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VISCOSITY SOLUTIONS OF FULLY NONLINEAR ELLIPTIC PATH DEPENDENT PDES

ZHENJIE REN

Abstract. This paper introduces a convenient solution space for the uniformly elliptic fully nonlinear path dependent PDEs. It provides a wellposedness result under standard Lipschitz-type assumptions on the nonlinearity and an additional assumption formulated on some partial differential equation defined locally by freezing the path.

Part 1. Introduction

Consider a Dirichlet problem for a fully nonlinear elliptic path-dependent partial differential equation (PPDE) defined on a set of continuous paths $\mathcal{Q} \subset \Omega = \{\omega \in C([0,T];\mathbb{R}^d) : \omega_0 = 0\}$:

$$\begin{aligned} -\mathcal{L}u(\omega) &:= -G\left(\cdot, u, \partial_\omega u, \partial^2_{\omega\omega} u\right)(\omega) = 0,\ t < T,\ \omega \in \mathcal{Q}, \\ u(\omega) &= \xi(\omega),\ \omega \in \partial\mathcal{Q}. \end{aligned} \tag{0.1}$$

for some progressively measurable nonlinearity G: $\Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$, where $\mathbb{S}^d$ is the set of symmetric matrices of size d with real entries. Ekren, Touzi and Zhang [1, 2, 3] have considered parabolic PPDE with a terminal condition. However, in the elliptic equation, the time variable is absent. Instead of working with the terminal condition, we need to work with the boundary condition.

When the nonlinearity G is semilinear, i.e. linear with respect to the $\partial^2_{\omega\omega}u$-component, the theory of backward stochastic differential equations with random terminal, studied by Darling and Pardoux [5], Briand and Hu [6], provides a wellposedness result for the Dirichlet problem. They also related the solution to the corresponding elliptic PDE in the Markovian case.

The first contribution of this paper is to find a convenient solution space. A series of observations lead to the space $\Omega^e := \{\omega \in \Omega : \text{for some } t \geq 0,\ \omega = \omega_{t\wedge\cdot}\}$ instead of the whole space Ω. Also, we define a distance $d^e(\cdot,\cdot)$ on Ω^e such that the corresponding regularity of the nonlinearity G and of the boundary condition ξ ensures that the solution is independent of time (elliptic). Then, we define the derivatives by the functional Itô formula. Finally, inspired by [3], we define the viscosity solution, by using the nonlinear expectation.

The paper contains the main wellposedness theory for the above PPDE. We mainly follow the framework of [3]. The extra technical difficulty comes from the boundary. The irregularity of the hitting time of the boundary makes some estimates more complex. To show the wellposedness, we start from the partial comparison result which states, under fairly general condition on the nonlinearity G and the boundary condition ξ, that for any bounded viscosity subsolution u^1 and

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supersolution u^2 with $u^1 \leq u^2$ on $\partial\mathcal{O}$, we have $u^1 \leq u^2$ in $\mathcal{O}$, provided that one of them is smooth. Then, we follow the spirit of the Perron's approach to construct a viscosity solution and prove that the comparison result of bounded viscosity subsolutions and supersolutions holds true without the requirement that one of them is smooth. In this Perron's approach, the constructed viscosity solution is obtained from the solutions to the path-frozen PDEs.

The rest of the paper is organized as follows. Part 2 introduces the solution space, defines the derivatives and recalls the nonlinear expectations. Part 3 gives the definition of the viscosity solutions to the elliptic PPDE, lists the basic assumptions of wellposedness, and introduces some tools which will be crucial in the following discussions. Part 4 explains the comparison principle so as to achieve the uniqueness. Part 5 verifies that the function constructed before is indeed a viscosity solution so that we get the existence. Part 6 gives some more technical proofs.

Part 2. Preliminaries

1. The canonical space

Let $\Omega := \left\{\omega \in C(\mathbb{R}^+, \mathbb{R}^d) : \omega_0 = 0\right\}$ be the set of continuous paths starting from the origin, B be the canonical process, $\mathbb{F}$ be the filtration generated by B, and $\mathbb{P}_0$ be the Wiener measure. Also, denote by $\mathbb{S}^d$ the set of $d \times d$ matrices and

$$x \cdot y = \sum_{i=1}^{d} x_i y_i \text{ for all } x, y \in \mathbb{R}^d,\ \gamma : \eta = \text{trace}[\gamma\eta] \text{ for all } \gamma, \eta \in \mathbb{S}^d.$$

Let Ω_b be the subset of Ω containing all the bounded paths. We define the supreme norm on Ω_b:

$$\|\omega\| := \sup_{s \geq 0} |\omega_s| \text{ for all } \omega \in \Omega_b.$$

For any $\omega \in \Omega$ and $s, t \in \mathbb{R}^+$, we denote

$$\|\omega\|_t := \sup_{s \leq t} |\omega_s|,\ \|\omega\|_s^t := \sup_{s \leq u \leq t} |\omega_u|.$$

Let $\omega, \omega^{'} \in \Omega$. The concatenation is defined as

$$(\omega \otimes_t \omega^{'})(s) := \omega_s 1_{[0,t)}(s) + (\omega_t + \omega^{'}_{s-t}) 1_{[t,\infty)}(s).$$

Also, given $\varphi : \Omega \to \mathbb{R}$, we define

$$\varphi^{t,\omega}(\omega^{'}) := \varphi(\omega \otimes_t \omega^{'}). \tag{1.1}$$

As in the work of ETZ [1], the theory of viscosity solutions to PPDEs is closely related to the BSDEs. In the viewpoint of BSDE, the canonical space is a convenient background for the probability set up. In the parabolic case the solution to a PPDE can be well defined on the same space. However, when talking about the elliptic equations, we need a slightly different solution space for the PPDEs.

2. Solution space for elliptic PPDEs

We define a subset of Ω (also in Ω_b):

$$\Omega^e := \{\omega \in \Omega : \omega = \omega_{t\wedge\cdot} \text{ for some } t \geq 0\},$$

a subset of paths with flat tails. We consider Ω^e as the solution space for the elliptic PPDEs. We denote

$$\bar{t}(\omega) := \inf\{t : \omega = \omega_{t\wedge\cdot}\} \text{ for all } \omega \in \Omega^e.$$

In this paper we treat the Dirichlet problem.

Definition 2.1. We denote by $\mathcal{R}$ the set of all open, bounded and convex subsets of $\mathbb{R}^d$.

Let $Q \in \mathcal{R}$. We define a subset of Ω^e in which the paths take value in Q:

$$\mathcal{Q} := \left\{\omega \in \Omega^e : \omega_t \in Q \text{ for all } t \in \mathbb{R}^+\right\}.$$

$\mathcal{Q}$ is the domain of our Dirichlet problem. Also we define the boundary and the closure of $\mathcal{Q}$ in the following sense:

$$\partial\mathcal{Q} := \{\omega \in \Omega^e : \bar{t}(\omega) = H_Q(\omega)\}, \ \mathrm{cl}(\mathcal{Q}) := \mathcal{Q} \cup \partial\mathcal{Q},$$

where $H_Q(\omega) := \inf\{t : \omega_t \notin Q\}$, the hitting time to the boundary of Q. Later, we will use other subsets of $\mathcal{R}$. In that case, if , for instance, $D \in \mathcal{R}$, then the corresponding subset $\mathcal{D}$, $\partial\mathcal{D}$, $\mathrm{cl}(\mathcal{D})$ in Ω^e and H_D are defined in a similar manner.

Define the concatenation in Ω^e:

$$(\omega\bar{\otimes}\omega^{'})(s) := (\omega \otimes_{\bar{t}(\omega)} \omega^{'})(s) \text{ for all } \omega \in \Omega^e, \omega^{'} \in \Omega.$$

Similar to (1.1), we define:

$$\varphi^{\omega}(\omega^{'}) := \varphi^{\bar{t}(\omega),\omega}(\omega^{'}) \text{ for all } \omega \in \Omega^e.$$

In particular, for the functions on $\mathbb{R}^d$:

$$\varphi^x(y) := \varphi(x+y) \text{ for all } x, y \in \mathbb{R}^d.$$

We denote by $\mathcal{T}$ the set of all $\mathbb{F}$-stopping times, and by $\mathcal{H} \subset \mathcal{T}$ the subset of the hitting times H_D, where $D \in \mathcal{R}$. Moreover, fixing $\omega \in \Omega^e$, we set $\mathcal{H}^{\omega}$ the subset of the stopping times H_D^{ω} with the form:

$$H_D^{\omega}(\omega^{'}) := \inf\left\{t \geq 0 : \omega_{\bar{t}} + \omega_t^{'} \notin D\right\},$$

that is, the hitting times for the paths starting from $\omega_{\bar{t}}$. For a function φ on Ω, we may denote

$$(\varphi_{H_D})^{\omega} := \varphi^{\omega}_{H_D^{\omega}} \text{ for all } \omega \in \mathcal{D}.$$

We also use the hitting times for the paths starting from a point $x \in \mathbb{R}^d$ denoted by

$$H_D^x(\omega) := \inf\{t : x + \omega_t \notin D\}.$$

Correspondingly, given a function φ on $\mathbb{R}^d$, we may denote

$$(\varphi_{H_D})^x := \varphi^x_{H_D^x}.$$

To give a further explanation of why we choose Ω^e to be our solution space, we explore the relation between the PPDEs and the BSDEs.

2.1. **Heuristic idea of choosing the space.** This section provides a heuristic idea of choosing the space Ω^e. The reasoning is not strict, and some notations are not defined precisely. The readers can find the rigorous proofs in the subsequent sections. Here, we prefer to make the motivation clear.

Darling and Pardoux [5] presented the relation between semilinear PDEs and Markovian BSDEs. In the markovian case, they showed that the solutions to the BSDEs are the viscosity solutions (in the classic sense) to the corresponding PDEs. In our path dependent context, we are going to generalize their approach.

Let a semilinear elliptic PPDE be of the form

$$-\frac{1}{2}\partial^2_{\omega\omega}u - F(\omega, u, \partial_\omega u) = 0 \text{ on } \mathcal{Q},\ u = \xi \text{ on } \partial\mathcal{Q}. \tag{2.1}$$

Adapting the idea of Darling and Pardoux, we are interested in connecting PPDE (2.1) to the following BSDE with a random terminal:

$$Y_t = \xi(B_{H_Q\wedge\cdot}) + \int_{t\wedge H_Q}^{H_Q} F(B_{s\wedge\cdot}, Y_s, Z_s)ds - \int_{t\wedge H_Q}^{H_Q} Z_s dB_s, \mathbb{P}_0\text{-a.s.}, \tag{2.2}$$

where $\mathbb{P}_0$ is Wiener measure. First, it is proved by Darling and Pardoux [5] or Briand and Hu [6] that the BSDE has a unique solution under some general conditions. We want to show that the solution to BSDE (2.2) is indeed a viscosity solution to PPDE (2.1), i.e. $u(t,\omega) = Y_t(\omega)$. For this purpose, we set the following group of BSDEs with a random terminal. For all $(t,\omega) \in \mathbb{R}^+ \times \Omega$ such that $\omega_{t\wedge\cdot} \in \mathcal{Q}$, we set

$$\begin{aligned} Y^{t,\omega}_u &= \xi^{t,\omega}(B_{H^{t,\omega}_Q\wedge\cdot}) + \int_{u\wedge H^{t,\omega}_Q}^{H^{t,\omega}_Q} F^{t,\omega}(B_{s\wedge\cdot}, Y^{t,\omega}_s, Z^{t,\omega}_s)ds \\ &\quad - \int_{u\wedge H^{t,\omega}_Q}^{H^{t,\omega}_Q} Z^{t,\omega}_s dB_s,\ \mathbb{P}_0\text{-a.s.} \end{aligned} \tag{2.3}$$

By the uniqueness of solution to a BSDE, we may show that

$$Y^{t,\omega}_0 = Y_t(\omega),\ \mathbb{P}_0\text{-a.s.} \tag{2.4}$$

Therefore, we are interested in verifying that

$$u(t,\omega) := Y^{t,\omega}_0 \tag{2.5}$$

is a viscosity solution to PPDE (2.1).

Since we are treating the elliptic equations, the first requirement is that the solutions should be independent of time, i.e. $\partial_t u = 0$, where ∂_t is the Dupire's time derivative. Clearly, this cannot be satisfied unless ξ and F have a particular structure. Assuming this requirement is satisfied, we may abuse the notation:

$$u(t,\omega) = u(\omega_{t\wedge\cdot}),$$

since we have (2.4) and Y is adapted as the solution to the BSDE (2.2). Recall that (t,ω) is a pair in $\mathbb{R}^+ \times \Omega$ such that $\omega_{t\wedge\cdot} \in \mathcal{Q} \subset \Omega^e$. Hence, we can indeed well define a function on Ω^e:

$$u(\omega) = Y^\omega_0 := Y^{\bar{t}(\omega),\omega}_0, \tag{2.6}$$

which justifies the choice of Ω^e as the solution space.

2.2. **Distance on Ω^e.** The example of the semilinear elliptic PPDE also helps to define a convenient distance on the space Ω^e. We want to equip Ω^e with a topology in which the regularity of the generator F and of the boundary condition ξ guarantees that the solution u is independent of time.

As we have mentioned, u defined in (2.5) needs to have the null t-derivative. Indeed, we have the following observation.

Proposition 2.2. *Let u be defined as in (2.5). Suppose that for any y and z, $F(\cdot, y, z)$ and ξ satisfy the following property (P) : for $\varphi : \Omega^e \to \mathbb{R}$,*

(P) for all $\omega \in \Omega$ and $\omega' \in \Omega^e$: $\varphi(\omega \otimes_s \omega') = \varphi(\omega \otimes_s \mathbf{0} \otimes_{s+h} \omega')$, $\forall s, h \geq 0$.

Then $\partial_t u(s, \omega) = 0$, for $\omega_{s\wedge\cdot} \in \mathcal{Q}$.

Proof. For any $h > 0$, we know that $u(s + h, \omega_{s\wedge\cdot}) = Y_0^{s+h,\omega_{s\wedge\cdot}}$. The Dupire's t-derivative is defined as

$$\partial_t u(s, \omega) = \lim_{h\to 0} \frac{u(s + h, \omega_{s\wedge\cdot}) - u(s, \omega)}{h}.$$

Since ξ satisfies the property (P), we have

$$\begin{aligned} \xi^{s,\omega}(B_{H_Q^{s,\omega}\wedge\cdot}) &= \xi\left((\omega \otimes_s B)_{H_Q\wedge\cdot}\right) \\ &= \xi\left((\omega \otimes_s \mathbf{0} \otimes_{s+h} B)_{H_Q\wedge\cdot}\right) = \xi^{s+h,\omega_{s\wedge\cdot}}(B_{H_Q^{s+h,\omega_{s\wedge\cdot}}\wedge\cdot}). \end{aligned}$$

In the same way, we may show that $F^{s,\omega}(B_\cdot, y, z) = F^{s+h,\omega_{s\wedge\cdot}}(B_\cdot, y, z)$. Finally, by the uniqueness of solution to a BSDE, we conclude that

$$u(s + h, \omega_{s\wedge\cdot}) = u(s, \omega).$$

Hence $\partial_t u(s, \omega) = 0$. □

The following example shows that if F or ξ does not satisfy property (P), it is possible that $\partial_t u \neq 0$. So the solution could no longer be elliptic.

