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PARTICLES AND FIELDS

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Abstract

The topic of the paper is the "duality" matter / force fields, studied from the way they are used in models in Physics. A systematic review of the concept of motion leads to a deep understanding of spinors, and to the introduction of a new formalism, using in full Clifford algebra, which is easily transposed in the General Relativity context. In a Yang-Mills model, and using theorems of Quantum Mechanics proven in a previous paper, it is then possible to give a general definition of matter fields, as sections of fiber bundles, and to the concept of bosons, for all fields, including the gravitational field.

Quantum Mechanics (QM) is immensely popular. Taught for decennies and used daily by millions of people, it is regularly lauded for its efficiency and its accordance with experiments. And praised by many for the charms of its mysteries. However it still raises many issues, practical, theoretical and philosophical. They are muddled because QM is not a consistent, unified theory (in spite of the numerous books which claim to achieve that). It is rather a patchwork of axioms (such as the Hilbert space, observable and eigen values ...), concepts (such as spin), physical assumptions (such as the Plank law), generally accepted methods (such as the minimal substitution rules), metaphysical views (such as the "Copenhagen interpretation" and the two physics), and practical computational tools (such as the path integrals). In a previous paper, revisiting the famous "axioms" of QM, I show that they are, actually, the consequence of the properties of most of the mathematical models used in physics and, as such, can be proven : they are not axioms, but mathematical artefacts. In particular they are true whatever the scale (and indeed their usual expression does not refer to any scale), as long as the model deals with variables which are vectors in infinite dimensional vector spaces. So a part of the veil is raised, but there are still many questions. Notably it appears that physical concepts, such as position, speed, momentum, which are clear at the macroscopic scale need to be adjusted when applied to particles. This is the topic of this paper.

This is not another philosophical discussion on the subject. As I did for the axioms of QM, I proceed to a systematic review of the concepts of particles, fields, trajectories, motion, as they are practically used in the mathematical models in physics. We see that the concept of motion - meaning a combination of translation and rotation - is not so obvious, even in the galilean geometry, and that there is a discrepancy between the "clean" point of view of mathematicians (through Lie algebra) and the practical way of physicists. And this discrepancy becomes indomitable in Relativity. The usual approach through the Poincaré group is loaded with too many issues to be a basis for a theory of relativist motion. The spinors are a convincing substitute. But to be efficient and consistent, they should be seen in the full formalism of Clifford algebras. In this framework a new method to

represent the motion of particles is introduced and tested, as well as many practical tools to deal with Clifford algebras. It can be immediately implemented in the General Relativity (GR) context when the, now more universally accepted, framework of fiber bundles is used. The concept of spin and its singular properties appear clearly as rooted in the symmetry breakdown introduced by the motion of particles and observers on their world lines.

Next are reviewed the concepts used to represent the "physical characteristics" of particles, in the common context of Yang-Mills models. A basic review of lagrangians and the difficulties to implement practically the principle of least action is done. Physicists are used, rightly, to distinguish continuous and discontinuous processes, and they are well aware of the intricacies of the concept of equilibrium. The implementation of the principle of least action leads, quite inevitably, to continuous representations which, intrinsically, give to particles the form of "matter fields". Quantization shows that there is a finite number of types of particles (meaning a set of elementary particles glued together), and that to each is associated a section of an associated bundle, defined at any point, with definite characteristics, such as a proper time. The rigorous application of the principle of least action, both for an isolated particle and a continuous distribution of particles, provides quite simple equations, which are similar in both cases. The new tools introduced previously supply a convenient way to deal with them, and computations in the GR context seem to be in easy reach. In particular, because the QM theorems used are valid whatever the scale, the formalism holds in astrophysics, even for rotating bodies, whenever their internal structure is not involved.

The concept of fields is quite clear in the Yang Mills approach, the Noether currents and the energy-momentum tensor let perceive how a force field can behave as a particle. However the concept of boson acquires a clear meaning in discontinuous models, either in the interaction of a single particle or in a gas of particles, in the Fock space formalism.

The paper brings new tools, and a better understanding of ideas and concepts of the "quantic world". It does not answer to all the questions, but also leads to a new approach to some old issues, such as the cosmological constant and the inconsistency between the previsions of GR and the motions of stars in astrophysics.

The first part is a reminder of some basic concepts and principles used in the paper, notably the theorems of Quantum Mechanics and the fiber bundle formalism. It gives a more precise idea of the aims of the paper, and the tools that are used.

The second part is a systematic review of the concept of motion in the galilean, Special Relativity (SR) and General Relativity (GR) context, with the fiber bundle approach of the gravitational field in GR. This is mainly there that the new formalism using Clifford algebra is introduced.

The third part is a complement to the model of particles, to account for the physical characteristics involved in the interactions with other force fields. Staying in the fiber bundle formalism, and using a usual Yang-Mills representation of interactions, the concept of "matter fields" is precised. The specifications for a consistent lagrangian are deduced from the covariance and equivariance requirements. Two continuous models, in a simplified version, are presented, with rigorous mathematical solutions, including for isolated interacting particles. This leads to the Noether currents and the energy-tensor momentum, which complements the understanding of the dual behaviour of particles and fields. A short introduction to discontinuous models and Fock spaces is given.

This paper uses many mathematical results, in areas that can be new to the reader, and it would be too cumbersome to recall all of them. The simplest way to deal with this issue is to refer the reader to my paper on "Mathematics for theroretical physics" which covers all these topics. The additional proofs are provided here, either in the core of the paper of in the annex (and recalled in

the list of formulas).

Part I
PRELIMINARIES

1 PRINCIPLES IN PHYSICS

1.1 Models

Almost all scientific theories use concepts to summarize the phenomena they study and symbols to represent these concepts. This meets the first requisite of any scientific theory, that it shall procure a common langage, precise, clearly related to the topics which are studied, based on some broad explanation of the phenomena, and accessible to anybody with the will to understand. The best illustration of such an endeavour is the system used in Chemistry. With a single line such as :

$H_2 + \frac{1}{2}O_2 \rightarrow H_2O + 286kJ/Mol$

a chemist knows almost everything that he needs about the reaction, and it is based on the solid grounds of thermodynamics and atomistic theory.

Physics use mathematics : concepts are represented by variables, and physical laws are relations between these variables. Physical laws must follow the required mathematical consistency of equations. This method provides the immense advantage of opening the powerful methods of computation of the Mathematics. But it is not neutral.

Each variable is a mathematical object of its own, it can be a scalar, a vector, a matrix,... It lives in a precise environment and presents the related properties. The collection of the data resulting from the measures of the variable shall follow these properties. And, as this is usually the case, the process of assigning significant values to the variables from the measures needs some statistical method which must be organized in accounting for these properties.

In order to have any physical meaning, the variables must be related to identified phenomena, and defined in a background which precises the conditions in which the measures can be done, including the procedures and their environment.

A physical theory is then represented by a **model**, which includes all these elements. However not all models are of physical interest (at least for a theorician). They should follow rules, given by some general principles of Physics, imposed by reason or by the experience. In what we will study in this paper, they stem from :

- gauge invariance
- quantum mechanics
- fundamental laws

2 GAUGE INVARIANCE

The criteria that a theory shall meet to be deemed "scientific" is a matter of deep philosophical arguments. However there is a large agreement, after Popper, that it should be falsifiable, or conversely, because physicists have the privilege that they can make experiments, it must be possible to check if the results of an experiment are in accordance with the predictions of the theory. Practically that means that, using the model representing the theory, taking the measures according to the prescribed protocols, computing the predictions, one can check the concordance.

So we must be able to compare the results of experiments done by different observers, at different times and at different places. Otherway the criterium would be meaningless.

2.0.1 Geometry

Almost all, if not all, measures rely on measures of lengths and times. These concepts are expressed in theories about the geometry of the universe, meaning of the container in which live the objects of physics. Geometry is borne from this quest of a theory of the universe, and has produced a rich of tools, starting from the study of space, and extended to the relation between space and time. The issue here is not a model of the Universe, seen in its totality, which is the topic of Cosmology, but a model which tells us how measures of lengths and times done by different observers are related. Such a model is a prerequisite to any physical theory. Of course it is a physical theory by itself, and one can conceive of geometries in which the "container" cannot be distinguished from the "content" (such as the Mach's point of view). However we will limit ourselves to well known theories : galilean geometry, Special Relativity and mostly General Relativity (often abreviated as SR and GR).

In each case measures of lengths and times are taken by observers, located at some place in the universe, using frames (which include clocks) and there are precise rules which relate the values of measures taken by different observers, according to their definition as mathematical objects (scalars, vectors, tensors,...).

What is true for vectorial quantities measured locally holds also for coordinates, used to locate a point in a frame or in a chart. The value of the variables should follow the rules for a change of coordinates.

The fact that the geometry is influenced by the distribution of matter and fields does not change these principles : they only change the parameters (the connection) through which the relations are expressed.

2.0.2 Gauge theories

This last remark takes a more profund meaning when it is extended to other measures. Indeed the requisite holds for any kind of measures, such as the strength of a force field. What do we mean when we say that a "field is constant over an area" ? By definition the value of a field can vary with the location. So, implicitly, we assume that the variable "strength of the field" follows some rules which tell how to compare the measures done by two obervers using different protocols. In "gauge theories" the model explicits these relations, usually by defining the variable in a fiber bundle.

Actually the situation is very similar to the previous one. In both cases we assume that there is (or could be) a network of observers who could take the measures at each point, according to consistent procedures, so that the results can be compared with the relevant rules. This network is "virtual", in the meaning that when the rules are known, there is no need to build it, a simple computation suffices. But the rules could be checked. And the fact that the rules depend on physical parameters is the manifestation of the field itself. So, in General Relativity the geometric rules depend on the gravitational field.

2.0.3 Gauge invariance

To be consistent, and meet the criterium above, the models used in physical theories must be gauge invariant : the variables must change according to some precise rules, called the rules in a change of gauge. In the case of geometric variables this is usually called covariance, but the meaning is the same. As a consequence the physical laws must comply with these rules. The simplest example of gauge invariance is the change of units used to take a measure : this is the well known rule of dimensions in any expression. But is has a much broader application. Actually whenever a variable is expressed as a map over an area, it should be possible to represent it as a geometrical quantity, that is independant of the choice of a frame. This is a more precise expression of a general principle of physics : the laws should not depend on the observer.

So the requisite leads to the adoption of a mathematical formalism which emphasizes gauge transformations.

2.0.4 Fiber bundle formalism

As seen above there is an intimate relation between geometric measures and measures of any quantity which is localized. Gauge theories are best expressed in the formalism of fiber bundles. The "container" (the universe) is represented in some geometric model based, in the 3 cases that we will consider, on a manifold (or an affine space, which is also a manifold). The mathematical objects which represent the other physical quantities then live "above" this manifold. This is an extension of the familiar tangent bundle of GR : a vector bundle is a copy of a vector space located over a manifold M (representing part of the "universe") at each of its points m, by a holonomic basis at m which defines a local frame. Usually one wants to add other characteristics to the vector spaces, such as a scalar product. This is done by defining preferred frames, and a gauge transformation rule, telling how one gets from one frame to another, based on a group. The set of preferred frames defines itself a principal bundle. To the same principal bundle different vector bundles can be associated. The derivatives of variables are then dealt with in the formalism of jets bundles. The r-jet prolongation of a vector bundle can be seen as the direct product of copies of the vector bundle, the derivative, for each combination of coordinates on M, is an independant variable.

In order to alleviate the presentation, as far as possible I will stick to some homogeneous notations, and I remind here the basics principles of the fiber bundle formalism.

A fiber bundle P with base M, standard fiber V and projection π_P is denoted $P(M, V, \pi_P)$. Its trivialization is a map :

$\varphi_P : M \times V \to P :: p = \varphi_P(m, v)$

and any element of P is projected on M : $\forall v \in V : \pi_P(\varphi_P(m, v)) = m$

All the fiber bundles that we will use are assumed to be trivial, so in the domain considered, they can be seen as the product of manifolds P = M $\times$ V. Then a section $\mathbf{p}$ on P is defined by a map : $v : M \to V$ and $\mathbf{p} = \varphi_P(m, v(m))$. The set of sections is denoted $\mathfrak{X}(P)$.

If V=G is a Lie group then P is a principal bundle. It defines a gauge at each point, denoted : $\mathbf{p}(m) = \varphi_P(m, 1)$. There is a right action of G on P : $p = \varphi_P(m, g) \to p.g' = \varphi_P(m, g.g')$

If V is a vector space then P is a vector bundle. Usually vector bundle are defined as associated vector bundles.

Whenever there is a manifold V, a left action λ of G on V, one can built an associated fiber bundle denoted $P[V,\lambda]$ comprised of couples :

$(p,v) \in P \times V$ with the equivalence relation : $(p,v) \sim \left(p \cdot g, \lambda\left(g^{-1}, v\right)\right)$

It is convenient to define these couples by a gauge on P:

$(\mathbf{p}(m), v) = (\varphi_P(m,1), v) \sim \left(\varphi_P(m,g), \lambda\left(g^{-1}, v\right)\right)$

If V is a vector space and $[V,\rho]$ a representation of the group G then we have an associated vector bundle $P[V,\rho]$ which has locally the structure of a vector space. It is convenient to define a holonomic basis $(\varepsilon_i(m))_{i=1}^n$ from a basis $(\varepsilon_i)_{i=1}^n$ of V by : $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$ then any vector of $P[V,\rho]$ reads :

$v_m = (\mathbf{p}(m), v) = \left(\mathbf{p}(m), \sum_{i=1}^n v^i \varepsilon_i\right) = \sum_{i=1}^n v^i \varepsilon_i(m)$

A fiber bundle can be defined by different trivializations, which play an essential role if the fiber bundles are trivial. In a change of trivialization the *same* element p is defined by a different map φ : this is very similar to the charts for manifold.

$p = \varphi_P(m,v) = \widetilde{\varphi}_P(m,\widetilde{v})$

and there is a necessary relation between v and $\widetilde{v}$ (m stays always the same) depending on the kind of fiber bundle.

If P is a principal bundle a change of trivialization is induced by a map : $\chi : M \to G$ and :

$p = \varphi_P(m,g) = \widetilde{\varphi}_P(m, \chi(m) \cdot g) \Leftrightarrow \widetilde{g} = \chi(m) \cdot g$ ($\chi(m)$ acts on the left)

$\chi(m)$ can be identical over M (the change is said to be global) or depend on m (the change is local).

A change of trivialization induces a change of gauge :

$\mathbf{p}(m) = \varphi_P(m,1) \to \widetilde{\mathbf{p}}(m) = \widetilde{\varphi}_P(m,1) = \mathbf{p}(m) \cdot \chi(m)^{-1}$

Which impacts all associated fiber bundles :

$v_p = (\mathbf{p}(m), v) = (\varphi_P(m,1), v) = (\widetilde{\mathbf{p}}(m), \widetilde{v}) = (\varphi_P(m, \chi(m)), \widetilde{v}) \sim (\varphi_P(m,1), \lambda(\chi(m), v))$

$\Leftrightarrow \widetilde{v} = \lambda(\chi(m), v)$

The holonomic basis of a vector bundle changes as :

$\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i) \to$

$\widetilde{\varepsilon}_i(m) = (\widetilde{\mathbf{p}}(m), \varepsilon_i) = \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \varepsilon_i\right) \sim \left(\mathbf{p}(m), \chi(m)^{-1}(\varepsilon_i)\right) = \chi(m)^{-1} \varepsilon_i(m)$

so that the components of a vector in the holonomic basis change as :

$v_m = \sum_{i=1}^n v^i \varepsilon_i(m) = \sum_{i=1}^n \widetilde{v}^i \widetilde{\varepsilon}_i(m) = \sum_{i=1}^n \widetilde{v}^i \chi(m)^{-1} \varepsilon_i(m)$

$\Rightarrow \widetilde{v}^i = [\chi(m)]_j^i v^j$

Similarly for the components of a section :

$\sigma \in \mathfrak{X}(P) :: \sigma = \varphi_P(m, \sigma(m)) = \widetilde{\varphi}_P(m, \widetilde{\sigma}(m)) = \varphi_P\left(m, \chi(m)^{-1} \widetilde{\sigma}(m)\right)$

$\Rightarrow \widetilde{\sigma}(m) = \chi(m) \cdot \sigma(m)$

$v \in \mathfrak{X}(P[V,\lambda]) :: v = (\mathbf{p}(m), v(m)) = (\widetilde{\mathbf{p}}(m), \widetilde{v}(m)) \Leftrightarrow \widetilde{v}(m) = \lambda(\chi(m), v(m))$

$v \in \mathfrak{X}(P[V,\rho]) :: v(m) = \sum_{i=1}^n v^i(m) \varepsilon_i(m) = \sum_{i=1}^n \widetilde{v}^i(m) \widetilde{\varepsilon}_i(m)$

$\Leftrightarrow \widetilde{v}^i(m) = [\chi(m)]_j^i v^j(m)$

The adjoint bundle to a principal bundle $P(M,G,\pi_P)$ is the associated vector bundle $P[T_1G, Ad]$ where T_1G is the Lie algebra of G and Ad the adjoint map : $Ad : G \times T_1P \to \mathcal{L}(T_1P; T_1P)$. A section $\kappa \in \mathfrak{X}(P_G[T_1G, Ad])$ of the adjoint bundle defines a family of change of gauge, parametrized by a scalar $\tau \in \mathbb{R} : \chi(m,\tau) = \exp \tau\kappa(m) \in G$

Whenever there is a scalar product (bilinear symmetric of hermitian two form) on a vector space V, so that (V,ρ) is a unitary representation of the group G : $\langle \rho(g) v, \rho(g) v' \rangle = \langle v, v' \rangle$, the scalar product can be extended on the associated vector bundle $P[V,\rho]$:

$\langle (\mathbf{p}(m), v), (\mathbf{p}(m), v') \rangle_{P[V,\rho]} = \langle v, v' \rangle_V$

2.0.5 Locality

In the most common cases, a physical system is represented by a model with variables which are smooth sections $\mathbf{Z} \in \mathfrak{X}(E)$ of a vector bundle E(Ω, V, π) with fiber V on an open subset Ω of the manifold M, and their derivatives up to the order r. So the model is built on the r-jet extension J^rE, which is itself a vector bundle $J^rE\left(\Omega, J_0^r\left(\mathbb{R}^m; V\right), \pi^r\right)$. Its elements have the coordinated expressions : $Z = \left(z^i_{\alpha_1 \ldots \alpha_s}, i = 1 \ldots n, s = 0, \ldots, r\right)$. The index i refers to the component in the vector bundle, the indices $\alpha_1, \ldots \alpha_s$ to the partial derivatives with respect to the coordinates in M. Whenever Z is a section on E, then $z^i_{\alpha_1 \ldots \alpha_s} = \frac{\partial^s z^i}{\partial \xi^1 \ldots \partial \xi^s}$ takes the value of the partial derivatives of the components z^i of z in a holonomic frame of E. E is the set which represents all the possible evolutions of the system (the configuration space) : this is just a mathematical representation of physical phenomena and objects which can be measured.

To follow the evolution of the system, we need to introduce the time. It requires the definition of a special chart φ_M on M, such that there is a folliation of Ω in submanifolds $\Omega_3(t)$, which are hypersurfaces representing the "space" of the observer at t. The initial data of the model are then the values of the variables, and their derivatives, on $\Omega_3(0)$.

Physical laws are expressed as mathematical relations between the variables and, because their derivatives are involved, they are differential equations. The most general mathematical definition of a set of differential equations is that it is a closed subbundle of J^rE. The solutions (meaning the values of the state of the system which are predicted) are sections of E such that their r-jet extension belong to this closed subbundle.

When the model is interpreted in its predictive capacity, it involves the use of the chart φ_M. The initial conditions are defined in the fiber $\pi^{-1}\left(\Omega_3\left(0\right)\right)$ of E over $\Omega_3(0)$. A subbundle of J^rE is :

- a submanifold Ω_0 of Ω
- a vector bundle $E_0\left(\Omega_0, V_0, \pi_0\right)$ such that $\pi|_{\Omega_0} = \pi_0$,
- and there is a vector bundle morphism $F : E_0 \to J^rE$, that is a map such that :

$\forall m \in \Omega_0 : F\left(\pi_0^{-1}\left(m\right)\right) \in \mathcal{L}\left(\pi_0^{-1}\left(m\right); \pi^{-1}\left(f\left(m\right)\right)\right)$

where : $f : \Omega_0 \to \Omega :: f \circ \pi_0 = \pi \circ F$

As E_0 is closed in J^rE and π is an open map, Ω_0 is a closed submanifold of Ω, this is obviously $\Omega_3(0)$ and E_0 is the set of initial conditions. To each set of initial conditions is associated (by F) a section of J^rE. f is not necessarily surjective, but it is continuous and for t' > t, the state of the system is in the fiber $\pi^{-1}\left(f\left(\Omega_3\left(0\right)\right) \cap S(t')\right)$ over $f\left(\Omega_3\left(0\right)\right) \cap \Omega_3(t')$, and is a continuously linear function of the initial conditions (by F). Thus the problem is "well posed" : the solutions depend continuously on the initial conditions, and moreover the relation is linear if we have a vector bundle. This is in accordance with the usual assumption that, at least at an elementary level, the state of the system at a time t depends only on its value at some initial time, and not on the values of Z at the intermediary times 0 < t' < t.

If the differential equations are expressed as differential operators : $D : J^rE \to E$, which are local maps where the value of Z at some point m depends only of the value of Z and its derivative at m, then the model is local. As the state at t depends only on the values of the derivatives at t and the initial conditions, if the model is not local it would imply that the value of Z(m) at t depends of the value of Z(m') for m' on the same spatial hypersurface at t, in violation of a generally assumed relativist postulate.

The principle of gauge invariance seems unavoidable, and if a "logic in physics" should come

one day it would probaby be one of its axioms. It leads to precise requirements in the specification of models, which moreover should, at least in the usual cases, satisfy basic properties such as locality and good behaviour of the solutions. However it is not without limits. In Relativity no two observers can get the same information about the totality of a system, and the issue is acute in astrophysics where a big (?) part of the universe will stay forever out of reach of our measures. It is worse in cosmology, as one can assume that fundamental laws are imposed, such that the fundamental constants. The speed of light has two different meanings. on one hand it is the speed of propagation of an electromagnetic signal and, as such, can be measured, depends on the medium and could possibly vary. On the other hand it is the ratio of the units of lengths (measured by rules) and units of time (measured by clocks) and as such the concept of a variation of c seems non sensical.

3 QUANTUM MECHANICS

"I think I can safely say that nobody understands quantum mechanics." Richard Feynman, in The Character of Physical Law (1965)

Quantum Mechanics (QM) as exposed usually encompasses some very general axioms, a few rules of thumb (such as the "minimal substitution" of symbols in equations), "generally accepted assumptions", and more or less standarized methods. However, starting from general and plausible assumptions about physical models, most of its "axioms" can actually be *proven*, which enables to use them in a safe a precise way. I will sum up here the theorems which will be used, the demonstration of which can be found in my paper (2013).

The starting point is a model, representing a system, with variables $X_1, ..., X_n$ collectively represented as Z. The variables have a dual purpose. On one hand they are the mathematical representation of phenomena observed in the system and as so have a precise definition as mathematical objects (they are vectors, tensors, maps,...). On the other hand they are related to the measures, by a set of procedures telling how to take the measure and assign the data which are collected to the value of the mathematical objects, usually through statistical methods.

The theorems have few requisites, the most important is that the variables must belong to Banach vector spaces $V_1, ...V_n$, one of them, at least, infinite dimensional (one can consider discrete variables but I will not use them). The variables X_k can be valued in $\mathbb{R}$, in $\mathbb{C}$, in vector spaces or vector bundles. Usually they are maps and there is no condition on their continuity or differentiability, as far as they belong to Banach vector spaces. The variables can be derivatives of other variables, they are then considered as independant variables (as in variational calculus : they belong to some jet prolongation of the original variables). The system can be "open", the actions ot the "outside" shall then be represented as variables, with the same status as the others (even if they are known their value can be measured).

So for each variable X_k there is some basis $\left(w_i^k\right)_{i\in I_k}$ of V_k (which usually are themselves maps), I_k an infinite set of indices, such that : $X_k = \sum_{i\in I_k} x_k^i w_i^k$ and only a finite number of components x_k^i are non null.

Collectively $Z = (X_1, ..., X_n)$ belongs to their product $W = V_1 \times ... \times V_n$ and Z can be written as : $Z = \sum_{i\in I} z^i w_i$ where :

$I = \oplus_{k=1}^n I_k$

$(w_i)_{i\in I} = \oplus_{k=1}^n \left(w_i^k\right)_{i\in I_k}$

3.1 Hilbert space

The state of the system is known when the value of Z is known, and the set of possible states is the set of all possible values of Z. Under a few general topological assumptions, the representation of the state of the system by the variables induces a structure of manifold on this set, diffeomorphic to an open, connected subset H_0 of an infinite dimensional, separable Hilbert space H, defined up to an isomorphism, with hermitian product $\langle\rangle$. Thus, to any state Z of the system, is associated a unique vector ψ belonging to $H_0 \subset H$.

i) There is a structure of separable Hilbert space on W, with an interior product $\langle\rangle_W$, and hilbertian basis $(w_i)_{i\in I}$

ii) There is a unique countable hilbertian basis $(\omega_i)_{i\in I}$ on H and an isometry $\Upsilon : W \to H$ such that

$\forall Z \in W : \Upsilon(Z) = \Upsilon\left(\sum_{i\in I} z^i w_i\right) = \sum_{i\in I} z^i \omega_i$

iii) To each variable $X_1, ..., X_n$ corresponds a subset of indices $I_1, ...I_n$ which define separable, mutually orthogonal, Hilbert subspaces $H_1, .., H_n$ of H and as a consequence, Hilbert spaces structures on the spaces $V_1, ...V_n$

$H = \oplus_{k=1}^n H_k \leftrightarrow W = \oplus_{k=1}^n V_k$

$X_k = \sum_{i\in I_k} x_k^i w_i^k$ and $x_k^i = \left\langle w_k^i, X_k\right\rangle_{V_k} = \left\langle \omega_k^i, \psi\right\rangle_H$

Then it is usually easy to find the scalar product and one can deal directly with X_k.

Whenever two systems, with the properties above and similar variables, interact, they can be represented in a unique system, accounting for the interactions, where the variables are the tensorial product of the similar variables, and the corresponding Hilbert space is the tensorial product of the Hilbert space.

The key point to understand this theorem is that the variables X are usually maps, so the vector ψ represents the total system (and particularly its evolution) and is not localized. We do not have "wave functions" $\psi(x)$. Moreover the representation of the system in H is not projective : H is defined up to isomorphism, but we do not need the usual "phase factor".

3.2 Observables

The values of the variables X, which are maps, are actually measured from a finite set of data, usually through a statistical procedure. Using the previous notations one can formally associate to any variable X_k an observable :

$Y_k : V_k \to V_k :: Y_k(X_k) = \sum_{i\in J_k} y_k^i x_k^i w_i^k$

where the set $J_k \subset I_k$ is finite, and the coefficients y_k^i are constant (they depend on the estimation process only). And to each observable Y_k is associated a operator $\widehat{Y}_k = \Upsilon \circ Y_k \circ \Upsilon^{-1} \in \mathcal{L}(H_k; H_k)$

Then it can be proven that the operators $\widehat{Y}_k$ on H_k, Y_k on V_k are self-adjoint, Hilbert-Schmidt, trace class and compact operators. Moreover the set of these observables is a commutative C*algebra.

Whenever 2 observables encompasses two disjoint subsets of I, their corresponding operators are orthogonal. For any two observables taking value in different Hilbert spaces : $\widehat{Y}_{k_1} \circ \widehat{Y}_{k_2} = \widehat{Y}_{k_2} \circ \widehat{Y}_{k_1} = 0$. But if they take their values in subsets of I which are not disjointed then they do not commute, which is usually the case for the components of vectorial functions (as far as they are considered together as components of a vector). However the composition of observables (what is usually undertood as their product) has no precise physical meaning, and does not play any role.

Because the operators $\widehat{Y}_k$ on H_k, Y_k on V_k are compact and self adjoint, their eigen values are real, their set is finite and the eigen spaces for distinct eigen values (except possibly for 0) are orthogonal. The result of any measure of the observable is equal to one of the eigen values with a probability proportional to the norm of the vector which is measured. This is easily understood : the measure $Y_k(X_k)$ is the projection of X_k on the vector subspace spanned by the eigen vectors $\left(y_k^i w_i^k\right)_{i\in J_k}$. The probability to find $\sum_{i\in J_k} y_k^i x_k^i w_i^k$ if $X_k = \sum_{i\in I_k} x_k^i w_i^k$ is $\frac{\sum_{i\in J_k}\left|x_k^i\right|^2}{\sum_{i\in I_k}\left|x_k^i\right|^2}$ because the error on the measure comes from the discrepancy between the infinite set I_k and the finite set J_k.

So we have the convenient, and usual in QM, way to define a basis of the Hibert space H as the eigen vectors of orthogonal observables, which can be easily identified through the measure that they provide. What we see is that actually this can be done on the space W itself, when the Hilbert space structure has been identified.

3.3 Wigner's theorem

Whenever there is a map U, such that the measures taken by two different observers, for the same system and the same state, are related by ; $Z \to \widetilde{Z} = U(Z)$, there is a unitary operator $\widehat{U} = \Upsilon \circ U \circ \Upsilon^{-1}$ on H such that the corresponding vectors in H are related by : $\widetilde{\psi} = \widehat{U}\ \psi$. The operators associated to the observable Y are related by : $\widehat{\widetilde{Y}} = \widehat{U} \circ \widehat{Y} \circ \widehat{U}^{-1}$.

$\widetilde{Y}\left(\widetilde{Z}\right) = U \circ Y(Z) \leftrightarrow \widehat{\widetilde{Y}}\left(\widetilde{\psi}\right) = \widehat{U} \circ \widehat{Y} \circ \widehat{U}^{-1}\left(\widetilde{\psi}\right) = \widehat{U} \circ \widehat{Y}(\psi)$

The vectors w_i of the basis of W do not change, neither the vectors ω_i, so :

$\widetilde{Y}\left(\widetilde{Z}\right) = \sum_{i\in I} y^i \widetilde{z}^i w_i \leftrightarrow \widetilde{\psi} = \sum_{i\in I} y^i \widetilde{z}^i \omega_i$

The map U is part of the model, and of the definition of the variables. It must be known. U does not need to be linear but it occurs that eventually this is a linear map and even an unitary operator on the vector space W with its structure of Hilbert space. This comes from the requisite that W is a vector space.

If ψ is eigen vector of $\widehat{Y}$ with eigen value λ then $\widehat{U}\psi$ is eigen vector of $\widehat{\widetilde{Y}}$ with the same eigen value.

If there is a continuous map : $G \to \mathcal{L}(W;W)$ such that (W,U) is a representation of the Lie group G, then $(H,\widehat{U})$ is a unitary representation of G and $(H,\widehat{U}'(1))$ is an anti-hermitian representation of the Lie algebra T_1G. Then there are self adjoint operators $\widehat{T}_a$ on H such that, for any vector $\kappa = \sum_a \kappa^a \overrightarrow{\kappa}_a$ of T_1G :

$\widehat{U}(\exp\kappa) = \exp\left(\frac{1}{i}\sum_a \kappa^a \widehat{T}_a\right)$ with $\widehat{T}_a = \widehat{U}'(1)\,\overrightarrow{\kappa}_a$.

If U is a uniformly continuous one parameter group of transformations $U(\theta)$ then $\widehat{U}$ is also a one parameter group of unitary transformations, and there is an operator $\widehat{S}$ on H such that :

$\frac{d\widehat{U}}{d\theta} = -i\widehat{S} \circ \widehat{U}(\theta) \Leftrightarrow \widehat{U}(\theta) = \exp\left(\frac{1}{i}\theta\widehat{S}\right) \circ \widehat{U}(0)$

and $\widehat{S}$ is self-adjoint.

A change of units for the measures is obviously such a map U. Because it is unitary we should have : $\left\langle\widetilde{\psi},\widetilde{\psi}'\right\rangle = |k|^2\langle\psi,\psi'\rangle = \langle\psi,\psi'\rangle \Rightarrow k = \pm 1$

So all the quantities used in a model must be dimensionless.

3.4 Change of gauge

An important case occurs with vector bundles. Let $P_G(M,G,\pi)$ be a principal bundle over a manifold (usually the universe) with Lie group G and trivialization $\varphi_G(m,g)$, and $P_G[E,\rho]$ an associated vector bundle. A section $\mathbf{S}$ of this vector bundle is defined by a map : $S : M \to E$ such that $\mathbf{S}(m) = (\varphi_G(m,1), S(m)) \sim \left(\varphi_G(m,g), \rho\left(g^{-1}\right) S(m)\right)$. In a change of gauge on P_G the same section $\mathbf{S}$ is defined by a map : $\widetilde{S}(m) = \chi(m) S(m)$.

If a variable of the system is represented by a section $\mathbf{S}$ belonging to some Banach vector subspace $\mathcal{S}$ of the space of sections $\mathfrak{X}(P_G[E,\rho])$ then the observable is the map S, which belongs to some Banach vector space $F \subset C(M;E)$. The space F is isometric to some Hilbert space H, and is a Hilbert space. S can be considered as a vector of this Hilbert space, representing the state of the system (for its part represented by S) and the value of S which is measured is $\widehat{S} \in F$. The Wigner's theorem applies.

Consider a global change of gauge : $\chi : M \to G :: \chi(m) = g$

Then $\widetilde{S}(g) = U(g)(S)$ with a map $U(g) : F \to F$ depending on g such that : $\forall m \in M$: $\widetilde{S}(g)(m) = U(g)(S(m))$. Moreover :

$U(g) \in \mathcal{L}(F;F)$.

$U(1) = Id$

$U(gg')(S) = U(g \cdot (U(g')S))$

U is a linear, unitary operator on $\mathcal{L}(F;F)$ so that (F,U) is a unitary representation of the Lie group G. And $(F, U'|_{g=1})$ is an anti-hermitian representation of the Lie algebra T_1G. If G is compact (F,U) is the sum of finite dimensional, orthogonal, irreducible representations of G. If G is not compact there are always unitary, infinite dimensional, functional representations.

For the observable $\widehat{S}$ we have : $\widetilde{\widehat{S}}(g) = U(g) \circ \widehat{S} \circ U(g)^{-1}$

U is usually built from ρ but formally they do not operate on the same vector spaces.

A section $\kappa \in \mathfrak{X}(P_G[T_1G, Ad])$ of the adjoint bundle defines a family of change of gauge, parametrized by a scalar $\tau \in \mathbb{R}$: $\chi(m,\tau) = \exp \tau\kappa(m)$ so that $\widetilde{S}(m,\tau) = \rho(\exp \tau\kappa(m)) S(m)$. Consider such a local change of gauge. Then $\widetilde{S}(\tau,\kappa) = V(\tau,\kappa)(S)$ with a map $V(\tau,\kappa) : F \to F ::$ $\widetilde{S}(\tau,\kappa)(m) = \rho(\exp \tau\kappa(m)) S(m)$ depending on τ, κ. Moreover :

$V(\tau,\kappa) \in \mathcal{L}(F;F)$.

$\widetilde{S}(\tau,\kappa)|_{\tau=0} = S$

$\widetilde{S}(\tau+\tau',\kappa) = V(\tau,\kappa)(V(\tau',\kappa)S) = V(\tau+\tau',\kappa)S$

so that we have a one parameter group of unitary operators (with τ) on $\mathcal{L}(F;F)$. The map : $V : \mathbb{R} \to \mathcal{L}(F;F)$ is continuous thus there is an infinitesimal self-adjoint generator $H(\kappa) \in \mathcal{L}(F;F)$ such that :

$\frac{dV}{d\tau} = \frac{1}{i} H(\kappa) V(\tau,\kappa)$

Whenever $g = \exp \tau\kappa$:

$V(\tau,\kappa) = U(\exp \tau\kappa)$ so $\frac{dV}{d\tau}|_{\kappa=0} = U'|_{g=1}\kappa = \frac{1}{i} H(\kappa)$

so that $(F, \frac{1}{i}H)$ is a representation of the Lie algebra T_1G.

3.5 Evolution of the system

The previous theorems hold for a system which is represented through its whole evolution : the time is just one parameter which enters into the domain of the variables (such as the space coordinates),

the vectors w_i, ω_i are constant as well as the components z^i. This corresponds to the Schrödinger picture. However if the evolution is followed during some period of time [0,T], meaning that one takes measures at times $t_1, t_2, ...t_N$ the theorems hold for each "snapshot" of the system, and we are in the Heisenberg picture. Both pictures are related. The relation depends on the geometry of the universe (time can be or not related to the space variables). However in the three cases that we will follow (including GR) the Schrödinger equation holds : there is an anti-hermitian operator $\widehat{H}$ such that :

$\psi(t) = \left(\exp \frac{1}{i\hbar} tH\right) \psi(0)$.

This is a direct consequence of the Wigner's theorem.

This presentation of QM is based upon the concept of models, the intimate relation between variables and measures, and the assignation process which, because it is based, in the most usual case, on statistical methods, introduces probability in physics. So it enters fully in our description of physics, and brings more results to what one can expect from any physical theory.

4 FUNDAMENTAL LAWS OF PHYSICS

So far there is no "model of everything", if such a thing could ever exist. Physicists use different concepts, models and computational methods according to the problem they try to solve. In each area of physics there are several models which are commonly used, and we will focus on the models representing particles and fields. They can be roughly classified according to two criteria : continuous / discontinuous, equilibrium / desequilibrium. In each case there is a dominant, non exclusive "principle" which has the status of a physical law, meaning a general rule that governs the solutions as they are represented in a model. These principles are very general, as they shall address a great diversity of cases, but they are essential in the formalisation of a model and of the mathematical equations which are the basis of its predictive capability. In the area that we will cover they are the principle of least action and the principle of maximal entropy.

4.1 Continuous models

In continuous models of particles and fields it is assumed that no discontinuous process occur in the system. As a consequence particles and fields conserve, under some rules which depend on their representation, their main characteristics over the evolution of the system. Of course there is some flexibility in this, imprecise, definition. In particular it depends on the scale at which the objects are observed. But, in a perhaps simplified but pertinent (in regard with our purpose) classification, we can say that a model is continuous if the physicist chooses to represent the phenomena by continuous variables : he assumes that what may happen which is not continuous is not relevant to his study. For instance in Hydrodynamics the motion of molecules are summarized in the motion of fluids currents, which are defined at an intermediate scale. These currents conserve their properties as vector fields.

Continuous models usually assume that some equilibrium has been achieved, but not necessarily. However as desequilibrium means going from one equilibrium to another one, it implies some kind of discontinuity in order to be fully explained.

4.1.1 The principle of least action

Continuous models of particles and fields are an extension of Mechanics. They aim at the description of the motion of bodies and the prediction of the value of the fields resulting from all the interactions. The laws of mechanics are then summarized in the principle of least action, which provides a general framework for the models.

It is convenient to use the fiber bundle formalism to explicit the principle, as it will be extensively used in this paper.

If the system is represented by a model, where the variables Z are sections of a vector bundle E over a n dimensional manifold M, and their derivatives at the order r, the principle states that there is a real valued n-form on O, defined on $J^rE : \mathcal{L} : J^rE \to \Lambda_n(M;\mathbb{R}) :: \mathcal{L}\left(z^i, z^i_\alpha\right)$ such that, *at the equilibrium* of the system, the action : $\int_\Omega \mathcal{L}$ is stationary. If there is a volume form $\varpi_n \in \Lambda_n(M;\mathbb{R})$ on M, then $\mathcal{L} = L\varpi_n$ where the scalar lagrangian L is a real function on J^rE.

In a continuous process the maps are assumed to be smooth. Then the sections $\mathbf{Z} \in \mathfrak{X}(E)$ for which the action is stationary follow the Euler-Lagrange equations. For r=1 they read :

$\forall i : \frac{\partial L}{\partial z^i} = \sum_{\alpha=1}^n \frac{d}{d\xi^\alpha}\left(\frac{\partial L}{\partial z^i_\alpha}\right)$

When the evolution of a system is followed by maps Z(t) depending on the time t from t_1 to $t_2$the action reads : $\int_{t_1}^{t_2} L\left(z^i(t), \dot{z}^i\right) dt$ with the derivatives with respect to t.

When the system considered is defined similarly over a submanifold of M, we should have an action defined on this submanifold, with the volume form induced by the metric if any.

So, much of the physics is incorporated in the specification of the lagrangian L which, as a consequence, should comprise all the variables of the model. But some conditions can be imposed on the side, such as the conservation of the characteristics of the particles.

L shall be a dimensionless quantity (as it is a prerequisite of QM), so it is not easy to link it to a precise physical quantity. The profound physical significance of the principle of least action is a matter of conjectures, notably when it addresses the most basic phenomena, but this principle has several crucial consequences.

i) It is expressed as a global condition : the stationarity of the action $\int_\Omega \mathcal{L}$ but, under the smoothness assumption, it leads to the Euler-Lagrange equations which are expressed through differential operators. So the model is determinist (the problem is well posed), and the principle comforts the assumption of locality of the laws of physics.

ii) The lagrangian must be equivariant in a change of gauge (for the principal or associated bundles) and covariant (equivariant in a change of charts on M). Mathematically these conditions lead to some constraints for the specification of the lagrangian itself (some variables are incompatible with others).

iii) In the usual cases the derivatives $\frac{\partial L}{\partial z^i}, \frac{\partial L}{\partial z^i_\alpha}$are tensorial. The momenta associated to a variable z are the partial derivatives $\frac{\partial L}{\partial z^i_\alpha}$., and the energy-momentum tensor is $T^\alpha_\beta = \sum_i \frac{\partial L}{\partial z^i_\alpha} z^i_\beta - L\delta^\beta_\alpha$. They lead to conserved geometric quantities which can be related to physical quantities and actually provides the basis of experimental measures. So, the abstract principle of least action and its obscure lagrangian can meet the requirements of experimental physics.

The principle seems to introduce a paradox in that the values taken by the variables at any moment depend on the values on the whole evolution of the system, that is on the values which will be taken in the future. But this paradox stems from the model itself : at the very beginning the physicist assumes that the variables which are measured or computed belong to some class of

objects which are defined all over Ω. So the procedure is not a step by step procedure, in which the model would be readjusted at every time : the mathematical objects which are used belong to some classes, chosen once and for all, of smooth maps. This is similar to the foundations of Quantum Mechanics : if the model specifies smooth maps as variable, it is logical to expect smooth maps as solutions, meaning that they are defined over the whole evolution of the system and estimated from a batch of data related to measures done at different times.

4.2 Discontinuous models

According our rough classification discontinuous models aknowledge the existence of a dicontinuous process, that they try to represent.

In the real world discontinuous processes are the rule : change of phase, collisions, chemical reactions, disintegration of nuclei,...However they are, in some way, the transition between two continuous states and there are two, related but distinct, kinds of phenomena which are addressed.

4.2.1 Reversible processes

As one goes from an equilibrium to another, it is often possible to conceive of the co-existence of the two states of equilibrium, and to study the resulting system. For instance in the kinetic of gas one studies the result of the collisions between molecules, or in thermodynamics systems exchanging work and heat. When discontinuous processes are involved, but in a way that at a certain scale the system itself is still in equilibrium, one can model the system by using smooth variables : the density, the concentration of species, the flows of heat,... are defined as the averadge of similar quantities. For this it is required that the system meets certain conditions, broadly defined as involving only reversible processes, which usually means with slow evolution, so that over a small (respective to the duration of the experiment) period of time an equilibrium as been reached between the two states.

Because one moves between equilibrium, the lagrangian is still a key component, and because time is essential, it is used in the form of hamiltonian. As such it integrates the physics involved in the equilibrium. Even if these models are extremely efficient, they have limits. First their key variables, for which predictions are done, represent, not the elementary processes, but their result for the whole system : they provide figures for either the total system, or averadges at an intermediary level. Second, because of this discrepancy, there is an inherent uncertainty afflicting any prediction. And this uncertainty grows as one goes closer to the scale at which the discontinuous process occurs. We are below the usual uncertainty linked to the precision of measures : it is part of the model itself, in which the physicist trades precision for efficiency.

So it is natural that statistical tools are part of the mathematics of these models. They come under various form, but they are summed up in the "principle of maximum entropy", stated explicitely by E.T.Jaynes. It has been the topic of many discussions (more, it seems, than the principle of least action) and has different formulations. For the most usual case it can be formulated as follows.

We have a system comprised of a=1...N (a large number) of identical microsystems which are represented by n variables $X = (X_i)_{i=1}^n$. So the state of a microsystem is known by the value x_i of X_i. The probability that any microsystem is in a state X is given by a law, identical for all : $\Pr(X = x) = \rho(x, \theta)$ depending of p parameters $(\theta_i)_{i=1}^p$ .There are m observables, macroscopic variables which can be measured for the whole system. The value y of Y depend on the value of the

X : $y_k = f_k(x_1, .., x_n)$ for a microsystem and $Y_k = g_k\left(y_k^1, ...y_k^N\right)$ for the total systemSo if we know the value $\left(\widehat{Y}_k\right)_{k=1}^m$ measured for the Y, the problem is to find the parameters θ. The principle of maximum entropy (called "MaxEnt") states that the parameters $(\theta_i)_{i=1}^p$ are such that the integral :

$S = \int_\Omega -\rho(x_1, ..x_n, \theta) \ln \rho(x_1, ..x_n, \theta)\, dx_1...dx_n$

over the domain Ω of the x_i is maximum, under the constraints given by the observed values $\widehat{Y}_k$ and that $\int \rho dx = 1$.

S is the information entropy. Expressed this way we have actually a statistical problem, and it is not too difficult to show that the MaxEnt principle gives indeed a good estimator under general conditions. So it does not come from out of the blue. But many authors give a more physical significance to the principle, in direct relation with the thermodynamic entropy.

Using the estimated parameters it is then possible to know the probability for a given microsystem to be in a given state : as we see the prediction is only probabilistic. If the processes are discontinuous the probability can be directly related to the transition from one state to another.

4.3 Desequilibrium

Discontinuous processes raise other questions : why the discontinuity occurs at first, which is related to the frequency of its occurence, and usually to the rate at which the whole system moves from one state to another.

Another way to look at discontinuous processes proceeds from their irreversibility. Irreversible processes are represented by equations which are not time reversible : if the sign of t in a solution is changed to -t the result is no longer a solution. And a disconcerting fact is that all the elementary processes are reversible. And this stays true in the relativist picture, as the 4 dimensional universe is anisotropic : there cannot be observers going backward in time.

There are many models, using phenomenological laws, to represent these phenomena, but no general model answering these questions, and the only principle available is the second principle of thermodynamics, which delimits what can occur, but do not tell when it occurs (see Röpke for more). So we know how to deal with discontinuous processes at a macroscopic level, but not at a microscopic one.

5 PARTICLES AND FIELDS

Our main purpose is to understand the duality fields / particles. The issue is not here to discuss what particles and fields are "really", whatever the meaning of reality. We are looking at the concepts : by particles and fields physicists mean physical objects with specific properties, represented by mathematical objects in a model. When one says that a particle can behave as a field and conversely, we need to look at the distinctive characteristics of the corresponding mathematical objects. In this section we give a short general description of the common representation of particles and fields and their interactions, which is then detailed in the next sections.

5.1 Particle

The concept of particles dates back to greek philosophers and has been confirmed in late XIX° century. It means some material object which cannot be further divided. In the following by particle we mean a material body, with no internal structure and negligible spatial extension, with respect to the area covered by the system. It could be an elementary particle, a nucleus or an atom. Actually many of the concepts presented here are used in astrophysics, where "particles" are ... galaxies. Even if "quantic behaviors", such as the two slits experiment, have been proven for molecules, it is clear that molecules have a well characterized internal structure, which is fundamental in any modelling. Studies of molecules and more generally of organized matter involve only the electromagnetic field (and rarely gravitation), and can proceed from a combination of classic mechanics and Quantum Mechanics, using these internal structures.

Particles are assumed to have geometric, kinematic as well as physical characteristics.

5.1.1 Geometric features

The first characteristic of a particle is that it has a trajectory, represented as a path in the universe : a map $\mu : I \to M :: \mu(t) = m$ from an interval of $\mathbb{R}$ to the universe for some parameter t. Its image is a curve, which is a one dimensional manifold. The nature, determinist of stochastic, of this trajectory is subject to conjecture, but the existence of this trajectory is a distinctive feature of particles, as opposed to fields. Particles can collide, but between collisions their path is assumed to be smooth. In almost all practical models it is represented by a smooth map belonging to some class of maps which can be easily estimated from a finite number of points (a straigth line, a parabol, an ellipse,...).

The trajectory can be described by different parameters, which is usually the time of some observer, and this choice matters in the relativist picture. The derivative of the map μ with respect to the parameter t used to represent the path is a vector of the tangent space to M at each point of the trajectory. The issue is not here if it is possible, or meaningful, to measure simultaneously momentum and position (we will come back on this later). Whenever we have a trajectory we can assign some "averadge speed" to a particle, and a vector which represents the velocity. And in most of the experiments the speed is actually not a direct measure but computed from the design of the apparatus itself. When a particle exits an accelerator it has a speed which is known, and it is assumed that it keeps it when it enters in a collision.

The usual picture of a particle is that of a small rigid body, and as such a particle is supposed to have also some kind of rotation, that we will call "spin" to keep it simple. In mathematics the concept of rotation addresses the relative disposition of two frames, located at the same point, in a vector space, and its derivative is expressed in the Lie algebra of a group. In Relativity, because time and space are entangled, the rotation of two 4 dimensional frames comprises a part which is related to the local translation of the frames (what is usually called the boost). The implementation of this concept in physics is not obvious : the body can have symmetries, which, by definition, means that some rotations cannot be observed, attached frames can be defined only for solid, which do no not exist in relativity and of course do not exist for bodies with no internal structure,...

The motion of a particle is the combination of its trajectory and its spin, both with a parametrization with respect to some observer. Their derivatives with respect to the time provides two vectorial

quantities, which usually do not belong to the same vector space but are subjected to precise rules in a change of gauge according to the model used to represent the geometry of the universe.

5.1.2 Kinematic

Particles show physical features which are directly linked to their motion : kinetic energy and momenta, which are both linked to the characteristics of the motion (velocity and spin). So their value depend on the frame in which these variables are expressed. If the trajectory has a clear physical meaning, the spin is best measured through its corresponding momentum, even for macroscopic bodies.

The momenta are represented by vectorial quantities, linked to the derivatives of the motion by scalars : the mass and the moment of inertia. Thus they belong to the same respective vector spaces. They can be seen as the reluctance of the particle to change its motion : its inertia.

5.1.3 Physical features

Particles show physical features which tell how they interact with force fields. For the most usual fields (gravitation and electromagnetic) these features are represented by scalars : the charges, which are defined with respect to specific physical units. When other fields are considered the situation is more complicated and actually the "charges" are defined with respect to standards provided by elementary particles (as the electric charge is defined with respect to the charge of the electron). The charges tell how the momenta change under the action of the fields. Conversely, combined with the characteristics of the motion, they define "currents" and "moments", which are the sources of the fields.

So there is a great similtude between the "mechanical" and "physical" characteristics. And indeed the identity between inertial mass and gravitational charge, measured with great accuracy, has lead to the General Relativity.

Mass, charge, ... are assumed to be intrinsic : they are characteristics of the particle, and conserved along its trajectory. This implies conservation laws along the trajectory of the particle. However particles can change "spontaneously" or through interaction with a field or other particles. Moreover there are stable, or short-lived, combinations of particles which share all the characteristics of particles. So any advanced physical theory should account for this flexibility.

5.2 Force fields

The concept of fields has appeared in the XIX° century, in the wake of the electromagnetism theory, to replace the old picture of "action at a distance" between particles. In the following by fields we mean one of the forces which interact with particles : the strong interaction, the weak and the electromagnetic forces combined in an electroweak interaction, gravitation being in one league by itself. The status of the Higgs field is still open.

Because particles are localized, the field must be able to act anywhere, that is to be present everywhere. So the first feature of force fields, as opposed to particles, is that, a priori, they are defined all over the universe, even if their action can decrease quickly with the distance.

Fields are basically defined through their interaction with particles. They are represented as first order differential operators (a connection) acting on both linear and rotational momenta, with an action proportional to the charges.

Fields propagate in the vacuum, meaning where there is no particle : this propagation is characterized by a speed and a range. We know very little about the propagation of the weak and strong interactions, which have singular features, and their range is small. For the electromagnetic and the gravitational field the propagation if defined through second order differential operators (the laplacian), involving the strength $\mathcal{F}$ of the field, which is a kind of derivative of the potential by the wave equation.

Particles are the sources of fields. This is represented by adding a term to the propagation equation, related to the value of the currents and magnetic momenta of the particles.

Fields (at least the electromagnetic field) store energy, and from their interaction with particles can be seen as storing momentum.

Any measure on the force field involves its interaction with known particles. So the representation of the field is related to the existence of rules telling how the measures change with the procedures, and this is exactly the description of the law that the field is assumed to follow as a mathematical object (a connection).

Any physical model of fields and particles relies on some representation of the universe. So we will review how these general concepts are implemented in the three usual models.

Part II
MOTION OF A PARTICLE

6 GALILEAN GEOMETRY

Galilean geometry is the basic model of the universe. It is well known, but some concepts such as rotation are less obvious than it seems. We will limit our review to particles, because fields cannot be fully represented in Galilean geometry (the Maxwell's equations are not covariant). But we will give a short presentation of the main consequences of QM on the model of the motion of a particle.

6.1 The Galilean model of geometry

6.1.1 Fundamental hypotheses of galilean geometry

The galilean geometry is based upon 3 hypotheses :

i) In the universe space and time are disconnected : the time, as measured by a clock, is the same for any observer.

ii) The space is modeled as a 3 dimensional oriented euclidean affine space E.

iii) There is a special class of observers, the galilean observers, who do not measure any inertial forces, whose frames are constant with respect to the time.

So, here, a gauge is a frame that is a point, which is the origin, and a set of 3 orthonormal vectors.

The orientability of the universe can be attested from the existence of stereoisomeric molecules, which have not the same chemical properties.

6.1.2 Gauge transformations

A gauge transformation is a change of coordinates : let $\left(O,(\varepsilon_i)_{i=1}^3\right)$ be the initial frame, and $\left(\widetilde{O},(\widetilde{\varepsilon}_i)_{i=1}^3\right)$ the new frame. The two orthonormal bases $(\varepsilon_i)_{i=1}^3$, $(\widetilde{\varepsilon}_i)_{i=1}^3$ are related by an orthogonal map, and because they have the same orientation this map belongs to a group which can be identified with the group SO(3) of 3×3 real matrices such that $R^tR=I, \det R=1$. The set of gauge transformations is the group of displacements D, semi-product of the group SO(3) of spatial rotations and the abelian group Θ of translations with the composition rule $(R,L)\times(R',L')=(R\circ R',R(L')+L)$ and inverse $(R,L)^{-1}=\left(R^{-1},-R^{-1}(L)\right)$.

A change of gauge is then given by a couple (R,L) such that:

$\left(\widetilde{O},(\widetilde{\varepsilon}_i)_{i=1}^3\right)=(R,L)\left(O,(\varepsilon_i)_{i=1}^3\right)$

$O\to\widetilde{O}::\overrightarrow{O\widetilde{O}}=\sum_{i=1}^3 L^i\varepsilon_i$

$(\varepsilon_i)_{i=1}^3\to(\widetilde{\varepsilon}_i)_{i=1}^3::\widetilde{\varepsilon}_i=\sum_{j=1}^3[R]_i^j\,\varepsilon_j$

where $[R]=\imath(R)$ is the matrix of R in the standard representation $\left(\mathbb{R}^3,\imath\right)$ of SO(3)

The coordinates of a point M then changes as:

$\overrightarrow{OM}=\sum_{i=1}^3 x^i\varepsilon_i\to\overrightarrow{\widetilde{O}M}=\sum_{i=1}^3\widetilde{x}^i\widetilde{\varepsilon}_i::\widetilde{x}^i=\sum_{j=1}^3\left([R]^{-1}\right)_j^i\left(x^j-L^j\right)$

Equivalently one can see the affine space E as a representation of the Lie group D given by the choice of a frame $\left(O,(\varepsilon_i)_{i=1}^3\right)$.

