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# Viscosity Solutions of Fully Nonlinear Elliptic Path Dependent Partial Differential Equations

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## Abstract

This paper extends the recent work on path dependent PDEs to elliptic equations with Dirichlet boundary conditions. We propose a notion of viscosity solution in the same spirit as [10, 11], relying on the theory of optimal stopping under nonlinear expectation. We prove a comparison result implying the uniqueness of viscosity solution, and the existence follows from a Perron-type construction using path-frozen PDEs. We also provide an application to a time homogeneous stochastic control problem motivated by an application in finance.

**Key words:** Viscosity solutions, optimal stopping, path-dependent PDEs, comparison principle, Perron's approach.

**AMS 2000 subject classifications:** 35D40, 35K10, 60H10, 60H30.

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# 1 Introduction

In this paper, we develop a theory of viscosity solutions of elliptic PDEs for non-Markovian stochastic processes, by extending the recent literature on path-dependent PDEs to this context.

Nonlinear path dependent PDEs appear in various applications as the stochastic control of non-Markovian systems [10], and the corresponding stochastic differential games [21]. They are also intimately related to the backward stochastic differential equations introduced by Pardoux and Peng [20], and their extension to the second order in [5, 24]. We also refer to the recent applications in [13] to establish a representation of the solution of a class of PPDEs in terms of branching diffusions, and to [16] for the small time large deviation results of path-dependent diffusions.

Our starting point is the BSDE approach of Darling and Pardoux [7] which can be viewed as a theory of Sobolev solutions for path-dependent PDEs. In particular, the Feynman-Kac representation provided in [7] shows a close connection between Markovian BSDEs with random terminal times and semilinear PDEs with Dirichlet boundary conditions.

Following the recent work of Ekren, Touzi and Zhang [10, 11], we would like to introduce a notion of viscosity solution for an elliptic path dependent PDE on the path space  $\Omega$  of the form:

$$G(\cdot, u, \partial_\omega u, \partial_{\omega\omega}^2 u)(\omega) = 0, \quad \omega \in \mathcal{Q}, \quad \text{and} \quad u = \xi, \quad \omega \in \partial\mathcal{Q},$$

where  $\omega \in \Omega$  is a path variable and  $\mathcal{Q} \subset \Omega$  is a domain of continuous paths. Our notions of the derivatives  $\partial_\omega$  and  $\partial_{\omega\omega}^2$  are inspired by the calculus developed in Dupire [8] as well as Cont and Fournie [2]. According to their work, considering the process  $u_t(\omega) := u(\omega_{t\wedge\cdot})$ , one may define the horizontal and vertical derivatives:

$$\partial_t u_t(\omega) := \lim_{h \rightarrow 0} \frac{u_{t+h}(\omega_{t\wedge\cdot}) - u_t(\omega)}{h} \quad \text{and} \quad \partial_\omega u_t(\omega) := \lim_{h \rightarrow 0} \frac{u_t(\omega) - u_t(\omega + h1_{[t,\infty)})}{h}. \quad (1.1)$$

It follows that a smooth process satisfies the functional Itô formula:

$$du_t = \partial_t u dt + \partial_\omega u d\omega_t + \frac{1}{2} \partial_{\omega\omega}^2 u d\langle \omega \rangle_t, \quad \mathbb{P}\text{-a.s. for all semimartingale measures } \mathbb{P}. \quad (1.2)$$

However, Definition (1.1) requires to extend the process  $u$  to the set of cadlag paths. Although this technical difficulty is addressed and solved in [2], it was observed by [9] that it is more convenient to define the derivatives by means of the Itô formula (1.2).

We next restrict our solution space so as to agree with the condition  $\partial_t u = 0$  for any potential solution of an elliptic path dependent PDE. A measurable function  $u : \Omega \rightarrow \mathbb{R}$  is called time invariant, if

$$u(\omega) = u(\omega_{\ell(\cdot)}) \quad \text{for all } \omega \text{ and all increasing bijection } \ell : \mathbb{R}^+ \rightarrow \mathbb{R}^+.$$

Loosely speaking, a time invariant map  $u$  is unchanged by any time scaling of path. This property implies the time-independence property, i.e.  $\partial_t u = 0$ , necessary for solutions of elliptic equations. Moreover, when reduced to the Markovian case, the time invariance property coincides exactly with time independence: any potential solution  $v(x)$  of an elliptic PDE corresponds to a map  $u$

defined on the space of stopped paths  $\Omega^e$  by  $u(\omega) := v(\omega_\infty) = u(\omega_{\ell(\infty)})$  for all increasing bijections  $\ell : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ .

Our existence and uniqueness results follow the line of argument in [11], but we have to address some new difficulties in the present elliptic context. One of the main difficulties is due to the boundary of Dirichlet problem. In general, the hitting time of the boundary  $H_Q(\omega)$  is not continuous in  $\omega$ . Consequently, it is non-trivial to verify the continuity of the viscosity solution constructed through path-frozen PDEs. Also, the transform:  $\tilde{u} := e^{\lambda t}u$ , repeatedly used in [11], clearly violates the time-independence of solution, so we need to introduce a convenient substitute to this argument.

We also provide an application of our result to the problem of superhedging a time invariant derivative security under uncertain volatility model. This is a classical time homogeneous stochastic control problem motivated by the application in financial mathematics.

The rest of paper is organized as follows. Section 2 introduces the main notations, as well as the notion of time-invariance, and recalls the result of optimal stopping under non-dominated measures. Section 3 defines the viscosity solution of the elliptic PPDEs. Section 4 presents the main results of this paper. In Section 5, we prove the comparison result which implies the uniqueness of viscosity solutions. In Section 6 we verify that a function constructed by a Perron-type approach is an viscosity solution, so the existence follows. We present in Section 7 an application of elliptic PPDE in the field of financial mathematics. Finally, we complete some proofs in Section 8.

## 2 Preliminary

Let  $\Omega := \{\omega \in C(\mathbb{R}^+, \mathbb{R}^d) : \omega_0 = 0\}$  be the set of continuous paths starting from the origin,  $B$  be the canonical process,  $\mathbb{F}$  be the filtration generated by  $B$ , and  $\mathbb{P}_0$  be the Wiener measure. We introduce the following notations:

- $\mathbb{S}^d$  denotes the set of  $d \times d$  symmetric matrices and  $\gamma : \eta = \text{Tr}[\gamma\eta]$  for all  $\gamma, \eta \in \mathbb{S}^d$ ;
- $\mathcal{R}$  denotes the set of all open, bounded and convex subsets of  $\mathbb{R}^d$  containing 0;
- $O_L := \{x \in \mathbb{R}^d : |x| < L\}$ ,  $\overline{O}_L$  denotes the closure of  $O$ ,  $[aI_d, bI_d] := \{\beta \in \mathbb{S}_d : aI_d \leq \beta \leq bI_d\}$ ;
- $\mathbb{H}^0(E)$  denotes the set of all  $\mathbb{F}$ -progressively measurable processes with values in  $E$ , and  $\mathbb{H}_L^0 := \mathbb{H}^0([\sqrt{2/L}I_d, \sqrt{2L}I_d])$  for  $L > 0$ ;
- $\mathcal{T}^t$  denote the set of all stopping times larger than  $t$ ; in particular,  $\mathcal{T} := \mathcal{T}^0$ ;
- $\|\omega\|_t := \sup_{s \leq t} |\omega_s|$ ,  $\|\omega\|_s^t := \sup_{s \leq u \leq t} |\omega_u|$  for  $\omega \in \Omega$  and  $s, t \in \mathbb{R}^+$ ;
- $(\omega \otimes_t \omega')(s) := \omega_s 1_{[0,t)}(s) + (\omega_t + \omega'_{s-t}) 1_{[t,\infty)}(s)$  for  $\omega, \omega' \in \Omega$  and  $s, t \in \mathbb{R}^+$ ;
- given  $\varphi : \Omega \rightarrow \mathbb{R}^d$ , we define  $\varphi^{t,\omega}(\omega') := \varphi(\omega \otimes_t \omega')$ .

In this paper, we focus on a subset of  $\Omega$  denoted as  $\Omega^e$  which will be considered as the solution space of elliptic path-dependent PDEs.

- $\Omega^e := \{\omega \in \Omega : \omega = \omega_{t \wedge \cdot} \text{ for some } t \geq 0\}$  denotes the set of all paths with flat tails;
- $\bar{t}(\omega) := \inf \{t : \omega = \omega_{t \wedge \cdot}\}$  for all  $\omega \in \Omega^e$ ;
- given  $\varphi : \Omega \rightarrow \mathbb{R}^d$ , we define the process  $\varphi_t(\omega) := \varphi(\omega_{t \wedge \cdot})$ .

Elliptic equations are devoted to model time-invariant phenomena, and in the path space the time-invariance property can be formulated mathematically as follows.

**Definition 2.1** *Define the distance on  $\Omega^e$ :*

$$d^e(\omega, \omega') := \inf_{t \in \mathcal{I}} \sup_{t \in \mathbb{R}^+} |\omega_{l(t)} - \omega'_t|, \text{ for } \omega, \omega' \in \Omega^e,$$

where  $\mathcal{I}$  is the set of all increasing bijections from  $\mathbb{R}^+$  to  $\mathbb{R}^+$ . We say  $\omega$  is equivalent to  $\omega'$ , if  $d^e(\omega, \omega') = 0$ . A function  $u$  on  $\Omega^e$  is time-invariant, if  $u$  is well defined on the equivalent class:

$$u(\omega) = u(\omega') \text{ whenever } d^e(\omega, \omega') = 0.$$

Further,  $C(\Omega^e)$  denotes the set of all random variables on  $\Omega^e$  continuous with respect to  $d^e(\cdot, \cdot)$ . We also use the notations  $C(\Omega^e; \mathbb{R}^d)$ ,  $C(\Omega^e; \mathbb{S}^d)$  when we need to emphasize the space where the functions take values. Finally, we say  $u \in \text{BUC}(\Omega^e)$  if  $u : \Omega^e \rightarrow \mathbb{R}$  is bounded and uniformly continuous with respect to  $d^e(\cdot, \cdot)$ , i.e. there exists a modulus of continuity  $\rho$  such that

$$|u(\omega^1) - u(\omega^2)| \leq \rho(d^e(\omega^1, \omega^2)) \text{ for all } \omega^1, \omega^2 \in \Omega^e. \quad (2.1)$$

In this paper, we assume  $\rho$  to be convex.

**Example 2.2** *We show some examples of time-invariant functions:*

- *Markovian case:* Assume that there exists  $\bar{u} : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $u(\omega) = \bar{u}(\omega_{\bar{t}(\omega)})$ . Since  $|\omega_{\bar{t}(\omega^1)}^1 - \omega_{\bar{t}(\omega^2)}^2| \leq d^e(\omega^1, \omega^2)$  for all  $\omega^1, \omega^2 \in \Omega^e$ ,  $u$  is time-invariant.
- *Maximum dependent case:* Assume that there exists  $\bar{u} : \mathbb{R} \rightarrow \mathbb{R}$  such that  $u(\omega) = \bar{u}(\|\omega\|_\infty)$ . Note that  $\|\omega\|_\infty = d^e(\omega, 0)$  and  $d^e(\omega^1, 0) - d^e(\omega^2, 0) \leq d^e(\omega^1, \omega^2)$ . Thus,  $\|\omega^1\|_\infty = \|\omega^2\|_\infty$  whenever  $d^e(\omega^1, \omega^2) = 0$ . Consequently,  $u$  is time-invariant.
- Let  $(t_i, x_i) \in \mathbb{R}^+ \times \mathbb{R}^d$  for each  $1 \leq i \leq n$ . We denote by

$$\text{Lin}\{(0, 0), (t_1, x_1), \dots, (t_n, x_n)\} \quad (2.2)$$

the linear interpolation of the points with a flat tail extending to  $t = \infty$ . Then, for the two paths defined as the interpolations as follows:

$$\omega^i := \text{Lin}\{(0, 0), (t_1^i, x_1), \dots, (t_n^i, x_n)\} \text{ for } i = 1, 2,$$

the distance between them is 0, i.e.  $d^e(\omega^1, \omega^2) = 0$ .

In this paper, we will prove the wellposedness result for time-invariant solutions to elliptic path-dependent PDEs.

Further, for all  $D \in \mathcal{R}$  we have the following notations:

- $\mathcal{D} := \{\omega \in \Omega^e : \omega_t \in D \text{ for all } t \geq 0\}$ ;
- $D^x := D - x := \{y : x + y \in D\}$  for  $x \in D$ , and  $D^\omega := D^{\bar{t}(\omega)}$  for  $\omega \in \mathcal{D}$ ;
- $\mathbb{H}_D := \inf\{t \geq 0 : \omega_t \notin D\}$  and  $\mathcal{H} := \{\mathbb{H}_D : D \in \mathcal{R}\}$ ;
- $\partial\mathcal{D} := \{\omega \in \Omega^e : \bar{t}(\omega) = \mathbb{H}_D(\omega)\}$  defines the boundary of  $\mathcal{D}$ , and  $\text{cl}(\mathcal{D}) := \mathcal{D} \cup \partial\mathcal{D}$  defines the closure of  $\mathcal{D}$ .

Also,  $C(\mathcal{D})$  denotes the set of all continuous functions defined on  $\mathcal{D}$ . For  $\omega \in \Omega^e$  and  $\omega' \in \Omega$ ,

- $(\omega \bar{\otimes} \omega')(s) := (\omega \otimes_{\bar{t}(\omega)} \omega')(s)$  defines the concatenation of the paths;
- given  $\varphi : \Omega \rightarrow \mathbb{R}^d$ , we define  $\varphi^\omega(\omega') := \varphi^{\bar{t}(\omega), \omega}(\omega') = \varphi(\omega \bar{\otimes} \omega')$ .

Similarly, for  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , we define that  $\phi^x(y) := \phi(x + y)$  for all  $x, y \in \mathbb{R}^d$ .

We next introduce the smooth functions on the space  $\Omega^e$ . First, as in [11], for every constant  $L > 0$ , we denote by  $\mathcal{P}_L$  the collection of all continuous semimartingale measures  $\mathbb{P}$  on  $\Omega$  whose drift and diffusion belong to  $\mathbb{H}^0(\bar{O}_L)$  and  $\mathbb{H}_L^0$ , respectively. More precisely, let  $\tilde{\Omega} := \Omega \times \Omega \times \Omega$  be an enlarged canonical space,  $\tilde{B} := (B, A, M)$  be the canonical process. A probability measure  $\mathbb{P} \in \mathcal{P}^L$  means that there exists an extension  $\mathbb{Q}^{\alpha, \beta}$  of  $\mathbb{P}$  on  $\tilde{\Omega}$  such that:

$$\begin{aligned} B &= A + M, \quad A \text{ is absolutely continuous, } M \text{ is a martingale,} \\ \|\alpha^\mathbb{P}\|_\infty &\leq L, \quad \beta^\mathbb{P} \in \mathbb{H}_L^0, \quad \text{where } \alpha_t^\mathbb{P} := \frac{dA_t}{dt}, \quad \beta_t^\mathbb{P} := \sqrt{\frac{d\langle M \rangle_t}{dt}}, \end{aligned} \quad \mathbb{Q}^{\alpha, \beta}\text{-a.s.} \quad (2.3)$$

Further, denote  $\mathcal{P}^\infty := \cup_{L>0} \mathcal{P}^L$ .