Example 2.3. Set the parameters of the BSDE (2.2) as:

$$F = 0,\ \xi(\omega) = \int_0^{\bar{t}(\omega)} \omega(s)ds,\ H(\omega) = \inf\{t : |\omega(t)| > 1\}.$$

Note that ξ does not satisfy the (P) property. We are going to calculate the t-derivative of u. We have

$$u(t, \omega) = Y^{t,\omega}(0) = \mathbb{E}\left[\xi\left((\omega \oplus_t B)_{H\wedge\cdot}\right)\right].$$

We divide the discussion into two cases.

(1) If $\|\omega_{t\wedge\cdot}\| > 1$, i.e. $H(\omega) \leq t$, then

$$\mathbb{E}\left[\xi\left((\omega \oplus_t B)_{H\wedge\cdot}\right)\right] = \int_0^{H(\omega)} \omega_s ds.$$

In this case, the t-derivative is equal to 0.

(2) Otherwise, we have

$$\mathbb{E}\left[\xi\left((\omega \oplus_t B)_{H\wedge\cdot}\right)\right] = \int_0^t \omega_s ds + \mathbb{E}\left[\int_0^{H^{t,\omega}} (\omega_t + B_s)ds\right],$$

The t-derivative is ω_t, which is not necessarily 0.

Finally, we conclude that ξ in the form $\xi(\omega) = \int_0^{\bar{t}(\omega)} \omega(s)ds$ leads to a solution which is not elliptic.

Now a natural question is how we can introduce a topology in which the continuous functions automatically satisfy property (P). For this purpose, we introduce the following distance ignoring the time.

Definition 2.4. On the space Ω^e, we define the following pseudometric. For any $\omega, \tilde{\omega} \in \Omega^e$,

$$d^e(\omega, \tilde{\omega}) := \inf_{l \in \mathcal{I}} \sup_{t \in [0,+\infty)} |\omega_{l(t)} - \tilde{\omega}_t|,$$

where $\mathcal{I}$ is the set of all increasing bijections from $[0, +\infty)$ to $[0, +\infty)$. Moreover, we denote by $C(\Omega^e)$ the set of continuous functionals on Ω^e with respect to $d^e(\cdot, \cdot)$. We also use the notations $C\left(\Omega^e; \mathbb{R}^d\right)$, $C\left(\Omega^e; \mathbb{S}^d\right)$ when we need to emphasize the space where the functionals takes values.

Note that in the definition of $d^e(\cdot, \cdot)$, we make use of the time scaling to remove the importance of the time.

Proposition 2.5. *The following statements are correct.*

(i) For all $\omega \in \Omega^e$, $d^e(\omega, \mathbf{0}) = \|\omega\|$.

(ii) For $t < t^{'}$ we have

$$d^e(\omega \oplus_t \omega^{'}, \omega \oplus_t \mathbf{0} \oplus_{t'} \omega^{'}) = 0, \ \forall \omega \in \Omega, \omega^{'} \in \Omega^e.$$

In particular, given $\varphi \in C_b(\Omega^e)$, we have $\varphi\left(\omega \oplus_t \omega^{'}\right) = \varphi\left(\omega \oplus_t \mathbf{0} \oplus_{t'} \omega^{'}\right)$, i.e. φ satisfies property (P).

(iii) Let $\{x_i\}_{i=1,\cdots,n}$ be a sequence in $\mathbb{R}^d$, and $\{t_i\}_{i=1,\cdots,n}$ and $\{\tilde{t}_i\}_{i=1,\cdots,n}$ be two different time sequences. Denote by ω the linear interpolation of $\{(t_i, x_i)\}_{i=1,\cdots,n}$ and by $\tilde{\omega}$ that of $\{(\tilde{t}_i, x_i)\}_{i=1,\cdots,n}$. To make both ω and $\tilde{\omega}$ be in the space Ω^e, add flat tails to them. By the definition of $d^e(\cdot, \cdot)$, we can easily verify that $d^e(\omega, \tilde{\omega}) = 0$, while $\|\omega - \tilde{\omega}\|$, the supremum norm, can be arbitrarily large.

The proof is easy and omitted. Indeed, in order for property (P) to be satisfied, we need, in a certain sense, to consider distance $d^e(\cdot, \cdot)$.

Proposition 2.6. *Let φ be defined on $\mathcal{Q}$. The following two statements are equivalent:*

(1) φ is continuous w.r.t. the distance $d^e(\cdot, \cdot)$, $\mathcal{P}^L$-q.s.;

(2) φ satisfies the property (P) and is continuous w.r.t. the supremum norm $\|\cdot\|$ $\mathcal{P}^L$-q.s..

The family of probabilities $\mathcal{P}^L$ will be introduced later in Section 3. Since the proof is neither trivial nor needed for our main results, we will present it in Appendix.

2.3. **Derivatives.** To give a precise as well as a brief definition of the derivative of a function defined on Ω^e, we turn to ETZ [2, 3], where they introduced a class of $C^{1,2}$ processes by using the functional Itô's formula.

Definition 2.7. (1) Denote by $\mathbb{H}^0(E)$ the set of all $\mathbb{F}$-progressively measurable processes with values in E. In particular, denote by $\mathbb{H}^0\left(\bar{B}_L^d(0)\right)$ (where $\bar{B}_L^d(0) := \left\{x \in \mathbb{R}^d : |x| \leq L\right\}$) the set of all d-dimensional $\mathbb{F}$-progressively measurable processes which are bounded by L.

(2) Similarly, $\mathbb{H}^0([aI_d, bI_d])$ is the set of all $\mathbb{S}^d$-valued $\mathbb{F}$-progressively measurable processes which satisfy:

$$aI_d \leq \beta_t \leq bI_d \text{ for all } t \in \mathbb{R}^+.$$

We define a family of probability:

$$(2.7)\ \mathcal{P}_0^L := \left\{ \mathbb{P} : B_t = \int_0^t \alpha_s ds + \int_0^t \beta_s dW_s^{\mathbb{P}},\ \mathbb{P}\text{-a.s., for some} \right.$$
$$\left. (\alpha, \beta) \in \mathbb{H}^0 \left(\bar{B}_L^d(0) \times \left[0, \sqrt{2L} I_d\right] \right),\ \mathbb{P}\text{-Browinan motion } W^{\mathbb{P}} \right\}$$

Furthermore, define

$$\mathcal{P}_0^\infty := \cup_{L>0} \mathcal{P}_0^L.$$

Definition 2.8 (The class $C^2(\Omega^e)$)**.** We say that $u \in C^2(\Omega^e)$, if $u \in C(\Omega^e)$ and there exist $\partial_\omega u \in C\left(\Omega^e; \mathbb{R}^d\right)$, $\partial^2_{\omega\omega} u \in C\left(\Omega^e; \mathbb{S}^d\right)$ such that, for any $\omega \in \Omega$ and any $\mathbb{P} \in \mathcal{P}_0^\infty$, $\{u(\omega_{t\wedge\cdot})\}_{t\geq 0}$ is a local $\mathbb{P}$-semimartingale with decomposition:

$$du(\omega_{t\wedge\cdot}) = \partial_\omega u(\omega_{t\wedge\cdot}) \cdot dB_t + \frac{1}{2}\partial^2_{\omega\omega} u(\omega_{t\wedge\cdot}) : d\langle B\rangle_t,\ \mathbb{P}\text{-a.s.} \tag{2.8}$$

By a direct localization argument, we see that the above $\partial_\omega u$ and $\partial^2_{\omega\omega} u$, if they exist, are unique. Consequently, we call them the first order and the second order derivatives of u, respectively.

Since we are discussing the Dirichlet problem, our solutions are defined only on the domain $\mathcal{Q}$.

Definition 2.9. For $D \in \mathcal{R}$ we say that $u \in C^2(\mathcal{D})$, if $u \in C(\mathrm{cl}(\mathcal{D}))$ and there exist $\partial_\omega u \in C\left(\mathcal{D}; \mathbb{R}^d\right)$, $\partial^2_{\omega\omega} u \in C\left(\mathcal{D}; \mathbb{S}^d\right)$ such that, for any $\omega \in \Omega$ and any $\mathbb{P} \in \mathcal{P}_0^\infty$, $\{u(\omega_{t\wedge\cdot})\}_{t\geq 0}$ is a local $\mathbb{P}$-semimartingale and the following property holds: $\forall t \leq H_D$,

$$u(\omega_{t\wedge\cdot}) - u(0) = \int_0^t \partial_\omega u(\omega_{s\wedge\cdot}) \cdot dB_s + \int_0^t \frac{1}{2}\partial^2_{\omega\omega} u(\omega_{s\wedge\cdot}) : d\langle B\rangle_s,\ \mathbb{P}\text{-a.s.} \tag{2.9}$$

2.4. **Class of solution.** To ensure the wellposedness of the viscosity solutions to the PPDEs, we need to set a requirement on the regularity of the solutions.

Definition 2.10. We denote by $\mathrm{BUC}(\mathcal{Q})$ the collection of functions $u : \mathrm{cl}(\mathcal{Q}) \to \mathbb{R}$ such that u is bounded and uniformly continuous, i.e. there exists a modulus of continuity ρ such that

$$\left| u(\omega) - u(\omega') \right| \leq \rho(d^e(\omega, \omega')) \text{ for all } \omega, \omega' \in \mathrm{cl}(\mathcal{Q}).$$

Since we are going to handle several uniform continuous functions (the generators, the boundary conditions, the solutions, etc.), many different moduli of continuity will be concerned. However, the moduli themselves are not essential in our discussion. Therefore, for simplification, we allow ourselves to abuse the notation of ρ, which can be different from line to line. In addition, without loss of generality, we assume that ρ is concave.

3. Capacity and nonlinear expectation

As in the framework of [3], the capacity and the nonlinear expectation play an important role in the definition of viscosity solutions to parabolic PPDEs. In the elliptic case, they will still be crucial.

Let $\mathcal{P}$ be a family of probabilities on Ω. The capacities and the nonlinear expectations in our discussion are always in the form:

$$\mathcal{C}[\cdot] := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{P}[\cdot], \ \overline{\mathcal{E}}[\cdot] := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\cdot], \ \underline{\mathcal{E}}[\cdot] := \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[\cdot] \tag{3.1}$$

3.1. **Notations of capacity and nonlinear expectation.** Define a family of probability measures

$$\mathcal{P}^L := \left\{ \mathbb{P} \in \mathcal{P}_0^L : d\langle B\rangle_t \geq \sqrt{\frac{2}{L}} I_d dt \right\}.$$

Lemma 3.1. *$\mathcal{P}^L$ is weakly compact.*

As in (3.1), we define $\mathcal{C}^L$, $\overline{\mathcal{E}}^L$ and $\underline{\mathcal{E}}^L$. Further, the conditional nonlinear expectation in this paper is denoted as:

$$\overline{\mathcal{E}}_t^L[X](\omega) := \sup_{\mathbb{P}\in\mathcal{P}^L} \mathbb{E}^{\mathbb{P}}\left[X^{t,\omega}\right]. \tag{3.2}$$

3.2. **Properties of the capacity and of the nonlinear expectation.** According to Nutz and Van Handel [7], the conditional nonlinear expectation defined in (3.2) satisfies the tower property:

$$\overline{\mathcal{E}}_\sigma^L[\cdot] = \overline{\mathcal{E}}_\sigma^L\left[\overline{\mathcal{E}}_\tau^L[\cdot]\right] \text{ for stopping times } \sigma \leq \tau. \tag{3.3}$$

Let $H_D \in \mathcal{H}$. For $X \in \mathbb{L}^0(\Lambda)$ and (t,ω) such that $\omega_{t\wedge\cdot} \in \mathcal{D}$, denote

$$\overline{\mathcal{S}}_t^L[X_{H_D\wedge\cdot}](\omega) := \sup_{\tau\in\mathcal{T}^t} \overline{\mathcal{E}}_t^L[X_{\tau\wedge H_D}](\omega),$$

$$\underline{\mathcal{S}}_t^L[X_{H_D\wedge\cdot}](\omega) := \inf_{\tau\in\mathcal{T}^t} \underline{\mathcal{E}}_t^L[X_{\tau\wedge H_D}](\omega).$$

We recall from ETZ [4] that the following characterization of the optimal stopping problems holds.

Theorem 3.2 (Snell envelope characterization)**.** *Let $H_D \in \mathcal{H}$ and $X \in \mathrm{BUC}(\mathcal{D})$. Define the Snell envelope and the corresponding first hitting time of the obstacles:*

$$Y := \overline{\mathcal{S}}^L[X_{H_D\wedge\cdot}], \ \tau^* := \inf\{t \geq 0 : Y_t = X_t\}.$$

Then $Y_{\tau^*} = X_{\tau^*}$. Y is an $\overline{\mathcal{E}}^L$-supermartingale on $[0, H_D]$ and an $\overline{\mathcal{E}}^L$-martingale on $[0,\tau^*]$. Consequently, τ^* is an optimal stopping time.

Apart from the tower property and the optimal stopping theorem, the capacity has some other properties which will be useful in our future arguments.

Fix any $D \in \mathcal{R}$. Let $O \subset D$ be also in $\mathcal{R}$. Then define a sequence of stopping times H_n:

$$H_0 = 0, \ H_{i+1} := \inf\{s \geq H_i : B_s - B_{H_i} \notin O\}, \ i \geq 0. \tag{3.4}$$

Proposition 3.3. *(1) For any $T \in \mathbb{R}^+$, $\lim_{n\to\infty} \mathcal{C}^L[H_n < T] = 0$, $i = 0, 1$.*

(2) We have $\overline{\mathcal{E}}^L[H_D] < \infty$.

(3) $\lim_{n\to\infty} \mathcal{C}^L[H_n < H_D] = 0$.

Proof. (1) and (2) are easy to show, so we omit the proof here. We prove (3) as follow.

First, we have

$$\mathcal{C}^L\left[H_D \geq T\right] \leq \frac{\overline{\mathcal{E}}^L\left[H_D\right]}{T}.$$

By (1), we have $\mathcal{C}^L\left[H_D \geq T\right] \leq \frac{C}{T}$. It follows

$$\lim_{T\to\infty} \mathcal{C}^L\left[H_D \geq T\right] = 0.$$

Furthermore,

$$\begin{aligned} \mathcal{C}^L\left[H_n < H_D\right] &\leq \mathcal{C}^L\left[H_n < H_D; H_D \leq T\right] + \mathcal{C}^L\left[H_n < H_D; H_D > T\right] \\ &\leq \mathcal{C}^L\left[H_n < T\right] + \mathcal{C}^L\left[H_D > T\right]. \end{aligned}$$

Finally, by (2), the result follows. □

Corollary 3.4. *We have*

$$\lim_{n\to\infty} \sup_{x\in D} \mathcal{C}^L\left[H_n < H_D^x\right] = 0. \tag{3.5}$$

Proof. For any $D \subset \mathbb{R}^d$, we put $D^x := \{y : \exists z \in D,\ y = z - x\}$. Take a bigger set in $\mathcal{R}$: $\hat{D} := \cup_{x\in D} D^x$. Then observe that

$$H_D^x \leq H_{\hat{D}} \text{ for all } x \in D.$$

Hence we have

$$\sup_{x\in D} \mathcal{C}^L\left[H_D^x \geq T\right] \leq \mathcal{C}^L\left[H_{\hat{D}} \geq T\right] \to 0.$$

□

4. Extension of $C^2(\mathcal{Q})$

It is important for us to propose the following extension of $C^2(\mathcal{Q})$, which is the class of the piecewise smooth functions.