Similarly the time t measured by O and t' measured by A are related by : $t = at' + b$ with some constants a,b.

The Lie algebra of SO(3) is the vector space of 3×3 real antisymmetric matrices. It has the same dimension as $\mathbb{R}^3$, so there is an isomorphism of vector spaces by the choice of the canonical basis $(\varepsilon_i)_{i=1}^3$ of $\mathbb{R}^3$ and any basis $([\kappa_i])_{i=1}^3$ of so(3) :

$\vartheta : so(3) \to \mathbb{R}^3 :: \vartheta\left(\sum_{i=1}^3 \kappa^i [\kappa_i]\right) = \sum_{i=1}^3 \kappa^i \varepsilon_i$

and this particularity is at the root of many computational tricks. Notice that this isomorphism is not canonical : it depends on the choice of a basis in both vector spaces.

Let us denote $j : \mathbb{R}^3 \to L(\mathbb{R},3) :: [j(r)] = \begin{bmatrix} 0 & -r^3 & r^2 \\ r^3 & 0 & -r^1 \\ -r^2 & r^1 & 0 \end{bmatrix}$

The operator j is very convenient to represent quantities which are rotated, and has many nice algebraic properties (see formulas in the annex).

For any vector u : $\sum_{ij=1}^3 [j(v)]_j^i u^j \varepsilon_i = \overrightarrow{v} \times \overrightarrow{u}$ with the cross product $\times$.

If we take the following matrices as basis of so(3) :

$\kappa_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}; \kappa_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}; \kappa_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

then any matrix of so(3) reads :

$\sum_{i=1}^3 \kappa^i [\kappa_i] = [j(\vartheta(\kappa))] = \begin{bmatrix} 0 & -\kappa^3 & \kappa^2 \\ \kappa^3 & 0 & -\kappa^1 \\ -\kappa^2 & \kappa^1 & 0 \end{bmatrix}$

Because SO(3) is a compact Lie group the exponential is surjective :

$\forall g \in SO(3), \exists \kappa = \sum_{i=1}^3 \kappa^i [\kappa_i] \in so(3) : g = \exp \kappa$

It is easy to show that :

$[g] = \exp[j(\vartheta(\kappa))] = I_3 + [j(\vartheta(\kappa))] \frac{\sin\sqrt{\langle\vartheta(\kappa),\vartheta(\kappa)\rangle}}{\sqrt{\langle\vartheta(\kappa),\vartheta(\kappa)\rangle}} + [j(\vartheta(\kappa))][j(\vartheta(\kappa))] \frac{1-\cos\sqrt{\langle\vartheta(\kappa),\vartheta(\kappa)\rangle}}{\langle\vartheta(\kappa),\vartheta(\kappa)\rangle}$

However the group D is not compact.

The axis of rotation is by definition the unique eigen vector of $[g]$ with eigen value 1 and norm 1 in the standard representation of SO(3). It is easy to see that its components are proportional to $[r] = \begin{bmatrix} \kappa^1 \\ \kappa^2 \\ \kappa^3 \end{bmatrix}$.

For any vector u of norm 1 : $\langle u, gu\rangle = \cos\theta$ where θ is an angle which depends on u. With the formula above, and using

$[j(\vartheta(\kappa))][j(\vartheta(\kappa))] = [\vartheta(\kappa)][\vartheta(\kappa)]^t - \langle\vartheta(\kappa),\vartheta(\kappa)\rangle I$ and $\langle u, [j(\vartheta(\kappa))]u\rangle = 0$ we get :

$\langle u, gu\rangle = 1 + \left(\langle u,\vartheta(\kappa)\rangle^2 - \langle\vartheta(\kappa),\vartheta(\kappa)\rangle\right) \frac{1-\cos\sqrt{\langle\vartheta(\kappa),\vartheta(\kappa)\rangle}}{\langle\vartheta(\kappa),\vartheta(\kappa)\rangle}$

which is minimum for $\langle u,\vartheta(\kappa)\rangle = 0$ that is for the vectors orthogonal to the axis, and :

$\cos\theta = \cos\sqrt{\langle\vartheta(\kappa),\vartheta(\kappa)\rangle}$

So we can define the angle of rotation by the scalar $\sqrt{\langle\vartheta(\kappa),\vartheta(\kappa)\rangle}$.

A constant rotation is a rotation with the same axis $\overrightarrow{r} = \frac{\vartheta(\kappa)}{\langle\vartheta(\kappa),\vartheta(\kappa)\rangle}$, and the angular speed : $\frac{d\theta}{dt} = \omega = Ct$ thus the rotation after the time t is : $\exp t\omega j(\overrightarrow{r})$. Equivalently one can represent

the rotation with axis $\overrightarrow{r}$ and angular speed $\varpi = \sqrt{\langle \vartheta(\kappa), \vartheta(\kappa) \rangle}$ by the vector $\vartheta(\kappa)$,

6.2 Motion of a body

6.2.1 Geometric definition

A motion is always relative, that is it is defined with respect to a frame (galilean or not). The geometric state of a rigid body, a solid, can be defined by the displacement D of a frame $\left(A(t), (e_i(t))_{i=1}^3\right)$ attached to it, with respect to a frame of reference. Because time and spatial coordinates are disconnected it is possible to assign a time to every state : $D(t) = (R(t), L(t))$.

The motion of a body is then defined by the left logarithmic derivative of the displacement, which belongs to the Lie algebra of the group :

$D(t)^{-1} \frac{d}{dt} D(t) = \left(R(t)^{-1}, -R(t)^{-1}(L(t))\right)\left(\frac{d}{dt}R(t), \frac{d}{dt}L(t)\right) = (\kappa(t), u(t))$

and comprises two vectors :

$\kappa(t) = R(t)^{-1} \frac{d}{dt} R(t) \in so(3), u(t) = R(t)^{-1}\left(\frac{d}{dt}L(t) - L(t)\right) \in \mathbb{R}^3$

In a change of frames :

$\left(O, (\varepsilon_i)_{i=1}^3\right) \to \left(\widetilde{O}(t), (\widetilde{\varepsilon}_i(t))_{i=1}^3\right) = (R_0(t), L_0(t))\left(O, (\varepsilon_i)_{i=1}^3\right) = D_0(t)\left(O, (\varepsilon_i)_{i=1}^3\right)$

$\widetilde{D}(t)^{-1} \frac{d}{dt}\widetilde{D}(t) = (\widetilde{\kappa}(t), \widetilde{u}(t))$

$\widetilde{u}(t) = R_0(t)\, u(t) + \left(I + \frac{dR_0}{dt} R_0(t)^{-1} - Ad_{R_0(t)}\left(R(t)^{-1}\frac{d}{dt}R(t)\right)\right) L_0(t) - \frac{dL_0}{dt}$

$\widetilde{\kappa}(t) = Ad_{R_0(t)}(\kappa(t)) - \frac{dR_0}{dt} R_0(t)^{-1}$

So we have two different cases:

- if $D_0(t) = D_0$ is a constant displacement, then :

$\widetilde{u}(t) = R_0 u(t) + \left(I - Ad_{R_0}\left(R(t)^{-1}\frac{d}{dt}R(t)\right)\right) L_0(t)$

$\widetilde{\kappa}(t) = Ad_{R_0}(\kappa(t))$

u is transformed by an affine map, and κ by a linear map.

- if not then the transformations laws are more complicated.

Usually only the first ones, the galilean change of frames, are considered, the others, which involve the motion $D_0(t)^{-1}\frac{d}{dt}D_0(t)$ of the frames are seen as defining a different motion.

However this definition is a bit abstract, and not obviously related to quantities which can be easily measured.

6.2.2 Physical definition

Actually most physicists would not be dogmatic with respect to the group of displacements. They would simply define the motion as a couple of vectors of the underlying vector space $\overrightarrow{E}$: $(\overrightarrow{r}(t), \overrightarrow{v}(t))$ where $\overrightarrow{v}(t) = \sum_{i=1}^3 \left(\frac{d}{dt}x^i(t)\right)\varepsilon_i$ is the speed and $\overrightarrow{r}(t)$ is the vector defined as the axis of rotation, normed to the instantaneous angular speed of rotation. Both vectors change as any other vector in a change of frame.

Fortunately both vectors $(\overrightarrow{r}(t), \overrightarrow{v}(t))$ are directly related to the vectors $(\kappa(t), u(t))$ of the Lie algebra of the group of displacements, and change according to the same rules in a galilean change of frames $(D_0 = Ct)$:

$\kappa(t) \rightarrow \widetilde{\kappa}(t) = Ad_{R_0}\kappa(t) = [R_0][j(\vartheta(\kappa(t)))][R_0]^{-1}$ which has the same eigen vectors as $[j(\vartheta(\kappa(t)))]$. Moreover the norm is unchanged in a change of gauge. So the "rotation vector" of $\widetilde{\kappa}(t)$ is the same as $\kappa(t)$ and its components transform as any other vector of $\overrightarrow{E}$.

Going from the purely geometric definition $(\kappa(t), u(t))$ to the practical definition $(\overrightarrow{r}(t), \overrightarrow{v}(t))$ can be seen mathematically as defining the motion by two vectors belonging to a vector space on which the group SO(3) is represented. This is obvious for $\overrightarrow{v}(t)$. $\overrightarrow{r}(t)$ is defined through the Lie algebra, but is not an element of the Lie algebra so(3) (it would be a matrix), this is just a vector of $\overrightarrow{E}$.

However by defining the instantaneous rotation through the Lie algebra so(3) we have introduced an ambiguity : SO(3) is not the only group which has so(3) as Lie algebra, the other one is the double cover of SO(3) which is the Spin group Spin(3). This can be understood from the relation : $R(t)^{-1}\frac{d}{dt}R(t)$ between elements of the group. In the Spin groups multiplication by the scalar -1 is valid : if g belongs to a Spin group, then -g still belongs to the group. So both +R(t) and -R(t) define equivalent instantaneous rotations. Similarly we see that the same vector $\overrightarrow{r}$can represent a rotation with axis $\overrightarrow{r}$ and speed $\|\overrightarrow{r}\|$ or the rotation with axis -$\overrightarrow{r}$ and speed -$\|\overrightarrow{r}\|$. Whenever we consider the motion of a body, the axis and speed of rotation should be defined by continuous maps, so the two rotations are not equivalent, and indeed the definition through the Spin group is more pertinent because it makes the distinction between the two rotations. These particularities lead often the physicists to say that $\overrightarrow{r}$ is an axial vector. Indeed it depends on the orientation of the basis, but the roots of these particularities are deeper than that.

This representation of a rotation is similar to the representation of a circular trajectory in a plane : the vector $\overrightarrow{r}$ is orthogonal to the plane and its length proportional to the angular speed on the trajectory.

6.2.3 Inertial characteristics

The inertia of a body, that is its tendency to resist at any change of its motion, is characterized by its linear momentum and its angular momentum.

With respect to any frame, the linear momentum of a particle of mass m and speed v it is represented by a vector : $\overrightarrow{p} = m\overrightarrow{v}$. This is an additive quantity which is well defined for any system $\overrightarrow{P} = \sum_{i\in I}\overrightarrow{p}_i = \sum_{i\in I} m_i\overrightarrow{v}_i$ and in a galilean change of frames it transforms as any other vector. The momentum $\overrightarrow{p}$ is, mathematically, a localized vector, as well as $\overrightarrow{v}$: it is defined at a point m. To incorporate this information it is convenient to introduce the angular momentum : $\overrightarrow{\ell} = \overrightarrow{x} \times \overrightarrow{p}$ where $\overrightarrow{x} = \overrightarrow{OM}$, with respect to a frame (O, ε_i). For a rotating solid it leads to the definition of a new variable.

If the particles belong to a solid, with center of mass G, attached frame $(G, e_i) = (R(t), L(t)) \times (O, \varepsilon_i)$ with coordinates y, comprised of particles with mass $(m_i)_{i\in I}$, the sum of the angular momenta with respect to O reads :

$\sum_i \overrightarrow{\ell}_i = \sum_i \overrightarrow{x}_i \times \overrightarrow{p}_i$

We have :

$x_i = R(t)\, y_i + L(t)$

$\sum_i y_i m_i = 0 \Rightarrow L(t) = \frac{1}{M}\sum_i x_i(t)\, m_i$

$\frac{dx_i}{dt} = \frac{dR}{dt}y_i + \frac{dL}{dt} = Rj(r)\, y_i + \frac{dL}{dt}$

$\overrightarrow{p}_i = m\overrightarrow{v}_i = m_i\frac{dx_i}{dt}$

$\sum_i \overrightarrow{x}_i \times \overrightarrow{p}_i = \sum_i j(x_i) m_i \frac{dx_i}{dt} = RJr + j(L) M \frac{dL}{dt}$
using the identity, for $R \in O(3) : j(Ru)Rv = Rj(u)v \Leftrightarrow Ru \times Rv = R(u \times v)$
So : $\sum_i \overrightarrow{\ell}_i = \overrightarrow{\ell} + L \times \overrightarrow{P}$
where :
$J = -\sum_i m_i j(y_i) j(y_i) = \sum_i m_i \left([y_i]^t [y_i] I - [y_i] [y_i]^t \right)$ is the moment of inertia of the solid. It does not depend of the choice of the orthonormal basis.

$\overrightarrow{\ell} = RJ\overrightarrow{r}$ is the rotational momentum of the solid (expressed in the frame O). So the total angular momentum due to the solid is the sum of the angular momentum $L \times \frac{d\overrightarrow{P}}{dt}$ related to its center of mass, and of the rotational momentum $\overrightarrow{\ell}$. It is represented by a vector in $\mathbb{R}^3$.

A rotational momentum can be similarly defined for particle with a circular orbit by taking : $L(t) = R(t) L_0$
$\overrightarrow{\ell} = L \times \overrightarrow{P} = R(t) L_0 \times mRj(r) L_0 = -mR(t) j(L_0) j(L_0) r \Rightarrow J = -j(L_0) j(L_0)$

For a particle the definition holds for the linear momentum and the angular momentum, but fails for the rotational momentum if the particle has no internal structure and no dimension. However one observes features similar to a rotational momentum (electrons orbiting a nucleus or with intrinsic spin) which are represented, by extension, as vectors in $\mathbb{R}^3$ proportional to the rotation vector $\overrightarrow{r}$.

The kinetic energy of a system of particle reads :
$\frac{1}{2} \sum_i \langle \overrightarrow{p}_i, \overrightarrow{v}_i \rangle$
If these particles belong to a solid :
$\frac{1}{2} \sum_i \langle \overrightarrow{p}_i, \overrightarrow{v}_i \rangle = \frac{1}{2} \sum_i m_i \left\langle Rj(r) y_i + \frac{dL}{dt}, Rj(r) y_i + \frac{dL}{dt} \right\rangle = \frac{1}{2} M \left\| \frac{dL}{dt} \right\|^2 + \frac{1}{2} r^t J r$

6.3 Quantum mechanics in Galilean geometry

6.3.1 The model

To represent the motion of a body in Quantum Mechanics, the model must satisfy the main requisite : the variables must be maps belonging to a Banach vector space. For a particle, we can take its position x(t) and the two vectors $(\overrightarrow{r}(t), \overrightarrow{v}(t))$, their components being measured in a galilean basis. But we must give up part of our freedom of gauge : x must satisfy linear transformations, affine transformations would not do. So the origin of the galilean frame is fixed. Every variable x, r, v is a smooth map from [0,T] in $\mathbb{R}$ in the vector space $\overrightarrow{E}$, defined by three components $\left(x^i, r^i, v^i\right)_{i=1}^3$ in a galilean basis, and belonging to Banach vector spaces W_x, W_v, W_r with product W. Moreover the variables must be dimensionless, so that they must be scaled by some fixed scalars. The first theorem of QM tells us that W is isometric to an open subset of a Hilbert vector space H. The state of the system is represented by a vector Z of W to which is associated by an isometry $\Upsilon : W \to H$ a unique vector ψ of H. To each variable is associated a self adjoint, compact operator acting on H.

6.3.2 Gauge transformations with SO(3)

The variables transform as vectors, W_x, W_v, W_r are representations of the group SO(3) with the gauge transformations. We can skip the usual machinery of commuting operators and implement directly the Wigner's theorem. The Hilbert space H is isomorphic to a unitary representation of SO(3). This is a compact group, so its unitary representations are sums of orthogonal, finite dimensional, unitary representations, which are representations over the spaces of degree 2N homogeneous

polynomials with two complex variables. These representations can be transformed into representations, denoted (P_N, D_N) over degree 2N homogeneous polynomials P with 3 real variables (ξ_1, ξ_2, ξ_3) and complex coefficients, which are harmonic $\Delta P = 0$, and D_N is the left action of SO(3) on the arguments of P.

From there, in the common interpretation of Quantum Mechanics, two fundamental assumptions are done :

i) ψ is a "wave function" : a complex scalar map $\psi(x)$ on $\mathbb{R}^3$ where x is a point of the physical space

ii) in the representation by harmonic polynomials the parameters (ξ_1, ξ_2, ξ_3) are the coordinates of the point x of the physical space.

There is nothing to substantiate these assumptions, which are the sources of many interpretations, and controversies. The first issue is that the time t does not appear, so it is difficult to conciliate this static function with the Schrödinger equation. The second is that if, by a mathematical trick it is possible to go from 2 complex variables to three real variables, there is no physical motivation to support the assumption that these parameters (ξ_1, ξ_2, ξ_3) are coordinates in the physical space.

We will not pursue further these issues here, which will be reviewed in more details later.

6.3.3 The Schrödinger equation

There is another group which acts on the variables : the translation in time.

If we stay in the Schrödinger picture the variables are estimated all together at the end of the process, from a batch of data. Time is not an observable, it is a coordinate (indeed the measures can be taken at different times for each variable). In galilean geometry it is defined up to an affine transformation, and one can take a translation.

If an observer uses the time t and another $\widetilde{t} = t - \theta$ with some fixed origin θ , for the same state of the system they will get the maps Z and $\widetilde{Z} = U(Z)$ for some map $U : W \to W$. Then the Wigner's theorem says that the corresponding vectors of H are related by an unitary linear map : $\widetilde{\psi}_\theta = \widehat{U}_\theta(\psi)$. Moreover if $\theta = 0$ then $\widehat{U}_\theta = Id$, and if we take $\theta + \theta'$ by the same procedure we have $\widetilde{\psi}_{\theta+\theta'} = \widehat{U}_\theta\left(\widehat{U}_{\theta'}(\psi)\right) = \widehat{U}_{\theta+\theta'}(\psi)$. So we have a one parameter group of unitary operators, and $\left(H, \widehat{U}\right)$ is a unitary representation of the abelian group $(\mathbb{R}, +)$. We assume that it is uniformly continuous, then :

i) there is a self-adjoint operator denoted $\widehat{H}$ such that : $\widehat{U}(\theta) = \exp\left(\frac{1}{i}\theta\widehat{H}\right)$

ii) there is a unique spectral measure P on the $\sigma-$algebra of $\mathbb{R}$ such that :

$\widehat{U}(\theta) = \int_{\mathbb{R}} \exp(-i\theta s) P(s)$

If $\widehat{U}$ is continuous, it is smooth, and the differentiation of :

$\left\langle \widehat{U}(\theta)\psi, \psi \right\rangle = \int_{\mathbb{R}} \exp(-i\theta s) \langle P(s)\psi, \psi \rangle$

with respect to θ at $\theta = 0$ gives :

$\left\langle \frac{1}{i}\widehat{H}\psi, \psi \right\rangle = -i \int_{\mathbb{R}} \langle P(s)\psi, \psi \rangle$

In the Heisenberg picture the state of the system is measured at different times t (with some frequency) and the time becomes an observable with an associated operator $\widehat{t}$.

In the Schrödinger picture :

$Z = \sum_{i \in I} c_i z_i \leftrightarrow \psi = \sum_{i \in I} c_i \varepsilon_i$ where c_i are fixed scalars and z_i are maps with domain [0,T]

In the Heisenberg picture :

$Z(t) = \sum_{i\in I} d_i(t)\,\zeta_i \leftrightarrow \psi(t) = \sum_{i\in I} d_i(t)\,\omega_i$ where d_i are maps with domain [0,T] and ζ_i are fixed scalars

One goes from one picture to the other by evaluations maps :

$\mathcal{E}(t) : W \to \left(\overrightarrow{E}\right)^3 :: \mathcal{E}(t)(Z) = Z(t)$

which commute with U : $\mathcal{E}(t) \circ U(\theta) = U(\theta) \circ \mathcal{E}(t) = \mathcal{E}(t+\theta)$

And similarly on H :

$\widehat{\mathcal{E}}(t) \circ \widehat{U}(\theta) = \widehat{U}(\theta) \circ \widehat{\mathcal{E}}(t) = \widehat{\mathcal{E}}(t+\theta)$

The equation above reads in the Heisenberg picture:

$\widehat{\mathcal{E}}(t)\,\widetilde{\psi}_\theta = \widehat{\mathcal{E}}(t) \circ \widehat{U}(\theta)(\psi) = \widehat{U}(\theta) \circ \widehat{\mathcal{E}}(t)\,\psi = \widehat{U}(\theta)\,\psi(t) = \widehat{\mathcal{E}}(t+\theta)\,\psi = \psi(t+\theta)$

By taking $\theta = 0$:

$\psi(t) = \exp\left(\frac{1}{i} t\widehat{H}\right)\psi(0) \Rightarrow \frac{d\psi}{dt}(t) = \frac{1}{i}\widehat{H}\psi(t)$

P is the spectral resolution of $\widehat{H}$ thus $P(\mathbb{R}) = Id_H$, $\widehat{H} = \int_{\mathbb{R}} tP(t)$ and $\widehat{H}$ can be seen as the operator associated to the observable $\widehat{t}$:

$\widehat{t}(Z(t)) = \left\langle \widehat{H}\psi(t), \psi(t)\right\rangle = \int_{\mathbb{R}} \langle P(s)\,\psi(t), \psi(t)\rangle$

The value of t which is measured is equal to one of the eigen value of P with the probability $\frac{|\langle P(s)\psi(t),\psi(t)\rangle|^2}{\|\psi(t)\|^2}$

6.3.4 The interpretation from the variables

Actually it is possible to implement directly the theorems of QM to the variables x,r,v.

One can safely assume that the maps belong to the Hilbert space $\mathcal{L}^2\left([0,T];\mathbb{R}^3;dt\right)$ with the scalar product :

$\langle x_1, x_2\rangle = \int_0^T \langle x_1(t), x_2(t)\rangle\, dt$

Indeed [0,T] is compact, the maps are smooth, so the value of their norm belong to a compact of $\mathbb{R}$ and the integral is well defined. The Hilbert space $\mathcal{L}^2\left([0,T];\mathbb{R}^3;dt\right)$ is separable, and has a countable Hilbertian basis $(w_n)_{n\in\mathbb{N}}$ of maps on [0,T].

W_x, W_r, W_v can be seen as closed, orthogonal, vector subspaces of $\mathcal{L}^2\left([0,T];\mathbb{R}^3;dt\right)$. Because they are unitary representations of SO(3) they are the hilbertian sum of finite dimensional Hilbert spaces, which are representations of the groups. Each of these vector spaces $(W^p)_{p=1}^N$ (N can be infinite) has a hilbertian basis comprised of maps of $\mathcal{L}^2\left([0,T];\mathbb{R}^3;dt\right)$. Thus one can write :

$x(t) = \sum_{N=1}^{\infty}\sum_{n=1}^{N} x^{N,n} X_{N,n}(t)$

and the measure of the map x is $\widehat{x}(t) = \sum_{N=1}^{\infty}\sum_{n=1}^{N} \widehat{x^{N,n}} X_{N,n}(t)$ with $\widehat{x^{N,n}}$ either 0 or $x^{N,n}$.

The probability to measure $\widehat{x}(t)$ is $\frac{\sum_{n,N}\left|\widehat{x^{N,n}}\right|^2}{\sum_{n,N}|x^{N,n}|^2}$.

The accuracy of the measure, and thus this probability, depends on the dimension N of the space and on the number of measures which are taken. N is characteristic of the diversity, real or assumed, of the evolution of the system. If the motion is periodic, of period T, the natural choice for the basis w_n is the Fourier series : $\left(\cos\left(2\pi n\frac{t}{T}\right)\right)_{n=-\infty}^{0}, \left(\sin\left(2\pi n\frac{t}{T}\right)\right)_{n=1}^{+\infty}$. It corresponds to the model of "plane waves" with a spectrum of frequencies given by the Fourier transformation. A more general model is given by "wavelets", with the sinc series $\left(\frac{\sin\pi\left(\left(t-\frac{1}{2}\right)\frac{1}{T}-n\right)}{\pi\left(\left(t-\frac{1}{2}\right)\frac{1}{T}-n\right)}\right)_{n=-\infty}^{+\infty}$. In both cases

there is a strong analogy with signal analysis. The precision of the measures is related to their frequency.

The Schrödinger equations read :

$x(t) = (\exp tL_x)\, x(0) \Rightarrow \frac{dx}{dt}(t) = L_x x(t)$

$v(t) = (\exp tL_v)\, v(0) \Rightarrow \frac{dv}{dt}(t) = L_v v(t)$

$r(t) = (\exp tL_r)\, r(0) \Rightarrow \frac{dr}{dt}(t) = L_r r(t)$

where L_x, L_v, L_r are antisymmetric operators on the respective Hilbert spaces :

$\langle L_x(\overrightarrow{x_1}), \overrightarrow{x_2}\rangle = -\langle \overrightarrow{x_1}, L_x(\overrightarrow{x_2})\rangle$

As : $v(t) = \frac{dx}{dt}$:

$v(t) = L_x x(t)$

$\frac{dv}{dt}(t) = L_v \circ L_x x(t) = \frac{d^2x}{dt^2} = L_x^2 x(t) \Rightarrow (L_v - L_x) \circ L_x x(t) = 0$

And indeed because x and v have the same range one can assume that the basis $X_{N,n}(t)$ is the same, and :

$\frac{dX_{N,n}(t)}{dt} = \sum_{N=1}^{\infty}\sum_{n=1}^{N} y^{N,n} X_{N,n}(t)$ for some fixed scalars, thus

$\frac{dx}{dt}(t) = L_x x(t)$

$\Leftrightarrow \sum_{N=1}^{\infty}\sum_{n=1}^{N} x^{N,n} y^{N,n} X_{N,n}(t) = \sum_{N=1}^{\infty}\sum_{n=1}^{N} L_N\left(x^{N,n}\right) X_{N,n}(t)$

$\Leftrightarrow L_x = L_v$

Going from the Schrödinger to the Heisenberg picture is actually a change of model and, as it happens often, this is not neutral. It entails important consequences.

i) In the model x and v are independant variables. They can be measured independantly, through different procedures. In the Schrödinger picture the maps x(t) and v(t) are simultaneously estimated from a batch of data, after the end of the experiment. Thus we can reasonably assume that the estimations account for their relation, and that the results are consistent : $v(t) = \frac{dx}{dt}$.

In the Heisenberg picture we take snapshots at different times $t_1, ..t_N$ and we estimate x(t), v(t) from the data. And there is no guarantee that the same consistency is met. Indeed the simplest trajectory that we could build from the data would be a succession of straight lines : hardly a smooth curve. Moreover there are physical limits to the possibility of measuring simultaneously the position and the speed of a particle. As Dirac wrote "we have to admit that there is a limit to the fineness of our powers of observation and the smallness of the accompanying disturbance - a limit which is inherent in the nature of things and can never be surpassed" (p.4). But these limits should lead logically to the conclusion that the Heisenberg picture is not physically realist at this scale. And not necessarily to assume some bizarre behaviours from particles.

ii) As noticed before, the variables must be unitless. So there should be some universal system of units to rescale all measures, based on natural phenomena (this is already a requisite of statistical mechanics). This need becomes clear in the Heisenberg picture, where the time becomes an observable, with the associated operator $\widehat{H}$ and enters into an exponential. The Schrödinger equation, which holds for any system, should read :

$\psi(t) = \exp\left(\frac{1}{ik} t\widehat{H}\right)\psi(0)$

with some constant k to absorb the unit. Moreover the conventions used in the definition of the rotation give a special meaning of the length of the vector $\overrightarrow{r}$. So one can see that all these quantities are related.

Of course the value of $\hbar$ has a direct impact on the model, in that the values of the rescaled variables are dramatically increased, and with them the dimension of the spaces W.

7 SPECIAL RELATIVITY

7.1 Geometry

7.1.1 Basic assumptions

The model of Special Relativity is a big leap from Galilean geometry. It can be introduced in different ways. The most useful for our purpose is based upon 5 hypotheses.

i) The universe is modeled as a 4 dimensional affine space M.

ii) The underlying vector space $\overrightarrow{M}$ *(the Minkowski space) is endowed with a scalar product (the Lorentz metric) which reads (with the signature (3,1)) in an orthonormal basis* $(\varepsilon_i)_{i=0}^3$ *:* $\langle u, v\rangle = \sum_{i=1}^3 u^i v^i - u^0 v^0$

The scalar product splits the affine space at each point according to the sign of $\langle u, u\rangle$ (with this signature the time like vectors have $\langle u, u\rangle < 0$ and the space like vectors $\langle u, u\rangle > 0$).

iii) All material bodies travel along a world line. The time, as measured by a clock (the proper time τ*), depends on the world line. The tangent to the world line, as parametrized by the proper time, is a 4-dimensional vector u (its proper velocity) which has a constant norm :* $\langle u, u\rangle = -k^2$*. Similarly the field forces, such as the electromagnetic field, propagate along trajectories such that* $\langle u, u\rangle = 0$ *(the light cone).*

iv) The space of each observer (the set of events which are simultaneous for the observer at any time) is an hyperplane orthogonal to his velocity.

v) There is a privileged class of observers, who do not measure any inertial force (the inertial observers), whose orthonormal frames $(\varepsilon_i)_{i=0}^3$ *are constant.*

From these hypotheses one can prove that :

i) The spatial speed of ligth c is the same for any observer

ii) the constant k=c the speed of light.

Thus the model is cleary rooted in the representation of physical measures.

One can see the galilean geometry as a 4 dimensional affine space in which all the observers have parallel velocity.

7.1.2 The Lorentz group

From these hypotheses one can deduce the formulas for the gauge transformations, which are a crucial part of Special Relativity, and have been verified with a great accuracy.

As in the galilean model, the only gauge transformations which are considered are change of coordinates. They constitute the group of displacements (the "Poincaré group"), semi-product of the group of isometries (linear transformations which preserve the scalar product) and the abelian group of translations in $\mathbb{R}^4$ with the same composition rule. In any given frame of reference the isometries are represented by the matrix group O(3,1), and the isometries which preserve also the orientation of frames by the subgroup SO(3,1) with determinant 1.

O(3,1) is the group of 4×4 real matrices such that :

$[g]^t [\eta] [g] = [\eta]$ where $[\eta] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

and moreover $[g] \in SO(3,1) \Rightarrow \det [g] = 1$.

Both O(3,1) and SO(3,1) have same Lie algebra o(3,1) (the condition det(g)=0 leads to Tr(h)=0 which is always met). This is a 6 dimensional real vector space : the subset of 4×4 real matrices such that : $[h]^t [\eta] + [\eta] [h] = [0]$, and we will use as basis of o(3,1) the following matrices :

$$[\kappa_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} ; [\kappa_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} ; [\kappa_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\kappa_4] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; [\kappa_5] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; [\kappa_6] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

so that any matrix of o(3,1) can be written :
$[\kappa] = [J(r)] + [K(w)]$ with

$$[J(r)] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r_3 & r_2 \\ 0 & r_3 & 0 & -r_1 \\ 0 & -r_2 & r_1 & 0 \end{bmatrix} ; [K(w)] = \begin{bmatrix} 0 & w_1 & w_2 & w_3 \\ w_1 & 0 & 0 & 0 \\ w_2 & 0 & 0 & 0 \\ w_3 & 0 & 0 & 0 \end{bmatrix}$$

The exponential of these matrices read :
$\exp [K(w)] = I_4 + \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} K(w) + \frac{\cosh \sqrt{w^t w} - 1}{w^t w} K(w) K(w)$
$= \begin{bmatrix} \cosh \sqrt{w^t w} & w^t \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} \\ w \frac{\sinh \sqrt{w^t w}}{\sqrt{w^t w}} & I_3 + \frac{\cosh \sqrt{w^t w} - 1}{w^t w} w w^t \end{bmatrix}$
$\exp [J(r)] = I_4 + \frac{\sin \sqrt{r^t r}}{\sqrt{r^t r}} J(r) + \frac{1 - \cos \sqrt{r^t r}}{r^r r} J(r) J(r) = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$

where R a 3x3 matrix of SO(3)

The group O(3) has two connected components : the subgroup SO(3) with determinant = 1, and the subset $O_1(3)$ with determinant -1.

O(3,1) has four connected components which can be distinguished according to the sign of the determinant and their projection under the compact subgroup $SO(3) \times \{I\}$.

- $SO_0(3,1)$, the connected component of the identity with determinant = 1. Its elements can be written :

$M = \exp K(w) \times \exp J(r) = \exp K(w) \times \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$

- $SO_1(3,1)$: with determinant 1: $M = \exp K(w) \times \begin{bmatrix} -1 & 0 \\ 0 & -R \end{bmatrix}$

- $SO_2(3,1)$ with determinant = -1: $M = \exp K(w) \times \begin{bmatrix} -1 & 0 \\ 0 & R \end{bmatrix}$

- $SO_3(3,1)$ with determinant = -1: $M = \exp K(w) \times \begin{bmatrix} 1 & 0 \\ 0 & -R \end{bmatrix}$

where R a 3x3 matrix of SO(3), so that $-R \in O_1(3)$

$SO_k(3,1), k = 1,2,3$ are generated by the product of any element of $SO_0(3,1)$ by either :

the time reversal matrix : $T = \begin{bmatrix} -1 & 0 \\ 0 & I_3 \end{bmatrix}$

or the space reversal matrix : $S = \begin{bmatrix} 1 & 0 \\ 0 & -I_3 \end{bmatrix}$

So, in the model of the geometry of Special Relativity, several questions, which were not encountered in Galilean Geometry, arise with regard to the choice of the goup.

7.1.3 The issues

Which signature for the metric ? Because $O(3,1) \equiv O(1,3)$ the same matrix $[\kappa_i]$ can be used to represent a basis vector in the standard representation of the Lie algebra. The only change when going from the signature (3,1) to the signature (1,3) is the replacement of $[\eta]$ by $-[\eta]$ (in the following $[\eta]$ will always refer to the signature (3,1)).

So any choice of signature is consistent with the hypotheses, with the obvious adjustments of the definitions of "space-like" and 'time-like" vectors.

However the Clifford algebras Cl(3,1) and Cl(1,3) are not isomorphic, and as we shall see later, this issue may have physical significance.

Is the universe orientable ? In a universe where all observers have the same time, the simple existence of stereoisomers which do not have the same chemical properties suffices. In a space-time universe one needs a process with an outcome which discriminates an orientation. All chemical reactions starting with a balanced mix of stereoisomers produce an equally balanced mix (stereoisomers have the same level of energy). However there are experiments involving the weak interactions which show the required property. So we can state that the 4 dimensional universe is orientable, and then we can distinguish orientation preserving gauge transformations. The right group to consider is SO(3,1).

The universe of Special Relativity is no longer isotropic : all directions at not equivalent. At any point one can discriminate the vectors v according to the value of the scalar product <v,v> . Moreover the subset of time like vectors has two disconnected components (this is no longer true in universes with more than one "time component"). This has two important consequences.

A change of gauge, physically, implies some transport of the frame (one does not jump from one point to another) : we have an isometry $\Phi : M \to M$ along a path $p : I \subset \mathbb{R} \to M$ and the path which is followed matters. In particular it is connected. The frame $(e_i)_{i=0}^3$ is transported by : $\widetilde{e}_i(\tau) = \Phi'(p(\tau))\, e_i(0) = g(\tau)\, e_i(0)$. So $g(\tau)$, image of the connected interval I by a continuous map is a connected subset of SO(3,1), and because g(0) = Id it must be the component of the identity. So the rigth group to consider is the connected component of the identity $SO_0(3,1)$.

Because the subset of time like vectors has two disconnected components one can discriminate these components and, in accordance with the assumptions about the velocity of material bodies, it is logical to consider that their velocity is "future oriented". By continuity the light cone can

be similarly oriented, and the fields are assumed to propagate towards the future. And one can distinguish gauge transformations which preserve this time orientation.

We will assume that the future orientation is given in a frame by the vector ε_0. So a vector u is time like and future oriented if :

$\langle u,u\rangle<0,\langle u,\varepsilon_0\rangle<0$

A matrix $[G]$ of $SO_0(3,1)$ preserves the time orientation iff $[G]_0^0>0$ and this will always happen if $[G]=\exp[K(w)]\exp[J(r)]$ that is if $[G]\in SO_0(3,1)$.

A gauge transformation which preserves both the time orientation, and the global orientation must preserve also the spatial orientation.

We can now express the formula for these gauge transformations.

7.1.4 Gauge transformation formulas

Let :

O be an inertial observer with frame $(O,(\varepsilon_i)_{i=0}^3)$ and proper time t, O is its position at t = 0 and $\frac{d\varepsilon_i}{dt}=0$ because he is inertial.

A be any observer, with frame $\left(A(\tau),(e_i(\tau))_{i=0}^3\right)$ and proper time τ.

For any observer an euclidean spatial frame is defined in his present (by definition he is at rest with respect to this frame) thus it is orthogonal to its velocity. And the velocity of A :

$u(\tau)=\frac{dA}{d\tau}=ce_0(\tau)$

because $\langle u,u\rangle=-c^2$ and $\langle e_0(\tau),e_0(\tau)\rangle=-1$ and $e_0(\tau),u(\tau)$ are future oriented.

Let $\left(L^i\right)_{i=0}^3$ be the coordinates of A in the frame of O. Then $L^0=ct$ because O observes A in his proper time. $u(\tau)$ is a vector so it can be measured by the observer in his own frame :

$u=\sum_{i=0}^3\frac{dL^i}{d\tau}\varepsilon_i=\frac{dt}{d\tau}\sum_{i=0}^3\frac{dL^i}{dt}\varepsilon_i=\frac{dt}{d\tau}\left(\overrightarrow{v}+\frac{dL^0}{dt}\varepsilon_0\right)=\frac{dt}{d\tau}\left(\overrightarrow{v}+c\varepsilon_0\right)$

with the usual relative spatial speed $\overrightarrow{v}$ of A with respect to O.

$\langle u,u\rangle=\left(\frac{dt}{d\tau}\right)^2\left(\|\overrightarrow{v}\|^2-c^2\right)=-c^2\Rightarrow\frac{dt}{d\tau}=\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}$

thus in the basis of O : $u=\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}\left(\overrightarrow{v}+c\varepsilon_0\right)=c\frac{1}{\sqrt{-\langle V,V\rangle}}V$ with $V=\frac{dA}{dt}=\overrightarrow{v}+c\varepsilon_0$

The gauge transformation to go from the frame of O to the frame of A is given by a displacement with a translation $\overrightarrow{L}(t)=\sum_{i=0}^3\varepsilon_iL^i(t)=\overrightarrow{OA}$, and a rotation $G(t)\in SO_0(3,1)$ such that: $e_i(\tau)=\sum_{j=0}^3[G(t)]_i^j\varepsilon_j$ and

$$[G(t)]=\begin{bmatrix}\cosh\sqrt{w^tw} & w^t\frac{\sinh\sqrt{w^tw}}{\sqrt{w^tw}}\\ w\frac{\sinh\sqrt{w^tw}}{\sqrt{w^tw}} & I_3+\frac{\cosh\sqrt{w^tw}-1}{w^tw}ww^t\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}$$

for some $w\in\mathbb{R}^3,R\in SO(3)$

The elements of the first column of G are the components of e_0, that is of u/c:

$\cosh\sqrt{w^tw}=\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}$

$w\frac{\sinh\sqrt{w^tw}}{\sqrt{w^tw}}=\frac{\overrightarrow{v}}{c}\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}$

$w=k\overrightarrow{v}\Rightarrow w^tw=k^2\|\overrightarrow{v}\|^2$

which leads to the classical formula with

$w = \frac{v}{\|v\|} \arg\tanh \left\| \frac{v}{c} \right\| = \frac{1}{2} \frac{v}{\|v\|} \ln \left(\frac{c+\left\|\overrightarrow{v}\right\|}{c-\left\|\overrightarrow{v}\right\|} \right)$:

$$[G] = \begin{bmatrix} \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}} & \frac{\frac{v^t}{c}}{\sqrt{1-\frac{\|v\|^2}{c^2}}} \\ \frac{\frac{v}{c}}{\sqrt{1-\frac{\|v\|^2}{c^2}}} & I_3 + \left(\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}} - 1 \right) \frac{vv^t}{\|v\|^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

Notice that the observer A is not necessarily inertial so the formula holds for any observer A. If A is also an inertial observer then, by definition, its frame is constant, which implies that $\overrightarrow{v} = Ct \Rightarrow w, R = Ct$.

7.2 Motion of a body

7.2.1 The issues with the group of displacements

Whenever it is possible to identify frames $\left(A(t), (e_i(t))_{0=1}^3\right)$ attached to a body A, the formulas above hold, and one can define the motion of the body itself by the displacement D(t) of A with respect to an inertial frame $O = (O, (\varepsilon_i)_{i=0}^3)$. The observer O moves along his world line with his proper time t, the origin O is fixed and represents the position of the observer at t = 0, the vectors ε_i stay the same. Then the motion of A is characterized by :

$D(t)^{-1} \frac{d}{dt} D(t) = \left(G(t)^{-1} \frac{d}{dt} G(t), R(t)^{-1} \left(\frac{d}{dt} L(t) - L(t) \right) \right) \in so(3,1) \ltimes \mathbb{R}^4$

that is by two vectors

$\kappa(t) = G(t)^{-1} \frac{d}{dt} G(t) \in so(3,1), u(t) = G(t)^{-1} \left(\frac{d}{dt} L(t) - L(t) \right) \in \mathbb{R}^4$

However in Relativist Geometry we meet some problems with this definition. The first, and obvious one, being that the time t depends now of the observer.

If we consider another inertial frame of reference

$\widetilde{O} = \left(\widetilde{O}(\widetilde{t}), (\widetilde{\varepsilon}_i(\widetilde{t}))_{i=1}^3 \right) = D_0(t)(O(t), (\varepsilon_i)_{i=0}^3)$,

$D_0(t) = (G_0(t), L_0(t))$

then it must be in constant translation with respect to O ($\overrightarrow{v} = Ct$), so the rotation G_0 is constant, $\widetilde{\varepsilon}_i(\widetilde{t}) = \widetilde{\varepsilon}_i = R_0 \varepsilon_i$, $L_0(t) = (ct, v^1 t, v^2 t, v^3 t)$ and the times are related by : $\widetilde{t} = t\sqrt{1 - \frac{\|v\|^2}{c^2}}$.

$\left(A(t), (e_i(t))_{i=1}^3\right) = D(t)(O(t), (\varepsilon_i)_{i=0}^3) = \widetilde{D}(\widetilde{t}) \left(\widetilde{O}(\widetilde{t}), (\widetilde{\varepsilon}_i)_{i=1}^3 \right) = \widetilde{D}(\widetilde{t}) D_0(t)(O(t), (\varepsilon_i)_{i=0}^3)$

$\widetilde{D}(\widetilde{t}) = D(t)(G_0, L_0(t))^{-1} = D(t) D_0(t)^{-1}$

$D(t)^{-1} \frac{d}{dt} D(t) \to \widetilde{D}(\widetilde{t})^{-1} \frac{d}{d\widetilde{t}} \widetilde{D}(\widetilde{t}) = D_0(t) D(t)^{-1} \frac{dt}{d\widetilde{t}} \frac{d}{dt} \left(D(t) D_0(t)^{-1} \right)$

$= \frac{dt}{d\widetilde{t}} \left(D_0(t) D(t)^{-1} \left(\frac{d}{dt} D(t) \right) D_0(t)^{-1} + D_0(t) D(t)^{-1} D(t) \frac{d}{dt} \left(D_0(t)^{-1} \right) \right)$

$= \frac{dt}{d\widetilde{t}} \left(Ad_{D_0(t)} \left(D(t)^{-1} \frac{d}{dt} D(t) \right) - \left(\frac{d}{dt} D_0(t) \right) D_0(t)^{-1} \right)$

$= \frac{dt}{d\widetilde{t}} \left(Ad_{D_0(t)} \left(D(t)^{-1} \frac{d}{dt} D(t) \right) - \left(0, \frac{dL_0}{dt}\right) \left(G_0^{-1}, -G_0^{-1}(L_0(t)) \right) \right)$

$= \frac{dt}{d\widetilde{t}} \left(Ad_{D_0(t)} \left(D(t)^{-1} \frac{d}{dt} D(t) \right) - \left(0, \frac{dL_0}{dt}\right) \right)$

So the transformation involves $\frac{dt}{d\widetilde{t}}$ which is not linearly related to $\kappa(t), u(t)$.

Another way to proceed would be to consider the displacement with respect to a frame attached to the body itself, parametrized by the proper time τ of A :

$\left(A(\tau),(e_i(\tau))_{0=1}^3\right) = D(\tau)\left(A(0),(e_i(0))_{0=1}^3\right)$

Then taking $D(\tau)^{-1}\frac{d}{d\tau}D(\tau)$

It is independant of the choice of an inertial frame of reference, however it will depend on the choice of the frame at $\tau = 0$. The only case when this does not happen is if $D(\tau)$ is a one parameter group, meaning that if $D(\tau)^{-1}\frac{d}{dt}D(\tau) = Ct$.

Moreover in this case we have to account for the condition $\left\|\frac{dA}{d\tau}\right\|^2 = -c^2$.

In both cases the motion is represented by two vectors $\kappa(t) \in so(3,1), u(t) \in \mathbb{R}^4$. But, as a consequence of the assumption iv), these two vectors are not independant , as the formulas in a change of gauge show : G(t) = expJ(r(t))expK(w(t)) with a complicated and non linear relation between w and u :

$u(t) = G(t)^{-1}\left(\frac{d}{dt}L(t) - L(t)\right)$ with $\frac{d}{dt}L(t) = \overrightarrow{v} + c\varepsilon_0$

$w = \frac{1}{2}\frac{v}{\|v\|}\ln\left(\frac{c+\left\|\overrightarrow{v}\right\|}{c-\left\|\overrightarrow{v}\right\|}\right)$

A somewhat laborious but straightforward computation starting from:

$\exp K(w) = I_4 + AK(w) + BK(w)K(w)$

$\exp J(r) = I_4 + CJ(r) + DJ(r)J(r)$

with $A = \frac{\sinh\sqrt{w^t w}}{\sqrt{w^t w}}, B = \frac{\cosh\sqrt{w^t w}-1}{w^t w}; C = \frac{\sin\sqrt{r^t r}}{\sqrt{r^t r}}, D = \frac{1-\cos\sqrt{r^t r}}{r^r r}; \sqrt{w^t w} = \frac{1}{2}\ln\left(\frac{c+\left\|\overrightarrow{v}\right\|}{c-\left\|\overrightarrow{v}\right\|}\right)$

gives the following formula for the element of the Lie algebra so(3,1) representing the rotation of the two frames :

$[G(t)]^{-1}\left[\frac{dG}{dt}\right] = K\left(-Aj(w)[X]\frac{dr}{dt} + \left(I + \frac{1-A}{\langle w,w\rangle}j(w)j(w)\right)\frac{dw}{dt}\right)$

$+J\left((1+\langle w,w\rangle)[X]\frac{dr}{dt} + Bj(w)\frac{dw}{dt} - B\left\langle w,[X]\frac{dr}{dt}\right\rangle w\right)$

with $[X] = \left(I - Dj(r) + \frac{1-C}{r^t r}j(r)j(r)\right)$ and

$\frac{dw}{dt} = \frac{1}{\|v\|}\frac{1}{2}\ln\left(\frac{c+\left\|\overrightarrow{v}\right\|}{c-\left\|\overrightarrow{v}\right\|}\right)\frac{dv}{dt} - \frac{1}{\|v\|^2}\ln\left(\frac{c+\left\|\overrightarrow{v}\right\|}{c-\left\|\overrightarrow{v}\right\|}\right)\left\langle v,\frac{dv}{dt}\right\rangle v - 2\frac{c}{\|v\|}\frac{c-\left\|\overrightarrow{v}\right\|}{\left(c+\left\|\overrightarrow{v}\right\|\right)^3}\left\langle v,\frac{dv}{dt}\right\rangle v$

So $\kappa(t)$ depends on $\overrightarrow{v}, r$ and their derivatives $\frac{dv}{dt}, \frac{dr}{dt}$

This is just the consequence of the formulas in a change of gauge between two observers (we remind that only one of the two must be inertial), formulas which have been checked with a good accuracy. Indeed this is not surprising. In galilean geometry the group of displacements is the semi-direct product of two groups which represent motions which are not related (translation and rotation). In relativity, by combining space and time together one extends the scope of rotation to incorporate the translation. A displacement is characterized by the parameters : L, r, and w which is related to the derivative of L.

There is a more general objection to the use of the group of displacement. As it has been noticed before, to base the comparison between two frames on a displacement assumes that we jump from one point to another (or at least that we follow a straight line). But in SR the universe is no longer isotropic, so the path which is followed to transport a frame from one point to the other matters (as can be seen in the formulas above, which involve $\overrightarrow{v}$). And this is more obvious in the GR picture. So, in the definition of the motion of a body we should consider independantly the translation part (L) and the rotation part (R) and give up the group of displacements. But as seen above, the rotation itself already incorporates (in the guise of w) the translation part.

We could consider, as in galilean geometry, the axis of rotation. However, with

$R(t) = \exp K(w(t)) \exp J(r(t))$ one can chek that, if the 3 dimensional axis of the spatial rotation is still an eigen vector of R, with eigen value 1, it may be not unique. If $w^t r = 0$, that is if the axis of spatial rotation is orthogonal to the spatial translation, then the eigen space of R for the eigen value 1 is two dimensional.

There are also physical objections to the use of the group of displacements :

i) there is no rigid body in relativity, and anyway the direct observation of a spinning particle is difficult to conceive

ii) particles show specific features (such as chirality) which seem difficult to represent in the group of displacements.

So we have to find another way, but for this it is necessary to precise how a physical system is modelled and observed. This is where enters the fiber bundle formalism.

7.2.2 Principal bundle of frames

The description of any physical system (out of cosmology) needs an observer and the definition of a bounded area of the universe. So let O be an inertial observer with the frame $\left(O, (\varepsilon_i)_{i=0}^3\right)$. He follows a path $(ct, 0, 0, 0)$ with his proper time t, assumed to go from 0 to T. A measure done at a point A, and assigned to the date t is not recorded "live" : it arrives at O at a time t_1 which is such that the spatial distance d at t between A and O(t) is $d < c(t_1 - t)$. Thus the area that can be observed is in the hypercone of M with apex (cT,0,0,0), axis ε_0 and spatial diameter 2cT. So two different observers, even inertial, will not be able to observe the same area. In Relativity the definition of the system depends on the observer and, if we consider a unique observer, the system is actually uniquely defined and the concept of gauge transformation is meaningless.

However one can imagine a network of observers, located at each point of the area of the system in the 4 dimensional universe M, and taking measures using coordinated frames. Actually these observers can be virtual : knowing the formulas in a gauge transformation (assuming that they are right), one can compute what would be the measures taken by observers belonging to the networks but using different inertial frames. This sums up to the definition of a principal bundle of orthonormal frames over the affine space M. To be precise, it is done in several steps.

In the following we will address the Minkowski space as if it were a manifold, this is no more complicated and so it will be easy to generalize all the results to the General Relativity.

i) There is a standard, inertial, observer, whose clocks are used to give a synchronized time t to all the observers of the network.

ii) The system is defined in the area Ω of the Minkovski space M, delimited by the hyperplanes $\Omega_3(0)$ t = 0 and $\Omega_3(T)$ t = T of the standard observer.

iii) A chart φ_M of M in Ω is provided by the frame $\left(O, (\varepsilon_i)_{i=0}^3\right)$, where O is the position of the standard observer at t=0, by :

$\varphi_M : \mathbb{R} \times \Omega_3(0) \to \Omega :: \varphi_M(t, x) = \overrightarrow{Om} = ct\varepsilon_0 + \sum_{i=0}^3 \xi^i \varepsilon_i$

with $x = \sum_{i=0}^3 \xi^i \varepsilon_i$

A point m of M has the coordinates $\left(\xi^i\right)_{i=0}^3$ in this chart and $\xi^0 = ct$, so that the coordinates are expressed in the same unity of length.

iv) The chart gives a folliation of Ω by hyperplanes
$\Omega_3(t) = \left\{m = \varphi_M(t,x), x \in \mathbb{R}^3\right\}$
corresponding to the "present space" of the observer at the time t.

At each point $m = \varphi_M(t,x)$ there is an inertial observer of the network, so that x can be seen as the position of this observer at t=0 in $\Omega_3(0)$. This observer uses a frame of reference, called a gauge, given by the point m and a basis, denoted $(\varepsilon_i(m))_{i=0}^3$, with the same vectors $(\varepsilon_i)_{i=0}^3$. So all the standard observers use the same time t.

From these frames, used by the standard observers, other frames can be defined at m by rotation of the orthonormal basis through a matrix of $SO_0(3,1)$. This defines the structure of a principal bundle $P_G(M, SO_0(3,1), \pi)$ with trivialization the map : $\varphi_G : M \times SO_0(3,1) \to P_G :: p = \varphi_G(m,g)$. The gauge is $\mathbf{p}(m) = \varphi_G(m,\mathbf{1})$ and a new orthonormal basis at m is given by $p = \varphi_G(m,g) = \mathbf{p}(m) \cdot g$. An observer using a different frame has another time, and we go from one to the other by the formula seen previoulsy.

A change of gauge is a change of frame of reference, that is a change of trivialization. It can be local or global and is defined by a smooth map : $\chi : M \to SO_0(3,1)$ so that the *same* frame is defined by : $p = \varphi_G(m,g) = \widetilde{\varphi}_G(m, \chi(m)g)$ and the new gauge becomes : $\widetilde{\mathbf{p}}(m) = \widetilde{\varphi}_G(m,1) = \varphi_G\left(m, \chi(m)^{-1}\right) = \mathbf{p}(m) \cdot \chi(m)^{-1}$.

