Artin-Schreier extensions in n-dependent fields

Nadja Hempel

Artin-Schreier extensions in NIP_n fields

Nadja Hempel

January 20, 2014

Abstract

A Baldwin-Saxl condition for groups without the n-independence property (NIP_n) is established. It follows that NIP_n fields are Artin-Schreier closed and certain properties of NIP valued fields extend to the NIP_n context.

1 Introduction

Superstable fields are algebraically closed (Macintyre [13] and Cherlin-Shelah [4]). Less is known for supersimple fields. Hrushovski showed that any infinite perfect bounded pseudo-algebraically closed (PAC) field is supersimple [10], conversly supersimple fields are perfect and bounded (Pillay and Poizat [14]), and it is conjectured that they are PAC. More is known about Artin-Schreier extensions of certain fields. Using a suitable chain condition for uniformly definable subgroups, Kaplan, Scanlon and Wagner showed in [12] that NIP fields of positive characteristic are Artin-Schreier closed and simple fields have only finitely many Artin-Schreier extensions. The latter result was generalized to fields of positive characteristic defined in a theory without the tree property of the second kind (NTP_2 fields) by Chernikov, Kaplan and Simon [6].

We study the existence of Artin-Schreier extensions of fields without the n-independence property. Theories without the n-independence property, or briefly NIP_n theories, were induced by Shelah in [16]. They are a natural generalization of NIP theories, and in fact both notions coincide when n equals to 1. For backgroud on NIP theories the reader may consult [19]. It is easy to see that any theory with the n-independence property has the $(n+1)$-independence property. On the other hand, as for any natural number n the random $(n+1)$-hypergraph is NIP_{n+1} but has the n-independence property, the classes of NIP_n theories form a proper hierarchy of classes. Additionally, since the random graph is simple, the previous example shows that there are simple unstable NIP_n theories for any n greater than 1. Hence the natural question arises: Which results of NIP theories can be generalized to NIP_n theories or more specifically which results of (super)stable theories remains true for (super)simple NIP_n theories? Beyarslan [2] constructed the random n-hypergraph in any pseudo-finite field or, more generally, in any e-free perfect PAC field (PAC fields whose absolute Galois group is the profinite completion of the free group on e generators). Thus, those fields lie outside of the hierarchy of NIP_n fields.

In this paper, we find a Baldwin-Saxl condition for NIP_n groups (Section 2). Using this and connectivity of a certain vector group established in Section 3 we deduce (Section 4) that NIP_n fields are Artin-Schreier closed. In Section 5 we note that our result implies that certain consequences found in [6] for strongly dependent valued fields as well as in [11] by Jahnke and Koenigsmann for NIP henselian valued field extend to the NIP_n context.

Acknowledgement

I would like to thank my supervisors Thomas Blossier and Frank O. Wagner for useful comments during the work on this article and on first versions of this paper. Also, I like to thank Artem Chernikov for bringing this problem to my attention and to Daniel Palacín for valuable disscusion around the topic.

2 Baldwin-Saxl condition for NIP_n theories

Definition 2.1. A theory has the *n-independence property* (IP_n) if there exists a formula $\psi(x_0,\ldots,x_{n-1};y)$ and some parameters $(a_i^j : i \in \omega, j \in n)$ and $(b_I : I \subset \omega^n)$ such that $\models \psi(a_{i_0}^0,\ldots,a_{i_{n-1}}^{n-1}, b_I)$ if and only if $(i_0,\ldots,i_{n-1}) \in I$.

A theory is called *NIP_n* if it does not have the IP_n.

We shall now prove a suitable version of the Baldwin-Saxl condition for NIP_n theories. By a *subarray* of ω^n of size at least m^n, we mean a set $I_0 \times \cdots \times I_{n-1}$ with $I_j \subset \omega$ an $|I_j| \geq m$ for $0 \leq j < n$.