Definition 4.1. Let $u : \mathcal{Q} \to \mathbb{R}$. We say that $u \in \overline{C}^2(\mathcal{Q})$, if u is bounded, $u(\omega_{t\wedge\cdot})$ is continuous in t, and there exists an increasing sequence of $\mathbb{F}$-stopping times $\{H_n; n \geq 1\}$, such that

(1) for each i and ω, $H_{i+1}^{H_i,\omega} \in \mathcal{H}^{\omega_{H_i\wedge\cdot}}$ whenever $H_i(\omega) < H_Q(\omega) < \infty$; for any $L > 0$ the set $\{i : H_i(\omega) < H_Q(\omega)\}$ is finite $\mathcal{P}^L$-q.s. and $\lim_{i\to\infty} \mathcal{C}_0^L\left[H_i^\omega < H_Q^\omega\right] = 0$ for any $\omega \in \mathcal{Q}$;

(2) for each i there exist $\partial_\omega u^i$, $\partial^2_{\omega\omega} u^i$ such that for all ω, $\left(\partial_\omega u^i\right)^{\omega_{H_i\wedge\cdot}}$ and $\left(\partial^2_{\omega\omega} u^i\right)^{\omega_{H_i\wedge\cdot}}$ are both continuous on $\mathcal{O}_i^\omega := \left\{\omega'_{t\wedge\cdot} \in \Omega^e : t \leq H_{i+1}^{H_i,\omega}\right\}$ and such that for all $t \in [H_i, H_{i+1})$

$$u(\omega_{t\wedge\cdot}) - u(\omega_{H_i\wedge\cdot}) = \int_{H_i}^t \partial_\omega u^i(\omega_{s\wedge\cdot}) \cdot dB_s + \int_{H_i}^t \frac{1}{2}\partial^2_{\omega\omega} u^i(\omega_{s\wedge\cdot}) : d\left\langle B\right\rangle_s,\ \mathcal{P}_0^\infty\text{-q.s.}$$

Since $H_{i+1}^{H_i,\omega} \in \mathcal{H}^{\omega_{H_i\wedge\cdot}}$, there are sets in $\mathcal{R}$ corresponding to the hitting times. We will call them O_i^ω.

Part 3. Fully nonlinear elliptic path dependent PDEs

5. Definition of viscosity solutions to uniformly elliptic PPDEs

In the previous discussion, we have mentioned the form of the semilinear elliptic PPDEs (2.1). However, at that moment the discussion was only heuristic. In this section, we will introduce the elliptic PPDEs in rigorous way.

Define

$$\begin{cases} \mathcal{L}u(\omega) := -G(\omega, u, \partial_\omega u, \partial^2_{\omega\omega} u) = 0, & \omega \in \mathcal{Q}; \\ u = \xi, & \omega \in \partial\mathcal{Q}. \end{cases} \tag{5.1}$$

The nonlinearity is denoted by G and the boundary condition by ξ.

Assumption 5.1. *The nonlinearity $G : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ satisfies:*

(i) For fixed (y, z, γ), $G(\cdot, y, z, \gamma) \in \mathbb{L}^0(\Omega)$ and $|G(\cdot, 0, z, 0)| \leq C_0$ uniformly in z.

(ii) G is uniformly elliptic, i.e., there exists $L_0 > 0$ such that for all (ω, y, z)

$$G(\omega, y, z, \gamma_1) - G(\omega, y, z, \gamma_2) \geq \frac{1}{L_0} I_d : (\gamma_1 - \gamma_2) \text{ for all } \gamma_1 \geq \gamma_2.$$

(iii) G is uniformly continuous on Ω^e with respect to $d^e(\cdot, \cdot)$, and is uniformly Lipschitz continuous in (y, z, γ) with a Lipschitz constant L_0.

(iv) G is uniformly decreasing in y, i.e. there exists a function $\lambda : \mathbb{R} \to \mathbb{R}$ strictly increasing and continuous ($\lambda(0) = 0$), and $y_2 \geq y_1$

$$G(\omega, y_1, z, \gamma) - G(\omega, y_2, z, \gamma) \geq \lambda(y_2 - y_1).$$

For any $u \in \mathrm{BUC}(\mathcal{Q})$, $\omega \in \mathcal{Q}$ and $L > 0$, define:

$$\underline{\mathcal{A}}^L u(\omega) := \left\{ \varphi \in C^2(\mathcal{D}) \text{ for some } D \in \mathcal{R} : (\varphi - u^\omega)_0 = 0 = \underline{\mathcal{S}}^L \left[(\varphi - u^\omega)_{H^\omega_D \wedge \cdot} \right] \right\},$$

$$\overline{\mathcal{A}}^L u(\omega) := \left\{ \varphi \in C^2(\mathcal{D}) \text{ for some } D \in \mathcal{R} : (\varphi - u^\omega)_0 = 0 = \overline{\mathcal{S}}^L \left[(\varphi - u^\omega)_{H^\omega_D \wedge \cdot} \right] \right\}.$$

Remark 5.2 (u on $\Omega \backslash \Omega^e$). In the above definition, H_D can possibly take the value of ∞. In that case, $u(B_{H_D \wedge \cdot})$ is not defined, since u is a function on Ω^e. However, it is not essential, because $H_D < \infty$ $\mathcal{P}^L$-q.s.. If necessary, we can define complementarily $u := 0$ on $\Omega \backslash \Omega^e$.

Definition 5.3. (i) Let $L > 0$. We say that $u \in \mathrm{BUC}(\mathcal{Q})$ is an L-viscosity subsolution (resp. L-supersolution) of PPDE (5.1) if, for $\omega \in \mathcal{Q}$ and any $\varphi \in \underline{\mathcal{A}}^L u(\omega)$ (resp. $\varphi \in \overline{\mathcal{A}}^L u(\omega)$):

$$-G(\omega, \varphi(0), \partial_\omega \varphi(0), \partial^2_{\omega\omega} \varphi(0)) \leq \ (\textit{resp.} \ \geq) \ 0.$$

(ii) We say that $u \in \mathrm{BUC}(\mathcal{Q})$ is a viscosity subsolution (resp. supersolution) of PPDE (5.1) if u is an L-viscosity subsolution (resp. L-supersolution) of PPDE (5.1) for some $L > 0$.

(iii) We say that $u \in \mathrm{BUC}(\mathcal{Q})$ is a viscosity solution of PPDE (5.1) if it is both a viscosity subsolution and a viscosity supersolution.

6. Consistency with the classical solutions

In this section, we verify that the definition of the viscosity solution is consistent with that of the classical solution.

Proposition 6.1. *Let Assumption 5.1 hold. Given a functional $u \in C^2(\mathcal{Q})$, then u is a viscosity solution to PPDE (5.1) if and only if u is a classical solution.*

Proof. First, suppose that u is an L-viscosity solution to the PPDE. Set $\varphi := u$. Obviously, $\varphi \in \overline{\mathcal{A}}^L u(\omega)$ (resp. $\varphi \in \underline{\mathcal{A}}^L u(\omega)$), for any $\omega \in \mathcal{Q}$ and $L > 0$. By the subsolution (resp. supersolution) property of u, we deduce that $\mathcal{L}u(\omega) \leq 0$ (resp. ≥ 0). Therefore, $\mathcal{L}u(\omega) = 0$.

On the other hand, suppose that u is a classical solution. Without loss of generality, we prove only that u is an L_0-viscosity supersolution at 0, where L_0 is the constant in Assumption 5.1. Assume the contrary, i.e. there exists $\varphi \in \overline{\mathcal{A}}^{L_0} u(0)$ such that $-c := \mathcal{L}\varphi(0) < 0$. Let H_D be the hitting time required in $\overline{\mathcal{A}}^{L_0} u(0)$. Since $\varphi \in C^2(\mathcal{D})$ and $u \in C^2(\mathcal{Q})$, without loss of generality we may assume that

$$\mathcal{L}\varphi(\omega_{t\wedge\cdot}) \leq -\frac{c}{2} \text{ and } |\varphi_t - \varphi_0| + |u_t - u_0| \leq \frac{c}{6L_0}, \text{ for all } t \leq H_D.$$

Since $\varphi \in \overline{\mathcal{A}}^{L_0} u(0)$, it implies that for all $\mathbb{P} \in \mathcal{P}^{L_0}$:

$$0 = (\varphi - u)_0 \geq \mathbb{E}^{\mathbb{P}}\left[(\varphi - u)_{H_D}\right] \tag{6.1}$$

Denote $\mathcal{G}^{\mathbb{P}}\phi := \alpha^{\mathbb{P}} \cdot \partial_\omega \phi + \frac{1}{2}(\beta^{\mathbb{P}})^2 : \partial^2_{\omega\omega}\phi$. Again, since $\varphi \in C^2(\mathcal{D})$ and $u \in C^2(\mathcal{Q})$, we have

$$\begin{aligned} 0 \quad &\geq \quad \mathbb{E}^{\mathbb{P}}\left[(\varphi - u)_{H_D} - (\varphi - u)_0\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^{H_D} \mathcal{G}^{\mathbb{P}}(\varphi - u)_s ds\right] \\ &\geq \quad \mathbb{E}^{\mathbb{P}}\Bigg[\int_0^{H_D} \Big(\frac{c}{2} - G(\omega_{s\wedge\cdot}, \varphi_s, \partial_\omega \varphi_s, \partial^2_{\omega\omega}\varphi_s) \\ &\qquad\qquad + G(\omega_{s\wedge\cdot}, u_s, \partial_\omega u_s, \partial^2_{\omega\omega} u_s) + \mathcal{G}^{\mathbb{P}}(\varphi - u)_s\Big) ds\Bigg], \end{aligned}$$

where the last inequality is due to the supersolution property of u and the monotonicity in y of G. Since $\varphi_0 = u_0$, we get

$$\begin{aligned} 0 \quad &\geq \quad \mathbb{E}^{\mathbb{P}}\Bigg[\int_0^{H_D} \Big(\frac{c}{3} - G(\omega_{s\wedge\cdot}, u_0, \partial_\omega \varphi_s, \partial^2_{\omega\omega}\varphi_s) \\ &\qquad\qquad + G(\omega_{s\wedge\cdot}, u_0, \partial_\omega u_s, \partial^2_{\omega\omega} u_s) + \mathcal{G}^{\mathbb{P}}(\varphi - u)_s\Big) ds\Bigg]. \end{aligned}$$

Now let $\eta = \frac{c}{6C}$. We may assume that

$$\begin{aligned} H_D \quad &\leq \quad \inf\Big\{t \geq 0 : \rho(d^e(\omega_{t\wedge\cdot}, 0)) + |\partial_\omega \varphi_t - \partial_\omega \varphi_0| + |\partial^2_{\omega\omega}\varphi_t - \partial^2_{\omega\omega}\varphi_0| \\ &\qquad + |\partial_\omega u_t - \partial_\omega u_0| + |\partial^2_{\omega\omega} u_t - \partial^2_{\omega\omega} u_0| \geq \eta\Big\}. \end{aligned}$$

Thus,

$$\begin{aligned} 0 &\geq (\frac{c}{3} - C\eta)\mathbb{E}^{\mathbb{P}}[H_D] + \mathbb{E}^{\mathbb{P}}\int_0^{H_D}\Big(-G(\cdot, u_0, \partial_\omega\varphi_0, \partial^2_{\omega\omega}\varphi_0) \\ &\qquad + G(\cdot, u_0, \partial_\omega u_0, \partial^2_{\omega\omega}u_0) + \mathcal{G}^{\mathbb{P}}(\varphi - u)_0\Big)ds \\ &= \frac{c}{6}\mathbb{E}^{\mathbb{P}}[H_D] + \mathbb{E}^{\mathbb{P}}\int_0^{H_D}\Big(\alpha\cdot\partial_\omega(u-\varphi)_0 + \frac{1}{2}\beta^2 : \partial^2_{\omega\omega}(u-\varphi)_0 \\ &\qquad + \mathcal{G}^{\mathbb{P}}(\varphi - u)_0\Big)ds, \end{aligned}$$

for some $\Big|\alpha\Big| \leq L_0$, $\sqrt{L/2}I_d \leq \beta \leq \sqrt{2L}I_d$. Now choose $\mathbb{P}^* \in \mathcal{P}^{L_0}$ such that $\alpha_t^{\mathbb{P}^*} = \alpha$, $\beta_t^{\mathbb{P}^*} = \beta$. Then $0 \geq \frac{c}{6}\mathbb{E}^{\mathbb{P}^*}[H_D]$, which is a contradiction. □

7. Path-frozen PDE

Given a nonlinearity G satisfying Assumption 5.1, define the following function on $\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$:

$$g^\omega(y,z,\gamma) := G(\omega, y, z, \gamma),\ \omega\in\mathcal{Q}. \tag{7.1}$$

For any $\epsilon > 0$, we denote

$$O_\epsilon := \left\{x\in\mathbb{R}^d : |x| < \epsilon\right\}. \tag{7.2}$$

Denote

$$O_\epsilon^Q := O_\epsilon\cap Q,\ O_\epsilon^{Q,x} = O_\epsilon\cap Q^x.$$

Also by abusing the previous notation, we introduce

$$O_\epsilon^{Q,\omega} := O_\epsilon\cap Q^{\omega_{\bar{t}(\omega)}},\ \text{for all } \omega\in\mathcal{Q}.$$

We introduce here a path-frozen PDE with the nonlinearity g^ω as in (7.1) :

$$(E)_\epsilon^\omega \quad \boldsymbol{L}^\omega v := -g^\omega(v, Dv, D^2v) = 0 \text{ on } O_\epsilon^{Q,\omega}.$$

Notice that for fixed $\omega\in\mathcal{Q}$, this is a standard deterministic partial differential equation for which we now assume the following wellposedness condition.

Assumption 7.1. *For $\epsilon > 0$, $\omega\in\mathcal{Q}$, and $h\in C(\partial O_\epsilon^{Q,\omega})$, we have $\overline{v} = \underline{v}$, where*

$$\overline{v}(x) := \inf\left\{w(x) : w\in C^2(O_\epsilon^{Q,\omega})\cap C(\mathrm{cl}(O_\epsilon^{Q,\omega})), \boldsymbol{L}^\omega w\geq 0 \text{ on } O_\epsilon^{Q,\omega}, w\geq h \text{ on } \partial O_\epsilon^{Q,\omega}\right\},$$

$$\underline{v}(x) := \sup\left\{w(x) : w\in C^2(O_\epsilon^{Q,\omega})\cap C(\mathrm{cl}(O_\epsilon^{Q,\omega})), \boldsymbol{L}^\omega w\leq 0 \text{ on } O_\epsilon^{Q,\omega}, w\leq h \text{ on } \partial O_\epsilon^{Q,\omega}\right\}.$$

Remark 7.2. This assumption implies two points. First, by applying the comparison principle of PDE, we may deduce that $\overline{v} = \underline{v}$ is the unique viscosity solution of $(E)_\epsilon^\omega$. On the other hand, there exist regular approximations of the solutions.