Vectors are represented in the Minkovski space $\overrightarrow{M}$, which is here the same as the tangent space to M. A vector $v \in \overrightarrow{M}$ located at m can be represented by its components in a frame located at m. These components change when the gauge change. So it can be equivalently represented by a couple $(p,v) \in P_G \times \overrightarrow{M}$ with the equivalence relation :
$(\varphi_G(m,\mathbf{1}), v) \sim \left(\varphi_G(m,g), g^{-1}(v)\right)$
And this defines the structure of an associated vector bundle $P_G\left[\mathbb{R}^4, \imath\right]$ with the holonomic basis : $(\mathbf{p}(m), \varepsilon_i) = \varepsilon_i(m)$.
In a change of gauge on P_G :
$\varepsilon_i(m) \to \widetilde{\varepsilon}_i(m) = (\widetilde{\mathbf{p}}(m), \varepsilon_i) = \left(\mathbf{p}(m) \cdot \chi(m)^{-1}, \varepsilon_i\right) \sim \left(\mathbf{p}(m), \chi(m)^{-1}\varepsilon_i\right) = \chi(m)^{-1}\varepsilon_i(m)$
So the components of a vector v located at m read in the two bases :
$\sum_{i=0}^3 v^i \varepsilon_i(m) = \sum_{i=0}^3 \widetilde{v}^i \widetilde{\varepsilon}_i(m) \Rightarrow \widetilde{v}^i = [\chi(m)]_j^i v^j$
In particular a change of gauge implies a change in the time used to identify a point in M.
The trajectory of a particle, as measured by the network, is a map $\mu : [0,T] \to M :: \mu(t) = \varphi_M(t, x(t))$. Its velocity, as measured by the standard observers, in the gauge $\mathbf{p}(m)$ is : $V(t) = \frac{d\mu}{dt} = c\varepsilon_0 + \overrightarrow{v}(t)$ that is a vector located at m(t), belonging to $P_G\left[\mathbb{R}^4, \imath\right]$: $V(t) = (\varphi_P(m,\mathbf{1}), c\varepsilon_0 + \overrightarrow{v}(t))$, thus its components change in a change of gauge as : $\widetilde{V}^i(t) = \sum [\chi(m(t))]_i^j V^j(t)$.
And we get back the formula seen previously for the change of coordinates.

7.2.3 The representation of the spin of a particle

We assume that a particle, meaning a body with no measurable internal structure, has a local motion which can be measured with respect to a local gauge. In the most general way, it is represented in a fiber bundle associated to P_G :
- the "motion" is an element S of a manifold E
- there is a left action $\gamma : SO_0(3,1) \times E \to E :: \gamma(g,S)$

- and an equivalence relation

such that : $(\varphi_G(m,\mathbf{1}),S) \sim \left(\varphi_G(m,g),\gamma\left(g^{-1},S\right)\right)$ represent the same "motion" of the particle with respect to a local frame of P_G.

The motion of the body with respect to the network of inertial observers is then represented by :

$(m(t),(\varphi_G(m(t),\mathbf{1}),S(t))) \sim \left(\varphi_P(m(t),g),\gamma\left(g^{-1},S(t)\right)\right)$

with a map : $S:[0,T]\to E$

If S represents some geometric characteristic of the particle, it should stay constant along its trajectory, and according to the hypothese iii) there is a priviledged parametrization $\tau(t)$ such that at each point $\widetilde{m}(\tau)=m(\tau(t))$ of the trajectory $S(\tau)$ is constant in some frame $\varphi_P(\widetilde{m}(\tau),g(\tau))$ of P_G defined by a map $g:\mathbb{R}\to SO_0(3,1)$.

So there is some constant $S_0\in E$ such that :

$(\varphi_P(\widetilde{m}(\tau),\mathbf{1}),S(\tau)) \sim (\varphi_G(\widetilde{m}(\tau),g(\tau)),S_0) \sim (\varphi_P(\widetilde{m}(\tau),\mathbf{1}),\gamma(g(\tau),S_0))$

$\Rightarrow S(\tau)=\gamma(g(\tau),S_0)$

which is equivalent to say that $S(t)=\gamma(g(t),S_0)$.

The relation $S(t)=\gamma(g(t),S_0)$ is just the consequence of our very general assumptions. But to get a full profit of this representation we have to adopt an entirely new point of view. We cannot any longer view the particle as "living" in M, the four dimensional affine space and "spinning" in it. Actually the particle lives in E, which happens to be associated to P_G. Its trajectory is a curve in E, which projects on a curve in M. E can be seen as the physical world (at least as part of it), that we can represent through networks of frames in M. So S(t) cannot be seen properly as a motion, it is only a characteristic of the particle (such as mass and charge). Experience shows that it can be measured through geometric frames, because it is related to its inertial features, but this does not imply the existence of a real spinning motion of the particle. In some way this is what physicists do, intuitively, in galilean geometry : the rotation, the rotational moment, are not represented as elements of the group or the Lie algebra, but as vectors (it happens that it is the same vector space as M, but this is fortuituous).

The value of S varies : locally according to the observer (the motion is relative), and along the trajectory (for the standard observer with holonomic frame S(t) changes).

The issue that we face is then to precise E. We will make the following, reasonnable, assumptions :

i) E is some vector space, so that we have an associated vector bundle $P_G[E,\gamma]$

ii) it implies that (E,γ) is a representation of SO(3,1). As we have seen in galilean geometry, the right group to consider should be Spin(3,1), so we have to look for a representation of Spin(3,1).

iii) this representation should be finite dimensional (we consider here the value of the spin at some point, not the maps S(t)).

iv) if (E,γ) is a representation of Spin(3,1), then $(E,\gamma'(1))$ is a representation of its Lie algebra, both are subsets of the Clifford algebra $Cl(\mathbb{R},3,1)$ so that, if γ is a linear map, then $\gamma'(1)=\gamma$ and this is not a big leap forward to assume that (E,γ) is a representation of the Clifford algebra itself.

7.2.4 Enter the spinors

"*No one fully understands spinors. Their algebra is formally understood but their general significance is mysterious. In some sense they describe the "square root" of geometry and, just as understanding the square root of -1 took centuries, the same might be true of spinors" (Michael Atiyah)*

Experience shows that the spin can be represented as spinors which, to keep it simple, can be defined as vectors S of a vector space E such that (E, γ) is a representation of a Clifford algebra.

Clifford algebras have been known for a long time in mathematics, even if more systematic studies are recent. Their introduction in physics has been indirect. First because the "spin representations", meaning non classical representations of the groups SO(3) and SO(3,1), are rooted in the Clifford algebras. Second as an interpretation of the solution to the Dirac equation, whose γ matrices had initially no precise meaning. Recently Clifford algebras have found a vibrant development in computer vision and robotics, as an efficient tool to model the movements and deformations of bodies, thus they seem to be pertinent to our purpose.

The main feature of Clifford algebras is the introduction of a specific operation : the product of vectors. This kind of operation appears often in galilean physics, whenever the rotation of a solid or a particle is involved, usually under the guise of a cross product, as we have seen for the angular and rotational momenta, and for the action of the magnetic field on particles with spin. The product of vectors in Clifford algebras is actually related to the exterior product in tensorial algebras, and so can be seen as an extension of an operation more usual in physics.

Several points shall be clear :

- in this representation the group of displacements is no more involved.

- the rules in a change of gauge from one observer to the other are not affected, and stay as expressed previously.

- this representation is specific to particles : whenever a body shows an internal structure which can be measured (such as a molecule), it must be accounted for, with the more conventional tools as the angular momentum. Indeed in such cases it makes sense to consider a local frame attached to the body.

It is often said that the "spin is a purely quantic phenomenon". Actually it is the unavoidable consequence of the principles of relativist geometry.

7.3 Spinors

We will start by a reminder of the basics of Clifford algebra, which is useful to establish some notations.

7.3.1 Clifford algebra and Spin groups

From any vector space endowed with a bilinear *symmetric* form $\langle\rangle$ one can define a Clifford algebra, and all Clifford algebras built on vector spaces on the same field, with same dimension and bilinear form with same signature are isomorphic. But the Clifford algebras $Cl\,(\mathbb{R}, 3, 1)$ and $Cl\,(\mathbb{R}, 1, 3)$ corresponding to 4 dimensional real vector spaces with signatures (3,1) and (1,3) respectively are not isomorphic. In the following we will state the results for $Cl\,(\mathbb{R}, 3, 1)$, and also for $Cl\,(\mathbb{R}, 1, 3)$only

when they are different. Clifford algebras are characterized by a product, denoted $\cdot$, with the property that for any two vectors u,v : $u \cdot v + v \cdot u = 2 \langle u, v \rangle$. Elements of a Clifford algebras are linear combinations of either scalars or vectors or product of vectors. In Clifford algebras it is convenient to use orthonormal basis, and $(\varepsilon_i)_{i=0}^{3}$ will denote an orthonormal basis on a 4 dimensional vector space, so that : $\varepsilon_i \cdot \varepsilon_j + \varepsilon_j \cdot \varepsilon_i = 2\eta_{ij}$ where $\eta_{ij} = \pm 1$ depending on the signature of the bilinear form.

The group Pin(3,1), is the subset of the Clifford algebra $Cl(\mathbb{R}, 3, 1)$:
$Pin(3,1) = \left\{ u_1 \cdot u_2 ... \cdot u_k, \langle u_p, u_p \rangle = \pm 1, u_p \in \mathbb{R}^4 \right\}$. Pin(3,1) is a Lie group,
Spin(3,1) is its subgroup where we have an even number of vectors :
$Spin(3,1) = \left\{ u_1 \cdot u_2 ... \cdot u_{2k}, \langle u_p, u_p \rangle = \pm 1, u_p \in \mathbb{R}^4 \right\}$
and similarly for Pin(1,3) and Spin(1,3).
Notice that the *scalars* ± 1 belong to the groups. The identity element is the scalar 1.
Pin(3,1) and Pin(1,3) are not isomorphic. Spin(3,1) and Spin(1,3) are isomorphic. Spin(3,1) is isomorphic to $SL(2, \mathbb{C})$
The map : $\mathbf{Ad} : (Pin(3,1), \cdot) \to (O(3,1), \langle \rangle) :: \mathbf{Ad}_s w = s \cdot w \cdot s^{-1}$ is a surjective group morphism : $\langle \mathbf{Ad}_s u, \mathbf{Ad}_s u' \rangle = \langle u, u' \rangle$ and O(3,1) is algebraically isomorphic to $Pin(3,1)/\{+1, -1\}$
Spin(3,1) is the double cover (as manifold) of SO(3,1) : for each element g of $SO(3,1)$ there are two elements +s and -s of Spin(3,1) such that $\mathbf{Ad}_s = \mathbf{Ad}_{-s} \simeq g$. Spin(3,1) has two connected components (which contains either +1 or -1) and its connected component $Spin_0(3,1)$is simply connected and is the universal cover group of $SO_0(3,1)$.

The Lie algebra $T_1 Spin(3,1)$ of Spin(3,1) is a subset of $Cl(\mathbb{R}, 3, 1)$.The map : $(\mathbf{Ad}_s)'_{s=1}$: $T_1 Spin(3,1) \to o(3,1)$ is an isomorphism of Lie algebras. Conversely, there is a map : $so(3,1) \to T_1 Spin(3,1)$ which is an isomorphism of Lie algebras.

Notation 1 $\upsilon(r, w) = \frac{1}{2}\left(w^1 \varepsilon_0 \cdot \varepsilon_1 + w^2 \varepsilon_0 \cdot \varepsilon_2 + w^3 \varepsilon_0 \cdot \varepsilon_3 + r^3 \varepsilon_2 \cdot \varepsilon_1 + r^2 \varepsilon_1 \cdot \varepsilon_3 + r^1 \varepsilon_3 \cdot \varepsilon_2\right)$ *for both* $Cl(\mathbb{R}, 3, 1), Cl(\mathbb{R}, 1, 3)$

Notation 2 $\varepsilon_5 = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

With this notation :
$so(3,1) \to T_1 Spin(3,1) :: K(w) + J(r) \to \upsilon(r, w) \in T_1 Spin(3,1) \subset Cl(\mathbb{R}, 3, 1)$.
$so(1,3) \to T_1 Spin(1,3) :: K(w) + J(r) \to -\upsilon(r, w) \in T_1 Spin(1,3) \subset Cl(\mathbb{R}, 1, 3)$.
We have the identities, which will be used quite often (see Annex for the proofs and a list of formulas) :
$\varepsilon_5 \cdot \varepsilon_5 = -1$
With signature (3,1) :
$\upsilon(r', w') \cdot \upsilon(r, w)$
$= \frac{1}{4}(w^t w' - r^t r') + \frac{1}{2}\upsilon(-j(r) r' + j(w) w', -j(w) r' - j(r) w') - \frac{1}{4}(w^t r' + r^t w') \varepsilon_5$
the bracket on the Lie algebra:
$[\upsilon(r, w), \upsilon(r', w')] = \upsilon(r, w) \cdot \upsilon(r', w') - \upsilon(r', w') \cdot \upsilon(r, w)$
$[\upsilon(r, w), \upsilon(r', w')] = \upsilon(j(r) r' - j(w) w', j(w) r' + j(r) w')$
$\varepsilon_5 \cdot \upsilon(r, w) = \upsilon(r, w) \cdot \varepsilon_5 = \upsilon(r, -w)$
With signature (1,3) :
$\upsilon(r, w) \cdot \upsilon(r', w')$

$= \frac{1}{4}\left(w^t w' - r^t r'\right) - \frac{1}{2}\upsilon\left(-j\left(r\right)r' + j\left(w\right)w', j\left(w\right)r' + j\left(r\right)w'\right) - \frac{1}{4}\left(w^t r' + r^t w'\right)\varepsilon_5$
$\left[\upsilon\left(r,w\right), \upsilon\left(r',w'\right)\right] = -\upsilon\left(j\left(r\right)r' - j\left(w\right)w', j\left(w\right)r' + j\left(r\right)w'\right)$
$\varepsilon_5 \cdot \upsilon\left(r,w\right) = \upsilon\left(r,w\right)\cdot\varepsilon_5 = \upsilon\left(w,r\right)$

There is a scalar product on a Clifford algebra, by defining its canonical basis as orthonormal : $\left\langle \varepsilon_{i_1}\cdot\varepsilon_{i_2}\cdot ...\cdot\varepsilon_{i_n}, \varepsilon_{j_1}\cdot\varepsilon_{j_2}\cdot ...\cdot\varepsilon_{j_n}\right\rangle = \eta_{i_1 j_1}...\eta_{i_n j_n l}\epsilon\left(i_1...,i_n,j_1,...j_n\right)$ the latter term is the signature of the permutation $\left(i_1...,i_n,j_1,...j_n\right)$
It is preserved by **Ad** .

7.3.2 Symmetry breakdown

The elements of the Spin groups read (see annex) in both signatures :
$s = a + \frac{1}{2}\left(w^1\varepsilon_0\cdot\varepsilon_1 + w^2\varepsilon_0\cdot\varepsilon_2 + w^3\varepsilon_0\cdot\varepsilon_3 + r^3\varepsilon_2\cdot\varepsilon_1 + r^2\varepsilon_1\cdot\varepsilon_3 + r^1\varepsilon_3\cdot\varepsilon_2\right) + b\varepsilon_5$
That we will write :

$$s = a + \upsilon\left(r,w\right) + b\varepsilon_5 \tag{1}$$

using the map υ in both signatures.
$a, \left(w^j, r^j\right)_{j=1}^3, b$ are real scalar which are related. We have the necessary identities :

$$a^2 - b^2 = 1 + \frac{1}{4}\left(w^t w - r^t r\right) \tag{2}$$

$$ab = -\frac{1}{4}r^t w \tag{3}$$

s corresponds to the element of SO(3,1) :
$\left[g\right] = I + a\left(K\left(w\right) + J\left(r\right)\right) + \frac{1}{2}\left(K\left(w\right) + J\left(r\right)\right)^2$
The product s.s' reads :
$\left(a + \upsilon\left(r,w\right) + b\varepsilon_5\right)\cdot\left(a' + \upsilon\left(r',w'\right) + b'\varepsilon_5\right) = a" + \upsilon\left(r",w"\right) + b"\varepsilon_5$
with :
$a" = aa' - b'b + \frac{1}{4}\left(w^t w' - r^t r'\right)$
$b" = ab' + ba' - \frac{1}{4}\left(w^t r' + r^t w'\right)$
and in Spin(R,3,1) :
$r" = \frac{1}{2}\left(j\left(r\right)r' - j\left(w\right)w'\right) + a'r + ar' - b'w - bw'$
$w" = \frac{1}{2}\left(j\left(w\right)r' + j\left(r\right)w'\right) + a'w + aw' + b'r + br'$
and in Spin(R,1,3) :
$r" = \frac{1}{2}\left(j\left(r\right)r' - j\left(w\right)w'\right) + a'r + ar' + b'w + bw'$
$w" = -\frac{1}{2}\left(j\left(w\right)r' + j\left(r\right)w'\right) + a'w + aw' + b'r + br'$
The inverse is :

$$\left(a + \upsilon\left(r,w\right) + b\varepsilon_5\right)^{-1} = a - \upsilon\left(r,w\right) + b\varepsilon_5 \tag{4}$$

In the gauge transformation formulas it is clear that the rotations in the Minkovski space are the product of spatial rotations (represented by expJ(r)) and boosts, linked to the velocity and represented by expK(w). This is more illuminating with Clifford algebras.

The subset of $Spin(3,1)$ of the elements s_r which commute with ε_0 is a maximal compact subgroup of $Spin(3,1)$. They leave ε_0 unchanged : $\mathbf{Ad}_{s_r}\varepsilon_0 = s_r \cdot \varepsilon_0 \cdot s_r^{-1} = \varepsilon_0 = [g]_0^i \varepsilon_i$. They have a special physical meaning : they are the "spatial rotations" for an observer with a velocity in the direction of ε_0. They read : $s_r = \epsilon \left(\sqrt{1-\frac{1}{4}r^t r} + \upsilon(r,0)\right)$ with $\epsilon = \pm 1, r^t r \leq 4$.

They are generated by vectors belonging to the subspace spanned by the vectors $(\varepsilon_i)_{i=1}^3$ so they belong to Spin(3) which can be seen as a subgroup of Spin(3,1). It has 2 connected components. The connected component of the identity is comprised of elements with $\epsilon = +1$. We have the same results for $Cl(\mathbb{R},1,3)$.

The quotient space $Spin(3,1)/Spin(3)$ is not a group but a 3 dimensional manifold, called a homogeneous space. It is characterized by the equivalence relation :

$s = a + \upsilon(r,w) + b\varepsilon_5 \sim s' = a' + \upsilon(r',w') + b'\varepsilon_5 \Leftrightarrow \exists s_r \in Spin(3) : s' = s \cdot s_r$

The projection : $\pi_w : Spin(3,1) \to Spin(3,1)/Spin(3)$ is a submersion, its derivative $\pi'_w(s)$ is surjective, $Spin(3,1)$ is a principal fiber bundle $Spin(3,1)(Spin(3,1)/Spin(3), Spin(3), \pi_w)$.

The elements of $Spin(3,1)/Spin(3)$ are coordinated by w, and the matrix $[g_w]$ corresponds to a gauge transformation for an observer moving with a spatial speed $\overrightarrow{v}$ parallel to w, without spatial rotation.

In each class of $Spin(3,1)/Spin(3)$ there are only two elements which can be written as : $s = a + \upsilon(0,w)$ and they have opposite sign : $\pm s$ belong to the same class.

Any s in the Spin groups can be written :

$$s = a + \upsilon(r,w) + b\varepsilon_5 = s_w \cdot s_r = (a_w + \upsilon(0,w_w)) \cdot (a_r + \upsilon(r_r,0)) \tag{5}$$

s_w, s_r are unique up to sign : $s = \epsilon s_w \cdot \epsilon s_r$

$\pm s_w$ belong to the class of equivalence of s in $Spin(\mathbb{R},3,1)/Spin(\mathbb{R},3)$. They are specific representatives of the projection of s on the homogeneous space.

There is a left action of $Spin(3,1)$ on $Spin(3,1)/Spin(3)$:

$\lambda : Spin(3,1) \times Spin(3,1)/Spin(3) \to Spin(3,1)/Spin(3)$:

$\lambda(s,s_w) = \pi_w(s \cdot s_w)$

Proof. Indeed we have :

$\lambda(s,s_w) = \pi_w(s \cdot s_w) \Leftrightarrow \exists s_r : s \cdot s_w = \pi_w(s \cdot s_w) \cdot s_r \Leftrightarrow \lambda(s,s_w) = s \cdot s_w \cdot s_r^{-1}$

$\lambda(s,\lambda(s',s_w)) = \lambda\left(s, s' \cdot s_w \cdot s_r'^{-1}\right) = s \cdot s' \cdot s_w \cdot s_r'^{-1} \cdot s_r^{-1}$

$\lambda(s \cdot s', s_w) = s \cdot s' \cdot s \cdot s_w \cdot s_r$"

$s \cdot s' \cdot s \cdot s_w \cdot s_r$" $\sim s \cdot s' \cdot s_w \cdot s_r'^{-1} \cdot s_r^{-1}$ ■

In the Minkowski space all rotations (given by $Spin(3,1)$) are on the same footing. But, because of our assumptions about the motion of observers (along time like lines), any observer introduces a "breakdown of symmetry" : some rotations are privileged. Indeed the spatial rotations (in a space which depends on each observer) are special, in that they are the ones for which the choice of the axis is manifest.

7.3.3 Representation of Clifford algebras

A real Clifford algebra can be complexified, using the same procedure as for any finite dimensional vector space. The complexified of both $Cl(\mathbb{R},3,1)$ and $Cl(\mathbb{R},1,3)$ is $Cl(\mathbb{C},4)$, the Clifford algebra on $\mathbb{C}^4$ with the bilinear *symmetric* form of signature $(+++ +)$.

There are real algebras morphisms (injective but not surjective) from the real Clifford algebras to $Cl(\mathbb{C},4)$.

Let $\widetilde{C}:\left(\mathbb{R}^4,\langle\rangle_L\right)\to Cl(\mathbb{C},4)$ be the real linear map defined by : $\widetilde{C}(\varepsilon_j)=\varepsilon_j, j=1,2,3;\widetilde{C}(\varepsilon_0)=i\varepsilon_0$ and the scalar product $\langle\rangle_L$ with signature (- + + +).

It is easy to check that : $\widetilde{C}(u)\cdot\widetilde{C}(v)+\widetilde{C}(v)\cdot\widetilde{C}(u)=2\langle u,v\rangle_L$ so, by the universal property of Clifford algebras, there is a unique real algebra morphism $C:Cl(\mathbb{R},3,1)\to Cl(\mathbb{C},4)$ such that $\widetilde{C}=C\circ\jmath$ where $\jmath$ is the canonical injection $\left(\mathbb{R}^4,\langle\rangle_L\right)\to Cl(\mathbb{R},3,1)$. We will denote for simplicity $\widetilde{C}=C$. The image $C(Cl(\mathbb{R},3,1))$ is a real subalgebra of $Cl(\mathbb{C},4)$

Similarly with $\widetilde{C}'(\varepsilon_j)=i\varepsilon_j, j=1,2,3;\widetilde{C}'(\varepsilon_0)=\varepsilon_0$ we have a real algebra morphism $C':Cl(\mathbb{R},1,3)\to Cl(\mathbb{C},4)$ and $C'(Cl(\mathbb{R},1,3))$ is a real subalgebra of $Cl(\mathbb{C},4)$. Moreover $C'(\varepsilon_j)=-i\eta_{jj}C(\varepsilon_j)$ (η always correspond to the signature - + + +).

An element of a Clifford algebra is a linear combination of products of vectors. A Clifford algebra has, up to isomorphism, a unique faithful algebraic representation in an algebra of matrices, thus mathematically it is equivalent to use either the Clifford elements or the matrices (the operations are similar). As can be expected the representations depend on the signature :

For $Cl(\mathbb{R},3,1)$ this is $\mathbb{R}(4)$ the 4×4 real matrices (the corresponding spinors are the "Majorana spinors")

For $Cl(\mathbb{R},1,3)$ this is $H(2)$ the 2×2 matrices with quaternionic elements

In both cases an element of the Clifford algebra is characterized by $2^4=16$ real parameters.

A representation is fully defined by the family of generators $(\gamma_i)_{i=1}^4$, matrices representing each vector $(\varepsilon_i)_{i=1}^4$ of an orthonormal basis. The choice of these matrices is not unique : the only condition is that $[\gamma_i][\gamma_j]+[\gamma_j][\gamma_i]=2\eta_{ij}[I]$ and any family of matrices deduced by conjugation with a fixed matrix gives an equivalent algebraic representation. An element of the Clifford algebra is then uniquely represented by a linear combination of generators :

$\gamma(w)=\gamma\left(\sum_{\{i_1...i_r\}}a^{i_1...i_r}\varepsilon_{i_1}\cdot...\cdot\varepsilon_{i_r}\right)=\sum_{\{i_1...i_r\}}a^{i_1...i_r}\gamma_{i_1}...\gamma_{i_r}$

A change of orthonormal basis in $\mathbb{R}^4$ is represented in the Clifford algebra by the action $\mathbf{Ad}_s$ for some element s of the Pin group. And the impact on the matrix representing an element is :

$w\to\widetilde{w}=\mathbf{Ad}_s w=s\cdot w\cdot s^{-1}$

$\gamma(w)\to\gamma(\widetilde{w})=\gamma(s)\cdot\gamma(w)\cdot\gamma(s)^{-1}$

These algebraic representations can be seen as a geometric representation (E,γ) of the Clifford algebra on a vector space E : $\gamma:Cl\to L(E;E)$ the matrices $\gamma(w)$ acting as the representation of the endomorphisms $\gamma(w)$ in any basis of E. In a change of basis in E represented by a matrix Q the components of a vector $u\in E$ changes according to : $[u]\to[\widetilde{u}]=Q^{-1}[u]$ and the matrices γ representing endomorphisms change as : $\gamma\to\widetilde{\gamma}=Q^{-1}\gamma Q$. So we have an equivalent representation of the Clifford algebra : the action of an element w of the Clifford algebra on E corresponds to a change of basis $Q=\gamma(w)^{-1}$ in E.

The spin is represented, not by γ matrices, but by vectors S of the space E, which are called spinors (or Dirac spinors). The action of an element s of the Spin group on a spinor S is then a change of basis $Q=\gamma(s)^{-1}$ in E, and the components of S change as :

$S^i\to\widetilde{S}^i=\sum_j[h(s)]_j^i S^j$

This is similar to the representation of rotations by vectors of $\overrightarrow{E}$ in galilean geometry : vectors representing rotations change in a gauge transformation as ordinary vectors. Higher orders spinors, tensorial products of vectors of E, have been studied, but do not seem to have a physical significance.

The geometry of the universe is based upon real structures. Thus we should consider representations of $Cl(\mathbb{R},3,1)$ or $Cl(\mathbb{R},1,3)$, which raises the issue of the signature. However it happens, from experience, that the vector space E must be complex.

The irreducible representation of $Cl(\mathbb{C},4)$ is by 4×4 matrices on complex numbers which must meet the condition : $\gamma_j\gamma_k+\gamma_k\gamma_j=2\delta_{jk}I_4$.

If (E,γ) is a complex representation of $Cl(\mathbb{C},4)$ then $(E,\gamma\circ C)$ is a real geometric representation of $Cl(\mathbb{R},3,1)$ on the complex vector space E: the map $\gamma\circ C : Cl(\mathbb{R},3,1)\rightarrow L(E;E)$ is a real morphism of algebras, and the maps $\gamma\circ C(w)$ are complex linear. The matrices of the real representation are $i\gamma_0$, $\gamma_j, j=1,2,3, i\gamma_0$. Similarly $(E,\gamma\circ C')$ is a real geometric representation of $Cl(\mathbb{R},1,3)$ with matrices $\gamma_0, i\gamma_j, j=1,2,3$.

Using this trick we see that we are fortunate, in that we have the same representation (E,γ) for both signatures, and a complex vector space E. Moreover it is easy to specify the representation through additional features of E (such as chirality as we will see). A spinor has 8 real components (vs 16 real components for elements of the real Clifford algebras) thus a spinor carries more information than a simple vector of $\mathbb{R}^4$ and this solves the issue of the number of parameters needed to represent the motion (both translation and rotation).

7.3.4 Chirality

Any Clifford algebra Cl is the direct sum of one Clifford subalgebra Cl_0 comprised of elements which are the sum of products of an even number of vectors, and a vector subspace Cl_1 comprised of elements which are the sum of products of an odd number of vectors. Moreover some Clifford algebras present a specific feature : they are the direct sum of two subalgebras which can be seen as algebras of "left handed" and "right handed" elements. This property depends on the existence of a volume element ϖ such that $\varpi\cdot\varpi=1$. This element exists in any complex algebra, but not in $Cl(\mathbb{R},1,3), Cl(\mathbb{R},3,1)$. As chirality is a defining feature of particles, this is an additional argument for using $Cl(\mathbb{C},4)$.

In this subsection and the next one the vectors $(\varepsilon_j)_{j=0}^3$ denote the canonical basis of $\mathbb{C}^4$ endowed with the bilinear symmetric form of signature $+\ +\ +\ +$. In $Cl(\mathbb{C},4)$ the volume element is : $\varpi=\pm\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\in Spin(\mathbb{C},4)$. Thus there is a choice and we will use : $\varpi=\varepsilon_5=\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$.

The Clifford algebra splits in two subalgebras :
$Cl(\mathbb{C},4)=Cl^R(\mathbb{C},4)\oplus Cl^L(\mathbb{C},4)$:
$Cl^R(\mathbb{C},4)=\{w\in Cl(\mathbb{C},4):\varepsilon_5\cdot w=w\}$,
$Cl^L(\mathbb{C},4)=\{w\in Cl(\mathbb{C},4):\varepsilon_5\cdot w=-w\}$
and any element of $Cl(\mathbb{C},4)$ can be uniquely written as : $w=w_R+w_L$
The projections from $Cl(\mathbb{C},4)$ on each subalgebra are the maps
$p_R=\frac{1}{2}(1+\varepsilon_5), p_L=\frac{1}{2}(1-\varepsilon_5)$:
$w_R=p_R\cdot w, w_L=p_L\cdot w$
$p_R\cdot p_L=p_L\cdot p_R=0, p_R^2=p_R, p_L^2=p_L, p_R+p_L=1$
We have similarly : $E=E^R\oplus E^L$ with
$E^R=\gamma_R(E), E^L=\gamma_L(E), \gamma_R=\gamma(p_R), \gamma_L=\gamma(p_L)\Rightarrow\gamma(\varepsilon_5)=\gamma_R-\gamma_L$
$u\in E : u=u_R+u_L : u_R=\gamma_R(u)=\frac{1}{2}(u+\gamma(\varepsilon_5)u); u_L=\gamma_L(u)=\frac{1}{2}(u-\gamma(\varepsilon_5)u)$
For any homogeneous element $w=v_1\cdot v_2...\cdot v_k, v_j\in\mathbb{C}^4$ we have $\varepsilon_5\cdot w=(-1)^k w\cdot\varepsilon_5$
$\forall w\in Cl(\mathbb{C},4), u\in E$:

$\gamma_R(\gamma(w)u_R) = \frac{1}{2}\left(1+(-1)^k\right)\gamma(w)u_R$
k even : $\gamma_R(\gamma(w)u_R) = \gamma(w)u_R$
k odd : $\gamma_R(\gamma(w)u_R) = 0$
For k even : $\gamma(w)$ preserves both E^R, E^L
For k odd : $\gamma(w)$ exchanges E^R, E^L
In particular the elements of the images $C(Spin(3,1))$ and $C'(Spin(1,3))$ by γ preserve both E^R, E^L. So we have reducible representations of these groups.

7.3.5 The choice of the representation γ

An algebraic representation is defined by the choice of its generators γ_i, and any set of generators conjugate by a fixed matrix gives an equivalent representation. We can specify the generators by the choice of a basis $(e_i)_{i=1}^4$ of E . The previous result leads to a natural choice : take $(e_i)_{i=1}^2$ as basis of E^R and $(e_i)_{i=3}^4$ as basis of E^L, then :
$\gamma_R = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \gamma_L = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}, \gamma_5 = \gamma(\varepsilon_5) = \gamma_R - \gamma_L = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix}$
Denote : $\gamma_j = \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}$ with four 2×2 complex matrices.

ε_5 belongs to the Spin group $Spin(\mathbb{C},4)$, commutes with any element of $Cl_0(\mathbb{C},4)$ and anticommutes with any vector, thus $\gamma_5\gamma_j = -\gamma_j\gamma_5$ which imposes the condition :
$\begin{bmatrix} A_j & -B_j \\ C_j & -D_j \end{bmatrix} = -\begin{bmatrix} A_j & B_j \\ -C_j & -D_j \end{bmatrix} \Rightarrow \gamma_j = \begin{bmatrix} 0 & B_j \\ C_j & 0 \end{bmatrix}$

The defining relations : $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{jk}I_4$ lead to :
$\begin{bmatrix} B_jC_k + B_kC_j & 0 \\ 0 & C_jB_k + C_kB_j \end{bmatrix} = 2\delta_{jk}I_4$
$j \neq k : B_jC_k + B_kC_j = C_jB_k + C_kB_j = 0$
$j = k : B_jC_j = C_jB_j = I_2 \Leftrightarrow C_j = B_j^{-1}$
thus $(\gamma_i)_{i=0}^3$ is fully defined by a set $(B_i)_{i=0}^3$ of 2×2 complex matrices
$\gamma_j = \begin{bmatrix} 0 & B_j \\ B_j^{-1} & 0 \end{bmatrix}$
meeting : $j \neq k : B_jB_k^{-1} + B_kB_j^{-1} = B_j^{-1}B_k + B_k^{-1}B_j = 0$
which reads :
$B_jB_k^{-1} = -\left(B_jB_k^{-1}\right)^{-1} \Leftrightarrow \left(B_jB_k^{-1}\right)^2 = -I_2$
$B_j^{-1}B_k = -\left(B_j^{-1}B_k\right)^{-1} \Leftrightarrow \left(B_j^{-1}B_k\right)^2 = -I_2$

Let us define : k=1,2,3 : $M_k = -iB_kB_0^{-1}$
The matrices $(M_k)_{k=1}^3$ are such that :
$M_k^2 = -\left(B_jB_0^{-1}\right)^2 = -I_2$
$M_jM_k + M_kM_j = -B_jB_0^{-1}B_kB_0^{-1} - B_kB_0^{-1}B_jB_0^{-1} = -\left(-B_jB_k^{-1}B_0 - B_kB_j^{-1}B_0\right)B_0^{-1}$
$= B_jB_k^{-1} + B_kB_j^{-1} = 0$
that is k=1,2,3 : $M_jM_k + M_kM_j = 2\delta_{jk}I_2$
Moreover : $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \Rightarrow$

$B_0 B_1^{-1} B_2 B_3^{-1} = I_2$
$B_0^{-1} B_1 B_2^{-1} B_3 = -I_2$
with $B_k = iM_k B_0, B_k^{-1} = -iB_0^{-1} M_k^{-1}$
$B_0 \left(-iB_0^{-1} M_1^{-1}\right) (iM_2 B_0) \left(-iB_0^{-1} M_3^{-1}\right) = I_2 = -iM_1^{-1} M_2 M_3^{-1}$
$B_0^{-1} (iM_1 B_0) \left(-iB_0^{-1} M_2^{-1}\right) (iM_3 B_0) = -I_2 = iB_0^{-1} M_1 M_2^{-1} M_3 B_0$
which reads :
$iM_2 = -M_1 M_3 = M_3 M_1$
$-M_1^{-1} M_3^{-1} = iM_2^{-1} \Leftrightarrow iM_2 = M_3 M_1$
$M_2 M_3 + M_3 M_2 = 0 = iM_1 M_3 M_3 + M_3 M_2 \Leftrightarrow iM_1 = -M_3 M_2 = M_2 M_3$
$M_1 M_2 + M_2 M_1 = 0 = iM_3 M_2 M_2 + M_2 M_1 \Rightarrow iM_3 = -M_2 M_1 = M_1 M_2$
The set of 3 matrices $(M_k)_{k=1}^3$ has the multiplication table :
$\begin{bmatrix} 1\backslash 2 & M_1 & M_2 & M_3 \\ M_1 & I & iM_3 & -iM_2 \\ M_2 & -iM_3 & I & iM_1 \\ M_3 & iM_2 & -iM_1 & I \end{bmatrix}$
which is the same as the set of Pauli's matrices :

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \tag{6}$$

$$\sigma_j \sigma_k = \epsilon (j, k, l)\, i\sigma_l \tag{7}$$

There is still some freedom in the choice of the γ_i matrices by the choice of B_0 and the simplest is : $B_0 = -iI_2 \Rightarrow B_k = \sigma_k$

Moreover, because scalars belong to Clifford algebras, one must have the identity matrix I_4 and $\gamma (z) = zI_4$

Thus with $\sigma_0 = I_2$:

$$\gamma_0 = \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix}; \gamma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}; \gamma_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}; \gamma_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix}; \tag{8}$$

The matrices γ_j are then unitary and hermitian :

$$\gamma_j = \gamma_j^* = \gamma_j^{-1} \tag{9}$$

which is extremely convenient.

We will use the following :

Notation 3 *j = 1,2,3 :* $\widetilde{\gamma}_j = \begin{bmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{bmatrix}$

$j \neq k, l = 1,2,3 : \gamma_j \gamma_k = -\gamma_k \gamma_j = i\epsilon(j,k,l)\widetilde{\gamma}_l$
$j = 1,2,3 : \gamma_j \gamma_0 = -\gamma_0 \gamma_j = i \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix} = i\gamma_5 \widetilde{\gamma}_j$

See the annex for more formulas.

Notice that the choice of the matrices is done in $Cl(\mathbb{C},4)$, so it is independant of the choice of signature. However we have the representations of the real algebras by the matrices $\gamma C(\varepsilon_j)$ and $\gamma C'(\varepsilon_j)$

$$Cl(\mathbb{R},3,1) : \gamma C(\varepsilon_j) = \gamma_j, j = 1,2,3; \gamma C(\varepsilon_0) = i\gamma_0; \gamma C(\varepsilon_5) = i\gamma_5 \tag{10}$$

$$Cl(\mathbb{R},1,3) : \gamma C'(\varepsilon_j) = i\gamma_j, j = 1,2,3; \gamma C'(\varepsilon_j) = \gamma_0; \gamma C'(\varepsilon_5) = \gamma_5 \tag{11}$$

The choice of $\varpi = -\varepsilon_5 = -\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$ would have lead to take $\widetilde{\gamma}_j = -\gamma_j$. In the standard model we have a representation of $Cl(\mathbb{R},1,3)$ by the matrices : $\widetilde{\gamma}_0 = i\gamma_0, \widetilde{\gamma}_j = \gamma_j, j = 1,2,3$ and $\widetilde{\gamma}_5 = -i\widetilde{\gamma}_0\widetilde{\gamma}_1\widetilde{\gamma}_2\widetilde{\gamma}_3$

Expression of the matrices for the Lie algebra and the Spin groups

The matrices $\gamma C(\upsilon(r,w)), \gamma C'(\upsilon(r,w))$ are of constant use.

In Cl(R,3,1) :

$$\gamma C(\upsilon(r,w)) = -i\tfrac{1}{2}\textstyle\sum_{a=1}^{3}(w^a\gamma_a\gamma_0 + r^a\widetilde{\gamma}_a) \tag{12}$$

In Cl(R,1,3) :

$$\gamma C'(\upsilon(r,w)) = -i\tfrac{1}{2}\textstyle\sum_{a=1}^{3}(w^a\gamma_a\gamma_0 - r^a\widetilde{\gamma}_a) \tag{13}$$

so one goes from one signature to the other by changing the sign of r.
The elements of the spin groups are represented by the matrices :

in Spin(3,1):

$$\gamma C(a + \upsilon(r,w) + b\varepsilon_5) = aI - i\tfrac{1}{2}\textstyle\sum_{a=1}^{3}(w^a\gamma_a\gamma_0 + r^a\widetilde{\gamma}_a) + b\gamma_5 \tag{14}$$

in Spin(1,3) :

$$\gamma C'(a + \upsilon(r,w) + b\varepsilon_5) = aI - i\tfrac{1}{2}\textstyle\sum_{a=1}^{3}(w^a\gamma_a\gamma_0 - r^a\widetilde{\gamma}_a) + b\gamma_5 \tag{15}$$

7.3.6 Space and time reversal

So the right group to consider in the representation of the spin is the Spin group (as it is actually in galilean geometry). It seems logical to consider the same group for all the geometry. This is possible if we replace, for usual vectors, the standard representation $\left(\mathbb{R}^4, \imath\right)$ by the representation $\left(\mathbb{R}^4, \mathbf{Ad}\right)$. Any element of the spin group acts as an element of SO(3,1) on vectors of $\mathbb{R}^4$, so the formulas for a gauge transformation do not change, but they act on the spinors by $\gamma C, \gamma C'$ and +s and -s have not the same impact.

Space reversal and time reversal are basis dependant operations in $\mathbb{R}^4$: they are defined with respect to an orthonormal basis, and depend on the signature. They are easily defined in the Clifford algebra.

In $Cl(\mathbb{R},3,1)$ time reversal is a reflexion with respect to $\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = -\varepsilon_0 \cdot \varepsilon_5 : \tau = \mathbf{Ad}_{-\varepsilon_0 \cdot \varepsilon_5}$
$\mathbf{Ad}_{\varepsilon_0 \cdot \varepsilon_5}\left(\sum_{j=0}^{3} u^j \varepsilon_j\right) = \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \cdot \left(u^0\varepsilon_0 + u^1\varepsilon_1 + u^2\varepsilon_2 + u^3\varepsilon_3\right) \cdot \left(-\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3\right)$
$= -u^0\varepsilon_0 + \sum_{j=1}^{3} u^j \varepsilon_j$
A vector of E with components $\begin{bmatrix} S_R \\ S_L \end{bmatrix}$ would have the components : $i\begin{bmatrix} S_L \\ S_R \end{bmatrix}$ in the new basis.
In $Cl(\mathbb{R},3,1)$ space reversal is a reflexion with respect to $\varepsilon_0 : \tau = \mathbf{Ad}_{\varepsilon_0}$
$\mathbf{Ad}_{\varepsilon_0}\left(\sum_{j=0}^{3} u^j \varepsilon_j\right) = \varepsilon_0 \cdot \left(u^0\varepsilon_0 + u^1\varepsilon_1 + u^2\varepsilon_2 + u^3\varepsilon_3\right) \cdot \left(-\varepsilon_0\right)$
$= u^0\varepsilon_0 - \sum_{j=1}^{3} u^j \varepsilon_j$
A vector of E with components $\begin{bmatrix} S_R \\ S_L \end{bmatrix}$ would have the components : $\begin{bmatrix} S_L \\ -S_R \end{bmatrix}$ in the new basis.
Notice that they cannot be represented by the action of the Spin group (or the Lie algebra).
We have similarly in $\mathrm{Cl}(\mathbb{R},1,3)$:
Time reversal : $\gamma C'\left(\varepsilon_0 \cdot \varepsilon_5\right) = -\gamma\left(\varepsilon_0 \cdot i\varepsilon_5\right) = -i\gamma\left(\varepsilon_0 \cdot \varepsilon_5\right) = -\begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}$:
$\begin{bmatrix} S_R \\ S_L \end{bmatrix} \to -\begin{bmatrix} S_L \\ S_R \end{bmatrix}$
Space reversal : $\gamma C'\left(\varepsilon_0\right) = \gamma\left(\varepsilon_0\right) = i\begin{bmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{bmatrix}$
$\begin{bmatrix} S_R \\ S_L \end{bmatrix} \to i\begin{bmatrix} -S_L \\ S_R \end{bmatrix}$

The combined spatial and time reversals are equivalent to a multiplication by -1 of the vectors of $\mathbb{R}^4$. Thus we have in both signatures : $\left[\gamma\left(s\right)\right]^{-1} = -I$ and : $\begin{bmatrix} S_R \\ S_L \end{bmatrix} \to \begin{bmatrix} -S_R \\ -S_L \end{bmatrix}$

7.3.7 Scalar product

We need a scalar product on E, preserved by a gauge transformation, that is by both Spin(3,1) and Spin(1,3). It is represented in the basis of E by a 4×4 hermitian matrix G such that :
$G = G^*$

$\forall s \in Spin(3,1) : [\gamma \circ C(s)]^* G [\gamma \circ C(s)] = G$
or $\forall s \in Spin(1,3) : [\gamma \circ C'(s)]^* G [\gamma \circ C'(s)] = G$
$[\gamma \circ C(s)]^* G = G [\gamma \circ C(s)]^{-1} = G \left[\gamma \circ C\left(s^{-1}\right)\right]$

$$\gamma C(s) = \begin{bmatrix} (a+ib)\sigma_0 + \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)\sigma_a & 0 \\ 0 & (a-ib)\sigma_0 - \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)\sigma_a \end{bmatrix}$$
$$\gamma C(s)^* = \begin{bmatrix} (a-ib)\sigma_0 + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)\sigma_a & 0 \\ 0 & (a+ib)\sigma_0 - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)\sigma_a \end{bmatrix}$$
$$\gamma C(s)^{-1} = \begin{bmatrix} (a+ib)\sigma_0 - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)\sigma_a & 0 \\ 0 & (a-ib)\sigma_0 + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)\sigma_a \end{bmatrix}$$
$$G = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, A = A^*, C = C^*$$
$[\gamma \circ C(s)]^* G$
$$= \begin{bmatrix} (a-ib)A + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)\sigma_a A & (a-ib)B + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)\sigma_a B \\ (a+ib)B^* - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)\sigma_a B^* & (a+ib)C - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)\sigma_a C \end{bmatrix}$$
$G\left[\gamma \circ C\left(s^{-1}\right)\right]$
$$= \begin{bmatrix} (a+ib)A - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)A\sigma_a & (a-ib)B + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)B\sigma_a \\ (a+ib)B^* - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)B^*\sigma_a & (a-ib)C + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)C\sigma_a \end{bmatrix}$$

$(a-ib)A + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)\sigma_a A = (a+ib)A - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)A\sigma_a$
$(a-ib)B + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)\sigma_a B = (a-ib)B + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)B\sigma_a$
$(a+ib)B^* - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)\sigma_a B^* = (a+ib)B^* - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)B^*\sigma_a$
$(a+ib)C - \frac{1}{2}\sum_{a=1}^{3}(w_a - ir_a)\sigma_a C = (a-ib)C + \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)C\sigma_a$
$\sum_{a=1}^{3}(w_a + ir_a)\sigma_a B = \sum_{a=1}^{3}(w_a + ir_a)B\sigma_a$
$\sum_{a=1}^{3}(w_a - ir_a)\sigma_a B^* = \sum_{a=1}^{3}(w_a - ir_a)B^*\sigma_a$
$2ibA$
$= \frac{1}{2}\sum_{a=1}^{3}(w_a + ir_a)\sigma_a A + (w_a - ir_a)A\sigma_a$
$= \frac{1}{2}\sum_{a=1}^{3} w_a(\sigma_a A + A\sigma_a) + ir_a((\sigma_a A - A\sigma_a))$
$2ibC$
$= \frac{1}{2}\sum_{a=1}^{3}((w_a + ir_a)C\sigma_a + (w_a - ir_a)\sigma_a C)$
$= \frac{1}{2}\sum_{a=1}^{3} w_a(\sigma_a C + C\sigma_a) + ir_a((C\sigma_a - \sigma_a C))$
By taking the adjoint on the two last equations :
$-2ibA$
$= \frac{1}{2}\sum_{a=1}^{3} w_a(A\sigma_a + \sigma_a A) - ir_a((A\sigma_a - \sigma_a A))$
$= -\frac{1}{2}\sum_{a=1}^{3} w_a(\sigma_a A + A\sigma_a) + ir_a((\sigma_a A - A\sigma_a)) \Rightarrow A = 0$
$-2ibC$
$= \frac{1}{2}\sum_{a=1}^{3} w_a(\sigma_a C + C\sigma_a) - ir_a((C\sigma_a - \sigma_a C))$
$= -\frac{1}{2}\sum_{a=1}^{3} w_a(\sigma_a C + C\sigma_a) + ir_a((C\sigma_a - \sigma_a C)) \Rightarrow C = 0$
We are left with :
$\forall w, r : \sum_{a=1}^{3}(w_a + ir_a)\sigma_a B = \sum_{a=1}^{3}(w_a + ir_a)B\sigma_a$
which implies that B commutes with all the Dirac matrices, which happens only for the scalar matrices : $B = k\sigma_0$.

$G = \begin{bmatrix} 0 & k\sigma_0 \\ \overline{k}\sigma_0 & 0 \end{bmatrix}$

The scalar product will never be definite positive, so we can take k=-i that is $G = \gamma_0$. And it is easy to check that it works also for the signature (1,3).

Any vector of E reads : $u = \sum_{i=1}^{4} u^i e_i = u_R + u_L$ with $u_R = \sum_{i=1}^{2} u^i e_i, u_L = \sum_{i=3}^{4} u^i e_i$
The scalar product of two vectors u,v of E is then:

$$\left\langle \sum_{i=1}^{4} u^i e_i, \sum_{i=1}^{4} v^i e_i \right\rangle = [u]^* [\gamma_0] [v] = i \left(u_L^* v_R - u_R^* v_L \right) \tag{16}$$

It is not definite positive. It is preserved both by Spin(3,1) and Spin(1,3).

The basis $(e_i)_{i=1}^4$ of E is not orthonormal : $\langle e_j, e_k \rangle = i \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

In a time reversal the scalar product takes the opposite sign, in a space reversal the scalar product is unchanged, the combined space and time reversal preserves the scalar product (for both signatures).

7.4 Models of particles in special relativity

7.4.1 Description of the system

Fiber bundles

As previously, the system is in some relatively compact area Ω of the Minkovski space M, delimited by the hyperplanes $\Omega_3(0)$ t = 0 and $\Omega_3(T)$ t = T of an inertial observer. The chart on M is
$\varphi_M : \mathbb{R} \times \Omega_3(0) \to \Omega :: \varphi_M(t,x) = \overrightarrow{Om} = ct\varepsilon_0 + \sum_{i=0}^{3} \xi^i \varepsilon_i$
But the model is adjusted to account for our additional findings :
1. The principal bundle $P_G(M, Spin_0(\mathbb{R},3,1), \pi_G)$ has for fiber the Spin group, for trivialization the map :
$\varphi_G : M \times Spin_0(\mathbb{R},3,1) \to P_G :: p = \varphi_G(m,s)$.
The gauge is $\mathbf{p}(m) = \varphi_G(m, \mathbf{1})$
A section $\sigma \in \mathfrak{X}(P_G)$ is defined by a map: $\sigma : M \to Spin$ such that : $\sigma(m) = \varphi_G(m, \sigma(m))$ and in a change of gauge :
$\sigma(m) = \varphi_G(m, \sigma(m)) = \widetilde{\varphi}_G(m, \widetilde{\sigma}(m)) \Rightarrow \widetilde{\sigma}(m) = \chi(m) \cdot \sigma(m)$
2. The vectors on M are represented in the associated vector bundle $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$ defined through the holonomic basis :
$\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$ with the equivalence relation :
$(\mathbf{p}(m), v) \sim \left(\varphi_G(m,g), \mathbf{Ad}_{g^{-1}} v\right)$
The Lorentz scalar product on $\mathbb{R}^4$ is preserved by $\mathbf{Ad}$ thus it can be extended to $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$.
3. A particle has a trajectory $\mu_O : \mathbb{R} \to M :: \mu_O(t) = \varphi_M(t, x(t))$ for the observers with velocity with respect to O :
$V = \frac{d\mu_O}{dt}, \langle V, V \rangle < 0, \langle V, \varepsilon_0(m) \rangle < 0$

4. The spin S is represented in the associated vector bundle : $P_G[E,\gamma C]$, defined through the holonomic basis : $\mathbf{e}_i(m) = (\mathbf{p}(m), e_i)$ with the equivalence relation : $(\mathbf{p}(m), S) \sim \left(\varphi_G(m,g), \gamma C\left(g^{-1}\right) S\right)$ so that in a change of gauge the holonomic basis becomes :

$\widetilde{\mathbf{e}}_i(m) = (\widetilde{\mathbf{p}}(m), e_i) = \left(\varphi_G\left(m, \chi(m)^{-1}\right), e_i\right) \sim \gamma C\left(\chi(m)^{-1}\right) \mathbf{e}_i(m)$

$\mathbf{S}_m = (\mathbf{p}(m), S_m) = \left(\widetilde{\mathbf{p}}(m), \widetilde{S}_m\right) \Rightarrow \widetilde{S}_m = \gamma C(\chi(m)) S_m$

The scalar product on E is preserved by γC thus it can be extended to $P_G[E,\gamma C]$.

The standard observer uses the frame $\left(O, (\varepsilon_i)_{i=0}^3\right)$ and the holonomic maps $(\mathbf{e}_i(m))_{i=0}^3$ to take his measures.

5. Moreover the symmetry breakdown of the Spin group induces a splitting of the principal bundle.

i) The velocities of the localized observers give rise to an associated fiber bundle

$P_W = P_G[Spin(\mathbb{R},3,1)/Spin(\mathbb{R},3), \lambda]$:

$(\mathbf{p}(m), s_w) = (\varphi_G(m,1), s_w) \sim \left(\varphi_G(m,s), \lambda\left(s^{-1}, s_w\right)\right)$

with the left action :

$\lambda : Spin(\mathbb{R},3,1) \times Spin(\mathbb{R},3,1)/Spin(\mathbb{R},3) \to Spin(\mathbb{R},3,1)/Spin(\mathbb{R},3) : \lambda(s, s_w) = \pi_w(s \cdot s_w)$

ii) On the manifold $P_G$there is a structure of principal fiber bundle

$P_R(P_W, Spin(\mathbb{R},3), \pi_R)$ with trivialization :

$\varphi_R : P_W \times Spin(\mathbb{R},3) \to P_G ::$

$\varphi_R((\mathbf{p}(m), s_w), s_r) = \varphi_G(m, s_w \cdot s_r) = \varphi_R\left(\left(\varphi_G(m,s), \lambda\left(s^{-1}, s_w\right)\right), s_r\right)$

As the latest trivialization shows, for a given s, s_r depends on s_w in that it is a part of $s \in Spin(3,1)$.

It sums up to define the local frame in two steps : first by choosing s_w second by choosing s_r

Our objective is now to understand the relation between sections $\mathbf{S} \in \mathbf{P}_G[E,\gamma C]$ of the vector bundle and motion of a particle, represented by a map : $S : \mathbb{R} \to E$. We will proceed in several steps. First we will focus on the trajectories and we will show that they can be matched with sections of P_W.

Trajectories and vector fields on M

A trajectory is more than a curve, it is a smooth map $\mu : \mathbb{R} \to M$. Not any such map can represent the trajectory of a material body. There are two cases :

i) If the parametrization is the proper time τ of the body the domain of the map $\mu_P : \mathbb{R} \to \Omega$ is different than [0,T], but one can always choose $\tau = 0$ at the origin of the trajectory on $\Omega_0(0)$. By definition the proper velocity $U = \frac{d\mu_P}{d\tau}$ is a time like, future oriented vector : $U(\tau)$ with $U^0(\tau) > 0, \langle U(\tau), U(\tau)\rangle = -c^2$.

ii) If the parametrization is the time t of the network of observers, then $\mu_O : [0,T] \to \Omega$ and the velocity is $V(t) = \frac{d\mu_O}{dt}$. V is a time like, future oriented vector : $\langle V(\tau), V(\tau)\rangle < 0, V^0(\tau) > 0$.

We will call μ_P a **world line** and μ_O a **trajectory**.

The two maps have the same image : a curve in M, and there is a bijective map :

$f : [0,T] \to \mathbb{R} :: \mu_O(t) = \mu_P(f(t))$ given by : $\frac{d\mu_O}{dt} = V(t) = \frac{d\mu_P}{d\tau}\frac{df}{dt} = U(f(t))\frac{df}{dt}$

$\Rightarrow \langle V, V\rangle = \langle U, U\rangle \left(\frac{df}{dt}\right)^2 = -c^2\left(\frac{df}{dt}\right)^2$

$\Rightarrow \frac{df}{dt} = -\frac{1}{c}\langle V, V\rangle \geq 0$

Any smooth curve can be represented as the integral curve of a vector field. By definition an integral curve of the vector field W is defined by its flow Φ_W and the condition : $\frac{\partial}{\partial\theta}\Phi_W(\theta, m)|_{\theta=\theta_0} =$

$W(\Phi_W(\theta_0, m))$. Because Ω is relatively compact the flow of smooth vector fields is defined for all values of the parameter θ.