**Definition 2.3 (Smooth time-invariant processes)** *Let  $D \in \mathcal{R}$ . We say that  $u \in C^2(\mathcal{D})$ , if  $u \in C(\mathcal{D})$  and there exist  $Z \in C(\mathcal{D}; \mathbb{R}^d)$ ,  $\Gamma \in C(\mathcal{D}; \mathbb{S}^d)$  such that*

$$du_t = Z_t \cdot dB_t + \frac{1}{2} \Gamma_t : \langle B \rangle_t \text{ for } t \leq \mathbb{H}_D, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}^\infty.$$

*By a direct localization argument, we see that the above  $Z$  and  $\Gamma$ , if they exist, are unique. Denote  $\partial_\omega u := Z$  and  $\partial_\omega^2 u := \Gamma$ .*

**Remark 2.4** *In the Markovian case mentioned in Example 2.2, if the function  $\bar{u} : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies  $\bar{u} \in C^2(D)$ , then by the Itô's formula it follows that  $u \in C^2(\mathcal{D})$ .*

**Remark 2.5** *In the path-dependent case, Dupire [8] defined derivatives,  $\partial_t u$  and  $\partial_\omega u$ , for process  $u : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^d$ . In particular, the  $t$ -derivative is defined as:*

$$\partial_t u(s, \omega) := \lim_{h \rightarrow 0^+} \frac{u(t+h, \omega_{t \wedge \cdot}) - u(t, \omega)}{h}.$$

Also, Dupire and other authors, for example [2], proved the functional Itô formula for the processes regular in Dupire's sense:

$$du_s = \partial_t u_s ds + \partial_\omega u_s \cdot dB_s + \frac{1}{2} \partial_{\omega\omega}^2 u_s : \langle B \rangle_s, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}^\infty,$$

Note that in the time-invariant case it always holds that  $\partial_t u = 0$ . Consequently, the processes with Dupire's derivatives in  $C(\mathcal{D})$  are also smooth according to our definition.

We next introduce the notations about nonlinear expectations. For a measurable set  $A \in \Omega$ , a random variable  $\xi$  and a process  $X$ , we define:

- $\mathcal{C}^L[A] := \sup_{\mathbb{P} \in \mathcal{P}^L} \mathbb{P}[A]$ ,  $\bar{\mathcal{E}}^L[\xi] := \sup_{\mathbb{P} \in \mathcal{P}^L} \mathbb{E}^\mathbb{P}[\xi]$  and  $\underline{\mathcal{E}}^L[\xi] := \inf_{\mathbb{P} \in \mathcal{P}^L} \mathbb{E}^\mathbb{P}[\xi]$ ;
- $\bar{\mathcal{E}}_t^L[\xi](\omega) := \sup_{\mathbb{P} \in \mathcal{P}^L} \mathbb{E}^\mathbb{P}[\xi^{t,\omega}]$  and  $\underline{\mathcal{E}}_t^L[\xi](\omega) := \inf_{\mathbb{P} \in \mathcal{P}^L} \mathbb{E}^\mathbb{P}[\xi^{t,\omega}]$ ;
- $\bar{\mathcal{S}}_t^L[X_{H_D \wedge \cdot}](\omega) := \sup_{\tau \in \mathcal{T}^t} \bar{\mathcal{E}}_t^L[X_{\tau \wedge H_D}](\omega)$  and  $\underline{\mathcal{S}}_t^L[X_{H_D \wedge \cdot}](\omega) := \inf_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L[X_{\tau \wedge H_D}](\omega)$ .

A process  $X$  is an  $\bar{\mathcal{E}}^L$ -supermartingale on  $[0, T]$ , if  $X_t(\omega) = \bar{\mathcal{S}}_t^L[X](\omega)$  for all  $(t, \omega) \in [0, T] \times \Omega$ ; similarly, we define the  $\underline{\mathcal{E}}^L$ -submartingales. The existing literature gives the following results.

**Lemma 2.6 (Tower property, Nutz and van Handel [19])** *For a bounded measurable process  $X$ , we have*

$$\bar{\mathcal{E}}_\sigma^L[X] = \bar{\mathcal{E}}_\sigma^L \left[ \bar{\mathcal{E}}_\tau^L[X] \right] \quad \text{for all stopping times } \sigma \leq \tau.$$

**Lemma 2.7 (Snell envelop characterization, Ekren, Touzi and Zhang [12])** *Let  $H_D \in \mathcal{H}$  and  $X \in \text{BUC}(\mathcal{D})$ . Define the Snell envelope and the corresponding first hitting time of the obstacles:*

$$Y := \bar{\mathcal{S}}^L[X_{H_D \wedge \cdot}], \quad \tau^* := \inf \{t \geq 0 : Y_t = X_t\}.$$

*Then  $Y_{\tau^*} = X_{\tau^*}$ .  $Y$  is an  $\bar{\mathcal{E}}^L$ -supermartingale on  $[0, H_D]$  and an  $\bar{\mathcal{E}}^L$ -martingale on  $[0, \tau^*]$ . Consequently,  $\tau^*$  is an optimal stopping time.*

Some other properties of the nonlinear expectation will be useful.

**Proposition 2.8** *Let  $D \in \mathcal{R}$  and  $O \subset D$  also in  $\mathcal{R}$ . Define a sequence of stopping times  $H_n$ :*

$$H_0 = 0, \quad H_{i+1} := \inf \{s \geq H_i : B_s - B_{H_i} \notin O\}, \quad i \geq 0. \quad (2.4)$$

*Then, it holds that*

$$\lim_{n \rightarrow \infty} \mathcal{C}^L[H_n < T] = 0 \quad \text{for all } T \in \mathbb{R}^+; \quad \bar{\mathcal{E}}^L[H_D] < \infty; \quad \text{and} \quad \lim_{n \rightarrow \infty} \sup_{x \in D} \mathcal{C}^L[H_n < H_D^x] = 0.$$

We report the proof in Appendix.

### 3 Fully nonlinear elliptic path-dependent PDEs

#### 3.1 Definition of viscosity solutions of uniformly elliptic PPDEs

Let  $Q \in \mathcal{R}$  and consider  $\mathcal{Q}$  as the domain of the path-dependent Dirichlet problem of the equation:

$$\mathcal{L}u(\omega) := -G(\omega, u, \partial_\omega u, \partial_{\omega\omega}^2 u) = 0 \text{ for } \omega \in \mathcal{Q}, \quad u = \xi \text{ on } \partial\mathcal{Q}, \quad (3.1)$$

with nonlinearity  $G$  and boundary condition by  $\xi$ .

**Assumption 3.1** *The nonlinearity  $G : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$  satisfies:*

- (i) *For fixed  $(y, z, \gamma)$ ,  $|G(\cdot, 0, 0, 0)| \leq C_0$ ;*
- (ii)  *$G$  is uniformly elliptic, i.e., there exists  $L_0 > 0$  such that for all  $(\omega, y, z)$*

$$G(\omega, y, z, \gamma_1) - G(\omega, y, z, \gamma_2) \geq \frac{1}{L_0} I_d : (\gamma_1 - \gamma_2) \text{ for all } \gamma_1 \geq \gamma_2.$$

- (iii)  *$G$  is uniformly continuous on  $\Omega^e$  with respect to  $d^e(\cdot, \cdot)$ , and is uniformly Lipschitz continuous in  $(y, z, \gamma)$  with a Lipschitz constant  $L_0$ ;*
- (iv)  *$G$  is uniformly decreasing in  $y$ , i.e. there exists a function  $\lambda : \mathbb{R} \rightarrow \mathbb{R}$  strictly increasing and continuous,  $\lambda(0) = 0$ , and*

$$G(\omega, y_1, z, \gamma) - G(\omega, y_2, z, \gamma) \geq \lambda(y_2 - y_1), \text{ for all } y_2 \geq y_1, (\omega, z, \gamma) \in \Omega^e \times \mathbb{R}^d \times \mathbb{S}^d.$$

For any time-invariant function  $u$  on  $\Omega^e$ ,  $\omega \in \mathcal{Q}$  and  $L > 0$ , we define the set of test functions:

$$\begin{aligned} \underline{\mathcal{A}}^L u(\omega) &:= \left\{ \varphi : \varphi \in C^2(\mathcal{O}_\varepsilon) \text{ and } (\varphi - u^\omega)_0 = \underline{\mathcal{S}}_0^L \left[ (\varphi - u^\omega)_{\mathbb{H}_{\mathcal{O}_\varepsilon}^\omega \wedge} \right] \text{ for some } \varepsilon > 0 \right\}, \\ \overline{\mathcal{A}}^L u(\omega) &:= \left\{ \varphi : \varphi \in C^2(\mathcal{O}_\varepsilon) \text{ and } (\varphi - u^\omega)_0 = \overline{\mathcal{S}}_0^L \left[ (\varphi - u^\omega)_{\mathbb{H}_{\mathcal{O}_\varepsilon}^\omega \wedge} \right] \text{ for some } \varepsilon > 0 \right\}. \end{aligned}$$

We call  $\mathbb{H}_{\mathcal{O}_\varepsilon}$  a localization of test function  $\varphi$ . Note that the stopping time  $\mathbb{H}_{\mathcal{O}_\varepsilon}$  can take the value of  $\infty$ , while  $u$  is only defined on  $\Omega^e$ . However, since  $\mathbb{H}_{\mathcal{O}_\varepsilon} < \infty$   $\mathcal{P}^L$ -q.s., it is not essential. If necessary, we can define complementarily  $u := 0$  on  $\Omega \setminus \Omega^e$ . Now, we define the viscosity solution to the elliptic path-dependent PDEs (3.1).

**Definition 3.2** *Let  $L > 0$  and  $\{u_t\}_{t \in \mathbb{R}^+}$  be a time-invariant progressively measurable process.*

- (i)  *$u$  is an  $L$ -viscosity subsolution (resp.  $L$ -supersolution) of PPDE (3.1) if for  $\omega \in \mathcal{Q}$  and any  $\varphi \in \underline{\mathcal{A}}^L u(\omega)$  (resp.  $\varphi \in \overline{\mathcal{A}}^L u(\omega)$ ):*

$$-G(\omega, u(\omega), \partial_\omega \varphi_0, \partial_{\omega\omega}^2 \varphi_0) \leq \text{ (resp. } \geq) 0.$$

- (ii)  *$u$  is a viscosity subsolution (resp. supersolution) of PPDE (3.1) if  $u$  is an  $L$ -viscosity subsolution (resp.  $L$ -supersolution) of PPDE (3.1) for some  $L > 0$ .*
- (iii)  *$u$  is a viscosity solution of PPDE (3.1) if it is both a viscosity subsolution and a viscosity supersolution.*

By very similar arguments as in the proof of Theorem 3.16 and Theorem 5.1 in [10], we may easily prove that:



**Theorem 3.3 (Consistency with classical solution)** *Let Assumption 3.1 hold. Given a function  $u \in C^2(\mathcal{Q})$ , then  $u$  is a viscosity supersolution (resp. subsolution, solution) to PPDE (3.1) if and only if  $u$  is a classical supersolution (resp. subsolution, solution).*

**Theorem 3.4 (Stability)** *Let  $L > 0$ ,  $G$  satisfies Assumption 3.1, and  $u \in \text{BUC}$ . Assume*

- (i) *for any  $\varepsilon > 0$ , there exist  $G^\varepsilon$  and  $u^\varepsilon \in \text{BUC}$  such that  $G^\varepsilon$  satisfies Assumption 3.1 and  $u^\varepsilon$  is a  $L$ -viscosity subsolution (resp. supersolution) of PPDE (3.1) with generator  $G^\varepsilon$ ;*
- (ii) *as  $\varepsilon \rightarrow 0$ ,  $(G^\varepsilon, u^\varepsilon)$  converge to  $(G, u)$  locally uniformly in the following sense: for any  $(\omega, y, z, \gamma) \in \Omega^e \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ , there exists  $\delta > 0$  such that*

$$\lim_{\varepsilon \rightarrow 0} \sup_{(\tilde{\omega}, \tilde{y}, \tilde{z}, \tilde{\gamma}) \in O_\delta(\omega, y, z, \gamma)} \left[ |(G^\varepsilon - G)^\omega(\tilde{\omega}, \tilde{y}, \tilde{z}, \tilde{\gamma})| + |(u^\varepsilon - u)^\omega(\tilde{\omega})| \right] = 0,$$

*where we abuse the notation  $O_\delta$  to denote a ball in the corresponding space. Then,  $u$  is a  $L$ -viscosity solution (resp. supersolution) of PPDE (3.1) with generator  $G$ .*

### 3.2 Equivalent definition by semijets

Following the standard theory of viscosity solutions for PDEs, we may also define viscosity solutions via semijets. Similar to [22] and [23], we introduce the notion of semijets in the context of PPDE. First, denote functions:

$$Q^{\alpha, \beta, \omega}(\omega') = \alpha(\omega'_{\bar{t}(\omega')} - \omega_{\bar{t}(\omega)}) + \frac{1}{2}\beta(\omega'_{\bar{t}(\omega')} - \omega_{\bar{t}(\omega)})^2.$$

Epecially, if  $\omega = 0$ , we denote  $Q^{\alpha, \beta} := Q^{\alpha, \beta, 0}$ . We next define:

$$\underline{\mathcal{J}}^L u(\omega) := \left\{ (\alpha, \beta) : Q^{\alpha, \beta, \omega} \in \underline{\mathcal{A}}^L u(\omega) \right\} \quad \text{and} \quad \overline{\mathcal{J}}^L u(\omega) := \left\{ (\alpha, \beta) : Q^{\alpha, \beta, \omega} \in \overline{\mathcal{A}}^L u(\omega) \right\}.$$

**Proposition 3.5** *Let  $u \in \text{BUC}(\mathcal{Q})$ . Then,  $u$  is an  $L$ -viscosity subsolution (resp. supersolution) of PPDE (3.1), if and only if for any  $\omega \in \mathcal{Q}$ ,*

$$-G(\omega, u(\omega), \alpha, \beta) \leq (\text{resp. } \geq) 0, \quad \text{for all } (\alpha, \beta) \in \underline{\mathcal{J}}^L u(\omega) \text{ (resp. } \overline{\mathcal{J}}^L u(\omega))$$

**Proof** The 'only if' part is trivial by the definitions. It remains to prove the 'if' part. We only show the result for  $L$ -viscosity subsolutions. Let  $\varphi \in \underline{\mathcal{A}}^L u(0)$  and  $O_\delta$  is the corresponding localization. Without loss of generality, we may assume that  $\omega = 0$  and  $\varphi(0) = u(0)$ . Define:

$$\alpha := \partial_\omega \varphi(0) \quad \text{and} \quad \beta := \partial_{\omega\omega}^2 \varphi(0).$$

Let  $\varepsilon > 0$ . Since  $\partial_\omega \varphi$  and  $\partial_{\omega\omega}^2 \varphi$  are both continuous, there exists  $\delta' \leq \delta$  such that on  $O_{\delta'}$  it holds:

$$|\partial_\omega \varphi - \alpha| \leq \varepsilon \quad \text{and} \quad |\partial_{\omega\omega}^2 \varphi - \beta| \leq \varepsilon.$$

Denote  $\beta_\varepsilon := \beta + (1 + 2L)\varepsilon$ . Then, for all  $\tau \in \mathcal{T}$  such that  $\tau \leq \text{H}_{O_{\delta'}}$ , we have

$$\begin{aligned} (Q^{\alpha, \beta_\varepsilon} - u)_0 - \underline{\mathcal{E}}^L[(Q^{\alpha, \beta_\varepsilon} - u)_\tau] &= \overline{\mathcal{E}}^L[(u - u_0 - Q^{\alpha, \beta_\varepsilon})_\tau] \leq \overline{\mathcal{E}}^L[(u - \varphi)_\tau] + \overline{\mathcal{E}}^L[(\varphi - \varphi_0 - Q^{\alpha, \beta_\varepsilon})_\tau] \\ &\leq \overline{\mathcal{E}}^L \left[ \int_0^\tau (\partial_\omega \varphi_s - \alpha) dB_s + \frac{1}{2} \int_0^\tau (\partial_{\omega\omega}^2 \varphi_s - \beta_\varepsilon) ds \right] \leq \overline{\mathcal{E}}^L \left[ \int_0^\tau (L|\partial_\omega \varphi_s - \alpha| + \frac{1}{2}(\partial_{\omega\omega}^2 \varphi_s - \beta_\varepsilon)) ds \right] \leq 0, \end{aligned}$$

where we used the fact that  $\varphi \in \underline{\mathcal{A}}^L u(0)$  and the definition of  $\mathcal{P}^L$  in (2.3). Consequently, we proved  $(\alpha, \beta_\varepsilon) \in \underline{\mathcal{J}}^L u(0)$ , and thus

$$-G(0, u(0), \alpha, \beta_\varepsilon) \leq 0.$$

Finally, thanks to the continuity of  $G$ , we obtain the desired result by sending  $\varepsilon \rightarrow 0$ .  $\blacksquare$

## 4 Main results

Following Ekren, Touzi and Zhang [11], we introduce the path-frozen PDEs:

$$(E)_\varepsilon^\omega \quad \mathbf{L}^\omega v := -G(\omega, v, Dv, D^2v) = 0 \text{ on } O_\varepsilon(\omega) := O_\varepsilon \cap Q^\omega. \quad (4.1)$$

Note that  $\omega$  is a parameter instead of a variable in the above PDE. Similar to [11], our wellposedness result relies on the following condition on the PDE  $(E)_\varepsilon^\omega$ .