8. Two important functions

Recall the class $\overline{C}^2(\mathcal{Q})$ in Definition 4.1. The following two functions will be essential in our future construction of solution:

$$\overline{u}(\omega) := \inf\left\{\psi(\omega) : \psi\in\overline{\mathcal{D}}_Q^\xi(\omega)\right\},\ \underline{u}(\omega) := \sup\left\{\psi(\omega) : \psi\in\underline{\mathcal{D}}_Q^\xi(\omega)\right\}, \tag{8.1}$$

where

$$\overline{\mathcal{D}}_Q^\xi(\omega) := \left\{\psi\in\overline{C}^2(\mathcal{Q}^\omega) : \psi \text{ is bounded, } \mathcal{L}^\omega\psi\geq 0 \text{ on } \mathcal{Q},\ \psi\geq\xi^\omega \text{ on } \partial\mathcal{Q}\right\},$$

$$\underline{\mathcal{D}}_Q^{\xi}(\omega) := \left\{\psi \in \overline{C}^2(\mathcal{Q}^\omega) : \psi \text{ is bounded, } \mathcal{L}^\omega \psi \le 0 \text{ on } \mathcal{Q},\ \psi \le \xi^\omega \text{ on } \partial\mathcal{Q}\right\}.$$

Indeed, we shall finally show that $\overline{u} = \underline{u}$ is the unique viscosity solution to PPDE (5.1).

Proposition 8.1. *Let Assumption 5.1 hold and $|\xi| \le C_0$. Then $\overline{u}$ is bounded from above and $\underline{u}$ is bounded from below.*

Proof. Set a constant function:

$$\psi := \lambda^{-1}(C_0) + C_0.$$

Note that $\psi \in \bar{C}^2$. Observe that $\psi_T \ge C_0 \ge \xi$. Also,

$$(\mathcal{L}\psi)_s^\omega = -G^\omega(\cdot, \psi_s, 0, 0) \ge C_0 - G^\omega(\cdot, 0, 0, 0) \ge 0.$$

It follows that $\psi \in \overline{\mathcal{D}}_Q^{\xi}(\omega)$, and thus $\overline{u}(\omega) \le \psi(0) = \lambda^{-1}(C_0) + C_0$. Similarly, one can show that $\underline{u}(\omega) \ge -\lambda^{-1}(C_0) - C_0$. □

9. Stochastic representation for HJB equations

The nonlinear expectation $\overline{\mathcal{E}}^L$ is closely related to optimal control problems and the corresponding HJB equations. In this section, we will explore this topic.

Recall the constants L_0 and C_0 in Assumption 5.1 and consider two functions:

$$\overline{g}(y, z, \gamma) := \frac{1}{2} \sup_{\beta \in \mathbb{H}^0\left(\left[\sqrt{2/L_0} I_d, \sqrt{2L_0} I_d\right]\right)} \beta^2 : \gamma + L_0 |z| + L_0 y^- + C_0, \tag{9.1}$$

$$\underline{g}(y, z, \gamma) := \frac{1}{2} \inf_{\beta \in \mathbb{H}^0\left(\left[\sqrt{2/L_0} I_d, \sqrt{2L_0} I_d\right]\right)} \beta^2 : \gamma - L_0 |z| - C_0, \tag{9.2}$$

Indeed, for all nonlinearities G satisfying Assumption 5.1, it holds

$$\underline{g} \le G \le \overline{g}.$$

Consider PDEs:

$$\overline{\mathcal{L}} u := -\overline{g}(u, Du, D^2 u) = 0 \text{ and } \underline{\mathcal{L}} u := -\underline{g}(u, Du, D^2 u) = 0.$$

Let $D \in \mathcal{R}$, and $h_D : \mathbb{R}^d \to \mathbb{R}$ a bounded and continuous function. We define for $x \in D$

$$\overline{w}(x) := \sup_{b \in \mathbb{H}^0([0, L_0])} \overline{\mathcal{E}}^{L_0}\left[h_D(B_{H_D^x}) e^{-\int_0^{H_D^x} b_r dr} + C_0 \int_0^{H_D^x} e^{-\int_0^t b_r dr} dt\right], \tag{9.3}$$

$$\underline{w}(x) := \underline{\mathcal{E}}^{L_0}\left[h_D(B_{H_D^x}) + C_0 H_D^x\right]. \tag{9.4}$$

It is not hard to verify that if $\overline{w}$ and $\underline{w}$ are continuous, then they are respectively the viscosity solutions to $\overline{\mathcal{L}} u = 0$ and $\underline{\mathcal{L}} u = 0$ with the boundary condition h_D on ∂D. We should pay attention to the continuity of $\overline{w}$ and $\underline{w}$.

Lemma 9.1. *There exists a modulus of continuity ρ, such that*

$$\overline{\mathcal{E}}^{L_0}\left[|H^{x_1} - H^{x_2}|\right] \le \rho(|x_1 - x_2|).$$

Moreover, define

$$h_D(x) := \overline{\mathcal{E}}^{L_0}\left[v(x, B_{H_D^x \wedge \cdot})\right] \tag{9.5}$$

for some $v \in BUC(\mathbb{R}^d \times \Omega^e)$. Then $h_D \in BUC\left(\mathbb{R}^d\right)$.

The proof can be found in Appendix. Indeed, we can go further to show the uniform continuity of the functions $\overline{w}$ and $\underline{w}$.

Proposition 9.2. *Given h_D in the form of (9.5), the functions $\overline{w}$ and $\underline{w}$ are uniformly continuous in* $\mathrm{cl}(D)$.

Proof. We only show the proposition for $\overline{w}$. Since h_D is bounded and b only takes non-negative values, we can easily estimate that for $x_1, x_2 \in D$,

$$\begin{aligned}|\overline{w}(x_1)-\overline{w}(x_2)| &\leq \sup_{b\in\mathbb{H}^0([0,L_0])} \overline{\mathcal{E}}^{L_0}\left[\left|h_D(B_{H_D^{x_1}})e^{-\int_0^{H_D^{x_1}} b_r dr} + C_0\int_0^{H_D^{x_1}} e^{-\int_0^t b_r dr}dt\right.\right.\\ &\qquad \left.\left. -h_D(B_{H_D^{x_2}})e^{-\int_0^{H_D^{x_2}} b_r dr} - C_0\int_0^{H_D^{x_2}} e^{-\int_0^t b_r dr}dt\right|\right]\\ &\leq \overline{\mathcal{E}}^{L_0}\left[\left|h_D(B_{H_D^{x_1}}) - h_D(B_{H_D^{x_2}})\right|\right] + C\overline{\mathcal{E}}^{L_0}\left[|H_D^{x_1}-H_D^{x_2}|\right].\end{aligned}$$

By applying Lemma 9.1, we finish the proof. □

This uniform continuity result will be useful to show the existence of the viscosity solution. Recall that we assume that the viscosity solution should be in the class $\mathrm{BUC}(\mathcal{Q})$.

Part 4. Uniqueness

10. Partial comparison principle

The first step to show the uniqueness is the following partial comparison principle.

Proposition 10.1. *Let Assumption 5.1 hold true. Let $u^2 \in \mathrm{BUC}(\mathcal{Q})$ be a viscosity supersolution of PPDE (5.1) and let $u^1 \in \overline{C}^2(\mathcal{Q})$ be bounded, satisfying $\mathcal{L}u^1(\omega) \leq 0$ for all $\omega \in \mathcal{Q}$. If $u^1 \leq u^2$ on $\partial\mathcal{Q}$, then $u^1 \leq u^2$ in $\mathcal{Q}$. Similar result holds if we exchange the roles of u^1 and u^2.*

In the proof of the proposition, we will use the result of the following lemma.

Lemma 10.2. *Given a domain $D \in \mathcal{R}$ such that $D \subset O_\delta$ (defined in (7.2)), there exists a function $l \in C^2(D) \cap C(\mathrm{cl}(D))$ such that*

$$l = 0 \text{ on } \partial D,\ l < 0 \text{ on } D \text{ and } l' \text{ bounded.}$$

In particular, for arbitrary $a > 0$, there exists a function l such that $l = 0$ on ∂D, $l < 0$ on D and $\left|l'\right| \leq a$.

Proof. Consider the Dirichlet problem for the Poisson equation in the domain D:

$$\Delta l = 1 \text{ on } D, \text{ and } l = 0 \text{ on } \partial D.$$

By the classical PDE theory [9] (Theorem 4.3, page 56), there exists a unique classical solution, i.e. $l \in C^2(D) \cap C(\mathrm{cl}(D))$. Moreover, the solution can be written explicitly as:

$$l(x) = \mathbb{E}^{\mathbb{P}_0}\left[H_D^x\right], \text{ for all } x \in D.$$

By Corollary 15.2 in Appendix, we have

$$|l(x_1) - l(x_2)| = \left|\mathbb{E}^{\mathbb{P}_0}\left[H_D^{x_1} - H_D^{x_2}\right]\right| \leq C\left|x_1 - x_2\right|.$$

Consequently, we get the estimate:

$$\left|l^{'}\right| \leq C.$$

It remains to verify that $l < 0$ on D. Indeed, by Itô's formula, we have that for all $x \in D$

$$l(x + B_{H_D^x}) - l(x) = \int_0^{H_D^x} l^{'}(x + B_s) \cdot dB_s + \frac{1}{2}\int_0^{H_D^x} \Delta l(x + B_s)ds, \ \mathbb{P}_0\text{-a.s.}$$

Observe that $l(x + B_{H_D^x}) \equiv 0$, that $l^{'}$ is bounded and that $\Delta l = 1$ in D. By taking the expectation on both sides, we have

$$0 - l(x) = \frac{1}{2}\mathbb{E}\left[H_D^x\right] > 0.$$

To construct a function satisfying the second statement, we only need to consider $l_a = \frac{a}{C} l$. □

Proof of Proposition 10.1. Recall the notation H_n and O_n^ω in Definition 4.1. We now prove the proposition in two steps.

Step 1. We first show that for all $i \geq 0$ and $\omega \in \mathcal{Q}$

$$(u^1 - u^2)^+_{H_i}(\omega) \leq \overline{\mathcal{E}}^L\left[\left((u^1_{H_{i+1}})^{H_i,\omega} - (u^2_{H_{i+1}})^{H_i,\omega}\right)^+\right].$$

Clearly it suffices to consider $i = 0$, i.e.

$$(u^1 - u^2)^+(\mathbf{0}) \leq \overline{\mathcal{E}}^L\left[\left(u^1 - u^2\right)^+_{H_1}\right]. \tag{10.1}$$

Assume the contrary, i.e.

$$2c := (u^1 - u^2)^+(\mathbf{0}) - \overline{\mathcal{E}}^L\left[\left(u^1 - u^2\right)^+_{H_1}\right] > 0.$$

Let l be the function defined in Lemma 10.2 with the domain $D := O_0^0$. Also, denote the minimum of l as $l_m := \min_{x \in \mathrm{cl}(O_0^0)} l(x)$. Define

$$X_t(\omega) := (u^1 - u^2)^+(\omega_{t\wedge\cdot}) - c\frac{l(\omega_t)}{l_m},$$

$$Y := \overline{\mathcal{S}}^L\left[X_{H_1\wedge\cdot}\right] \ \textit{and} \ \tau^* := \inf\left\{t \geq 0 : Y_t = X_t\right\}.$$

Note that $0 < \frac{l(x)}{l_m} \leq 1$, for all $x \in O_0^0$. Applying the optimal stopping result (Theorem 3.2), we get

$$\begin{aligned}\overline{\mathcal{E}}^L\left[X_{\tau^*}\right] &= Y_0 \geq X_0 = (u^1 - u^2)^+(\mathbf{0}) - c\frac{l(0)}{l_m} \\ &\geq c + \overline{\mathcal{E}}^L\left[\left(u^1 - u^2\right)^+_{H_1}\right] \geq c + \overline{\mathcal{E}}^L\left[X_{H_1}\right].\end{aligned}$$

Therefore, there exists $\omega^* \in \mathcal{O}_0^0$, such that $t^* := \bar{t}(\omega^*) < H_1(\omega^*)$. Next, by the definition of Y, we have

$$\left((u^1 - u^2)^+(\omega^*) - c\frac{l(\omega^*_{t^*})}{l_m}\right) = X_{t^*}(\omega^*) = Y_{t^*}(\omega^*) \geq \overline{\mathcal{E}}^L\left[\left(X_{H_1}\right)^{t^*,\omega^*}\right] \geq 0.$$

Note that $\frac{l(\omega^*_{t^*})}{l_m} > 0$. Hence, $0 < \left(u^1 - u^2\right)^+(\omega^*) = \left(u^1 - u^2\right)(\omega^*)$. Then $X_t^{t^*,\omega^*}(\omega) = \varphi(\omega_{t\wedge\cdot}) - \left(u^2\right)^{\omega^*}(\omega_{t\wedge\cdot})$ for all $t \in \left[0, H_1^{\omega^*}\right]$, where $\varphi(\omega) := (u^1)^{\omega^*}(\omega) -$

$c\frac{l(\omega_{\bar{t}(\omega)})}{l_m}$ for all $\omega \in \Omega^e$. Observe that $\varphi \in C^2\left((\mathcal{O}_0^0)^{\omega^*}\right)$, since u^1 is assumed to be in $\bar{C}^2(\mathcal{Q})$. Using the $\overline{\mathcal{E}}$-supermartingale property of Y, we see that for all $\tau \in \mathcal{T}^{t^*}$:

$$\begin{aligned}\left(\varphi - (u^2)^{\omega^*}\right)_{t^*} &= X_{t^*}(\omega^*) = Y_{t^*}(\omega^*) \geq \overline{\mathcal{E}}^L\left[Y^{t^*,\omega^*}_{\tau\wedge H_1}\right] \\ &\geq \overline{\mathcal{E}}^L\left[X^{t^*,\omega^*}_{\tau\wedge H_1}\right] \geq \overline{\mathcal{E}}^L\left[\left(\varphi - (u^2)^{\omega^*}\right)_{\tau\wedge H_1}\right],\end{aligned}$$

i.e. $\varphi \in \overline{\mathcal{A}}^L u^2(\omega^*)$. Finally, by the L-viscosity supersolution property of u^2 and the assumption on the function G, we have

$$\begin{aligned}0 &\leq -G\left(\cdot, u^2, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi\right)(\omega^*_{t^*\wedge\cdot}) \\ &= -G\left(\cdot, u^2, \partial_\omega u^1 - \frac{c}{l_m}l', \partial^2_{\omega\omega}u^1 - \frac{c}{l_m}a\right)(\omega^*_{t^*\wedge\cdot}) \\ &\leq -G\left(\cdot, u^1, \partial_\omega u^1, \partial^2_{\omega\omega}u^1\right)(\omega^*_{t^*\wedge\cdot}) - \lambda\left(\left(u^1 - u^2\right)(\omega^*_{t^*\wedge\cdot})\right) + Ca.\end{aligned}$$

Since $\left(u^1 - u^2\right)(\omega^*) > 0$ and a is arbitrary, we have

$$-G\left(\cdot, u^1, \partial_\omega u^1, \partial^2_{\omega\omega}u^1\right)(\omega^*_{t^*\wedge\cdot}) > 0.$$

This is in contradiction with the subsolution property of u^1.

Step 2. We are going to show that

$$(u^1 - u^2)^+(\mathbf{0}) \leq \overline{\mathcal{E}}^L\left[\left(u^1 - u^2\right)^+_{H_Q}\right].$$

By the result of Step 1 and the tower property of $\overline{\mathcal{E}}^L$ (as mentioned before, it is a result from [7]), we get

$$(u^1 - u^2)^+(\mathbf{0}) \leq \overline{\mathcal{E}}^L\left[\left(u^1 - u^2\right)^+_{H_i}\right] \text{ for all } i \geq 1.$$

Next we get that for any $T > 0$,

$$\overline{\mathcal{E}}^L\left[\left(u^1 - u^2\right)^+_{H_i}\right] - \overline{\mathcal{E}}^L\left[\left(u^1 - u^2\right)^+_{H_Q}\right] \leq \overline{\mathcal{E}}^L\left[\left(u^1 - u^2\right)^+_{H_i} - \left(u^1 - u^2\right)^+_{H_Q}\right].$$

Note that $\mathcal{C}^L\left[H_i < H_Q\right] \to 0$ as $i \to \infty$. Therefore,

$$(u^1 - u^2)^+(\mathbf{0}) = 0.$$

□

11. Consistency of the Perron approach

We are going to give a constructive proof of the fact that $\overline{u} = \underline{u}$ (as defined in (8.1)). Since we follow the framework of ETZ [3], the lemmas in this section resemble to the corresponding work in their paper. However, there are technical differences, because we are dealing with the Dirichlet problem.