Proposition 2.2. *Fix a group G defined in an NIP_n theory, a formula $\psi(x;y_0,\ldots,y_{n-1})$ and an array of parameters $(a_{i,j} : i < n,\ j < \omega)$. We suppose that*

$$\{H_\eta := \psi(G; a_{0,i_0},\ldots,a_{n-1,i_{n-1}}) : \eta = (i_0,\ldots,i_{n-1}) \in \omega^n\}$$

is a family of uniformly definable subgroups of G. Then there exists a natural number m such that for every subarray $I \subseteq \omega^n$ of size at least m^n there is $\nu \in m^n$ such that

$$\bigcap_{\eta\in I} H_\eta = \bigcap_{\eta\in I,\eta\neq\nu} H_\eta.$$

Proof. Suppose, towards a contradiction, that for arbitrarily large m there is a subarray $I \subseteq \omega^n$ of size m^n such that $\bigcap_{\eta\in I} H_\eta$ is strictly contained in any of its proper subintersections. Hence, for every $\nu \in I$ there is $c_\nu \in \bigcap_{\eta\neq\nu} H_\eta \setminus \bigcap_\eta H_\eta$.

Now, for every subset J of I, we let $c_J := \prod_{\eta\in J} c_\eta$ (multiplied in lexicographical order). Note that $c_J \in H_\nu$ whenever $\nu \notin J$. On the other hand, if $\nu \in J$, all factors of the product except c_ν belong to H_ν, whence $c_J \notin H_\nu$. By compactness, this formula $\psi(x;y_0,\ldots,y_{n-1})$ has the IP_n property contradicting the assumption. □

3 A special vector group

For this section, we fix an algebraically closed field $\mathbb{K}$ of characteristic $p > 0$ and we let $\wp(x)$ be the additive homomorphism $x \mapsto x^p - x$ on $\mathbb{K}$.

We analyze the following algebraic subgroups of $(\mathbb{K},+)^n$:

Definition 3.1. For a singelton a in $\mathbb{K}$, we let G_a be equal to $(\mathbb{K},+)$, and for a tuple $\bar{a} = (a_0,\ldots,a_{n-1}) \in \mathbb{K}^n$ with $n > 1$ we define:

$$G_{\bar{a}} = \{(x_0,\ldots,x_{n-1}) \in \mathbb{K}^n |\ a_0 \cdot \wp(x_0) = a_i \cdot \wp(x_i) \text{ for } 0 \leq i < n\}.$$

Our aim is to show that $G_{\bar{a}}$ is connected for certain choices of $\bar{a}$.

Lemma 3.2. *Let k be an algebraically closed subfield of $\mathbb{K}$, the group G be a k-definable connected algebraic subgroup of $(\mathbb{K}^n, +)$ and f be a k-definable algebraic homomorphism from G to $(\mathbb{K}, +)$ which is locally represented by rational functions. Then f is an additive polynomial in $k[X_0, \dots, X_{n-1}]$. In fact, it is of the form $\sum_{i=0}^{m_0} a_{i,0} x_0^{p^i} + \dots + \sum_{i=0}^{m_n} a_{i,n} x_n^{p^i}$ with coefficients $a_{i,j}$ in k.*

Proof. By compactness, one can find finitely many definable subsets D_i of G such that f is represented by a rational function on D_i. Using [3, Lemma 3.8] we can extend f to a k-definable homomorphism $F : (\mathbb{K}^n, +) \to (\mathbb{K}, +)$ which is also locally rational. Now, the functions

$$F_0(x) := F(x, 0, \dots, 0), \ \dots \ , F_{n-1}(x) := F(0, \dots, 0, x)$$

are k-definable homomorphisms of $(\mathbb{K}, +)$ to itself. Additionally, it is rational on a finite definable decomposition of $\mathbb{K}$. Hence every F_i is an additive polynomial in $k[X]$. Thus

$$F(X_0, \dots, X_{n-1}) = F_0(X_0) + \dots + F_{n-1}(X_{n-1})$$

is an additive polynomial in $k[X_0, \dots, X_{n-1}]$ as it is a sum of additive polynomials and by [8, Proposition 1.1.5] it is of the desired form. □

Lemma 3.3. *Let $\bar{a} = (a_0, \dots, a_n)$ be a tuple in $\mathbb{K}^\times$ for which the set $\left\{\frac{1}{a_0}, \dots, \frac{1}{a_n}\right\}$ is linearly $\mathbb{F}_p$-independent. Then $G_{\bar{a}}$ is connected.*

The beginning of the proof follows the one of [12, Lemma 2.8].