$\mu_O(t)$ is the integral curve of W_O if :

$\mu_O(t) = \Phi_{W_O}(t, \mu_O(0))$

$\frac{d\mu_O}{dt} = V(t) = \frac{\partial}{\partial t}\Phi_{W_O}(t, \mu_O(0)) = W_O(\Phi_{W_O}(t, \mu_O(0))) = W_O(\mu_O(t))$

And W must be a time like and future oriented vector field.

$\mu_P(\tau)$ is the integral curve of W_P if :

$\mu_P(\tau) = \Phi_{W_P}(\tau, \mu_P(0))$

$\frac{d\mu_P}{d\tau} = U(\tau) = \frac{\partial}{\partial \tau}\Phi_{W_P}(\tau, \mu_P(0)) = W_P(\Phi_{W_P}(\tau, \mu_P(0))) = W_P(\mu_P(\tau))$

And W must be a time like, future oriented vector field with constant norm $\langle W_P, W_P\rangle = -c^2$.

There are always smooth vector field meeting these conditions. The integral curve are then precisely defined by the origin $x = \mu_O(0) = \mu_P(0)$ on $\Omega_3(0)$.

The two vector fields W_P, W_O are different but have the same integral curve. If we have the curve defined by a vector field W_P the point on the curve which is observed at t is just the intersection of this curve with the hypersurface $\Omega_3(t)$.

One goes from one integral curve to the other by :

$\mu_P(f(t)) = \Phi_{W_P}(f(t), x) = \mu_O(t) = \Phi_{W_O}(t, x) = \varphi_M(t, x)$

$\Rightarrow U(f(t))\frac{df}{dt} = V(t)$

so the observer needs an additional information (the function f) to locate precisely the particle on the curve.

Definition 4 *A vector field W on M is said to be* ***admissible*** *if it is smooth, its support is a compact subset of M and at each point its value W(m) is a time like, future oriented vector.*

The integral curves of admissible vector fields can represent trajectories of particles. It moreover their norm is -c^2 they can represent proper world lines.

An intriguing question is : for a given curve, do all particles have the same velocity V ? It is generally assumed that "free particles", meaning which are subjected to the gravitational field only, follow geodesics. As there is a unique geodesic at a given point for a given velocity, at least for free particles the answer is yes. In this case the curve defines the world line and the trajectory, and the function f results from the comparison between the curves followed respectively by the observer and the particle : this is a purely geometric quantity. But of course the curve, and so the vector field itself, depends on the initial conditions, which includes the velocity.

Trajectories and sections of P_W

We know how to link trajectories and vector fields. Now we see how to link vector fields and sections of P_w.

Proposition 5 *For any time like, future oriented vector V at any point m of M there are two elements*

$\pm\sigma_w = \pm(a_w + \upsilon(0, w)) \in Spin(3,1)$ *such that* $V = \sqrt{-\langle V, V\rangle}\mathbf{Ad}_{\sigma_w}\varepsilon_0(m)$.

Then $\forall\sigma_r \in Spin(3) : V = \sqrt{-\langle V, V\rangle}\mathbf{Ad}_{\sigma_w\cdot\sigma_r}\varepsilon_0(m)$

Conversely for any element $\sigma \in Spin(3,1)$ *the vectors V such that at any point m :*

$V=\sqrt{|\langle V,V\rangle|}\mathbf{Ad}_{\sigma}\varepsilon_0(m)$ *are time like and future oriented. They are given by :*
$V=\sqrt{-\langle V,V\rangle}\left(\sqrt{2a_w^2-1}\varepsilon_0(m)+a_w\sum_{j=1}^{3}w_j\varepsilon_j(m)\right)$
where $\sigma_w=\pm(a_w+\upsilon(0,w))$ *belongs to the projection of* σ *to the homogeneous space* $Spin(3,1)/Spin(3)$

Proof. i) $V=\sum_{i=0}^{3}V_i\varepsilon_i(m)$
Define $u_m=\frac{1}{\sqrt{-\langle V,V\rangle}}V$
Then : $\langle u_m,u_m\rangle=-1,\langle u(m),\varepsilon_0(m)\rangle<0$
In the holonomic basis of $P_G\left[\mathbb{R}^4,\mathbf{Ad}\right]$ $\mathbf{u}$ reads :
$u_m=(\mathbf{p}(m),u)$ with $u=u_0\varepsilon_0+u_1\varepsilon_1+u_2\varepsilon_2+u_3\varepsilon_3,\langle u,u\rangle=-1,u_0>0$
With $w=w_1\varepsilon_1+w_2\varepsilon_2+w_3\varepsilon_3,\sigma_w=a_w+\upsilon(0,w)$, $\mathbf{Ad}_{\sigma_w}\varepsilon_0(m)=u_m$ reads :
$\left(a_w+\frac{1}{2}\varepsilon_0\cdot w\right)\cdot\varepsilon_0\cdot\left(a_w-\frac{1}{2}\varepsilon_0\cdot w\right)=\left(a_w+\frac{1}{2}\varepsilon_0\cdot w\right)\cdot\left(a_w\varepsilon_0+\frac{1}{2}w\right)$
$=a_w^2\varepsilon_0+\frac{1}{2}a_ww+\frac{1}{2}a_w\varepsilon_0\cdot w\cdot\varepsilon_0+\frac{1}{4}\varepsilon_0\cdot w\cdot w$
$=a_w^2\varepsilon_0+\frac{1}{2}a_ww+\frac{1}{2}a_ww+\frac{1}{4}w^tw\varepsilon_0=\left(2a_w^2-1\right)\varepsilon_0+a_ww$
$\left(2a_w^2-1\right)\varepsilon_0+a_w(w_1\varepsilon_1+w_2\varepsilon_2+w_3\varepsilon_3)=u_0\varepsilon_0+u_1\varepsilon_1+u_2\varepsilon_2+u_3\varepsilon_3$
$\Leftrightarrow$
$w_j=\frac{1}{a_w}u_j$
$a_w^2=\frac{1}{2}(u_0+1)>0$
One can check that :
$w^tw=\frac{1}{a_w^2}\sum_{j=1}^{3}(u_j)^2=\frac{1}{a_w^2}\left(-1+u_0^2\right)=\frac{1}{a_w^2}\left(-1+\left(2a_w^2-1\right)^2\right)=4\left(a_w^2-1\right)$
$a_w=\epsilon\sqrt{\frac{1}{2}(u_0+1)}=\epsilon\sqrt{\frac{1}{2}\left(\frac{V_0}{\sqrt{-\langle V,V\rangle}}+1\right)}$ with $\epsilon=\pm1$
j= 1,2,3 : $w_j=\frac{1}{a_w}u_j=\epsilon\frac{1}{\sqrt{\frac{1}{2}\left(\frac{V_0}{\sqrt{-\langle V,V\rangle}}+1\right)}}\frac{V_j}{\sqrt{-\langle V,V\rangle}}$
$\sigma_w=\epsilon\left(\sqrt{\frac{1}{2}\left(\frac{V_0}{\sqrt{-\langle V,V\rangle}}+1\right)}+\upsilon\left(0,\frac{1}{\sqrt{\frac{1}{2}\left(\frac{V_0}{\sqrt{-\langle V,V\rangle}}+1\right)}}\frac{V_j}{\sqrt{-\langle V,V\rangle}}\right)\right)$
ii) $Ad_{\sigma_w\cdot\sigma_r}\varepsilon_0(m)=Ad_{\sigma_w}Ad_{\sigma_r}\varepsilon_0(m)=Ad_{\sigma_w}\varepsilon_0(m)$
with $Ad_{\sigma_r}\varepsilon_0(m)=\varepsilon_0(m)$ by definition of $\mathrm{Spin}(\mathbb{R},3)$
iii) Conversely, if V is such that : $V=\sqrt{|\langle V,V\rangle|}\mathbf{Ad}_{\sigma_w}\varepsilon_0(m)$
with $u_m=\frac{1}{\sqrt{|\langle V,V\rangle|}}V=\mathbf{Ad}_{\sigma_w}\varepsilon_0(m)$
$\langle u_m,u_m\rangle=\langle\mathbf{Ad}_{\sigma_w}\varepsilon_0(m),\mathbf{Ad}_{\sigma_w}\varepsilon_0(m)\rangle=\langle\varepsilon_0(m),\varepsilon_0(m)\rangle=-1=\frac{\langle V,V\rangle}{|\langle V,V\rangle|}$
$\Rightarrow\langle V,V\rangle=-|\langle V,V\rangle|$
Any element of $Spin(3,1)$ can be written :
$\sigma=\epsilon(a_w+\upsilon(0,w_w))\cdot\epsilon(a_r+\upsilon(0,r_r))$
Let us denote $\sigma_w=a_w+\upsilon(0,w)$ the solution such that $a_w>0$ and $-\sigma_w$ the other one.
$u_m=\mathbf{Ad}_{\sigma}\varepsilon_0(m)=\mathbf{Ad}_{\pm\sigma_w}\varepsilon_0(m)=\mathbf{Ad}_{\sigma_w}\varepsilon_0(m)$
So for both solutions we have :
$2\langle u_m,\varepsilon_0\rangle=u_m\cdot\varepsilon_0+\varepsilon_0\cdot u_m$
$=\mathbf{Ad}_{\sigma_w}\varepsilon_0\cdot\varepsilon_0+\varepsilon_0\cdot\mathbf{Ad}_{\sigma_w}\varepsilon_0$
$=\left(a_w+\frac{1}{2}\varepsilon_0\cdot w\right)\cdot\varepsilon_0\cdot\left(a_w-\frac{1}{2}\varepsilon_0\cdot w\right)\cdot\varepsilon_0+\varepsilon_0\cdot\left(a_w+\frac{1}{2}\varepsilon_0\cdot w\right)\cdot\varepsilon_0\cdot\left(a_w-\frac{1}{2}\varepsilon_0\cdot w\right)$
$=\left(a_w\varepsilon_0+\frac{1}{2}w\right)\cdot\left(a_w\varepsilon_0-\frac{1}{2}w\right)+\left(a_w\varepsilon_0-\frac{1}{2}w\right)\cdot\left(a_w\varepsilon_0+\frac{1}{2}w\right)$
$=\left(a_w\varepsilon_0+\frac{1}{2}w\right)\cdot\left(a_w\varepsilon_0-\frac{1}{2}w\right)+\left(a_w\varepsilon_0-\frac{1}{2}w\right)\cdot\left(a_w\varepsilon_0+\frac{1}{2}w\right)$

$= -a_w^2 + \frac{1}{2}a_w w \cdot \varepsilon_0 - \frac{1}{2}a_w \varepsilon_0 \cdot w - \frac{1}{4}w \cdot w - a_w^2 - \frac{1}{2}a_w w \cdot \varepsilon_0 + \frac{1}{2}a_w \varepsilon_0 \cdot w - \frac{1}{4}w \cdot w$
$= -2a_w^2 - \frac{1}{2}w \cdot w = -2\left(\frac{1}{4}w^t w + 1\right) - \frac{1}{2}w^t w = -w^t w - 2 < 0$
thus u_m is future oriented
The computation of $u_m = \mathbf{Ad}_{\sigma_w}\varepsilon_0(m)$ gives as above :
$a_w^2 = \frac{1}{2}(u_0 + 1) \Rightarrow u_0 = 2a_w^2 - 1 = 2\left(1 + \frac{1}{4}w^t w\right) - 1 = 1 + \frac{1}{2}w^t w > 0$
j= 1,2,3 : $w_j = \frac{1}{a_w}u_j \Rightarrow u_j = a_w w_j = (-a_w)(-w_j)$
$u_m = \left(2a_w^2 - 1\right)\varepsilon_0(m) + a_w \sum_{j=1}^3 w_j \varepsilon_j(m)$ for both solutions
$\langle u_m, u_m\rangle = a_w^2 w^t w - \left(2a_w^2 - 1\right)^2 = a_w^2\left(4\left(a_w^2 - 1\right)\right) - \left(2a_w^2 - 1\right)^2 = -1$
So both elements $\pm\sigma_w$define the same, time like, future oriented, vector:
$V = \sqrt{-\langle V, V\rangle}\left(\left(2a_w^2 - 1\right)\varepsilon_0(m) + a_w \sum_{j=1}^3 w_j \varepsilon_j(m)\right)$ ■
Notice that, if V is past oriented ($u_0 < 0$) or null ($\langle V, V\rangle = 0$)there is no solution :
$u_0^2 = 1 + \sum_{j=1}^3 u_j^2 > 1 \Rightarrow u_0 + 1 < 0.$

Proposition 6 *To any admissible vector field V can be associated two sections* $\sigma_w = \pm(a_w + \upsilon(0, w)) \in \mathfrak{X}(P_G)$ *. They are such that*

Proposition 7 $V(m) = \sqrt{-\langle V(m), V(m)\rangle}Ad_{\sigma_w(m)}\varepsilon_0(m)$

Proof. The implementation of the previous proposition at each point gives $\sigma_w(m)$
The result is a section of P_G : in a change of gauge $\sigma_w(m)$ transforms as a section of P_G :
$\mathbf{Ad}_{\widetilde{\sigma}_w}\widetilde{\varepsilon}_0(m) = u = \mathbf{Ad}_{\widetilde{\sigma}_w}\mathbf{Ad}_{\chi^{-1}}\varepsilon_0(m) = \mathbf{Ad}_{\sigma_w^{-1}}\varepsilon_0(m)$
$\sigma_w = \widetilde{\sigma}_w \cdot \chi^{-1}$
$\widetilde{\sigma}_w(m) = \chi(m) \cdot \sigma_w(m)$ ■

Proposition 8 *Conversely any section* $\sigma \in \mathfrak{X}(P_G)$ *defines for any positive function* $f \in C_\infty(\Omega; \mathbb{R}_+)$ *a unique admissible vector field V such that*
$\langle V(m), V(m)\rangle = -f^2(m) : V(m) = f(m)\left(\left(2a_w^2 - 1\right)\varepsilon_0(m) + a_w \sum_{j=1}^3 w_j \varepsilon_j((m))\right)$

Proof. i) For any section $\sigma \in \mathfrak{X}(P_G)$ there is at each point a unique projection to P_w. In this class there are two elements $\epsilon\sigma_w = \pm(a_w + \upsilon(0, w))$ such as $\sigma = \epsilon\sigma_w \cdot \epsilon\sigma_r$
ii) Both sections define at each point the same vector u_m :
$u_m = \left(2a_w^2 - 1\right)\varepsilon_0(m) + a_w \sum_{j=1}^3 w_j \varepsilon_j(m)$
of norm : $\langle u_m, u_m\rangle = -1$ which is future oriented.
iii) If we take $V(m) = f(m)u_m$ we have an admissible vector field. ■

So the trajectory of any particle, as followed by the observers, can be associated to two opposite sections $\sigma_w = \pm(a_w + \upsilon(0, w))$ which belong to the same class of $\mathfrak{X}(P_W)$:

$$\sigma_w = \epsilon\left(\sqrt{\frac{1}{2}\left(\frac{V_0}{\sqrt{-\langle V,V\rangle}} + 1\right)} + \upsilon\left(0, \frac{1}{\sqrt{\frac{1}{2}\left(\frac{V_0}{\sqrt{-\langle V,V\rangle}} + 1\right)}}\frac{V_j}{\sqrt{-\langle V,V\rangle}}\right)\right)$$

and correspond either to w with the same orientation as the spatial speed ($\epsilon = 1$) or the opposite orientation ($\epsilon = -1$).

Conversely to a section $\sigma \in \mathfrak{X}(P_G)$ and any function $f \in C_\infty(\Omega; \mathbb{R}_+)$ one can associate a unique admissible vector field V :

$$V(m) = f(m)\left(\left(2a_w^2 - 1\right)\varepsilon_0(m) + a_w \sum_{j=1}^{3} w_j \varepsilon_j((m))\right) \tag{17}$$

whose integral curves are the trajectories of particles, as seen by the observers, with $\mu_O(t) = \Phi_V(t, \mu_O(0))$.

The proper world line of a particle can be associated to two opposite sections $\sigma_w = \pm(a_w + \upsilon(0, w))$ which belong to the same class of $\mathfrak{X}(P_W)$.

$\sigma_w = \epsilon\left(\sqrt{\frac{1}{2}\left(\frac{1}{c}U_0 + 1\right)} + \upsilon\left(0, \frac{1}{\sqrt{\frac{1}{2}\left(\frac{1}{c}U_0+1\right)}}\frac{1}{c}U_j\right)\right)$

And conversely to any section $\sigma \in \mathfrak{X}(P_G)$ one can associate a unique admissible vector field U

$$U(m) = c\left(\left(2a_w^2 - 1\right)\varepsilon_0(m) + a_w \sum_{j=1}^{3} w_j \varepsilon_j((m))\right) \tag{18}$$

whose integral curves are proper world lines, with $\mu_P(\tau) = \Phi_V(\tau, \mu_P(0))$.

Notice that :

- a given section $\sigma \in \mathfrak{X}(P_G)$ defines a family of proper world lines, but for a family of trajectories as measured by the observers we need a function f.

- a map $\sigma : [0, T] \to P_G$ is projected on M as a curve, which is not necessarily time like or defines a world line.

Motions of particles and sections of the fiber bundles

Our basic hypothesis is that the Spin is a dynamic characteristic of the geometric state of the particle. Physically it means that :

i) The Spin sums up the motion of the particle. The motion is measured with respect to a frame and the Spin measured at m is a vector of the associated bundle : $\mathbf{S} = (\varphi_G(m, 1), S(m)) \in P_G[E, \gamma C]$.

ii) At each point m(t) of the trajectory there is some frame, $\sigma(t) \in P_G(m(t))$ such that the particle is at rest : there is no motion, thus measured in this frame : $S = S_0 = Ct \in E$

iii) So the motion is characterized by a map : $\mathbf{S} : [0, T] \to P_G[E, \gamma C] :: \mathbf{S}(t) = \gamma C(\sigma(t)) S_0$ with some fixed $S_0 \in E$

The value of S_0 depends on the inital conditions (position and velocity) and the characteristics of the particle.

If we have a map : $\mathbf{S} : [0, T] \to P_G[E, \gamma C] :: \mathbf{S}(t) = (\varphi_G(m(t), 1), S(t))$ then :

i) the trajectory is defined by the projection $\mu_O(t) = \pi_G(\mathbf{S}(t))$ from $P_G[E, \gamma C]$ onto M. We assume that it is admissible.

ii) the trajectory defines, at each point, two sections $\sigma_w = \pm(a_w + \upsilon(0, w)) \in Spin(\mathbb{R}, 3, 1)$ with w aligned in the direction of the spatial speed or the opposite

iii) σ has a unique, up to sign, decomposition as : $\sigma = \sigma_w \cdot \sigma_r$ with $\sigma_r = a_r + \upsilon(0, r)$ belonging to one of the two connected components of Spin(3) (according to the sign of a_r, it is + for the component of the identity). So $\sigma_r(t) = \sigma_w(t)^{-1} \cdot \sigma(t)$

iv) the **spatial spin** is then defined as :

$$\mathbf{S}_r(t) = \gamma C\left(\sigma_w^{-1}\right)\mathbf{S}(t) = \gamma C(\sigma_r(t)) S_0 \tag{19}$$

The "spatial spin" is the representation of a spatial rotation. It is measured usually through the magnetic moment. And for a given trajectory there are two possible, opposite, values of the spatial spin : $S_r(t) = \pm\gamma C(\sigma_r(t)) S_0$. In all cases $S(t) = \gamma C(\sigma(t)) S_0$: the total spin stays the same, the distinction between the two opposite values is the consequence of the breakdown of symmetry induced by the observer. It is related to a trajectory (the orientation of w with respect to the spatial speed) so one can speak of spin "up" or "down" with respect to the trajectory.

If we assume that the spatial spin is, by itself, an intrinsic feature of the motion of the particle, then one must assume that the map : $\sigma_r : \mathbb{R} \to P_R$ is continuous, thus σ_r must belong and stay in one of the two connected components of P_R. Normally the decomposition $\sigma = \epsilon\sigma_w \cdot \epsilon\sigma_r$ is continuous, and the passage to the opposite sign is, for the spatial spin, a discontinuity.

If the trajectory is circular, with a constant rotational speed ω , radius R, in a plane orthogonal to ε_3

$w = k(-\sin\omega t, \cos\omega t; 0)$

$\sigma_w = a + k\upsilon(0, (-\sin\omega t, \cos\omega t; 0))$

with $\nu = \frac{R\varpi}{c}; a = \epsilon\sqrt{\frac{1}{2}\left(\frac{1}{\sqrt{1-\upsilon^2}} + 1\right)}, k = \epsilon\nu\frac{\sqrt{2}}{\sqrt{1+\sqrt{1-\nu^2}}}$

$\gamma C(\sigma_w) = aI - i\frac{1}{2}k(-(\sin\omega t)\gamma_1 + (\cos\omega t)\gamma_2)\gamma_0$

7.4.2 Quantization

Model

In the model of any physical system involving particles, the motion of a particle is then represented by a map :

$\mathbf{S} : [0, T] \to P_G[E, \gamma C] :: \mathbf{S}(t) = (\varphi_G(m(t), 1), S(t))$

which is practically split :

- in a trajectory $\mu_O(t) = \pi_G(\mathbf{S}(t))$ defined by the projection of $\mathbf{S}(t)$ on M.

- a spatial spin: $S_r : [0, T] \to E :: S_r(t) = \gamma C(\sigma_r(t)) S_0$ where S_r belongs to the Banach vector space F_r of maps with the norm : $\|S_r\|^2 = \int_0^T \sum_{i=1}^4 |S_r(t)|^2 dt < \infty$

So we assume that a choice of ϵ has been made.

In a global change of gauge the maps change as :

$\forall s \in Spin(\mathbb{R}, 3) : S_r \to \widetilde{S}_r :: \widetilde{S}_r(t) = \gamma C(s) S_r(t) = \gamma C(s \cdot \sigma(t)) S_0$

The application of the theorems of QM tells us that F_r is isometric to a Hilbert space, which is a unitary representation of $Spin(3)$. Because it is a compact group, its unitary representations are hilbertian sums of finite dimensional unitary representations.

The scalar product on F_r :

$\langle S_r, S_r'\rangle = \int_0^T \langle S_r(t), S_r'(t)\rangle_E dt = \int_0^T \langle \gamma C(\sigma_r(t)) S_0, \gamma C_r(\sigma'(t)) S_0'\rangle_E dt$

can be definite positive if the scalar product on E is definite positive.

So there should be some vector subspace E_0 of E such that :

- it is invariant by $\gamma C(\sigma_r)$ for $\sigma_r \in Spin(\mathbb{R}, 3) : \forall S_0 \in E_0, s_r \in Spin(\mathbb{R}, 3) : \gamma C(s_r) S_0 \in E_0$

- on which the scalar product is definite positive :
$\forall S_0 \in E_0 : \langle \gamma C(s_r(t)) S_0, \gamma C(s_r(t)) S_0 \rangle_E = \langle S_0, S_0 \rangle_E = 0 \Rightarrow S_0 = 0$

Quantization of the spatial spin

Proposition 9 *The only vector subspace of E invariant by γC on $Spin(3)$ and over which the scalar product is definite positive is*
$E_0 = \left\{ S = \begin{bmatrix} u_R \\ u_L \end{bmatrix} = \begin{bmatrix} v \\ iv \end{bmatrix}, v \in \mathbb{C}^2 \right\}$

Proof. i) The scalar product on E (which does not depend on the signature) reads :
$u = \begin{bmatrix} u_R \\ u_L \end{bmatrix} \in E : \begin{bmatrix} u_R^* & u_L^* \end{bmatrix} \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix} \begin{bmatrix} u_R \\ u_L \end{bmatrix} = i(u_L^* u_R - u_R^* u_L)$
So it is definite positive for any u such that :
$u_L^* u_R = -iv^* v$
$u_R^* u_L = iw^* w$
which implies :
$i(-iv^* v - iw^* w) = v^* v + w^* w \geq 0$
$v^* v = w^* w \geq 0 \Rightarrow v = w$
Thus the only vector subspaces E_0 of E on which $(E_0, \gamma C)$ is an unitary representation of $Spin(3)$ are :
$S = \begin{bmatrix} u_R \\ u_L \end{bmatrix} = \begin{bmatrix} v \\ iv \end{bmatrix}$, or $S = \begin{bmatrix} u_R \\ u_L \end{bmatrix} = \begin{bmatrix} iv \\ -v \end{bmatrix} = i \begin{bmatrix} v \\ iv \end{bmatrix}$,$v \in \mathbb{C}^2$
that is :
$E_0 = \left\{ S = \begin{bmatrix} u_R \\ u_L \end{bmatrix} = \begin{bmatrix} v \\ iv \end{bmatrix}, v \in \mathbb{C}^2 \right\}$
Then $\langle S, S \rangle = i(u_L^* u_R - u_R^* u_L) = i(-iv^* v - v^*(iv)) = i(-2iv^* v) = 2v^* v$
ii) The vector subspace must be invariant by $\gamma C(s_r)$. Which is equivalent to $S_L = iS_R$
For any $S_0 \in E_0, s \in Spin(3,1)$
$\gamma C(a + \upsilon(r,w) + b\varepsilon_5) S_0$
$= \begin{bmatrix} (a+ib)\sigma_0 + \frac{1}{2}\sum_{a=1}^3 (w_a - ir_a)\sigma_a & 0 \\ 0 & (a-ib)\sigma_0 - \frac{1}{2}\sum_{a=1}^3 (w_a + ir_a)\sigma_a \end{bmatrix} \begin{bmatrix} v \\ iv \end{bmatrix} = \begin{bmatrix} S_R \\ S_L \end{bmatrix}$
$S_R = \left((a+ib)\sigma_0 + \frac{1}{2}\sum_{a=1}^3 (w_a - ir_a)\sigma_a \right) v$
$S_L = \left((a-ib)\sigma_0 - \frac{1}{2}\sum_{a=1}^3 (w_a + ir_a)\sigma_a \right) iv$
and $S \in E_0 \Leftrightarrow S_L = iS_R$
$\Leftrightarrow \left((a-ib)\sigma_0 - \frac{1}{2}\sum_{a=1}^3 (w_a + ir_a)\sigma_a \right) v = \left((a+ib)\sigma_0 + \frac{1}{2}\sum_{a=1}^3 (w_a - ir_a)\sigma_a \right) v$
$\Leftrightarrow \left(2ib\sigma_0 + \sum_{a=1}^3 w_a \sigma_a \right) v = 0$
This condition is met for w=0 that is $s \in Spin(3)$.
iii) It is easy to see that the result does not depend on the signature:
$\gamma C'(s) \begin{bmatrix} v \\ iv \end{bmatrix} = \begin{bmatrix} v \\ iv \end{bmatrix} \Rightarrow$
$S_R = \left((a+b)\sigma_0 + \frac{1}{2}\sum_{a=1}^3 (w_a + ir_a)\sigma_a \right) v$
$S_L = \left((a-b)\sigma_0 + \frac{1}{2}\sum_{a=1}^3 (-w_a + ir_a)\sigma_a \right) iv$

$\left(2b\sigma_0 + \sum_{a=1}^{3} w_a\sigma_a\right) v = 0$ ■

Proposition 10 *The vectors of E_0 are the only eigen vectors of γ_0 with eigenvalue 1*

$$\gamma_0 = \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -iv \\ iu \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow v = iu$$

In a time reversal a vector of E_0 changes as:
$\begin{bmatrix} u \\ iu \end{bmatrix} \to \begin{bmatrix} iu \\ -u \end{bmatrix}$ with the signature (3,1) and $\begin{bmatrix} u \\ iu \end{bmatrix}$ with the signature (1,3)
In a space reversal a vector of E_0 changes as:
$\begin{bmatrix} u \\ iu \end{bmatrix} \to \begin{bmatrix} -u \\ iu \end{bmatrix}$ with the signature (3,1) and $\begin{bmatrix} -iu \\ -u \end{bmatrix}$ with the signature (1,3)

The vectors :
$\mathcal{E}_1 = \frac{1}{\sqrt{2}}(e_1 + ie_3)$
$\mathcal{E}_2 = \frac{1}{\sqrt{2}}(e_2 + ie_4)$
constitute an orthonormal basis of E_0 so that any vector reads:
$S = u_1\mathcal{E}_1 + u_2\mathcal{E}_2$
and its image by $\gamma C(a_r + \upsilon(r,0))$ is :
$\gamma C(a_r + \upsilon(r,0))(u_1\mathcal{E}_1 + u_2\mathcal{E}_2)$
$= a_r(u_1\mathcal{E}_1 + u_2\mathcal{E}_2) + \frac{1}{2}(r_2 - ir_1)(u_2\mathcal{E}_1 + u_1\mathcal{E}_2) + \frac{1}{2}ir_3(u_2\mathcal{E}_2 - u_1\mathcal{E}_1)$

The set of maps :
$F_r = \{S_r : [0,T] \to E_0 :: S_r = \gamma C(\sigma_r(t)) S_0, \sigma_r(t) \in Spin(3), S_0 \in E_0\}$
endowed with the scalar product :
$\langle S_r, S_r' \rangle = \int_0^T \langle \gamma C(\sigma_r(t)) S_0, \gamma C(\sigma_r'(t)) S_0' \rangle dt$
is a Hilbert space and the couple $(F_r, \gamma C)$ is a unitary representation of Spin(3). Let $(S_n)_{n\in\mathbb{N}}$ be a Hilbert basis of F_r.
$S_r = \sum_{n\in\mathbb{N}} c_n S_n = \sum_{n\in\mathbb{N}} c_n \left(s_n^1\mathcal{E}_1 + s_n^2\mathcal{E}_2\right)$
It can be seen in two, equivalent (through evaluation maps) ways : either the coordinates c_n are constant, and the S_n depends on t, or the other way around. We choose the second option, then :
$c_n \in C_\infty([0,T];\mathbb{C})$
$(S_n)_{n\in\mathbb{N}}$ are fixed orthogonal vectors of E_0 :
$\langle S_n, S_p \rangle = \delta_{np} \Rightarrow S_n = s_n^1\mathcal{E}_1 + s_n^2\mathcal{E}_2$ with $\overline{s_n^1}s_p^1 + \overline{s_n^2}s_p^2 = \delta_{np}$ and $s_n^k \in \mathbb{C}$
$S_r(t) = \sum_{n\in\mathbb{N}} c_n(t)\left(s_n^1\mathcal{E}_1 + s_n^2\mathcal{E}_2\right)$
The action of $Spin(3)$ reads :
$\gamma C(g) S_n = \gamma C(g)\left(s_n^1\mathcal{E}_1 + s_n^2\mathcal{E}_2\right)$
$= \left(s_n^1\left(a - \frac{1}{2}ir_3\right) - \frac{1}{2}s_n^2(ir_1 + r_2)\right)\mathcal{E}_1 + \left(\frac{1}{2}s_n^1(r_2 - ir_1) + s_n^2\left(a + \frac{1}{2}ir_3\right)\right)\mathcal{E}_2$
$\gamma C(g)\left(s_n^1\mathcal{E}_1 + s_n^2\mathcal{E}_2\right) = \left(\widetilde{s_n^1}\mathcal{E}_1 + \widetilde{s_n^2}\mathcal{E}_2\right)$
$\widetilde{s_n^1} = s_n^1\left(a - \frac{1}{2}ir_3\right) - \frac{1}{2}s_n^2(ir_1 + r_2)$
$\widetilde{s_n^2} = \frac{1}{2}s_n^1(r_2 - ir_1) + s_n^2\left(a + \frac{1}{2}ir_3\right)$
The measure $\widehat{S}_r$ of the spatial spin gives :

$\widehat{S}_r(t) = \sum_{n\in\mathbb{N}} \widehat{c}_n(t) S_n$ where $\widehat{c}_n(t) = 0$, or $\widehat{c}_n(t) = c_n(t)$ for a finite number of n, with a probability : $\frac{\sum_{n\in\mathbb{N}}|\widehat{c}_n|^2}{\sum_{n\in\mathbb{N}}|c_n|^2}$

Spin(3) is isomorphic to SU(2), and, up to isomorphism, the irreducible, unitary, representations of SU(2), denoted (P_N, D_N) are over order 2N homogeneous polynomials P with 3 real variables and complex coefficients, which are harmonic $\Delta P = 0$, and D_N is the left action of Spin(3) on the arguments of P. N is an integer of half an integer (it is said that particles have a spin 1/2). So P(x) is valued in $\mathbb{C}$ and depends on 3 real scalars ξ_1, ξ_2, ξ_3.

The vectors of a basis of an irreducible vector space reads :

$S_{Np} = p^1_{Np} P_{Np} \mathcal{E}_1 + p^2_{Np} P_{Np} \mathcal{E}_2$

where :

P_{Np} is an order 2N harmonic, homogeneous polynomials P with 3 real variables and complex coefficients,

$(P_{Np})_{p=-N}^{p=+N}$ is an orthonormal basis of D_N :

$\langle P_{Np}, P_{Nq}\rangle_{D_N} = \delta_{pq} = \int_{\mathbb{R}^3} \overline{P_{Np}(\xi)} P_{Nq}(\xi)\, d\xi$

$\left(p^1_{Np}, p^2_{Np}\right)_{p=-N}^{p=+N}$ are complex scalars

So for a given system and k = 1, 2 :

$s^k_n = \sum_{N\in\mathbb{N}/2}^{\infty} \sum_{p=-N}^{p=+N} p^{kp}_N P_{Np}$

$\Rightarrow S_r(t) = \sum_{n\in\mathbb{N}} c_n(t) \sum_{N\in\mathbb{N}/2}^{\infty} \sum_{p=-N}^{p=+N} P_{Np} \left(p^{1p}_N \mathcal{E}_1 + p^{2p}_N \mathcal{E}_2\right)$

The action D_N of Spin(3) is the left action on the arguments $\xi = (\xi_1, \xi_2, \xi_3)$ of the polynomial :

$D_N(g)(P)(\xi) = P\left([g]^{-1}[\xi]\right)$

$\widetilde{s^k_n} = \sum_{N\in\mathbb{N}/2}^{\infty} \sum_{p=-N}^{p=+N} \widetilde{p^{pk}_N} D_N(g) P_{Np} = \sum_{N\in\mathbb{N}/2}^{\infty} \sum_{p,q=-N}^{p,q=+N} [K_N]^p_q p^{kq}_N P_{Np}$

$p^{1p}_N \left(a - \frac{1}{2} i r_3\right) - \frac{1}{2} p^{2p}_N (i r_1 + r_2) = \sum_{q=-N}^{q=+N} [K_N]^p_q p^{1q}_N$

$\frac{1}{2} p^1_N (r_2 - i r_1) + p^2_N \left(a + \frac{1}{2} i r_3\right) = \sum_{q=-N}^{q=+N} [K_N]^p_q p^{2q}_N$

So far we have done only mathematics, using basic representation theory. Indeed ξ can be any parameter and the results stand whatever the interpretation of the variables. Now if we take a physicist point of view, S_r is the spatial spin, measured along the trajectory of a particle. What can be the meaning of ξ? We have already met this situation in Galilean geometry. In the common interpretation of QM there is no qualms : there is an object called a wave function, which is localized and so it is assumed that ξ are the coordinates of m. The rule in a change of gauge supports this interpretation of ξ as coordinates but, even assuming that there is such a localized wave function, there is no reason why ξ should be the coordinates of the point where the spatial spin is observed.

In our picture we have a more logical explanation. The principal bundle P_R depends on the choice of a section σ_w, and similarly the set F_r is subordinated to the choice of a trajectory, or to the choice of an admissible vector field. Given such a vector field, the trajectory is then defined by the choice of an origin x in $\Omega_3(0)$. So it makes sense that, for a family of trajectories defined by the same vector field, the functions s_{kn} are parametrized by x. If we assume that this is the case ;

$S_r(t) = \sum_{n\in\mathbb{N}} c_n(t) \sum_{N\in\mathbb{N}/2}^{\infty} \sum_{p=-N}^{p=+N} P_{Np}(x) \left(p^{1p}_N \mathcal{E}_1 + p^{2p}_N \mathcal{E}_2\right)$

Then we have the sections of the vector bundle $P_R[E_0, \gamma C]$

k=1,2: $\mathcal{S}_{k,N,p,n}(\Phi_V(t,x)) = \sum_{n\in\mathbb{N}} c_n(t) \sum_{N\in\mathbb{N}/2}^{\infty} \sum_{p=-N}^{p=+N} P_{Np}(x) \mathcal{E}_k$

and $S_r(t) = \sum_{n\in\mathbb{N}} \sum_{N\in\mathbb{N}/2}^{\infty} \sum_{p=-N}^{p=+N} \sum_{k=1}^{2} d^{k,N,p,n} \mathcal{S}_{k,N,p,n}(\Phi_V(t, x(\xi)))$

However, because F_r is separable, we must have a simpler basis :
$\mathcal{S}_{Np}\left(\Phi_V\left(t,x\right)\right)=\sum_{k=1}^{2}c_{Np}^{k}\left(t\right)P_{Np}\left(x\right)\mathcal{E}_k$
and

$$S_r\left(t\right)=\sum_{N\in\mathbb{N}/2}^{\infty}\sum_{p=-N}^{p=+N}d_{Np}\mathcal{S}_{Np}\left(\Phi_V\left(t,x\right)\right) \tag{20}$$

So $S_r\left(t\right)$ is quantized as the sum (possibly infinite) of orthogonal sections of the vector bundle $P_R\left[E_0,\gamma C\right]$. These sections depend on the vector field V by the formula above. The action of Spin(3) is then a change of trivialization on $\Omega_3\left(0\right)$: the same section is read in a new basis and its coordinates adjust accordingly. We will see more on the subject when we account for the physical characteristics of the particles.

Total spin

The set
$\hat{E}_0=\left\{\gamma C\left(\sigma_w\right)S_0,\sigma_w\in Spin\left(3,1\right)/Spin\left(3\right),S_0\in E_0\right\}=\left\{a_wS_0-i\frac{1}{2}\sum_{a=1}^{3}w_a\gamma_a\gamma_0S_0,S_0\in E_0\right\}$
is a subset $\hat{E}_0$ of E larger than E_0. This is not a vector space ($a_w=\epsilon\sqrt{1+\frac{1}{4}w^tw}$) it is a real manifold, embedded in E, with real dimension 10. For a given S_0 a chart of $\hat{E}_0$ has for coordinates the vectors r (with $r^tr\leq 4$) and w, so it has the real dimension 6 which is the same as $Spin(3,1)$. By definition it is invariant by $Spin\left(3,1\right)$.

Proposition 11 *The action γC of $Spin\left(3,1\right)$ on E_0. is free and effective and the representation $\left(E,\gamma C\right)$ is faithful on $\widehat{E}_0$*

Proof. a) The propositions :
i) the action γC is effective : $\forall S_0\in E_0:\gamma C\left(s\right)S_0=\gamma C\left(s'\right)S_0\Rightarrow s=s'$
ii) the action γC is free : $\gamma C\left(s\right)S_0=S_0\Rightarrow s=1$
iii) the representation $\left(E,\gamma C\right)$ is faithful on $\widehat{E}_0$: $\forall S\in\widehat{E}_0,S_0\in E_0:\exists s\in E_0$ unique : $S=\gamma C\left(s\right)S_0$
are equivalent
i) $\Rightarrow\gamma C\left(s\right)S_0=S_0\Rightarrow\gamma C\left(s^{-1}\right)\gamma C\left(s\right)S_0=\gamma C\left(s^{-1}\right)S_0=\gamma C\left(1\right)S_0\Rightarrow s=1$
ii) $\gamma C\left(s\right)S_0=\gamma C\left(s'\right)S_0\Rightarrow S_0=\gamma C\left(s^{-1}\right)\gamma C\left(s'\right)S_0\Rightarrow s^{-1}\cdot s'=1$
so i) $\Leftrightarrow$ ii)
By definition $\forall S\in\widehat{E}_0,S_0\in E_0:\exists s\in E_0$ and i) or ii) $\Rightarrow$ s is unique
and conversely : iii) $\Rightarrow S_0=\gamma C\left(1\right)S_0=\gamma C\left(s\right)S_0\Rightarrow s=1$ that is ii)
b) $\gamma C\left(s\right)S_0=S_0\Leftrightarrow$
$S_R=\left(\left(a+ib\right)\sigma_0+\frac{1}{2}\sum_{a=1}^{3}\left(w_a-ir_a\right)\sigma_a\right)v=v$
$S_L=\left(\left(a-ib\right)\sigma_0-\frac{1}{2}\sum_{a=1}^{3}\left(w_a+ir_a\right)\sigma_a\right)iv=iv$
$\Rightarrow$
$\left(2a\sigma_0-i\sum_{a=1}^{3}r_a\sigma_a\right)v=2v$
$\left(2ib\sigma_0+\sum_{a=1}^{3}w_a\sigma_a\right)v=0$
$\left(\sum_{a=1}^{3}r_a\sigma_a\right)v=2i\left(1-a\right)v$

$\left(\sum_{a=1}^{3} w_a\sigma_a\right) v = -2ibv$
$\Rightarrow$
$\sum_{a=1}^{3} r_a v^*\sigma_a v = 2i\left(1-a\right) v^* v$
$\sum_{a=1}^{3} w_a v^*\sigma_a v = -2ibv^* v$
The scalar $v^*\sigma_a v$ are real because the Dirac matrices are hermitian, as is $v^* v$,so
$\Rightarrow b = 0, a = 1$
$\Rightarrow r = w = 0$
and the only solution is $\sigma = 1$. ■

We have the left action of $Spin\left(3,1\right)$:
$\gamma C : Spin\left(3,1\right) \times \widehat{E}_0 \to \widehat{E}_0 :: \gamma C\left(s\right) S$
so we can define the structure of associated fiber bunde (not a vector bundle) $P_G\left[\widehat{E}_0, \gamma C\right]$.
Its elements are couples : $\left(\mathbf{p}\left(m\right), S\right) \sim \left(\varphi_G\left(m,s\right), \gamma C\left(s^{-1}\right) S\right)$
Its sections $\mathbf{S} \in \mathfrak{X}\left(P_G\left[\widehat{E}_0, \gamma C\right]\right)$ are defined by maps : $S : M \to \widehat{E}_0$
In the holonomic frame : $S\left(m\right)$ defines a vector $S_0\left(m\right) \in \widehat{E}_0$ from which equivalent vectors are deduced by
$\left(\mathbf{p}\left(m\right), S_0\left(m\right)\right) \sim \left(\varphi_G\left(m,s\right), \gamma C\left(s^{-1}\right) S_0\left(m\right)\right)$
As the representation is faithful : $P_G\left[\widehat{E}_0, \gamma C\right] \simeq P_G \times E_0$

Schrödinger equation

A fixed family $F\left(\mu_O\right)$ of maps :$\mathbf{S} : \left[0,T\right] \to P_G\left[\widehat{E}_0, \gamma C\right]$ which share the same projection : $\mu_O\left(t\right) = \pi_P\left(\mathbf{S}\left(t\right)\right)$ have in common σ_W up to sign, and is then a vector space :
$\forall \alpha, \beta \in \mathbb{C} : \alpha\gamma C\left(\sigma_w\left(t\right)\right)\gamma C\left(\sigma_r\left(t\right)\right) S_0 + \beta\gamma C\left(\sigma_w\left(t\right)\right)\gamma C\left(\sigma'_r\left(t\right)\right) S'_0$
$= \gamma C\left(\sigma_w\left(t\right)\right)\left(\alpha\gamma C\left(\sigma_r\left(t\right)\right) S_0 + \beta\gamma C\left(\sigma'_r\left(t\right)\right) S'_0\right)$
and $\alpha\gamma C\left(\sigma_r\left(t\right)\right) S_0 + \beta\gamma C\left(\sigma'_r\left(t\right)\right) S'_0 \in E_0$
It is also a Hilbert space with the previous scalar product.
So we can implement the theorems of QM.

In the Schrödinger picture the motion is defined for the whole evolution of the system, and measured from a batch of data. The definition of the system is geometric : the area is defined by two hyperplanes, and fiber bundles. The holonomic frame $\mathbf{p}\left(m\right)$ does not depend on the chart used on M. As seen in the formulas for a change of coordinates only the derivative (that is the spatial speed) is involved, so if we keep the same inertial observer and change only the origin of the time : $\widetilde{t} = t + \theta$ with some fixed θ then $m = \varphi_M\left(t,x\right) = \widetilde{\varphi}_M\left(t+\theta, x\right)$.

The new map will be : $\widetilde{S}\left(t+\theta\right) = S\left(t\right)$ that is : $\widetilde{S} = T^*_\theta S$ the pull back of S by the translation operator. We can enlarge $F\left(\mu_O\right)$ to the maps $I_0 \to E :: S\left(t\right) = \gamma C\left(\sigma\left(t\right)\right) S_0$ where I_0 is any closed interval of $\mathbb{R}$ with the scalar product :
$\left\langle S, S'\right\rangle = \int_{I_0} \left\langle S\left(t\right), S'\left(t\right)\right\rangle_E dt$
T^*_θ is a linear, unitary operator on $F\left(\mu_O\right)$. It defines an action of $\left(\mathbb{R},+\right)$ on $F\left(\mu_O\right)$, and a one parameter group. There is a self-adjoint operator H on $F\left(\mu_O\right)$ such that $T^*_\theta = \exp\frac{1}{i\hbar}\theta H$.
One goes from the Schrödinger picture to the Heisenberg picture by evaluations maps :
$\mathcal{E}\left(t\right) : F\left(\mu_O\right) \to \hat{E}_0 :: \mathcal{E}\left(t\right)\left(S\right) = S\left(t\right)$
and $\mathcal{E}\left(t\right)$ commutes with T^*_θ : $\mathcal{E}\left(t\right) \circ T^*_\theta = T^*_\theta \circ \mathcal{E}\left(t\right) = \mathcal{E}\left(t+\theta\right)$

$\mathcal{E}(t) T_\theta^* S = \mathcal{E}(t) \circ T_\theta^*(S) = T_\theta^* \circ \mathcal{E}(t) S = T_\theta^* S(t) = \mathcal{E}(t+\theta) S = S(t+\theta)$
By taking $\theta = 0$ we have the Schrödinger equation :
$\forall S \in F(\mu_O) : S(t) = \exp\left(\frac{1}{i\hbar} tH\right) S(0) \Rightarrow \frac{dS}{dt}(t) = \frac{1}{i\hbar} HS(t)$
For a given fixed S_0, there is a unique
$S(t) = \exp\left(\frac{1}{i\hbar} tH\right) \gamma C(\sigma(0)) S_0 = \gamma C(\sigma(t)) S_0$
The derivation applies actually to $\sigma(t)$: $\frac{d}{dt} S(t) = \gamma C\left(\frac{d\sigma}{dt}\right) S_0$
H depends on the vector field which defines the trajectories, but not on the origin of the trajectories.

If the trajectory is periodic, of period T, then any smooth map S has a representation as a Fourier series, they are a combination of $\left(\cos\left(2\pi n \frac{t}{T}\right)\right)_{n=-\infty}^{0}, \left(\sin\left(2\pi n \frac{t}{T}\right)\right)_{n=1}^{+\infty}$. It corresponds to the model of plane waves with a spectrum of frequencies given by the Fourier transformation.

7.4.3 Fields of particles

Proposition 12 *Any section $\sigma \in P_G$ define a unique family of world lines on M, there is a unique world line going through $x \in \Omega_3(0)$, and for any vector $S_0 \in E_0$ it defines on the world line two opposite spatial spins.*

Proof. i) at each point $m \in \Omega$ the section $\sigma(m)$ has a unique projection on Spin(3,1)/Spin(3) and in this class two elements : $\sigma_w = \epsilon(a_w + \upsilon(0, w)), \epsilon = \pm 1$
ii) the spatial spin is $S_r(t) = \gamma C\left(\sigma_w^{-1}(m)\right) \gamma C(\sigma(m)) S_0$
iii) each σ_w defines uniquely the same family of admissible vector fields
$U(m) = c\left(\left(2a_w^2 - 1\right) \varepsilon_0(m) + a_w \sum_{j=1}^{3} w_j \varepsilon_j((m))\right)$ with $\langle U, U\rangle = -c^2$
iv) the world line of the particle is $\mu_P(\tau) = \Phi_U(\tau, x)$ ■

Proposition 13 *Any section $\sigma \in P_G$ and positive function $f \in C_\infty(\Omega; \mathbb{R}_+)$ define a unique family of trajectories on M, there is a unique trajectory going through $x \in \Omega_3(0)$, and for any vectors $S_0 \in E_0$it defines on the trajectory two opposite spatial spins.*

Proof. We define the vector field by :
$V(m) = f(m)\left(\left(2a_w^2 - 1\right) \varepsilon_0(m) + a_w \sum_{j=1}^{3} w_j \varepsilon_j((m))\right)$
then V is admissible and the trajectory is : $\mu_O(t) = \Phi_V(t, x)$ ■

So it is logical to represent particles, for their geometric features, as sections of $\mathbf{S} \in \mathfrak{X}\left(P_G\left[\widehat{E}_0, \gamma C\right]\right)$. This representation gives a precise understanding of how a particle can "behave as field" and of the singular features of the spin. A section is defined at each point, it propagates along definite curves, and it is clearly related to the type of particle considered.

The construct used here is mathematically complicated, because we have to represent one dimensional manifolds in a 4 dimensional space, but it is more obvious and intuitive on the physical level. Actually there is a more direct and geometric approach, by considering the families of curves generated by time-like, future oriented vector fields, the relation of equivalence given by the appartenance to the same curve, and the quotient set over M. But we would be left with the spatial spin issue.

However it is clear that the trajectories are a crucial component of the physical representation of particles, and the assumed random behaviour of particles is not the only possible answer to the facts seen at the atomic scale.

7.4.4 Kinetic features of particles

In the usual frame work of relativist physics, each matter particle has a mass, which is a fixed scalar m. The linear momentum of a particle with proper velocity u and mass m is just the quadri-vector mu=p, which breaks down for any observer in a part related to its 3 dimensional momentum $\overrightarrow{p} = m\overrightarrow{v}\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}$ and its energy $\frac{1}{c}E = mc\frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}$ and $\langle p,p\rangle = P^2 = -m^2c^4 = \|\overrightarrow{p}\|^2 - \frac{E^2}{c^2} \Leftrightarrow E^2 = P^2c^2 + m^2c^4$

There is no strict equivalent to the rotational momentum, only the angular momentum J is considered. The proxy for the description of the motion of a particle in QFT would be the Pauli-Lubanski tensor : $W_\alpha = \epsilon_{\alpha\beta\gamma\delta}J^{\beta\gamma}P^\delta$ which is such that $W^\alpha W_\alpha = -m^2 s\,(s+1)$ where s is the spin number.

In our picture it is legitimate to relate the mass to the positive scalar $s_0^2 = \langle S_0, S_0\rangle$ which is unchanged in any gauge transformation and is a characteristic of the particle. This leads to see the momentum of a particle as the vector S itself. It clearly depends on the particle (by S_0) and the observer (by σ). It is represented in the associated fiber bundle $P_G\left[\widehat{E}_0, \gamma C\right]$, so this is a complex quantity, which is vectorial (it transforms as a vector) but $\widehat{E}_0$ is not a vector space.

Linear and rotational momenta are mixed, and cannot any longer be distinguished :

$S\,(t) = \gamma C\,(\sigma_w \cdot \sigma_r)\,S_0 = \left(a_r a_w I - i\frac{1}{2}\sum_{a=1}^{3}(a_r w_a \gamma_a \gamma_0 + a_w r_a \widetilde{\gamma}_a) - \frac{1}{2}i\,(w^t r)\,\gamma_5\right) S_0$

but $S\,(t)$ is proportional to s_0.

Along its trajectory by definition $S_0 = Ct$, and it implies the conservation of its mass.

However we should expect from momenta to be additive quantities. But S,S_r do not add, for two strong reasons : $\widehat{E}_0$ is not a vector space, and S is essentially a localized variable : summing S + S' would mean that the two particles are at the same position.

We will see more presisely later how all this relate to the usual quantities.

8 GENERAL RELATIVITY

8.1 The Geometry of General Relativity

8.1.1 The principles

General Relativity is, in many ways, an extension of Special Relativity to deal with the equivalence between gravitational mass and inertial mass.

Its geometry is based on five assumptions :

i) The universe is a four dimensional manifold M.

ii) There is a lorentzian metric represented by a symmetric bicovariant tensor g so that the tangent space at any point is the Minkovski space.

iii) All material bodies travel along a world line, future oriented, parametrized by their proper time. The proper time τ of an observer A (as measured by a clock) is such that $g\left(\frac{dA}{d\tau}, \frac{dA}{d\tau}\right) = -c^2$ with the speed of light c. Similarly it is assumed that the field forces propagate, as the electromagnetic field, along trajectories such that g(u,u)=0 (the light cone).

iv) There is a linear metric connection which transports orthonormal frames as orthonormal frames.

v) It should be possible to define in a consistent and continuous manner a past and a future. This is an additional condition, specific to GR, imposed upon M, which must be time orientable.

Not any 4 dimensional manifold structure can accept a Lorentz metric and be time orientable. There are topological obstructions which are not easily expressed.

The assumption iv) replaces physically the assumption on the inertial observers of Special Relativity. The connection defines preferred lines, the goedesics, and observers who travel along geodesics do not measure inertial forces. Usually the connection is assumed to be the Lévi-Civita connection, which is the only torsionfree connection which preserves the metric. In this case the only variable in the model is the metric, which characterizes the gravitational field.

As we have actually represented the universe of Special Relativity as a manifold (an affine space is a manifold), almost all that has been said in the framework of Special Relativity extends easily to General Relativity, this is the most obvious when using the fiber bundles formalism. However it is necessary to address three points :

- what is a chart in GR ? How can we build one ?
- what are the orthonormal frames ?
- what are the gauge transformations ?

8.1.2 Gaussian charts

General Relativity is at the foundation of cosmology, and then involves several additional hypotheses, based on astrophysical observations (mainly isotropy at large scale), which, with the additional requirements cited above, restrict the choice of admissible manifold structure. However, when one considers only physical systems, meaning the set of physical objects included in a delimited region of the universe, the picture is simpler. And it is possible to define a special chart (a gaussian chart) through a network of observers which gives back most of the properties of Special Relativity.

The starting point is a connected space-like hypersurface (its normal are time like) $\Omega_3(0)$. It represents the present of an observer at his proper time t=0. The choice of this hypersurface is arbitrary but crucial, because it defines completely the system. The metric induced over $\Omega_3(0)$ is riemannian, so it is possible to define by classical means (such as the radar coordinates of Einstein) a system of coordinates and 3 dimensional orthogonal frames. The fourth coordinate t is given by the clocks of the observers which are synchronized in the network.

Over each point x of $\Omega_3(0)$ there is a unique unitary, time like, future oriented vector n(x) normal to $\Omega_3(0)$. This vector is "virtual" for an observer located at x, but it defines the tangent to its own world line and the 4th vector of its orthonormal basis. We assume that there is a linear connection on M, so that it defines geodesics : in a neighborhood of x there is a unique geodesic with has n(x) as tangent. The observer can stay on the geodesic by checking that there is no change in the inertial forces. He can similarly transport the frame from x along this world line. So we can define a family of geodesics $\nu(t,x)$ tangent to n(x) and a vector field n in the future of $\Omega_3(0)$, which is the infinitesimal generator of diffeomorphisms which maps $\Omega_3(0)$ to a hypersurface $\Omega_3(t)$ for each $t \geq 0$ .One can prove that the vector field n is orthonormal to each $\Omega_3(t)$. A chart is given by the time $t = \frac{\xi^0}{c}$ and coordinates $\left(\xi^i\right)_{i=1}^3$ of x on $\Omega_3(0)$. The frame $(\varepsilon_i(m))_{i=0}^3$ of an observer at any point m of $\Omega_3(t)$ is then an orthonormal basis, with $\varepsilon_0(m)$ orthonormal to $\Omega_3(t)$. It is the holonomic frame $(\varepsilon_i(m))_{i=0}^3$ that we have defined in SR.

