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On n -dependent groups and fields

Nadja Hempel*

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Abstract

First, an example of a 2-dependent group without a minimal subgroup of bounded index is given. Second, all infinite n -dependent fields are shown to be Artin-Schreier closed. Furthermore, the theory of any non separably closed PAC field has the IP_n property for all natural numbers n and certain properties of dependent (NIP) valued fields extend to the n -dependent context.

1 INTRODUCTION

Superstable fields are algebraically closed (Macintyre [16] and Cherlin-Shelah [5]). Less is known for supersimple fields. Hrushovski showed that any infinite perfect bounded pseudo-algebraically closed (PAC) field is supersimple [12], conversely supersimple fields are perfect and bounded (Pillay and Poizat [17]), and it is conjectured that they are PAC. More is known about Artin-Schreier extensions of certain fields. Using a suitable chain condition for uniformly definable subgroups, Kaplan, Scanlon and Wagner showed in [14] that NIP fields of positive characteristic are Artin-Schreier closed and simple fields have only finitely many Artin-Schreier extensions. The latter result was generalized to fields of positive characteristic defined in a theory without the tree property of the second kind (NTP₂ fields) by Chernikov, Kaplan and Simon [7].

We study groups and fields without the n -independence property. Theories without the n -independence property, briefly n -dependent or NIP_n theories, were induced by Shelah in [19] and are defined as follows:

Definition 1.1. A theory has the n -independence property (IP_n) if there exists a formula $\psi(x_0, \dots, x_{n-1}; y)$ and some parameters $(a_i^j : i \in \omega, j \in n)$ and $(b_I : I \subset \omega^n)$ such that $\models \psi(a_{i_0}^0, \dots, a_{i_{n-1}}^{n-1}, b_I)$ if and only if $(i_0, \dots, i_{n-1}) \in I$.

A theory is called n -dependent if it does not have the IP_n .

They are a natural generalization of NIP theories, and in fact both notions coincide when n equals to 1. For background on NIP theories the reader may consult [22]. It is easy to see that any theory with the $(n+1)$ -independence property has the n -independence property. On the other hand, as for any natural number n the random $(n+1)$ -hypergraph

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is $n + 1$ -dependent but has the n -independence property, the classes of n -dependent theories form a proper hierarchy of classes. Additionally, since the random graph is simple, the previous example shows that there are simple unstable n -dependent theories for n greater than 1. Hence one might ask if there are any non combinatorial examples of n -dependent theories which have the independence property? And furthermore, which results of NIP theories can be generalized to n -dependent theories or more specifically which results of (super)stable theories remains true for (super)simple n -dependent theories? Beyarslan [2] constructed the random n -hypergraph in any pseudo-finite field or, more generally, in any e -free perfect PAC field (PAC fields whose absolute Galois group is the profinite completion of the free group on e generators). Thus, those fields lie outside of the hierarchy of n -dependent fields.

In this paper, we first give an example of a group with a simple 2-dependent theory which has the independence property. Additionally, in this group the A -connected component does depend on the parameter set A . This establish on the hand a non combinatorial example of a proper 2-dependent theory and on the other hand shows that the existence of an absolute connected component in any NIP group cannot be generalized to 2-dependent groups. Secondly, we find a Baldwin-Saxl condition for n -dependent groups (Section 3). Using this and connectivity of a certain vector group established in Section 4 we deduce (Section 5) that n -dependent fields are Artin-Schreier closed. Furthermore, we show in Section 6 that the theory of any non separably closed PAC field has in fact the IP_n property for all n which was establish by Duret for the case n equals to 1 [8]. In Section 7 we extend certain consequences found in [7] for strongly dependent valued fields as well as in [13] by Jahnke and Koenigsmann for NIP henselian valued field to the n -dependent context.

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2 EXAMPLE OF A 2-DEPENDENT GROUP WITHOUT A MINIMAL SUBGROUP OF BOUNDED INDEX

Let G be $\mathbb{F}_p^{(\omega)}$ where \mathbb{F}_p is the finite field with p elements. We consider the structure \mathcal{M} defined as $(G, \mathbb{F}_p, 0, +, \cdot)$ where 0 is the neutral element, $+$ is addition in G , and \cdot is the bilinear form $(a_i)_i \cdot (b_i)_i = \sum_i a_i b_i$ from G to \mathbb{F}_p . This example in the case p equals 2 has been studied by Wagner in [23, Example 4.1.14]. He shows that it is simple and that the connected component G_A^0 for any parameter set A is equal to $\{g \in G : \bigcap_{a \in A} g \cdot a = 0\}$. Hence, it's getting smaller and smaller while enlarging A and whence the absolute connected component G^0 does not exists which holds in any NIP group.