**Assumption 4.1** *For  $\varepsilon > 0$ ,  $\omega \in \mathcal{Q}$  and  $h \in C(\partial O_\varepsilon(\omega))$ , we have  $\bar{v} = \underline{v}$ , where*

$$\begin{aligned} \bar{v}(x) &:= \inf \{w(x) : w \in C_0^2(O_\varepsilon(\omega)), \mathbf{L}^\omega w \geq 0 \text{ on } O_\varepsilon(\omega), w \geq h \text{ on } \partial O_\varepsilon(\omega)\}, \\ \underline{v}(x) &:= \sup \{w(x) : w \in C_0^2(O_\varepsilon(\omega)), \mathbf{L}^\omega w \leq 0 \text{ on } O_\varepsilon(\omega), w \leq h \text{ on } \partial O_\varepsilon(\omega)\}, \end{aligned}$$

and  $C_0^2(O_\varepsilon(\omega)) := C^2(O_\varepsilon(\omega)) \cap C(\text{cl}(O_\varepsilon(\omega)))$ .

In this paper, we use PDEs as tools to study PPDEs. Note that by viscosity solutions of PDEs, we mean the classical definition in the PDE literature, for example in [4].

**Remark 4.2** *If Equation  $(E)_t^\omega$  has the classical solution, then Assumption 4.1 holds. For example, let function  $G(\omega, \cdot) : \mathbb{S}^d \rightarrow \mathbb{R}$  be convex, and assume that the elliptic PDE:*

$$-G(\omega, D^2v) = 0 \text{ on } O, \quad v = h \text{ on } \partial O$$

*has a unique viscosity solution. Then, according to Caffarelli and Cabre [3], the viscosity solution has the interior  $C^2$ -regularity.*

The rest of the paper will be devoted to prove the following two main results.

**Theorem 4.3 (Comparison result)** *Let Assumptions 3.1 and 4.1 hold true, and let  $\xi \in \text{BUC}(\partial \mathcal{Q})$ . Let  $u$  be a BUC viscosity subsolution and  $v$  be a BUC viscosity supersolution to the path-dependent PDE (3.1). Then, if  $u_{\text{H}\mathcal{Q}} \leq \xi \leq v_{\text{H}\mathcal{Q}}$ , we have  $u \leq v$  on  $\mathcal{Q}$ .*

**Theorem 4.4 (Wellposedness)** *Let Assumptions 3.1 and 4.1 hold, and let  $\xi \in \text{BUC}(\partial \mathcal{Q})$ . Then, the path-dependent PDE (3.1) has a unique viscosity solution in  $\text{BUC}(\mathcal{Q})$ .*

## 5 Comparison result

### 5.1 Partial comparison

Similar to [11], we introduce the class of piecewise smooth processes in our time-invariant context.

**Definition 5.1** Let  $u : \mathcal{Q} \rightarrow \mathbb{R}$ . We say that  $u \in \overline{C}^2(\mathcal{Q})$ , if  $u$  is bounded, process  $\{u_t\}_{t \in \mathbb{R}^+}$  is pathwise continuous, and there exists an increasing sequence of  $\mathbb{F}$ -stopping times  $\{H_n; n \geq 1\}$  such that

- (1) for each  $i$  and  $\omega$ ,  $\Delta H_{i,\omega} := H_{i+1}^{\mathbb{H}_i(\omega),\omega} - H_i(\omega) \in \mathcal{H}$  whenever  $H_i(\omega) < H_Q(\omega) < \infty$ , i.e. there is  $O_{i,\omega} \in \mathcal{R}$  such that  $\Delta H_{i,\omega} = \inf\{t : \omega_t \notin O_{i,\omega}\}$ ;
- (2)  $\{i : H_i(\omega) < H_Q(\omega)\}$  is finite  $\mathcal{P}^\infty$ -q.s. and  $\lim_{i \rightarrow \infty} \mathcal{C}_0^L [H_i^\omega < H_Q^\omega] = 0$  for all  $\omega \in \mathcal{Q}$  and  $L > 0$ ;
- (3) for each  $i$ ,  $u^{\omega_{H_i \wedge \cdot}} \in \text{BUC}(O_{i,\omega})$ , and there exist  $\partial_\omega u^i$ ,  $\partial_{\omega\omega}^2 u^i$  such that for all  $\omega$ ,  $(\partial_\omega u^i)^{\omega_{H_i \wedge \cdot}}$  and  $(\partial_{\omega\omega}^2 u^i)^{\omega_{H_i \wedge \cdot}}$  are both continuous on  $O_{i,\omega}$  and

$$u_t^{\omega_{H_i \wedge \cdot}} - u_0^{\omega_{H_i \wedge \cdot}} = \int_0^t (\partial_\omega u^i)_s^{\omega_{H_i \wedge \cdot}} \cdot dB_s + \frac{1}{2} \int_0^t (\partial_{\omega\omega}^2 u^i)_s^{\omega_{H_i \wedge \cdot}} : d\langle B \rangle_s \text{ for all } t \leq \Delta H_{i,\omega}, \mathcal{P}^\infty\text{-q.s.}$$

The rest of the subsection is devoted to the proof of the following partial comparison result.

**Proposition 5.2** Let Assumption 3.1 hold. Let  $u^2 \in \text{BUC}(\mathcal{Q})$  be a viscosity supersolution of PPDE (3.1) and let  $u^1 \in \overline{C}^2(\mathcal{Q})$  satisfies  $\mathcal{L}u^1(\omega) \leq 0$  for all  $\omega \in \mathcal{Q}$ . If  $u^1 \leq u^2$  on  $\partial\mathcal{Q}$ , then  $u^1 \leq u^2$  in  $\text{cl}(\mathcal{Q})$ . A similar result holds if we exchange the roles of  $u^1$  and  $u^2$ .

In preparation to the proof of Proposition 5.2, we prove the following lemma.

**Lemma 5.3** Let  $D \in \mathcal{R}$  and  $X \in \text{BUC}(\mathcal{D})$  and non-negative. Assume that  $X_0 > \overline{\mathcal{E}}^L[X_{H_D}]$ , then there exists  $\omega^* \in \mathcal{D}$  and  $t^* := \bar{t}(\omega^*)$  such that

$$X_{t^*}(\omega^*) = \overline{\mathcal{S}}_{t^*}^L[X_{H_D \wedge \cdot}](\omega^*) \quad \text{and} \quad X_{t^*}(\omega^*) > 0.$$

**Proof** Denote  $Y$  as the Snell envelop of  $X_{H_D \wedge \cdot}$ , i.e.  $Y_t := \overline{\mathcal{S}}_t^L[X_{H_D \wedge \cdot}]$ . By Lemma 2.7, we know that  $\tau^* := \inf\{t : X_t = Y_t\}$  defines an optimal stopping rule. So, we have

$$\overline{\mathcal{E}}^L[X_{\tau^*}] = Y_0 \geq X_0 > \overline{\mathcal{E}}^L[X_{H_D}].$$

Hence  $\{\tau^* < H_D\} \neq \emptyset$ . Suppose that  $X_{\tau^*} = 0$  on  $\{\tau^* < H_D\}$ . Then,

$$0 = X_{\tau^*} 1_{\{\tau^* < H_D\}} = Y_{\tau^*} 1_{\{\tau^* < H_D\}} \geq \overline{\mathcal{E}}_{\tau^*}^L[X_{H_D}] 1_{\{\tau^* < H_D\}}.$$

Since  $X$  is non-negative, we obtain that  $X_{H_D} 1_{\{\tau^* < H_D\}} = 0$ . So,  $X_{\tau^*} = X_{H_D}$  on  $\{\tau^* < H_D\}$ . Thus, we conclude that

$$X_0 \leq Y_0 = \overline{\mathcal{E}}^L[X_{\tau^*}] = \overline{\mathcal{E}}^L[X_{H_D}] < X_0.$$

This contradiction implies that  $\{\tau^* < H_D, X_{\tau^*} > 0\} \neq \emptyset$ . Finally, take  $\omega' \in \{\tau^* < H_D, X_{\tau^*} > 0\}$ , and then  $\omega^* := \omega'_{\tau^*}(\omega')_{\wedge \cdot}$  is the desired path.  $\blacksquare$

**Proof of Proposition 5.2** Recall the notation  $H_n$  and  $O_n^\omega$  in Definition 5.1. We decompose the proof in two steps.

Step 1. We first show that for all  $i \geq 0$  and  $\omega \in \mathcal{Q}$

$$(u^1 - u^2)_{H_i}^+(\omega) \leq \bar{\mathcal{E}}^L \left[ \left( (u_{H_{i+1}}^1)^{H_i, \omega} - (u_{H_{i+1}}^2)^{H_i, \omega} \right)^+ \right].$$

Clearly it suffices to consider  $i = 0$ . Assume the contrary, i.e.

$$c := (u^1 - u^2)^+(\mathbf{0}) - \bar{\mathcal{E}}^L \left[ (u^1 - u^2)_{H_1}^+ \right] > 0.$$

Denote  $X := (u^1 - u^2)^+$ . Then, by Lemma 5.3, there exists  $\omega^* \in \mathcal{O}_0^0$  and  $t^* := \bar{t}(\omega^*)$  such that

$$X_{t^*}(\omega^*) = \bar{\mathcal{S}}_{t^*}^L[X_{H_D \wedge \cdot}] \text{ and } X_{t^*}(\omega^*) > 0.$$

Since  $u^1$  is smooth on  $\mathcal{O}_0^0$ , we have  $\varphi := (u_{H_{\mathcal{O}_0^0 \wedge \cdot}}^1)^{\omega^*} \in \bar{\mathcal{A}}^L u^2(\omega^*)$ . By the  $L$ -viscosity supersolution property of  $u^2$  and the assumption on the function  $G$ , this implies that

$$\begin{aligned} 0 &\leq -G(\cdot, u^2, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi)(\omega_{t^* \wedge \cdot}^*) \\ &\leq -G(\cdot, u^1, \partial_\omega u^1, \partial_{\omega\omega}^2 u^1)(\omega_{t^* \wedge \cdot}^*) - \lambda((u^1 - u^2)_{t^*}(\omega^*)) \\ &< -G(\cdot, u^1, \partial_\omega u^1, \partial_{\omega\omega}^2 u^1)(\omega_{t^* \wedge \cdot}^*). \end{aligned}$$

This is in contradiction with the classical subsolution property of  $u^1$ .

Step 2. By the result of Step 1 and the tower property of  $\bar{\mathcal{E}}^L$  stated in Lemma 2.6, we have

$$(u^1 - u^2)^+(\mathbf{0}) \leq \bar{\mathcal{E}}^L \left[ (u^1 - u^2)_{H_i}^+ \right] \leq \bar{\mathcal{E}}^L \left[ (u^1 - u^2)_{H_Q}^+ \right] + \bar{\mathcal{E}}^L \left[ (u^1 - u^2)_{H_i}^+ - (u^1 - u^2)_{H_Q}^+ \right] \text{ for all } i \geq 1.$$

By Proposition 2.8, we have  $\mathcal{C}^L[H_i < H_Q] \rightarrow 0$  as  $i \rightarrow \infty$ . Therefore,

$$(u^1 - u^2)^+(\mathbf{0}) \leq \bar{\mathcal{E}}^L \left[ (u^1 - u^2)_{H_Q}^+ \right] = 0. \quad \blacksquare$$

## 5.2 The Perron type construction

Define the following two functions:

$$\bar{u}(\omega) := \inf \left\{ \psi(\omega) : \psi \in \bar{\mathcal{D}}_Q^\xi(\omega) \right\}, \quad \underline{u}(\omega) := \sup \left\{ \psi(\omega) : \psi \in \underline{\mathcal{D}}_Q^\xi(\omega) \right\}, \quad (5.1)$$

where

$$\bar{\mathcal{D}}_Q^\xi(\omega) := \left\{ \psi \in \bar{\mathcal{C}}^2(\mathcal{Q}^\omega) : \mathcal{L}^\omega \psi \geq 0 \text{ on } \mathcal{Q}, \psi \geq \xi^\omega \text{ on } \partial \mathcal{Q} \right\},$$

$$\underline{\mathcal{D}}_Q^\xi(\omega) := \left\{ \psi \in \bar{\mathcal{C}}^2(\mathcal{Q}^\omega) : \mathcal{L}^\omega \psi \leq 0 \text{ on } \mathcal{Q}, \psi \leq \xi^\omega \text{ on } \partial \mathcal{Q} \right\}.$$

As a direct corollary of Proposition 5.2, we have:

**Corollary 5.4** *Let Assumption 3.1 hold. For all BUC viscosity supersolutions (resp. subsolution)  $u$  such that  $u \geq \xi$  (resp.  $u \leq \xi$ ) on  $\partial\mathcal{Q}$ , it holds that  $u \geq \underline{u}$  (resp.  $u \leq \bar{u}$ ) on  $\mathcal{Q}$ .*

In order to prove the comparison result of Theorem 4.3, it remains to show the following result.

**Proposition 5.5** *Let  $\xi \in \text{BUC}(\partial\mathcal{Q})$ . Under Assumptions 3.1 and 4.1, we have  $\bar{u} = \underline{u}$ .*

The proof of this proposition is reported in Subsection 5.4, and requires the preparations in Subsection 5.3.