Proof. We prove this lemma by induction on the length of the tuple $\bar{a}$ which we denote by n. Let $n = 1$, then $G_{\bar{a}}$ is equal to $(\mathbb{K}, +)$ and thus connected since the additive group of an algebraically closed field is always connected.

Let $\bar{a}$ be an $(n+1)$-tuple such that $\left\{\frac{1}{a_0}, \dots, \frac{1}{a_n}\right\}$ is linearly $\mathbb{F}_p$-independent and suppose that the lemma holds for tuples of length n. Define $\bar{a}'$ to be the restriction of $\bar{a}$ to the first n coordinates. Observe that the natural map $\pi : G_{\bar{a}} \to G_{\bar{a}'}$ is surjective since $\mathbb{K}$ is algebraically closed and that

$$[G_{\bar{a}'} : \pi(G_{\bar{a}}^0)] = [\pi(G_{\bar{a}}) : \pi(G_{\bar{a}}^0)] \leq [G_{\bar{a}} : G_{\bar{a}}^0] < \infty.$$

Hence the definable group $\pi(G_{\bar{a}}^0)$ has finite index in $G_{\bar{a}'}$. As $\left\{\frac{1}{a_0}, \dots, \frac{1}{a_{n-1}}\right\}$ is also linearly $\mathbb{F}_p$-independent, the group $G_{\bar{a}'}$ is connected by assumption. Therefore $\pi(G_{\bar{a}}^0) = G_{\bar{a}'}$.

Now, suppose that $G_{\bar{a}}$ is not connected.

Claim. *For every $\bar{x} \in G_{\bar{a}'}$, there exists a unique $x_n \in \mathbb{K}$ such that $(\bar{x}, x_n) \in G_{\bar{a}}^0$.*

Proof of the Claim. Assume there exists $\bar{x} \in \mathbb{K}^{n+1}$ and two distinct elements x_n^0 and x_n^1 of $\mathbb{K}$ such that $(\bar{x}, x_n^0)$ and $(\bar{x}, x_n^1)$ are elements of $G_{\bar{a}}^0$. As $G_{\bar{a}}^0$ is a group, their difference $(\bar{0}, x_n^0 - x_n^1)$ belongs also to $G_{\bar{a}}^0$. Thus, by definition of $G_{\bar{a}}$, its last coordinate $x_n^0 - x_n^1$

lies in $\mathbb{F}_p$. So $(\bar{0}, \mathbb{F}_p)$ is a subgroup of $G^0_{\bar{a}}$. Take an arbitrary element $(\bar{x}, x_n)$ in $G_{\bar{a}}$. As $\pi(G^0_{\bar{a}}) = G_{\bar{a}'}$, there exists $x'_n \in \mathbb{K}$ with $(\bar{x}, x'_n) \in G^0_{\bar{a}}$. Again, the difference of the last coordinate $x'_n - x_n$ lies in $\mathbb{F}_p$. So

$$(\bar{x}, x_n) = (\bar{x}, x'_n) - (\bar{0}, x'_n - x_n) \in G^0_{\bar{a}}.$$

This leads to a contradiction, as $G^0_{\bar{a}}$ is assumed to be a proper subgroup of $G_{\bar{a}}$. □