Lemma 2.1. *\mathcal{M} eliminates quantifiers.*

Proof. A quantifier free formula $\varphi(x, \bar{y})$ is a finite disjunction of formulas of the form

$$\phi(x; \bar{y}) = \psi(\bar{y}) \wedge x \cdot x = \epsilon \wedge \bigwedge_{i \in I_0} x = t_i^0(\bar{y}) \wedge \bigwedge_{i \in I_1} x \neq t_i^1(\bar{y}) \wedge \bigwedge_{i \in I_2} x \cdot t_i^2(\bar{y}) = \epsilon_i$$

where $t_i^j(\bar{y})$ are group terms and ϵ, ϵ_i are elements of \mathbb{F}_p . If I_0 is nonempty, the formula $\exists x \varphi(x, \bar{y})$ is equivalent to

$$\bigwedge_{j, l \in I_0} t_j^0(\bar{y}) = t_l^0(\bar{y}) \wedge \psi(\bar{y}) \wedge t_i^0(\bar{y}) \cdot t_i^0(\bar{y}) = \epsilon \wedge \bigwedge_{j \in I_1} t_i^0(\bar{y}) \neq t_j^1(\bar{y}) \wedge \bigwedge_{j \in I_2} t_i^0(\bar{y}) \cdot t_j^2(\bar{y}) = \epsilon_j$$

for any $i \in I_0$. Now, we assume that I_0 is the empty set. If there exists an element x' such that $x' \cdot z_i = \epsilon_i$ for given z_0, \dots, z_m in G and $\epsilon_i \in \mathbb{F}_p$, one can always find an element x such that $x \cdot x = \epsilon$ and $x \neq v_j$ for given v_0, \dots, v_q in G which still satisfies $x \cdot z_i = \epsilon_i$ by modifying x' at a large enough coordinate. Hence, it is enough to find a quantifier free condition which is equivalent to $\exists x \bigwedge_{i \in I_2} x \cdot t_i^2(\bar{y}) = \epsilon_i$. For $i \in \mathbb{F}_p$, let

$$Y_i = \{j \in I_2 : \epsilon_j = i\}.$$

Then $\exists x \bigwedge_{i \in I_2} x \cdot t_i^2(\bar{y}) = \epsilon_i$ is equivalent to

$$\bigwedge_{i=0}^{p-1} \bigwedge_{j \in Y_i} t_j^2(\bar{y}) \notin \left\{ \sum_{k \in Y_0} \lambda_k^0 t_k^2(\bar{y}) + \dots + \sum_{k \in Y_i \setminus j} \lambda_k^{i-1} t_k^2(\bar{y}) : \lambda_k^l \in \mathbb{F}_p, \sum_{l=1}^i \sum_{k \in Y_l} l \cdot_{\mathbb{F}_p} \lambda_k^l \neq i \right\}$$

which finishes the proof. \square

Remark 2.2. If the formula $\varphi(x, \bar{y}, \bar{z})$ has the 2-independence property then any formula $\psi(x, \bar{y}, \bar{z})$ such that for any tuple \bar{a} and \bar{b} , the cardinality of the realizations of the symmetric difference of $\varphi(x, \bar{a}, \bar{b})$ and $\psi(x, \bar{a}, \bar{b})$ is finite, has it as well.

Lemma 2.3. *The structure \mathcal{M} is 2-dependent.*

Proof. Towards a contradiction, we suppose that there exists a formula $\phi(x, \bar{y}, \bar{z})$ with $|x| = 1$ together with $(\bar{a}_i : i \in \omega)$ and $(\bar{b}_j : j \in \omega)$ witnessing the 2-independence property.

By Remark 2.2 and an inspection of the formulas we may assume that $\phi(x, \bar{y}, \bar{z})$ is of the following form:

$$\phi(x; \bar{y}, \bar{z}) = \bigvee_{m=1}^n \left[\bigwedge_{\mu=1}^{n_m} [x \cdot (t_{m,\mu}(\bar{y}, \bar{z})) = \epsilon_{m,\mu}] \wedge x \cdot x = \epsilon_m \right]$$