### 5.3 Preliminary: HJB equations

In this part of the paper, we will recall the relation between HJB equations and stochastic control problems. Recall the constants  $L_0$  and  $C_0$  in Assumption 3.1 and consider two functions:

$$\begin{aligned}\bar{g}(y, z, \gamma) &:= C_0 + L_0 |z| + L_0 y^- + \sup_{\beta \in [\sqrt{2/L_0}I_d, \sqrt{2L_0}I_d]} \frac{1}{2}\beta^2 : \gamma, \\ \underline{g}(y, z, \gamma) &:= -C_0 - L_0 |z| - L_0 y^+ + \inf_{\beta \in [\sqrt{2/L_0}I_d, \sqrt{2L_0}I_d]} \frac{1}{2}\beta^2 : \gamma,\end{aligned}\tag{5.2}$$

Then for all nonlinearities  $G$  satisfying Assumption 3.1, it holds  $\underline{g} \leq G \leq \bar{g}$ . Consider the HJB equations:

$$\bar{\mathbf{L}}u := -\bar{g}(u, Du, D^2u) = 0 \quad \text{and} \quad \underline{\mathbf{L}}u := -\underline{g}(u, Du, D^2u) = 0.$$

In the rest of this subsection, we show that the explicit solutions of the Dirichlet problems of the above equations on a set  $D \in \mathcal{R}$  are given in terms of boundary condition  $h_D$ , i.e.

$$\begin{aligned}\bar{w}(x) &:= \sup_{b \in \mathbb{H}^0([0, L_0])} \bar{\mathcal{E}}^{L_0} \left[ h_D(B_{\mathbb{H}_D^x}) e^{-\int_0^{\mathbb{H}_D^x} b_r dr} + C_0 \int_0^{\mathbb{H}_D^x} e^{-\int_0^t b_r dr} dt \right], \\ \underline{w}(x) &:= \inf_{b \in \mathbb{H}^0([0, L_0])} \underline{\mathcal{E}}^{L_0} \left[ h_D(B_{\mathbb{H}_D^x}) e^{-\int_0^{\mathbb{H}_D^x} b_r dr} + C_0 \int_0^{\mathbb{H}_D^x} e^{-\int_0^t b_r dr} dt \right].\end{aligned}$$

**Lemma 5.6** *Let  $h_D(x) := \bar{\mathcal{E}}^{L_0} [v(x, B_{\mathbb{H}_D^x \wedge \cdot})]$  for some  $v \in \text{BUC}(\mathbb{R}^d \times \Omega^e)$ . Then  $\bar{w}$  and  $\underline{w}$  are the unique viscosity solutions in  $\text{BUC}(\text{cl}(D))$  of the equations  $\bar{\mathbf{L}}u = 0$  and  $\underline{\mathbf{L}}u = 0$ , respectively, with boundary condition  $u = h_D$  on  $\partial D$ .*

**Proof** According to Proposition 8.1 in Appendix, there exists a modulus of continuity  $\rho$  such that

$$\bar{\mathcal{E}}^{L_0} [|H_D^{x_1} - H_D^{x_2}|] \leq \rho(|x_1 - x_2|).\tag{5.3}$$

Since  $v \in \text{BUC}(\mathbb{R} \times \Omega^e)$ , we obtain that

$$|h_D(x_1) - h_D(x_2)| \leq \bar{\mathcal{E}}^{L_0} [|v(x_1, B_{\mathbb{H}_D^{x_1} \wedge \cdot}) - v(x_2, B_{\mathbb{H}_D^{x_2} \wedge \cdot})|] \leq \rho(|x_1 - x_2| + \bar{\mathcal{E}}^{L_0} [|B_{\mathbb{H}_D^{x_1}} - B_{\mathbb{H}_D^{x_2}}|])\tag{5.4}$$

where we used the convexity of  $\rho$  and the Jensen's inequality. We next estimate:

$$\bar{\mathcal{E}}^{L_0} [|B_{\mathbb{H}_D^{x_1}} - B_{\mathbb{H}_D^{x_2}}|] \leq \left( \bar{\mathcal{E}}^{L_0} [|B_{\mathbb{H}_D^{x_1}} - B_{\mathbb{H}_D^{x_2}}|^2] \right)^{\frac{1}{2}} \leq \left( 2L_0 \bar{\mathcal{E}}^{L_0} [|H_D^{x_1} - H_D^{x_2}|] \right)^{\frac{1}{2}}.\tag{5.5}$$

In view of (5.3), we conclude that  $h_D$  is bounded and uniformly continuous. Further, since  $h_D$  is bounded and  $b$  only takes non-negative values, we can easily obtain that for  $x_1, x_2 \in D$ ,

$$|\bar{w}(x_1) - \bar{w}(x_2)| \leq \bar{\mathcal{E}}^{L_0} [|h_D(B_{\mathbb{H}_D^{x_1}}) - h_D(B_{\mathbb{H}_D^{x_2}})|] + C \bar{\mathcal{E}}^{L_0} [|H_D^{x_1} - H_D^{x_2}|].$$

Since  $h_D \in \text{BUC}(\mathbb{R}^d)$ , by the same arguments in (5.4) and (5.5), we conclude that  $\bar{w} \in \text{BUC}(\text{cl}(D))$ . Then, by standard argument, one can easily verify that  $\bar{w}$  is the viscosity solution to  $\bar{\mathbf{L}}u = 0$  with the boundary condition  $h_D$  on  $\partial D$ . Similarly, we may prove the corresponding result for  $\underline{w}$ .  $\blacksquare$

#### 5.4 Proof of $\bar{u} = \underline{u}$

Recall the two functions  $\bar{u}, \underline{u}$  defined in (5.1). In the next lemma, we will use the path-frozen PDEs to construct the functions  $\theta_n^\epsilon$ , which will be needed to construct the approximations of  $\bar{u}$  and  $\underline{u}$  defined in (5.1). Recall the notation of linear interpolation in (2.2). Then

- let  $(x_i; 1 \leq i \leq n) \in (\bar{B}_\epsilon^d)^n$  and denote  $\pi_n := \text{Lin}\{(0, 0), (1, x_1), \dots, (n, x_n)\}$ ; in particular, note that  $\pi_n \in \Omega^\epsilon$ ;
- denote  $\pi_n^x := \text{Lin}\{\pi_n, (n+1, x)\}$  for all  $x \in \bar{B}_\epsilon^d$ ; clearly, we have  $\pi_n^x \in \Omega^\epsilon$ ;
- define a sequence of stopping times: for  $i \geq 1$

$$H_0^{\pi_n^x} := 0, \quad H_1^{\pi_n^x} := \inf \{t \geq H_i : x + B_t \notin O_\epsilon\} \wedge H_Q^{\pi_n^x}, \quad H_{i+1}^{\pi_n^x} := \inf \left\{ t \geq H_i^{\pi_n^x} : B_t - B_{H_i} \notin O_\epsilon \right\} \wedge H_Q^{\pi_n^x}$$

- given  $\omega \in \Omega$ , we define for  $m > 0$ :

$$\begin{aligned} \pi_n^1(x, \omega) &:= \text{Lin}\left\{ \pi_n, (n+1, x + \omega_{H_1^{\pi_n^x}}) \right\} \quad \text{and} \\ \pi_n^m(x, \omega) &:= \text{Lin}\left\{ \pi_n, (n+1, x + \omega_{H_1^{\pi_n^x}}), (n+j, \omega_{H_j^{\pi_n^x}} - \omega_{H_{j-1}^{\pi_n^x}})_{2 \leq j \leq m} \right\}, \quad \text{for all } m > 1. \end{aligned}$$

**Lemma 5.7** *Let Assumption 3.1 hold, and assume that  $|\xi| \leq C_0$ . Let  $\omega \in \mathcal{Q}$ ,  $|x_i| = \epsilon$  for all  $i \geq 1$ ,  $\pi_n := \{x_i\}_{1 \leq i \leq n}$ , and  $\omega \otimes \pi_n^x \in \mathcal{Q}$ . Then*

- (i) *there exists a sequence of continuous functions  $(\pi_n, x) \mapsto \theta_n^{\omega, \epsilon}(\pi_n, x)$ , bounded uniformly in  $(\epsilon, n)$ , such that*

$$\theta_n^{\omega, \epsilon}(\pi_n; \cdot) \text{ is a viscosity solution of } (E)_\epsilon^{\omega \otimes \pi_n},$$

*with boundary conditions:*

$$\begin{cases} \theta_n^{\omega, \epsilon}(\pi_n; x) = \xi(\omega \otimes \pi_n^x), & |x| < \epsilon \text{ and } x \in \partial Q^{\omega \otimes \pi_n}, \\ \theta_n^{\omega, \epsilon}(\pi_n; x) = \theta_{n+1}^{\omega, \epsilon}(\pi_n^x; 0), & |x| = \epsilon. \end{cases}$$

- (ii) *Moreover, there is a modulus of continuity  $\rho_\epsilon$  such that for any  $\omega^1, \omega^2 \in \mathcal{Q}$ ,*

$$\left| \theta_0^{\omega^1, \epsilon}(0; 0) - \theta_0^{\omega^2, \epsilon}(0; 0) \right| \leq \rho_\epsilon(d^\epsilon(\omega^1, \omega^2)). \quad (5.6)$$

For the domain  $O_\epsilon(\pi_n)$  defined in (4.1), a part of the boundary belongs to  $\partial Q^{\pi_n}$ , while the other belongs to  $\partial O_\epsilon$ . On  $\partial Q^{\pi_n}$ , we should set the solution to be equal to the boundary condition of the path-dependent PDE. Otherwise, on  $\partial O_\epsilon$ , the value of the solution should be consistent with that of the next piece of the path-frozen PDEs. The proof of Lemma 5.7 is similar to that of Lemma

6.2 in [11]. However, the stochastic representations and the estimates that we will use are all in the context of the elliptic equations. So it is necessary to explain the proof in detail.

In preparation of the proof of Lemma 5.7, we give the following estimate on the viscosity solutions to the path-frozen PDEs. The proof is reported in Appendix, Subsection 8.

**Lemma 5.8** *Fix  $D \in \mathcal{R}$ . Let  $h^i : \partial D \rightarrow \mathbb{R}$  be continuous ( $i = 1, 2$ ),  $G$  satisfy Assumption 3.1, and  $v^i$  be the viscosity solutions to the following PDEs:*

$$G(\omega^i, v^i, Dv^i, D^2v^i) = 0 \text{ on } D, \quad v^i = h^i \text{ on } \partial D.$$

Then, we have

$$(v^1 - v^2)(x) \leq \bar{\mathcal{E}}^{L_0} \left[ \left( (h^1 - h^2)_{\mathbb{H}_D}^+ \right)^x \right] + C \|G(\omega^1, \cdot) - G(\omega^2, \cdot)\|_\infty.$$

In particular, if  $\omega^1 = \omega^2$ , then we have  $(v^1 - v^2)(x) \leq \bar{\mathcal{E}}^{L_0} \left[ \left( (h^1 - h^2)_{\mathbb{H}_D}^+ \right)^x \right]$ .

**Proof of Lemma 5.7** Since  $\varepsilon$  is fixed, to simplify the notation, we omit  $\varepsilon$  in the superscript in the proof. We decompose the proof in five steps.

Step 1. We first prove (i) in the case of  $G := \bar{g}$ , where  $\bar{g}$  is defined in (5.2). For any  $N$ , denote

$$\bar{\theta}_{N,N}^\omega(\pi_N; 0) := \bar{\mathcal{E}}^{L_0} \left[ (\xi_{\mathbb{H}_Q})^{\omega \otimes \pi_N} \right].$$

Thanks to Lemma 5.6, we may define  $\bar{\theta}_{N,n}^\omega(\pi_n; \cdot)$  as the viscosity solution of the following PDE:

$$-\bar{g}(\theta, D\theta, D^2\theta) = 0 \text{ on } O_\varepsilon(\omega \otimes \pi_n), \quad \theta(x) = \bar{\theta}_{N,n+1}^\omega(\pi_n^x; 0) \text{ on } \partial O_\varepsilon(\omega \otimes \pi_n), \quad \text{for all } n \leq N-1, \quad (5.7)$$

and we know

$$\begin{aligned} \bar{\theta}_{N,n}^\omega(\pi_n; x) = & \\ & \sup_{b \in \mathbb{H}^0([0, L_0])} \bar{\mathcal{E}}^{L_0} \left[ e^{-\int_0^{\mathbb{H}_{N-n}^{\omega \otimes \pi_n^x}} b_r dr} \xi \left( \omega \otimes \pi_n^{N-n}(x, B) \otimes (B_{\mathbb{H}_Q^{\omega \otimes \pi_n^x \wedge}})^{\mathbb{H}_{N-n}^{\omega \otimes \pi_n^x}} \right) + C_0 \int_0^{\mathbb{H}_{N-n}^{\omega \otimes \pi_n^x}} e^{-\int_0^s b_r dr} ds \right]. \end{aligned} \quad (5.8)$$

Lemma 5.6 also implies that  $\bar{\theta}_{N,n}^\varepsilon(\pi_n; x)$  is continuous in both variables  $(\pi_n, x)$ , and clearly, they are uniformly bounded. We next define

$$\begin{aligned} \bar{\theta}_n^\omega(\pi_n; x) := & \\ & \sup_{b \in \mathbb{H}^0([0, L_0])} \bar{\mathcal{E}}^{L_0} \left[ e^{-\int_0^{\mathbb{H}_Q^{\omega \otimes \pi_n^x}} b_r dr} \overline{\lim}_{N \rightarrow \infty} \xi \left( \omega \otimes \pi_n^{N-n}(x, B) \otimes (B_{\mathbb{H}_Q^{\omega \otimes \pi_n^x \wedge}})^{\mathbb{H}_{N-n}^{\omega \otimes \pi_n^x}} \right) + C_0 \int_0^{\mathbb{H}_Q^{\omega \otimes \pi_n^x}} e^{-\int_0^s b_r dr} ds \right]. \end{aligned} \quad (5.9)$$

Then, it is easy to estimate that

$$|\bar{\theta}_n^\omega(\pi_n; x) - \bar{\theta}_{N,n}^\omega(\pi_n; x)| \leq CC^{L_0} \left[ \mathbb{H}_{N-n}^{\omega \otimes \pi_n^x} < \mathbb{H}_Q^{\omega \otimes \pi_n^x} \right] \rightarrow 0, \quad N \rightarrow \infty.$$

By Proposition 2.8, the convergence is uniform in  $(\pi_n, x)$ . This implies that  $\bar{\theta}_n^\omega(\pi_n; x)$  is uniformly bounded and continuous in  $(\pi_n, x)$ . Moreover, by the stability of viscosity solutions we see that  $\bar{\theta}_n^\omega(\pi_n; \cdot)$  is the viscosity solution of PDE (5.7) in  $O_\varepsilon(\omega \bar{\otimes} \pi_n)$ , with the boundary condition:

$$\begin{cases} \bar{\theta}_n^\omega(\pi_n; x) = \xi(\omega \bar{\otimes} \pi_n^x), & |x| < \varepsilon \text{ and } x \in \partial Q^{\omega \bar{\otimes} \pi_n}, \\ \bar{\theta}_n^\omega(\pi_n; x) = \bar{\theta}_{n+1}^\omega(\pi_n^x; 0), & |x| = \varepsilon. \end{cases}$$

Hence, we have showed the desired result in the case  $G = \bar{g}$ . Similarly, we may show that  $\underline{\theta}_n^\omega$  defined below is the viscosity solution to the path-frozen PDE when the nonlinearity is  $\underline{g}$ :

$$\begin{aligned} \underline{\theta}_n^\omega(\pi_n; x) := \\ \inf_{b \in \mathbb{H}^0([0, L_0])} \underline{\mathcal{E}}^{L_0} \left[ e^{-\int_0^{\mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x}} b_r dr} \overline{\lim}_{N \rightarrow \infty} \xi \left( \omega \bar{\otimes} \pi_n^{N-n}(x, B) \bar{\otimes} (B_{\mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x \wedge \cdot}})^{\mathbb{H}_{N-n}^{\omega \bar{\otimes} \pi_n^x}} \right) + C_0 \int_0^{\mathbb{H}_Q^{\omega \bar{\otimes} \pi_n^x}} e^{-\int_0^s b_r dr} ds \right]. \end{aligned}$$