Thus, we can fix a function $f : G_{\bar{a}'} \to \mathbb{K}$ that sends every tuple to this unique element. Note that $G_{\bar{a}}$ is defined over $\bar{a}$, hence $G^0_{\bar{a}}$ is defined over $\bar{a}$, as is f. Now, let $\bar{x} = (x_0, \ldots, x_{n-1})$ be any tuple in $G^0_{\bar{a}}$. Set $L := \mathbb{F}_p(a_0, \ldots, a_n)$. Then:

$$x_n := f(\bar{x}) \in \mathrm{dcl}(\bar{a}, \bar{x})$$

In other words, x_n is definable over $L(x_0, \ldots, x_{n-1})$ which simply means in this context that it is a root of an inseparable polynomial over $L(x_0, \ldots, x_{n-1})$. Since there exists an $l \in L(x_0)$ such that $x_n^p - x_n - a_n^{-1} l = 0$, the element x_n is separable over $L(x_0, \ldots, x_{n-1})$. So it belongs to $L(x_0, \ldots, x_{n-1})$ which implies that there exists some mutually prime polynomials $g, h \in L[X_0, \ldots, X_n]$ such that $x_n = h(x_0, \ldots, x_n)/g(x_0, \ldots, x_n)$. Thus, by Lemma 3.2 the definable function $f(X_0, \ldots, X_{n-1})$ we started with is an additive polynomial in n variables over $\mathbb{F}_p(a_0, \ldots, a_n)^{\mathrm{alg}}$ and there exists $c_{i,j}$'s in $\mathbb{F}_p(a_0, \ldots, a_n)^{\mathrm{alg}}$ such that

$$f(X_0, \ldots, X_{n-1}) = \sum_{i=0}^{m_0} c_{0,i} X_0^{p^i} + \cdots + \sum_{i=0}^{m_{n-1}} c_{n-1,i} X_{n-1}^{p^i}.$$

Using the identities $X_i^p - X_i = \frac{a_0}{a_i}(X_0^p - X_0)$ in $G^0_{\bar{a}}$, the function f can be rewritten as follows:

$$f(X_0, \ldots, X_{n-1}) = g(X_0) + \sum_{j=0}^{n-1} \beta_j \cdot X_j$$

with $g(X_0) = \sum_{i=1}^{m_0} d_i X_0^{p^i}$ an additive polynomial in $\mathbb{F}_p(a_0, \ldots, a_n)[X_0]$ with summands of powers of X_0 higher or equal to p. Since the image under f of any unitary vector of $\mathbb{K}^n$ has to be in $\mathbb{F}_p$, for $0 < i < n$ the β_i's have to be elements of $\mathbb{F}_p$. On the other hand, for any element $(x_0, \ldots, x_n)$ of $G^0_{\bar{a}}$ we have $a_n(x_n^p - x_n) = a_0(x_0^p - x_0)$. Replacing x_n by $f(x_0, \ldots, x_{n-1})$ we obtain

$$\begin{aligned} 0 &= a_n \left[f(x_0, \ldots, x_{n-1})^p - f(x_0, \ldots, x_{n-1}) \right] - a_0(x_0^p - x_0) \\ &= a_n \left[g(x_0)^p - g(x_0) + (\beta_0^p x_0^p - \beta_0 x_0) + \sum_{j=1}^{n-1} \beta_j (x_j^p - x_j) \right] - a_0(x_0^p - x_0). \end{aligned}$$

Using again the identities $x_i^p - x_i = \frac{a_0}{a_i}(x_0^p - x_0)$ in $G^0_{\bar{a}}$ we obtain a polynomial in one variable

$$P(X) = a_n \left[g(X)^p - g(X) + (\beta_0^p X^p - \beta_0 X) + \sum_{j=1}^{n-1} \beta_j \frac{a_0}{a_j}(X^p - X) \right] - a_0(X^p - X)$$

which vanishes for all elements x_0 of $\mathbb{K}$ such that there exists $x_1, \ldots, x_{n-1}$ in $\mathbb{K}$ with $(x_0, \ldots, x_{n-1}) \in G_{\bar{a}'}$, these are all elements of $\mathbb{K}$. Hence, P is the zero polynomial. Notice that $g(X)$ appears in a pth power. Since it contains only summands of power of X higher or equal to p, the polynomial $g(X)^p$ contains only summands of power of X