As $t_{m,\mu}(\bar{y}, \bar{z})$ is just a sum of the y_i 's and z_j 's we may write this formula as follows

$$\phi(x; \bar{y}, \bar{z}) = \bigvee_{m=1}^n \left[\bigwedge_{\mu=1}^{n_m} [x \cdot (t_{m,\mu}^a(\bar{y}) + t_{m,\mu}^b(\bar{z})) = \epsilon_{m,\mu}] \wedge x \cdot x = \epsilon_m \right]$$

in which the term $t_{m,\mu}^a(\bar{y})$ (resp. $t_{m,\mu}^b(\bar{z})$) is a sum of the element of the tuple \bar{y} (resp. \bar{z}). Let

$$S_m^{ij} := \left\{ x : \bigwedge_{\mu=1}^{n_m} x \cdot (t_{m,\mu}^a(\bar{a}_i) + t_{m,\mu}^b(\bar{b}_j)) = \epsilon_{m,\mu} \text{ and } x \cdot x = \epsilon_m \right\}$$

and note that the set of realisations of $\phi(x; \bar{a}_i, \bar{b}_j)$ is equal to $\bigcup_{m=1}^n S_m^{ij}$. For any m less or equal to n , an element c belongs to S_m^{ij} if and only if for all $\mu \leq n_m$, we have

$$e_{m,\mu}^{ij}(c) = c \cdot \left(t_{m,\mu}^a(\bar{a}_i) + t_{m,\mu}^b(\bar{b}_j) \right) = \epsilon_{m,\mu} \text{ and } c \cdot c = \epsilon_m.$$

Let i, l, j , and k be arbitrary natural numbers.

$$\begin{aligned} e_{m,\mu}^{ij}(c) &= c \cdot \left(t_{m,\mu}^a(\bar{a}_i) + t_{m,\mu}^b(\bar{b}_j) \right) \\ &= c \cdot \left((t_{m,\mu}^a(\bar{a}_i) + t_{m,\mu}^b(\bar{b}_k)) + (p-1)(t_{m,\mu}^a(\bar{a}_l) + t_{m,\mu}^b(\bar{b}_k)) + (t_{m,\mu}^a(\bar{a}_l) + t_{m,\mu}^b(\bar{b}_j)) \right) \\ &= e_{m,\mu}^{ik}(c) + (p-1)e_{m,\mu}^{lk}(c) + e_{m,\mu}^{lj}(c) \end{aligned}$$

If the element c belongs to $S_{m,\mu}^{ik} \cap S_{m,\mu}^{lk} \cap S_{m,\mu}^{lj}$, the terms $e_{m,\mu}^{ik}$, $e_{m,\mu}^{lk}(c)$, and $e_{m,\mu}^{lj}(c)$ are all equal to $\epsilon_{m,\mu}$ and $c \cdot c = \epsilon_m$. By the equality above we get that $e_{m,\mu}^{ij}(c)$ is also equals to $\epsilon_{m,\mu}$ and so c also belongs to $S_{m,\mu}^{ij}$.

Let $I = \{(i, j) \subset \omega^2 : i \leq j\}$. Using the hypothesis that the formula $\phi(x, \bar{y}, \bar{z})$ together with $(a_i : i \in \omega)$ and $(b_j : j \in \omega)$ witnesses the 2-independence property, we can choose an element c_I such that $\phi(c_I; a_i, b_j)$ holds if and only if (i, j) is in I .

By omitting certain a_i 's and b_j 's, we may choose $\{m_r : r \leq n+1\}$ such for r less or equal to $n+1$, the element $c_I \in S_{m_r, \mu}^{rs}$ if and only if $s \geq r$. Thus there exists $i < j \leq n+1$ such that $m_i = m_j = m$. Then $c_I \in S_{m,\mu}^{i,i} \cap S_{m,\mu}^{i,j} \cap S_{m,\mu}^{j,j}$ but it does not belong to $S_{m,\mu}^{j,i}$ which yields the desired contradiction and whence \mathcal{M} is 2-dependent. \square

3 BALDWIN-SAXL CONDITION FOR NIP_n THEORIES

We shall now prove a suitable version of the Baldwin-Saxl condition for n -dependent theories. By a *subarray* I of ω^n of size at least m^n , we mean that I contains a set $I_0 \times \cdots \times I_{n-1}$ with $I_j \subset \omega$ and $|I_j| \geq m$ for $0 \leq j < n$.

Proposition 3.1. *Fix a group G defined in an n -dependent theory, an array of parameters $(a_{i,j} : i < n, j < \omega)$ and a formula $\psi(x; y_0, \dots, y_{n-1})$. We suppose that*

$$\{H_\eta := \psi(G; a_{0,i_0}, \dots, a_{n-1,i_{n-1}}) : \eta = (i_0, \dots, i_{n-1}) \in \omega^n\}$$

is a family of uniformly definable subgroups of G . Then there exists a natural number m such that for every subarray $I \subseteq \omega^n$ of size at least m^n there is $\nu \in m^n$ such that

$$\bigcap_{\eta \in I} H_\eta = \bigcap_{\eta \in I, \eta \neq \nu} H_\eta.$$

Proof. Suppose, towards a contradiction, that for arbitrarily large m there is a subarray $I \subseteq \omega^n$ of size m^n such that $\bigcap_{\eta \in I} H_\eta$ is strictly contained in any of its proper subintersections. Hence, for every $\nu \in I$ there is $c_\nu \in \bigcap_{\eta \neq \nu} H_\eta \setminus \bigcap_{\eta} H_\eta$.