Step 2. We next prove (ii) in the case of  $G = \bar{g}$ . Considering  $\pi_n^x \in \mathcal{Q}^{\omega^1} \cap \mathcal{Q}^{\omega^2}$ , we have the following estimate:

$$\begin{aligned} \left| \bar{\theta}_{N,n}^{\omega^1}(\pi_n; x) - \bar{\theta}_{N,n}^{\omega^2}(\pi_n; x) \right| \leq C \bar{\mathcal{E}}^{L_0} \left[ \left| \mathbb{H}_{N-n}^{\omega^1 \bar{\otimes} \pi_n^x} - \mathbb{H}_{N-n}^{\omega^2 \bar{\otimes} \pi_n^x} \right| \right] \\ + C \bar{\mathcal{E}}^{L_0} \left[ \xi \left( \omega^1 \bar{\otimes} \pi_n^{N-n}(x, B) \bar{\otimes} (B_{\mathbb{H}_Q^{\omega^1 \bar{\otimes} \pi_n^x \wedge \cdot}})^{\mathbb{H}_{N-n}^{\omega^1 \bar{\otimes} \pi_n^x}} \right) - \xi \left( \omega^2 \bar{\otimes} \pi_n^{N-n}(x, B) \bar{\otimes} (B_{\mathbb{H}_Q^{\omega^2 \bar{\otimes} \pi_n^x \wedge \cdot}})^{\mathbb{H}_{N-n}^{\omega^2 \bar{\otimes} \pi_n^x}} \right) \right], \end{aligned}$$

where  $\bar{\theta}_{N,n}^{\omega^i}(\pi_n; x)$ ,  $i = 1, 2$ , are defined in (5.8). Note that  $\left| \mathbb{H}_{N-n}^{\omega^1 \bar{\otimes} \pi_n^x} - \mathbb{H}_{N-n}^{\omega^2 \bar{\otimes} \pi_n^x} \right| \leq \left| \mathbb{H}_Q^{\omega^1 \bar{\otimes} \pi_n^x} - \mathbb{H}_Q^{\omega^2 \bar{\otimes} \pi_n^x} \right|$ . As in Lemma 5.6, one may easily show that

$$\left| \bar{\theta}_{N,n}^{\omega^1} - \bar{\theta}_{N,n}^{\omega^2} \right| \leq \rho_\varepsilon(d^e(\omega^1, \omega^2)),$$

where  $\rho_\varepsilon$  is independent of  $N$ . Considering  $\bar{\theta}_n^{\omega^i}$  defined in (5.9), we obtain by sending  $N \rightarrow \infty$  that

$$\left| \bar{\theta}_n^{\omega^1} - \bar{\theta}_n^{\omega^2} \right| \leq \rho_\varepsilon(d^e(\omega^1, \omega^2)).$$

A similar argument provides the same estimate for  $\underline{\theta}_n^\omega$ :

$$\left| \underline{\theta}_n^{\omega^1} - \underline{\theta}_n^{\omega^2} \right| \leq \rho_\varepsilon(d^e(\omega^1, \omega^2)). \quad (5.10)$$

Step 3. We now prove (i) for general  $G$ . Given the construction of Step 1, define:

$$\bar{\theta}_m^{\omega, m}(\pi_m; x) := \bar{\theta}_m^\omega(\pi_m; x), \quad \underline{\theta}_m^{\omega, m}(\pi_m; x) := \underline{\theta}_m^\omega(\pi_m; x); \quad m \geq 1.$$

For  $n \leq m - 1$ , we may define  $\bar{\theta}_n^{\omega, m}$  and  $\underline{\theta}_n^{\omega, m}$  as the unique viscosity solution of the path-frozen PDE  $(E)_\varepsilon^{\omega \bar{\otimes} \pi_n}$  with boundary conditions

$$\bar{\theta}_n^{\omega, m}(\pi_n; x) = \bar{\theta}_{n+1}^{\omega, m}(\pi_n^x; 0), \quad \underline{\theta}_n^{\omega, m}(\pi_n; x) = \underline{\theta}_{n+1}^{\omega, m}(\pi_n^x; 0) \quad \text{for } x \in \partial O_\varepsilon(\omega \bar{\otimes} \pi_n).$$



Since  $\underline{g} \leq G \leq \bar{g}$ , it is easy to deduce that  $\bar{\theta}_m^{\varepsilon,m}$  and  $\underline{\theta}_m^{\varepsilon,m}$  are respectively viscosity supersolution and subsolution to the path-frozen PDE  $(E)_\varepsilon^{\pi_m}$ . By the comparison result for viscosity solutions of PDEs, we obtain that

$$\bar{\theta}_m^{\omega,m}(\pi_m; \cdot) \geq \bar{\theta}_m^{\omega,m+1}(\pi_m; \cdot) \geq \underline{\theta}_m^{\omega,m+1}(\pi_m; \cdot) \geq \underline{\theta}_m^{\omega,m}(\pi_m; \cdot) \text{ on } O_\varepsilon(\omega \otimes \pi_m),$$

Using the comparison argument again, we obtain

$$\bar{\theta}_n^{\omega,m}(\pi_n; \cdot) \geq \bar{\theta}_n^{\omega,m+1}(\pi_n; \cdot) \geq \underline{\theta}_n^{\omega,m+1}(\pi_n; \cdot) \geq \underline{\theta}_n^{\omega,m}(\pi_n; \cdot) \text{ on } O_\varepsilon(\omega \otimes \pi_n) \text{ for all } n \leq m. \quad (5.11)$$

Denote  $\delta\theta_n^{\omega,m} := \bar{\theta}_n^{\omega,m} - \underline{\theta}_n^{\omega,m}$ . Applying Lemma 5.8 repeatedly and using the tower property of  $\bar{\mathcal{E}}^{L_0}$  stated in Lemma 2.6, we obtain that

$$|\delta\theta_n^{\omega,m}(\pi_n; x)| \leq \bar{\mathcal{E}}^{L_0} [|\delta\theta_m^{\omega,m}(\pi_n^{m-n}(x, B); 0)|].$$

Note that  $\delta\theta_m^{\omega,m}(\pi_n^{m-n}(x, B); 0) = 0$  as  $\pi_n^{m-n}(x, B) \in \partial\mathcal{Q}^\omega$ . Then, by Proposition 2.8, we have

$$|\delta\theta_n^{\omega,m}(\pi_n; x)| \leq CC^{L_0} \left[ \mathbb{H}_{m-n}^{\omega \otimes \pi_n^x} < \mathbb{H}_Q^{\omega \otimes \pi_n^x} \right] \rightarrow 0, \quad \text{as } m \rightarrow \infty.$$

Together with (5.11), this implies the existence of  $\theta_n^\omega$  such that

$$\bar{\theta}_n^{\omega,m} \downarrow \theta_n^\omega, \quad \underline{\theta}_n^{\omega,m} \uparrow \theta_n^\omega, \quad \text{as } m \rightarrow \infty. \quad (5.12)$$

Clearly  $\theta_n^\omega$  is uniformly bounded and continuous. Finally, it follows from the stability of viscosity solutions that  $\theta_n^\omega$  satisfies the statement of (i).

Step 4. We next prove (ii) for a general nonlinearity  $G$ . For the simplicity of notation, we denote the stopping times:

$$\mathbb{H}^i := \mathbb{H}_Q^{i \otimes \pi_n^x} \text{ for } i = 1, 2, \quad \mathbb{H}^{1,2} := \mathbb{H}^1 \wedge \mathbb{H}^2,$$

$$\mathbb{H}_0 = 0, \quad \mathbb{H}_1 := \inf\{t \geq 0 : x + B_t \notin O_\varepsilon\}, \quad \mathbb{H}_{i+1} := \inf\{t \geq \mathbb{H}_i : B_t - B_{\mathbb{H}_i} \notin O_\varepsilon\} \text{ for } i \geq 1.$$

First, considering  $\bar{\theta}_n^{\omega,m}$  defined in Step 3, we claim that for  $\pi_n^x \in \mathcal{Q}^{\omega^1} \cap \mathcal{Q}^{\omega^2}$

$$\begin{aligned} (\bar{\theta}_n^{\omega^1,m} - \underline{\theta}_n^{\omega^2,m})(\pi_n; x) &\leq \bar{\mathcal{E}}^{L_0} \left[ (\bar{\theta}_m^{\omega^1} - \underline{\theta}_m^{\omega^2})(\pi_n^{m-n}(x, B); 0) 1_{\{\mathbb{H}_{m-n} \leq \mathbb{H}^{1,2}\}} \right] \\ &\quad + I_1 + I_2 + C(m-n)\rho(d^e(\omega^1, \omega^2)), \end{aligned} \quad (5.13)$$

where

$$\begin{aligned} I_1 &:= \sum_{k=0}^{m-n-1} \bar{\mathcal{E}}^{L_0} \left[ \left( \bar{\theta}_{n+k+1}^{\omega^1,m}(\pi_n^{k+1}(x, B); 0) - \underline{\theta}_{n+k}^{\omega^2,m}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}^k}) \right) 1_{\{\mathbb{H}^1 < \mathbb{H}_{k+1} \leq \mathbb{H}^2\}} \right], \\ I_2 &:= \sum_{k=0}^{m-n-1} \bar{\mathcal{E}}^{L_0} \left[ \left( \bar{\theta}_{n+k}^{\omega^1,m}(\pi_n^k(x, B); B_{\mathbb{H}^2} - B_{\mathbb{H}^k}) - \underline{\theta}_{n+k+1}^{\omega^2,m}(\pi_n^{k+1}(x, B); 0) \right) 1_{\{\mathbb{H}^2 < \mathbb{H}_{k+1} \leq \mathbb{H}^1\}} \right] \end{aligned}$$

This claim will be proved in Step 5. We next focus on the term in  $I_1$ :

$$\left( \bar{\theta}_{n+k+1}^{\omega^1,m}(\pi_n^{k+1}(x, B); 0) - \underline{\theta}_{n+k}^{\omega^2,m}(\pi_n^k(x, B); B_{\mathbb{H}^1} - B_{\mathbb{H}^k}) \right) 1_{\{\mathbb{H}^1 < \mathbb{H}_{k+1} \leq \mathbb{H}^2\}}.$$

Note that as  $H^1 < H_{k+1} \leq H^2$ , we have  $\pi_n^{k+1} = \pi_n^k \bar{\otimes} \text{Lin}\{((0,0), (1, B_{H^1} - B_{H_k}))\}$ , and thus

$$\bar{\theta}_{n+k+1}^{\omega^1, n+k+1}(\pi_n^{k+1}(x, B); 0) = \underline{\theta}_{n+k+1}^{\omega^1, n+k+1}(\pi_n^{k+1}(x, B); 0) = \underline{\theta}_{n+k}^{\omega^1, n+k}(\pi_n^k(x, B); B_{H^1} - B_{H_k}).$$

So, by using (5.10), we obtain

$$\begin{aligned} & \left| \bar{\theta}_{n+k+1}^{\omega^1, n+k+1}(\pi_n^{k+1}(x, B); 0) - \underline{\theta}_{n+k}^{\omega^2, n+k}(\pi_n^k(x, B); B_{H^1} - B_{H_k}) \right| \\ & \leq \left| \underline{\theta}_{n+k}^{\omega^1, n+k}(\pi_n^k(x, B); B_{H^1} - B_{H_k}) - \underline{\theta}_{n+k}^{\omega^2, n+k}(\pi_n^k(x, B); B_{H^1} - B_{H_k}) \right| \leq \rho_\varepsilon(d^e(\omega^1, \omega^2)). \end{aligned} \quad (5.14)$$

Further, as in Step 3, the comparison result implies that

$$\begin{aligned} \underline{\theta}_{n+k}^{\omega^2, m}(\pi_n^k(x, B); B_{H^1} - B_{H_k}) & \geq \underline{\theta}_{n+k}^{\omega^2, n+k}(\pi_n^k(x, B); B_{H^1} - B_{H_k}); \\ \bar{\theta}_{n+k+1}^{\omega^1, m}(\pi_n^{k+1}(x, B); 0) & \leq \bar{\theta}_{n+k+1}^{\omega^1, n+k+1}(\pi_n^{k+1}(x, B); 0). \end{aligned}$$

Therefore, (5.14) implies that

$$\bar{\theta}_{n+k+1}^{\omega^1, m}(\pi_n^{k+1}(x, B); 0) - \underline{\theta}_{n+k}^{\omega^2, m}(\pi_n^k(x, B); B_{H^1} - B_{H_k}) \leq \rho_\varepsilon(d^e(\omega^1, \omega^2)).$$

Similarly, we may prove the same estimate for the term in  $I_2$ . Then, by (5.13) we conclude that

$$(\bar{\theta}_n^{\omega^1, m} - \underline{\theta}_n^{\omega^2, m})(\pi_n; x) \leq CC^L [H_{m-n} < H^{1,2}] + C(m-n+1)\rho_\varepsilon(d^e(\omega^1, \omega^2)).$$

Recalling (5.12), we obtain that

$$(\theta_n^{\omega^1} - \theta_n^{\omega^2})(\pi_n; x) \leq \rho_\varepsilon(d^e(\omega^1, \omega^2)).$$

By exchanging the roles of  $\omega^1$  and  $\omega^2$ , we have  $|(\theta_n^{\omega^1} - \theta_n^{\omega^2})(\pi_n; x)| \leq \rho_\varepsilon(d^e(\omega^1, \omega^2))$ .

Step 5. We prove Claim (5.13). Suppose that  $m \geq n+1$ . It suffices to show that

$$\begin{aligned} (\bar{\theta}_n^{\omega^1, m} - \underline{\theta}_n^{\omega^2, m})(\pi_n; x) & \leq \bar{\mathcal{E}}^{L_0} \left[ (\bar{\theta}_{n+1}^{\omega^1, m} - \underline{\theta}_{n+1}^{\omega^2, m})(\pi_n^1(x, B); 0) 1_{\{H_1 \leq H^{1,2}\}} \right] \\ & \quad + \bar{\mathcal{E}}^{L_0} \left[ (\bar{\theta}_{n+1}^{\omega^1, m}(\pi_n^1(x, B); 0) - \underline{\theta}_n^{\omega^2, m}(\pi_n; x + B_{H^1})) 1_{\{H^1 < H_1 \leq H^2\}} \right] \\ & \quad + \bar{\mathcal{E}}^{L_0} \left[ (\bar{\theta}_n^{\omega^1, m}(\pi_n; x + B_{H^2}) - \underline{\theta}_{n+1}^{\omega^2, m}(\pi_n^1(x, B); 0)) 1_{\{H^2 < H_1 \leq H^1\}} \right] \\ & \quad + C\rho_\varepsilon(d^e(\omega^1, \omega^2)). \end{aligned}$$

Then the claim can be easily proved by induction. Recall that  $\bar{\theta}_n^{\omega^1, m}$  (resp.  $\underline{\theta}_n^{\omega^2, m}$ ) is a solution to the PDE with generator  $G(\omega^1, \cdot)$  (resp.  $G(\omega^2, \cdot)$ ). Now we study those two PDEs on the domain:

$$O_\varepsilon \cap Q^{\omega^1} \cap Q^{\omega^2}.$$

The boundary of this set can be divided into three parts which belong to  $\partial O_\varepsilon$ ,  $\partial Q^{\omega^1}$  and  $\partial Q^{\omega^2}$  respectively. We denote them by  $\text{Bd}_1$ ,  $\text{Bd}_2$  and  $\text{Bd}_3$ .

(i) On  $\text{Bd}_1$ , we have  $H_1 \leq H^{1,2}$ , and thus

$$\bar{\theta}_n^{\omega^1, m}(\pi_n, x) = \bar{\theta}_{n+1}^{\omega^1, m}(\pi_n^x, 0) \quad \text{and} \quad \underline{\theta}_n^{\omega^2, m}(\pi_n, x) = \underline{\theta}_{n+1}^{\omega^2, m}(\pi_n^x, 0).$$

(ii) On  $\text{Bd}_2$ , we have  $H^1 < H_1 \leq H^2$ , so we only have  $\bar{\theta}_n^{\omega^1, m}(\pi_n, x) = \bar{\theta}_{n+1}^{\omega^1, m}(\pi_n^x, 0)$ .