The following lemma gives an estimate on the difference between the supersolutions and the subsolutions to the PDEs. It will be useful when we analyze the path-frozen PDEs.

Lemma 11.1. *Fix $D \in \mathcal{R}$. Let $h^i : \partial D \to \mathbb{R}$ be continuous ($i = 1, 2$), g^i satisfy Assumption 5.1, and v^i be the viscosity solutions to the following PDEs:*

$$g^i(v, Dv, D^2v) = 0 \text{ on } D, \ v^i = h^i \text{ on } \partial D.$$

Assume that there exists a constant c_0, such that

$$|g^1(y, z, \gamma) - g^2(y, z, \gamma)| \leq c_0 \text{ for any } (y, z, \gamma).$$

Then, denoting $\delta v = v^1 - v^2$, $\delta h := h^1 - h^2$, *we have*

$$\delta v(x) \le \overline{\mathcal{E}}^{L_0}\left[\left((\delta h)^+_{H_D}\right)^x\right] + Cc_0.$$

In particular, if $g^1 = g^2$, *then we have* $\delta v(x) \le \overline{\mathcal{E}}^{L_0}\left[\left((\delta h)^+_{H_D}\right)^x\right]$.

Proof. By the standard argument, function $w(x) := \overline{\mathcal{E}}^{L_0}\left[\left((\delta h)^+_{H_D}\right)^x + \int_0^{H^x_D} c_0 dt\right]$ is a viscosity solution of the nonlinear PDE:

$$-c_0 - L_0|Dw| - \frac{1}{2}\sup_{\sqrt{\frac{2}{L_0}}I_d \le \sigma \le \sqrt{2L_0}I_d} \sigma^2 : D^2 w = 0 \text{ on } D,$$

$$\text{and } w = (\delta h)^+ \text{ on } \partial D.$$

Let K be a smooth nonnegative kernel with unit total mass. For all $\eta > 0$, we define the mollification $w^\eta := w * K^\eta$ of w. Then w^η is smooth, and it follows from a convexity argument that w^η is a classic supersolution of

$$-c_0 - L_0|Dw^\eta| - \frac{1}{2}\sup_{\sqrt{\frac{2}{L_0}}I_d \le \sigma \le \sqrt{2L_0}I_d} \sigma^2 : D^2 w^\eta \ge 0 \text{ on } D, \tag{11.1}$$

$$\text{and } w^\eta = (\delta h)^+ * K^\eta \text{ on } \partial D.$$

We claim that

$$\bar{w}^\eta + v^2 \text{ is a supersolution to the PDE with generator } g^1,$$

where $\bar{w}^\eta := w^\eta + \|w^\eta - (\delta h)^+\|_{L^\infty}$. Then we note that $\bar{w}^\eta + v^2 \ge w^\eta + h^2 + \|w^\eta - (\delta h)^+\|_{L^\infty} \ge h^1 = v^1$ on ∂D. By comparison principle in PDEs, we have $\bar{w}^\eta + v^2 \ge v^1$ on $\mathrm{cl}(D)$. Setting $\eta \to 0$, we obtain that $\delta v \le w$.

It remains to prove that $\bar{w} + v^2$ is a supersolution of the PDE with generator g^1. Let $x_0 \in D$, $\phi \in C^2(D)$ be such that $0 = (\phi - \bar{w}^\eta - v^2)(x_0) = \max\left(\phi - \bar{w}^\eta - v^2\right)$. Then, it follows from the viscosity supersolution property of v^2 that $\boldsymbol{L}^2(\phi - \bar{w}^\eta)(x_0) \ge 0$. Hence, at the point x_0, by (11.1) we have

$$\begin{aligned}
\boldsymbol{L}^1\phi &\ge \boldsymbol{L}^1\phi - \boldsymbol{L}^2(\phi - \bar{w}^\eta)\\
&= -g^1(\phi, D\phi, D^2\phi) + g^2(\phi - \bar{w}^\eta, D(\phi - \bar{w}^\eta), D^2(\phi - \bar{w}^\eta))\\
&\ge -g^1(\phi, D\phi, D^2\phi) + g^2(\phi, D(\phi - \bar{w}^\eta), D^2(\phi - \bar{w}^\eta))\\
&\ge c_0 + L_0|Dw^\eta| + \frac{1}{2}\sup_{\sqrt{\frac{2}{L_0}}I_d \le \sigma \le \sqrt{2L_0}I_d} \sigma^2 : D^2 w^\eta - c_0 - \alpha \cdot Dw^\eta - \frac{1}{2}\gamma : D^2 w^\eta\\
&\ge 0,
\end{aligned}$$

where $|\alpha| \le L_0$ and $\frac{2}{L_0}I_d \le \gamma \le 2L_0 I_d$. □

In the next lemma, we will use the path-frozen PDEs to construct the functions θ^ϵ_n, which will be needed to construct close approximations of $\overline{u}$ and $\underline{u}$ defined in (8.1). Before looking into the proof, we define some useful notations.

We first introduce the space of linear interpolations. For all $\epsilon > 0$, $n \ge 0$, denote

$$\Pi^\epsilon_n := \left\{\pi_n = (x_i)_{1 \le i \le n} : |x_i| \le \epsilon\right\}.$$

Example 11.2. Recall O_ϵ defined as in (7.2). We denote a sequence of stopping times by:

$$H_0 := 0,\ H_{i+1} := \inf\{t \geq H_i : B_t - B_{H_i} \notin O_\epsilon\} \wedge H_Q,\ i \geq 0.$$

Then we see that $(B_{H_i})_{1\leq i\leq n} \in \Pi_n^\epsilon$.

For all $\pi_n \in \Pi_n^\epsilon$, we denote by $\omega^{\pi_n} \in \Omega^e$ the linear interpolation of $(0,0)$, $(i, \sum_{j=1}^i x_j)_{1\leq i\leq n}$ with a flat tail extending to $t = \infty$. For simplicity, we use π_n instead of ω^{π_n} as superscript (e.g. $O^{\pi_n} := O^{\omega^{\pi_n}}$).

We also denote :

$$\pi_n^x := (\pi_n, x) \in \Pi_{n+1}^\epsilon \text{ for all } |x| \leq \epsilon.$$

The sequence of stopping times $H_i^{\pi_n^x}$ is defined as:

$$H_0^{\pi_n^x} := 0,\ H_1^{\pi_n^x} := \inf\{t \geq H_i : x + B_t \notin O_\epsilon\} \wedge H_Q^{\pi_n^x},$$

$$H_{i+1}^{\pi_n^x} := \inf\left\{t \geq H_i^{\pi_n^x} : B_t - B_{H_i} \notin O_\epsilon\right\} \wedge H_Q^{\pi_n^x},\ i \geq 1.$$

Given the canonical process B , $|x| < \epsilon$ and $\pi_n \in \Pi_n$, we define for all $m > n$

$$\omega^{\pi_n^m(x,B)} := \omega^{\left(\pi_n, x+B_{H_1^{\pi_n^x}}, (B_{H_j^{\pi_n^x}} - B_{H_{j-1}^{\pi_n^x}})_{1\leq j\leq m}\right)},\ \omega^{\pi_n^m(x,B,H_Q)} := \omega^{\pi_n^m(x,B)} \bar{\otimes} B^{H_m^{\pi_n^x}}_{H_Q^{\pi_n^x}\wedge\cdot}.$$

Lemma 11.3. *Let Assumptions 5.1 and 7.1 hold. Suppose that $|\xi| \leq C_0$. Let $|x_i| = \epsilon$, $i \geq 1$ and $\pi_n := \{x_i\}_{1\leq i\leq n}$. Assume that $\omega^{\pi_n^x} \in \mathcal{Q}$. Then, there exists a sequence of continuous functions $(\pi_n, x) \mapsto \theta_n^\epsilon(\pi_n, x)$, bounded uniformly in (ϵ, n), such that*

$$\theta_n^\epsilon(\pi_n; \cdot) \text{ is a viscosity solution of } (E)_\epsilon^{\omega^{\pi_n}},$$

and

$$\begin{cases} \theta_n^\epsilon(\pi_n; x) = \xi(\omega^{\pi_n^x}), & |x| < \epsilon \text{ and } x \in \partial Q^{\pi_n}, \\ \theta_n^\epsilon(\pi_n; x) = \theta_{n+1}^\epsilon(\pi_n^x; 0), & |x| = \epsilon. \end{cases}$$

Remark 11.4. The boundary is composed of two parts, because for the domain O_ϵ^{Q,π_n} one part of the boundary is that of Q^{π_n}, while the other is that of O_ϵ. When the variable touches ∂Q^{π_n}, we should set the solution to be equal to the boundary condition of the PPDE. Otherwise, when ∂O_ϵ is touched, the value of the solution should be consistent with that of the next piece of the path-frozen PDE.

Proof. Step 1. Recall the definition of $\overline{g}$ and $\underline{g}$ in (9.1) and (9.2). Also recall that

$$\underline{g} \leq G \leq \overline{g}.$$

We first prove the lemma in the case $G := \overline{g}$ and $G := \underline{g}$. It is easier than the general case, because Bellman equations can be represented explicitly as optimal control problems (as we discussed in Section 9).

We first study $\overline{g}$. For any N, denote

$$\overline{\theta}_{N,N}^\epsilon(\pi_N, 0) := \overline{\mathcal{E}}^{L_0}\left[\xi_{H_Q}^{\pi_N}\right],$$

and for $n \leq N-1$, $\theta := \overline{\theta}_{N,n}^\epsilon(\pi_n; \cdot)$ is a viscosity solution of the following PDE:

$$-\overline{g}(\theta, D\theta, D^2\theta) = 0 \text{ on } O_\epsilon^{Q,\pi_n}, \tag{11.2}$$

$$\theta(x) = \overline{\theta}_{N,n+1}^\epsilon(\pi_n^x; \mathbf{0}) \text{ on } \partial O_\epsilon^{Q,\pi_n}.$$

Note that $\overline{\theta}^{\epsilon}_{N,n}(\pi_n,x)$ can be represented as the solution of the following optimal control problem:

(11.3)
$$\overline{\theta}^{\epsilon}_{N,n}(\pi_n,x)=\sup_{b\in\mathbb{H}^0([0,L_0])}\overline{\mathcal{E}}^{L_0}\left[e^{-\int_0^{H^{\pi^x_n}_{N-n}}b_rdr}\xi\left(\omega^{\pi^{N-n}_n(x,B,H_Q)}\right)+C_0\int_0^{H^{\pi^x_n}_{N-n}}e^{-\int_0^s b_rdr}ds\right].$$

As we have discussed in Section 9, we may prove that $\overline{\theta}^{\epsilon}_{N,n}(\pi_n,x)$ is continuous in both variables (π_n,x). Meanwhile, observe that they are uniformly bounded.

Define
$$\overline{\theta}^{\epsilon}_{n}(\pi_n,x):=\sup_{b\in\mathbb{H}^0([0,L_0])}\overline{\mathcal{E}}^{L_0}\left[e^{-\int_0^{H^{\pi^x_n}_{Q}}b_rdr}\overline{\lim_{N\to\infty}}\xi\left(\omega^{\pi^{N-n}_n(x,B,H_Q)}\right)+C_0\int_0^{H^{\pi^x_n}_{Q}}e^{-\int_0^s b_rdr}ds\right].$$

Then,
$$|\overline{\theta}^{\epsilon}_{n}(\pi_n,x)-\overline{\theta}^{\epsilon}_{N,n}(\pi_n,x)|\quad\leq\quad C\mathcal{C}^{L_0}\left[H^{\pi^x_n}_{N-n}<H^{\pi^x_n}_{Q}\right]\to 0,\ N\to\infty.$$

As we have argued in Corollary 3.4, the convergence is uniform in (π_n,x). This implies that $\overline{\theta}^{\epsilon}_{n}(\pi_n;x)$ is uniformly bounded and continuous in (π_n,x). Moreover, by the stability of viscosity solutions we see that $\overline{\theta}^{\epsilon}_{n}(\pi_n;\cdot)$ is the viscosity solution of PDE (11.2) in O^{Q,π_n}_{ϵ}, with the boundary condition:
$$\begin{cases}\bar{\theta}^{\epsilon}_{n}(\pi_n;x)=\xi(\omega^{\pi^x_n}), & |x|<\epsilon \text{ and } x\in\partial Q^{\pi_n},\\ \bar{\theta}^{\epsilon}_{n}(\pi_n;x)=\bar{\theta}^{\epsilon}_{n+1}(\pi^x_n;0), & |x|=\epsilon.\end{cases}$$

Hence, we have shown the result in the case $G=\overline{g}$.