Now, for every subset J of I , we let $c_J := \prod_{\eta \in J} c_\eta$ (multiplied in lexicographical order). Note that $c_J \in H_\nu$ whenever $\nu \notin J$. On the other hand, if $\nu \in J$, all factors of the product except c_ν belong to H_ν , whence $c_J \notin H_\nu$. By compactness, this formula $\psi(x; y_0, \dots, y_{n-1})$ has the IP_n property contradicting the assumption. \square

4 A SPECIAL VECTOR GROUP

For this section, we fix an algebraically closed field \mathbb{K} of characteristic $p > 0$ and we let $\wp(x)$ be the additive homomorphism $x \mapsto x^p - x$ on \mathbb{K} .

We analyze the following algebraic subgroups of $(\mathbb{K}, +)^n$:

Definition 4.1. For a singleton a in \mathbb{K} , we let G_a be equal to $(\mathbb{K}, +)$, and for a tuple $\bar{a} = (a_0, \dots, a_{n-1}) \in \mathbb{K}^n$ with $n > 1$ we define:

$$G_{\bar{a}} = \{(x_0, \dots, x_{n-1}) \in \mathbb{K}^n \mid a_0 \cdot \wp(x_0) = a_i \cdot \wp(x_i) \text{ for } 0 \leq i < n\}.$$

Our aim is to show that $G_{\bar{a}}$ is connected for certain choices of \bar{a} .

Lemma 4.2. *Let k be an algebraically closed subfield of \mathbb{K} , the group G be a k -definable connected algebraic subgroup of $(\mathbb{K}^n, +)$ and f be a k -definable algebraic homomorphism from G to $(\mathbb{K}, +)$ which is locally represented by rational functions. Then f is an additive polynomial in $k[X_0, \dots, X_{n-1}]$. In fact, it is of the form $\sum_{i=0}^{m_0} a_{i,0}x_0^{p^i} + \dots + \sum_{i=0}^{m_n} a_{i,n}x_n^{p^i}$ with coefficients $a_{i,j}$ in k .*

Proof. By compactness, one can find finitely many definable subsets D_i of G such that f is represented by a rational function on D_i . Using [3, Lemma 3.8] we can extend f to a k -definable homomorphism $F : (\mathbb{K}^n, +) \rightarrow (\mathbb{K}, +)$ which is also locally rational. Now, the functions

$$F_0(x) := F(x, 0, \dots, 0), \dots, F_{n-1}(x) := F(0, \dots, 0, x)$$

are k -definable homomorphisms of $(\mathbb{K}, +)$ to itself. Additionally, they are rational on a finite definable decomposition of \mathbb{K} . Hence every F_i is an additive polynomial in $k[X]$. Thus

$$F(X_0, \dots, X_{n-1}) = F_0(X_0) + \dots + F_{n-1}(X_{n-1})$$

is an additive polynomial in $k[X_0, \dots, X_{n-1}]$ as it is a sum of additive polynomials and by [10, Proposition 1.1.5] it is of the desired form. \square

Lemma 4.3. *Let $\bar{a} = (a_0, \dots, a_n)$ be a tuple in \mathbb{K}^\times for which the set $\left\{\frac{1}{a_0}, \dots, \frac{1}{a_n}\right\}$ is linearly \mathbb{F}_p -independent. Then $G_{\bar{a}}$ is connected.*

The beginning of the proof follows the one of [14, Lemma 2.8].

Proof. We prove this lemma by induction on the length of the tuple \bar{a} which we denote by n . Let $n = 1$, then $G_{\bar{a}}$ is equal to $(\mathbb{K}, +)$ and thus connected since the additive group of an algebraically closed field is always connected.

Let \bar{a} be an $(n+1)$ -tuple such that $\left\{\frac{1}{a_0}, \dots, \frac{1}{a_n}\right\}$ is linearly \mathbb{F}_p -independent and suppose that the lemma holds for tuples of length n . Define \bar{a}' to be the restriction of \bar{a} to the first n coordinates. Observe that the natural map $\pi : G_{\bar{a}} \rightarrow G_{\bar{a}'}$ is surjective since \mathbb{K} is algebraically closed and that

$$[G_{\bar{a}'} : \pi(G_{\bar{a}}^0)] = [\pi(G_{\bar{a}}) : \pi(G_{\bar{a}}^0)] \leq [G_{\bar{a}} : G_{\bar{a}}^0] < \infty.$$

Hence the definable group $\pi(G_{\bar{a}}^0)$ has finite index in $G_{\bar{a}'}$. As $\left\{\frac{1}{a_0}, \dots, \frac{1}{a_{n-1}}\right\}$ is also linearly \mathbb{F}_p -independent, the group $G_{\bar{a}'}$ is connected by assumption. Therefore $\pi(G_{\bar{a}}^0) = G_{\bar{a}'}$.