(iii) On  $\text{Bd}_3$ , we have  $H^2 < H_1 \leq H^1$ , so we only have  $\underline{\theta}_n^{\omega^2, m}(\pi_n, x) = \underline{\theta}_{n+1}^{\omega^2, m}(\pi_n^x, 0)$ .

Finally, Lemma 5.8 completes the proof.  $\blacksquare$

The previous lemma shows the existence of the viscosity solution to the path-frozen PDEs. We now use Assumption 4.1 to construct smooth super- and sub-solutions to the PPDE from the solution to the path-frozen PDEs.

Denote

$$\theta_n^\epsilon := \theta_n^{0, \epsilon}, \quad H_n := H_n^0 \quad \text{and} \quad \hat{\pi}_n := \text{Lin}\{(H_i, \omega_{H_i}); 0 \leq i \leq n\}.$$

**Lemma 5.9** *There exists  $\psi^\epsilon \in \bar{C}^2(\mathcal{Q})$  such that*

$$\psi^\epsilon(0) = \theta_0^\epsilon(0) + \epsilon, \quad \psi^\epsilon \geq h \quad \text{on} \quad \mathcal{Q} \quad \text{and} \quad \mathcal{L}^{\hat{\pi}_n} \psi^\epsilon \geq 0 \quad \text{on} \quad \mathcal{O}_\epsilon(\hat{\pi}_n) \quad \text{for all } n \in \mathbb{N}.$$

**Proof** For simplicity, in the proof we omit the superscript  $\epsilon$ . Set  $\delta_n := 2^{-n-2}\epsilon$ . First, since PDE  $(E)_\epsilon^0$  satisfies Assumption 4.1 and  $G(\omega, y, z, \gamma)$  is decreasing in  $y$ , there exists a function  $v_0 \in C_0^2(\mathcal{O}_\epsilon(0))$  such that

$$v_0(0) = \theta_0(0) + \frac{\epsilon}{2}, \quad \mathbf{L}^0 v_0 \geq 0 \quad \text{on} \quad \mathcal{O}_\epsilon(0) \quad \text{and} \quad v_0 \geq \theta_0 \quad \text{on} \quad \partial\mathcal{O}_\epsilon(0).$$

Define

$$\psi(\omega) := v_0(\omega_{\bar{t}}) + \sum_{i \geq 0} \delta_i \quad \text{on} \quad \text{cl}(\mathcal{O}_\epsilon(0)).$$

By the monotonicity of  $G$ , it is clear that

$$\psi(0) - \theta_0(0) = \frac{\epsilon}{2} + \sum_{i \geq 0} \delta_i = \epsilon, \quad \psi \in C^2(\mathcal{O}_\epsilon(0)) \quad \text{and} \quad \mathcal{L}^0 \psi \geq \mathbf{L}^0 v_0 \geq 0 \quad \text{on} \quad \mathcal{O}_\epsilon(0).$$

Next, applying again Assumption 4.1 to PDE  $(E)_{\hat{\pi}_1}^{\hat{\pi}_1}$ , we can find a function  $v_1(\hat{\pi}_1; \cdot) \in C_0^2(\mathcal{O}_\epsilon(\hat{\pi}_1))$  such that

$$v_1(\hat{\pi}_1; 0) = v_0(x_1) + \delta_0, \quad \mathbf{L}^{\hat{\pi}_1} v_1 \geq 0 \quad \text{on} \quad \mathcal{O}_\epsilon(\hat{\pi}_1), \quad v_1(\hat{\pi}_1; \cdot) \geq \theta_1(\hat{\pi}_1; \cdot) \quad \text{on} \quad \partial\mathcal{O}_\epsilon(\hat{\pi}_1).$$

Then, define

$$\psi(\omega) := v_1(\hat{\pi}_1; \omega_{\bar{t}} - \omega_{H_1}) + \sum_{i \geq 1} \delta_i, \quad \text{for } \omega \in \mathcal{O}_\epsilon(\hat{\pi}_1)$$

It is clear that the updated  $\psi$  is in  $\bar{C}^2$  and  $\mathcal{L}^{\hat{\pi}_1} \psi \geq 0$  on  $\mathcal{O}_\epsilon(\hat{\pi}_1)$ . Repeating the procedure, we may find a sequence of functions  $v_n$  and thus construct the desired  $\psi \in \bar{C}^2(\mathcal{Q})$ .  $\blacksquare$

Finally, we have done all the necessary constructions and are ready to show the main result of the section.

**Proof of Proposition 5.5** For any  $\epsilon > 0$ , let  $\psi^\epsilon$  be as in Lemma 5.9, and  $\bar{\psi}^\epsilon := \psi^\epsilon + \rho(2\epsilon) + \lambda^{-1}(\rho(2\epsilon))$ , where  $\rho$  is the modulus of continuity of  $\xi$  and  $G$  and  $\lambda^{-1}$  is the inverse of the function in Assumption 3.1. Then clearly  $\bar{\psi}^\epsilon \in \bar{C}^2(\mathcal{Q})$  and bounded. Also,

$$\bar{\psi}^\epsilon(\omega) - \xi(\omega) \geq \psi^\epsilon(\omega) + \rho(2\epsilon) - \xi(\omega) \geq \xi(\omega^\epsilon) - \xi(\omega) + \rho(2\epsilon) \geq 0 \text{ on } \partial\mathcal{Q}.$$

Moreover, when  $\bar{t}(\omega) \in [H_n(\omega), H_{n+1}(\omega))$ , we have that

$$\begin{aligned} \mathcal{L}\bar{\psi}^\epsilon(\omega) &= -G\left(\omega, \bar{\psi}^\epsilon, \partial_\omega \psi^\epsilon, \partial_{\omega\omega}^2 \psi^\epsilon\right) \\ &\geq -G\left(\hat{\pi}_n, \psi^\epsilon + \lambda^{-1}(\rho(2\epsilon)), \partial_\omega \psi^\epsilon, \partial_{\omega\omega}^2 \psi^\epsilon\right) - \rho(2\epsilon) \\ &\geq -G\left(\hat{\pi}_n, \psi^\epsilon, \partial_\omega \psi^\epsilon, \partial_{\omega\omega}^2 \psi^\epsilon\right) \geq 0. \end{aligned}$$

Then by the definition of  $\bar{u}$  we see that

$$\bar{u}(0) \leq \bar{\psi}^\epsilon(0) = \psi^\epsilon + \rho(2\epsilon) + \lambda^{-1}(\rho(2\epsilon)) \leq \theta_0^\epsilon(0) + \epsilon + \rho(2\epsilon) + \lambda^{-1}(\rho(2\epsilon)). \quad (5.15)$$

Similarly,  $\underline{u}(0) \geq \theta_0^\epsilon(0) - \epsilon - \rho(2\epsilon) - \lambda^{-1}(\rho(2\epsilon))$ . That implies that

$$\bar{u}(0) - \underline{u}(0) \leq 2\epsilon + 2\rho(2\epsilon) + 2\lambda^{-1}(\rho(2\epsilon)).$$

Since  $\epsilon$  is arbitrary, this shows that  $\bar{u}(0) = \underline{u}(0)$ . Similarly, we can show that  $\bar{u}(\omega) = \underline{u}(\omega)$  for all  $\omega \in \mathcal{Q}$ . ■

## 6 Existence

In this section, we verify that

$$u := \bar{u} = \underline{u} \quad (6.1)$$

is the unique BUC viscosity solution to the path-dependent PDE (3.1). We will prove that  $u$  is BUC in Subsection 6.1 and  $u$  satisfies the viscosity property in Subsection 6.2.

### 6.1 Regularity

The non-continuity of the hitting time  $H_Q(\omega)$  brings difficulty to the proof of the regularity of  $u$ . One cannot adapt the method used in [11]. In our approach, we make use of the uniform continuity of the solution of the path-frozen PDEs proved in (ii) of Lemma 5.7.

First, it is easy to show that  $u$  is bounded.

**Proposition 6.1** *Let Assumption 3.1 hold and  $\xi \in \text{BUC}(\partial\mathcal{Q})$ . Then  $\bar{u}$  is bounded from above and  $\underline{u}$  is bounded from below.*

**Proof** Assume that  $|\xi| \leq C_0$ . Define:

$$\psi := \lambda^{-1}(C_0) + C_0.$$

Note that  $\psi \in \bar{C}^2$ . Observe that  $\psi_T \geq C_0 \geq \xi$ . Also,

$$\mathcal{L}^\omega \psi_s = -G^\omega(\cdot, \psi_s, 0, 0) \geq C_0 - G^\omega(\cdot, 0, 0, 0) \geq 0.$$

It follows that  $\psi \in \bar{\mathcal{D}}_Q^\xi(\omega)$ , and thus  $\bar{u}(\omega) \leq \psi(0) = \lambda^{-1}(C_0) + C_0$ . Similarly, one can show that  $\underline{u}(\omega) \geq -\lambda^{-1}(C_0) - C_0$ .  $\blacksquare$

The rest of the subsection is devoted to prove the uniform continuity of  $u$ .

**Proposition 6.2** *The function  $u$  defined in (6.1) is uniformly continuous in  $\mathcal{Q}$ .*

**Proof** Recall (5.15), i.e. for  $\omega^1, \omega^2 \in \mathcal{Q}$ , it holds that

$$\bar{u}(\omega^1) \leq \theta_0^{\omega^1}(0) + \epsilon + \rho(2\epsilon) \quad \text{and} \quad \underline{u}(\omega^2) \geq \theta_0^{\omega^2}(0) - \epsilon - \rho(2\epsilon).$$

Hence, it follows from Lemma 5.7 that

$$u(\omega^1) - u(\omega^2) = \bar{u}(\omega^1) - \underline{u}(\omega^2) \leq \theta_0^{\omega^1}(0) - \theta_0^{\omega^2}(0) + 2(\epsilon + \rho(2\epsilon)) \leq \rho_\epsilon(d^e(\omega^1, \omega^2)) + \rho(2\epsilon).$$

By exchanging the roles of  $\omega^1$  and  $\omega^2$ , we obtain that  $u$  is uniformly continuous.  $\blacksquare$

## 6.2 Viscosity property

After having shown that  $u$  is uniformly continuous, we need to verify that it indeed satisfies the viscosity property. The following proof is similar to that of Proposition 4.3 in [11].

**Proposition 6.3**  *$u$  is the viscosity solution to PPDE (3.1).*

**Proof** Without loss of generality, we prove only that  $\bar{u}$  is a  $L_0$ -viscosity supersolution at 0. Assume the contrary, i.e. there exists  $\varphi \in \bar{\mathcal{A}}^{L_0} \bar{u}(0)$  such that  $-c := \mathcal{L}\varphi(0) < 0$ . For any  $\psi \in \bar{\mathcal{D}}_Q^\xi(0)$  and  $\omega \in \mathcal{Q}$  it is clear that  $\psi^\omega \in \bar{\mathcal{D}}_Q^\xi(\omega)$  and  $\psi(\omega) \geq \bar{u}(\omega)$ . Now by the definition of  $\bar{u}$ , there exists  $\psi^n \in \bar{C}^2(\mathcal{Q})$  such that

$$\delta_n := \psi^n(0) - \bar{u}(0) \downarrow 0 \text{ as } n \rightarrow \infty, \quad \mathcal{L}\psi^n(\omega) \geq 0, \quad \omega \in \mathcal{Q}. \quad (6.2)$$

Let  $\mathbb{H}_{O_\epsilon}$  be a localization of test function  $\varphi$ . Since  $\varphi \in C^2(\mathcal{O}_\epsilon)$  and  $\bar{u} \in \text{BUC}(\mathcal{Q})$ , without loss of generality we may assume that

$$\mathcal{L}\varphi(\omega_{t \wedge \cdot}) \leq -\frac{c}{2} \text{ and } |\varphi_t - \varphi_0| + |\bar{u}_t - \bar{u}_0| \leq \frac{c}{6L_0} \text{ for all } t \leq \mathbb{H}_{O_\epsilon}.$$

Since  $\varphi \in \bar{\mathcal{A}}^{L_0} \bar{u}(0)$ , this implies for all  $\mathbb{P} \in \mathcal{P}^{L_0}$  that :

$$0 \geq \mathbb{E}^\mathbb{P} [(\varphi - \bar{u})_{\mathbb{H}_{O_\epsilon}}] \geq \mathbb{E}^\mathbb{P} [(\varphi - \psi^n)_{\mathbb{H}_{O_\epsilon}}]. \quad (6.3)$$

Denote  $\mathcal{G}^\mathbb{P}\phi := \alpha^\mathbb{P} \cdot \partial_\omega \phi + \frac{1}{2}(\beta^\mathbb{P})^2 : \partial_{\omega\omega}^2 \phi$ . Then, since  $\varphi \in C^2(\mathcal{O}_\varepsilon)$  and  $\psi^n \in \overline{C}^2(\mathcal{Q})$ , it follows from (6.2) that:

$$\begin{aligned} \delta_n &\geq \mathbb{E}^\mathbb{P} [(\varphi - \psi^n)_{\mathbb{H}O_\varepsilon} - (\varphi - \psi^n)_0] = \mathbb{E}^\mathbb{P} \left[ \int_0^{\mathbb{H}O_\varepsilon} \mathcal{G}^\mathbb{P}(\varphi - \psi^n) ds \right] \\ &\geq \mathbb{E}^\mathbb{P} \left[ \int_0^{\mathbb{H}O_\varepsilon} \left( \frac{c}{2} - G(\omega_{s\wedge\cdot}, \varphi, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) + G(\omega_{s\wedge\cdot}, \psi^n, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) + \mathcal{G}^\mathbb{P}(\varphi - \psi^n) \right) ds \right] \\ &\geq \mathbb{E}^\mathbb{P} \left[ \int_0^{\mathbb{H}O_\varepsilon} \left( \frac{c}{2} - G(\omega_{s\wedge\cdot}, \varphi, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) + G(\omega_{s\wedge\cdot}, \bar{u}, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) + \mathcal{G}^\mathbb{P}(\varphi - \psi^n) \right) ds \right], \end{aligned}$$

where the last inequality is due to the monotonicity in  $y$  of  $G$ . Since  $\varphi_0 = \bar{u}_0$ , we get

$$\begin{aligned} \delta_n &\geq \mathbb{E}^\mathbb{P} \left[ \int_0^{\mathbb{H}O_\varepsilon} \left( \frac{c}{3} - G(\omega_{s\wedge\cdot}, \bar{u}_0, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \right. \right. \\ &\quad \left. \left. + G(\omega_{s\wedge\cdot}, \bar{u}_0, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) + \mathcal{G}^\mathbb{P}(\varphi - \psi^n) \right) ds \right]. \end{aligned}$$

We next let  $\eta > 0$ , and for each  $n$ , define  $\tau_0^n := 0$  and

$$\begin{aligned} \tau_{j+1}^n : &= \mathbb{H}O_\varepsilon \wedge \inf \{ t \geq \tau_j^n : \rho(d^e(\omega_{t\wedge\cdot}, \omega_{\tau_j^n \wedge\cdot})) + |\partial_\omega \varphi(\omega_{t\wedge\cdot}) - \partial_\omega \varphi(\omega_{\tau_j^n \wedge\cdot})| \\ &\quad + |\partial_{\omega\omega}^2 \varphi(\omega_{t\wedge\cdot}) - \partial_{\omega\omega}^2 \varphi(\omega_{\tau_j^n \wedge\cdot})| + |\partial_\omega \psi^n(\omega_{t\wedge\cdot}) - \partial_\omega \psi^n(\omega_{\tau_j^n \wedge\cdot})| \\ &\quad + |\partial_{\omega\omega}^2 \psi^n(\omega_{t\wedge\cdot}) - \partial_{\omega\omega}^2 \psi^n(\omega_{\tau_j^n \wedge\cdot})| \geq \eta \}. \end{aligned}$$