Similarly, we may show that $\underline{\theta}^{\epsilon}_{n}$, defined below, is the viscosity solution to the path-frozen PDE when the nonlinearity is $\underline{g}$:
$$\underline{\theta}_{n}(\pi_n,x):=\inf_{b\in\mathbb{H}^0([0,L_0])}\underline{\mathcal{E}}^{L_0}\left[e^{-\int_0^{H^{\pi^x_n}_{Q}}b_rdr}\overline{\lim_{i\to\infty}}\xi\left(\omega^{\pi^{N-n}_n(x,B,H)}\right)+C_0\int_0^{H^{\pi^x_n}_{Q}}e^{-\int_0^s b_rdr}ds\right].$$

Step 2. We now prove the lemma for G. Given the construction of Step 1, define:
$$\overline{\theta}^{\epsilon,m}_{m}(\pi_m;x):=\overline{\theta}^{\epsilon}_{m}(\pi_m;x),\ \underline{\theta}^{\epsilon,m}_{m}(\pi_m;x):=\underline{\theta}^{\epsilon}_{m}(\pi_m;x);\ m\geq 1.$$

For $n\leq m-1$, we may define $\overline{\theta}^{\epsilon,m}_{n}$ and $\underline{\theta}^{\epsilon,m}_{n}$ as the unique viscosity solution of the PDE $(E)^{\pi_n}_{\epsilon}$ with boundary conditions
$$\overline{\theta}^{\epsilon,m}_{n}(\pi_n;x)=\overline{\theta}^{\epsilon,m}_{n+1}(\pi^x_n;0),\ \underline{\theta}^{\epsilon,m}_{n}(\pi_n;x)=\underline{\theta}^{\epsilon,m}_{n+1}(\pi^x_n;0)$$
on $\partial O^{Q,\pi_n}_{\epsilon}$. Note that, for $x\in\partial O^{Q,\pi_m}_{\epsilon}$,
$$\overline{\theta}^{\epsilon,m}_{m}(\pi_m;x)=\overline{\theta}^{\epsilon,m+1}_{m+1}(\pi^x_m;0),\ \underline{\theta}^{\epsilon,m}_{m}(\pi_m;x)=\underline{\theta}^{\epsilon,m+1}_{m+1}(\pi^x_m;0).$$

By the comparison result for PDE $(E)^{\pi_n}_{\epsilon}$, we also have that
$$\overline{\theta}^{\epsilon,m}_{m}(\pi_m;\cdot)\geq\overline{\theta}^{\epsilon,m+1}_{m}(\pi_m;\cdot)\geq\underline{\theta}^{\epsilon,m+1}_{m}(\pi_m;\cdot)\geq\underline{\theta}^{\epsilon,m}_{m}(\pi_m;\cdot)\text{ on }O^{Q,\pi_m}_{\epsilon},$$

and therefore, by the same comparison argument:

(11.4)
$$\overline{\theta}^{\epsilon,m}_{n}(\pi_n;\cdot)\geq\overline{\theta}^{\epsilon,m+1}_{n}(\pi_n;\cdot)\geq\underline{\theta}^{\epsilon,m+1}_{n}(\pi_n;\cdot)\geq\underline{\theta}^{\epsilon,m}_{n}(\pi_n;\cdot)\text{ on }O^{Q,\pi_n}_{\epsilon}\text{ for all }n\leq m.$$

Denote $\delta\theta_n^{\epsilon,m} := \overline{\theta}_n^{\epsilon,m} - \underline{\theta}_n^{\epsilon,m}$. Applying Lemma 11.1 several times and using the tower property of $\overline{\mathcal{E}}^L$, we get that

$$|\delta\theta_n^{\epsilon,m}(\pi_n;x)| \le \overline{\mathcal{E}}^{L_0}\left[\left|\delta\theta_m^{\epsilon,m}\left(\pi_n^{m-n}(x,B);0\right)\right|\right].$$

Note that $\delta\theta_n^{\epsilon,m}(\pi_n;x) = 0$ when $x \in \partial Q^{\pi_n}$. Then, since $\mathcal{C}^{L_0}[H_n < H_Q] \to 0$, as $m \to \infty$, we have that

$$|\delta\theta_n^{\epsilon,m}(\pi_n;x)| \le C\mathcal{C}^{L_0}[H_{m-n} < H_Q] \to 0.$$

Together with (11.4), this implies the existence of θ_n^ϵ such that

$$\overline{\theta}_n^{\epsilon,m} \downarrow \theta_n^\epsilon,\ \underline{\theta}_n^{\epsilon,m} \uparrow \theta_n^\epsilon,\ \text{as } m \to \infty.$$

Clearly θ_n^ϵ is uniformly bounded and continuous. Finally, it follows from the stability of viscosity solutions that θ_n^ϵ satisfies the requirement of the lemma. □

The previous lemma shows the existence of the viscosity solution to the path-frozen PDE. Indeed, we can construct smooth super- and sub-solutions to the PPDE from the solution to the path-frozen PDE. Let $\hat{\pi}_n$ denote the sequence $(H_i, B_{H_i})_{1\le i\le n}$. Denote $\omega^\epsilon := \lim_{n\to\infty}\omega^{\hat{\pi}_n}$.

Lemma 11.5. *There exists $\psi^\epsilon \in \overline{C}^2(\mathcal{Q})$, such that*

$$\psi^\epsilon(0) = \theta_0^\epsilon(0) + \epsilon,\ \psi^\epsilon \ge h \text{ on } \mathcal{Q},\ \mathcal{L}^{\omega^{\hat{\pi}_n}}\psi^\epsilon \ge 0 \text{ on } \mathcal{O}_\epsilon^{Q,\hat{\pi}_n}.$$

Proof. For simplicity, in this proof we omit the superscript ϵ. Set $\delta_n := 2^{-n-2}\epsilon$.

First, by applying Assumption 7.1 to PDE $(E)_\epsilon^0$, we obtain that there exists a function $v_0 \in C^2(\mathrm{cl}(O_\epsilon^Q))$, such that

$$v_0(0) = \theta_0(0) + \frac{\epsilon}{2},\ \boldsymbol{L}^0 v_0 \ge 0 \text{ on } O_\epsilon^Q,\ v_0 \ge \theta_0 \text{ on } \partial O_\epsilon^Q.$$

Set

$$\psi(\omega) := v_0(\omega_{\bar{t}}) + \sum_{i\ge 0}\delta_i \text{ on } \mathrm{cl}(\mathcal{O}_\epsilon^Q).$$

By the monotonicity of G, it is clear that

$$\psi(0) - \theta_0(0) = \frac{\epsilon}{2} + \sum_{i\ge 0}\delta_i = \epsilon \text{ and } \mathcal{L}^0\psi \ge \boldsymbol{L}^0 v_0 \ge 0 \text{ on } \mathrm{cl}(\mathcal{O}_\epsilon^Q).$$

Moreover, it is clear that $\psi \in C(\mathcal{O}_\epsilon^Q)$.

Next, fix $\pi_1 := (x_1)$ such that $x_1 \in \partial O_\epsilon^Q$. Applying again Assumption 7.1 to PDE $(E)_\epsilon^{\omega^{\pi_1}}$, we obtain that there exists a function $v_1(\pi_1;\cdot) \in C^2(\mathrm{cl}(O_\epsilon^{Q,\pi_1}))$ such that

$$v_1(\pi_1;0) = v_0(x_1) + \delta_0,\ \boldsymbol{L}^{\omega^{\pi_1}} v_1 \ge 0 \text{ on } O_\epsilon^{Q,\pi_1},\ v_1(\pi_1;\cdot) \ge \theta_1(\pi_1;\cdot) \text{ on } \partial O_\epsilon^{Q,\pi_1}.$$

Set

$$\psi(\omega) := v_1(\pi_1;\omega_{\bar{t}} - \omega_{H_1}) + \sum_{i\ge 1}\delta_i \text{ on } \mathcal{O}_\epsilon^{Q.\pi_1},$$

$$\text{otherwise } \psi(\omega) := v_1(\pi_1;\omega_{H_2} - \omega_{H_1}) + \sum_{i\ge 1}\delta_i.$$

It is clear that the updated ψ is in $\overline{C}^2(\mathcal{Q})$. Repeating the above arguments we may construct a sequence of functions v_n and thus construct the desired $\psi \in \overline{C}^2(\mathcal{Q})$. □

Finally, we have done all the necessary constructions and are ready to show the main result in this section.

Proposition 11.6. *Suppose $|\xi| \leq C_0$. Under Assumptions 5.1 and 7.1, we have $\overline{u} = \underline{u}$.*

Proof. For any $\epsilon > 0$, let ψ^ϵ be as in Lemma 11.5, and $\overline{\psi}^\epsilon := \psi^\epsilon + \rho(2\epsilon) + \lambda^{-1}(\rho(2\epsilon))$, where ρ is the modulus of continuity of ξ and G and λ^{-1} is the inverse of the function in Assumption 5.1. Then clearly $\overline{\psi}^\epsilon \in \overline{C}^2(\mathcal{Q})$ and bounded. Also,

$$\overline{\psi}^\epsilon(\omega) - \xi(\omega) \geq \psi^\epsilon(\omega) + \rho(2\epsilon) - \xi(\omega) \geq \xi(\omega^\epsilon) - \xi(\omega) + \rho(2\epsilon) \geq 0 \text{ on } \partial\mathcal{Q}.$$

Moreover, when $\bar{t}(\omega) \in [H_n(\omega), H_{n+1}(\omega))$, we have that

$$\begin{aligned}
\mathcal{L}\overline{\psi}^\epsilon(\omega) &= -G\left(\omega, \overline{\psi}^\epsilon, \partial_\omega \psi^\epsilon, \partial^2_{\omega\omega}\psi^\epsilon\right) \\
&\geq -G\left(\omega^{\hat{\pi}_n}, \psi^\epsilon + \lambda^{-1}(\rho(2\epsilon)), \partial_\omega \psi^\epsilon, \partial^2_{\omega\omega}\psi^\epsilon\right) - \rho(2\epsilon) \\
&\geq -G\left(\omega^{\hat{\pi}_n}, \psi^\epsilon, \partial_\omega \psi^\epsilon, \partial^2_{\omega\omega}\psi^\epsilon\right) \geq 0.
\end{aligned}$$

Then by the definition of $\overline{u}$ we see that

$$\overline{u}(0) \leq \overline{\psi}^\epsilon(0) = \psi^\epsilon + \rho(2\epsilon) + \lambda^{-1}(\rho(2\epsilon)) \leq \theta^\epsilon_0(0) + \epsilon + \rho(2\epsilon) + \lambda^{-1}(\rho(2\epsilon)).$$

Similarly, $\underline{u}(0) \geq \theta^\epsilon_0(0) - \epsilon - \rho(2\epsilon) - \lambda^{-1}(\rho(2\epsilon))$. That implies that

$$\overline{u}(0) - \underline{u}(0) \leq 2\epsilon + 2\rho(2\epsilon) + 2\lambda^{-1}(\rho(2\epsilon)).$$

Since ϵ is arbitrary, this shows that $\overline{u}(0) = \underline{u}(0)$. Similarly, we can show that $\overline{u}(\omega) = \underline{u}(\omega)$ for all $\omega \in \mathcal{Q}$. □

Combing the fact that $\overline{u} = \underline{u}$ with the partial comparison principle, we finish the proof of the uniqueness of the viscosity solutions to the PPDEs.

Theorem 11.7. *Suppose $|\xi| \leq C_0$. Let Assumptions 5.1 and 7.1 hold. If a viscosity solution to the PPDE (5.1) exists, it is unique.*

Part 5. Existence

12. Regularity

Our objective is to show that $u := \overline{u} = \underline{u}$ is indeed the unique viscosity solution of PPDE (5.1). The uniform continuity of u will be a by-product of our approximation procedure. We have following corollary of Lemma 11.3.

Corollary 12.1. *Let $|x_i| = \epsilon$, $i \geq 1$. Fix an $\omega \in \mathcal{Q}$. For $\epsilon > 0$ small enough, there exists a sequence of continuous functions $(\pi_n, x) \mapsto \theta^\omega_n(\pi_n, x)$, bounded uniformly in (ϵ, n), such that*

$$\theta^\omega_n(\pi_n; \cdot) \text{ is a viscosity solution of } (E)^{\omega\bar{\otimes}\omega^{\pi_n}}_\epsilon,$$

with boundary conditions:

$$\begin{cases} \theta^\omega_n(\pi_n; x) = \xi(\omega^{\omega\bar{\otimes}\pi^x_n}), & |x| < \epsilon \text{ and } x \in \partial Q^{\omega\bar{\otimes}\omega^{\pi_n}}, \\ \theta^\omega_n(\pi_n; x) = \theta^\omega_{n+1}(\pi^x_n; 0), & |x| = \epsilon. \end{cases}$$

Moreover, we can prove the following lemma which is important for the uniform continuity of u.

Lemma 12.2. *Let θ_n^ω be defined as in Corollary 12.1. Then there is a modulus of continuity ρ such that for any $\omega^1, \omega^2 \in \mathcal{Q}$,*

$$\left|\theta_0^{\omega^1}(0;0) - \theta_0^{\omega^2}(0;0)\right| \leq \rho\left(d^e(\omega^1,\omega^2)\right). \tag{12.1}$$

Proof. Similarly as in the proof of Lemma 11.3, we divide the proof in two parts.

Step 1. We again first prove the result for the case when $G := \overline{g}$ or $G := \underline{g}$. The detailed proof is only given for $\overline{g}$.

Review the proof of Lemma 11.3 and note the definition of $\overline{\theta}^\epsilon_{N,n}$. Here, we define correspondingly $\overline{\theta}^\omega_{N,n}$ which depends on ω (the superscript ϵ is omitted for simplicity, since the dependence on ω is more essential here). For any N, denote

$$\overline{\theta}^\omega_{N,N}(\pi_N, 0) := \overline{\mathcal{E}}^{L_0}\left[\xi_H^{\omega\bar{\otimes}\omega^{\pi_N}}\right].$$

For $n \leq N-1$, $\theta := \overline{\theta}^\omega_{N,n}(\pi_n;\cdot)$ is a viscosity solution of the following PDE:

$$\begin{aligned} -\overline{g}(\theta, D\theta, D^2\theta) &= 0 \text{ on } O_\epsilon^{Q,\omega\bar{\otimes}\omega^{\pi_n}} \\ \theta(x) &= \overline{\theta}^\omega_{N,n+1}(\pi_n^x;\mathbf{0}) \text{ on } \partial O_\epsilon^{Q,\omega\bar{\otimes}\omega^{\pi_n}}. \end{aligned} \tag{12.2}$$

Function $\overline{\theta}^\omega_{N,n}(\pi_n, x)$ can be represented as the solution of the following optimal control problem:

(12.3)

$$\overline{\theta}^\omega_{N,n}(\pi_n, x) = \sup_{b\in\mathbb{H}^0([0,L_0])} \overline{\mathcal{E}}^{L_0}\left[e^{-\int_0^{H^{\omega,\pi_n^x}_{N-n}} b_r dr}\xi\left(\omega\bar{\otimes}\omega^{\pi_n^{N-n}(x,B,H_Q^\omega)}\right) + C_0\int_0^{H^{\omega,\pi_n^x}_{N-n}} e^{-\int_0^s b_r dr}ds\right],$$

where $H_Q^{\omega,\pi_n^x} := H_Q^{\omega\bar{\otimes}\omega^{\pi_n^x}}$. Then, we have the following estimate: for $\omega^{\pi_n^x} \in \mathcal{Q}^{\omega^1} \cap \mathcal{Q}^{\omega^2}$,

$$\begin{aligned} \left|\overline{\theta}^{\omega^1}_{N,n}(\pi_n,x) - \overline{\theta}^{\omega^2}_{N,n}(\pi_n,x)\right| \quad &\leq \quad C\overline{\mathcal{E}}^{L_0}\left[\left|H^{\omega^1,\pi_n^x}_{N-n} - H^{\omega^2,\pi_n^x}_{N-n}\right|\right] + C\overline{\mathcal{E}}^{L_0}\left[\xi\left(\omega^1\bar{\otimes}\omega^{\pi_n^{N-n}(x,B,H_Q^{\omega^1})}\right)\right. \\ &\qquad \left. -\xi\left(\omega^2\bar{\otimes}\omega^{\pi_n^{N-n}(x,B,H_Q^{\omega^2})}\right)\right]. \end{aligned}$$

Note that $\left|H^{\omega^1,\pi_n^x}_{N-n} - H^{\omega^2,\pi_n^x}_{N-n}\right| \leq \left|H_Q^{\omega^1,\pi_n^x} - H_Q^{\omega^2,\pi_n^x}\right|$. Based on Lemma 9.1 , we may show that

$$\left|\overline{\theta}^{\omega^1}_{N,n} - \overline{\theta}^{\omega^2}_{N,n}\right| \leq \rho(d^e(\omega^1,\omega^2)), \tag{12.4}$$

where ρ is independent of N. Then define $\overline{\theta}^\omega_n$ similarly to $\overline{\theta}^\epsilon_n$ in Lemma 11.3. Then $|\overline{\theta}^\omega_n(\pi_n,x) - \overline{\theta}^\omega_{N,n}(\pi_n,x)| \to 0$, and

$$\left|\overline{\theta}^{\omega^1}_n - \overline{\theta}^{\omega^2}_n\right| \leq \rho(d^e(\omega^1,\omega^2)). \tag{12.5}$$

A similar argument provides the same estimate for $\underline{\theta}^\omega_n$:

$$\left|\underline{\theta}^{\omega^1}_n - \underline{\theta}^{\omega^2}_n\right| \leq \rho(d^e(\omega^1,\omega^2)). \tag{12.6}$$

Step 2. We now show the result in the general case.