higher than p. As X only appears in powers less or equal to p in all other summand of P, the polynomial $g(X)$ has to be the zero polynomial itself. By the same argument as for the other β_j, the coefficient β_0 has to belong to $\mathbb{F}_p$ as well. Dividing by $a_0 a_n$ yields

$$\sum_{j=0}^{n} \beta_j \frac{1}{a_j}(X^p - X)$$

with $\beta_n := -1$ is the zero polynomial. Thus

$$\sum_{j=0}^{n} \beta_j \frac{1}{a_j} = 0$$

As β_n is different from 0 and all β_i are elements of $\mathbb{F}_p$, this contradicts the assumption and the lemma is established. □

Using Lemma 3.3, a stronger version of [12, Lemma 2.8] together with [12, Corollary 2.6], we obtain the following corollary in the same way as Kaplan, Scanlon and Wagner obtain [12, Corollary 2.9].

Corollary 3.4. *Let k be a perfect subfield of $\mathbb{K}$ and $\bar{a} \in k^n$ be as in the previous lemma. Then $G_{\bar{a}}$ is isomorphic over k to $(\mathbb{K}, +)$. In particular, for any field $K \geq k$ with $K \leq \mathbb{K}$, the group $G_{\bar{a}}(K)$ is isomorphic to $(K, +)$.*

4 Artin-Schreier extensions

Definition 4.1. Let K be a field of characteristic $p > 0$ and $\wp(x)$ the additive homomorphism $x \mapsto x^p - x$. A field extension L/K is called an *Artin-Schreier extension* if $L = K(a)$ with $\wp(a) \in K$. We say that K is *Artin-Schreier closed* if it has no proper Artin-Schreier extension i. e. $\wp(K) = K$.

In the following Remark, we produce elements from an algebraically independent array of size m^n which fit the condition of Lemma 3.3.

Remark 4.2. Let $\{\alpha_{i,j} : i \in n, j \in m\}$ be a set of algebraically independent elements in $\mathbb{K}$. Then the tuple $(a_{(i_0,\ldots,i_{n-1})} : (i_0, \ldots, i_{n-1}) \in m^n)$ with $a_{(i_0,\ldots,i_{n-1})} = \prod_{l=0}^{n-1} \alpha_{l,i_l}$ and ordered lexicographically satisfies the condition of Lemma 3.3.

Proof. Suppose that there exists a tuple of elements $(\beta_{(i_0,\ldots,i_{n-1})} : (i_0, \ldots, i_{n-1}) \in m^n)$ in $\mathbb{F}_p$ not all equal to zero such that

$$\sum_{(i_0,\ldots,i_{n-1}) \in m^n} \beta_{(i_0,\ldots,i_{n-1})} \frac{1}{a_{(i_0,\ldots,i_{n-1})}} = 0$$

Then the $\alpha_{i,j}$ satisfy:

$$\sum_{(i_0,\ldots,i_{n-1}) \in m^n} \beta_{(i_0,\ldots,i_{n-1})} \cdot \left(\prod_{\{(k,l) \neq (j,i_j) : j \leq n-1\}} \alpha_{k,l} \right) = 0$$

which contradicts the algebraically independence of the $\alpha_{i,j}$. □

We can now follow the proof in [12] that an infinite NIP field is Artin-Schreier closed to obtain the same result for a NIP_n field.

Theorem 4.3. *An infinite NIP_n field is Artin-Schreier closed.*

Proof. Let K be an infinite NIP_n field that we may assume to be $\aleph_0$-saturated. We work in a big algebraically closed field $\mathbb{K}$ that contains all objects we will consider. Let $k = \bigcap_{l\in\omega} K^{p^l}$, which is a type-definable infinite perfect subfield of K. We consider the formula $\psi(x; y_0, \ldots, y_{n-1}) := \exists t \ (x = \prod_{I=0}^{n-1} y_i \cdot \wp(t))$ which for every tuple $(a_0, \ldots, a_{n-1})$ in k^n defines an additive subgroup of $(K, +)$. Let $m \in \omega$ be the natural number given by Proposition 2.2 for this formula. Now, we fix an array of size m^n of algebraically independent elements $\{\alpha_{i,j} : i \in n, j \in m\}$ and set $a_{(i_0,\ldots,i_{n-1})} = \prod_{l=0}^{n} \alpha_{l,i_l}$. By choice of m, there exists $(j_0, \ldots, j_{n-1}) \in m^n$ such that