In any region of the universe with no singularity this construct is always possible (but certainly not at a cosmological scale because there is always some point where the geodesics cross or vanish). The system is then defined as the area enclosed in a region Ω generated by some open bounded domain of $\Omega_3(0)$. Each slice of Ω intersecting $\Omega_3(t)$ is considered by the observer as in the system at the time t. We see that the choice of another hypersurface $\Omega_3(0)$ defines another system. Geometrically Ω is a 4 dimensional manifold and a trivial fiber bundle with base $\mathbb{R}$ and trivialization : $m = \Phi_n(x,t)$ where Φ_n is the flow of the vector field n and x a point of $\Omega_3(0)$.

Spatial coordinates in $\Omega_3(0)$ can be established by any conventional method and transported along the vector field n which are geodesics. Notice that events occuring on $\Omega_3(t)$ cannot be reported live to the observer, but can be reported with a known delay. A bundle of orthonormal bases is built in each point, in a consistent manner, defining a principal bundle structure on M.

A particle which is located at some point x of $\Omega_3(0)$ (it enters the system at t=0) follows its own world line $\mu_P(\tau)$. Because M is a fiber bundle, for each point $\mu_P(\tau)$ of its world line there is a unique time $t = \pi(\mu_P(\tau))$ (consistent with the time of the observer). The 4 velocity $u = \frac{d\mu_P}{d\tau}$ of the particle is a future oriented, time like vector, which is projected on the base $\mathbb{R}$ as a positive scalar $\pi'(\mu_P(\tau))\, u = \frac{dt}{d\tau} > 0$. So the map $t(\tau)$ is injective : at any time t the particle is in a unique hypersurface $\Omega_3(t)$ and a particle which enters the system stays in the system (if Ω is "spatially" large enough).

This construct seems a bit abstract and farfetched, but it is very similar to the one used in the Global Positioning System (see Ashby) which accounts for General Relativity. And indeed it seems to be the only way to implement practically the hypotheses of GR at a local level.

8.1.3 Gauge transformations

There are two kinds of gauge transformations to consider in General Relativity. A change of chart of the manifold and a change of orthonomal bases, which, in GR, are always local. The variables used to represent physical quantities should change accordingly : they should be covariant (change of chart) and equivariant (change of orthonormal basis). In all cases the choice of $\Omega_3(0)$ defines the system thus it is a constant. It defines the normal and the geodesics.

By analogy with Special Relativity, where the center piece is taken by displacements, a great emphasize is commonly given to isometries, that is maps : $f : M \to M$ such that the derivative $f'(m)$ preserves the scalar product. This focus is misguided. As we have already seen in Special Relativity, a change of gauge is not done by jumping from a point to another, but by transporting a frame along a path. And this is all the more important when the universe is not isotropic : the path which is chosen matters. And the main ingredient to be involved is the connection.

One can change the chart on $\Omega_3(0)$, which is a riemannian manifold, and one can change how to measure the coordinate t (for instance in changing the unit or the origin). These changes of charts are defined by bijective, smooth maps :

$m = \varphi_M(t,x) = \widetilde{\varphi}_M\left(\widetilde{t},\widetilde{x}\right)$ with $\widetilde{t} = \chi_t(t), \widetilde{x} = \chi_x(x)$

The observers are the same, $\widetilde{t}$ is always related to the proper time of the observer, which comes from a physical measure given by a clock. They are "pure", covariant, changes of charts.

At any point one can consider another observer A who is travelling along his world line : he does not belong to the "standard network" established above, but anyway A and the standard observer O, located at the same point, using their own clock and frames, can compare their measures.

If A has a trajectory $\mu_P(\tau)$ in M, parametrized by its proper time, his velocity reads in the chart with coordinates $(\xi^\alpha)_{\alpha=.}^3$:

$u = \sum_{\alpha=0}^3 \frac{d\xi^\alpha}{d\tau}\partial\xi_\alpha = \frac{dt}{d\tau}\sum_{\alpha=0}^3 \frac{d\xi^\alpha}{dt}\partial\xi_\alpha = \frac{dt}{d\tau}\left(\sum_{\alpha=0}^3 v^\alpha\partial_\alpha\xi + c\varepsilon_0(t)\right)$

and we still have the relation : $\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}$

At the point m the observer O of the network has an orthonormal basis $(\varepsilon_i(m))_{i=0}^3$, the observer A has the orthonormal basis $(e_i(\tau))_{i=0}^3$ and the map between the two orthonormal basis is given by a matrix of SO(3,1) characterized by two parameters (r,w).

We still have $u = ce_0(\tau)$ The same demonstration as for Special Relativity shows that $w = \frac{v}{\|v\|}\arg\tanh\left\|\frac{v}{c}\right\|$ where v is the spatial speed of A with respect to the observer of the network located at m, and r is the axis of spatial rotation between the frames.

So the gauge transformation is given by :

i) the coordinates of A : $\left(\xi^i\right)_{i=0}^3$ with $\xi^0 = ct$ and t the time in the network

ii) the matrix $[G] = \exp[K(w)]\exp[J(r)]$ where (r,w) are two vectors in $\mathbb{R}^3$

iii) and we have the relations :

$v = \sum_{\alpha=1}^3 \frac{d\xi^\alpha}{dt}\partial\xi_\alpha$

$w = \frac{v}{\|v\|}\arg\tanh\left\|\frac{v}{c}\right\|$

With this coordinates system, a gauge transformation between an observer A and the network is characterized by two maps : $(\overrightarrow{r},\overrightarrow{v}) : \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3$ and an initial value (G(0),x(0)).

Actually we can consider a third case : the observer is the same, travelling along the same world line, so the times on the clock are the same, but A uses a spatial orthonormal frame which is different (always orthogonal to ε_0). This is a special case of the previous one : w = 0, expK(w) = I and only a classic rotation of the axes is involved.

8.2 The fiber bundle model

8.2.1 Principles

The usual geometric model of GR is actually a formal extension of the model of Special Relativity, where a manifold replaces an affine space. The main purpose of this model is to deal with gravitation, and the focus is placed on the metric, as it is (through the Christoffel symbols) the manifestation of the gravitational forces. However, at the view of the gaussian chart, the same principles can be represented in a more geometrical way, which is more suited to the usual tools used in theoretical physics, and more closely linked to the practical measures, without any loss of generality.

The implementation of the basic principles of General Relativity requires :

a) a four dimensional vector bundle $\mathfrak{M}$ over the manifold M, $\mathfrak{M}$ represents the local Minkovski space time

b) the existence of a network of frames on the tangent space TM which are deemed orthonormal with a lorentzian scalar product

c) a linear connection on $\mathfrak{M}$, which enables to transport the frames and preserves the metric

d) M must be "time orientable"

There are topological obstructions to the existence of a network of orthonormal frames for a lorentzian metric. If the universe meets these constraints is, by large, a matter of speculation. However, out of cosmology this does not seem a very huge assumption.

At each point there is a frame of reference $(\varepsilon_i(m))_{i=0}^3$ which is orthonormal for the lorentz scalar product : the vectors i=1,2,3 constitute an orthogonal euclidean frame, and the fourth vector $\varepsilon_0(m)$ is deemed orthogonal to the others with $\langle \varepsilon_0(m), \varepsilon_0(m) \rangle = -1$. As, locally, it is always possible to change the frame by a rotation of a group G which preserves the scalar product, the network defines a principal bundle P_G over M, with fiber G. The gauge $\mathbf{p}(m)$ of the principal bundle is then given by the frames of the network (they are assigned the value 1_G). Vectors and tensors at any point are defined in the basis $(\varepsilon_i(m))_{i=0}^3$. So mathematically the tangent space to M is defined as an associated vector bundle $P_G\left[\mathbb{R}^4, \rho\right]$, corresponding to a unitary representation $(\mathbb{R}^4, \rho)$ of G, with a scalar product defined through the vectors of the frame. The measures done at different points are compared by parallel transport, using a connection on P_G. As any principal connection on P_G is metric, it preserves the scalar product. The condition d) is easily translated by requiring that G is restricted to its connected component of the identity.

This model, which is a variant of the method of tetrads, is more geometric and physical, as it emphasizes the role of the frames and of a network of observers. It is increasingly popular as it is more suited to the framework used in the other fields of theoretical physics.

In this picture the choice of the group G is more open. The set of maps $\{\rho(g), g \in G\}$ must preserve the Lorentz product, and also the orientation (as the network of frames defines a continuous

orientation on the vector bundle) so it must belong to the connected component $SO_0(3,1)$. Thus we need a group morphism : $\rho : G \to SO_0(3,1)$ and the natural candidate for G is the Spin group. And there is a 4 dimensional representation of the Spin group on $\mathbb{R}^4$ which gives the Minkowski space. So we will assume that M can hold the structure of a principal Spin bundle.

In General Relativity the Universe is a 4 dimensional manifold, so there are charts in which any point can be identified by 4 real parameters. From the mathematical point of view, the same manifold can be described by infinitely many charts, as far as they are compatible, that is there is a way to pass in a consistent way from one to another. These charts carry the topological content of the universe, its "shape". The celebrated Einstein's equations of RG involve, not these charts, but the metric (or, in our case the connection). What is changing with the distribution of matter and energy is not the chart or the topology of the universe, but the metric. However the structure of the manifold and the physics it contains are not independant because they must be compatible. This is a strong issue for the interpretation of blackholes, as they are usually local singularities of the manifold (and the chart). But as said before we will not consider these extreme cases.

From general mathematical and physical considerations (such as the quasi isotropic distribution of matter at large scale) one can restrict the scope of the possible topologies of the universe, but it is still very large (in fact it could possibly be ...a 4 dimensional affine space). The cosmological models are based usually on a warped universe, which is actually a fiber bundle with base $\mathbb{R}$: there is a folliation of M in 3 dimensional riemannian hypersurfaces, and a universal time which is the same on each hypersurface. This is an extension of the gaussian chart, but with a critical point at an end. The expansion of the universe refers to the variation of the euclidean metric with respect to the universal time, and not to the manifold itself which stays the same.

If the choice of a chart is arbitrary, it must gives a way to physically identify a point. So its mathematical definition is a representation of physical procedures, and involves itself physical laws. For instance the existence of a proper time is based upon the assumption that there are ideal clocks which run at a constant rate for an observer. And the measure of lengths require either some surveying (with constant standards) or assumptions about the speed of light. We will not address these issues here, but we must always keep in mind that a model is based on assumptions on the procedures which can be implemented to do the measures.

8.2.2 Hypotheses

So in this alternate model, that we will adopt in the following of this paper, the assumptions are, formally, the following :

i) The universe is a four dimensional manifold M and the system covers an open, relatively compact, domain Ω of M.

ii) There is a structure of principal fiber bundle $P_G(M, Spin(3,1), \pi)$ which is trivial over Ω, with trivialization $\varphi_G : \Omega \times Spin(3,1) \to P_G$ and a gauge of reference $\mathbf{p}(m) = \varphi_G(m,1)$.

iii) Vectors in the tangent space of M are represented in the associated vector bundle $\mathfrak{M} = P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$. The holonomic basis $(\varepsilon_i(m))_{i=0}^3$ of $\mathfrak{M}$ defined in $m \in M$ by : $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$ is a frame of reference for the observers. It defines at each point a lorentzian scalar product on M.

iv) There is a principal connection, represented by its potential G, on P_G

v) *The motion of any observer in M is such that : $\frac{d\mu_P}{d\tau} = c\widetilde{\varepsilon}_0(\mu_P(\tau))$ where τ is his time, c a constant, and $\widetilde{\varepsilon}_0(\mu_P(\tau))$ the 4th vector of his frame.*

vi) There is a chart of Ω : $\varphi_M : \Omega_3(0) \times [0,T] \to \Omega :: m = \varphi_M(t,x)$ where t is the proper time of an observer, $\Omega_3(t) = \varphi_M(t, \Omega_3(0))$ are hypersurfaces of Ω, such that the metric induced on $\Omega_3(t)$ is riemannian and the frame of reference is parallel transported by the connection along the trajectory of the observer.

8.2.3 The fiber bundles

As we have already used the fiber bundle formalism in Special Relativity, we stay on familiar grounds, the only innovation is about the metric.

In any chart of M the coordinates of a point m of M will be denoted $(\xi^\alpha)_{\alpha=0}^3$, the holonomic basis is denoted $(\partial\xi_\alpha)_{\alpha=0}^3$ and its dual $(d\xi^\alpha)_{\alpha=0}^3$. In the chart φ_M the coordinates of $x \in \Omega_3(0)$ are $(\xi^\alpha)_{\alpha=1}^3$ and $\xi^0 = ct$.

A pure covariant change of chart is then given by two bijective maps : $\chi_x : \mathbb{R}^3 \to \mathbb{R}^3, \chi_t : \mathbb{R} \to \mathbb{R}$ such that :

$\widetilde{t} = \chi_t(t), \widetilde{x}\left(\widetilde{\xi}^1, \widetilde{\xi}^2, \widetilde{\xi}^3\right) = x\left(\chi_x\left(\xi^1, \xi^2, \xi^3\right)\right)$

thus χ_x is a change of map on $\Omega_3(0)$

The assumption v) deals with the basic issue of the measure of the "fourth coordinate", and the definition of a 4 dimensional frame in a three dimensional physical space. As t is measured by clocks, and the other coordinates as lengths, we need a constant c, which has the dimension of a speed, to relate units of time and space. As a consequence the "physical space" of any observer is orthogonal to its trajectory.

Fo any observer its velocity, expressed in his proper time, is such that

$\left\langle \frac{d\mu_P}{d\tau}, \frac{d\mu_P}{d\tau} \right\rangle = \langle c\widetilde{\varepsilon}_0(\mu_P(\tau)), c\widetilde{\varepsilon}_0(\mu_P(\tau))\rangle = -c^2.$

But we do no make any general assumption about the velocity of material bodies (observers have clocks, which give physical measures, there is nothing equivalent for a body).

The velocity of an observer A who travels along $\varphi_M(t(\tau), x(\tau))$ where τ is the time on his clock, and uses another frame $(\widetilde{\varepsilon}_i)_{i=0}^3$ is :

$V = \frac{d\mu_A(\tau)}{d\tau} = c\widetilde{\varepsilon}_0(\mu_A(\tau))$ in his frame

$V = \frac{dt}{d\tau}\frac{d\mu_A(\tau)}{dt} = \frac{dt}{d\tau}\left(\overrightarrow{v}(t) + c\varepsilon_0(t)\right)$ with respect to the observer of the network

$\langle V, V\rangle = \left(\frac{dt}{d\tau}\right)^2\left(\|\overrightarrow{v}\|^2 - c^2\right) = -c^2$

$V = \frac{1}{\sqrt{1-\frac{\|v\|^2}{c^2}}}\left(\overrightarrow{v} + c\varepsilon_0\right) = c\widetilde{\varepsilon}_0(\mu_A(\tau))$

And by the same reasoning as in Special Relativity we retrieve the usual coordinates transformations.

The assumption ii) tells that, at each point m of Ω, there is a frame of reference (a tetrad) which is an orthonormal basis for the Lorentz product and the existence of the principal bundle structure asserts that the observers can change their frames with respect to this frame of reference (they have freedom of gauge).

The frame of reference defines a gauge on the principal bundle $P_G : \mathbf{p}(m) = \varphi_G(m, 1)$ and a holonomic basis $(\varepsilon_i(m))_{i=0}^3$ for the associated bundle :

$\mathfrak{M} = P_G\left[\mathbb{R}^4, \mathbf{Ad}\right] \simeq TM$ which defines the vector space used at each point to measure all vectorial quantities :

$\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i) \sim (\mathbf{p}(m) \cdot s, \mathbf{Ad}_{s^{-1}}\varepsilon_i)$

A change of gauge in P_G is represented with a map : $\chi : M \to Spin_0(\mathbb{R}, 3, 1)$ by :

$p = \varphi_G(m, g) = \widetilde{\varphi}_G(m, \chi(m) g)$

$\mathbf{p}(m) \to \widetilde{\mathbf{p}}(m) = \widetilde{\varphi}_G(m, 1) = \mathbf{p}(m) \cdot \chi(m)^{-1}$

And its impact on the vectors of the frame is :

$\varepsilon_i(m) \to \widetilde{\varepsilon}_i(m) = (\widetilde{\mathbf{p}}(m), \varepsilon_i) \sim \left(\mathbf{p}(m), \mathbf{Ad}_{\chi^{-1}}\varepsilon_i\right) = \mathbf{Ad}_{\chi^{-1}}\varepsilon_i(m)$

A section $\sigma \in \mathfrak{X}(P_G)$ is defined by a map : $\sigma : M \to Spin :: \sigma(m) = \varphi_G(m, \sigma(m))$ and in a change of gauge :

$\sigma(m) = \varphi_G(m, \sigma(m)) = \widetilde{\varphi}_G(m, \chi(m)\sigma(m)) \Rightarrow \widetilde{\sigma}(m) = \chi(m)\sigma(m)$

A section $\mathbf{u} \in \mathfrak{X}\left(P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]\right)$ of $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$ is defined by : $\mathbf{u}(m) = \sum_{i=0}^3 u^i(m)\varepsilon_i(m)$

In a change of gauge :

$\mathbf{u}(m) = \sum_{i=0}^3 u^i(m)\varepsilon_i(m) = \sum_{i=0}^3 \widetilde{u}^i(m)\widetilde{\varepsilon}_i(m) \Leftrightarrow \widetilde{u}^i = \sum_{j=0}^3 [h(\chi)]_j^i u^j$ where $[h(\chi)]$ is the matrix of $\mathbf{Ad}_\chi$ in the standard representation $\left(\mathbb{R}^4, \imath\right)$ of SO(3,1).

Every local frame of reference $\varepsilon_i(m)$ has components (called "vierbein") in any chart of M : $\varepsilon_i(m) = \sum_{\alpha=0}^3 P_i^\alpha(m)\partial\xi_\alpha$ where $[P]$ is a real invertible matrix and we denote $[P'] = [P]^{-1} = \left[P_\alpha'^i\right]$. The quantities $(P_i^\alpha(m))_{i=1}^3$ can be measured. They are given for a network but, because they are the physical realization of an experiment, they are themselves subject to the laws of physics, and must be considered as variables in the model.

In a change of gauge :

$\partial\xi_\alpha = \sum_{i=0}^3 \widetilde{P}_\alpha'^i(m)\widetilde{\varepsilon}_i(m) = \sum_{i=0}^3 \widetilde{P}_\alpha'^i(m)\mathbf{Ad}_{\chi^{-1}}\varepsilon_i(m) = \sum_{i=0}^3 \widetilde{P}_\alpha'^i(m)[h(\chi)]_i^j\varepsilon_j(m)$

$\widetilde{P}_\alpha'^i(m) = \sum_{j=0}^3 [h(\chi)]_j^i P_\alpha'^j(m)$

$\left[\widetilde{P}'\right] = [h(\chi)][P'] \Leftrightarrow \left[\widetilde{P}\right] = [P]\left[h\left(\chi^{-1}\right)\right]$ with $\left[h\left(\chi^{-1}\right)\right] \in SO_0(3,1)$

Any observer who travels along $\varphi_M(t, a)$ where $a \in \Omega_3(0)$ is fixed where t is the time on his clock, and uses the frame of reference has for velocity:

$V = \frac{d\mu_A(t)}{dt} = c\varepsilon_0(\mu_A(t)) = \partial\xi_0$

$\Rightarrow \Omega_3(t)$ is a space like hypersurface.

$\alpha = 1, 2, 3 : \partial\xi_\alpha(m) = \sum_{i=1}^3 P_\alpha'^i\varepsilon_i(m) \Leftrightarrow P_\alpha'^0 = 0$

$$P_0'^0 = P_0^0 = 1 \tag{21}$$

Ad preserves the scalar product, thus the Lorentz scalar product extends in a scalar product on the associated vector bundle $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$. The orthonormal basis $(\varepsilon_i)_{i=0}^3$ of $\mathbb{R}^4$ gives the orthonormal basis $\varepsilon_i(m) = (\mathbf{p}(m), \varepsilon_i)$ on $\mathfrak{M} = P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$. with $\langle\varepsilon_0(m), \varepsilon_0(m)\rangle = -1$

So the traditional metric g on M is defined by the components of the tetrad :

$\varepsilon_i(m) = \sum_{\alpha=0}^3 P_i^\alpha(m)\partial\xi_\alpha \Leftrightarrow \partial\xi_\alpha = \sum_{\alpha=0}^3 P_\alpha'^i(m)\varepsilon_i(m)$ with $[P'] = [P]^{-1}$

$g_{\alpha\beta}(m) = \langle\partial\xi_\alpha, \partial\xi_\beta\rangle = \sum_{ij=0}^3 \eta_{ij}[P']_\alpha^i[P']_\beta^j$

$g^{\alpha\beta}(m) = \sum_{ij=0}^3 \eta^{ij}[P]_i^\alpha[P]_j^\beta$

$$[g]^{-1} = [P]^t[\eta][P] \Leftrightarrow [g] = [P']^t[\eta][P'] \tag{22}$$

$[\eta]$ is to be replaced by -$[\eta]$ fo the signature (1,3) so g becomes -g.

The metric defines a volume form on M :

$\varpi_4 = \sqrt{|\det[g]|} d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$ which reads also : $\varpi_4 = \varepsilon_0 \wedge \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$ with the holonomic basis of $\mathfrak{M}$. Because :

$[g] = [P']^t [\eta] [P'] \Rightarrow \det[g] = -(\det[P'])^2 \Rightarrow \sqrt{|\det[g]|} = \det[P']$

assuming that the holonomic basis of $\mathfrak{M}$ is direct.

$$\varpi_4 = \det[P'] d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 \tag{23}$$

The principal bundle defines a Clifford bundle Cl(M) on M. At any point m, the holonomic frame $(\varepsilon_i)_{i=0}^4$ is orthonormal and is the basis of a Clifford algebra Cl(m) isomorphic to $Cl(\mathbb{R}, 3, 1)$. So the product of vectors is computed locally, and all the properties of Clifford algebras extend to Cl(m). Any other orthonormal frame deduced by a local gauge transformation gives rise to a Clifford algebra isomorphic to Cl(m).

As a consequence :

- the tangent vector space can be seen as a subspace of Cl(m) and the action of the Spin group on vectors of $\mathfrak{M} = P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$ as the action in Cl(m) : $\mathbf{Ad}_g v = g \cdot v \cdot g^{-1}$.

- the principal bundle P_G can be seen as a subbundle of the Clifford bundle Cl(M) : an element of the spin group at m can be written :

$s = a + \upsilon(r, w) + b\varepsilon_5(m)$ with

$\upsilon(r, w) = \frac{1}{2}\left(w^1\varepsilon_0 \cdot \varepsilon_1 + w^2\varepsilon_0 \cdot \varepsilon_2 + w^3\varepsilon_0 \cdot \varepsilon_3 + r^3\varepsilon_2 \cdot \varepsilon_1 + r^2\varepsilon_1 \cdot \varepsilon_3 + r^1\varepsilon_3 \cdot \varepsilon_2\right)$,

$\varepsilon_5 = \varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

- M is a spin bundle : for any representation $(E, \gamma C)$ of the Clifford algebra there is an associated vector bundle structure $P_G[E, \gamma C]$ on M such that $(E(m), \gamma C)$ is a representation of Cl(m).

- The scalar product on the Clifford algebra is preserved by **Ad** so it extends to Cl(M) :

$\langle \varepsilon_{i_1} \cdot \varepsilon_{i_2} \cdot ... \cdot \varepsilon_{i_n}, \varepsilon_{j_1} \cdot \varepsilon_{j_2} \cdot ... \cdot \varepsilon_{j_n} \rangle = \eta_{i_1 j_1} ... \eta_{i_n j_n l} \epsilon(i_1 ..., i_n, j_1, ...j_n)$

The symmetry breakdown of the Spin group leads, as in Special Relativity, to an associated fiber bundle $P_W = P_G[Spin(3,1)/Spin(3), \lambda]$:

$(\mathbf{p}(m), s_w) = (\varphi_G(m, 1), s_w) \sim \left(\varphi_G(m, s), \lambda\left(s^{-1}, s_w\right)\right)$

with the left action :

$\lambda : Spin(3,1) \times Spin(3,1)/Spin(3) \to Spin(3,1)/Spin(3) ::$

$\lambda(s, s_w) = \pi_w(s \cdot s_w)$

And on the manifold P_G to the structure of principal fiber bundle

$P_R(P_W, Spin(3), \pi_R)$

with trivialization :

$\varphi_R : P_W \times Spin(3) \to P_G ::$

$\varphi_R((\mathbf{p}(m), s_w), s_r) = \varphi_G(m, s_w \cdot s_r) = \varphi_R\left(\left(\varphi_G(m, s), \lambda\left(s^{-1}, s_w\right)\right), s_r\right)$

8.2.4 The connection

The true innovation of GR is the connection, which is the manifestation of the gravitational field. The crux of this topic is the transport of gauge, meaning the way we compare orthonormal frames at different locations. So we need to address the vector space tangent to P_G (which is a manifold) itself.

The principal bundle P_G is a manifold and its tangent space T_pP_G is isomorphic to the product $T_mM \times T_gSpin$: a vector of T_pP_G is defined by a couple $(v_m, v_g) \in T_mM \times T_gSpin$ located at m.. It is easily understood by differentiation of the trivialisation :

$p = \varphi_G(m,g) \in P_G \rightarrow v_p = \varphi'_{Gm}(m,g)\, v_m + \varphi'_{Gg}(m,g)\, v_g \in T_pP_G$

The vertical bundle VP_G is comprised of vectors $\varphi'_{Gg}(m,g)\, v_g$. It is isomorphic to the Lie algebra, and it is convenient to use the fundamental vectors : $\zeta(X)(p) = \varphi'_{Gg}(m,g)\, L'_g 1(X)$ where X is a vector of the Lie algebra : $X = \upsilon(X_r, X_w)$. When using the holonomic frame $\zeta(X)(\mathbf{p}) = \varphi'_{Gg}(m,1)(X)$ so that fundamental vectors are just vectors of the Lie algebra located at m, and thus belong to Cl(m). Any vector of the tangent bundle T_pP_G can be written : $v_p = \varphi'_{Gm}(m,g)\, v_m + \zeta(X)(p) \in T_pP_G$ with $X \in T_1Spin$

A principal connection $\mathbf{G}$ on P_G is an equivariant projection from T_pP_G to V_pP_G. So in the holonomic frame it can be seen as a map from T_pP_G to the Lie algebra, that is as a one form G over M valued in the Lie algebra, called the potential G of the connection $\mathbf{G} : G \in \Lambda_1(M; T_1Spin(3,1))$. It can be expressed in two different ways, that we will use.

The generic way, valid whatever the Lie algebra :

$G(m) = \sum_{\alpha=0}^{3} \sum_{a=1}^{6} G^a_\alpha \overrightarrow{\kappa}_a \otimes d\xi^\alpha$ where $(\overrightarrow{\kappa}_a)_{a=1}^{6}$ is a basis of the Lie algebra $T_1Spin(3,1)$.

Or, using the decomposition of the components in two parts, $G^a_{r\alpha}, G^a_{w\alpha}$:

$G(m) = \sum_{\alpha=0}^{3} \upsilon(G_{r\alpha}(m), G_{w\alpha}(m))\, d\xi^\alpha$

We go from one to the other by :

$\overrightarrow{\kappa}_1 = \frac{1}{2}\varepsilon_3 \cdot \varepsilon_2$, $\overrightarrow{\kappa}_2 = \frac{1}{2}\varepsilon_1 \cdot \varepsilon_3$, $\overrightarrow{\kappa}_3 = \frac{1}{2}\varepsilon_2 \cdot \varepsilon_1$,

$\overrightarrow{\kappa}_4 = \frac{1}{2}\varepsilon_0 \cdot \varepsilon_1$, $\overrightarrow{\kappa}_5 = \frac{1}{2}\varepsilon_0 \cdot \varepsilon_2$, $\overrightarrow{\kappa}_6 = \frac{1}{2}\varepsilon_0 \cdot \varepsilon_3$

Notice that in both cases the vectors ε_i are fixed (they do not depend on m).

In a change of gauge the potential transforms by an affine map :

$\mathbf{p}(m) \rightarrow \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1}$:

$G(m) \rightarrow \widetilde{G}(m) = Ad_\chi \left(G(m) - L'_{\chi^{-1}}(\chi)\, \chi'(m) \right)$

The adjoint operator Ad on the Lie algebra so(3,1) is just $\mathbf{Ad}|_{Spin}$ (see Annex).

$\widetilde{G}(m) = \chi(m) \cdot (\upsilon(G_{r\alpha}, G_{w\alpha}) - \upsilon(X_{r\alpha}, X_{w\alpha})) \cdot \chi(m)^{-1}$

$= (a + \upsilon(r,w) + b\varepsilon_5) \cdot \upsilon(G_{r\alpha} - X_{r\alpha}, G_{w\alpha} - X_{w\alpha}) \cdot (a - \upsilon(r,w) + b\varepsilon_5)$

$Ad_g \upsilon(G_{r\alpha} - X_{r\alpha}, G_{w\alpha} - X_{w\alpha}) = \upsilon\left(\widetilde{G}_r(m), \widetilde{G}_w(m)\right)$

Principal connections live in an affine bundle on M, with the associated vector bundle $P_G[T_1Spin, Ad]$, which is a subbundle of the Clifford bundle Cl(M).

The connection acts on sections of the principal bundle, and the covariant derivative of $\sigma = \varphi_G(m, \sigma(m)) \in \mathfrak{X}(P_G)$ is :

$$\nabla^G : \mathfrak{X}(P_G) \rightarrow \Lambda_1(M; T_1Spin) :: \nabla^G\sigma = \sigma^{-1} \cdot \sigma' + \mathbf{Ad}_{\sigma^{-1}} G = \sum_{\alpha=0}^{3} \sigma^{-1} \cdot \partial_\alpha \sigma + \mathbf{Ad}_{\sigma^{-1}} G_\alpha \quad (24)$$

The covariant derivative is invariant in a change of gauge.

Proof. $\varphi_G(m,g) = \widetilde{\varphi}_G(m, \chi(m)\, g)$

$G(m) \rightarrow \widetilde{G}(m) = Ad_\chi \left(G(m) - L'_{\chi^{-1}}(\chi)\, \chi'(m) \right)$

$\sigma(m) \rightarrow \widetilde{\sigma}(m) = \chi(m) \cdot \sigma(m)$

$\nabla^G\sigma \rightarrow \widetilde{\nabla^G\sigma} = \widetilde{\sigma}^{-1} \cdot \widetilde{\sigma}' + \mathbf{Ad}_{\widetilde{\sigma}^{-1}} \widetilde{G}$

$= \sigma^{-1} \cdot \chi^{-1} \cdot (\chi' \cdot \sigma + \chi \cdot \sigma') + \mathbf{Ad}_{\sigma^{-1}} \mathbf{Ad}_{\chi^{-1}} \left(Ad_{\chi} \left(G - \chi^{-1} \cdot \chi' \right) \right)$
$= \sigma^{-1} \cdot \chi^{-1} \cdot \chi' \cdot \sigma + \sigma^{-1} \cdot \chi^{-1} \cdot \chi \cdot \sigma' + \mathbf{Ad}_{\sigma^{-1}} \left(G - \chi^{-1} \cdot \chi' \right)$
$= \sigma^{-1} \cdot \chi^{-1} \cdot \chi' \cdot \sigma + \sigma^{-1} \cdot \sigma' + \mathbf{Ad}_{\sigma^{-1}} G - \sigma^{-1} \cdot \chi^{-1} \cdot \chi' \cdot \sigma$
$= \sigma^{-1} \cdot \sigma' + \mathbf{Ad}_{\sigma^{-1}} G = \nabla^G \sigma$ ■

And the potential is the covariant derivative of the gauge : $G = \nabla^G \mathbf{p}$.

Then a gauge is parallel transported along a vector field V on TM if : $\nabla^G_V p = \mathbf{0}$. This defines a differential equation in p(t).

The connection on P_G induces a linear connection ∇^M on $\mathfrak{M} = P_G \left[\mathbb{R}^4, \mathbf{Ad} \right]$ with Christoffell symbols :

$\Gamma_M (m) = (\mathbf{Ad}_s)'_{s=1} (G(m))$
$= \sum_{\alpha, ij=0}^{3} (K(G_{w\alpha}) + J(G_{r\alpha}))^i_j \, \varepsilon_i (m) \otimes \varepsilon^j (m) \otimes d\xi^\alpha$

$$\Gamma_{M\alpha} (m) = \begin{bmatrix} 0 & G^1_{w\alpha} & G^2_{w\alpha} & G^3_{w\alpha} \\ G^1_{w\alpha} & 0 & -G^3_{r\alpha} & G^2_{r\alpha} \\ G^2_{w\alpha} & G^3_{r\alpha} & 0 & -G^1_{\alpha} \\ G^3_{w\alpha} & -G^2_{r\alpha} & G^1_{r\alpha} & 0 \end{bmatrix} = K(G_{w\alpha}) + J(G_{r\alpha})$$

It can also be written with the product of vectors in Cl(m):

$v = \sum_{j=0}^{3} v^j \varepsilon_j (m) \to \sum_{i,j=0}^{3} [\Gamma_G (m)]^j_i \, v^i \varepsilon_j (m) = \upsilon (G_{w\alpha}, G_{r\alpha}) \cdot v - v \cdot \upsilon (G_{w\alpha}, G_{r\alpha})$

The covariant derivative of a section $V \in \mathfrak{X} \left(P_G \left[\mathbb{R}^4, \mathbf{Ad} \right] \right)$ is then :

$\nabla^M V = \sum_{\alpha i=0}^{3} \left(\partial_\alpha V^i + \sum_{j=0}^{3} \Gamma_G (m)^i_{\alpha j} V^j \right) \varepsilon_i (m) \otimes d\xi^\alpha$

Similarly a vector U is parallel transported along a vector field V of TM if : $\nabla^M_V U = \mathbf{0}$. This defines a differential equation in U(t).

8.2.5 The motion of a particle

All that has been said in Special Relativity still holds.

For any particle there is a vector S_0 of E_0 :

$S = u_1 \mathcal{E}_1 + u_2 \mathcal{E}_2 = s_0 \begin{bmatrix} v_0 \\ i v_0 \end{bmatrix} \in E_0$

$\langle S_0, S_0 \rangle = s_0^2 > 0$

The motion of a particle is represented in the vector space E, and to the trajectory : $\mu_O : [0, T] \to M$ is associated a vector $S(t) = \gamma C (\sigma (t)) S_0$.

Because P_G is a spin bundle, there is an associated vector bundle $P_G [E, \gamma C]$ and an associated fiber bundle $\mathcal{S} = P_G \left[\widehat{E}_0, \gamma C \right]$ so that the motion is represented as a path $[0, T] \to P_G \left[\widehat{E}_0, \gamma C \right]$

The holonomic basis is : $\mathbf{e}_i (m) = (\mathbf{p}(m), e_i)$ with the equivalence relation :

$(\mathbf{p}(m), S) \sim \left(\varphi_G (m, g), \gamma C \left(g^{-1} \right) S \right)$

so that in a change of gauge the holonomic basis becomes :

$\widetilde{\mathbf{e}}_i (m) = (\widetilde{\mathbf{p}}(m), e_i) = \left(\varphi_G \left(m, \chi (m)^{-1} \right), e_i \right) \sim \gamma C \left(\chi (m)^{-1} \right) \mathbf{e}_i (m)$

In a change of gauge a section $\mathbf{S} \in \mathcal{S} = P_G \left[\widehat{E}_0, \gamma C \right]$ transform as :

$\sum_j S^j (m) \varepsilon_j (m) = \sum_j \widetilde{S}^j (m) \widetilde{\varepsilon}_j (m) \Rightarrow \widetilde{S} (m) = \gamma C (\chi (m)) S (m)$

and the maps r,w transform accordingly.

The scalar product on E is preserved by γC thus it can be extended to $P_G [E, \gamma C]$.

To any map $S:[0,T]\rightarrow P_G\left[\widehat{E}_0,\gamma C\right]$ one can associate an admissible vector field V by the projection of S(t) to M, at each point two elements σ_w such that : $V(m)=\sqrt{-\langle V(m),V(m)\rangle}Ad_{\sigma_w(m)}\varepsilon_0(m)$, and each defines a spatial spin $S_r(t)=\gamma C\left(\sigma_w^{-1}\right)S(t)$.

Conversely any section $\sigma\in\mathfrak{X}(P_G)$, function $f\in C([0,T];\mathbb{R}_+)$ defines uniquely a family of admissible vector fields and trajectories, and for any $S_0\in E_0$ a pair of opposite spatial spin.

8.3 Gravitational field

8.3.1 Principles

The frames of reference are not a feature of the universe. They are continuously defined, but otherwise arbitrary. In order to conceive coherent networks we need some way to compare the frames chosen at different points. This is the purpose of the hypothesis vi) which precises two points.

The chart φ_M is a generalization of the gaussian chart. It defines the trajectory of "privileged observers", whose time t, on their clock, is identical and world line is given by $\varphi_M(t,a)$. So we have at any time a privileged observer located in the area Ω of the system. Actually it defines M as a trivial fiber bundle with base $\mathbb{R}$ and $\Omega_3(0)$ as standard fiber.

The holonomic frames $(\varepsilon_i(m))_{i=0}^3$ of these observers are parallel transported using the connection : thus we have a consistent basis of frames of reference at any point.

From these frames of reference other frames can be defined, for the observers of the network or for any other observer.

This implies that, for the preferred observers :

$\forall i:\nabla_{\varepsilon_0}^M\varepsilon_i(m)=0=\sum_{j=0}^3\sum_{\alpha=0}^3\Gamma_M(m)_{\alpha i}^j P_0^\alpha\varepsilon_j(m)$

which leads to $\sum_{\alpha=0}^3 G_{w\alpha}P_0^\alpha=0;\sum_{\alpha=0}^3 G_{r\alpha}P_0^\alpha=0$

So we have 6 constraints for 4 components. The solution is

$$P_0^\alpha=\delta_0^\alpha;G_{r0}=0;G_{w0}=0 \tag{25}$$

It is clear that we need a physical description of these privileged observers. In Special Relativity they are the inertial observers, who do not feel any change in the inertial forces. As it is impossible to distinguish a gravitational force and an inertial force, in General Relativity the privileged observers do not feel any change in the gravitational or the inertial forces. Which states that these forces are equivalent, as attested by the strict equality between gravitational mass and inertial mass. And as a consequence the connection itself is the representation of the gravitational field.

As the formulas above show : $G_{r0}=0;G_{w0}=0$ the preferred observers travel along a geodesic and do not feel any change in the inertial or gravitational force.

It is worth to notice that this equivalence cannot be done in Special Relativity because inertial observers must necessarily be in constant translation with respect to a fixed frame (ε_i).

In the traditional picture of GR the key variable is the metric, defined by 10 coefficients $g_{\alpha\beta}$ which then defines the Lévy-Civitta connection. In the present model the gravitational field is treated as in any gauge field theory and the potential is the key variable. The metric on one hand is defined through the components of the tetrad (which are variables), and on the other hand the

connection is a separate variable. So we have more degrees of freedom , and the choice of the Lévy-Civita connection is optional. Of course the coefficients of the tetrad are defined up to a matrix of SO(3,1), this will be dealt with in the specification of the lagrangian.

The gravitational field acts on the motion S of a particle. The action is represented through a lagrangian. If there is no field present, for a single particle the action is just $\int_0^T \langle S(t), S(t)\rangle\, dt = \int_A^B \langle S_0, S_0\rangle\, d\ell$ where $d\ell$ is the measure induced by the metric on any curve between two points A,B. The trajectories are the curves with extremal length, that is geodesics, which are no longer straight lines. If we want to go further, and notably to understand how the gravitational field propagates and is impacted by the presence of matter, we need to introduce two terms, which represent the interaction particle / field and field / field. In both cases they should involve some kind of derivative of S and G.

8.3.2 Action on a particle

The action of the gravitational field on a particle is represented by a differential operator D involving G and acting on S. As G is a one form D can be of first order, and it would be better if it were linear. DS is then incorporated in a lagrangian, usually as a scalar product such as : $\langle S, DS\rangle$, which implies that DS belongs to the same space as S, and that D is self-adjoint so that the scalar product is real.

In the differential geometry and fiber bundle context any linear first order differential operator is necessarily a linear combination of Lie derivatives and covariant derivatives. The Lie derivative is of no interest here, and thus the basic tool is the covariant derivative. The Dirac operator, which appears in QTF, is deduced from the covariant derivative. We will just review their properties here, and we will come back on the subject later, as the choice of a differential operator is closely linked with the lagrangian.

Covariant derivative

The connection on P_G induces a linear connection on the associated fiber bundle $P_G\left[\widehat{E}_0, \gamma C\right]$ and a covariant derivative which acts on sections **S** in $\mathcal{S} = P_G\left[\widehat{E}_0, \gamma C\right]$ represented in the holonomic basis by a map : $S : M \to \widehat{E}_0$:

$$\nabla^S S = \sum_{\alpha=0}^{3} \left(\partial_\alpha S + \gamma C\left(G_\alpha\right) S\right) d\xi^\alpha = \sum_{\alpha=0}^{3} \left(\partial_\alpha S + \gamma C\left(\upsilon\left(G_{r\alpha}, G_{w\alpha}\right)\right) S\right) d\xi^\alpha \tag{26}$$

$$\gamma C\left(\upsilon\left(G_{r\alpha}, G_{w\alpha}\right)\right) = -i\frac{1}{2}\sum_{a=1}^{3}\left(G_{w\alpha}\gamma_a\gamma_0 + G_{r\alpha}\widetilde{\gamma}_a\right) \tag{27}$$

$$\gamma C'\left(\upsilon\left(G_{r\alpha}, G_{w\alpha}\right)\right) = -i\frac{1}{2}\sum_{a=1}^{3}\left(G_{w\alpha}\gamma_a\gamma_0 - G_{r\alpha}\widetilde{\gamma}_a\right) \tag{28}$$

So we go from the signature (3,1) to (1,3) by a change of the sign of $G_{r\alpha}$.

G_α being valued in $T_1 Spin(3,1)$ and γC being a representation of the Clifford algebra the expression makes sense. Its coordinates expression is with rigth and left chiral parts:

$\nabla^S S = \sum_{\alpha=0}^{3} \left[\begin{array}{c} \partial_\alpha S_R + \frac{1}{2}\sum_{a=1}^{3}\left(G^a_{w\alpha} - iG^a_{r\alpha}\right)\sigma_a S_R \\ \partial_\alpha S_L - \frac{1}{2}\sum_{a=1}^{3}\left(G^a_{w\alpha} + iG^a_{r\alpha}\right)\sigma_a S_L \end{array} \right] d\xi^\alpha$

It preserves the chirality : if S_R or S_L is null, so it is for $\nabla^S S$.

For any vector field V of TM, $\nabla^S_V S$ is valued in the vector space $T_{S(m)}\widehat{E}_0$ tangent to the manifold $\widehat{E}_0$ at S(m), which can be seen as E itself. So $\nabla^S S$ can be seen as belonging to $*_1\left(M; P_G\left[E, \gamma C\right]\right)$.

In a change of gauge :

$\mathbf{p}(m) \to \widetilde{\mathbf{p}}(m) = \mathbf{p}(m) \cdot \chi(m)^{-1}$

a section on $\mathcal{S}$ transforms as :

$\mathbf{S}(m) = \left(\mathbf{p}(m), S(m)\right) = \left(\widetilde{\mathbf{p}}(m), \widetilde{S}(m)\right) \sim \left(\mathbf{p}(m), \gamma C\left(\chi(m)^{-1}\right)\widetilde{S}(m)\right)$

$\Rightarrow \widetilde{S}(m) = \gamma C\left(\chi(m)\right) S(m)$

The covariant derivative transforms as a section of $P_G\left[E, \gamma C\right]$:

Proof. $\nabla^S S \to \widetilde{\nabla^S S} = \sum_{\alpha=0}^{3}\left(\partial_\alpha \widetilde{S} + \gamma C\left(\widetilde{G}_\alpha\right)\widetilde{S}\right) d\xi^\alpha$

$= \sum_{\alpha=0}^{3}\left(\gamma C\left(\partial_\alpha \chi\right) S + \gamma C\left(\chi\right)\partial_\alpha S + \gamma C\left(Ad_\chi\left(G - \chi^{-1}\partial_\alpha\chi\right)\right)\gamma C\left(\chi\right) S\right) d\xi^\alpha$

$= \sum_{\alpha=0}^{3}\left(\gamma C\left(\partial_\alpha \chi\right) S + \gamma C\left(\chi\right)\partial_\alpha S + \gamma C\left(\chi\left(G - \chi^{-1}\partial_\alpha\chi\right)\chi^{-1}\right)\gamma C\left(\chi\right) S\right) d\xi^\alpha$

$= \sum_{\alpha=0}^{3}\left(\gamma C\left(\partial_\alpha \chi\right) S + \gamma C\left(\chi\right)\partial_\alpha S + \gamma C\left(\chi\right)\gamma C\left(G\right) S - \gamma C\left(\partial_\alpha\chi\right) S\right) d\xi^\alpha$

$= \sum_{\alpha=0}^{3}\gamma C\left(\chi\right)\left(\partial_\alpha S + \gamma C\left(G\right) S\right) d\xi^\alpha = \gamma C\left(\chi\right)\nabla^S S$ ■

so the operator reads: $\nabla^S : \mathfrak{X}\left(J^1\mathcal{S}\right) \to \wedge_1\left(M; \mathfrak{X}\left(P_G\left[E, \gamma C\right]\right)\right)$

The isomorphism $so(3,1) \to T_1 Spin\left(3,1\right) \subset Cl\left(\mathbb{R}, 3, 1\right)$ reads :

$\left[J(r) + K(w)\right] \to \upsilon\left(r, w\right) = \frac{1}{4}\sum_{i=0}^{3}\left(\left[J(r) + K(w)\right]\left[\eta\right]\right)^i_j \varepsilon_i \cdot \varepsilon_j$

thus in matrix form the Christoffell coefficient of the connexion reads :

$\left[\Gamma_{S\alpha}\left(m\right)\right] = \left[\gamma C\left(G_\alpha\right)\right] = \frac{1}{4}\sum_{ijpq=0}^{3}\left(\left[J(G_{r\alpha}) + K(G_{w\alpha})\right]\left[\eta\right]\right)^i_j\left(\left[\gamma C\left(\varepsilon_i\right)\right]\left[\gamma C\left(\varepsilon_j\right)\right]\right)^p_q \varepsilon_p\left(m\right)\otimes\varepsilon^q\left(m\right)$

But on the other hand the Christoffel coefficient of the connexion on $P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]$ is:

$\left[\Gamma_{M\alpha}\left(m\right)\right] = \sum_{ij=0}^{3}\left[K\left(G_{w\alpha}\right) + J\left(G_{r\alpha}\right)\right]^i_j \varepsilon_i\left(m\right)\otimes\varepsilon^j\left(m\right)$

thus :

$\left[\Gamma_{S\alpha}\left(m\right)\right] = \frac{1}{4}\sum_{ij=0}^{3}\left(\left[\Gamma_{M\alpha}\left(m\right)\right]\left[\eta\right]\right)^i_j\left[\gamma C\left(\varepsilon_i\right)\right]\left[\gamma C\left(\varepsilon_j\right)\right]$

$P_G\left[E, \gamma C\right]$ is a spin bundle, and we have the identity between the derivatives :

$\forall V \in \mathfrak{X}\left(P_G\left[\mathbb{R}^4, \mathbf{Ad}\right]\right), S \in \mathfrak{X}\left(P_G\left[E, \gamma C\right]\right)$:

$\nabla^S\left(\gamma C\left(V\right) S\right) = \gamma C\left(\nabla^M V\right) S + \gamma C\left(V\right)\nabla^S S$

which makes of **G** a Clifford connection.

If the section represents some particle then : $S = \gamma C\left(\sigma\left(m\right)\right) S_0$ with $\sigma \in \mathfrak{X}\left(P_G\right)$ and $S_0 \in E_0$

$\nabla^S_\alpha S = \partial_\alpha S + \gamma C\left(\upsilon\left(G_{r\alpha}, G_{w\alpha}\right)\right) S$

$= \partial_\alpha \gamma C\left(\sigma\right) S_0 + \gamma C\left(\upsilon\left(G_{r\alpha}, G_{w\alpha}\right)\right)\gamma C\left(\sigma\right) S_0$

$= \gamma C\left(\partial_\alpha\sigma + \upsilon\left(G_{r\alpha}, G_{w\alpha}\right)\cdot\sigma\right) S_0$

$= \gamma C\left(\sigma\right)\gamma C\left(\sigma^{-1}\cdot\partial_\alpha\sigma + \sigma^{-1}\cdot\upsilon\left(G_{r\alpha}, G_{w\alpha}\right)\cdot\sigma\right) S_0$

$= \gamma C\left(\sigma\right)\gamma C\left(\sigma^{-1}\cdot\partial_\alpha\sigma + \mathbf{Ad}_{\sigma^{-1}}\upsilon\left(G_{r\alpha}, G_{w\alpha}\right)\right) S_0$

$$\nabla^S_\alpha S = \gamma C\left(\sigma\right)\gamma C\left(\nabla^G_\alpha\sigma\right) S_0 \tag{29}$$

where $\nabla^G_\alpha\sigma = \sigma^{-1}\cdot\partial_\alpha\sigma + \mathbf{Ad}_{\sigma^{-1}}\upsilon\left(G_{r\alpha}, G_{w\alpha}\right)$ is the covariant derivative over P_G

The explicit formula is the following :

With :

$\sigma = a + \upsilon\left(r, w\right) + b\varepsilon_5$

$\sigma^{-1} = a - \upsilon(r,w) + b\varepsilon_5$
$\sigma^{-1} \cdot \partial_\alpha \sigma = (a - \upsilon(r,w) + b\varepsilon_5) \cdot (\partial_\alpha a + \upsilon(\partial_\alpha r, \partial_\alpha w) + \partial_\alpha b\varepsilon_5)$
$\sigma^{-1} \cdot \partial_\alpha \sigma =$
$\upsilon(-(\partial_\alpha(a+b))r + (a+b)\partial_\alpha r + \frac{1}{2}(j(r)\partial_\alpha r - j(w)\partial_\alpha w),$
$-\partial_\alpha(a-b)w + (a-b)\partial_\alpha w + \frac{1}{2}(j(w)\partial_\alpha r + j(r)\partial_\alpha w)$
The formula for the Adjoint map is given in the annex.
All together we have :

$$\nabla^G_\alpha \sigma = \sigma^{-1} \cdot \partial_\alpha \sigma + \mathbf{Ad}_{\sigma^{-1}} \upsilon(G_{r\alpha}, G_{w\alpha}) = \upsilon(X_\alpha, Y_\alpha) \tag{30}$$

with :
$X_\alpha = -(\partial_\alpha(a+b))r + (a+b)\partial_\alpha r + \frac{1}{2}(j(r)\partial_\alpha r - j(w)\partial_\alpha w)$
$+\left[(a+b)^2 - ab + \frac{1}{4}r^t r - (a+b)j(r) + \frac{1}{4}(j(w)j(r) - 2j(w)j(w) + 3j(r)j(r))\right] G_{r\alpha}$
$+\left[ab + \frac{1}{4}r^t r - (b-a)j(w) - \frac{1}{4}(3j(w)j(r) + 2j(r)j(w) - j(r)j(r))\right] G_{w\alpha}$
$Y_\alpha = -\partial_\alpha(a-b)w + (a-b)\partial_\alpha w + \frac{1}{2}(j(w)\partial_\alpha r + j(r)\partial_\alpha w)$
$+\left[\frac{1}{4}w^t w - ab - (a+b)j(w) + \frac{1}{4}(j(w)j(w) + 2j(w)j(r) + 3j(r)j(w))\right] G_{r\alpha}$
$+\left[(a-b)^2 - \frac{1}{4}w^t w - ab - (a-b)j(r) - \frac{1}{4}(3j(w)j(w) - j(r)j(w) - 2j(r)j(r))\right] G_{w\alpha}$
For the signature (1,3) change the sign of r.

Dirac operator

The Dirac operator uses the isomorphim of TM with TM* provided by the metric g. It is a differential operator and no longer a 1-form on M (so it "absorbs" the α of the covariant derivative). The mechanism is the following :

i) using the isomorphism between TM* and TM provided by the metric g to each covector $\omega = \sum_{\alpha=0}^3 \omega_\alpha d\xi^\alpha$ one can associate a vector :
$\omega^* = \sum_{\alpha\beta=0}^3 g^{\alpha\beta}\omega_\alpha \partial\xi_\beta$
ii) vectors of TM can be seen as elements of Cl(M) and as such acts on $\mathcal{S}(m)$ by :
$v = \sum_{\alpha=0}^3 v^\alpha \partial\xi^\alpha = \sum_{\alpha j=0}^3 v^\alpha P'^j_\alpha \varepsilon_j(m)$
$T_m M \times \mathcal{S}(m) \to \mathcal{S}(m) :: \left(\mathbf{p}(m), \sum_{\alpha j=0}^3 v^\alpha P'^j_\alpha \gamma C(\varepsilon_j) S\right)$
iii) there is an action of TM* on $\mathcal{S}(m)$:
$T_m M^* \times \mathcal{S}(m) \to \mathcal{S}(m) ::$
$(\mathbf{p}(m), \gamma C(\omega^*) S) = \left(\mathbf{p}(m), \sum_{\alpha\beta j=0}^3 g^{\alpha\beta}\omega_\alpha P'^j_\beta \gamma C(\varepsilon_j) S\right)$
and :
$\sum_\beta g^{\alpha\beta} P'^j_\beta = \sum_{\beta kl} \eta^{kl} P^\alpha_k P^\beta_l P'^j_\beta = \sum_k \eta^{kj} P^\alpha_k$
thus : $\sum_{\alpha\beta j=0}^3 g^{\alpha\beta}\omega_\alpha P'^j_\beta \gamma C(\varepsilon_j) S = \sum_{\alpha k} \eta^{kk} P^\alpha_k \omega_\alpha \gamma C(\varepsilon_k) S$
iv) the covariant derivative is a one form on M, valued in $\mathcal{S}$ and the Dirac operator is :
$D : J_1\mathcal{S} \to \mathcal{S} :: DS = \sum_{\alpha k} \eta^{kk} P^\alpha_k \gamma C(\varepsilon_k)\left(\nabla^S_\alpha S\right)$
When acting on a section representing the spin of a particle we get :
$DS = \sum_{\alpha k} \eta^{kk} P^\alpha_k \gamma C(\varepsilon_k) \gamma C\left(\sigma \cdot \nabla^G_\alpha \sigma\right) S_0$
$= \gamma C\left(\sum_{\alpha k} \eta^{kk} P^\alpha_k \varepsilon_k \cdot \sigma \cdot \nabla^G_\alpha \sigma\right) S_0$
$= \gamma C\left(\sum_\alpha d\xi^\alpha \cdot \sigma \cdot \nabla^G_\alpha \sigma\right) S_0$

So the Dirac operator can be seen as the "trace" of the covariant operator, which "averages" the action of the covariant derivative along the directions $\alpha = 0...3$.

Geodesics

A geodesic is a path along a curve such that its tangent is parallel transported by the connection. Which reads for a vector field V :

$\mu_P(\tau) = \Phi_V(\tau, x)$

$\nabla_V^M V = 0 = \frac{dV}{d\tau} + \sum_{\alpha=0}^{3} (\upsilon(G_{r\alpha}, G_{w\alpha}) \cdot V - V \cdot \upsilon(G_{r\alpha}, G_{w\alpha})) V^\alpha \partial\xi_\alpha$

It depends of the parametrization, and this requires $\langle V, V\rangle = Ct$ so that we can take $\langle V, V\rangle = -1$ for a time like geodesic.