We follow Lemma 11.3 and define $\overline{\theta}^{\omega,m}_n$ similarly to $\overline{\theta}^{\epsilon,m}_n$. Define the stopping times:

$$H^i := H_Q^{\omega^i,\pi_n^x},\ i=1,2;\ H^{1,2} := H^1 \wedge H^2,$$

$$H_0 = 0,\ H_{i+1} := \inf\{t \geq 0 : B_t - B_{H_i} \notin O_\epsilon\},\ i \geq 0.$$

By comparison arguments, we claim that for $\omega^{\pi_n^x} \in \mathcal{Q}^{\omega^1} \cap \mathcal{Q}^{\omega^2}$

$$\begin{aligned}(\overline{\theta}_n^{\omega^1,m} - \underline{\theta}_n^{\omega^2,m})(\pi_n;x) \quad &\leq \quad \overline{\mathcal{E}}^{L_0}\left[(\overline{\theta}_m^{\omega^1} - \underline{\theta}_m^{\omega^2})(\pi_n^{m-n}(x,B)),0)1_{\{H_{m-n}\leq H^{1,2}\}}\right] \\ &\quad +I_1 + I_2 + C(m-n)\rho(d^e(\omega^1,\omega^2)),\end{aligned} \tag{12.7}$$

where

$$I_1 := \sum_{k=0}^{m-n-1} \overline{\mathcal{E}}^{L_0}\left[\left(\overline{\theta}_{n+k+1}^{\omega^1,m}(\pi_n^{k+1}(x,B),0) - \underline{\theta}_{n+k}^{\omega^2,m}(\pi_n^k(x,B),B_{H^1}-B_{H_k})\right)1_{\{H^1<H_{k+1}\leq H^2\}}\right],$$

$$I_2 := \sum_{k=0}^{m-n-1} \overline{\mathcal{E}}^{L_0}\left[\left(\overline{\theta}_{n+k}^{\omega^1,m}(\pi_n^k(x,B),B_{H^2}-B_{H_k}) - \underline{\theta}_{n+k+1}^{\omega^2,m}(\pi_n^{k+1}(x,B),0)\right)1_{\{H^2<H_{k+1}\leq H^1\}}\right].$$

(We will prove the claim later.) Next, we focus on the term: $\overline{\theta}_{n+k+1}^{\omega^1,m}(\pi_n^{k+1}(x,B),0) - \underline{\theta}_{n+k}^{\omega^2,m}(\pi_n^k(x,B),B_{H^1}-B_{H_k})$. Note the results of the comparison:

$$\underline{\theta}_{n+k}^{\omega^2,m}(\pi_n^k(x,B),B_{H^1}-B_{H_k}) \geq \underline{\theta}_{n+k}^{\omega^2}(\pi_n^k(x,B),B_{H^1}-B_{H_k});$$

$$\overline{\theta}_{n+k+1}^{\omega^1,m}(\pi_n^{k+1}(x,B),0) \leq \overline{\theta}_{n+k+1}^{\omega^1}(\pi_n^{k+1}(x,B),0).$$

By (12.6), we have

$$\left|\underline{\theta}_{n+k}^{\omega^2}(\Pi_k(\pi_n^x),B_{H^1}-B_{H_k}) - \underline{\theta}_{n+k}^{\omega^1}(\Pi_k(\pi_n^x),B_{H^1}-B_{H_k})\right| \leq \rho(d^e(\omega^1,\omega^2)).$$

Also, notice that $\overline{\theta}_{n+k+1}^{\omega^1}(\pi_n^{k+1}(x,B),0) = \underline{\theta}_{n+k+1}^{\omega^1}(\pi_n^{k+1}(x,B),0) = \underline{\theta}_{n+k}^{\omega^1}(\pi_n^k(x,B),B_{H^1}-B_{H_k})$. As a result, we have

$$\overline{\theta}_{n+k+1}^{\omega^1,m}(\Pi_{k+1}^{H_1}(\pi_n^x),0) - \underline{\theta}_{n+k}^{\omega^2,m}(\Pi_k(\pi_n^x),B_{H^1}) \leq \rho(d^e(\omega^1,\omega^2)).$$

Moreover, turn back to (12.7). We conclude that

$$(\overline{\theta}_n^{\omega^1,m} - \underline{\theta}_n^{\omega^2,m})(\pi_n;x) \leq C\mathcal{C}^L\left[H_{m-n} < H^{1,2}\right] + C(m-n+1)\rho(d^e(\omega^1,\omega^2)).$$

Note that

$$\overline{\theta}_n^{\omega^1,m} \geq \theta_n^{\omega^1} \text{ and } \theta_n^{\omega^2} \geq \underline{\theta}_n^{\omega^2,m}.$$

It follows that

$$(\theta_n^{\omega^1} - \theta_n^{\omega^2})(\pi_n;x) \leq \rho(d^e(\omega^1,\omega^2)).$$

(Recall that ρ is different from line to line.) By exchanging the roles of ω^1 and ω^2, we may show that

$$\left|(\theta_n^{\omega^1} - \theta_n^{\omega^2})(\pi_n;x)\right| \leq \rho(d^e(\omega^1,\omega^2)).$$

Proof of the claim (12.7). Suppose that $m \geq n+1$. It suffices to show that

$$\begin{aligned}(\overline{\theta}_n^{\omega^1,m} - \underline{\theta}_n^{\omega^2,m})(\pi_n;x) \quad &\leq \quad \overline{\mathcal{E}}^{L_0}\left[(\overline{\theta}_{n+1}^{\omega^1,m} - \underline{\theta}_{n+1}^{\omega^2,m})(\pi_n^1(x,B)),0)1_{\{H_1\leq H^{1,2}\}}\right] \\ &\quad +\overline{\mathcal{E}}^{L_0}\left[(\overline{\theta}_{n+1}^{\omega^1,m}(\pi_n^1(x,B),0) - \underline{\theta}_n^{\omega^2,m}(\pi_n,x+B_{H^1}))1_{\{H^1<H_1\leq H^2\}}\right] \\ &\quad +\overline{\mathcal{E}}^{L_0}\left[(\overline{\theta}_n^{\omega^1,m}(\pi_n,x+B_{H^2}) - \underline{\theta}_{n+1}^{\omega^2,m}(\pi_n^1(x,B),0))1_{\{H^2<H_1\leq H^1\}}\right] \\ &\quad +C\rho(d^e(\omega^1,\omega^2)).\end{aligned}$$

Then the claim can be easily shown by induction. The proof is similar to the proof of the estimate in Lemma 11.1. Recall that $\overline{\theta}_n^{\omega^1,m}$ (resp. $\underline{\theta}_n^{\omega^2,m}$) is a solution to the PDE with generator g^{ω^1} (resp. g^{ω^2}). Now we study those two equations on the domain:

$$O_\epsilon \cap Q^{\omega^1} \cap Q^{\omega^2}.$$

Therefore, the boundary can be divided into three parts which belong to ∂O_ϵ, ∂Q^{ω^1} and ∂Q^{ω^2} respectively. We denote them by Bd_1, Bd_2 and Bd_3.

(a) On Bd_1, we have $H_1 \leq H^{1,2}$. Also, on this part of the boundary, $\overline{\theta}_n^{\omega^1,m} = \overline{\theta}_{n+1}^{\omega^1,m}$ and $\underline{\theta}_n^{\omega^2,m} = \underline{\theta}_{n+1}^{\omega^2,m}$.

(b) On Bd_2, we have $H^1 < H_1 \leq H^2$. Here $\overline{\theta}_n^{\omega^1,m} = \overline{\theta}_{n+1}^{\omega^1,m}$. However, on this part of the boundary, $\underline{\theta}_n^{\omega^2,m}$ does not take the value of $\underline{\theta}_{n+1}^{\omega^2,m}$.

(c) On Bd_3, we have $H^2 < H_1 \leq H^1$. Similar to the case of Bd_2, we find that $\underline{\theta}_n^{\omega^2,m} = \underline{\theta}_{n+1}^{\omega^2,m}$.

After clarifying all the boundary conditions of the two equations on the different parts of the boundary, we can then use Lemma 11.1 to finish the proof. □

Corollary 12.3. *u is uniformly continuous in $\mathcal{Q}$.*

Proof. We have already shown in Proposition 11.6 that for $\omega^1, \omega^2 \in \mathcal{Q}$

$$\overline{u}(\omega^1) \leq \theta_0^{\omega^1}(0) + \epsilon + \rho(2\epsilon) \text{ and } \underline{u}(\omega^2) \geq \theta_0^{\omega^2}(0) - \epsilon - \rho(2\epsilon).$$

Hence, it follows from Lemma 12.2 that

$$u(\omega^1) - u(\omega^2) = \overline{u}(\omega^1) - \underline{u}(\omega^2) \leq \theta_0^{\omega^1}(0) - \theta_0^{\omega^2}(0) + 2(\epsilon + \rho(2\epsilon)) \leq \rho(d^e(\omega^1, \omega^2)) + \rho(2\epsilon).$$

By exchanging the roles of ω^1 and ω^2, we show that u is uniformly continuous. □

13. Viscosity property

After having shown that $u = \overline{u} = \underline{u}$ is uniformly continuous, we are ready to verify that it is indeed the unique viscosity solution to PPDE (5.1). The following proof is similar to the corresponding work in ETZ [3].

Proposition 13.1. *u is the viscosity solution to PPDE (5.1).*

Proof. Without loss of generality, we prove only that $\overline{u}$ is a L_0-viscosity supersolution at 0. Assume the contrary, i.e. there exists $\varphi \in \overline{\mathcal{A}}^{L_0}\overline{u}(0)$ such that $-c := \mathcal{L}\varphi(0) < 0$. For any $\psi \in \overline{\mathcal{D}}_Q^\xi(0)$ and $\omega \in \mathcal{Q}$ it is clear that $\psi^\omega \in \overline{\mathcal{D}}_Q^\xi(\omega)$ and $\psi(\omega) \geq \overline{u}(\omega)$. Now by the definition of $\overline{u}$, there exists $\psi^n \in \overline{C}^2(\mathcal{Q})$ such that

$$\delta_n := \psi^n(0) - \overline{u}(0) \downarrow 0 \text{ as } n \to \infty,\ \mathcal{L}\psi^n(\omega) \geq 0,\ \omega \in \mathcal{Q}. \tag{13.1}$$

Let H_D be the hitting time required in $\overline{\mathcal{A}}^{L_0}\overline{u}(0)$. Since $\varphi \in C^2(\mathcal{D})$ and $\overline{u} \in \mathrm{BUC}(\mathcal{Q})$, without loss of generality we may assume that

$$\mathcal{L}\varphi(\omega_{t\wedge\cdot}) \leq -\frac{c}{2} \text{ and } |\varphi_t - \varphi_0| + |\overline{u}_t - \overline{u}_0| \leq \frac{c}{6L_0} \text{ for all } t \leq H_D.$$

We emphasize that the above H_D is independent of n. Now let $\{H_i^n, i \geq 1\}$ correspond to $\psi^n \in \overline{C}^2(\mathcal{Q})$. Since $\varphi \in \overline{\mathcal{A}}^{L_0}\overline{u}(0)$, this implies for all $\mathbb{P} \in \mathcal{P}^{L_0}$ and $n,\ i$ that :

$$0 \geq \mathbb{E}^{\mathbb{P}}\left[(\varphi - \overline{u})_{H_D \wedge H_i^n}\right] \geq \mathbb{E}^{\mathbb{P}}\left[(\varphi - \psi^n)_{H_D \wedge H_i^n}\right]. \tag{13.2}$$

Denote $\mathcal{G}^{\mathbb{P}}\phi := \alpha^{\mathbb{P}} \cdot \partial_\omega \phi + \frac{1}{2}(\beta^{\mathbb{P}})^2 : \partial^2_{\omega\omega}\phi$. Then, since $\varphi \in C^2$ and ψ^n is a semi-martingale on $[0, H^n_i]$, it follows from (13.1) that:

$$\begin{aligned}
\delta_n &\geq \mathbb{E}^{\mathbb{P}}\left[(\varphi-\psi^n)_{H_D\wedge H^n_i} - (\varphi-\psi^n)_0\right] = \mathbb{E}^{\mathbb{P}}\left[\int_0^{H_D\wedge H^n_i} \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n)ds\right] \\
&\geq \mathbb{E}^{\mathbb{P}}\Bigg[\int_0^{H_D\wedge H^n_i} (\frac{c}{2} - G(\omega_{s\wedge\cdot}, \varphi, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi) \\
&\qquad\qquad + G(\omega_{s\wedge\cdot}, \psi^n, \partial_\omega\psi^n, \partial^2_{\omega\omega}\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n))ds\Bigg] \\
&\geq \mathbb{E}^{\mathbb{P}}\Bigg[\int_0^{H_D\wedge H^n_i} (\frac{c}{2} - G(\omega_{s\wedge\cdot}, \varphi, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi) \\
&\qquad\qquad + G(\omega_{s\wedge\cdot}, \overline{u}, \partial_\omega\psi^n, \partial^2_{\omega\omega}\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n))ds\Bigg],
\end{aligned}$$

where the last inequality is due to the monotonicity in y of G. Since $\varphi_0 = \overline{u}_0$, we get

$$\begin{aligned}
\delta_n &\geq \mathbb{E}^{\mathbb{P}}\Bigg[\int_0^{H_D\wedge H^n_i} (\frac{c}{3} - G(\omega_{s\wedge\cdot}, \overline{u}_0, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi) \\
&\qquad\qquad + G(\omega_{s\wedge\cdot}, \overline{u}_0, \partial_\omega\psi^n, \partial^2_{\omega\omega}\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n))ds\Bigg].
\end{aligned}$$

Now let $\eta > 0$ be a small number. For each n, define $\tau^n_0 := 0$, and

$$\begin{aligned}
\tau^n_{j+1} :\ &= H_D \wedge \inf\{t \geq \tau^n_j : \rho(d^e(\omega_{t\wedge\cdot}, \omega_{\tau^n_j\wedge\cdot})) + |\partial_\omega\varphi(\omega_{t\wedge\cdot}) - \partial_\omega\varphi(\omega_{\tau^n_j\wedge\cdot})| \\
&\quad + |\partial^2_{\omega\omega}\varphi(\omega_{t\wedge\cdot}) - \partial^2_{\omega\omega}\varphi(\omega_{\tau^n_j\wedge\cdot})| + |\partial_\omega\psi^n(\omega_{t\wedge\cdot}) - \partial_\omega\psi^n(\omega_{\tau^n_j\wedge\cdot})| \\
&\quad + |\partial^2_{\omega\omega}\psi^n(\omega_{t\wedge\cdot}) - \partial^2_{\omega\omega}\psi^n(\omega_{\tau^n_j\wedge\cdot})| \geq \eta\}.
\end{aligned}$$

Since ψ^n is regular on $[0, H^n_i]$ for each n and so is φ, one can easily check that $\tau^n_j \uparrow H_D$ $\mathcal{P}^{L_0}$-q.s. as $j \to \infty$. Thus,

$$\begin{aligned}
\delta_n &\geq (\frac{c}{3} - C\eta)\mathbb{E}^{\mathbb{P}}\left[H_D \wedge H^n_i\right] + \sum_{j\geq 0}\mathbb{E}^{\mathbb{P}}\int_{\tau^n_j\wedge H^n_i}^{\tau^n_{j+1}\wedge H^n_i} \Big(-G(\cdot, \overline{u}_0, \partial_\omega\varphi, \partial^2_{\omega\omega}\varphi) \\
&\qquad\qquad + G(\cdot, \overline{u}_0, \partial_\omega\psi^n, \partial^2_{\omega\omega}\psi^n) + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n)\Big)_{\tau^n_j} ds \\
&= (\frac{c}{3} - C\eta)\mathbb{E}^{\mathbb{P}}\left[H_D \wedge H^n_i\right] + \sum_{j\geq 0}\mathbb{E}^{\mathbb{P}}\int_{\tau^n_j\wedge H^n_i}^{\tau^n_{j+1}\wedge H^n_i} \Big(\alpha^n_j \cdot \partial_\omega(\psi^n - \varphi) \\
&\qquad\qquad + \frac{1}{2}(\beta^n_j)^2 : \partial^2_{\omega\omega}(\psi^n - \varphi) + \mathcal{G}^{\mathbb{P}}(\varphi-\psi^n)\Big)_{\tau^n_j} ds,
\end{aligned}$$

for some appropriate α^n_j, β^n_j. Note that α^n_j and β^n_j are both $\mathcal{F}_{\tau^n_j}$-measurable. Now choose $\mathbb{P}_n \in \mathcal{P}^{L_0}$ such that $\alpha^{\mathbb{P}_n}_t = \alpha^n_j$, $\beta^{\mathbb{P}_n}_t = \beta^n_j$, for $t \in [\tau^n_j, \tau^n_{j+1})$. Then

$$\delta_n \geq (\frac{c}{3} - C\eta)\mathbb{E}^{\mathbb{P}_n}\left[H_D \wedge H^n_i\right].$$

Set $\eta := \frac{c}{6C}$. Since $\lim_{i\to\infty}\mathbb{P}_n\left[H^n_i < H_D\right] = 0$, we have $\underline{\mathcal{E}}^{L_0}\left[H_D\right] \leq \mathbb{E}^{\mathbb{P}_n}\left[H_D\right] \leq \delta_n$. By putting $n \to \infty$, we get $\underline{\mathcal{E}}^{L_0}\left[H_D\right] = 0$. This is a contradiction. □

Part 6. Appendix

14. Property of the distance $d^e(\cdot,\cdot)$

In this section, we discuss the relation between property (P) and the quasi metric $d^e(\cdot,\cdot)$.