Since  $\varphi \in C^2(\mathcal{O}_\varepsilon)$  and  $\psi^n \in \overline{C}^2(\mathcal{Q})$ , one can easily check that  $\tau_j^n \uparrow \mathbb{H}_D$   $\mathcal{P}^{L_0}$ -q.s. as  $j \rightarrow \infty$ . Thus,

$$\begin{aligned} \delta_n &\geq \left( \frac{c}{3} - C\eta \right) \mathbb{E}^\mathbb{P} [\mathbb{H}O_\varepsilon] + \sum_{j \geq 0} \mathbb{E}^\mathbb{P} \int_{\tau_j^n}^{\tau_{j+1}^n} \left( -G(\cdot, \bar{u}_0, \partial_\omega \varphi, \partial_{\omega\omega}^2 \varphi) \right. \\ &\quad \left. + G(\cdot, \bar{u}_0, \partial_\omega \psi^n, \partial_{\omega\omega}^2 \psi^n) + \mathcal{G}^\mathbb{P}(\varphi - \psi^n) \right)_{\tau_j^n} ds \\ &= \left( \frac{c}{3} - C\eta \right) \mathbb{E}^\mathbb{P} [\mathbb{H}O_\varepsilon] + \sum_{j \geq 0} \mathbb{E}^\mathbb{P} \int_{\tau_j^n}^{\tau_{j+1}^n} \left( \alpha_j^n \cdot \partial_\omega (\psi^n - \varphi) \right. \\ &\quad \left. + \frac{1}{2}(\beta_j^n)^2 : \partial_{\omega\omega}^2 (\psi^n - \varphi) + \mathcal{G}^\mathbb{P}(\varphi - \psi^n) \right)_{\tau_j^n} ds, \end{aligned}$$

for some  $\alpha_j^n, \beta_j^n$  such that  $|\alpha_j^n| \leq L$  and  $\beta_j^n \in \mathbb{H}_L^0$ . Note that  $\alpha_j^n$  and  $\beta_j^n$  are both  $\mathcal{F}_{\tau_j^n}$ -measurable. Take  $\mathbb{P}_n \in \mathcal{P}^{L_0}$  such that  $\alpha_t^{\mathbb{P}_n} = \alpha_j^n, \beta_t^{\mathbb{P}_n} = \beta_j^n$  for  $t \in [\tau_j^n, \tau_{j+1}^n)$ . Then

$$\delta_n \geq \left( \frac{c}{3} - C\eta \right) \mathbb{E}^{\mathbb{P}_n} [\mathbb{H}O_\varepsilon].$$

Let  $\eta := \frac{c}{6C}$ , and it follows  $\underline{\mathcal{E}}^{L_0} [\mathbb{H}O_\varepsilon] \leq \mathbb{E}^{\mathbb{P}_n} [\mathbb{H}O_\varepsilon] \leq \delta_n$ . By putting  $n \rightarrow \infty$ , we get  $\underline{\mathcal{E}}^{L_0} [\mathbb{H}O_\varepsilon] = 0$ , contradiction.  $\blacksquare$

## 7 Path-dependent time-invariant stochastic control

In this section, we present an application of fully nonlinear elliptic PPDE. An important question which is most relevant since the recent financial crisis is the risk of model mis-specification. The

uncertain volatility model (see Avellaneda, Levy and Paras [1], Lyons [15] or Nutz [18]) provides a conservative answer to this problem.

In the present application, the canonical process  $B$  represents the price process of some primitive asset, and our objective is the hedging of the derivative security defined by the payoff  $\xi(B.)$  at some maturity  $H_Q$  defined as the exiting time from some domain  $Q$ .

In contrast with the standard Black-Scholes modeling, we assume that the probability space  $(\Omega, \mathcal{F})$  is endowed with a family of probability measures  $\mathcal{P}^{\text{UVM}}$ . In the uncertain volatility model, the quadratic variation of the canonical process is assumed to lie between two given bounds,

$$\underline{\sigma}^2 dt \leq d\langle B \rangle_t \leq \bar{\sigma}^2 dt, \quad \mathbb{P}\text{-a.s. for all } \mathbb{P} \in \mathcal{P}^{\text{UVM}}.$$

Then, by the possible frictionless trading of the underlying asset, it is well known that the non-arbitrage condition is characterized by the existence of an equivalent martingale measure. Consequently, we take

$$\mathcal{P}^{\text{UVM}} := \{ \mathbb{P} \in \mathcal{P}^\infty : B \text{ is } \mathbb{P}\text{-martingale and } \frac{d\langle B \rangle_t}{dt} \in [\underline{\sigma}^2, \bar{\sigma}^2], \mathbb{P}\text{-a.s.} \}.$$

The superhedging problem under model uncertainty was initially formulated by Denis&Martini [6] and Neufeld&Nutz [17], and involves delicate quasi-sure analysis. Their main result expresses the cost of robust superhedging as

$$u_0 := \bar{\mathcal{E}}^{\text{UVM}}[e^{-rH_Q} \xi(B_{H_Q \wedge \cdot})] := \sup_{\mathbb{P} \in \mathcal{P}^{\text{UVM}}} \mathbb{E}^{\mathbb{P}}[e^{-rH_Q} \xi(B_{H_Q \wedge \cdot})],$$

where  $r$  is the discount rate. Further, define  $u$  on  $\Omega^e$  as:

$$u(\omega) := \bar{\mathcal{E}}^{\text{UVM}}[e^{-rH_Q} \xi((B_{H_Q \wedge \cdot})^\omega)], \quad \text{for all } \omega \in \Omega^e. \quad (7.1)$$

We are interested in characterizing  $u$  as the viscosity solution of the corresponding fully nonlinear elliptic PPDE.

**Assumption 7.1** *Assume that*

$$\xi \in \text{BUC}(\Omega^e), \quad \underline{\sigma} > 0, \quad \text{and the discount rate } r > 0.$$

**Proposition 7.2** *Under Assumption 7.1, the function  $u$  defined in (7.1) is BUC and is a viscosity solution to the elliptic path-dependent HJB equation:*

$$ru - L|\partial_\omega u| - \sup_{\beta \in [\underline{\sigma}, \bar{\sigma}]} \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 u = 0 \text{ on } \mathcal{Q}, \quad \text{and } u = \xi \text{ on } \partial\mathcal{Q}$$

In preparation to the proof of the proposition, we first show two lemmas.

**Lemma 7.3** *The function  $u$  defined in (7.1) is BUC.*

**Proof** By (5.3) and the fact that  $\xi \in \text{BUC}(\partial\mathcal{Q})$ , one may easily obtain the desired result. ■

**Lemma 7.4** *Let  $\xi \in \text{BUC}(\partial Q)$ . There exists a probability measure  $\mathbb{P}^* \in \mathcal{P}^{\text{UVM}}$  such that  $e^{-r \cdot} u(B_{t \wedge \cdot})$  is a  $\mathbb{P}^*$ -martingale.*

**Proof** Denote  $t_i^n := \frac{i}{2^n}$  for  $i \leq n2^n$  and  $n \in \mathbb{N}$ . Define process  $\tilde{u}_t := e^{-rt} u_t$ . Note that the tower property implies that  $\tilde{u}_0 = \bar{\mathcal{E}}^{\text{UVM}}[\tilde{u}_{t_1^n}]$ . Since  $\mathcal{P}^{\text{UVM}}$  is weakly compact and  $u$  is BUC, there exists a probability measure  $\mathbb{P}_0^n$  such that

$$\tilde{u}_0 = \bar{\mathcal{E}}^{\text{UVM}}[\tilde{u}_{t_1^n}] = \mathbb{E}^{\mathbb{P}_0^n}[\tilde{u}_{t_1^n}].$$

Since the space  $\Omega$  is separable, we may find a countable  $\mathbb{F}_{t_1^n}$ -measurable partition  $\{E_i\}_{i \in \mathbb{N}}$  such that  $\|\omega - \omega'\|_{t_1^n} < \varepsilon$  for all  $\omega, \omega' \in E_i$ . Fix an  $\omega^i \in E_i$ . As before, there exists probability measures  $\mathbb{P}_{1,i}^{n,\varepsilon}$  such that

$$\tilde{u}_{t_1^n}(\omega^i) = \mathbb{E}^{\mathbb{P}_{1,i}^{n,\varepsilon}}[(\tilde{u}_{t_2^n})^{t_1^n, \omega^i}].$$

For  $\omega \in E_i$ , we have  $|u_{t_1^n}(\omega) - u_{t_1^n}(\omega^i)| \leq \rho(\varepsilon)$  and  $\|(u_{t_2^n})^{t_1^n, \omega} - (u_{t_2^n})^{t_1^n, \omega^i}\| \leq \rho(\varepsilon)$ , where  $\rho$  is the modulus of continuity of  $u$ . Thus, we obtain that

$$\tilde{u}_{t_1^n}(\omega) \leq \mathbb{E}^{\mathbb{P}_{1,i}^{n,\varepsilon}}[(\tilde{u}_{t_2^n})^{t_1^n, \omega}] + 2\rho(\varepsilon). \quad (7.2)$$

Define  $\mathbb{P}_1^{n,\varepsilon}(A) := \mathbb{E}^{\mathbb{P}_0^n}[\sum_i \mathbb{P}_{1,i}^{n,\varepsilon}(A^{t_1^n, B}) \mathbf{1}_{\{B_{t_1^n \wedge \cdot} \in E_i\}}]$ . Clearly, still  $\mathbb{P}_1^{n,\varepsilon} \in \mathcal{P}^{\text{UVM}}$ . Note that (7.2) implies that  $\tilde{u}_{t_1^n} \leq \mathbb{E}^{\mathbb{P}_1^{n,\varepsilon}}[\tilde{u}_{t_2^n} | \mathcal{F}_{t_1^n}] + 2\rho(\varepsilon)$ ,  $\mathbb{P}_1^{n,\varepsilon}$ -a.s. Again, since  $\mathcal{P}^{\text{UVM}}$  is weakly compact and  $u$  is BUC, there exists  $\mathbb{P}_1^n \in \mathcal{P}^{\text{UVM}}$  such that  $\tilde{u}_{t_1^n} \leq \mathbb{E}^{\mathbb{P}_1^n}[\tilde{u}_{t_2^n} | \mathcal{F}_{t_1^n}]$ ,  $\mathbb{P}_1^n$ -a.s.. On the other hand, by Theorem 2.3 in [19], it holds that  $\tilde{u}_{t_1^n} \geq \mathbb{E}^{\mathbb{P}_1^n}[\tilde{u}_{t_2^n} | \mathcal{F}_{t_1^n}]$ ,  $\mathbb{P}_1^n$ -a.s. It follows that

$$\tilde{u}_{t_1^n} = \mathbb{E}^{\mathbb{P}_1^n}[\tilde{u}_{t_2^n} | \mathcal{F}_{t_1^n}], \quad \mathbb{P}_1^n\text{-a.s.}$$

Note that by the definition of  $\mathbb{P}_1^{n,\varepsilon}$ , we know that  $\mathbb{P}_1^n = \mathbb{P}_0^n$  on  $\mathcal{F}_{t_1^n}$ . So, it also holds that  $\tilde{u}_0 = \mathbb{E}^{\mathbb{P}_0^n}[\tilde{u}_{t_1^n}]$ . Repeating the construction, we may find a sequence of probability measures  $\mathbb{P}_0^n, \dots, \mathbb{P}_{n2^n}^n$ . Denote  $\mathbb{P}^n := \mathbb{P}_{n2^n}^n$ . It holds that for all  $m \leq n$

$$\tilde{u}_{t_i^m} = \mathbb{E}^{\mathbb{P}^n}[\tilde{u}_{t_j^n} | \mathcal{F}_{t_i^m}], \quad \mathbb{P}^n\text{-a.s. for } i \leq j \leq m2^m.$$

Finally, since  $\mathcal{P}^{\text{UVM}}$  is weakly compact and  $u$  is BUC, there exists  $\mathbb{P}^* \in \mathcal{P}^{\text{UVM}}$  such that  $\tilde{u}$  is a  $\mathbb{P}^*$ -martingale.  $\blacksquare$

**Proof of Proposition 7.2** *Step 1.* We first verify the viscosity supersolution property. Take  $L > 0$  such that  $L \geq \bar{\sigma}$  and  $\frac{1}{L} \leq \underline{\sigma}$ . Without loss of generality, we only verify it at the point 0. Let  $\varphi \in \bar{\mathcal{A}}^L u(0)$  i.e.  $(\varphi - u)_0 = \max_\tau \bar{\mathcal{E}}^L[(\varphi - u)_{\text{H}_{O_\varepsilon} \wedge \tau}]$ . Then, for all  $\mathbb{P} \in \mathcal{P}^L$ , we obtain that for all  $h > 0$

$$\begin{aligned} 0 &\geq \mathbb{E}^{\mathbb{P}}[\varphi_{\text{H}_{O_\varepsilon} \wedge h} - \varphi_0 - u_{\text{H}_{O_\varepsilon} \wedge h} + u_0] \\ &\geq \mathbb{E}^{\mathbb{P}}\left[\int_0^{\text{H}_{O_\varepsilon} \wedge h} \left(\frac{1}{2} \partial_{\omega\omega}^2 \varphi_s d\langle B \rangle_s + \partial_\omega \varphi_s dB_s\right) \right. \\ &\quad \left. + \mathbb{E}^{\mathbb{P}}\left[(e^{-r(\text{H}_{O_\varepsilon} \wedge h)} - 1)u_{\text{H}_{O_\varepsilon} \wedge h}\right] - \mathbb{E}^{\mathbb{P}}\left[e^{-r(\text{H}_{O_\varepsilon} \wedge h)}u_{\text{H}_{O_\varepsilon} \wedge h}\right] + u_0\right] \\ &\geq \mathbb{E}^{\mathbb{P}}\left[\int_0^{\text{H}_{O_\varepsilon} \wedge h} \left(\frac{1}{2} \partial_{\omega\omega}^2 \varphi_s d\langle B \rangle_s + \partial_\omega \varphi_s dB_s\right) \right] + \mathbb{E}^{\mathbb{P}}\left[(e^{-r(\text{H}_{O_\varepsilon} \wedge h)} - 1)u_{\text{H}_{O_\varepsilon} \wedge h}\right]. \end{aligned}$$



Now, we take  $\mathbb{P}_{\lambda,\beta} \in \mathcal{P}^{\text{UVM}}$  such that there exists a  $\mathbb{P}_{\lambda,\beta}$ -Brownian motion  $W$  such that  $B_t = \lambda t + \beta W_t$ ,  $\mathbb{P}_{\lambda,\beta}$ -a.s. It follows that

$$0 \geq \frac{1}{h} \mathbb{E}^{\mathbb{P}_{\lambda,\beta}} \left[ \int_0^{\mathbb{H}_{O_\varepsilon} \wedge h} \left( \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_s + \lambda \partial_\omega \varphi_s \right) ds + \left( e^{-r(\mathbb{H}_{O_\varepsilon} \wedge h)} - 1 \right) u_{\mathbb{H}_{O_\varepsilon} \wedge h} \right].$$

