  $a_0 = -\alpha_0 a_2$  and  $a_1 = -\alpha_1 a_2$ , since  $1, \xi_1, \xi_2$  span a  $\mathbb{Q}$ -vector space of dimension 2—there is exactly one  $\mathbb{Q}$ -linear relation among them, up to proportionality. As both

$$D_n a_1 \quad \text{and} \quad A \frac{D_{2n}}{D_n} a_1 = -A \alpha_1 \cdot \frac{D_{2n}}{D_n} a_2$$

are integral, we have  $\delta_n a_1 \in \mathbb{Z}$ , where  $\delta_n = \gcd(D_n, AD_{2n}/D_n) = e^{o(n)}$  as  $n \rightarrow \infty$ . If  $a_1 \neq 0$ , the latter asymptotics leads to the contradiction with  $\delta_n \geq |a_1|^{-1} \geq e^{(\tau+\varepsilon)n} \geq e^{\varepsilon n}$ , since  $\tau \geq 0$ . Therefore,  $a_1 = 0$ , implying  $a_2 = 0$  because  $\xi_2 \notin \mathbb{Q}$ ; finally,  $0 = \eta = |a_0|$ , which is impossible as  $(a_0, a_1, a_2) \neq \mathbf{0}$ . This completes the proof of the claim that  $\eta \neq 0$ .

We write now  $\eta = |\hat{a}_0 + \hat{a}_1 \xi_1|$  with  $\hat{a}_0 = a_0 + \alpha_0 a_2$  and  $\hat{a}_1 = a_1 + \alpha_1 a_2$ .

If  $\hat{a}_1 \neq 0$ , we have  $AD_n^2 D_{2n} \hat{a}_0 \in \mathbb{Z}$  and  $AD_{2n} \hat{a}_1 \in \mathbb{Z}$ . Set  $\tilde{a}_0 = -\text{sign}(\hat{a}_1) AD_n^2 D_{2n} \hat{a}_0 \in \mathbb{Z}$  and  $\tilde{a}_1 = AD_n^2 D_{2n} |\hat{a}_1| \in D_n^2 \mathbb{N}$ . By the assumption,  $\tilde{a}_1 > 0$  implying  $e^{2n+o(n)} \leq \tilde{a}_1 \leq e^{(4-\tau-\varepsilon)n+o(n)} \leq e^{(4-\tau)n}$ . Thus,  $(\log \tilde{a}_1)/(4-\tau) \leq n$  which ensures that  $\psi(\tilde{a}_1) |D_n^2| \tilde{a}_1$ . Since  $\tilde{a}_1 \geq e^{2n+o(n)}$  and  $n$  is sufficiently large in terms of  $\mu > \mu_\psi(\xi_1)$ , we deduce

$$AD_n^2 D_{2n} \eta = |\tilde{a}_0 - \tilde{a}_1 \xi_1| > \frac{1}{\tilde{a}_1^{\mu-1}} > e^{-(\mu-1)(4-\tau)n},$$

so that  $\eta > e^{-(4+(\mu-1)(4-\tau)+\nu)n}$  for  $n$  sufficiently large.

If  $\hat{a}_1 = 0$ , we get  $\eta = |a'_0|$ . Since  $\eta \neq 0$ , this implies  $AD_n^2 D_{2n} \eta \in \mathbb{N}^*$  and thus  $\eta > e^{-4n+o(n)}$ . Furthermore, from  $\gamma_\psi \in [0, 1)$  we deduce that  $\mu_\psi(\xi_1) \geq 2 - \gamma_\psi > 1$ , so that  $(\mu_\psi(\xi_1) - 1)(4 - \tau) > 0$ . Thus, we have  $\eta > e^{-(4+(\mu-1)(4-\tau)+\nu)n}$  for  $n$  sufficiently large in this case as well.

Therefore, in both cases  $4 + (\mu - 1)(4 - \tau) + \nu \in E_\tau(\xi_1, \xi_2)$  for all  $\mu > \mu_\psi(\xi_1)$  and all  $\nu > 0$ . Taking the infimum of  $E_\tau(\xi_1, \xi_2)$  we obtain the desired inequality for  $i = 1$ , and also for  $i = 2$  in view of  $\mu_\psi(\xi_1) = \mu_\psi(\xi_2)$ .

Finally,  $\mu_\psi(\xi) = 2 - \gamma_\psi = 2 - 2/(4 - \tau)$  for almost all  $\xi \in \mathbb{R}$  with respect to the Lebesgue measure, completing the proof.  $\square$

**4.4. Rational approximation to  $\zeta(3)$  only.** Combining Theorem 2 with Proposition 6, we deduce the following result.

**Proposition 8.** *For  $\psi(q) = D_n^3$  with  $n = \lfloor (\log q)/(4 - \tau_0) \rfloor$  and  $\tau_0$  defined in (1), we have the upper bound*

$$\mu_\psi(\zeta(3)) \leq 1.92357696 \dots$$

Let us conclude with a few remarks on this result.

As shown in [Fis09], Apéry's proof of the irrationality of  $\zeta(3)$  leads to the estimate  $\mu_{\psi'}(\zeta(3)) \leq 2$ , where  $\psi'(q) = D_n^3$  with  $n = \lfloor (\log q)/(4 \log(1 + \sqrt{2})) \rfloor$ . Since  $4 \log(1 + \sqrt{2}) > 4 - \tau_0$ , this implies  $\mu_\psi(\zeta(3)) \leq 2$  with the function  $\psi$  in Proposition 8. Therefore, Proposition 8 is slightly sharper than what follows from Apéry's construction.

Proposition 8 can be adapted to  $\zeta(2)$ ; namely, we have  $\mu_{\tilde{\psi}}(\zeta(2)) \leq 1.92$ , where  $\tilde{\psi}(q) = D_n D_{2n}$  with  $n = \lfloor (\log q)/(4 - \tau_0) \rfloor$  and  $\tau_0$  as before. However, this result follows directly from Apéry's construction [Fis09]: Apéry's proof yields  $\mu_{\tilde{\psi}'}(\zeta(2)) \leq 2$ , where  $\tilde{\psi}'(q) = D_n^2$  with  $n = \lfloor (\log q)/(5(\log(1 + \sqrt{5}) - \log 2)) \rfloor$ . Using elementary methods (see [Dau14]), this upper bound can be shown to imply  $\mu_{\tilde{\psi}}(\zeta(2)) \leq 1.26$ , which is sharper than the given one.

In the notation above, the upper bound of Proposition 8 and its analogue for  $\zeta(2)$  imply  $\mu_{\tilde{\psi}'}(\zeta(2)) \leq 15.33$  and  $\mu_{\tilde{\psi}'}(\zeta(3)) \leq 8.73$ : these upper bounds are worse than the ones followed from Apéry's construction.

Proposition 8 means that  $\zeta(3)$  does not belong to the set of  $\xi \in \mathbb{R} \setminus \mathbb{Q}$  satisfying  $\mu_{\psi}(\xi) > 1.92 \dots$ . This set has Hausdorff dimension equal to  $0.0681457 \dots$ ; this is smaller than the one obtained after Corollary 5 in [Fis09].

Finally, for the function  $\psi \in \mathcal{E}$  in Proposition 8 we have that  $\mu_{\psi}(\xi) = 1.03 \dots$  for almost all  $\xi \in \mathbb{R}$ . Therefore, Proposition 8 is still quite far from being optimal, since  $\zeta(3)$  is presumably a 'generic' real number.

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