Proposition 14.1. *Let h be defined on $\mathcal{Q}$. The following two statements are equivalent:*

(1) h is continuous with respect to $d^e(\cdot,\cdot)$ $\mathcal{P}^L$-q.s.;

(2) h satisfies (P), and h is continuous with respect to the supermum norm $\mathcal{P}^L$-q.s..

Proof. Without loss of generality, we only show the result for the one-dimensional case.

(1)⇒(2) is trivial, since $d^e(\omega,\tilde{\omega}) \leq \|\omega-\tilde{\omega}\|$.

(2)⇒(1). For $\omega \in \mathcal{Q}$, define the stopping times

$$H_0^n(\omega)=0,\ H_{i+1}^n(\omega)=\inf\left\{t\geq H_i^n : |\omega_t-\omega_{H_i}|\geq \frac{1}{n}\right\}\wedge H_Q, i\geq 0.$$

Set a piecewise constant path:

$$\omega_t^n := \sum_i \omega_{H_i^n} 1_{\{H_i^n\leq t<H_{i+1}^n\}} + \omega_{H_Q} 1_{\{t\geq H_Q\}}.$$

Then by Proposition 3.3, we know that

$$\omega^n \ has\ finite\ jumps\ \mathcal{P}^L-q.s. \tag{14.1}$$

Since now, we fix an $\omega\in\mathcal{Q}$ such that (14.1) holds and h is continuous in $\|\cdot\|$ at point ω.

Let $\epsilon>0$ and $\tilde{\omega}\in\Omega^e$, such that $d^e(\omega,\tilde{\omega})<\epsilon$. According to the definition of $d^e(\cdot,\cdot)$, there is a time scaling function $\phi\in\mathcal{I}$, such that $\left\|\omega_{\phi(\cdot)}-\tilde{\omega}_\cdot\right\|<\epsilon$. Note that

$$\left\|\omega^n_{\phi(\cdot)}-\omega_{\phi(\cdot)}\right\|\leq\frac{1}{n}\ \text{and}\ \|\omega^n_\cdot-\omega_\cdot\|\leq\frac{1}{n}. \tag{14.2}$$

Next, we are going to adjust the paths ω^n and ω^n_ϕ to be continuous. For these two piecewise constant paths, the jumps occur at $\{H_i^n\}_{i\geq 0}$ and at $\left\{\phi^{-1}(H_i^n)\right\}_{i\geq 0}$ respectively. Assume that the shortest constant piece of ω^n and $\omega^n_{\phi(\cdot)}$ has the length of δ, i.e.

$$\delta := \min_{i\geq 0, H_i^n<H_Q}\left\{|H_{i+1}^n-H_i^n|, |\phi^{-1}(H_{i+1}^n)-\phi^{-1}(H_i^n)|\right\}.$$

Then, define two continuous paths:

$$\begin{aligned}\omega_t^{n,c} &:= \omega_{H_Q}1_{\{t\geq H_Q\}}+\sum_i\Big(\omega_{H_i^n}1_{\{H_i^n\leq t<H_{i+1}^n\}}\\ &\quad+\frac{t-H_{i+1}^n+\delta}{\delta}(\omega_{H_{i+1}^n}-\omega_{H_i^n})1_{\{H_{i+1}^n-\delta\leq t\leq H_{i+1}^n\}}\Big),\end{aligned}$$

$$\begin{aligned}\omega_{\phi(t)}^{n,c} &:= \omega_{H_Q}1_{\{t\geq H_Q\}}+\sum_i\Big(\omega_{H_i^n}1_{\{\phi^{-1}(H_i^n)\leq t<\phi^{-1}(H_{i+1}^n)\}}\\ &\quad+\frac{t-\phi^{-1}(H_{i+1}^n)+\delta}{\delta}(\omega_{H_{i+1}^n}-\omega_{H_i^n})1_{\{\phi^{-1}(H_{i+1}^n)-\delta\leq t\leq\phi^{-1}(H_{i+1}^n)\}}\Big).\end{aligned}$$

Ignoring the constant pieces of $\omega^{n,c}$ and $\omega^{n,c}_{\phi(\cdot)}$, we may see that these two paths are exactly the same. Since h satisfies property (P), we have that

$$h(\omega^{n,c}) = h(\omega^{n,c}_{\phi(\cdot)}). \tag{14.3}$$

Recall that we have (14.2). Hence, by continuity of h in $\|\cdot\|$ and (14.3), we know that $h(\omega) = h(\omega_{\phi(\cdot)})$. Finally, we have

$$|h(\omega) - h(\tilde{\omega})| = |h(\omega_{\phi(\cdot)}) - h(\tilde{\omega})|.$$

The continuity in $d^e(\cdot,\cdot)$ at point ω follows. □

15. Estimate on the hitting time

Here, we compare two stopping times H^{ω^1} and H^{ω^2}, where the paths ω^1 and ω^2 are both supposed to be in $\mathcal{Q}$. The aim is to give some estimates of the form:

$$\exists \rho,\ s.t.\ \overline{\mathcal{E}}^L\left[|H^{\omega^1}_Q - H^{\omega^2}_Q|^n\right] \le \rho\left(d^e(\omega^1,\omega^2)\right) \text{ for all } n \ge 1. \tag{15.1}$$

To simplify notations, we put $H^1 := H^{\omega^1}_Q$ and $H^2 := H^{\omega^2}_Q$. First, fix a probability $\mathbb{P}^{0,\beta} \in \mathcal{P}^L$ such that there exists a $\mathbb{P}^{0,\beta}$-Brownian motion W and

$$B_t = \int_0^t \beta_s dW_s,\ \mathbb{P}^{0,\beta}\text{-a.s.}$$

(under $\mathbb{P}^{0,\beta}$ the canonical process is without drift a.s.) We will show that there exists a modulus of continuity ρ such that

$$\mathbb{E}^{\mathbb{P}^{0,\beta}}\left[|H^{\omega^1} - H^{\omega^2}|^n\right] \le \rho\left(d^e(\omega,\tilde{\omega})\right). \tag{15.2}$$

Further, by a simple application of Girsanov theorem, we may prove (15.1). The following discussion is to prove (15.2).

First of all, denote the starts of the flat tails of $\omega^1$and ω^2 as $x^1 := \omega^1_{\bar{t}}$ and $x^2 := \omega^2_{\bar{t}}$ respectively. Since $\omega^1, \omega^2 \in \mathcal{Q}$, it is clear that H^1 and H^2 depend only on x^1 and x^2, respectively. Since the roles of ω^1 and ω^2 are symmetric here, we only consider the case when $H^1 \ge H^2$. In this case the path $\omega^2 \bar{\oplus} B$ touches the boundary ∂Q earlier than $\omega^1 \bar{\oplus} B$. Also, we observe that

$$\omega^1 \bar{\oplus} B_{H^2} - \omega^2 \bar{\oplus} B_{H^2} = x^1 - x^2.$$

Denote by y^1 the point $\omega^1 \bar{\oplus} B_{H^2}$ and by y^2 the point $\omega^2 \bar{\oplus} B_{H^2}$. Notice that $y^2 \in \partial Q$ and $y^1 \in Q$. Furthermore, there exists a point $y^* \in \partial Q$ such that

$$\left|y^1 - y^*\right| = \min_{x \in \partial Q}\left|y^1 - x\right| \le \left|x^1 - x^2\right|.$$

From now on, we are interested in the one-dimensional subspace $\mathbb{R}_1$, the line passing through y^1 and y^*. We take the segment

$$l := \text{proj}_{\mathbb{R}_1}(Q)$$

to be the projection of Q on $\mathbb{R}_1$. Then we observe that $y^1 \in l$ and that y^* is one of the end points of l. Denote by y_* the other end point of l. Since Q is bounded (i.e. there exists a constant M such that $|x| \le M$ for any $x \in Q$), we deduce that

$$\left|y^1 - y_*\right| \le 2M.$$

Further we define the projection of the canonical process on $\mathbb{R}_1$:

$$B^l_t = \text{proj}_{\mathbb{R}_1}\left(y^1 + B_t\right).$$

It is easy to show that B^l is a one-dimensional martingale departing from y^1, whose quadratic variation is equal to $\int_0^{\cdot} \text{trace}(\beta_s^2)ds$. Observe that

$$H^1 - H^2 \leq \tau^l,$$

where $\tau^l := \inf\left\{t : B_t^l \notin l\right\}$.

Now, set a probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$. Let X be a 1-dimensional local martingale starting from 0 with the quadratic variation $\int_0^{\cdot} \text{trace}(\beta_s^2)ds$ under $\mathbb{P}$. Define the stopping time:

$$\eta_{\delta x,-2M} = \eta_{\delta x} \wedge \eta_{-2M},$$

where $\delta x := |x^1 - x^2|$, $\eta_{\delta x} := \inf\{t : X(t) > \delta x\}$ and $\eta_{-2M} := \inf\{t : X(t) < -2M\}$. In particular, note that

$$\delta x \leq d^e(\omega^1, \omega^2).$$

Finally, we have

$$\mathbb{E}^{\mathbb{P}^{0,\beta}}\left[|\tau^l|^n\right] \leq \mathbb{E}\left[\eta_{\delta x,-2M}^n\right].$$

Consequently, we will focus on estimating $\mathbb{E}\left[\eta_{\delta x,-2M}^n\right]$. The following lemma leads to (15.2).

Lemma 15.1. *Suppose that $n \geq 1$ and $\delta x \leq 2M$, then $\mathbb{E}\left[\eta_{\delta x,-2M}^n\right] \leq C\delta x$. In particular, C does not depend on β.*

Proof. Denote $X_t^* := \max_{s\leq t}|X_s|$, $\overline{X}_t := \max_{s\leq t} X_s^+$ and $\underline{X}(t) := \max_{s\leq t} X_s^-$.

Make use of the Burkholder-Davis-Gundy inequality:

$$\mathbb{E}\left[\eta_{\delta x,-2M}^n\right] \leq \left(\frac{Ld}{2}\right)^n \mathbb{E}\left[\left(\int_0^{\eta_{\delta x,-2M}} \beta_s^2 ds\right)^n\right] \leq C\mathbb{E}\left[\left(X_{\eta_{\delta x,-2M}}^*\right)^{2n}\right],$$

since $\beta \geq \sqrt{\frac{2}{L}} I_d$. Then observe

$$\begin{aligned}\mathbb{E}\left[\left(X_{\eta_{\delta x,-2M}}^*\right)^{2n}; \eta_{\delta x} \geq \eta_{-2M}\right] &= (2M)^{2n}\mathbb{P}\left[\eta_{\delta x} \geq \eta_{-2M}\right] \\ &= C\frac{\delta x}{\delta x + 2M} \leq \frac{C}{2M}\delta x.\end{aligned}$$

On the other hand, we can estimate:

$$\begin{aligned}\mathbb{E}\left[\left(X_{\eta_{\delta x,-2M}}^*\right)^{2n}; \eta_{\delta x} < \eta_{-2M}\right] &\leq \mathbb{E}\left[\overline{X}_{\eta_{\delta x,-2M}}^{2n}\right] + \mathbb{E}\left[\underline{X}_{\eta_{\delta x,-2M}}^{2n}; \eta_{\delta x} < \eta_{-2M}\right] \\ &\leq (\delta x)^{2n} + \mathbb{E}\left[\underline{X}_{\eta_{\delta x,-2M}}^{2n}; \eta_{\delta x} < \eta_{-2M}\right].\end{aligned}$$

To estimate the second item on the right side, we note that it is dominated by the following sum:

$$\begin{aligned}&\frac{2M}{N}\sum_{k=1}^N \mathbb{E}\left[2n\left(2M\frac{k}{N}\right)^{2n-1}; \{\eta_{\delta x} < \eta_{-2M}\} \cap \left\{\underline{X}(\eta_{\delta x,-2M}^{2n}) \geq 2M\frac{k-1}{N}\right\}\right] \\ &\qquad \leq \frac{2M}{N}\sum_{k=1}^N \mathbb{E}\left[2n\left(2M\frac{k}{N}\right)^{2n-1}; \left\{\eta_{\delta x} < \eta_{-2M\frac{k}{N}}\right\}\right] \\ &\qquad = \frac{2M}{N}\sum_{k=1}^N 2n\left(2M\frac{k}{N}\right)^{2n-1}\frac{\delta x}{\delta x + 2M\frac{k}{N}}.\end{aligned}$$

Let $N \to +\infty$, the sum converges to the following integral.

$$I := \int_0^{2M} 2nt^{2n-1} \frac{\delta x}{\delta x + t} dt.$$

Now, since $n \geq 1$, we have

$$I \leq \delta x \int_0^{2M} 2nt^{2n-2} dt = C\delta x.$$

□

Corollary 15.2. *For any $n \geq 1$, there exists a constant C such that*

$$\mathbb{E}^{\mathbb{P}^{0,\beta}} \left[|H_Q^{\omega^1} - H_Q^{\omega^2}|^n \right] \leq C d^e(\omega^1, \omega^2).$$

Furthermore, there exists a modulus of continuity ρ such that

$$\overline{\mathcal{E}}^L \left[|H_Q^{\omega^1} - H_Q^{\omega^2}|^n \right] \leq \rho \left(d^e(\omega^1, \omega^2) \right).$$

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