To an admissible vector field of geodesics it is possible to associate a section $\sigma_w \in P_W$:

$U = \left((2a_w^2 - 1)\varepsilon_0(m) + a_w \sum_{j=1}^{3} w_j \varepsilon_j(m)\right)$

where $V^\alpha = \sum_j P_j^\alpha U^j$

and $\mathbf{U}(m) = \mathbf{Ad}_{\sigma_w}\varepsilon_0(m)$

The formalism of vector bundles enables us to give a useful description of these geodesics, through the value of σ_w with respect to G.

Proposition 14 *Geodesics are represented by sections $\sigma_w \in \mathfrak{X}(P_G)$ such that $\nabla_\alpha^G \sigma_w \in T_1 Spin(\mathbb{R}, 3)$.*

Proof. In the standard basis :

$\nabla_V^M U = \frac{dU}{d\tau} + \sum_{\alpha=0}^{3} (V^\alpha (\upsilon(G_{r\alpha}, G_{w\alpha}) \cdot U - U \cdot \upsilon(G_{r\alpha}, G_{w\alpha})))$

Let us denote the value of the gravitational field along any trajectory :

$\widehat{G}_r = \sum_{\alpha=0}^{3} G_{r\alpha} V^\alpha; \widehat{G}_w = \sum_{\alpha=0}^{3} G_{w\alpha} V^\alpha$

$\frac{dU}{d\tau} = \frac{d}{d\tau}\mathbf{Ad}_{\sigma_w}\varepsilon_0$

$= \frac{d}{d\tau}\left(\sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1}\right) = \left(\frac{d}{d\tau}\sigma_w\right) \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} \cdot \frac{d}{d\tau}\sigma_w \cdot \sigma_w^{-1}$

$\sum_{\alpha=0}^{3} (V^\alpha (\upsilon(G_{r\alpha}, G_{w\alpha}) \cdot U - U \cdot \upsilon(G_{r\alpha}, G_{w\alpha})))$

$= \upsilon\left(\widehat{G}_r, \widehat{G}_w\right) \cdot U - U \cdot \upsilon\left(\widehat{G}_r, \widehat{G}_w\right)$

$= \upsilon\left(\widehat{G}_r, \widehat{G}_w\right) \cdot \mathbf{Ad}_{\sigma_w}\varepsilon_0 - \mathbf{Ad}_{\sigma_w}\varepsilon_0 \cdot \upsilon\left(\widehat{G}_r, \widehat{G}_w\right)$

$= \upsilon\left(\widehat{G}_r, \widehat{G}_w\right) \cdot \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} \cdot \upsilon\left(\widehat{G}_r, \widehat{G}_w\right)$

$\nabla_u^M U = \left(\frac{d}{d\tau}\sigma_w\right) \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} \cdot \frac{d}{d\tau}\sigma_w \cdot \sigma_w^{-1}$

$+\upsilon\left(\widehat{G}_r, \widehat{G}_w\right) \cdot \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \sigma_w^{-1} \cdot \upsilon\left(\widehat{G}_r, \widehat{G}_w\right)$

$= \sigma_w \cdot \left(\sigma_w^{-1} \cdot \left(\frac{d}{d\tau}\sigma_w\right) + \sigma_w^{-1} \cdot \upsilon\left(\widehat{G}_r, \widehat{G}_w\right) \cdot \sigma_w\right) \cdot \varepsilon_0 \cdot \sigma_w^{-1}$

$-\sigma_w \cdot \varepsilon_0 \cdot \left(\sigma_w^{-1} \cdot \frac{d}{d\tau}\sigma_w + \sigma_w^{-1} \cdot \upsilon\left(\widehat{G}_r, \widehat{G}_w\right) \cdot \sigma_w\right) \cdot \sigma_w^{-1}$

$= \sigma_w \cdot \left(\nabla_u^G \sigma_w\right) \cdot \varepsilon_0 \cdot \sigma_w^{-1} - \sigma_w \cdot \varepsilon_0 \cdot \left(\nabla_u^G \sigma_w\right) \cdot \sigma_w^{-1}$

$= \mathbf{Ad}_{\sigma_w}\left(\left(\nabla_u^G \sigma_w\right) \cdot \varepsilon_0 - \varepsilon_0 \cdot \left(\nabla_u^G \sigma_w\right)\right)$

So, with the covariant derivative on the principal bundle P_G. We have a geodesic iff :

$\nabla_u^G \sigma_w \cdot \varepsilon_0 - \varepsilon_0 \cdot \nabla_u^G \sigma_w = 0$

that is iff $\nabla_u^G \sigma_w$ commutes with ε_0.

For any element $\upsilon(r, w)$of $T_1 Spin(\mathbb{R}, 3, 1)$ we have the identity :

$\upsilon(r, w) \cdot \varepsilon_0 - \varepsilon_0 \cdot \upsilon(r, w) = w$ (see Annex for the proof)

So : $\upsilon(r, w) \in T_1 Spin(\mathbb{R}, 3) \Leftrightarrow \upsilon(r, w) \cdot \varepsilon_0 - \varepsilon_0 \cdot \upsilon(r, w) = 0 \Leftrightarrow w = 0$

And the geodesics are represented by sections such that $\nabla_\alpha^G \sigma_w \in T_1 Spin(3)$. ■

$\nabla_\alpha^G \sigma_w$ has been computed :

$\nabla^G_\alpha \sigma_w = \upsilon(-\frac{1}{2} j(w)\partial_\alpha w + \left(a_w^2 + \frac{1}{4} - 2j(w)j(w)\right) G_{r\alpha} + a_w j(w) G_{w\alpha},$
$-w\partial_\alpha a_w + a_w \partial_\alpha w + \left(a_w^2 - 1 - a_w j(w) + \frac{1}{4} j(w) j(w)\right) G_{r\alpha} + \left(1 - \frac{3}{4} j(w) j(w)\right) G_{w\alpha}$
So geodesic fields are associated to the sections such that :
$\sum_\alpha V^\alpha(-w\partial_\alpha a_w + a_w \partial_\alpha w + \left(a_w^2 - 1 - a_w j(w) + \frac{1}{4} j(w) j(w)\right) G_{r\alpha}$
$+ \left(1 - \frac{3}{4} j(w) j(w)\right) G_{w\alpha}) = 0$
$w\frac{da_w}{d\tau} - a_w \frac{dw}{d\tau} = \left(a_w^2 - 1 - a_w j(w) + \frac{1}{4} j(w) j(w)\right) \widehat{G}_r + \left(1 - \frac{3}{4} j(w) j(w)\right) \widehat{G}_w$
By left multiplication with w^t :
$w^t w \frac{da_w}{d\tau} - a_w w^t \frac{dw}{d\tau} = \left(a_w^2 - 1\right) w^t \widehat{G}_r + w^t \widehat{G}_w$
$a_w^2 = 1 + \frac{1}{4} w^t w \Rightarrow \left(w^t \frac{dw}{dt}\right) = 4a_w \frac{da_w}{d\tau}$
$w^t w \frac{da_w}{d\tau} - 4a_w^2 \frac{da_w}{d\tau} = \left(a_w^2 - 1\right) w^t \widehat{G}_r + w^t \widehat{G}_w$
$4\left(a_w^2 - 1 - a_w^2\right) \frac{da_w}{d\tau} = \left(a_w^2 - 1\right) w^t \widehat{G}_r + w^t \widehat{G}_w$
$\frac{da_w}{d\tau} = \frac{1}{4}\left(1 - a_w^2\right) w^t \widehat{G}_r - \frac{1}{4} w^t \widehat{G}_w$
$\frac{1}{4}\left(1 - a_w^2\right) w w^t \widehat{G}_r - \frac{1}{4} w w^t \widehat{G}_w - a_w \frac{dw}{d\tau}$
$= \left(a_w^2 - 1 - a_w j(w) + \frac{1}{4} j(w) j(w)\right) \widehat{G}_r + \left(1 - \frac{3}{4} j(w) j(w)\right) \widehat{G}_w$
$\left(-\left(a_w^2 - 1\right) a_w^2 + a_w j(w) - \frac{1}{4} a_w^2 j(w) j(w)\right) \widehat{G}_r - \left(4\left(a_w^2 - 1\right) + 1 - \frac{1}{2} j(w) j(w)\right) \widehat{G}_w = a_w \frac{dw}{d\tau}$
w is solution of the first order differential equation :

$$a_w \frac{dw}{d\tau} = \left(-\left(a_w^2 - 1\right) a_w^2 + a_w j(w) - \frac{1}{4} a_w^2 j(w) j(w)\right) \widehat{G}_r - \left(4\left(a_w^2 - 1\right) + 1 - \frac{1}{2} j(w) j(w)\right) \widehat{G}_w \tag{31}$$

8.3.3 The propagation of the gravitational field

A field (a field force) is essentially defined through its interaction with particles. So the gravitational field is naturally represented by the connection **G** on the principal bundle $P_G(M, Spin(3,1), \pi)$, acting on the motion of particles represented by S through the operator ∇^S. We will see in in more details in the next part all the issues regarding the implementation of the principle of least action. We will focus now on the propagation of the gravitational field. This propagation occurs in the vacuum, where no particles are present, so we need some kind of derivative of G, and a representation of the interaction of the field with itself.

Strength of the connection

Because $\mathbf{G} \in \Lambda_1(M; T_1 Spin)$ any derivative is a second order differential operator. There are several mathematical objects which can be considered, related to the curvature of the connection, but the "strength of the connection" seems to be the most pertinent. This a 2-form $\mathcal{F}$ on M valued in the Lie algebra, which can be seen as the exterior covariant derivative of the potential and is a good estimate of its rate of change. Its value is with the basis $(\overrightarrow{\kappa}_a)_{a=1}^6$ of $T_1 Spin$

$\mathcal{F}_G \in \Lambda_2(M; T_1 Spin)$

$$\mathcal{F}_G = \sum_{a,\alpha,\beta} \mathcal{F}^a_{G\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \otimes \overrightarrow{\kappa}_a = \sum_{a,\alpha,\beta} \left(\partial_\alpha G^a_\beta - \partial_\beta G^a_\alpha + [G_\alpha, G_\beta]^a\right) d\xi^\alpha \wedge d\xi^\beta \otimes \overrightarrow{\kappa}_a \tag{32}$$

where [] is the bracket in $T_1Spin(3,1)$. Notice that the indices $\alpha\beta$ are not ordered, and that it involves only the principal bundle, and not the associated vector bundles, and is valued in a fixed vector space.

The definition above uses the generic expression. With the basis of the Lie algebra in the Clifford algebra we can distinguish the two parts, $\mathcal{F}_r, \mathcal{F}_w$:

$$\mathcal{F}_G = \sum_{\alpha,\beta} \upsilon\left(\mathcal{F}_{r\alpha\beta}, \mathcal{F}_{w\alpha\beta}\right) d\xi^\alpha \wedge d\xi^\beta \tag{33}$$

$\mathcal{F}_G =$
$\sum_{\alpha,\beta} \frac{1}{2}\left(\mathcal{F}^1_{w\alpha}\varepsilon_0 \cdot \varepsilon_1 + \mathcal{F}^2_{w\alpha}\varepsilon_0 \cdot \varepsilon_2 + \mathcal{F}^3_{w\alpha}\varepsilon_0 \cdot \varepsilon_3 + \mathcal{F}^3_{r\alpha}\varepsilon_2 \cdot \varepsilon_1 + \mathcal{F}^2_{w\alpha}\varepsilon_1 \cdot \varepsilon_3 + \mathcal{F}^1_{r\alpha}\varepsilon_3 \cdot \varepsilon_2\right) \otimes d\xi^\alpha \wedge d\xi^\beta$
with the signature (3,1) :
$[\upsilon(r,w), \upsilon(r',w')] = \upsilon(j(r)r' - j(w)w', j(w)r' + j(r)w')$
$[G_\alpha, G_\beta] = [\upsilon(G_{r\alpha}, G_{w\alpha}), \upsilon(G_{r\beta}, G_{w\beta})] = \upsilon(j(G_{r\alpha})G_{r\beta} - j(G_{w\alpha})G_{w\beta}, j(G_{w\alpha})G_{r\beta} + j(G_{r\alpha})G_{w\beta})$

$$\mathcal{F}_{r\alpha\beta} = \upsilon\left(\frac{\partial G_{r\beta}}{\partial \xi^\alpha} - \frac{\partial G_{r\alpha}}{\partial \xi^\beta} + j(G_{r\alpha})G_{r\beta} - j(G_{w\alpha})G_{w\beta}, 0\right) \tag{34}$$

$$\mathcal{F}_{w\alpha\beta} = \upsilon\left(0, \frac{\partial G_{w\beta}}{\partial \xi^\alpha} - \frac{\partial G_{w\alpha}}{\partial \xi^\beta} + j(G_{w\alpha})G_{r\beta} + j(G_{r\alpha})G_{w\beta}\right) \tag{35}$$

With the signature (1,3):
$[\upsilon(r,w), \upsilon(r',w')] = -\upsilon(j(r)r' - j(w)w', j(w)r' + j(r)w')$
$[G_\alpha, G_\beta] = -\left(\upsilon\left(j(G_{r\alpha})G_{r\beta} - j(G_{w\alpha})G_{w\beta}, j(G_{w\alpha})G_{r\beta} + j(G_{r\alpha})G_{w\beta}\right)\right)$

$$\mathcal{F}_{r\alpha\beta} = -\upsilon\left(\frac{\partial G_{r\beta}}{\partial \xi^\alpha} - \frac{\partial G_{r\alpha}}{\partial \xi^\beta} + j(G_{r\alpha})G_{r\beta} - j(G_{w\alpha})G_{w\beta}, 0\right) \tag{36}$$

$$\mathcal{F}_{w\alpha\beta} = -\upsilon\left(0, \frac{\partial G_{w\beta}}{\partial \xi^\alpha} - \frac{\partial G_{w\alpha}}{\partial \xi^\beta} + j(G_{w\alpha})G_{r\beta} + j(G_{r\alpha})G_{w\beta}\right) \tag{37}$$

As $G_{r0} = 0; G_{w0} = 0$:
$\mathcal{F}_{r\alpha 0} = -\frac{\partial G_{r\alpha}}{\partial \xi^0}$;
$\mathcal{F}_{w\alpha 0} = -\frac{\partial G_{w\alpha}}{\partial \xi^0}$;

A key point is that in a change of gauge $\mathcal{F}$ changes as :
$\mathcal{F}^a_{G\alpha\beta} \to \widetilde{\mathcal{F}}_{G\alpha\beta}(m) = Ad_{\chi(m)}\mathcal{F}_{G\alpha\beta}$
so this is a linear map, and not an affine map. And we have similarly in the Clifford context:
$\mathcal{F}^a_{G\alpha\beta} \to \widetilde{\mathcal{F}}_{G\alpha\beta}(m) = \mathbf{Ad}_{\chi(m)}\mathcal{F}_{\alpha\beta}$

Thanks to this property, $\mathcal{F}_G$ is a two form on M valued in the adjoint bundle : $P_G[T_1Spin, Ad]$, which is a vector bundle.

In any model involving the gravitational field we should consider the variables $\upsilon(G_{r\alpha}, G_{w\beta})$, $\upsilon(\mathcal{F}_{r\alpha\beta}, \mathcal{F}_{w\alpha\beta})$. But the potential G has several drawbacks : several theorems (and we will give a proof in the next sections) show that it cannot figure explicitely in a lagrangian, and because it lives in an affine bundle it fails to meet the most important prerequisite of QM. So practically the key variable if $\mathcal{F}_G$. And actually in electromagnetism the "field" is represented by $\mathcal{F}$ and the potential has a dubious status (even if the Aharonov-Bohm effect shows the physical reality of potential).

Scalar product

As $\mathcal{F}_G$ figures explicitely in lagrangians, the simplest specification uses a scalar product. From the properties of $\mathcal{F}_G$ there is a natural scalar product, defined in 3 steps.

i) The Lorentz scalar product on $\mathbb{R}^4$ can be extended to the Clifford algebra by taking the basis of the Clifford algebra as orthonormal.

That is : $\langle \varepsilon_{i_1}.....\varepsilon_{i_k}, \varepsilon_{j_1}.....\varepsilon_{j_l} \rangle_{Cl} = \delta_{kl}\eta_{i_1 j_1}...\eta_{i_k j_k}\epsilon(i_1...i_k)\epsilon(j_1...j_l)$

So we can take the scalar product on $Cl(\mathbb{R},3,1)$ for $\alpha,\beta,\lambda,\mu,m$ fixed :

$\left\langle \mathcal{F}_{G\alpha\beta}(m), \mathcal{F}'_{G\lambda\mu}(m) \right\rangle_{Cl} = \left\langle \upsilon(\mathcal{F}_{r\alpha\beta}, \mathcal{F}_{w\alpha\beta}), \upsilon\left(\mathcal{F}'_{r\lambda\mu}, \mathcal{F}'_{w\lambda\mu}\right) \right\rangle_{Cl}$

$= \frac{1}{4}\{\mathcal{F}^1_{w\alpha\beta}\mathcal{F}'^1_{w\lambda\mu}\langle \varepsilon_0\cdot\varepsilon_1, \varepsilon_0\cdot\varepsilon_1\rangle + \mathcal{F}^2_{w\alpha\beta}\mathcal{F}'^2_{w\lambda\mu}\langle \varepsilon_0\cdot\varepsilon_2, \varepsilon_0\cdot\varepsilon_2\rangle + \mathcal{F}^3_{w\alpha\beta}\mathcal{F}'^3_{w\lambda\mu}\langle \varepsilon_0\cdot\varepsilon_3, \varepsilon_0\cdot\varepsilon_3\rangle$

$+\mathcal{F}^3_{r\alpha\beta}\mathcal{F}'^3_{r\lambda\mu}\langle \varepsilon_2\cdot\varepsilon_1, \varepsilon_2\cdot\varepsilon_1\rangle + \mathcal{F}^2_{r\alpha\beta}\mathcal{F}'^2_{r\lambda\mu}\langle \varepsilon_1\cdot\varepsilon_3, \varepsilon_1\cdot\varepsilon_3\rangle + \mathcal{F}^1_{r\alpha\beta}\mathcal{F}'^1_{r\lambda\mu}\langle \varepsilon_3\cdot\varepsilon_2, \varepsilon_3\cdot\varepsilon_2\rangle\}$

$= \frac{1}{4}\left(-\mathcal{F}^1_{w\alpha\beta}\mathcal{F}'^1_{w\lambda\mu} - \mathcal{F}^2_{w\alpha\beta}\mathcal{F}'^2_{w\lambda\mu} - \mathcal{F}^3_{w\alpha\beta}\mathcal{F}'^3_{w\lambda\mu} + \mathcal{F}^3_{r\alpha\beta}\mathcal{F}'^3_{r\lambda\mu} + \mathcal{F}^2_{r\alpha\beta}\mathcal{F}'^2_{r\lambda\mu} + \mathcal{F}^1_{r\alpha\beta}\mathcal{F}'^1_{r\lambda\mu}\right)$

$= \frac{1}{4}\left(\mathcal{F}^t_{r\alpha\beta}\mathcal{F}'_{r\lambda\mu} - \mathcal{F}^t_{w\alpha\beta}\mathcal{F}'_{w\lambda\mu}\right)$

$$\left\langle \mathcal{F}_{G\alpha\beta}(m), \mathcal{F}'_{G\lambda\mu}(m) \right\rangle_{Cl} = \frac{1}{4}\left(\mathcal{F}^t_{r\alpha\beta}\mathcal{F}'_{r\lambda\mu} - \mathcal{F}^t_{w\alpha\beta}\mathcal{F}'_{w\lambda\mu}\right) \tag{38}$$

The result does not depend on the signature.

ii) With the metric g on TM* one can lift a 2-form by :

$\mathcal{F}^{\alpha\beta}_G = \sum_{\lambda\mu} g^{\alpha\lambda}g^{\beta\mu}\mathcal{F}_{G\lambda\mu}$

and compute point wise :

$\left\langle \mathcal{F}_{G\alpha\beta}, \mathcal{F}'^{\alpha\beta}_G \right\rangle_{Cl} = \frac{1}{4}\sum_{\lambda\mu} g^{\alpha\lambda}g^{\beta\mu}\left(\mathcal{F}^t_{r\alpha\beta}\mathcal{F}'_{r\lambda\mu} - \mathcal{F}^t_{w\alpha\beta}\mathcal{F}'_{w\lambda\mu}\right)$

iii) The scalar product is then taken by integration over the area of the system :

$\langle \mathcal{F}_G, \mathcal{F}'_G \rangle = \int_\Omega \langle \mathcal{F}_G(m), \mathcal{F}'^G(m) \rangle \varpi_4(m)$

with the volume form on M induced by the metric.

This scalar product is invariant in a change of gauge because the scalar product in the Clifford algebra is invariant by **Ad**. It is non degenerate but not definite positive.

This leads to the definition of the Hodge dual of $\mathcal{F}$:

$*\mathcal{F} \in \Lambda_2(M; T_1Spin(\mathbb{R},3,1)) :: \forall \mu \in \Lambda_2(M; T_1Spin(\mathbb{R},3,1)) :$

$*\mathcal{F}\wedge\mu = \langle \mathcal{F}, \mu \rangle \varpi_4$ and $\langle \mathcal{F}, \mathcal{F} \rangle = \int_\Omega *\mathcal{F}\wedge\mathcal{F}$

There is a useful property which is actually more general.

Proposition 15 *On the Lie algebra T_1U of a Lie group U, endowed with a symmetric scalar product $\langle\rangle_{T_1U}$ which is preserved by the adjoint map :*

$\forall X,Y,Z \in T_1U : \langle X, [Y,Z] \rangle = \langle [X,Y], Z \rangle$

Proof. $\forall g \in U : \langle Ad_gX, Ad_gY \rangle = \langle X, Y \rangle$

take the derivative with respect to g at g = 1 for $Z \in T_1U$:

$(Ad_gX)'(Z) = ad(Z)(X) = [Z,X]$

$\langle [Z,X], Y \rangle + \langle X, [Z,Y] \rangle = 0 \Leftrightarrow \langle X, [Y,Z] \rangle = \langle [Z,X], Y \rangle$

exchange X,Z:

$\Rightarrow \langle Z, [Y,X] \rangle = \langle [X,Z], Y \rangle = -\langle [Z,X], Y \rangle = -\langle X, [Y,Z] \rangle = -\langle Z, [X,Y] \rangle$ ■

So

$\langle G_\alpha, [G_\beta, G_\gamma] \rangle = \langle [G_\alpha, G_\beta], G_\gamma \rangle$

$\langle \mathcal{F}_{G\alpha\beta}, [\mathcal{F}_{G\lambda\mu}, \mathcal{F}_{G\zeta\eta}] \rangle = \langle [\mathcal{F}_{G\alpha\beta}, \mathcal{F}_{G\lambda\mu}], \mathcal{F}_{G\zeta\eta} \rangle$

Scalar curvature

The traditional representation of General Relativity relies on the Ricci tensor, and the scalar curvature which defines the Hilbert action. It is necessary to see how it relates to the scalar product above.

The connection on P_G induces a linear connection on $P_G\left[\mathbb{R}^4,\mathbf{Ad}\right]$ with Christoffell symbols, expressed in the matrix form in so(3,1) :

$[\Gamma_{G\alpha}(m)] = [K(G_{w\alpha})] + J(G_{r\alpha})$

The covariant derivative of a section $V \in \mathfrak{X}\left(P_G\left[\mathbb{R}^4,\mathbf{Ad}\right]\right)$ is then :

$\nabla^M V = \sum_{\alpha i=0}^{3}\left(\partial_\alpha V^i + \sum_{j=0}^{3}\Gamma_G(m)^i_{\alpha j}V^j\right)\varepsilon_i(m)\otimes d\xi^\alpha$

This connection defines an exterior covariant derivative ∇^M_e, acting on forms on M valued in $P_G\left[\mathbb{R}^4,\mathbf{Ad}\right]$, and so acting on ∇^M :

$\nabla^M_e\left(\nabla^M\right) = \sum_{\{\alpha\beta\}ij} R^i_{\alpha\beta j}X d\xi^\alpha\wedge d\xi^\beta\otimes\varepsilon_i(m)\otimes\varepsilon^j(m)$

where R is the Riemann curvature of the connection **G** . This is a 2 form on M, valued in $P_G\left[\mathbb{R}^4,\mathbf{Ad}\right]\otimes P_G\left[\mathbb{R}^4,\mathbf{Ad}\right]^*$:

$R^i_{\alpha\beta j} = \partial_\alpha\Gamma^i_{G\beta j} - \partial_\beta\Gamma^i_{G\alpha j} + \sum_k\left(\Gamma^i_{G\alpha k}\Gamma^k_{M\beta j} - \Gamma^i_{G\beta k}\Gamma^k_{G\alpha j}\right)$

which can be expressed in the holonomic basis of a chart on M:

$R = \sum_{\{\alpha\beta\}ijkl\gamma\eta} R^i_{\alpha\beta j}d\xi^\alpha\wedge d\xi^\beta\otimes P^\gamma_i\partial\xi_\gamma\otimes P'^j_\eta d\xi^\eta$

Denoting : $[\mathcal{F}_{G\alpha\beta}] = [K(\mathcal{F}_{w\alpha\beta})] + [J(\mathcal{F}_{r\alpha\beta})] = \sum_{a1}^{6}\mathcal{F}^a_{G\alpha\beta}[\kappa_a]$

$[R_{\alpha\beta}]^i_j = ([\partial_\alpha\Gamma_{G\beta}] - [\partial_\beta\Gamma_{G\alpha}] + [\Gamma_{G\alpha}][\Gamma_{G\beta}] - [\Gamma_{G\beta}][\Gamma_{G\alpha}])^i_j = [\mathcal{F}_{\alpha\beta}]^i_j$

$R = \sum_{\{\alpha\beta\}ij}[\mathcal{F}_{G\alpha\beta}]^i_j\, d\xi^\alpha\wedge d\xi^\beta\otimes\varepsilon_i(m)\otimes\varepsilon^j(m)$

To the connection **G** is associated an usual affine connection, which has for Christoffel symbols :

$\widehat{\Gamma}^\gamma_{\alpha\beta} = P^\gamma_i\left(\partial_\alpha P'^i_\beta + \Gamma^i_{G\alpha j}P'^j_\beta\right)$

so that the related covariant derivative $\widehat{\nabla}$ of a vector field on M has the same value either if it is expressed in the holonomic basis in a chart, or as a section of $P_G\left[\mathbb{R}^4,\mathbf{Ad}\right]$

In matrix form :

$\left[\widehat{\Gamma}_\alpha\right] = [P]([\partial_\alpha P'] + [\Gamma_{G\alpha}][P']) \Leftrightarrow [\Gamma_{G\alpha}] = \left([P']\left[\widehat{\Gamma}_\alpha\right] - [\partial_\alpha P']\right)[P]$

It is symmetric, and is the Lévy-Civita connection, iff :

$\forall\alpha,\beta,\gamma : \left[\widehat{\Gamma}_\alpha\right]^\gamma_\beta = ([P]([\partial_\alpha P'] + [\Gamma_{G\alpha}][P']))^\gamma_\beta = \left[\widehat{\Gamma}_\beta\right]^\gamma_\alpha = ([P]([\partial_\beta P'] + [\Gamma_{G\beta}][P']))^\gamma_\alpha$

$\Leftrightarrow \nabla_\alpha(\partial\xi_\beta) = \nabla_\beta(\partial\xi_\alpha)$

The usual Riemann curvature of this affine connection is the multilinear map :

$\widehat{R} : \mathfrak{X}(TM)^3 \to \mathfrak{X}(TM) :: R(X,Y,Z) = \widehat{\nabla}_X\widehat{\nabla}_Y Z - \widehat{\nabla}_Y\widehat{\nabla}_X Z - \widehat{\nabla}_{[X,Y]}Z$

In the basis of a chart it reads :

$\widehat{R} = \sum_{\{\alpha\beta\}}\sum_{\gamma\eta}\widehat{R}^\gamma_{\alpha\beta\eta}d\xi^\alpha\wedge d\xi^\beta\otimes\partial\xi_\gamma\otimes d\xi^\eta$

with : $\widehat{R}^\gamma_{\alpha\beta\eta} = \left[\widehat{R}_{\alpha\beta}\right]^\gamma_\eta$

$\left[\widehat{R}_{\alpha\beta}\right] = \left[\partial_\alpha\widehat{\Gamma}_\beta\right] - \left[\partial_\beta\widehat{\Gamma}_\alpha\right] + \left[\widehat{\Gamma}_\alpha\right]\left[\widehat{\Gamma}_\beta\right] - \left[\widehat{\Gamma}_\beta\right]\left[\widehat{\Gamma}_\alpha\right]$

Expressing the value of $\left[\widehat{\Gamma}_\alpha\right]$ with respect to $[\Gamma_{G\alpha}]$:

$\left[\widehat{R}_{\alpha\beta}\right] = [P][\mathcal{F}_{G\alpha\beta}][P']$

Proof. $\left[\widehat{R}_{\alpha\beta}\right]$

$= [\partial_\alpha P]\,[\partial_\beta P'] + [\partial_\alpha P]\,[\Gamma_{G\beta}]\,[P'] + [P]\left[\partial^2_{\beta\alpha} P'\right] + [P]\,[\partial_\alpha \Gamma_{G\beta}]\,[P']$
$+ [P]\,[\Gamma_{G\beta}]\,[\partial_\alpha P'] - [\partial_\beta P]\,[\partial_\alpha P'] - [\partial_\beta P]\,[\Gamma_{G\alpha}]\,[P'] - [P]\left[\partial^2_{\alpha\beta} P'\right]$
$- [P]\,[\partial_\beta \Gamma_{G\alpha}]\,[P'] - [P]\,[\Gamma_{G\alpha}]\,[\partial_\beta P'] + [P]\,[\partial_\alpha P']\,[P]\,[\partial_\beta P']$
$+ [P]\,[\Gamma_{G\alpha}]\,[P']\,[P]\,[\partial_\beta P'] + [P]\,[\partial_\alpha P']\,[P]\,[\Gamma_{G\beta}]\,[P']$
$+ [P]\,[\Gamma_{G\alpha}]\,[P']\,[P]\,[\Gamma_{G\beta}]\,[P'] - [P]\,[\partial_\beta P']\,[P]\,[\partial_\alpha P']$
$- [P]\,[\Gamma_{G\beta}]\,[P']\,[P]\,[\partial_\alpha P'] - [P]\,[\partial_\beta P']\,[P]\,[\Gamma_{G\alpha}]\,[P']$
$- [P]\,[\Gamma_{G\beta}]\,[P']\,[P]\,[\Gamma_{G\alpha}]\,[P']$
$= + [P]\left([\partial_\alpha \Gamma_{G\beta}] - [\partial_\beta \Gamma_{G\alpha}] + [\Gamma_{G\alpha}]\,[\Gamma_{G\beta}] - [\Gamma_{G\beta}]\,[\Gamma_{G\alpha}]\right)[P']$
$+ [\partial_\alpha P]\,[\partial_\beta P'] - [\partial_\beta P]\,[\partial_\alpha P'] + [P]\,[\partial_\alpha P']\,[P]\,[\partial_\beta P']$
$- [P]\,[\partial_\beta P']\,[P]\,[\partial_\alpha P'] + [\partial_\alpha P]\,[\Gamma_{G\beta}]\,[P'] - [\partial_\beta P]\,[\Gamma_{G\alpha}]\,[P']$
$+ [P]\,[\Gamma_{G\beta}]\,[\partial_\alpha P'] - [P]\,[\Gamma_{G\alpha}]\,[\partial_\beta P'] + [P]\,[\Gamma_{G\alpha}]\,[\partial_\beta P']$
$- [P]\,[\Gamma_{G\beta}]\,[\partial_\alpha P'] + [P]\,[\partial_\alpha P']\,[P]\,[\Gamma_{G\beta}]\,[P'] - [P]\,[\partial_\beta P']\,[P]\,[\Gamma_{G\alpha}]\,[P']$
$= [P]\,[\mathcal{F}_{G\alpha\beta}]\,[P'] + [\partial_\alpha P]\,[\partial_\beta P'] - [\partial_\beta P]\,[\partial_\alpha P']$
$- [\partial_\alpha P]\,[P']\,[P]\,[\partial_\beta P'] + [\partial_\beta P]\,[P']\,[P]\,[\partial_\alpha P'] + [\partial_\alpha P]\,[\Gamma_{G\beta}]\,[P']$
$- [\partial_\beta P]\,[\Gamma_{G\alpha}]\,[P'] + [P]\,[\Gamma_{G\beta}]\,[\partial_\alpha P'] - [P]\,[\Gamma_{G\alpha}]\,[\partial_\beta P'] + [P]\,[\Gamma_{G\alpha}]\,[\partial_\beta P']$
$- [P]\,[\Gamma_{G\beta}]\,[\partial_\alpha P'] - [\partial_\alpha P]\,[P']\,[P]\,[\Gamma_{G\beta}]\,[P'] + [\partial_\beta P]\,[P']\,[P]\,[\Gamma_{G\alpha}]\,[P']$
$= [P]\,[\mathcal{F}_{G\alpha\beta}]\,[P']$
with $[P]\,[\partial_\alpha P'] + [\partial_\alpha P]\,[P'] = 0$ ■
that is :
$\widehat{R} = \sum_{\{\alpha\beta\}\gamma\eta} \sum_{ij=0}^{3} [P]^\gamma_i\,[\mathcal{F}_{G\alpha\beta}]^i_j\,[P']^j_\eta\, d\xi^\alpha \wedge d\xi^\beta \otimes \partial\xi_\gamma \otimes d\xi^\eta$
$\widehat{R} = \sum_{\{\alpha\beta\}\gamma\eta} \sum_{ij=0}^{3} [\mathcal{F}_{G\alpha\beta}]^i_j\, d\xi^\alpha \wedge d\xi^\beta \otimes [P]^\gamma_i\,\partial\xi_\gamma \otimes [P']^j_\eta\, d\xi^\eta$
$\widehat{R} = \sum_{\{\alpha\beta\}\gamma\eta} \sum_{ij=0}^{3} [\mathcal{F}_{G\alpha\beta}]^i_j\, d\xi^\alpha \wedge d\xi^\beta \otimes \varepsilon_i(m) \otimes \varepsilon^j(m) = R$
So the two Riemann curvatures are the same object, expressed in different bases.

$$\sum_{ij=0}^{3} [\mathcal{F}_{\alpha\beta}]^i_j\,[P]^\gamma_i\,[P']^j_\eta = \widehat{R}^\gamma_{\alpha\beta\eta} \Leftrightarrow [\mathcal{F}_{\alpha\beta}]^i_j = \sum_{\gamma\eta} \widehat{R}^\gamma_{\alpha\beta\eta}\,[P']^i_\gamma\,[P]^\eta_j = \sum_{a=1}^{6} \mathcal{F}^a_{G\alpha\beta}\,[\kappa_a]^i_j \tag{39}$$

The Ricci tensor is the contraction on the two indices γ, β of $\widehat{R}$:
$Ric = \sum_{\alpha\eta} Ric_{\alpha\eta} d\xi^\alpha \otimes d\xi^\eta$
$Ric_{\alpha\eta} = \sum_\beta \widehat{R}^\beta_{\alpha\beta\eta} = \sum_{ij\beta} R^i_{\alpha\beta j} P^\beta_i P'^j_\eta = \sum_\beta \left([P]\,[R_{\alpha\beta}]\,[P']\right)^\beta_\eta$
$= \sum_\beta \left([P]\left([\partial_\alpha \Gamma_{G\beta}] + [\Gamma_{G\alpha}]\,[\Gamma_{G\beta}]\right)[P']\right)^\beta_\eta$
$[\partial_\alpha \Gamma_{G\beta}] + [\Gamma_{G\alpha}]\,[\Gamma_{G\beta}] = \sum_a \partial_\alpha G^a_\beta\,[\kappa_a] + \sum_{ab} G^a_\alpha G^b_\beta\,[\kappa_a]\,[\kappa_b] = [\mathcal{F}_{G\alpha\beta}]$
$[\mathcal{F}_{G\alpha\beta}] - [\mathcal{F}_{G\beta\alpha}]$
$= \sum_a \left(\partial_\alpha G^a_\beta - \partial_\beta G^a_\alpha\right)[\kappa_a] + \sum_{ab} G^a_\alpha G^b_\beta\,[\kappa_a]\,[\kappa_b] - G^a_\beta G^b_\alpha\,[\kappa_a]\,[\kappa_b]$
$= \sum_a \left(\partial_\alpha G^a_\beta - \partial_\beta G^a_\alpha + [G_\alpha, G_\beta]^a\right)[\kappa_a] = \sum_a \mathcal{F}^a_{\alpha\beta}\,[\kappa_a]$
$Ric_{\alpha\eta} = \sum_\beta \left([P]\,[\mathcal{F}_{\alpha\beta}]\,[P']\right)^\beta_\eta$
The Ricci tensor is symmetric with the Lévy-Civita connection:
$\sum_\beta \left([P]\,[\mathcal{F}_{G\alpha\beta}]\,[P']\right)^\beta_\eta = \sum_\beta \left([P]\,[\mathcal{F}_{G\eta\beta}]\,[P']\right)^\beta_\alpha$
The scalar curvature is defined through the metric g :

$$\mathbf{R} = \sum_{\alpha\eta} g^{\alpha\eta} Ric_{\alpha\eta} = \sum_{\alpha\beta\eta} g^{\alpha\eta} \widehat{R}^{\beta}_{\alpha\beta\eta} = \sum_{\alpha\beta\eta} \left[\widehat{R}_{\alpha\beta}\right]^{\beta}_{\eta} \left[g^{-1}\right]^{\eta}_{\alpha} = \sum_{\alpha\beta} \left(\left[\widehat{R}_{\alpha\beta}\right] \left[g^{-1}\right]\right)^{\beta}_{\alpha}$$

$$\mathbf{R} = \sum_{\alpha\beta\eta} g^{\alpha\eta} \left([P] \left[\mathcal{F}_{G\alpha\beta}\right] [P']\right)^{\beta}_{\eta} = \sum_{\alpha\beta\eta} \left([P] \left[\mathcal{F}_{G\alpha\beta}\right] [P']\right)^{\beta}_{\eta} \left([P] [\eta] [P]^t\right)^{\eta}_{\alpha} = \sum_{\alpha\beta} \left([P] \left[\mathcal{F}_{G\alpha\beta}\right] [\eta] [P]^t\right)^{\beta}_{\alpha}$$

$$\mathbf{R} = \sum_{\alpha\beta} \left([P] \left[\mathcal{F}_{G\alpha\beta}\right] [\eta] [P]^t\right)^{\beta}_{\alpha} \tag{40}$$

In a change of gauge $\mathbf{R}$ does not change:
$[P] \to \left[\widetilde{P}\right] = [P] \left[h\left(\chi^{-1}\right)\right]$
$\left[\mathcal{F}_{G\alpha\beta}\right] \to \left[Ad_{\chi} \mathcal{F}_{\alpha\beta}\right] = [h(\chi)] \left[\mathcal{F}_{G\alpha\beta}\right] \left[h\left(\chi^{-1}\right)\right]$
$\left[\widetilde{P}'\right] = [h(\chi)] [P'] \Leftrightarrow \left[\widetilde{P}\right] = [P] \left[h\left(\chi^{-1}\right)\right]$
$\mathbf{R} \to$
$\widetilde{R} = \sum_{\alpha\beta} \left([P] \left[h\left(\chi^{-1}\right)\right] [h(\chi)] \left[\mathcal{F}_{\alpha\beta}\right] \left[h\left(\chi^{-1}\right)\right] [\eta] \left[h\left(\chi^{-1}\right)\right]^t [P]^t\right)^{\beta}_{\alpha}$
$= \sum_{\alpha\beta} \left([P] \left[\mathcal{F}_{\alpha\beta}\right] [P]^t\right)^{\beta}_{\alpha} = \mathbf{R}$
The scalar curvature does not depend on the signature :
$\mathcal{F}_{G\alpha\beta} \to -\mathcal{F}_{G\alpha\beta}, [\eta] \to -[\eta]$

The Killing form on so(3,1) is :
$B\left([K(w)] + J[r], [K(w')] + [J(r')]\right) = 2Tr\left(\left([K(w)] + J[r]\right)\left([K(w')] + J[r']\right)\right) = 4\left(w^t w' - r^t r'\right)$
thus the scalar product reads:
$\left\langle \mathcal{F}_{G\alpha\beta}(m), \mathcal{F}'_{G\lambda\mu}(m) \right\rangle_{Cl} = \frac{1}{4}\left(\mathcal{F}^t_{r\alpha\beta} \mathcal{F}'_{r\lambda\mu} - \mathcal{F}^t_{w\alpha\beta} \mathcal{F}'_{w\lambda\mu}\right)$
$= \frac{1}{8} Tr\left(\left([K(\mathcal{F}_{w\alpha\beta})] + J[\mathcal{F}_{r\alpha\beta}]\right)\left(\left[K\left(\mathcal{F}'_{w\lambda\mu}\right)\right] + J\left[\mathcal{F}'_{r\lambda\mu}\right]\right)\right)$
$= \frac{1}{8} Tr\left([\mathcal{F}_{\alpha\beta}]\left[\mathcal{F}'_{\lambda\mu}\right]\right)$
$\left[\widehat{R}_{\alpha\beta}\right] = [P]\left[\mathcal{F}_{\alpha\beta}\right][P'] \Leftrightarrow \left[\mathcal{F}_{\alpha\beta}\right] = [P']\left[\widehat{R}_{\alpha\beta}\right][P]$
$\left\langle \mathcal{F}_{G\alpha\beta}(m), \mathcal{F}'_{G\lambda\mu}(m) \right\rangle_{Cl} = \frac{1}{8} Tr\left([P']\left[\widehat{R}_{\alpha\beta}\right][P][P']\left[\widehat{R}'_{\lambda\mu}\right][P]\right)$
$= \frac{1}{8} Tr\left(\left[\widehat{R}_{\alpha\beta}\right]\left[\widehat{R}'_{\lambda\mu}\right]\right)$
$\langle \mathcal{F}_G(m), \mathcal{F}'_G(m) \rangle$
$= \sum_{\{\alpha\beta\}} \sum_{\lambda\mu=0}^{3} \left\langle \mathcal{F}_{G\alpha\beta}(m), \mathcal{F}'_{G\lambda\mu}(m) \right\rangle_{Cl} g^{\alpha\lambda} g^{\beta\mu}$
$= \frac{1}{8} \sum_{\{\alpha\beta\}} \sum_{\lambda\mu pq=0}^{3} g^{\alpha\lambda} g^{\beta\mu} \widehat{R}^{p}_{\lambda\mu q} \widehat{R}^{q}_{\alpha\beta p}$

So the scalar curvature can be computed in the framework of fiber bundle and connection. As seen above it makes explicit the tetrad, but leaves open the choice of the connection, which is not necessarily torsion free. In this respect this representation is an improvement on the Einstein-Cartan models, and should be easier to study. But it also rises a question : we have two possible, consistent, ways to investigate the interaction gravitational field / matter : through the scalar product (the "Yang–Mills way"), and the scalar curvature (the "Hilbert action" way). Is there a "right solution" ? At first it seems that the second, which has been experimentally checked, is the obvious answer. But we look here at particles, for which the action of the gravitational field is not really known. Moreover in astrophysics it is clear that GR does not provides a satisfying solution for the motion of stars and galaxies, which needs the addition of a problematic "dark matter". The theorems of QM,

as expressed in my paper, are valid whatever the scale : the only prerequisite is the formulation of the model. As long as we do not consider any internal, physically identifiable structure (as it is the case for stars in a galaxy), the motion can be represented by a Spin, and it adds an ingredient which is absent of the usual GR : the rotation. Intuitively we can see that the rotation of stars adds a significant amount of energy, which should be accounted for in the computation of the gravitational field. And of course the Spin model works only with the scalar product $\langle \mathcal{F}, \mathcal{F}' \rangle$.

Cosmological constant

One of the hot topics of GR is the cosmological constant. It appears usually in the action as a scalar added to the scalar curvature, which is consistent with the scalar Hilbert action. The Einstein equation in the vacuum is, with a cosmological constant Λ:

$Ric_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} (R + \Lambda) = 0$

It acts as a "pressure", positive or negative, to impact the expansion of the Universe in Cosmological models.

Even if Cosmology is out of the topics of this paper, there are some issues which stand in particles Physics :

i) as we will see in the next sections, in a Yang-Mills model a particle has no "mass at rest" in the absence of a gravitational field

ii) a material body is never "at rest" because it must travel along its world line, intuitively one can see its "energy at rest" as its kinetic energy in this fundamental motion,

iii) but the "engine" of this motion should be somewhat linked to the gravitational field itself

iv) if all this hints at the existence of a fundamental gravitational field, of cosmological nature, it should show some privileged direction : at any given point the light cone has a tangible physical meaning.

In a fiber bundle framework one can be a bit more sophisticated than a single scalar to deal with this issue. The simplest way would be to assume the existence of a constant (meaning which is not affected by the local distribution of matter and energy) field $\mathcal{F}_{0G}$ so that the observable gravitational field would be $\mathcal{F}_G + \mathcal{F}_{0G}$. This is just a generalization of the scalar Λ added to the curvature. However this specification is not compatible with the change of gauge. An alternate solution is to assume that the observable field is a deformation of a fundamental field $\mathcal{F}_{0G}$: $\mathcal{F}_G = Ad_\sigma \mathcal{F}_{0G}$ where $\sigma \in \mathfrak{X}(P_G)$ is a section which sums up the effects of the local distribution of matter and fields. In some way this is similar to the treatment of the spin of a particle : $S = \gamma C(\sigma) S_0$.

Because $\mathcal{F}_G$ is valued in the vector bundle $P_G[T_1 Spin, Ad]$ if, in a system, the variable $\mathcal{F}_G \in \Lambda_2(M; P_G[T_1 Spin, Ad])$ belongs to some Banach vector space $\digamma$, it is liable to the theorems of QM. We have a representation $(T_1 Spin, Ad)$ of the Spin group, and there is a structure of Hilbert space on $\digamma$, and so a scalar product, denoted ( ), definite positive, preserved by Ad, on $\digamma$. Thus $(\mathcal{F}_G, \mathcal{F}_G) = (\mathcal{F}_{0G}, \mathcal{F}_{0G}) \neq 0$ and as a consequence $\mathcal{F}_G$ is always non null. However it is clear that this scalar product is not the one which has been defined previously, which is non degenerate but not definite positive.

Then we could expect that there is some privileged motions, and notably translational (r=0).

The full expression of $\widetilde{\mathcal{F}}_{G\alpha\beta}$ is with $\sigma = a + \upsilon(r, w) + b\varepsilon_5 \in \mathfrak{X}(P_G)$

$Ad_\sigma \mathcal{F}_{G\alpha\beta} = \widetilde{\mathcal{F}}_{G\alpha\beta} = \upsilon\left(\widetilde{\mathcal{F}}_{r\alpha\beta}, \widetilde{\mathcal{F}}_{w\alpha\beta}\right)$

$\widetilde{\mathcal{F}}_{r\alpha\beta} =$

$\left[\left(a^2 + b^2 + ab + \frac{1}{4} r^t r\right) + (a+b) j(r) - \frac{1}{4}\left[j(w) j(w) - j(r) j(r) - j(w) j(r)\right]\right] \mathcal{F}_{r\alpha\beta}$

$- \left[-\left(ab + \frac{1}{4} r^t r\right) + (a-b) j(w) + \frac{1}{4}\left[2 j(w) j(r) + j(r) j(w)\right]\right] \mathcal{F}_{w\alpha\beta}$

$\widetilde{\mathcal{F}}_{w\alpha\beta} = \left[(w^t w + 4ab) + (a+b)\, j(w) + \frac{1}{4}\left[j(w)\, j(r) + 2j(r)\, j(w) \right] \right] \mathcal{F}_{r\alpha\beta}$
$+ \left[\left(a^2 + b^2 - ab - \frac{1}{4} w^t w \right) + (a-b)\, j(r) - \frac{1}{4}\left[j(w)\, j(w) - j(r)\, j(r) + j(r)\, j(w) \right] \right] \mathcal{F}_{w\alpha\beta}$
Thus if If $\sigma = a_w + \upsilon(0, w)$
$\widetilde{\mathcal{F}}_{r\alpha\beta} = \left[a_w^2 I - \frac{1}{4} j(w)\, j(w) \right] \mathcal{F}_{r\alpha\beta} - \left[a_w j(w) \right] \mathcal{F}_{w\alpha\beta}$
$\widetilde{\mathcal{F}}_{w\alpha\beta} = \left[4\left(a_w^2 - 1 \right) I + a_w j(w) \right] \mathcal{F}_{r\alpha\beta} + \left[I - \frac{1}{4} j(w)\, j(w) \right] \mathcal{F}_{w\alpha\beta}$

The 6×6 matrix $\mathcal{F}_G \to \widetilde{\mathcal{F}}_G$ has 3 eigen values, notably 1 and $a_w^2 > 1$. The eigen vector for 1 is $(0, 0, 0, w_1, w_2, w_3)$.

$\upsilon\left(\widetilde{\mathcal{F}}_{r\alpha\beta}, \widetilde{\mathcal{F}}_{w\alpha\beta} \right) = \upsilon\left(\mathcal{F}_{r\alpha\beta}, \mathcal{F}_{w\alpha\beta} \right)$ if : $\mathcal{F}_{r\alpha\beta} = 0, \mathcal{F}^a_{w\alpha\beta} = k w^a$

So if

$\mathcal{F}_{0r\alpha\beta} = 0, \mathcal{F}^a_{0w\alpha\beta} = \epsilon(\alpha, \beta) \sum_j P'^j_\alpha X^j,\ X^0 = 2a_w^2 - 1, X^j = a_w W_j, j = 1, 2, 3$

then the gravitational field would not change in a gauge transformation $\sigma = a + \upsilon(0, W)$. In some way it would be the "cosmological expansion" of the universe.

Of course this is very speculative. An experimental verification would need the comparison of the gravitational field, at the same location, for observers with a relative velocity. Anyway for a more practical purpose in the following we can only consider the variable $\mathcal{F}_G$: the introduction of an additional section σ would be of no use.

There is much to study on the gravitational model, but we need to clarify two points :

- how to represent the fields other than the gravitational field ?
- how to treat particles and fields in an integrated model based on the principle of least action ? This encompasses notably the problem of "point size particles".

This is the main subject of the next sections.

Part III
MODELS OF PARTICLES AND FIELDS

9 THE OTHER FORCES FIELDS

9.1 The standard Model

There are 3 force fields (the status of the "Higgs field" is open) which interact with particles, besides the gravitational field :

- the electromagnetic field
- weak interactions
- strong interactions

They are the main topic of Quantum Theory of Fields (QTF) and their representation is summarized in the Standard model.

9.1.1 The group representation

To each field force is associated a compact real Lie group :

- SU(3) for the strong force
- U(1)×SU(2) for the electroweak force
- U(1) for the electromagnetic force

Thus for all of them together the group is SU(3)×SU(2)×U(1) but because of their range and characteristics we have three "layers" : U(1) , U(1)×SU(2), U(1)×SU(2)×SU(3) which can be considered. The Higgs field is represented through a complex valued function.

The Standard Model is a version of the "Yang-Mills" model, adaptated to the Special Relativity geometry :

i) each of the groups or product of groups defines a principal bundle over M.

ii) the physical characteristics of the particles are vectors of a space which is a linear representation of the group, so they are vectors of vector bundle associated to the principal bundles

iii) the field itself is represented by a principal connection, which acts on the vector bundles

Because the groups are compact, the representations are sum of finite dimensional complex representations, usually the standard representation, but the combination of the groups give a picture which is quite complicated. The elementary particles show as vectors of the basis in these representations.

The state of the particles is then represented in a tensorial bundle, combining the spin (for the geometric characteristics) and the charges (for the physical characteristics). The masses are defined separately, because it is necessary to distinguish the proper mass and an apparent mass resulting from the screening by virtual particles.

The lagrangian is then built from scalar products and the Dirac's operator.

9.1.2 Fermions and bosons

Fermions

The interaction of matter particles, that we will call "fermions" [1], with force fields involve "charges", which together with their spin fully characterize the particles.The fermions are organized in 3 "generations", comprised each of 2 leptons and 2 quarks :

- First generation : quarks "up" and "down", electron, neutrino.
- Second generation : quarks "charm" and "strange", muon, muon neutrino
- Third generation : quarks "top" and "bottom", tau and tau neutrino

The stability decreases with each generation, the first generation constitute the usual matter. Each type of particle is called a "flavour".

The "charges" are :

- colour (strong interactions) : each type of quark can have one of 3 different colors (blue, green, red) and they are the only ones to interact with the strong field
- hypercharge (electroweak interaction) : all elementary particles have an hypercharge (-2,-1,0,1,2) and interact with the weak field
- electric charge (electromagnetic interactions) : except the neutrinos all elementary particles have an electric charge and interact with the electromagnetic field

Each fermion (as it seems also true for the neutrinos) has a mass and so interacts also with the gravitational field.

Each particle is represented by a spinor with 4 components, and in weak and strong interactions the left and right component interact differently with the fields.

Moreover each fermion has an associated antiparticle, which is represented by "conjugation" of the particle. In the process the charge changes (color becomes "anticolor" which are different, hypercharge takes the opposite sign), left handed spinors are exchanged with right handed spinors, but the mass is the same.

Linear combination of these fermions give resonances which have usually a very short life. Stable elementary particles (such as the proton and the neutron) are "bound states" of elementary particles, represented as tensorial combinations of these fermions.

Bosons

The "bosons" are particles which carry the forces. They are :

- 8 gluons (corresponding each to one of the basis vector space of the Lie algebra su(3)) : they have no electric charge but each of them carries a color and an anticolor. They are massless. They are their own antiparticles.
- 3 bosons W^j (corresponding each to one of the basis vector space of the Lie algebra su(2)) which carry weak hypercharge and have a mass.
- 1 boson B (corresponding to one of the basis vector space of the Lie algebra u(1)) carries an hypercharge and a mass.
- 1 Higgs boson, which has two binded components, is its own antiparticle and has a mass but no charge or color

The bosons W,B combine linearly to give the photon, the neutral boson Z and the charged bosons $W^{\pm}$. The photon and Z are their antiparticle, $W^{\pm}$ are the antiparticle of each other. So

[1] Fermions and bosons have a more general definition, related to the "Fermi statistic". However in the present framework of elementary particles there is no risk of confusion.

in the Standard Model photons are not elementary particles (at least when eletroweak interactions are considered).

Bosons are represented as potential of the force fields, that is 1-forms over M

9.1.3 The issues

The Standard Model does not sum up all of QTF, which encompasses many other aspects of the interactions between fields and particles. However there are several open issues in the Standard Model.

1. The Standard Model, built in the Special Relativity geometry, ignores gravitation. Considering the discrepancy between the forces at play, this is not really a problem for a model dedicated to the study of elementary particles. QTF is rooted in the Poincaré's algebra, and the localized state vectors, so it has no tool to handle trajectories, which are a key component of differential geometry.

2. The Higgs boson, celebrated recently, raises almost as many questions than it gives answers. It has been introduced in what can be considered as a "patch", needed to solve the issue of masses for fermions and bosons. The Dirac's operator, as it is used for the fermions, does not give a definite positive scalar product and is null (and so their mass) whenever the particles are chiral. And as for the bosons, the equivariance in a change of gauge forbids the explicit introduction of the potential, which is assumed to be their correct representation. The Higgs boson solves these problems, but at the cost of many additional parameters, and the introduction of a "fifth force" which it should carry.

3. From a semi-classic lagrangian, actually most of the implementation of the Standard Model relies on "particles to particles" interactions, detailed by Feynmann's diagram and computed through pertubatives methods. In many ways the actual QTF is, under a sophisticated guise, a return to the action at a distance physics, the bosons and other virtual particles making the link. Of course this is consistent with a discrete representation of the world, but also difficult to conciliate with the wave functions representing particles.