Now, suppose that $G_{\bar{a}}$ is not connected.

Claim. *For every $\bar{x} \in G_{\bar{a}'}$, there exists a unique $x_n \in \mathbb{K}$ such that $(\bar{x}, x_n) \in G_{\bar{a}}^0$.*

Proof of the Claim. Assume there exists $\bar{x} \in \mathbb{K}^{n+1}$ and two distinct elements x_n^0 and x_n^1 of \mathbb{K} such that (\bar{x}, x_n^0) and (\bar{x}, x_n^1) are elements of $G_{\bar{a}}^0$. As $G_{\bar{a}}^0$ is a group, their difference $(\bar{0}, x_n^0 - x_n^1)$ belongs also to $G_{\bar{a}}^0$. Thus, by definition of $G_{\bar{a}}$, its last coordinate $x_n^0 - x_n^1$ lies in \mathbb{F}_p . So $(\bar{0}, \mathbb{F}_p)$ is a subgroup of $G_{\bar{a}}^0$. Take an arbitrary element (\bar{x}, x_n) in $G_{\bar{a}}$. As $\pi(G_{\bar{a}}^0) = G_{\bar{a}'}$, there exists $x_n' \in \mathbb{K}$ with $(\bar{x}, x_n') \in G_{\bar{a}}^0$. Again, the difference of the last coordinate $x_n' - x_n$ lies in \mathbb{F}_p . So

$$(\bar{x}, x_n) = (\bar{x}, x_n') - (\bar{0}, x_n' - x_n) \in G_{\bar{a}}^0.$$

This leads to a contradiction, as $G_{\bar{a}}^0$ is assumed to be a proper subgroup of $G_{\bar{a}}$. \square

Thus, we can fix a function $f : G_{\bar{a}'} \rightarrow \mathbb{K}$ that sends every tuple to this unique element. Note that $G_{\bar{a}}$ is defined over \bar{a} , hence $G_{\bar{a}}^0$ is defined over \bar{a} , as is f . Now, let $\bar{x} = (x_0, \dots, x_{n-1})$ be any tuple in $G_{\bar{a}}^0$. Set $L := \mathbb{F}_p(a_0, \dots, a_n)$. Then:

$$x_n := f(\bar{x}) \in \text{dcl}(\bar{a}, \bar{x})$$

In other words, x_n is definable over $L(x_0, \dots, x_{n-1})$ which simply means that it belongs to the inseparable closure of $L(x_0, \dots, x_{n-1})$. Since there exists an $l \in L(x_0)$ such that $x_n^p - x_n - a_n^{-1}l = 0$, the element x_n is separable over $L(x_0, \dots, x_{n-1})$. So it belongs to $L(x_0, \dots, x_{n-1})$ which implies that there exists some mutually prime polynomials $g, h \in L[X_0, \dots, X_n]$ such that $x_n = h(x_0, \dots, x_n)/g(x_0, \dots, x_n)$. Thus, by Lemma 4.2 the definable function $f(X_0, \dots, X_{n-1})$ we started with is an additive polynomial in n variables over $\mathbb{F}_p(a_0, \dots, a_n)^{\text{alg}}$ and there exists $c_{i,j}$'s in $\mathbb{F}_p(a_0, \dots, a_n)^{\text{alg}}$ such that

$$f(X_0, \dots, X_{n-1}) = \sum_{i=0}^{m_0} c_{0,i} X_0^{p^i} + \dots + \sum_{i=0}^{m_{n-1}} c_{n-1,i} X_{n-1}^{p^i}.$$