$$\bigcap_{(i_0,\ldots,i_{n-1})\in m^n} a_{(i_0,\ldots,i_{n-1})} \cdot \wp(K) = \bigcap_{(i_0,\ldots,i_{n-1})\neq(j_0,\ldots,j_{n-1})} a_{(i_0,\ldots,i_{n-1})} \cdot \wp(K).(*)$$

By reordering the elements, we may assume that $(j_0, \ldots, j_{n-1}) = (m, \ldots, m)$. Let $\bar{a}$ be the tuple $(a_{(i_0,\ldots,i_{n-1})} : (i_0, \ldots, i_{n-1}) \in m^n)$ ordered lexicographically and $\bar{a}'$ the restriction the $m^n - 1$ coordinates (one coordinate less).

We consider the groups $G_{\bar{a}}$ and respectively $G_{\bar{a}'}$ defined as in Definition 3.1. Using Remark 4.2 and Corollary 3.4 we obtain the following commuting diagram.

$$\begin{array}{ccc} G_{\bar{a}} & \xrightarrow{\pi} & G_{\bar{a}'} \\ \downarrow \simeq & & \downarrow \simeq \\ (\mathbb{K}, +) & \xrightarrow{\rho} & (\mathbb{K}, +) \end{array}$$

It can be restricted to K. Note that π, whence ρ stays onto for this restriction by $(*)$. Using the fact that the size of $\ker(\rho)$ has to be p, we may assume that its kernel is $\mathbb{F}_p$. Then [12, Remark 4.2] ensures that ρ is of the form $a \cdot (x^p - x)^{p^n}$. Finally, let $l \in K$ be arbitrary. Since $\rho \restriction K$ is onto and X^{p^n} is an inseparable polynomial in characteristic p, there exists $h \in K$ with $l = h^p - h$. As $l \in K$ was arbitrary, we get that $\wp(K) = K$ and we can conclude. □

The proof of [12, Corollary 4.4] adapts immediately and yields the following corollary.

Corollary 4.4. *If K is an infinite NIP_n field of characteristic $p > 0$ and L/K is a finite separable extension, then p does not divide $[L : K]$.*

5 Applications to valued fields

First, we generalize a result for strong depended valued fields to strong valued fields without the n-independence property.

Definition 5.1. Let T be a complete theory. An *inp-pattern of depth κ* is a sequence $(\bar{a}_\alpha, \psi_\alpha(x; y_\alpha), k_\alpha)_{\alpha\in\kappa}$ consisting of tuples $\bar{a}_\alpha = (a_{\alpha,j} : j \in \omega)$, formulas $\psi_\alpha(x, y_\alpha)$ and natural numbers k_α such that:

- $\{\psi_\alpha(x; a_{\alpha,j}) : j \in \omega\}$ is k_α-inconsistent for every $\alpha \in \kappa$;
- $\{\psi_\alpha(x; a_{\alpha,f(\alpha)}) : \alpha \in \kappa\}$ is consistent for every function $f : \kappa \to \omega$.

A theory is called *strong* if there exists no inp-pattern of infinite depth.

In [6] the authors show that an infinite strong field is perfect [6, Proposition 4.7]. Additionally, they prove that a valued field of characteristic $p > 0$ which has at most finitely many Artin-Schreier extensions has a p-divisible value group [6, Proposition 3.2]. Hence, this is the case for any NIP$_n$ valued field. So we can conclude the following analogue to [6, Corollary 4.9].