4. The range of the weak and strong interactions is not well understood. Formally it is represented by the introduction of a "Yukawa potential" (which appears as a "constant coupling" in the Standard Model), proportional to $\frac{1}{r}\exp(-km)$ which implies that if the mass m of the carrier boson is not null the range decreases quickly with the distance r. Practically, as far as the system which is studied is limited to few particles, this is not a big issue.

5. We could wish to incorporate the three groups in a single one, meanwhile encompassing the gravitational field and explaining the hierarchy between the forces. This is the main topic of the Great Unification Theories (GUT) (see Sehbatu for a review of the subject). The latest, undergone by Garrett Lisi, invokes the exceptional Lie group E8. Its sheer size (it has 248 dimensions) enables to account for every thing, but also requires the introduction of as many parameters.

An option, which has been studied by Trayling and Lisi, would be to start, not from Lie groups, but from Clifford algebras as we have done for the gravitational field. The real dimension of SU(3)×SU(2)×U(1) is $12 = 8 + 3 + 1$ which implies to involve at least a Clifford algebra (dimension 2^n) on a four dimensional vector space and it makes sense to look at its complexified $Cl(\mathbb{C}, 4)$. The groups would then be Spin subgroups of the Clifford algebra. We have the following isomorphisms :

$U(1) \sim Spin(\mathbb{R}, 2)$

$SU(2) \sim Spin(\mathbb{R}, 3)$

but there is no simple isomorphism for SU(3).

Albeit all together they are part of $Cl(\mathbb{R}, 10)$.

9.2 Model of particles and fields

In the following we present a "basic model", which does not pretend to be any kind of GUT, but a tool to study our main concern regarding the representation of particles and fields.

9.2.1 The hypotheses

We have already the representation of the motion of particles, including their kinematical characteristics. In addition to the hypotheses already made in General Relativity, we assume the following :

i) There is a m dimensional connected real compact Lie group U. Its Lie algebra T_1U has for basis $\left(\overrightarrow{\theta}_a\right)_{a=1}^m$

ii) The physical features of particles are represented in a complex n dimensional vector space F, with basis $(f_j)_{i=1}^n$ and (F, ϱ) is a faithful representation of U

iii) For any system there is a principal bundle structure $P_U(M, U, \pi_U)$ over M, with trivialization $\varphi_U : M \times U \to P_U$ such that the value of the physical characteristics of a particle at m as measured by an observer is a vector : $(\varphi_U(m, 1), \phi) \sim \left(\varphi_U(m, g), \varrho\left(g^{-1}\right)\phi\right)$ of the associated vector bundle $\mathfrak{F}$=$P_U[F, \varrho]$.

iv) There is a principal connection with potential $\grave{\mathbf{A}} \in \Lambda_1(M, T_1U)$ on P_U such that the holonomic basis $f_j(m) = (\varphi_U(m, 1), f_j)$ is parallel transported on the world line of the standard observers (the same as for P_G)

9.2.2 Particles

Physical state

U can be SU(3), SU(2), U(1) or any of their product. As U is compact, it has finite dimensional unitary representations. We will assume that the principal bundle is trivial (which makes sense in the bounded area of a system).

The "physical state" of the particle is represented by a vector of $P_U[F, \varrho]$ which is measured by comparing the interaction of the particle with known fields to the interaction of other particles. These particles, which provide the standard, are the "elementary particles". So we can see the vectors $(f_j)_{i=1}^n$ as a representation of these standards. In this context the "charges" that the particle carry are represented in the vector $\phi \in F$, so the elementary particles have all distinct, unitary, charges. So there is not a quantity such as the charge of the electron, which could be measured in some units. What we can say is that, with respect to the electromagnetic field, a particle behave "like an electron". And indeed, in which unit could we measure the colour ? So one can expect that n $\geq$ m. The group SU(3)×SU(2)×U(1) has the dimension 12 and we have 12 fermions, but which are further differentiated by these charges (each quark can have one of 3 colors, and one of two distinct hypercharge), and moreover to each particle is associated an antiparticle (the anticolors are distinct from the colors).

In the representation $P_U[F,\varrho]$ there is no geometry involved, so there is no concept of time and stability of the particle. However in the concept of particle there is the concept of a localized quantity, that we will see below.

$(F,\varrho'(1))$ is a representation of T_1U. U is a real Lie group, and the connection is also real : $\grave{A}\in\Lambda_1(M;T_1U)$ is valued in the real Lie algebra T_1U. The potential À lives in an affine bundle over the adjoint bundle $P_U[T_1U,Ad]$ where Ad is the usual adjoint map : $Ad:U\to\mathcal{L}(T_1U;T_1U)$.

To a trivialization is associated a gauge : $\mathbf{p}_U(m)=\varphi_U(m,1)$ and we assume that the standard observers use this gauge. A change of gauge in P_U is defined by a section $\chi\in\mathfrak{X}(P_U)$:

$p=\varphi_U(m,g)=\widetilde{\varphi}_U(m,\chi(m)g)$

$\mathbf{p}_U(m)\to\widetilde{\mathbf{p}}_U(m)=\widetilde{\varphi}_U(m,1)=\varphi_U\left(m,\chi(m)^{-1}\right)$

A section $\mathbf{g}=\varphi_U(m,g(m))\in\mathfrak{X}(P_U)$ becomes :

$\mathbf{g}=\varphi_U(m,g(m))=\widetilde{\varphi}_U(m,\widetilde{g}(m))\Rightarrow\widetilde{g}(m)=\chi(m)g(m)$

In the vector bundle $P_U[F,\varrho]$

$\mathbf{f}_i(m)\to\widetilde{\mathbf{f}}_i(m)=(\widetilde{\mathbf{p}}_U(m),f_i)=\left(\mathbf{p}_U(m)\cdot\chi(m)^{-1},f_i\right)\sim\varrho\left(\chi(m)^{-1}\right)\mathbf{f}_i(m)$

A section $\phi(m)=(\varphi_U(m,1),\phi(m))\in\mathfrak{X}(P_U[F,\varrho])$ becomes :

$\mathbf{f}(m)=(\varphi_U(m,1),\phi(m))=\left(\varphi_U\left(m,\chi(m)^{-1}\right),\widetilde{\phi}(m)\right)\Rightarrow\widetilde{f}(m)=\varrho(\chi(m))\phi(m)$

$\phi(m)=\sum_{j=1}^n\phi^j(m)\mathbf{f}_j(m)=\sum_{j=1}^n\widetilde{\phi}^j(m)\widetilde{\mathbf{f}}_j(m)$

$\Rightarrow\widetilde{\phi}^j(m)=\sum_{k=1}^n[\varrho(\chi(m))]_k^i\phi^k(m)$

The potential becomes :

$\mathbf{p}_U(m)\to\grave{A}(m)\to\widetilde{\grave{A}}(m)=Ad_\chi\left(\grave{A}(m)-L'_{\chi^{-1}}(\chi)\chi'(m)\right)$

ϕ may change over the trajectory of a particle. The changes occur, either because the value of the fields change, or because the frames used to measure it change. In both cases the changes go through the representation ϱ. If there were no field, one can assume that ϕ would be fixed for the gauge. As ϕ is a characteristic of the particle, it seems logical that it should keep some feature of ϕ over its life. So we make the assumption that the representation $[F,\varrho]$ is faithful (or equivalently effective, or free) :

$\forall\phi\in F,g,g'\in U:\{\varrho(g)\phi=\varrho(g')\phi\Rightarrow g=g'\}\Leftrightarrow\{\varrho(g)\phi=\phi\Rightarrow g=1\}$

Then for any map : $\phi:[0,T]\to F$ and $\phi_0\in F$ there is a map: $\Theta:[0,T]\to P_U$ such that : $\phi(t)=\varrho(\Theta(t))\phi_0$

And conversely, for ϕ_0 fixed, ϕ is uniquely defined by a map $\Theta:[0,T]\to P_U$.

Quantization - Physical state

The physical state of a particle is fully defined by the value of $\phi\in P_U[F,\varrho]$ and in the Schrödinger picture the map $\phi:[0,T]\to F$ sums up this state in a system. Any model including this map as variable is liable to the theorems of QM.

$P_U[F,\varrho]$ is a representation of the compact group U, which is completely reducible in a sum of unitary, irreducible, orthogonal finite dimensional representations of U. The basis of F gives the standards to measure ϕ, so it is legitimate to assume that (F,ϱ) is itself a unitary representation.

As a consequence :

i) there is a definite positive hermitian scalar product on F, which is denoted $\langle\rangle_U$

ii) this scalar product is preserved by $\varrho:\langle\varrho(g)\phi,\varrho(g)\phi'\rangle_U=\langle\phi,\phi'\rangle_U$

iii) there is an orthonormal basis and we will assume that $(f_i)_{i=1}^n$ is orthonormal

iv) This scalar product can be extended to $P_U[F,\varrho]$ by :
$\mathbf{F},\mathbf{F'}\in\mathfrak{X}(P_U[F,\varrho]):\langle\mathbf{F},\mathbf{F'}\rangle=\int_\Omega\langle f(m),f'(m)\rangle\varpi_4$
and $(\mathbf{f}_i(m))_{i=1}^n$ is orthonormal.
v) $\varrho'(1)$ is anti-unitary and the matrices $\left[\varrho'(1)\overrightarrow{\theta}_a\right]=[\theta_a]$ are antihermitian :

$$[\theta_a]=-[\theta_a]^* \tag{41}$$

$\frac{d}{dg}\langle\varrho(g)\phi,\varrho(g)\phi'\rangle_U|_{g=1}$
$=\left\langle\frac{d}{dg}\varrho(g)\phi,\phi'\right\rangle_U+\left\langle\phi,\frac{d}{dg}\varrho(g)\phi'\right\rangle_U$
$=\langle\varrho'(1)\phi,\phi'\rangle_U+\left\langle\phi,\frac{d}{dg}\varrho(g)\phi'\right\rangle_U=0$

Full model for particles

We have two distinct representations for particles : the geometric one with the spin which represents the motion, and the "physical" one which represents the interaction with the other fields. In order to account for the interactions between the two features, a theorem of QM says that it is possible to replace the two separate representations by their tensorial product.

The principal bundle is $Q(M,Spin(3,1)\times U,\pi_Q)$ corresponding to the choice of frames both for the geometrical and the physical parts, with trivialization : $\varphi_Q:M\times(Spin(3,1)\times U)\to Q::$ $q=\varphi_Q(m,(s,g))$
and gauge : $\mathbf{q}(m)=\varphi_Q(m,(1,1))$
A change of gauge is given by a map :
$\chi:M\to(Spin(3,1)\times U):q=\varphi_Q(m,(s,g))=\widetilde{\varphi}_Q(m,\chi(m)(s,g))$

One can extend the action of the Spin group to the action of the Clifford algebra. We define the action ϑ of $Cl(\mathbb{R},3,1)\times U$ on $E\otimes F$ by the unique linear extension of:
$\vartheta:Cl(\mathbb{R},3,1)\times U\to L(E\otimes F;E\otimes F)::\vartheta(s,g)(S\otimes\phi)=\gamma C(s)(S)\otimes\varrho(g)(\phi)$
to all tensors on $E\otimes F$
This is a morphism from $Cl(\mathbb{R},3,1)$ on $L(E\otimes F;E\otimes F):\vartheta$ is linear and preserves the Clifford product
Proof. $\vartheta(s\cdot s',gg')(S\otimes\phi)=\gamma C(s\cdot s')(S)\otimes\varrho(gg')(\phi)$
$=\gamma C(s)\circ\gamma C(s')(S)\otimes\chi(g)\circ\varrho(g')(\phi)$
$=\gamma C(s)(\gamma C(s')(S))\otimes\varrho(g)(\varrho(g')(\phi))$
$=(\gamma C(s)\otimes\varrho(g))(\gamma C(s')(S)\otimes\varrho(g')(\phi))$
$=\gamma C(s)\otimes\varrho(g)(\gamma C(s')\otimes\varrho(g')(S\otimes\phi))$
$=(\gamma C(s)\otimes\varrho(g))\circ(\gamma C(s')\otimes\varrho(g'))(S\otimes\phi)$
$=(\vartheta(s,g)\circ\vartheta(s',g'))(S\otimes\phi)$
$\vartheta(1,1)(S\otimes\phi)=\gamma_{\mathbb{C}}C(1)(S)\otimes\varrho(1)(\phi)=S\otimes\phi$ ■
$\vartheta(\sigma,1)\psi=\gamma C(\sigma)\psi=\sum_{jkl}[\gamma C(\sigma)]_k^j\psi^{kl}e_j\otimes f_l$
The map ϑ defines a representation of $Cl(\mathbb{R},3,1)\times U$ on $E\otimes F$. So we have an action of the Clifford bundle Cl(M) on the vector bundle $Q[E\otimes F,\vartheta]\simeq P_G[E,\gamma C]\otimes P_U[F,\varrho]$.

A particle is localized, the state of a particle at each point is represented by a tensor ψ of the bundle $Q[E\otimes F,\vartheta]$. The state of the particle is represented by ψ and this is the variable which enters the lagrangian. It reads ;

$$\psi = \sum_{i=1}^{4}\sum_{j=1}^{n} \psi^{ij}(m)\,\mathbf{e}_i(m)\otimes \mathbf{f}_j(m) \text{ in matrix form : } [\psi]_{4\times n} \tag{42}$$

In a change of gauge:
$\widetilde{\psi}^{ij} = \sum_{k=1}^{4}\sum_{l=1}^{n} [\gamma C(\kappa)]_k^i\,[\varrho(\Theta)]_l^j\,\psi^{kl}$

So a particle can appear to be composed of several "elementary particles" localized at the same point (a proton is composed of 3 quarks udd and a neutron uud) which moves together and give its physical and spin characteristics.

The scalar product on $E\otimes F$ is necessarily defined as :

$$\langle\psi,\psi'\rangle = \sum_{ijq}[\gamma_0]_k^i\,\delta_{jq}\overline{\psi}^{ij}\psi'^{kq} = \sum_{ijk}[\gamma_0]_k^i\,\overline{\psi}^{ij}\psi'^{kj} = Tr\left([\psi]^*[\gamma_0][\psi']\right) \tag{43}$$

It is preserved by ϑ :
$\widetilde{\psi}^{ij} = \sum_{k=1}^{4}\sum_{l=1}^{n} [\gamma C(\kappa)]_k^i\,[\varrho(\Theta)]_l^j\,\psi^{kl}$
$\left\langle\widetilde{\psi},\widetilde{\psi}'\right\rangle = \sum [\gamma_0]_k^i\,\overline{[\gamma C(\kappa)]}_p^i\,\overline{[\varrho(\Theta)]}_q^j\,\overline{\psi}^{pq}\,[\gamma C(\kappa)]_r^k\,[\varrho(\Theta)]_s^j\,\psi^{rs}$
$= \sum\left([\gamma C(\kappa)]^*[\gamma_0][\gamma C(\kappa)]\right)_r^p\left([\varrho(\Theta)]^*[\varrho(\Theta)]\right)_s^q\,\overline{\psi}^{pq}\psi'^{rs}$
$= \sum [\gamma_0]_r^p\,[\gamma C(\kappa)]_r^k\,\overline{\psi}^{ps}\psi'^{rs}$

Because the representations $\left(\widehat{E}_0,\gamma C\right),(F,\varrho)$ are faithful it is the same for $\left(\widehat{E}_0\otimes F,\vartheta\right)$. The state of a particle is then represented by a map : $\psi : [0,T]\to Q\left[\widehat{E}_0\otimes F,\vartheta\right]$.The trajectory is defined by the projection $\mu_O(t) = \pi_G(\psi(t))$ from $Q\left[\widehat{E}_0\otimes F,\vartheta\right]$ onto M. We assume that it is admissible.

Quantization

The vector space $E_0\otimes F$ is globally invariant by ϑ for $\sigma_r\times\Theta\in Spin(3)\times U$. The set of maps :
$\mathcal{F} = \{\psi_r : [0,T]\to E_0\otimes F :: \psi_r(t) = \theta(\sigma_r,\Theta)\,\psi_0, \sigma_r\times\Theta\in Spin(3)\times U, \psi_0\in E_0\otimes F\}$
is a vector space and $(\mathcal{F},\vartheta)$ is a representation of the compact group $Spin(3)\times U$.
Endowed with the scalar product :
$\langle\psi_r,\psi_r'\rangle = \int_0^T\langle\vartheta(\sigma_r(t),\Theta(t))\,\psi_0,\vartheta(\sigma_r'(t),\Theta'(t))\,\psi_0'\rangle\,dt$
it is a Hilbert space. So we can extend the results of Special Relativity.

The principal bundle Q has a sub-bundle Q_r defined by restriction of Spin(3,1) to Spin(3) and the associated bundle $Q\left[\widehat{E}_0\otimes F,\vartheta\right]$ has a sub-bundle $Q_r[E_0\otimes F,\vartheta_r]$.

For trajectories which are defined by the same vector field V and origin x : $\mu_O(t) = \Phi_V(t,x)$ the map :
$\psi_r : [0,T]\to Q_r[E_0\otimes F,\vartheta_r] :: \psi_r(t) = \sum_{ij}\psi_r^{ij}(t)\,\mathbf{e}_i(\mu_O(t))\otimes\mathbf{f}_j(\mu_O(t))$
is such that $\psi_r\in\mathcal{F}$
There is a basis of orthonormal sections (ψ_{Np}) of the Hilbert space $\mathcal{F}$ defined by :
$\psi_{Rp}(\Phi_V(t,x)) = \sum_{N\in\mathbb{N}/2}^{\infty}\sum_{q=-R}^{q=+R}P_{Rq}(x)\sum_{i=1}^{2}\sum_{j=1}^{n}[\omega_{Rq}(t)]^{ij}\,\mathcal{E}_i\otimes f_j$ where P_{Rq} is a harmonic polynomial
with $[\omega_{Rp}]^*[\omega_{Rq}] = \delta_{pq}$

and $\psi_r(t) = \sum_{N\in\mathbb{N}/2}^{\infty} \sum_{p=-R}^{p=+R} d_{Rp}\psi_{Rp}(\mu_O(t))$ where $d_{Rp} \in \mathbb{C}$

The vectors ψ_{Rp} are defined all over Ω , so they define sections on $Q_r[E_0 \otimes F, \vartheta_r]$.

The action of the groups is then such that :

$\vartheta(g_r, g_u)\psi_r(t) = \sum_{R\in\mathbb{N}/2}^{\infty} \sum_{p=-R}^{p=+R} \widetilde{d_{Rp}}\varrho(g_u)\psi_{Rp}\left(\Phi_V\left(t, [g_r]^{-1}x\right)\right)$

$= \sum_{ij}\sum_{kl}[\gamma C(g_r)]_k^i [\varrho(g_u)]_j^l [\psi_r(t)]_j^i e_i \otimes f_j$

with $P_{Rp}\left([g_r]^{-1}x\right) = \sum_{q=-R}^{q=+R}[K_N(g_r)]_q^p P_{Np}(x)$ where $[K_N(g_r)]$ is the matrix of the action on harmonic polynomials.

$\psi_{Rp}\left(\Phi_V\left(t, [g_r]^{-1}x\right)\right) = \vartheta(g_r, 1)\sum_{q=-R}^{q=+R}[K_N(g_r)]_q^p \psi_{Rp}(\Phi_V(t,x))$

The value $\widehat{\psi_r}$ which is measured for ψ_r is any combination : $\widehat{\psi}_r(t) = \sum_{N\in\mathbb{N}/2}^{\infty}\sum_{p=-R}^{p=+R}\widehat{d_{Np}}\psi_{Np}(\mu_P(t))$ with $\widehat{d_{Np}} = 0$ or for a finite number of components $\widehat{d_{Np}} = d_{Np}$, with the probability : $\frac{\sum_{Rp}|\widehat{d_{Rp}}|^2}{\sum_{Rp}|d_{Rp}|^2}$.

Each physical state $\widehat{\psi_r}$ which is observed correspond to some finite linear combinations of the ψ_{Rp}. Because we observe only different types of particles, with similar spatial states, we can assume that there are only a finite number of privileged combinations, denoted $(\psi_{rp})_{p=1}^N$ of these sections in $Q_r[E_0 \otimes F]$. Each particle of the same type has, in a given system, the same behaviour for the same initial conditions (x and initial velocity) so their world lines are defined by the same vector field u_p :$\mu_p(\tau) = \Phi_{u_p}(\tau, x)$. This vector defines itself a each point an element $\sigma_w(\tau)$, and by $\psi_p(\tau) = \vartheta(\sigma_w(\tau), 1)\psi_{rp}(\tau)$ a section $\psi_p \in \mathfrak{X}\left(Q\left[\widehat{E}_0 \otimes F\right]\right)$. For a system with a network of observers the trajectories are then defined by a function : $f_p \in C_\infty(M;\mathbb{R}_+)$ by $V_p(m) = f_p(m)u_p$.

So we can make the assumptions :

i) There are a finite number N of particles

ii) *In a given system , for each type p of particle, there is a section* $\psi_p \in \mathfrak{X}\left(Q\left[\widehat{E}_0 \otimes F\right]\right)$ *and for each initial conditions a vector* $\psi_{0p} \in E_0 \otimes F$ *and a function (depending on the observer)* $f_p \in C_\infty(M;\mathbb{R}_+)$*such that :*

$\psi_p(m) = \vartheta(\sigma(m), \Theta(m))\psi_{0p}$ *defines uniquely the sections* $\sigma \in P_G, \Theta \in P_U$

$\sigma_w(m) = a_w + \upsilon(0, w)$ *belongs to the projection of* $\sigma(m)$ *on Spin(3,1)/Spin(3)*

the world line of the particle is defined by the vector field : $U_p(m) = cAd_{\sigma_w}\varepsilon_0(m)$

the trajectory of the particle is : $\mu_p(t) = \pi_Q(\psi_p(t)) = \Phi_{Vp}(t,x)$ *where x is the initial position on* $\Omega_3(0)$ *and* $V_p(m) = f_p(m)U_p(m)$

Remarks :

i) The section ψ_p alone does not define a set of curves, it does when associated to ψ_{0p} because the representation is faithful. So any particle of the same type with the same initial conditions follow the same curve. For instance all free particles are of the same type, and they follow the same curves if the initial conditions are identical.

ii) But to define a trajectory we need more, whence the vector field V_p, which depends on the observer and is defined by a function f_p.

$$f_p(m) = \frac{1}{c}\sqrt{-\langle V_p, V_p\rangle} = \frac{d\tau_p}{dt} \tag{44}$$

iii) We could expect that each elementary particle figures in the set $(\psi_p)_{p=1}^N$ but not necessarily (quarks have never been observed individually).

v) The spatial spin is no longer defined by a vector S_r but by the sign of $\sigma_w(t)$ (or $\sigma_r(t)$) with respect to V in the decomposition. Any type of particle can have a spin up or down.

The Schrödinger equation holds for a particle on its trajectory. So there is a self-adjoint operator H such that

$\psi_p(t) = \left(\exp \frac{1}{i\hbar} tH\right) \psi_p(0)$

$\psi_p(\Phi_{Vp}(t,x)) = \left(\exp \frac{1}{i\hbar} tH\right) \psi_p(\Phi_{Vp}(0,x))$

The sections ψ_p can be seen as "matter fields" : they are defined all over Ω and depend on the characteristics of the particles. They are "virtual" in that they take the form of particles only when initial conditions are defined. So we have achieved one part of our endeavour. Of course there is not a particle present at each point and we will see how to deal with this issue.

Example : electromagnetism

The group is U(1), the complex numbers of module 1, and its Lie algebra is $\mathbb{R}$. This is an abelian group, so its irreducible representations are unidimensional. Assuming that it is finite then :

$\phi = \sum_{j=1}^n \phi^j f_j$ where each vector f_j corresponds to some definite value of the charge, and the action of U(1) is: $\varrho\left(e^{i\theta}\right) = e^{i\theta}\phi$ where $\theta \in \mathbb{R}$.

Our assumptions about ϕ gives that this is a vector $\phi = e^{i\theta}\phi_0$ defined up to a scalar, each type of particles corresponding to a specific charge. The scalar product on F is : $\langle \phi, \phi \rangle = \sum_{j=1}^p \overline{\phi}^j \phi^j$ which is invariant by U(1) and $\langle \phi, \phi \rangle = Ct$ is assumed constant for each type of particle. The tensor ψ is then : $\psi = \gamma C(\sigma_w)\psi_r$ with $\psi_r = e^{i\theta}\gamma C(\sigma_r) S_0 \otimes \phi_0 = e^{i\theta}\gamma C(\sigma_r)\psi_{p0}$

9.2.3 Fields

Other fields than gravity

The frames defined by the principal bundle P_U are no more than a set of procedures to measure the values of the characteristics ϕ at each point. In order to built this network the observers need a consistent way to compare their measures from a point to another. This involves the parallel transport of frames, and as the procedures stem from the action of the field on particles, the connection **A** is necessarily representative of the field itself.

The connection **A** induces a covariant derivative acting on sections of the associated vector bundle

$\nabla_\alpha^A \phi = \sum_{\alpha=0}^3 \left(\partial_\alpha \phi^i + \left[\varrho'(1)\left(\sum_{a=1}^m \grave{A}_\alpha^a\right)\right]_j^i \phi^j\right) \mathbf{f}_i(m) \otimes d\xi^\alpha \in \Lambda_1(M, P_U[F,\varrho])$

Notation 16 $\grave{A} = \sum_{a=1}^m \sum_{\alpha=0}^3 \grave{A}_\alpha^a \overrightarrow{\theta}_a \otimes d\xi^\alpha$ *is the potential of the connection* ***A***

Notation 17 $[\theta_a]$ *is the* $n \times n$ *matrix* $\left[\varrho'(1)\left(\overrightarrow{\theta}_a\right)\right]$

Notation 18 $\left[\grave{A}_\alpha\right] = \sum_{a=1}^m \grave{A}_\alpha^a [\theta_a] o$

The parallel transport along a path $\mu_O(t)$ in M is defined by : $\nabla_V^A \phi = \mathbf{0}$ with $V = \frac{d\mu_O}{dt}$. If a particle preserves its physical characteristics along its trajectory $\mu_O(t)$ then we must have : $\nabla_V^A \phi(\mu_O(t)) = \mathbf{0}$.

The strength of the connection is, with non ordered indices :
$\mathcal{F}_A = \sum_{\alpha,\beta} \left(\mathcal{F}^a_{A\alpha\beta} d\xi^\alpha \wedge d\xi^\beta \right) \otimes \overrightarrow{\theta}_a \in \Lambda_2(M; T_1U)$ where [] is the bracket in T_1U

$$\mathcal{F}_{A\alpha\beta} = \frac{\partial \grave{A}_{r\beta}}{\partial \xi^\alpha} - \frac{\partial \grave{A}_{r\alpha}}{\partial \xi^\beta} + \left[\grave{A}_\alpha, \grave{A}_\beta \right]^a \tag{45}$$

In a change of gauge $\mathcal{F}$ changes as :
$\mathcal{F}^a_{A\alpha\beta} \to \widetilde{\mathcal{F}}_{A\alpha\beta}(m) = Ad_{\chi(m)} \mathcal{F}_{A\alpha\beta}$
For any model involving the fields, the variable $\mathcal{F}_A$ represents the field, we have a representation (T_1U, Ad) of the group U and the usual theorems of QM tell us that $\mathcal{F}_A$ belongs to a vector space which is a unitary representation of U. So there is a definite positive scalar product on T_1U, preserved by the adjoint map Ad and we define the scalar product between sections of $\Lambda_2(M; T_1U)$ as for the gravitational field :
i) point wise : $\left\langle \mathcal{F}_{A\alpha\beta}(m), \mathcal{F}'_{A\lambda\mu}(m) \right\rangle_{T_1G}$
ii) $\langle \mathcal{F}_A(m), \mathcal{F}'_A(m) \rangle = \sum_{\alpha\beta} \sum_{\lambda\mu=0}^3 \left\langle \mathcal{F}_{A\alpha\beta}(m), \mathcal{F}'_{A\lambda\mu}(m) \right\rangle_{T_1U} g^{\alpha\lambda} g^{\beta\mu} = \sum_{\alpha\beta} \left\langle \mathcal{F}_{\alpha\beta}(m), \mathcal{F}'^{\alpha\beta}(m) \right\rangle_{T_1U}$
iii) over M : $\langle \mathcal{F}_A, \mathcal{F}'_A \rangle = \int_\Omega \langle \mathcal{F}_A(m), \mathcal{F}'_A(m) \rangle \varpi_4(m) = \int_\Omega \mathcal{F}^t_{A\alpha\beta}(m) \mathcal{F}_A^{\alpha\beta}(m) \varpi_4(m)$
This scalar product is definite positive, invariant in a change of gauge and so we have a unitary representation of U. As T_1U is a real vector space the scalar product is a bilinear symmetric form. And because it is preserved by the adjoint map we have :
$\forall X, Y, Z \in T_1U : \langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$
This leads to the definition of the Hodge dual of $\mathcal{F}_A$:
$*\mathcal{F}_A \in \Lambda_2(M; T_1U) :: \forall \mu \in \Lambda_2(M; T_1U) : *\mathcal{F}_A \wedge \mu = \langle \mathcal{F}_A, \mu \rangle \varpi_4$
so that : $\langle \mathcal{F}_A, \mathcal{F}_A \rangle = \int_\Omega *\mathcal{F}_A \wedge \mathcal{F}_A$
Because we have a unitary representation of a compact group $\mathcal{F}_A$ belongs to a direct sum of finite dimensional ireducible representations, that is : $\mathcal{F}_A = \sum_k \lambda_k \mathcal{F}_{Ak}$ where λ_k are fixed real scalar, and $\mathcal{F}_{Ak}$ fixed two-forms. The field is quantized.

Example : electromagnetic field. The Lie algebra of U(1) is $\mathbb{R}$, there is a unique matrix $[\theta_a] = i$. A connection is then a real valued one form on M : $\grave{A} = \sum_{\alpha=0}^3 \grave{A}_\alpha d\xi^\alpha \in \Lambda_1(M; \mathbb{R})$ and the strength $\mathcal{F}_A$ is the two form : $\mathcal{F}_A = d\grave{A}$. Which gives immediately the first Maxwell law : $d\mathcal{F}_A = 0$.
$\mathcal{F}_A$ can be written :
$\frac{1}{c}\left(E_1 d\xi^1 + E_2 d\xi^2 + E_3 d\xi^3\right) \wedge d\xi^0 + B_1 d\xi^2 \wedge d\xi^3 + B_2 d\xi^3 \wedge d\xi^1 + B_3 d\xi^1 \wedge d\xi^2$
with the usual components of the electric E and magnetic B fields. And the potentials are defined up to the differential of a function.

Some comments.
i) In QTF, because the groups are comprised of matrices with complex coefficients, and the elements of the Lie algebra $T_1U$are operators in the Hilbert spaces, it is usual to introduce the imaginary i everywhere, and to consider the complexified of the Lie algebra T_1U. However it is clear that the potential $\grave{A}_\alpha$ belongs to the real algebra. For instance the group associated with the

electromagnetic field is U(1), the Lie algebra is $\mathbb{R}$ and the potential is a one form on M valued in $\mathbb{R}$ (usually seen as a vector field). And there are as many force carriers bosons (12) as fermions.

ii) It can be shown that, for simple groups of matrices, the only scalar products on their Lie algebra which are invariant by the adjoint map are of the kind : $\langle [X], [Y] \rangle = kTr\left([X]^* [Y]\right)$. In the standard model, because several groups are involved, three different constants are involved, the quantity $\sqrt{\frac{2}{k}}$ is called the "gauge coupling". Here we consider only one group, and we can take the basis $\left(\overrightarrow{\theta}_a\right)_{a=1}^m$ as orthonormal.

Force fields all together

The two connections **G,À** act together on the tensorial product. The group which is involved is the direct product : $Spin(3,1) \times U$ (or $Spin(1,3) \times U$). Its Lie algebra is the direct product $T_1 Spin(\mathbb{R},3,1) \times T_1 U$.

The covariant derivative of sections of $Q[E \otimes F, \vartheta]$ is :
$\nabla_\alpha \psi \in \Lambda_1 (M, P_U [F, \varrho])$
$\nabla_\alpha \psi = \sum_{\alpha=0}^3 \left(\partial_\alpha \psi^{ij} + \sum_{a=1}^6 [\gamma C (G_\alpha^a)] [\psi] + \sum_{a=1}^m \grave{A}_\alpha^a [\psi] [\theta_a]\right)^{ij} \mathbf{e}_i (m) \otimes \mathbf{f}_i (m) \otimes d\xi^\alpha$

$$[\nabla_\alpha \psi] = \sum_{\alpha=0}^3 [\partial_\alpha \psi] + [\gamma C (G_\alpha)] [\psi] + [\psi] \left[\grave{A}_a\right] \tag{46}$$

with $[\gamma C (G_\alpha)] = [\gamma C (\upsilon G_{r\alpha}, G_{w\alpha})]$
which, for separable tensors, reads :
$\nabla_\alpha (S \otimes F) = \left(\nabla_\alpha^S S\right) \otimes \phi + S \otimes \left(\nabla_\alpha^A \phi\right)$

The total field at any point m is represented by a couple $(\mathcal{F}_G (m), \mathcal{F}_A (m))$ belonging to the vector bundle $Q[T_1 Spin \times T_1 U, \mathbf{Ad} \times Ad]$

Propagation of the fields

One of major tenets of Relativity is that the fields (at least the electromagnetic and gravitational fields) propagate in the vacuum at the speed of light, whatever the observer. How can we interpret this statement ?

This not an easy topic. Actually the electromagnetic field, the only one for which accurate data are available, propagates at different speed with respect to the medium, gravitation impacts the trajectories, and supraluminal propagation has been speculated. The propagation of the electromagnetic field in the context of GR has been the topic of many studies (see Kopeikin), generally in the "post Newtonian" linear approximation, fueled by the increasing precision of astronomical measures. The electromagnetic field is represented as waves, propagating along null vector fields, as special solutions of the Maxwell equations. The propagation of more general "fields" in the context of QTF is also the topic of some studies, in order to understand the "EPR paradox". Actually in QTF there is only one time, that of the observer, as in Galilean geometry, so the locality of events is biased. Schnaid introduces an additional observable, "the travel time", to represent the delayed effect of the propagation of a perturbation, which is then introduced in a modified Schrödinger equation. Thus the difference between local time and observer time is accounted for.

In all cases the propagation at the speed of light is assumed : it shall be considered as one the hypothesis of any model about fields.

The geometry of the Universe is at the heart of the problem, and the propagation of fields has a clear meaning only in the Relativist picture. The value of the field at a given 4 dimensional location is fixed, so propagation means that there is a special relation between the values of the field at different locations, depending on their "distance", measured along some special vector because of the anisotropy of the universe. Moreover we should account for an attenuation factor for the weak and strong interactions, to represent their short ranges.

To compare the value of the field along a curve the most logical way is to use the Lie derivative. The principle is the following. With the diffeomorphim induced by the flow of a vector field W, we push forward the value of the potential from $\Phi_W(-\tau, m)$ to m, so that they belong to the same tangent space and can be compared. The result of the transport is $\Phi_W(\tau, m)_* \grave{A}(\Phi_W(-\tau, m))$ (we would have the same procedure for the gravitational field).

In components :

$\Phi_W(\tau, m)_* \grave{A}(\Phi_W(-\tau, m)) = \sum_{a\alpha\beta} [K(\tau, m)]_\alpha^\beta \grave{A}_\beta^a (\Phi_W(-\tau, m)) d\xi^\alpha \otimes \overrightarrow{\theta}_a$

with $[J(\tau, m)]_\beta^\alpha = \left[\frac{\partial \eta^\alpha}{\partial \xi^\beta}\right]$ the matrix of the derivative of $\Phi_W(\tau, m)$ with respect to m, and $[K(\tau)] = [J(\tau)]^{-1}$

The derivative of this operator at $\tau = 0$ is the Lie derivative :

$\pounds_W \grave{A} = \frac{d}{d\tau} \Phi_W(\tau, m)_* \grave{A}(\Phi_W(-\tau, m))|_{\tau=0}$

If $\Phi_W(\tau, m)_* \grave{A}(\Phi_W(-\tau, m)) = \grave{A}(m)$ then $\pounds_W \grave{A} = 0$. The value of the potential at $\Phi_V(\tau, m)$ is the value it has at m . So we can say that it has propagated along W. The four dimensional velocity of the propagation is W, and its 4 dimensional speed is $\langle W, W\rangle$. This is the speed of light if $\langle W, W\rangle = 0$. If there is an attenuation factor, depending on the distance (such as r^{-2}) it suffices to incorporate a factor $f(\tau)$ and the propagation reads :

$\Phi_W(\tau, m)_* \grave{A}(\Phi_W(-\tau, m)) = f(\tau) \grave{A}(m) \Rightarrow \pounds_W \grave{A} = f'(\tau)|_{\tau=0} \grave{A}.$

The Lie derivative can be expressed as:

$\pounds_V \grave{A} = i_V d\grave{A} + d \circ i_V \grave{A}$

$\pounds_V \grave{A} = \sum_\alpha \left(\sum_\beta V^\beta \left(\partial_\alpha \grave{A}_\beta - \partial_\beta \grave{A}_\alpha\right) + \partial_\alpha \left(V^\beta \grave{A}_\beta\right)\right) d\xi^\alpha$

so the condition reads :

$\forall a, \alpha : \sum_\beta V^\beta \left(\partial_\alpha \grave{A}_\beta - \partial_\beta \grave{A}_\alpha\right) + \partial_\alpha \left(V^\beta \grave{A}_\beta\right) = \sum_b [B]_b^a \grave{A}_\alpha^b (m)$

Because the potential is valued in the Lie algebra the procedure is more complicated. We must consider vector fields and paths in the fiber bundles where the connection lives.

The principal fiber bundle $P_U(M, U, \pi_U)$ is a manifold. A vector of $T_p P_U$ at $p = \varphi_U(m, g)$is defined by a couple $(v_m, v_g) \in T_m M \times T_g U$.

$v_p = \sum_\alpha v_m^\alpha \partial m_\alpha + \sum_a v_g^a \partial \theta_a \in T_p P_U$

The vertical bundle VP_U is comprised of vectors $\varphi'_{Ug}(m, g) v_g$. It is isomorphic to the Lie algebra, and it is convenient to use the fundamental vectors : $\zeta(v_g)(p) = \sum_{a=1}^m \left[L'_g 1(v_g)\right]^a \partial \theta_a$ where v_g is a vector of the Lie algebra.

We will need a vector field W on TP_U which is projected in a vector field V on M, called a projectable vector field. It is such that $\pi'_U(p) W(p) = V(\pi_U(p))$ and reads :

$W(p) = \sum_{\alpha=0}^3 V^\alpha(m) \partial m_\alpha + \zeta(X(p))(p)$ where $X(p) \in T_1 U$

We want also that the frames along an integral curve of W stay the same, that is $\mathbf{A}(p)(W(p)) = 0$: W is the horizontal lift of V and this condition reads :

$W(p) = \sum_{\alpha=0}^3 V^\alpha(m) \partial m_\alpha - \zeta\left(Ad_{g^{-1}}\left(\sum_\alpha V^\alpha \grave{A}_\alpha\right)\right)(p)$

Moreover if we keep the standard gauge $\mathbf{p}(m) = \varphi_U(m, 1)$:

$W(\mathbf{p}(m)) = \sum_{\alpha=0}^{3} V^{\alpha}(m)\left(\partial m_{\alpha} - \sum_{a=1}^{m} \grave{A}_{\alpha}^{a}\partial\theta_{a}\right)$

The 1-jet prolongation J^1P_U of the principal bundle P_U is an affine bundle over P_U modelled on $TM^* \otimes VP_U$. An element of J^1P_U has for coordinates $(\xi^{\alpha}, \theta_a, \theta_{\alpha}^{a}, \alpha = 0..3, a = 1..m)$ and for trivialization :

$\Upsilon(\xi^{\alpha}, \theta_a, \theta_{\alpha}^{a}, \alpha = 0..3, a = 1..m) = (\varphi_U(m,g), \sum_{am} \theta_{\alpha}^{a}\partial\theta_{a}^{\alpha}) = (p, \sum_{am} \theta_{\alpha}^{a}\partial\theta_{a}^{\alpha})$

The Christoffel symbol of the connection is : $\Gamma(p) = \sum_a \left[R_g' 1\left(\grave{A}_{\alpha}\right)\right]^a \partial\theta_a \otimes d\xi^{\alpha}$ and it can be seen as a section of J^1P_U :

$\left(p, \sum_a \left[R_g' 1\left(\grave{A}_{\alpha}\right)\right]^a \partial\theta_{a}^{\alpha}\right) \sim \left(\mathbf{p}(m), \sum_a \grave{A}_{\alpha}^{a}\partial\theta_{a}^{\alpha}\right) \in J^1P_U$

W is the infinitesimal generator of a one parameter group of base preserving diffeomorphism (m is unchanged) on P_U by $\Phi_W(\tau, \Theta(\Phi_V(-\tau, m)))$ for any section $\Theta \in \mathfrak{X}(P_U)$

$\pi_U(\Phi_W(\tau, p)) = \Phi_V(\tau, \pi_U(p))$

$\pi_U(\Phi_W(\tau, \Theta(\Phi_V(-\tau, m)))) = m$

This diffeomorphism can be extended to J^1P_U :

$J^1\Phi_W : J^1P_U \to J^1P_U ::$

$J^1\Phi_W\left(\tau, J^1\Theta\right) = j^1_{\Phi_W(\tau,\Theta(\Phi_V(-\tau,m)))}\Phi_W(\tau, \Theta(\Phi_V(-\tau, m)))$

The prolongation J^1W of W on J^1P_U is the infinitesimal generator of this one parameter group of diffeomorphism on J^1P_U : $\Phi_{J^1W} = J^1\Phi_W$. It is a a projectable vector field with components (Kolar p.360):

$J^1W(\xi^{\alpha}, \theta_a, \theta_{\alpha}^{a}) =$

$\sum_{\alpha} V^{\alpha}(m)\partial\xi_{\alpha} + \sum_{a=1}^{m} X^a(p)\partial\theta_a + \sum_{a\alpha}\left(\partial_{\alpha}X^a(p) + \sum_b \theta_{\alpha}^{b}\partial_b X^a - \sum_{\beta}\theta_{\beta}^{a}\partial_{\alpha}V^{\beta}\right)\partial\theta_{a}^{\alpha}$

which gives here with $X^a(p) = -\sum_{\alpha} V^{\alpha}(m)\grave{A}_{\alpha}^{a}(m)$,

$J^1W(\xi^{\alpha}, \theta_a, \theta_{\alpha}^{a}) = \sum_{\alpha} V^{\alpha}\partial\xi_{\alpha} - \sum_{\alpha a} V^{\alpha}\grave{A}_{\alpha}^{a}\partial\theta_a - \sum_{a\alpha\beta}\left(\partial_{\alpha}\left(V^{\beta}\grave{A}_{\beta}^{a}\right) + \theta_{\beta}^{a}\partial_{\alpha}V^{\beta}\right)\partial\theta_{a}^{\alpha}$

The Lie derivative of a section Z along $J^1W$is :

$\pounds_{J^1W}(Z) = \frac{\partial}{\partial\tau}\Phi_{J^1W}(\tau, Z(\Phi_V(-\tau, m)))|_{\tau=,0}$

It belongs to the vertical bundle VJ^1P_U, so it reads : $\pounds_{J^1W}(Z) = \sum_{a\alpha} Y_{\alpha}^{a}\partial\theta_{a}^{\alpha}$

$\pounds_{J^1W}\left(\mathbf{p},\grave{A}\right) = \frac{\partial}{\partial\tau}\Phi_{J^1W}\left(\tau, \left(\mathbf{p}(\Phi_V(-\tau, m)), \grave{A}(\Phi_V(-\tau, m))\right)\right)|_{\tau=0}$

$= \frac{\partial}{\partial Z}\Phi_{J^1W}\left(\tau, \left(\mathbf{p}(\Phi_V(-\tau, m)), \grave{A}(\Phi_V(-\tau, m))\right)\right)|_{\tau=0}\left(-p'V, -\sum_{\beta} V^{\beta}\partial_{\beta}\grave{A}_{\alpha}\right) + J^1W\left(\mathbf{p},\grave{A}\right)$

and we have in components :

$\pounds_{J^1W}\grave{A}$

$= \sum_{a\alpha}\left(-\sum_{b\beta\gamma}[K_1]_{\alpha}^{\beta}[K_2]_{a}^{b}V^{\gamma}\partial_{\gamma}\grave{A}_{\beta}^{b} - \sum_{\beta}\partial_{\alpha}\left(V^{\beta}\grave{A}_{\beta}^{a}\right) + \grave{A}_{\beta}^{a}\partial_{\alpha}V^{\beta}\right)\partial\theta_{a}^{\alpha}$

The condition reads :

$\pounds_{J^1W}\grave{A} = \sum_{a\alpha}\left(\sum_{b\beta}[B_1]_{b}^{a}[B_2]_{\alpha}^{\beta}\grave{A}_{\beta}^{b}\right)\partial\theta_{a}^{\alpha}$

that is :

$\exists V \in \mathfrak{X}(TM)$ with $\langle V, V\rangle = 0$:

$$\forall a, \alpha, m : \sum_{b\beta\gamma}[K_1(m)]_{\alpha}^{\beta}[K_2(m)]_{a}^{b}V^{\gamma}\partial_{\gamma}\grave{A}_{\beta}^{b} + \sum_{\beta}\partial_{\alpha}\left(V^{\beta}\grave{A}_{\beta}^{a}\right) + \grave{A}_{\beta}^{a}\partial_{\alpha}V^{\beta} + [B_1]_{b}^{a}[B_2]_{\alpha}^{\beta}\grave{A}_{\beta}^{b} = 0 \quad (47)$$

So we have similar, but not identical, differential equations as seen above.

Remark : propagation can be studied from partial differential equations (such as the wave equation), or more generally from pseudo-differential operators. But in both cases these methods

require scalar functions defined on $\mathbb{R}^4$. So they apply in the Special Relativity context, but are not easily transposed to the GR context, where the needs are more crucial, for theoretical and practical purposes.

10 LAGRANGIAN

The lagrangian is a key element in the implementation of the principle of least action, that is for continuous models and to study equilibrium of a system. However, in their hamiltonian form they are aso fundamental in many other models. In many ways they sum up the physics of a system in a single real function. In this section we will study the specification of the lagrangian.

As the variables related to different particles and fields have the same mathematical properties, in this section we will denote simply the lagrangian as a function :

$L\left(\psi^i, G^a_\alpha, \grave{A}^a_\alpha, P^\alpha_i, \partial_\beta\psi^i, \partial_\beta G^a_\alpha, \partial_\beta \grave{A}^a_\alpha, \partial_\beta P^\alpha_i\right)$ in an action such as : $\int_\Omega L \det P' \varpi_4$

Thus we assume that we have a first order lagrangian, which is consistant with first order connections. If a "density function", such as N_p below, is introduced it is assumed to be a scalar function invariant by a change of gauge or charts, and incorporated in L. Similarly the function f, used to compute the trajectory, is defined over M and invariant, so we will ignore it here.

All variables are represented by their coordinates in relevant frames, by real or complex scalars. L is not supposed to be holomorphic, so the real and imaginary part of the variables $\psi^{ij}, \partial_\alpha\psi^{ij}$ must appear explicitely. We will use the convenient notation for complex variables z and their conjugates $\overline{z}$, by introducing the holomorphic complex valued functions :

$\frac{\partial L}{\partial z} = \frac{1}{2}\left(\frac{\partial L}{\partial \operatorname{Re} z} + \frac{1}{i}\frac{\partial L}{\partial \operatorname{Im} z}\right)$

$\frac{\partial L}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial L}{\partial \operatorname{Re} z} - \frac{1}{i}\frac{\partial L}{\partial \operatorname{Im} z}\right)$

$\Leftrightarrow$

$\frac{\partial L}{\partial \operatorname{Re} z} = \frac{\partial l}{\partial z} + \frac{\partial L}{\partial \overline{z}}$

$\frac{\partial L}{\partial \operatorname{Im} z} = i\left(\frac{\partial L}{\partial z} - \frac{\partial L}{\partial \overline{z}}\right)$

The partial derivatives $\frac{\partial L}{\partial \operatorname{Re} z}, \frac{\partial L}{\partial \operatorname{Im} z}$ are real valued functions, so $\frac{\partial L}{\partial \overline{z}} = \overline{\frac{\partial L}{\partial z}}$. And we have the identities for any complex valued function u :

$\frac{\partial L}{\partial \operatorname{Re} z}\operatorname{Re} u + \frac{\partial L}{\partial \operatorname{Im} z}\operatorname{Im} u = 2\operatorname{Re}\frac{\partial L}{\partial z}u$

$-\frac{\partial L}{\partial \operatorname{Re} z}\operatorname{Im} u + \frac{\partial L}{\partial \operatorname{Im} z}\operatorname{Re} u = -2\operatorname{Im}\frac{\partial L}{\partial z}u$

To implement the Lagrange equations we need the explicit presence of the variables and their partial derivatives. But as our goal is to precise the specification of L, we can, without loss of generality, make the replacements :

$\partial_\alpha\psi^{ij} \to \nabla_\alpha\psi^{ij} = \partial_\alpha\psi^{ij} + \sum_{k=1}^4\sum_{a=1}^6\left[\gamma C\left(G^a_\alpha\right)\right]^i_k\psi^{kj} + \sum_{k=1}^n\psi^{ik}\left[\grave{A}_\alpha\right]^k_j$

$\partial_\beta G^a_\alpha \to \mathcal{F}^a_{G\alpha\beta} = \partial_\alpha G^a_\beta - \partial_\beta G^a_\alpha + [G_\alpha, G_\beta]$ and $F_{G\alpha\beta} = \partial_\alpha G^a_\beta + \partial_\beta G^a_\alpha$

$\partial_\beta \grave{A}^a_\alpha \to \mathcal{F}^a_{A\alpha\beta} = \partial_\alpha \grave{A}^a_\beta - \partial_\beta \grave{A}^a_\alpha + \left[\grave{A}_\alpha, \grave{A}_\beta\right]$ and $F_{G\alpha\beta} = \partial_\alpha \grave{A}^a_\beta + \partial_\beta \grave{A}^a_\alpha$

And the lagrangian is then a function :

$\mathcal{L}\left(\psi^{ij}, G^a_\alpha, \grave{A}^a_\alpha, P^\alpha_i, \nabla_\alpha\psi^{ij}, \mathcal{F}_{G\alpha\beta}, F^a_{G\alpha\beta}, \mathcal{F}_{A\alpha\beta}, F_{A\alpha\beta}, \partial_\beta P^\alpha_i\right)$

Most of the variables above are defined up to some transformation : for instance the components of the tetrad are defined up to a matrix of SO(3,1). The function L should be intrinsic, meaning invariant by :

- a change of gauge in the principal bundles P_G, P_U and their associated bundles

- a change of chart in the manifold M

The operations below will give the relations which must exist consequently between the variables and the partial derivatives of L, and some precious information about the presence or the absence of some variables.

10.1 Equivariance in a change of gauge

10.1.1 One parameter groups of change of gauge

One parameter groups of change of trivialization on a principal bundle are defined by sections of their adjoint bundle :

$\kappa \in \mathfrak{X}\left(P_G\left[T_1 Spin(\mathbb{R},3,1), Ad\right]\right)$

$\theta \in \mathfrak{X}\left(P_U\left[T_1 U, Ad\right]\right)$

The gauges become :

$\mathbf{p}_G(m) \to \widetilde{\mathbf{p}}_G(m,\tau) = \varphi_G(m, \exp(-\tau\kappa(m)))$ with $\chi(m,\tau) = \exp(\tau\kappa(m))$

$\mathbf{p}_U(m) \to \widetilde{\mathbf{p}}_U(m) = \varphi_U(m, \exp(-\tau\theta(m)))$ with $\chi(m,\tau) = \exp(\tau\theta(m))$

The variables become :

$P_i^\alpha \to \widetilde{P}_i^\alpha(m,\tau) = \sum_{j=0}^{3}\left[h(\exp(\tau\kappa))\right]_i^j P_j^\alpha$ where $[h]$ is the SO(3,1) corresponding matrix

$\psi^{ij} \to \widetilde{\psi}^{ij}(m,\tau) = \sum_{k=1}^{4}\left[\gamma C(\exp(\tau\kappa))\right]_k^i \psi^{kj} + \sum_{k=1}^{n}\left[\varrho(\exp(\tau\theta))\right]_k^j \psi^{ik}$

$\partial_\beta P_i^\alpha \to \widetilde{\partial_\beta P}_i^\alpha(m,\tau) = \sum_{j=0}^{3}\left(\left[h\left(\exp(\tau\kappa)'\partial_\beta\kappa\right)\right]_i^j P_j^\alpha + \left[h(\exp(\tau\kappa))\right]_i^j \partial_\beta P_i^\alpha\right)$

$G_\alpha(m) \to \widetilde{G}_\alpha(m) = Ad_{\exp(\tau\kappa(m))}\left(G_\alpha - \exp(-\tau\kappa)(\exp\tau\kappa)'\tau\partial_\alpha\kappa\right)$

$\grave{A}_\alpha \to \widetilde{\grave{A}}_\alpha(m,\tau) = Ad_{\exp(\tau\theta(m))}\left(\grave{A}_\alpha - \exp(-\tau\theta)\exp(\tau\theta)'\tau\partial_\alpha\theta\right)$

$\nabla_\alpha\psi \to \widetilde{\nabla_\alpha\psi^{ij}}(m,\tau) = \sum_{k=1}^{4}\left[\gamma C(\exp(\tau\kappa))\right]_k^i \nabla_\alpha\psi^{kj} + \sum_{k=1}^{n}\left[\varrho(\exp(\tau\theta))\right]_k^j \nabla_\alpha\psi^{ik}$

$\partial_\beta\widetilde{G}_\alpha(m,\tau)$

$= \left[(\exp -\tau\kappa)(\exp\tau\kappa)'\tau\partial_\beta\kappa, G_\alpha - \tau\partial_\alpha\kappa\right]$

$+Ad_{\exp\tau\kappa}\{\partial_\beta G_\alpha -$

$\left((\exp -\tau\kappa)'\tau\partial_\beta\kappa \circ (\exp\tau\kappa)'\tau\partial_\alpha\kappa + \exp(-\tau\kappa)\circ(\exp\tau\kappa)''(\tau\partial_\beta\kappa,\tau\partial_\alpha\kappa) + \exp(-\tau\kappa)\circ\exp(\tau\kappa)'\tau\partial^2_{\alpha\beta}\kappa\right)\}$

$\partial_\beta\widetilde{\grave{A}}_\alpha(m,\tau) = \left[(\exp -\tau\theta)(\exp\tau\theta)'\tau\partial_\beta\theta, \grave{A}_\alpha - \tau\partial_\alpha\theta\right]$

$+Ad_{\exp\tau\theta}(\partial_\beta\grave{A}_\alpha - \left(\exp(-\tau\theta)'\tau\partial_\beta\theta\circ(\exp\tau\theta)'\tau\partial_\alpha\theta\right)$

$+\exp(-\tau\theta)\circ(\exp\tau\kappa)''(\tau\partial_\beta\theta,\tau\partial_\alpha\theta) + \exp(-\tau\theta)\circ\exp(\tau\theta)'\tau\partial^2_{\alpha\beta}\theta)$

$\mathcal{F}^a_{G\alpha\beta} \to \widetilde{\mathcal{F}}_{G\alpha\beta}(\tau) = Ad_{\exp\tau\kappa}\mathcal{F}_{G\alpha\beta}$

$\mathcal{F}^a_{A\alpha\beta} \to \widetilde{\mathcal{F}}_{A\alpha\beta}(\tau) = Ad_{\exp\tau\theta}\mathcal{F}_{A\alpha\beta}$

$F_{G\alpha\beta} \to Ad_{\exp\tau\kappa}F_{G\alpha\beta}$

$+\left[(\exp -\tau\kappa)(\exp\tau\kappa)'\tau\partial_\beta\kappa, G_\alpha - t\partial_\alpha\kappa\right] + \left[(\exp -\tau\kappa)(\exp\tau\kappa)'\tau\partial_\alpha\kappa, G_\beta - \tau\partial_\beta\kappa\right]$

$-Ad_{\exp\tau\kappa}((\exp -\tau\kappa)'\tau\partial_\beta\kappa\circ(\exp\tau\kappa)'\tau\partial_\alpha\kappa + \exp(-\tau\kappa)\circ(\exp\tau\kappa)''(\tau\partial_\beta\kappa,\tau\partial_\alpha\kappa)$

$+\exp(-\tau\kappa)\circ\exp(\tau\kappa)'\tau\partial^2_{\alpha\beta}\kappa)$

$-Ad_{\exp\tau\kappa}((\exp -\tau\kappa)'\tau\partial_\alpha\kappa\circ(\exp\tau\kappa)'\tau\partial_\beta\kappa + \exp(-\tau\kappa)\circ(\exp\tau\kappa)''(\tau\partial_\alpha\kappa,\tau\partial_\beta\kappa)$

$+\exp(-\tau\kappa)\circ\exp(\tau\kappa)'\tau\partial^2_{\alpha\beta}\kappa)$

$F_{A\alpha\beta} \to Ad_{\exp(\tau\theta}F_{A\alpha\beta}$

$+\left[(\exp -\tau\theta)(\exp\tau\theta)'\tau\partial_\beta\theta, \grave{A}_\alpha - \tau\partial_\alpha\theta\right] + \left[(\exp -\tau\theta)(\exp\tau\theta)'\tau\partial_\alpha\theta, \grave{A}_\beta - \tau\partial_\beta\theta\right]$

$-Ad_{\exp\tau\theta}(\exp(-\tau\theta)'\tau\partial_\beta\theta\circ(\exp\tau\theta)'\tau\partial_\alpha\theta + \exp(-\tau\theta)\circ(\exp\tau\kappa)''(\tau\partial_\beta\theta,\tau\partial_\alpha\theta)$

$+\exp(-\tau\theta)\circ\exp(\tau\theta)'\,\tau\partial^2_{\alpha\beta}\theta)$
$-Ad_{\exp\tau\theta}(\exp(-\tau\theta)'\,\tau\partial_\alpha\theta\circ(\exp\tau\theta)'\,\tau\partial_\beta\theta+\exp(-\tau\theta)\circ(\exp\tau\kappa)''\,(\tau\partial_\alpha\theta,\tau\partial_\beta\theta)$
$+\exp(-\tau\theta)\circ\exp(\tau\theta)'\,\tau\partial^2_{\alpha\beta}\theta)$

The determinant $\det P'$ is invariant, because we have a change of orthonormal basis, so the scalar lagrangian L is invariant :

$\mathcal{L}\left(\psi^{ij},G^a_\alpha,\grave{A}^a_\alpha,P^\alpha_i,\nabla_\beta\psi^{ij},\mathcal{F}_{G\alpha\beta},F_{G\alpha\beta},\mathcal{F}_{A\alpha\beta},F_{A\alpha\beta},\partial_\beta P^\alpha_i\right)$
$=\mathcal{L}\left(\widetilde{\psi}^{ij},\widetilde{G}^a_\alpha,\widetilde{\grave{A}}^a_\alpha,\widetilde{P}^\alpha_i,\widetilde{\nabla_\beta\psi^{ij}},\widetilde{\mathcal{F}_{G\alpha\beta}},\widetilde{F_{G\alpha\beta}},\widetilde{\mathcal{F}_{A\alpha\beta}},\widetilde{F_{A\alpha\beta}},\widetilde{\partial_\beta P^\alpha_i}\right)$

If we take the derivative of this identity for $\tau=0$ we get an identity which must hold whatever $\kappa(m),\theta(m)$ and then identities between the partial derivatives of $\mathcal{L}$. From a mathematical point of view this derivative with respect to τ is the Lie derivative of the lagrangian along the vertical vector fields generated by the derivative $\frac{dz^i_\alpha}{d\tau}|_{\tau=0}$ for each variable. These vector fields define the "Noether currents".