Using the identities $X_i^p - X_i = \frac{a_0}{a_i}(X_0^p - X_0)$ in $G_{\bar{a}}^0$, the function f can be rewritten as follows:

$$f(X_0, \dots, X_{n-1}) = g(X_0) + \sum_{j=0}^{n-1} \beta_j \cdot X_j$$

with $g(X_0) = \sum_{i=1}^{m_0} d_i X_0^{p^i}$ an additive polynomial in $\mathbb{F}_p(a_0, \dots, a_n)[X_0]$ with summands of powers of X_0 higher or equal to p . Since the image under f of any unitary vector of \mathbb{K}^n has to be in \mathbb{F}_p , for $0 < i < n$ the β_i 's have to be elements of \mathbb{F}_p . On the other hand, for any element (x_0, \dots, x_n) of $G_{\bar{a}}^0$ we have $a_n(x_n^p - x_n) = a_0(x_0^p - x_0)$. Replacing x_n by $f(x_0, \dots, x_{n-1})$ we obtain

$$\begin{aligned} 0 &= a_n [f(x_0, \dots, x_{n-1})^p - f(x_0, \dots, x_{n-1})] - a_0(x_0^p - x_0) \\ &= a_n \left[g(x_0)^p - g(x_0) + (\beta_0^p x_0^p - \beta_0 x_0) + \sum_{j=1}^{n-1} \beta_j (x_j^p - x_j) \right] - a_0(x_0^p - x_0). \end{aligned}$$

Using again the identities $x_i^p - x_i = \frac{a_0}{a_i}(x_0^p - x_0)$ in $G_{\bar{a}}^0$ we obtain a polynomial in one variable

$$P(X) = a_n \left[g(X)^p - g(X) + (\beta_0^p X^p - \beta_0 X) + \sum_{j=1}^{n-1} \beta_j \frac{a_0}{a_j} (X^p - X) \right] - a_0 (X^p - X)$$

which vanishes for all elements x_0 of \mathbb{K} such that there exists x_1, \dots, x_{n-1} in \mathbb{K} with $(x_0, \dots, x_{n-1}) \in G_{\bar{a}}$, these are all elements of \mathbb{K} . Hence, P is the zero polynomial. Notice that $g(X)$ appears in a p th power. Since it contains only summands of power of X higher or equal to p , the polynomial $g(X)^p$ contains only summands of power of X higher than p . As X only appears in powers less or equal to p in all other summand of P , the polynomial $g(X)$ has to be the zero polynomial itself. By the same argument as for the other β_j , the coefficient β_0 has to belong to \mathbb{F}_p as well. Dividing by $a_0 a_n$ yields

$$\sum_{j=0}^n \beta_j \frac{1}{a_j} (X^p - X)$$

with $\beta_n := -1$ is the zero polynomial. Thus

$$\sum_{j=0}^n \beta_j \frac{1}{a_j} = 0$$

As β_n is different from 0 and all β_i are elements of \mathbb{F}_p , this contradicts the assumption and the lemma is established. \square

Using Lemma 4.3, a stronger version of [14, Lemma 2.8] together with [14, Corollary 2.6], we obtain the following corollary in the same way as Kaplan, Scanlon and Wagner obtain [14, Corollary 2.9].

Corollary 4.4. *Let k be a perfect subfield of \mathbb{K} and $\bar{a} \in k^n$ be as in the previous lemma. Then $G_{\bar{a}}$ is isomorphic over k to $(\mathbb{K}, +)$. In particular, for any field $K \geq k$ with $K \leq \mathbb{K}$, the group $G_{\bar{a}}(K)$ is isomorphic to $(K, +)$.*

5 ARTIN-SCHREIER EXTENSIONS

Definition 5.1. Let K be a field of characteristic $p > 0$ and $\wp(x)$ the additive homomorphism $x \mapsto x^p - x$. A field extension L/K is called an *Artin-Schreier extension* if $L = K(a)$ with $\wp(a) \in K$. We say that K is *Artin-Schreier closed* if it has no proper Artin-Schreier extension i. e. $\wp(K) = K$.

In the following Remark, we produce elements from an algebraically independent array of size m^n which fit the condition of Lemma 4.3.

Remark 5.2. Let $\{\alpha_{i,j} : i \in n, j \in m\}$ be a set of algebraically independent elements in \mathbb{K} . Then the tuple $(a_{(i_0, \dots, i_{n-1})} : (i_0, \dots, i_{n-1}) \in m^n)$ with $a_{(i_0, \dots, i_{n-1})} = \prod_{l=0}^{n-1} \alpha_{l, i_l}$ and ordered lexicographically satisfies the condition of Lemma 4.3.

Proof. Suppose that there exists a tuple of elements $(\beta_{(i_0, \dots, i_{n-1})} : (i_0, \dots, i_{n-1}) \in m^n)$ in \mathbb{F}_p not all equal to zero such that

$$\sum_{(i_0, \dots, i_{n-1}) \in m^n} \beta_{(i_0, \dots, i_{n-1})} \frac{1}{a_{(i_0, \dots, i_{n-1})}} = 0$$

Then the $\alpha_{i,j}$ satisfy:

$$\sum_{(i_0, \dots, i_{n-1}) \in m^n} \beta_{(i_0, \dots, i_{n-1})} \cdot \left(\prod_{\{(k,l) \neq (j,i_j) : j \leq n-1\}} \alpha_{k,l} \right) = 0$$

which contradicts the algebraic independence of the $\alpha_{i,j}$. \square

We can now follow the proof in [14] that an infinite NIP field is Artin-Schreier closed to obtain the same result for a n -dependent field.