Corollary 5.2. *Every strong valued field of characteristic $p > 0$ without the n-independence property for some $n \in \omega$ is Kaplansky, i.e.*

- *the value group is p-divisible.*
- *The residue field is perfect and does not admit a finite separable extension whose degree is divisible by p.*

Now, we turn to the question whether a NIP$_n$ henselian valued field can carry a definable henselian valuation. Note that by a definable henselian valuation v on K we mean that the valuation ring of (K, v), i. e. the set of elements of K with non-negative value, is a definable set in the language of rings. We need the following definition:

Definition 5.3. Let K be a field. We say that its absolute Galois group is *universal* if for every finite group G there exist finite Galois extensions $L \subseteq M$ of K such that $\mathrm{Gal}(M/L) = G$.

As any finite extension of an NIP$_n$ field K is still NIP$_n$, one cannot find any finite Galois extensions $L \subseteq M$ of K such that their Galois group $\mathrm{Gal}(M/L)$ is of order p. Hence any NIP$_n$ field of positive characteristic has a non-universal absolute Galois group. Note that Jahnke and Koenigsmann show in [11, Theorem 3.15] that a henselian valued field whose absolute value group is non universal and which is neither separably nor real closed admits a non-trivial definable henselian valuation. Hence this gives the following result which is a generalization of [11, Corollary 3.18]:

Proposition 5.4. *Let (K, v) be a non-trivially henselian valued field where K is neither separably nor real closed. If K is NIP$_n$ and of positive characteristic then K admits a non-trivial definable henselian valuation.*

REFERENCES

[1] John T. Baldwin, Jan Saxl: *Logical stability in group theory*, J. Austral. Math. Soc. Ser. A 21(3), pages 267-276, (1976)

[2] Özlem Beyarslan: *Random hypergraphs in pseudofinite fields*, Journal of the Inst. of Math. Jussieu 9, pages 29-47, (2010)

[3] Thomas Blossier: *Subgroups of the additive group of a separably closed field*, Annals of Pure and Applied Logic 13, pages 169-216, (2005)

[4] Gregory Cherlin, Saharon Shelah: *Superstable fields and groups*, Annals Math Logic 18, pages 227-270, (1980)

[5] Artem Chernikov: *Theories without the tree property of the second kind*, Annals of Pure and Applied Logic, accepted (2012)

[6] Artem Chernikov, Itay Kaplan, Pierre Simon: *Groups and fields with* NTP_2, Preprint, arXiv:1212.6213v1 (2013)

[7] Pierre Gabiel Michel Demazure: *Groupes Algébriques*, Tome I, North-Holland (1970)

[8] David Goss: *Basic structures of function field arithmetic*, Springer-Verlag, Berlin, (1996)

[9] James E. Humphreys: *Linear Algebraic Groups*, Springer Verlag, second edition, (1998)

[10] Ehrud Hrushovski: *Pseudo-finite fields and related structures*, Model theory and applications, Quad. Mat. 11, pages 151-212, (2002)

[11] Franziska Jahnke, Jochen Koenigsmann: *Definable henselian valuations*, Preprint, (2012)

[12] Itay Kaplan, Thomas Scanlon, Frank O. Wagner: *Artin Schreier extensions in NIP and simple fields*, Israel J. Math., 185:141-153, (2011)

[13] Angus Macintyre: ω_1*-categorical fields*, Fundamenta Mathematicae 70, no. 3, pages 253 - 270, (1971) .

[14] Anand Pillay, Bruno Poizat: *Corps et Chirurgie*, J. Symb. Log., vol. 60(2), pages 528-533, (1995)

[15] Bruno P. Poizat: *Groupe Stables*, Nur Al-Mantiq Wal-Ma'rifah, Villeurbanne, France, (1987)

[16] Saharon Shelah: *Strongly dependent theories*, preprint (2012)

[17] Jean-Pierre Serre: *Corps Locaux*, Hermann, (1962)

[18] Thomas Scanlon: *Infinite stable fields are Artin-Schreier closed*, note (http://math.berkeley.edu/ scanlon/papers/ASclos.pdf), (1999)

[19] Pierre Simon: *Lecture notes on NIP theories*, http://arxiv.org/abs/1208.3944, (2012)