Let  $h \rightarrow 0$ , we obtain that  $0 \geq -ru_0 + \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_0 + \lambda \partial_\omega \varphi_0$ . Since  $\lambda \in [-L, L], \beta \in [\underline{\sigma}, \bar{\sigma}]$  can be arbitrary, we finally have

$$ru_0 - L |\partial_\omega \varphi_0| - \sup_{\beta \in [\underline{\sigma}, \bar{\sigma}]} \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_0 \geq 0.$$

*Step 2.* Now we verify the viscosity subsolution property. Without loss of generality, we only verify it at the point 0. Let  $\varphi \in \mathcal{A}^L$ , i.e.  $(\varphi - u)_0 = \min_\tau \mathcal{E}^L[(\varphi - u)_{\mathbb{H}_{O_\varepsilon} \wedge \tau}]$ . Take the probability measure  $\mathbb{P}^* \in \mathcal{P}^{\text{UVM}}$  in Lemma 7.4, so that  $u$  is a  $\mathbb{P}^*$ -martingale. Then it holds that for all  $h > 0$

$$\begin{aligned} 0 &\leq \mathbb{E}^{\mathbb{P}^*} [\varphi_{\mathbb{H}_{O_\varepsilon} \wedge h} - \varphi_0 - u_{\mathbb{H}_{O_\varepsilon} \wedge h} + u_0] \\ &\leq \mathbb{E}^{\mathbb{P}^*} \left[ \int_0^{\mathbb{H}_{O_\varepsilon} \wedge h} \left( \frac{1}{2} \partial_{\omega\omega}^2 \varphi_s d\langle B \rangle_s + \partial_\omega \varphi_s dB_s \right) \right] \\ &\quad + \mathbb{E}^{\mathbb{P}^*} \left[ \left( e^{-r(\mathbb{H}_{O_\varepsilon} \wedge h)} - 1 \right) u_{\mathbb{H}_{O_\varepsilon} \wedge h} \right] - \mathbb{E}^{\mathbb{P}^*} \left[ e^{-r(\mathbb{H}_{O_\varepsilon} \wedge h)} u_{\mathbb{H}_{O_\varepsilon} \wedge h} \right] + u_0 \\ &\leq \mathbb{E}^{\mathbb{P}^*} \left[ \int_0^{\mathbb{H}_{O_\varepsilon} \wedge h} \left( L |\partial_\omega \varphi_s| + \sup_{\beta \in [\underline{\sigma}, \bar{\sigma}]} \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_s \right) ds + \left( e^{-r(\mathbb{H}_{O_\varepsilon} \wedge h)} - 1 \right) u_{\mathbb{H}_{O_\varepsilon} \wedge h} \right]. \end{aligned}$$

Divide the right side by  $h$ , and let  $h \rightarrow 0$ . Finally, we get

$$ru_0 - L |\partial_\omega \varphi_0| - \sup_{\beta \in [\underline{\sigma}, \bar{\sigma}]} \frac{1}{2} \beta^2 \partial_{\omega\omega}^2 \varphi_0 \leq 0. \quad \blacksquare$$

## 8 Appendix

**Proof of Proposition 2.8** The first result is easy, and we omit its proof. We decompose the proof in two steps.

*Step 1.* We first prove that  $\bar{\mathcal{E}}^L[\mathbb{H}_D] < \infty$ . Without loss of generality, we may assume that  $D = O_r$ . Further, since

$$\mathbb{H}_{O_r} \leq \mathbb{H}_r^1 := \inf\{t \geq 0 : |B_t^1| \geq r\},$$

without loss of generality, we may assume that the dimension  $d = 1$ .

We first consider the following Dirichlet problem of ODE:

$$-L |\partial_x u| - \frac{1}{L} \partial_{xx}^2 u - 1 = 0, \quad u(r) = u(-r) = 0. \quad (8.1)$$

It is easy to verify that Equation (8.1) has a classical solution:

$$u(x) = \frac{1}{L^3} \left( e^{L^2 r} - e^{L^2 x} \right) - \frac{1}{L} (R - x) \text{ for } 0 \leq x \leq r, \quad \text{and } u(x) = u(-x) \text{ for } -r \leq x \leq 0.$$

Further, it is clear that  $u$  is concave, so  $u$  is also a classical solution to the equation:

$$-L|\partial_x u| - \frac{1}{2} \sup_{\frac{2}{L} \leq \beta \leq 2L} \beta \partial_{xx}^2 u - 1 = 0, \quad u(r) = u(-r) = 0. \quad (8.2)$$

Then by Itô's formula, we obtain

$$0 = u(B_{\mathbb{H}_{O_r}}) = u_0 + \int_0^{\mathbb{H}_{O_r}} \partial_x u(B_t) dB_t + \frac{1}{2} \int_0^{\mathbb{H}_{O_r}} \partial_{xx}^2 u(B_t) d\langle B \rangle_t.$$

Taking the nonlinear expectation on both sides and recalling the definition of  $\mathbb{Q}^{\alpha, \beta}$  in (2.3), we have

$$0 = u_0 + \mathbb{E}^{\mathbb{Q}^{\alpha, \beta}} \left[ \int_0^{\mathbb{H}_{O_r}} (\alpha_t \partial_x u(B_t) + \frac{1}{2} \beta_t^2 \partial_{xx}^2 u(B_t)) dt \right] \quad \text{for all } \|\alpha\| \leq L, \frac{2}{L} \leq \beta \leq 2L \quad (8.3)$$

Since  $u$  is a solution of Equation (8.2), we have

$$\mathbb{E}^{\mathbb{Q}^{\alpha, \beta}} \left[ \int_0^{\mathbb{H}_{O_r}} (\alpha_t \partial_x u(B_t) + \frac{1}{2} \beta_t^2 \partial_{xx}^2 u(B_t)) dt \right] \leq -\mathbb{E}^{\mathbb{Q}^{\alpha, \beta}} [\mathbb{H}_{O_r}]$$

Hence  $u_0 \geq \bar{\mathcal{E}}^L [\mathbb{H}_{O_r}]$ . On the other hand, taking  $\alpha^* := L \text{sgn}(\partial_x u(B_t))$  and  $\beta^* := \sqrt{\frac{2}{L}}$ , we obtain from (8.2) and (8.3) that

$$u_0 = \mathbb{E}^{\mathbb{Q}^{\alpha^*, \beta^*}} [\mathbb{H}_{O_r}].$$

So, we have proved that  $u_0 = \bar{\mathcal{E}}^L [\mathbb{H}_{O_r}]$ . Consequently,  $\bar{\mathcal{E}}^L [\mathbb{H}_{O_r}] < \infty$ .

Step 2. Note that

$$\mathcal{C}^L [\mathbb{H}_D \geq T] \leq \frac{\bar{\mathcal{E}}^L [\mathbb{H}_D]}{T}.$$

By the result of Step 1, we have  $\mathcal{C}^L [\mathbb{H}_D \geq T] \leq \frac{C}{T}$ , and then  $\lim_{T \rightarrow \infty} \mathcal{C}^L [\mathbb{H}_D \geq T] = 0$ . Further,

$$\begin{aligned} \mathcal{C}^L [\mathbb{H}_n < \mathbb{H}_D] &\leq \mathcal{C}^L [\mathbb{H}_n < \mathbb{H}_D; \mathbb{H}_D \leq T] + \mathcal{C}^L [\mathbb{H}_n < \mathbb{H}_D; \mathbb{H}_D > T] \\ &\leq \mathcal{C}^L [\mathbb{H}_n < T] + \mathcal{C}^L [\mathbb{H}_D > T]. \end{aligned}$$

We conclude that  $\lim_{n \rightarrow \infty} \mathcal{C}^L [\mathbb{H}_n < \mathbb{H}_D] = 0$

Denote  $D^x := \{y : \exists z \in D, y = z - x\}$ . Then, define  $\hat{D} := \cup_{x \in D} D^x$ . Note that

$$\mathbb{H}_D^x \leq \mathbb{H}_{\hat{D}} \quad \text{for all } x \in D.$$

Hence we have

$$\sup_{x \in D} \mathcal{C}^L [\mathbb{H}_D^x \geq T] \leq \mathcal{C}^L [\mathbb{H}_{\hat{D}} \geq T] \rightarrow 0. \quad \blacksquare$$

**Proof of Lemma 5.8** Denote  $\delta h := h^1 - h^2$ . By standard argument, one can easily verify that function  $w(x) := \bar{\mathcal{E}}^{L_0} \left[ \left( (\delta h)_{\mathbb{H}_D}^+ \right)^x + \int_0^{\mathbb{H}_D^x} c_0 dt \right]$  is a viscosity solution of the nonlinear PDE:

$$-c_0 - L_0 |Dw| - \frac{1}{2} \sup_{\sqrt{\frac{2}{L_0}} I_d \leq \sigma \leq \sqrt{2L_0} I_d} \sigma^2 : D^2 w = 0 \text{ on } D, \quad \text{and } w = (\delta h)^+ \text{ on } \partial D.$$

Let  $K$  be a smooth nonnegative kernel with unit total mass. For all  $\eta > 0$ , we define the mollification  $w^\eta := w * K^\eta$  of  $w$ . Then  $w^\eta$  is smooth, and it follows from a convexity argument as in [14] that  $w^\eta$  is a classic supersolution of

$$-c_0 - L_0|Dw^\eta| - \frac{1}{2} \sup_{\sqrt{\frac{2}{L_0}}I_d \leq \sigma \leq \sqrt{2L_0}I_d} \sigma^2 : D^2w^\eta \geq 0 \text{ on } D, \quad \text{and } w^\eta = (\delta h)^+ * K^\eta \text{ on } \partial D. \quad (8.4)$$

We claim that

$$\bar{w}^\eta + v^2 \text{ is a supersolution to the PDE with generator } g^1,$$

where  $\bar{w}^\eta := w^\eta + \|w^\eta - (\delta h)^+\|_{L^\infty}$ . Then we note that  $\bar{w}^\eta + v^2 \geq w^\eta + h^2 + \|w^\eta - (\delta h)^+\|_{L^\infty} \geq h^1 = v^1$  on  $\partial D$ . By comparison principle in PDEs, we have  $\bar{w}^\eta + v^2 \geq v^1$  on  $\text{cl}(D)$ . Setting  $\eta \rightarrow 0$ , we obtain that  $v^1 - v^2 \leq w$ . The desired result follows.

It remains to prove that  $\bar{w} + v^2$  is a supersolution of the PDE with generator  $g^1$ . Let  $x_0 \in D$ ,  $\phi \in C^2(D)$  be such that  $0 = (\phi - \bar{w}^\eta - v^2)(x_0) = \max(\phi - \bar{w}^\eta - v^2)$ . Then, it follows from the viscosity supersolution property of  $v^2$  that  $\mathbf{L}^2(\phi - \bar{w}^\eta)(x_0) \geq 0$ . Hence, at the point  $x_0$ , by (8.4) we have

$$\begin{aligned} \mathbf{L}^1\phi &\geq \mathbf{L}^1\phi - \mathbf{L}^2(\phi - \bar{w}^\eta) \\ &= -g^1(\phi, D\phi, D^2\phi) + g^2(\phi - \bar{w}^\eta, D(\phi - \bar{w}^\eta), D^2(\phi - \bar{w}^\eta)) \\ &\geq -g^1(\phi, D\phi, D^2\phi) + g^2(\phi, D(\phi - \bar{w}^\eta), D^2(\phi - \bar{w}^\eta)) \\ &\geq c_0 + L_0|Dw^\eta| + \frac{1}{2} \sup_{\sqrt{\frac{2}{L_0}}I_d \leq \sigma \leq \sqrt{2L_0}I_d} \sigma^2 : D^2w^\eta - c_0 - \alpha \cdot Dw^\eta - \frac{1}{2}\gamma : D^2w^\eta \\ &\geq 0, \end{aligned}$$

where  $|\alpha| \leq L_0$  and  $\frac{2}{L_0}I_d \leq \gamma \leq 2L_0I_d$ . ■

**Proposition 8.1** *For all  $n \geq 1$ , there exists a modulus of continuity  $\rho$  such that*

$$\bar{\mathcal{E}}^L \left[ |\mathbb{H}_Q^{\omega^1} - \mathbb{H}_Q^{\omega^2}| \right] \leq \rho(d^e(\omega^1, \omega^2)). \quad (8.5)$$

**Proof** By the tower property, we have

$$\begin{aligned} \bar{\mathcal{E}}^L \left[ |\mathbb{H}_Q^{\omega^1} - \mathbb{H}_Q^{\omega^2}| \right] &\leq \bar{\mathcal{E}}^L \left[ |\mathbb{H}_Q^{\omega^1} - \mathbb{H}_Q^{\omega^2}| 1_{\{\mathbb{H}_Q^{\omega^1} \leq \mathbb{H}_Q^{\omega^2}\}} \right] + \bar{\mathcal{E}}^L \left[ |\mathbb{H}_Q^{\omega^1} - \mathbb{H}_Q^{\omega^2}| 1_{\{\mathbb{H}_Q^{\omega^1} > \mathbb{H}_Q^{\omega^2}\}} \right] \\ &\leq \bar{\mathcal{E}}^L \left[ \bar{\mathcal{E}}^L \left[ \mathbb{H}_Q^{\omega^2 \otimes B_{\mathbb{H}_Q^{\omega^1} \wedge}} \right] 1_{\{\mathbb{H}_Q^{\omega^1} \leq \mathbb{H}_Q^{\omega^2}\}} \right] + \bar{\mathcal{E}}^L \left[ \bar{\mathcal{E}}^L \left[ \mathbb{H}_Q^{\omega^1 \otimes B_{\mathbb{H}_Q^{\omega^2} \wedge}} \right] 1_{\{\mathbb{H}_Q^{\omega^1} > \mathbb{H}_Q^{\omega^2}\}} \right]. \end{aligned}$$

So, it suffices to show that there exists a modulus of continuity  $\rho$  such that

$$\bar{\mathcal{E}}^L \left[ \mathbb{H}_Q^{\omega^2 \otimes \omega'_{\mathbb{H}_Q^{\omega^1} \wedge}} \right] \leq \rho(d^e(\omega^1, \omega^2)), \text{ for all } \omega' \text{ such that } \mathbb{H}_Q^{\omega'}(\omega') \leq \mathbb{H}_Q^{\omega^2}(\omega').$$

Further, without loss of generality, we may assume that the dimension  $d = 1$  and  $Q = (a, a + h)$ .

Denote  $x_i := \omega_{\mathbb{H}_Q^{\omega^i}}$  and  $y_i := x_i + \omega'_{\mathbb{H}_Q^{\omega^i}}$  for  $i = 1, 2$ . Note that

$$|y_1 - y_2| = |x_1 - x_2|, \quad y_1 \in \partial Q, \quad y_2 \in Q, \quad \text{and} \quad \mathbb{H}_Q^{\omega^2 \otimes \omega'_{\mathbb{H}_Q^{\omega^1} \wedge}} = \mathbb{H}_Q^{y_2}.$$

In particular,  $d(y_2, \partial Q) = |x_1 - x_2|$ . We next consider the Dirichlet problem of ODE:

$$-L|\partial_x u| - \frac{1}{2} \sup_{\frac{2}{L} \leq \beta \leq 2L} \beta \partial_{xx}^2 u - 1 = 0 \quad \text{and} \quad u(-\frac{h}{2}) = u(\frac{h}{2}) = 0 \quad (8.6)$$

Then, as in the proof of Proposition 2.8 in Section 8, we can prove that Equation (8.6) has a classical solution and

$$\bar{\mathcal{E}}^L[\mathbb{H}Q^{y_2}] = u\left(\frac{h}{2} - |x_1 - x_2|\right) \leq \rho(|x_1 - x_2|),$$

where  $\rho$  is the modulus of continuity of  $u$ . The proof is complete. ■

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