Theorem 5.3. *Any infinite n -dependent field is Artin-Schreier closed.*

Proof. Let K be an infinite n -dependent field that we may assume to be \aleph_0 -saturated. We work in a big algebraically closed field \mathbb{K} that contains all objects we will consider. Let $k = \bigcap_{l \in \omega} K^{p^l}$, which is a type-definable infinite perfect subfield of K . We consider the formula $\psi(x; y_0, \dots, y_{n-1}) := \exists t (x = \prod_{i=0}^{n-1} y_i \cdot \wp(t))$ which for every tuple (a_0, \dots, a_{n-1}) in k^n defines an additive subgroup of $(K, +)$. Let $m \in \omega$ be the natural number given by Proposition 3.1 for this formula. Now, we fix an array of size m^n of algebraically independent elements $\{\alpha_{i,j} : i \in n, j \in m\}$ and set $a_{(i_0, \dots, i_{n-1})} = \prod_{l=0}^n \alpha_{l,i_l}$. By choice of m , there exists $(j_0, \dots, j_{n-1}) \in m^n$ such that

$$\bigcap_{(i_0, \dots, i_{n-1}) \in m^n} a_{(i_0, \dots, i_{n-1})} \cdot \wp(K) = \bigcap_{(i_0, \dots, i_{n-1}) \neq (j_0, \dots, j_{n-1})} a_{(i_0, \dots, i_{n-1})} \cdot \wp(K). (*)$$

By reordering the elements, we may assume that $(j_0, \dots, j_{n-1}) = (m, \dots, m)$. Let \bar{a} be the tuple $(a_{(i_0, \dots, i_{n-1})} : (i_0, \dots, i_{n-1}) \in m^n)$ ordered lexicographically and \bar{a}' the restriction to the $m^n - 1$ coordinates (one coordinate less).

We consider the groups $G_{\bar{a}}$ and respectively $G_{\bar{a}'}$ defined as in Definition 4.1. Using Remark 5.2 and Corollary 4.4 we obtain the following commuting diagram.

$$\begin{array}{ccc} G_{\bar{a}} & \xrightarrow{\pi} & G_{\bar{a}'} \\ \downarrow \simeq & & \downarrow \simeq \\ (\mathbb{K}, +) & \xrightarrow{\rho} & (\mathbb{K}, +) \end{array}$$

It can be restricted to K . Note that π , whence ρ stays onto for this restriction by $(*)$. Using the fact that the size of $\ker(\rho)$ has to be p , we may assume that its kernel is \mathbb{F}_p . Then [14, Remark 4.2] ensures that ρ is of the form $a \cdot (x^p - x)^{p^n}$. Finally, let $l \in K$ be arbitrary. Since $\rho \upharpoonright K$ is onto and X^{p^n} is an inseparable polynomial in characteristic p , there exists $h \in K$ with $l = h^p - h$. As $l \in K$ was arbitrary, we get that $\wp(K) = K$ and we can conclude. \square

The proof of [14, Corollary 4.4] adapts immediately and yields the following corollary.

Corollary 5.4. *If K is an infinite n -dependent field of characteristic $p > 0$ and L/K is a finite separable extension, then p does not divide $[L : K]$.*

6 NON SEPARABLY CLOSED PAC FIELD

The goal of this section is to generalize a result of Duret [8], namely that the theory of a non separably closed PAC field has the IP property. To do so we need the following fact.

Fact 6.1. [8, Lemme 6.2] *Let K be a PAC field and let p be a prime number which does not coincide with the characteristic of K such that K contains all p th roots of unity and there exists an element in K that doesn't have a p th root in K . Let $(a_i : i \in \omega)$ be a set of pairwise different elements of K , I and J finite disjoint subsets of ω , then there exists k in K such that*

$$\{\exists x(x^p = k + a_i) : i \in I\} \cup \{\neg(\exists x(x^p = k + a_j)) : j \in J\}$$

Theorem 6.2. *Let K be a field and k be a subfield of K that is a non separably closed PAC field and relatively closed in K . Then, the theory of K has the n -independence property.*

Proof. Assume, as we may, that K is \aleph_1 -saturated. As in the proof of Duret [8, Théorème 6.4] we may pass to a separable extension k' of k which admits a radical extension l . We pick α such that $k' = k(\alpha)$ and let $K' = K(\alpha)$. We may distinguish two cases:

- (1) l is an Artin-Schreier extension of k . As k' is separable over k , it is algebraically closed in K' by [15, p.59]. Hence K' admits an Artin-Schreier extension and consequently its theory has the IP_n property. As it is an algebraic extension of K , thus interpretable in K , $\text{Th}(K)$ has IP_n as well.
- (2) l is a Kummer extension of k' of degree p . Let $\{a_{i,j} : j < n, i \in \omega\}$ be a set of algebraically independent elements of K' which exists by saturation of K . This ensures that $\prod_{l=0}^{n-1} a_{i_l,l} \neq \prod_{l=0}^{n-1} a_{j_l,l}$ if $(i_0, \dots, i_{n-1}) \neq (j_0, \dots, j_{n-1})$. Thus we may apply Fact 6.1 to the infinite set $\{\prod_{l=0}^{n-1} a_{i_l,l} : (i_0, \dots, i_{n-1}) \in \mathbb{N}^n\}$ and deduce that for the formula $\varphi(y; x_0, \dots, x_{n-1}) = \exists z(z^p = y + \prod_{i=0}^{n-1} x_i)$ and for any disjoint subsets I and J of \mathbb{N}^n there exists an element in K' that realizes

$$\{\varphi(y; a_{i_0,0}, \dots, a_{i_{n-1},n-1})\}_{(i_0, \dots, i_{n-1}) \in I} \cup \{\neg(\varphi(y; a_{j_0,0}, \dots, a_{j_{n-1},n-1}))\}_{(j_0, \dots, j_{n-1}) \in J}$$

Thus $\text{Th}(K')$ has the IP_n property by compactness. As again K' is interpretable in K , we can conclude that the theory of K has the IP_n property as well. \square

Corollary 6.3. *The theory of any non separably closed PAC field has the IP_n property.*

7 APPLICATIONS TO VALUED FIELDS

First, we generalize a result for strong depended valued fields to strong valued fields without the n -independence property.

Definition 7.1. Let T be a complete theory. An *inp-pattern of depth κ* is a sequence $(\bar{a}_\alpha, \psi_\alpha(x; y_\alpha), k_\alpha)_{\alpha \in \kappa}$ consisting of tuples $\bar{a}_\alpha = (a_{\alpha,j} : j \in \omega)$, formulas $\psi_\alpha(x, y_\alpha)$ and natural numbers k_α such that:

- $\{\psi_\alpha(x; a_{\alpha,j}) : j \in \omega\}$ is k_α -inconsistent for every $\alpha \in \kappa$;
- $\{\psi_\alpha(x; a_{\alpha,f(\alpha)}) : \alpha \in \kappa\}$ is consistent for every function $f : \kappa \rightarrow \omega$.

A theory is called *strong* if there exists no inp-pattern of infinite depth.

In [7] the authors show that an infinite strong field is perfect [7, Proposition 4.7]. Additionally, they prove that a valued field of characteristic $p > 0$ which has at most finitely many Artin-Schreier extensions has a p -divisible value group [7, Proposition 3.2]. Hence, this is the case for any NIP_n valued field. So we can conclude the following analogue to [7, Corollary 4.9].

Corollary 7.2. *Every strong valued field of characteristic $p > 0$ without the n -independence property for some $n \in \omega$ is Kaplansky, i.e.*

- *the value group is p -divisible.*
- *The residue field is perfect and does not admit a finite separable extension whose degree is divisible by p .*

Now, we turn to the question whether a n -dependent henselian valued field can carry a definable henselian valuation. Note that by a definable henselian valuation v on K we mean that the valuation ring of (K, v) , i. e. the set of elements of K with non-negative value, is a definable set in the language of rings. We need the following definition:

Definition 7.3. Let K be a field. We say that its absolute Galois group is *universal* if for every finite group G there exist finite Galois extensions $L \subseteq M$ of K such that $\text{Gal}(M/L) = G$.

As any finite extension of an n -dependent field K is still n -dependent, one cannot find any finite Galois extensions $L \subseteq M$ of K such that their Galois group $\text{Gal}(M/L)$ is of order p . Hence any n -dependent field of positive characteristic has a non-universal absolute Galois group. Note that Jahnke and Koenigsmann show in [13, Theorem 3.15] that a henselian valued field whose absolute value group is non universal and which is neither separably nor real closed admits a non-trivial definable henselian valuation. Hence this gives the following result which is a generalization of [13, Corollary 3.18]:

Proposition 7.4. *Let (K, v) be a non-trivially henselian valued field where K is neither separably nor real closed. If K is n -dependent and of positive characteristic then K admits a non-trivial definable henselian valuation.*

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