10.1.2 Equivariance on P_G

The computation for $\chi(m,\tau)=\exp(\tau\kappa(m))$ gives :

$\frac{d}{d\tau}\widetilde{P}^\alpha_i(m,\tau)|_{\tau=0}=-\sum_a\kappa^a([P][\kappa_a])^\alpha_i$
$\frac{d}{d\tau}\operatorname{Re}\widetilde{\psi}^{ij}(m,\tau)|_{\tau=0}=\sum_a\kappa^a\sum_{k=1}^4\left(\operatorname{Re}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Re}\psi^{kj}-\operatorname{Im}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Im}\psi^{kj}\right)$
$\frac{d}{d\tau}\operatorname{Im}\widetilde{\psi^{ij}}(m,\tau)|_{\tau=0}=\sum_a\kappa^a\sum_{k=1}^4\left(\operatorname{Re}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Im}\psi^{kj}+\operatorname{Im}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Re}\psi^{kj}\right)$
$\frac{d}{d\tau}\partial_\beta\widetilde{P}(m,t)^\alpha_j|_{\tau=0}=-\sum_a\kappa^a([\partial_\beta P][\kappa_a])^\alpha_i+\partial_\beta\kappa^a([P][\kappa_a])^\alpha_i$
$\frac{d}{d\tau}\operatorname{Re}\widetilde{\nabla_\alpha\psi}^{ij}(m,\tau)|_{\tau=0}$
$=\sum_a\kappa^a\sum_{k=1}^4\left(\operatorname{Re}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Re}\nabla_\alpha\psi^{kj}-\operatorname{Im}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Im}\nabla_\alpha\psi^{kj}\right)$
$\frac{d}{d\tau}\operatorname{Im}\widetilde{\nabla_\alpha\psi^{ij}}(m,\tau)|_{\tau=0}$
$=\sum_a\kappa^a\sum_{k=1}^4\left(\operatorname{Re}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Im}\nabla_\alpha\psi^{kj}+\operatorname{Im}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Re}\nabla_\alpha\psi^{kj}\right)$
$\frac{d}{d\tau}\widetilde{G^a_\alpha}(m)|_{\tau=0}=\sum_b\kappa^b[\overrightarrow{\kappa}_b,G_\alpha]^a-\partial_\alpha\kappa^a$
$\frac{d}{d\tau}\partial_\beta\widetilde{G}^a_\alpha(m,\tau)|_{\tau=0}=\sum_b\kappa^b[\overrightarrow{\kappa}_b,\partial_\beta G_\alpha]^a+\partial_\beta\kappa^b[\overrightarrow{\kappa}_b,G_\alpha]^a-\partial_{\alpha\beta}\kappa^a$
$\frac{d}{d\tau}\widetilde{\mathcal{F}}^a_{G\alpha\beta}(\tau)|_{\tau=0}=\sum_b\kappa^b[\overrightarrow{\kappa}_b,\mathcal{F}_{G\alpha\beta}]^a$
$\frac{d}{d\tau}\widetilde{F}^a_{G\alpha\beta}|_{\tau=0}=\sum_b\kappa^b[\overrightarrow{\kappa}_b,F_{G\alpha\beta}]^a+\partial_\beta\kappa^b[\overrightarrow{\kappa}_b,G_\alpha]^a+\partial_\alpha\kappa^b[\overrightarrow{\kappa}_b,G_\beta]^a-2\partial_{\alpha\beta}\kappa^a$

So we have the identity :

$0=$
$\sum_{ij}\frac{\partial\mathcal{L}}{\partial\operatorname{Re}\psi^{ij}}\left(\sum_a\kappa^a\sum_{k=1}^4\left(\operatorname{Re}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Re}\psi^{ij}-\operatorname{Im}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Im}\psi^{kj}\right)\right)$
$+\frac{\partial\mathcal{L}}{\partial\operatorname{Im}\psi^{ij}}\left(\sum_a\kappa^a\sum_{k=1}^4\left(\operatorname{Re}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Im}\psi^{kj}+\operatorname{Im}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Re}\psi^{kj}\right)\right)$
$+\sum_{\alpha ij}\frac{\partial\mathcal{L}}{\partial\operatorname{Re}\nabla_\alpha\psi^{ij}}\left(\sum_a\kappa^a\sum_{k=1}^4\left(\operatorname{Re}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Re}\nabla_\alpha\psi^{ij}-\operatorname{Im}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Im}\nabla_\alpha\psi^{kj}\right)\right)$
$+\frac{\partial\mathcal{L}}{\partial\operatorname{Im}\nabla_\alpha\psi^{ij}}\left(\sum_a\kappa^a\sum_{k=1}^4\left(\operatorname{Re}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Im}\nabla_\alpha\psi^{kj}+\operatorname{Im}\left([\gamma C(\kappa_a)]^i_k\right)\operatorname{Re}\nabla_\alpha\psi^{kj}\right)\right)$
$+\sum_{i\alpha}\frac{\partial\mathcal{L}}{\partial P^\alpha_i}\left(-\sum_a\kappa^a([P][\kappa_a])^\alpha_i\right)+\sum_{i\alpha\beta}\frac{\partial\mathcal{L}}{\partial\partial_\beta P^\alpha_i}\left(-\sum_a\kappa^a([\partial_\beta P][\kappa_a])^\alpha_j+\partial_\beta\kappa^a([P][\kappa_a])^\alpha_j\right)$
$+\sum_{a\alpha}\frac{\partial\mathcal{L}}{\partial G^a_\alpha}\left(\sum_b\kappa^b[\overrightarrow{\kappa}_b,G_\alpha]^a-\partial_\alpha\kappa^a\right)$

$+\sum_{a\alpha\beta}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^a_{G\alpha\beta}}\left(\sum_b\kappa^b\left[\overrightarrow{\kappa}_b,\mathcal{F}_{G\alpha\beta}\right]^a\right)$
$+\frac{\partial\mathcal{L}}{\partial F^a_{G\alpha\beta}}\left(\sum_b\kappa^b\left[\overrightarrow{\kappa}_b,F_{G\alpha\beta}\right]^a+\partial_\beta\kappa^b\left[\overrightarrow{\kappa}_b,G_\alpha\right]^a+\partial_\alpha\kappa^b\left[\overrightarrow{\kappa}_b,G_\beta\right]^a-2\partial_{\alpha\beta}\kappa^a\right)$

With the component in $\partial_{\alpha\beta}\kappa^a$ we have immediately : $\forall a,\alpha,\beta:\frac{\partial\mathcal{L}}{\partial F^a_{G\alpha\beta}}=0$

With the component in $\partial_\alpha\kappa^a:\forall a,\alpha:\sum_{\beta i}\frac{\partial L}{\partial\partial_\alpha P^\beta_i}\left(\left[P\right]\left[\kappa_a\right]\right)^\beta_i=-\frac{\partial L}{\partial G^a_\alpha}$

And we are left with :
$\forall a=1..6:2\sum_{ij}\mathrm{Re}\left(\frac{\partial\mathcal{L}}{\partial\psi^{ij}}\left(\left[\gamma C\left(\kappa_a\right)\right]\psi\right)^{ij}+\sum_\alpha\frac{\partial\mathcal{L}}{\partial\nabla_\alpha\psi^{ij}}\left(\left[\gamma C\left(\kappa_a\right)\right]\nabla_\alpha\psi\right)^{ij}\right)+$
$+\sum_{i\alpha}\frac{\partial\mathcal{L}}{\partial P^\alpha_i}\left(-\left(\left[P\right]\left[\kappa_a\right]\right)^\alpha_i\right)+\sum_{i\alpha\beta}\frac{\partial\mathcal{L}}{\partial\partial_\beta P^\alpha_i}\left(-\left(\left[\partial_\beta P\right]\left[\kappa_a\right]\right)^\alpha_j\right)$
$+\sum_{b\alpha}\frac{\partial\mathcal{L}}{\partial G^b_\alpha}\left(\left[\overrightarrow{\kappa}_a,G_\alpha\right]^b\right)+\sum_{b\alpha\beta}\frac{\partial\mathcal{L}}{\partial\mathcal{F}^b_{G\alpha\beta}}\left(\left[\overrightarrow{\kappa}_a,\mathcal{F}_{G\alpha\beta}\right]^b\right)=0$

Moreover, by taking the derivative with respect to the initial variables we get :
$\sum_{k=1}^4\left[\gamma C\left(\exp\left(\tau\kappa\left(m\right)\right)\right)\right]^k_i\frac{\partial\widetilde{L}}{\partial\widetilde{\psi^{kj}}}=\frac{\partial L}{\partial\psi^{ij}}$
$\sum_{k=1}^4\left[\gamma C\left(\exp\left(\tau\kappa\left(m\right)\right)\right)\right]^k_i\frac{\partial\widetilde{L}}{\partial\widetilde{\nabla_\alpha\psi^{kj}}}=\frac{\partial L}{\partial\nabla\psi^{ij}}$
$\sum_j\left[h\left(\exp\left(-\tau\kappa\left(m\right)\right)\right)\right]^j_i\frac{\partial\widetilde{L}}{\partial\widetilde{P^\alpha_j}}=\frac{\partial L}{\partial P^\alpha_i}$
$\mathcal{L}\left(\left[Ad_{\exp\tau\kappa}\right]^a_b\mathcal{F}^b_{G\alpha\beta}\right)=\mathcal{L}\left(\mathcal{F}_{G\alpha\beta}\right)$
$\sum_b\left[Ad_{\exp\tau\kappa}\right]^b_a\frac{\partial\widetilde{L}}{\partial\widetilde{\mathcal{F}^b_{G\alpha\beta}}}=\frac{\partial\mathcal{L}}{\partial\mathcal{F}^a_{G\alpha\beta}}$
and other similar identities, which show that the partial derivatives are tensors, with respect to the vector bundles :
$\sum_i\frac{\partial L}{\partial\psi^{ij}}\mathbf{e}^i,\sum_i\frac{\partial L}{\partial\nabla_\alpha\psi^{ij}}\mathbf{e}^i,\sum_i\frac{\partial L}{\partial P^\alpha_i}\varepsilon_i,\sum_i\frac{\partial L}{\partial\partial_\beta P^\alpha_i}\varepsilon_i,\sum_a\frac{\partial L}{\partial\mathcal{F}^a_{G\alpha\beta}}\overrightarrow{\kappa}^a$
with $\overrightarrow{\kappa}^a$ the basis vector of the dual of T_1Spin : $\overrightarrow{\kappa}^a\left(\overrightarrow{\kappa}_b\right)=\delta^a_b$

10.1.3 Equivariance on P_U

We have similarly :

$\frac{d}{d\tau}\widetilde{\psi^{ij}}\left(m,\tau\right)|_{\tau=0}=\sum_{k=1}^n\sum_{a=1}^m\theta^a\psi^{ik}\left[\theta_a\right]^k_j$
$\frac{d}{d\tau}\mathrm{Re}\,\widetilde{\psi^{ij}}\left(m,\tau\right)|_{\tau=0}=\sum_{a=1}^m\theta^a\,\mathrm{Re}\left(\psi\left[\theta_a\right]\right)^{ij}$
$\frac{d}{d\tau}\mathrm{Im}\,\widetilde{\psi^{ij}}\left(m,\tau\right)|_{\tau=0}=\sum_{a=1}^m\theta^a\,\mathrm{Im}\left(\psi\left[\theta_a\right]\right)^{ij}$
$\frac{d}{d\tau}\widetilde{\grave{A}}^a_\alpha\left(m,\tau\right)|_{\tau=0}=\sum_{b=1}^m\theta^b\left[\overrightarrow{\theta}_b,\grave{A}_\alpha\right]^a-\partial_\alpha\theta^a$
$\frac{d}{d\tau}\mathrm{Re}\,\widetilde{\nabla_\alpha\psi^{ij}}\left(m,\tau\right)|_{\tau=0}=\sum_{a=1}^m\theta^a\,\mathrm{Re}\left(\nabla_\alpha\psi\left[\theta_a\right]\right)^{ij}$
$\frac{d}{d\tau}\mathrm{Im}\,\widetilde{\nabla_\alpha\psi^{ij}}\left(m,\tau\right)|_{\tau=0}=\sum_{a=1}^m\theta^a\,\mathrm{Im}\left(\nabla_\alpha\psi\left[\theta_a\right]\right)^{ij}$
$\frac{d}{d\tau}\partial_\beta\widetilde{\grave{A}}^a_\alpha\left(m,\tau\right)|_{\tau=0}=\sum_{b=1}^m\theta^b\left[\theta_b,\partial_\beta\grave{A}_\alpha\right]^a+\partial_\beta\theta^b\left[\theta_b,\grave{A}_\alpha\right]^a-\partial_{\alpha\beta}\theta^a$
$\frac{d}{d\tau}\widetilde{\mathcal{F}}_{A\alpha\beta}\left(\tau\right)|_{\tau=0}=\sum_{b=1}^m\theta^b\left[\overrightarrow{\theta}_b,\mathcal{F}_{A\alpha\beta}\right]^a$

$\frac{d}{d\tau}\widetilde{F}_{A\alpha\beta}|_{\tau=0} = \sum_{b=1}^{m} \theta^b \left[\theta_b, F_{A\alpha\beta}\right]^a + \partial_\beta\theta^b \left[\theta_b, \grave{A}_\alpha\right]^a + +\partial_\alpha\theta^b \left[\theta_b, \grave{A}_\beta\right]^a - 2\partial_{\alpha\beta}\theta^a$

$\sum_{ij} \frac{\partial\mathcal{L}}{\partial \operatorname{Re}\psi^{ij}} \sum_{a=1}^{m} \theta^a \operatorname{Re}\left(\psi\left[\theta_a\right]\right)^{ij} + \frac{\partial\mathcal{L}}{\partial \operatorname{Im}\psi^{ij}} \sum_{a=1}^{m} \theta^a \operatorname{Im}\left(\psi\left[\theta_a\right]\right)^{ij}$
$+ \sum_{ij\alpha} \frac{\partial\mathcal{L}}{\partial \operatorname{Re}\nabla_\alpha\psi^{ij}} \sum_{a=1}^{m} \theta^a \operatorname{Re}\left(\psi\left[\theta_a\right]\right)^{ij} + \sum_{ij\alpha} \frac{\partial\mathcal{L}}{\partial \operatorname{Im}\nabla_\alpha\psi^{ij}} \sum_{a=1}^{m} \theta^a \operatorname{Im}\left(\psi\left[\theta_a\right]\right)^{ij}$
$+ \sum_{a\alpha} \frac{\partial\mathcal{L}}{\partial \grave{A}^a_\alpha} \left(\sum_{b=1}^{m} \theta^b \left[\overrightarrow{\theta}_b, \grave{A}_\alpha\right]^a - \partial_\alpha\theta^a\right) + \sum_{a\alpha\beta} \frac{\partial\mathcal{L}}{\partial \mathcal{F}^a_{A\alpha\beta}} \left(\sum_{b=1}^{m} \theta^b \left[\overrightarrow{\theta}_b, \mathcal{F}_{A\alpha\beta}\right]^a\right)$
$+ \sum_{a\alpha\beta} \frac{\partial\mathcal{L}}{\partial F^a_{A\alpha\beta}} \left(\sum_{b=1}^{m} \theta^b \left[\theta_b, F_{A\alpha\beta}\right]^a + \partial_\beta\theta^b \left[\theta_b, \grave{A}_\alpha\right]^a + \partial_\alpha\theta^b \left[\theta_b, \grave{A}_\beta\right]^a - 2\partial_{\alpha\beta}\theta^a\right) = 0$

Which implies :
$\forall a, \alpha, \beta : \frac{\partial\mathcal{L}}{\partial F^a_{A\alpha\beta}} = 0, \frac{\partial\mathcal{L}}{\partial \grave{A}^a_\alpha} = 0$
$\forall a = 1..m :$
$2\sum_{ij} \operatorname{Re} \frac{\partial\mathcal{L}}{\partial\psi^{ij}} \left(\sum_{j=1}^{n} \left(\psi\left[\theta_a\right]\right)^{ij} + \sum_\alpha \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\psi^{ij}} \left(\nabla_\alpha\psi\left[\theta_a\right]\right)^{ij}\right) + \sum_{b\alpha\beta} \frac{\partial\mathcal{L}}{\partial \mathcal{F}^b_{A\alpha\beta}} \left[\overrightarrow{\theta}_a, \mathcal{F}_{A\alpha\beta}\right]^b = 0$

By taking the derivative with respect to the initial variables we check that the partial derivatives are tensors, with respect to the vector bundles : $\sum_i \frac{\partial\mathcal{L}}{\partial\psi^{ij}}\mathbf{f}^j, \sum_i \frac{\partial\mathcal{L}}{\partial\nabla_\alpha\psi^{ij}}\mathbf{f}^j, \sum_a \frac{\partial L}{\partial \mathcal{F}^a_{A\alpha\beta}} \overrightarrow{\theta}^a$ with $\overrightarrow{\theta}^a$ the basis vector of the dual of T_1U : $\overrightarrow{\theta}^a \left(\overrightarrow{\theta}_b\right) = \delta^a_b$

10.2 Covariance

In a change of charts with the jacobian : $J = \left[J^\alpha_\beta\right] = \left[\frac{\partial\widetilde{\xi}^\alpha}{\partial\xi^\beta}\right]$ and $K = J^{-1}$ the 4-form on M which defines the action changes as :
$L \det\left[P\right] d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3 = \widetilde{L} \det\left[\widetilde{P}\right] d\widetilde{\xi}^0 \wedge d\widetilde{\xi}^1 \wedge d\widetilde{\xi}^2 \wedge d\widetilde{\xi}^3$
and because :
$\det\left[\widetilde{P}\right] d\widetilde{\xi}^0 \wedge d\widetilde{\xi}^1 \wedge d\widetilde{\xi}^2 \wedge d\widetilde{\xi}^3 = \det\left[P\right] d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$
the scalar lagrangian should be invariant.

The variables change as :
ψ^{ij} do not change
The covariant derivatives are one form :
$\nabla_\alpha\psi^{ij} \to \widetilde{\nabla_\alpha\psi^{ij}} = \sum_\beta K^\beta_\alpha \nabla_\beta\psi^{ij}$
P are vectors, but their components are functions :
$P^\alpha_i \to \widetilde{P}^\alpha_i = \sum_\gamma J^\alpha_\gamma P^\gamma_i$
$\widetilde{\partial_\beta P^\alpha_i} = \frac{\partial}{\partial\widetilde{\xi}^\beta} \left(\sum_\gamma J^\alpha_\gamma\left(\xi\right) P^\gamma_i\left(\xi\right)\right) = \sum_\gamma \left(\frac{\partial}{\partial\widetilde{\xi}^\beta} J^\alpha_\gamma\left(\xi\right)\right) P^\gamma_i\left(\xi\right) + J^\alpha_\gamma\left(\xi\right) \frac{\partial}{\partial\widetilde{\xi}^\beta} P^\gamma_i\left(\xi\right)$
$\widetilde{\partial_\beta P^\alpha_i} = \sum_{\gamma\eta} \left(\partial_\eta J^\alpha_\gamma\right) K^\eta_\beta P^\gamma_i + \left(\left(\partial_\eta P^\gamma_i\right) J^\alpha_\gamma K^\eta_\beta\right)$
The potentials are 1-form :
$G^a_\alpha \to \widetilde{G}^a_\alpha = \sum_\beta K^\beta_\alpha G^a_\beta$
$\grave{A}^a_\alpha \to \widetilde{\grave{A}}^a_\alpha = \sum_\beta K^\beta_\alpha \grave{A}^a_\beta$
The strengths of the fields are 2-forms :
$\mathcal{F}^a_{G\alpha\beta} \to \widetilde{\mathcal{F}}^a_{G\alpha\beta} = \sum_{\gamma\eta} K^\gamma_\alpha K^\eta_\beta \mathcal{F}^a_{G\gamma\eta}$
$\mathcal{F}^a_{A\alpha\beta} \to \widetilde{\mathcal{F}}^a_{A\alpha\beta} = \sum_{\gamma\eta} K^\gamma_\alpha K^\eta_\beta \mathcal{F}^a_{A\gamma\eta}$

So we have the identity, for any change of chart :
$\mathcal{L}\left(P_i^\alpha, \psi^{ij}, G_\alpha^a, \partial_\beta P_i^\alpha, \nabla_\alpha \psi^{ij}, \mathcal{F}_{A\alpha\beta}^a, \mathcal{F}_{G\alpha\beta}^a\right) = \widetilde{\mathcal{L}}\left(\widetilde{P_i^\alpha}, \widetilde{\psi^{ij}}, \widetilde{G_\alpha^a}, \widetilde{\partial_\beta P_i^\alpha}, \widetilde{\nabla_\alpha \psi^{ij}}, \widetilde{\mathcal{F}_{A\alpha\beta}^a}, \widetilde{\mathcal{F}_{G\alpha\beta}^a}\right)$
$= \mathcal{L}(\sum_\gamma J_\gamma^\alpha P_i^\gamma, \psi^{ij}, \sum_\beta K_\alpha^\beta G_\beta^a, \sum_{\gamma\eta}\left(\partial_\eta J_\gamma^\alpha\right) K_\beta^\eta P_i^\gamma + \left(\left(\partial_\eta P_i^\gamma\right) J_\gamma^\alpha K_\beta^\eta\right),$
$\sum_\beta K_\alpha^\beta \nabla_\beta \psi^{ij}, \sum_{\gamma\eta} K_\alpha^\gamma K_\beta^\eta \mathcal{F}_{G\gamma\eta}^a, \sum_{\gamma\eta} K_\alpha^\gamma K_\beta^\eta \mathcal{F}_{A\gamma\eta}^a)$

If we take the derivative of this identity with respect to $\left(\partial_\eta J_\mu^\lambda\right)$:
$0 = \sum_{i\alpha\beta} \frac{\partial \mathcal{L}}{\partial \partial_\beta P_i^\alpha} \sum_{\gamma\eta} K_\beta^\eta P_i^\gamma \delta_\lambda^\alpha \delta_\gamma^\mu = \sum_{\alpha\beta\eta i} \frac{\partial \mathcal{L}}{\partial \partial_\beta P_i^\lambda} P_i^\mu K_\beta^\eta$
take $J_\mu^\lambda = \delta_\mu^\lambda \Rightarrow K_\beta^\eta = \delta_\beta^\eta$
$\sum_i \frac{\partial \mathcal{L}}{\partial \partial_\eta P_i^\lambda} P_i^\mu = 0$
$\forall \alpha, \beta, \gamma : \sum_i \frac{\partial \mathcal{L}}{\partial \partial_\alpha P_i^\beta} P_i^\gamma = 0$
by product with $P_\gamma^{j\prime}$ and summation : $\forall \alpha, \beta, j : \frac{\partial L}{\partial \partial_\alpha P_j^\beta} = 0$
and as we had :
$\forall a, \alpha : \sum_{\beta i} \frac{\partial L}{\partial \partial_\alpha P_i^\beta} \left([P]\left[\kappa_a\right]\right)_i^\beta = -\frac{\partial L}{\partial G_\alpha^a} \Rightarrow \forall a, \alpha : \frac{\partial L}{\partial G_\alpha^a} = 0$

The derivative with respect to J_μ^λ :
$\sum_{i\alpha} \frac{\partial \mathcal{L}}{\partial P_i^\alpha} \sum_\gamma P_i^\gamma \delta_\alpha^\lambda \delta_\gamma^\mu + \sum_{i\alpha} \frac{\partial \mathcal{L}}{\partial \operatorname{Re} \nabla_\alpha \psi^{ij}} \sum_\beta \left(\frac{\partial}{\partial J_\mu^\lambda} K_\alpha^\beta\right) \operatorname{Re} \nabla_\beta \psi^{ij}$
$+ \sum_{i\alpha} \frac{\partial \mathcal{L}}{\partial \operatorname{Im} \nabla_\alpha \psi^{ij}} \sum_\beta \left(\frac{\partial}{\partial J_\mu^\lambda} K_\alpha^\beta\right) \operatorname{Im} \nabla_\beta \psi^{ij}$
$+ \sum_{a\alpha\beta} \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{G\alpha\beta}^a} \sum_{\gamma\eta} \left(\left(\frac{\partial}{\partial J_\mu^\lambda} K_\alpha^\gamma\right) K_\beta^\eta + K_\alpha^\gamma \frac{\partial}{\partial J_\mu^\lambda} K_\beta^\eta\right) \mathcal{F}_{G\gamma\eta}^a$
$+ \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{A\alpha\beta}^a} \sum_{\gamma\eta} \left(\left(\frac{\partial}{\partial J_\mu^\lambda} K_\alpha^\gamma\right) K_\beta^\eta + K_\alpha^\gamma \frac{\partial}{\partial J_\mu^\lambda} K_\beta^\eta\right) \mathcal{F}_{A\gamma\eta}^a = 0$
with $\frac{\partial}{\partial J_\mu^\lambda} K_\alpha^\beta = -K_\lambda^\beta K_\alpha^\mu$
$\sum_{i\alpha} \frac{\partial \mathcal{L}}{\partial P_i^\lambda} P_i^\mu + 2 \operatorname{Re} \sum_{ij\alpha} \frac{\partial \mathcal{L}}{\partial \nabla_\alpha \psi^{ij}} \sum_\beta \left(-K_\lambda^\beta K_\alpha^\mu\right) \nabla_\beta \psi^{ij}$
$+ \sum_{a\alpha\beta} \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{G\alpha\beta}^a} \sum_{\gamma\eta} \left(\left(\left(-K_\lambda^\gamma K_\alpha^\mu\right)\right) K_\beta^\eta + K_\alpha^\gamma \left(-K_\lambda^\eta K_\beta^\mu\right)\right) \mathcal{F}_{G\gamma\eta}^a$
$+ \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{A\alpha\beta}^a} \sum_{\gamma\eta} \left(\left(\left(-K_\lambda^\gamma K_\alpha^\mu\right)\right) K_\beta^\eta + K_\alpha^\gamma \left(-K_\lambda^\eta K_\beta^\mu\right)\right) \mathcal{F}_{A\gamma\eta}^a = 0$
Let us take $J_\mu^\lambda = \delta_\mu^\lambda \Rightarrow K_\mu^\lambda = \delta_\mu^\lambda$
$\sum_i \frac{\partial \mathcal{L}}{\partial P_i^\lambda} P_i^\mu - 2 \sum_{i\alpha} \operatorname{Re} \frac{\partial \mathcal{L}}{\partial \nabla_\mu \psi^{ij}} \nabla_\lambda \psi^{ij} - \sum_{a\eta} \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{G\mu\eta}^a} \mathcal{F}_{G\lambda\eta}^a - \sum_{a\gamma} \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{G\gamma\mu}^a} \mathcal{F}_{G\gamma\lambda}^a - \sum_{a\eta} \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{A\mu\eta}^a} \mathcal{F}_{A\lambda\eta}^a -$
$\sum_{a\gamma} \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{A\gamma\mu}^a} \mathcal{F}_{A\gamma\lambda}^a = 0$
that is :
$\forall \alpha, \beta : 2 \sum_{ij} \operatorname{Re}\left(\frac{\partial \mathcal{L}}{\partial \nabla_\beta \psi^{ij}} \nabla_\alpha \psi^{ij}\right) + 2 \sum_{a\gamma} \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{G\beta\gamma}^a} \mathcal{F}_{G\alpha\gamma}^a + \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{A\beta\gamma}^a} \mathcal{F}_{A\alpha\gamma}^a = \sum_i \frac{\partial \mathcal{L}}{\partial P_i^\alpha} P_i^\beta$

The derivative with respect to the initial variables gives the identities :
$\frac{\partial L}{\partial P_i^\alpha} = \sum_\beta J_\alpha^\beta \frac{\partial \widetilde{L}}{\partial \widetilde{P_i^\beta}}$
$\frac{\partial L}{\partial \nabla_\beta \psi^{ij}} = \sum_\gamma K_\gamma^\beta \frac{\partial \widetilde{L}}{\partial \widetilde{\nabla_\gamma \psi^{ij}}};$
$\frac{\partial L}{\partial \mathcal{F}_{A\alpha\beta}^a} = \sum_{\gamma\eta} K_\gamma^\alpha K_\eta^\beta \frac{\partial L}{\partial \widetilde{\mathcal{F}_{A\gamma\eta}^a}}; \frac{\partial L}{\partial \mathcal{F}_{G\alpha\beta}^a} = \sum_{\gamma\eta} K_\gamma^\alpha K_\eta^\beta \frac{\partial L}{\partial \widetilde{\mathcal{F}_{G\gamma\eta}^a}}$
which show that the corresponding quantities are tensors.

All units should be deduced from units of length and time. A change of unit for the coordinates is expressed by taking $\widetilde{\xi}^{\alpha} = k\xi^{\alpha}, J = k\left[\delta^{\alpha}_{\beta}\right]$.The previous identity reads:

$\forall k : 2\left[k^2 - 1\right]\sum_{ij} \mathrm{Re}\, \frac{\partial \mathcal{L}}{\partial \nabla_{\mu}\psi^{ij}} \nabla_{\lambda}\psi^{ij} + 2\left(k-1\right)\sum_{a\gamma}\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}^{a}_{G\mu\gamma}}\mathcal{F}^{a}_{G\lambda\gamma} + \frac{\partial \mathcal{L}}{\partial \mathcal{F}^{a}_{A\mu\gamma}}\mathcal{F}^{a}_{A\lambda\gamma}\right) = 0$

$\Rightarrow k = 1$

So the quantities must be dimensionless.

10.3 Conclusion

i) The potentials À,G, and the derivatives $\partial_{\beta}P^{\alpha}_{i}$ do not figure explicitely, the derivatives of the potential À,G factor in the strength.

The lagrangian is a function :

$$\mathcal{L} = \mathcal{L}\left(\psi, \nabla_{\alpha}\psi_{p}, P^{\alpha}_{i}, \mathcal{F}_{G\alpha\beta}, \mathcal{F}_{A\alpha\beta}\right) \tag{48}$$

ii) We have the identities

$\forall a = 1..6 :$

$2\sum_{ij} \mathrm{Re}\left(\frac{\partial \mathcal{L}}{\partial \psi^{ij}}\left[\gamma C\left(\kappa_a\right)\right]\psi + \sum_{\alpha}\frac{\partial \mathcal{L}}{\partial \nabla_{\alpha}\psi^{ij}}\left[\gamma C\left(\kappa_a\right)\right]\nabla_{\alpha}\psi\right)^{ij}$

$-\sum_{i\alpha}\frac{\partial \mathcal{L}}{\partial P^{\alpha}_{i}}\left(\left[P\right]\left[\kappa_a\right]\right)^{\alpha}_{i} + \sum_{b\alpha\beta}\frac{\partial \mathcal{L}}{\partial \mathcal{F}^{b}_{G\alpha\beta}}\left[\overrightarrow{\kappa}_{a}, \mathcal{F}_{G\alpha\beta}\right]^{b} = 0$

$\forall a = 1..m :$

$2\sum_{ij} \mathrm{Re}\left(\frac{\partial \mathcal{L}}{\partial \psi^{ij}}\psi\left[\theta_a\right] + \sum_{\alpha}\frac{\partial \mathcal{L}}{\partial \nabla_{\alpha}\psi^{ij}}\nabla_{\alpha}\psi\left[\theta_a\right]\right)^{ij} + \sum_{b\alpha\beta}\frac{\partial \mathcal{L}}{\partial \mathcal{F}^{b}_{A\alpha\beta}}\left[\overrightarrow{\theta}_{a}, \mathcal{F}_{A\alpha\beta}\right]^{b} = 0$

$\forall \alpha, \beta : 2\sum_{ij} \mathrm{Re}\left(\frac{\partial \mathcal{L}}{\partial \nabla_{\beta}\psi^{ij}}\nabla_{\alpha}\psi^{ij}\right) + 2\sum_{a\gamma}\frac{\partial \mathcal{L}}{\partial \mathcal{F}^{a}_{G\beta\gamma}}\mathcal{F}^{a}_{G\alpha\gamma} + \frac{\partial \mathcal{L}}{\partial \mathcal{F}^{a}_{A\beta\gamma}}\mathcal{F}^{a}_{A\alpha\gamma} = \sum_{i}\frac{\partial \mathcal{L}}{\partial P^{\alpha}_{i}}P^{\beta}_{i}$

These identities are minimal necessary conditions for the lagrangian : the calculations could be continued to higher derivatives. They do not depend on the signature.

iii) The following quantities are tensors :

$\sum_{ij}\frac{\partial L}{\partial \psi^{ij}}\mathbf{e}^{i}\otimes\mathbf{f}^{i}$,

$\sum_{\alpha}\frac{\partial L}{\partial P^{\alpha}_{i}}d\xi^{\alpha}\otimes\varepsilon_{i}$;

$\sum_{\alpha}\frac{\partial L}{\partial \nabla_{\alpha}\psi^{ij}}\partial\xi_{\alpha}\otimes\mathbf{e}^{i}\otimes\mathbf{f}^{i}$

$\sum_{\alpha\beta}\frac{\partial L}{\partial \mathcal{F}^{a}_{A\alpha\beta}}\partial\xi_{\alpha}\wedge\partial\xi_{\beta}\otimes\overrightarrow{\theta}^{a}$;

$\sum_{\alpha\beta}\frac{\partial L}{\partial \mathcal{F}^{a}_{G\alpha\beta}}\partial\xi_{\alpha}\wedge\partial\xi_{\beta}\otimes\overrightarrow{\kappa}^{a}$ and similarly $\sum_{\alpha\beta}\upsilon\left(\frac{\partial L}{\partial \mathcal{F}_{r\alpha\beta}}, \frac{\partial L}{\partial \mathcal{F}_{w\alpha\beta}}\right)\partial\xi_{\alpha}\wedge\partial\xi_{\beta}$

Notice that these quantities, when det$[P]$ is added to L, are no longer covariant.

11 CONTINUOUS MODELS

We have now all the material necessary to study the models of interacting particles and fields. As said in the preliminaries we have to distinguish continuous and discontinuous models. We start with the continuous models.

The system is comprised of particles and fields interacting in a bounded area Ω of M. The processes are assumed to be continuous, which means here that :

- the particles keep their physical characteristic, represented here by their type p. In our model this does not imply a specific constraint on the covariant derivative of ψ.

- we do not consider collisions, so the trajectories do not cross and, for each type of particles, they are fully defined by a section $\sigma_{wp} \in \mathfrak{X}(P_G)$ and an origin $x \in \Omega_3(0)$. Actually the mathematical model is still valid if the trajectories cross (each of the section σ_w is distinct) but of course physically it should involve other processes.

- the fields can be represented by smooth sections

$(\mathcal{F}_G, \mathcal{F}_A) \in \mathfrak{X}(Q[T_1 Spin \times T_1 U, Ad \times Ad])$

The principle of least action applies and defines the solutions for an equilibrium of the system. However its implementation raises some issues, which are one face of the duality particles / fields.

11.1 The point particle issue

11.1.1 Particle moving in known fields

The system comprises one particle. The fields **G,À** and the tetrad P are supposed to be known over the trajectory. We focus on the state of the particle, the variables are the trajectory $\mu_O(t) = \varphi_M(t, x(t))$ in the chart, the state $\psi(t)$ and their derivatives along the trajectories , they are all functions defined over $[0, T]$.

The lagrangian is then : $L_1\left(\psi, \frac{d\psi}{dt}\right)$ and the action :

$\int_0^T L_1\left(\psi, \frac{d\psi}{dt}\right)\sqrt{|g(V,V)|}dt$ with $V = \frac{d\mu_O}{dt}$.

The integral is taken along the trajectory, with the metric $\sqrt{|g(V,V)|}$ induced on them by g. For covariance reasons the coordinates of $\mu_O(t)$ cannot appear explicitly in the scalar lagrangian L_1, but we need V as a variable.

If the fields induced by the particle are negligible we can expect a solution, but it will be at best expressed as general conditions that the trajectories must meet.

The main example is the trajectory of free particles. With the simple lagrangian $L_1 = 1$ and the Lévy-Civita connection one finds that the trajectory must be a geodesic, and there is a unique geodesic passing through any point x with a given tangent V(0). But the equation does not give by itself the coordinates of the geodesic (which require the knowledge of g) or the value of the field.

For the electromagnetic field, if we know the value of the field and we neglect the field induced by the particle, we get similarly a solution,

$\nabla_u u = \mu_0 \frac{q}{mc} \sum_\alpha \mathcal{F}^{\alpha\beta} u_\beta$ with u = $\frac{c}{\sqrt{-\langle V,V\rangle}} V$

which gives a general definition of the type of trajectory knowing the field, the mass m and the charge q of the particle.

These equations, usually called the "equations of motion", come from the Euler-Lagrange equations with respect to the field.

If we want to account for the field induced by the particle we have a problem. As the field propagates, we need to know the field out of the trajectory. It could be computed by the more general model, and the results reintegrated in the single particle model. The resulting equation for the trajectory is known, for the electromagnetic field, as the "Lorentz-Dirac equation" (see Poisson and Quinn). The procedure is not simple, and there are doubts about the physical meaning of the equation itself (as can be expected it involves the derivative of u).

So the situation is not satisfying. However, from these classic examples, two results seem quite clear :

- the trajectories should belong to some family of curves, defined by the interactions
- the initial conditions, that is the beginning x of the curve and its initial tangent, should determine the curve in the family.

They are consistent with our description of the motions by sections of P_G and their associated vector fields.

11.1.2 The propagation of fields

If we consider a system without any particle, focus on the fields and aim at knowing their propagation in Ω, the variables are just the components of the tetrad P, the potentials G,$\grave{A}$ and their derivatives $\partial_\alpha P_i^\beta, \partial_\alpha G_\beta^a, \partial_\alpha \grave{A}_\beta^a$. The principle of least action prescribes a scalar lagrangian

$L_4\left(P_i^\alpha, G_{w\alpha}^a, G_{r\alpha}^a, \grave{A}_\alpha^a, \partial_\beta G_{r\alpha}^a, \partial_\beta G_{w\alpha}^a, \partial_\beta \grave{A}_\alpha^a\right)$ and the action $\int_\Omega L_4(m)\,\varpi_4$

The Euler-Lagrange equations then give general solutions which are matched to the initial conditions, that is the value of P and the fields on $\Omega_3(0)$, with their derivatives.

The classic examples are the Einstein equation of General Relativity (with the Lévy-Civita connection) :

$Ric_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}(R+\Lambda) = 0$

and the Maxwell equations :

$\sum_{\alpha\beta} \partial_\alpha \left(\mathcal{F}^{\alpha\beta}\sqrt{|\det P'|}\right) = 0$

with the lagrangian : $L_4 = \sum_{\alpha\beta} G g^{\alpha\beta} Ric_{\alpha\beta} + \mu_0 \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$

So the situation is better, but we must notice that it requires the knowledge of the fields and their derivatives over all $\Omega_3(0)$, which is not a simple requirement. As said before, the propagation at the speed of ligth is postulated, and introduced separately in the linearized approximation.

11.1.3 The general problem

Now we consider the most general problem, with N particles interacting together and with the fields.

The principle of least action gives a procedure, which applies at equilibrium, but, by itself, it does not tell us precisely what should be the action. The simplest solution is to consider the action :

$\int_\Omega L_4 \left(P_i^\alpha, G_{w\alpha}^a, G_{r\alpha}^a, \grave{A}_\alpha^a, \partial_\beta G_{r\alpha}^a, \partial_\beta G_{w\alpha}^a, \partial_\beta \grave{A}_\alpha^a\right) \varpi_4$

$+\sum_{p=1}^N \int_0^T L_1 \left(\psi_p, , \frac{d\psi_p}{dt}, P_i^\alpha, G_{w\alpha}^a, G_{r\alpha}^a, \grave{A}_\alpha^a, \frac{d}{dt}G_{r\alpha}^a, \frac{d}{dt}G_{w\alpha}^a, \frac{d}{dt}\grave{A}_\alpha^a\right) \sqrt{|g\left(V_p, V_p\right)|}dt$

where L_4 is the lagrangian of the previous problem, and the second lagrangian is evaluated along the trajectory of each of the particles. For covariance reason, the coordinates of the path of each particle cannot figure in L_1.

If the trajectories of the particle are known, a direct computation gives usually the field that they induce. This is useful for particles which are bounded (such as in condensed matter). But in the general case the induced field modifies the trajectories, even for a single particle, and we have several issues.

The mathematical ones are obvious : the fields and the particles are defined over domains which are manifolds of different dimensions, and moreover the trajectories of the particles are themselves a variable, which cannot be explicitely present in the lagrangian. Thus the Euler-Lagrange equations cannot be implemented. Let us discard at once the naive use of Dirac's function : this is just a formal way to rewrite the same integral without any added value, the coordinates cannot figure in the lagrangian.

However thanks to the formalism used here with the representation of vector fields through sections of vector bundles, and a generalization of distributions to vector bundle it is posssible to solve rigorously the problem by functional derivatives, as we will see.

These difficulties have physical roots. The concept of field is aimed at removing the idea of action at a distance, but, as the example of the motion of a single particle in its own field shows, it seems difficult to circumvent the direct consideration of mutual interactions between particles, which needs to identify separately each of them.

The practical solutions which have been implemented with the principle of least action have many variants, but share the same assumptions :

- they assume that the particles follow some kind of continuous trajectories and keep their physical characteristics (this condition adds usually a separated constraint)
- the trajectory is the key variable, but the model gives up the concept of point particle, replaced by some form of "density of particles".

These assumptions makes sense when we are close to the equilibrium, and we are concerned not by the by the behaviour of each individual particle but by global results about distinguished populations, measured as "cross sections" over an hypersurface. They share many characteristics with the model used in fluid mechanics. In the usual QM interpretation the density of particles can be seen as a "probability of presence", but these models are used in the classical picture, and actually the "state" of the particles is represented as sections of the vector bundle TM (with a constraint imposed by the mass), combined with a density function. So the density has a direct, classic interpretation.

The simplest solution is, assuming that the particles have the same physical characteristics, to take as key variable a density $\rho\varpi_4$. Then the application of the principle of least action as a 4 dimensional integral gives the equations relating the fields and the density of charge.

The classical examples are :
- the 2nd Maxwell equation in GR :
$\nabla^{\beta}\mathcal{F}_{\beta\alpha} = -\mu_0 J_\alpha \Leftrightarrow \mu_0 J^\alpha \sqrt{-\det g} = \sum_\beta \partial_\beta \left(\mathcal{F}^{\alpha\beta}\sqrt{-\det g}\right)$
with the current : $J = \rho(m)\, qu$ and the lagrangian
$L_4 = \mu_0 \sum_\alpha \grave{A}_\alpha J^\alpha + \frac{1}{2}\sum_{\alpha\beta} \mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}$
- the Einstein Equation in GR :
$Ric_{\alpha\beta} - \frac{1}{2}(R+\Lambda)\, g_{\alpha\beta} = \frac{8\pi G}{\sqrt{c}} T_{\alpha\beta}$
with the momentum energy tensor $T_{\alpha\beta} = \frac{\partial T}{\partial g^{\alpha\beta}} - \frac{1}{2} g_{\alpha\beta} T$
and the lagrangian $L_4 = T\left(g, z^i, z^i_\alpha\right) + \frac{\sqrt{c}}{8\pi G}(R+\Lambda)$
The distribution of "charges" is defined independantly, but it must met a conservation law. In the examples above we must have :
$\sum \partial_\alpha J^\alpha = 0; \nabla^\alpha T_{\alpha\beta} = 0$

The "Einstein-Vlasov" systems are also based on a distribution function f(m,p) depending on the localization m and the linear momentum p, which must follow a conservation law, expressed as a differential equation (the Vlasov equation). The particles are generaly assumed to have the same mass, so there is an additional constraint as above. When only the gravitational field is considered the particles follow geodesics, to which the conservation law is adjusted. These systems have been extensively studied for plasmas and astrophysics (see Andréasson).

This kind of model has been adjusted to Yang-Mills fields (Choquet-Bruhat) : the particles have different physical characteristics (similar to the vector ϕ seen previously), and must follow an additional conservation law given by $\nabla_V \phi = 0$ (the "Wong equation").

In all these solutions the 4 dimensional action, with a lagrangian adaptated to the fields considered, gives an equation relating the field and the distribution of charges.

Using the material exposed previously we will give the outlines of a model, used both for a single particle and for a distribution of particles interacting with the fields . For this we need first to adapt the lagrangian.

11.2 Perturbative lagrangian

In a perturbative approach, meaning close to the equilibrium, which are anyway the conditions in which the principle of least action applies, the lagrangian can be estimated by polynomials. The most natural way is to look for scalar products. We have three parts to consider : the fields only, the particles only, and the interactions.

11.2.1 Fields

For the fields we have the obvious actions :
$\int_\Omega \langle \mathcal{F}_G(m), \mathcal{F}_G(m)\rangle \varpi_4(m) = \int_\Omega \frac{1}{4}\sum_{\zeta\eta\lambda\mu=0}^{3} g^{\alpha\lambda} g^{\beta\mu}\left(\mathcal{F}^t_{r\alpha\beta}\mathcal{F}_{r\lambda\mu} - \mathcal{F}^t_{w\alpha\beta}\mathcal{F}_{w\lambda\mu}\right)\varpi_4(m)$
$\int_\Omega \langle \mathcal{F}_A(m), \mathcal{F}'_A(m)\rangle \varpi_4(m) = \int_\Omega \sum_{\alpha\beta\lambda\mu=0}^{3} g^{\alpha\lambda} g^{\beta\mu}\mathcal{F}^t_{A\alpha\beta}\mathcal{F}_{A\lambda\mu}\varpi_4(m)$
The scalar curvature can be considered for the gravitational field but, with a general connection **G,** it is no more difficult and more interesting to keep a general definition.

11.2.2 Particles

In a continuous model the type of particle p is assumed to be fixed, and so ψ_{0p} is constant along the trajectory. Notice that we do not require any other conservation law such as $\nabla_u \psi = 0$.

Single particle

In a system comprised of a single particle interacting with the fields, the part of the lagrangian related to the particle only is simply $\langle \psi_0, \psi_0 \rangle$ and the action must be accounted for the trajectory $\mu_P(t) = \Phi_V(t,x)$ for some compatible, admissible, vector field V:

$C_P \int_0^T \langle \psi_0, \psi_0 \rangle \sqrt{-\langle V, V \rangle} dt$

using the metric induced on any curve, with a constant real scalar C_P. By definition $\sqrt{-\langle V, V \rangle} = f(\Phi_V(t,x))$.

Distribution of particles

In a given system, each type of particle is assumed to be associated with a unique section of $Q\left[\widehat{E}_0 \otimes F, \vartheta\right]$ and as ψ_{0p} is fixed by the initial conditions, there are unique sections : $\sigma_p \in \mathfrak{X}(P_G), \Theta_p \in \mathfrak{X}(P_U)$ such that : $\psi_p(m) = \vartheta(\sigma_p, \Theta_p)\psi_{0p}$. So the vector field u_p is defined by the projection of σ_p on the homogeneous space, and it is the world line of the particle in its proper time :

$u_p = c\left(\left(2a_w^2 - 1\right)\varepsilon_0(m) + a_w \sum_{j=1}^3 w_{pj}\varepsilon_j(m)\right)$

The trajectories, in the time of the observer, are defined by a vector field V_p, the same for all the particles of the same type, and we have the relation : $V_p(m) = f_p(m)u_p$ with $\sqrt{-\langle V_p, V_p \rangle} = cf_p(m)$

So we have the natural action, for a particle of type p, with the scalar product $\langle \psi_p, \psi_p \rangle$:

$\int_0^T \left\langle \psi_p\left(\Phi_{V_p}(t,x)\right), \psi_p\left(\Phi_{V_p}(t,x)\right)\right\rangle \sqrt{-\langle V_p, V_p \rangle} dt$

By definition :

$\psi_p\left(\Phi_{V_p}(t,x)\right) = \psi_p\left(\Phi_{u_p}(\tau_p, x)\right) = \vartheta\left(\sigma\left(\Phi_{u_p}(\tau_p,x)\right), \Theta\left(\Phi_{u_p}(\tau_p,x)\right)\right)\psi_{0p}$

thus :

$\left\langle \psi_p\left(\Phi_{V_p}(t,x)\right), \psi_p\left(\Phi_{V_p}(t,x)\right)\right\rangle = \langle \psi_{0p}, \psi_{0p} \rangle$

$\int_0^T \left\langle \psi_p\left(\Phi_{V_p}(t,x)\right), \psi_p\left(\Phi_{V_p}(t,x)\right)\right\rangle f_p\left(\Phi_{V_p}(t,x)\right) dt = \int_0^T \langle \psi_{0p}, \psi_{0p} \rangle f_p\left(\Phi_{V_p}(t,x)\right) dt$

$= \langle \psi_{0p}, \psi_{0p} \rangle \int_0^T f_p\left(\Phi_{V_p}(t,x)\right) dt$

$f\left(\Phi_{V_p}(t,x)\right) = \frac{d\tau_p}{dt}$

So : $\langle \psi_{0p}, \psi_{0p} \rangle \int_0^T f_p\left(\Phi_{V_p}(t,x)\right) dt = \langle \psi_{0p}, \psi_{0p} \rangle T_p(x)$ where $T_p(x)$ is the proper time at the arrival point in $\Omega_3(T)$.

11.2.3 Interactions

We have several choices to make :

- how to introduce the derivative of ψ : it should involve the covariant derivative
- which scalar function to consider : it should involve the scalar product $\langle \psi, \nabla\psi \rangle$ but must be at least a real quantity
- how to absorb the α in the covariant derivative : it can be done by taking the Dirac operator, which has been defined for the gravitational field, or by choosing the integration over a vector field.

Proposition 19 *The scalar product $\langle \psi, \nabla_\alpha \psi \rangle$ is purely imaginary.*

Proof. $[\nabla_\alpha \psi] = [\partial_\alpha \psi] + [\psi] \left[\grave{A}_\alpha \right] - \frac{i}{2} \sum_{a=1}^{3} G^a_{w\alpha} \gamma_a \gamma_0 [\psi] + G_{r\alpha} \widetilde{\gamma}_a [\psi]$

$\langle \psi, \nabla_\alpha \psi \rangle = Tr [\psi]^* \gamma_0 [\partial_\alpha \psi] + Tr [\psi]^* \gamma_0 [\psi] \left[\grave{A}_a \right] - \frac{i}{2} \sum_{a=1}^{3} G^a_{w\alpha} Tr [\psi]^* \gamma_0 \gamma_a \gamma_0 [\psi] + G_{r\alpha} Tr [\psi]^* \gamma_0 \widetilde{\gamma}_a [\psi]$

$= Tr [\psi]^* \gamma_0 [\partial_\alpha \psi] + Tr [\psi]^* \gamma_0 [\psi] \left[\grave{A}_a \right] - \frac{i}{2} \sum_{a=1}^{3} -G^a_{w\alpha} Tr [\psi]^* \gamma_a [\psi] + G_{r\alpha} Tr [\psi]^* \gamma_0 \widetilde{\gamma}_a [\psi]$

$Tr [\psi]^* \gamma_a [\psi], Tr [\psi]^* \gamma_0 \widetilde{\gamma}_a [\psi]$ are real, $Tr [\psi]^* \gamma_0 [\psi] [\theta_a]$ is imaginary :

$\overline{\left(Tr [\psi]^* \gamma_a [\psi] \right)} = Tr \left([\psi]^* \gamma_a [\psi] \right)^* = Tr [\psi]^* \gamma_a [\psi]$

$\overline{Tr [\psi]^* \gamma_0 \widetilde{\gamma}_a [\psi]} = Tr \left([\psi]^* \gamma_0 \widetilde{\gamma}_a [\psi] \right)^* = Tr [\psi]^* \left(\gamma_0 \widetilde{\gamma}_a \right)^* [\psi] = Tr [\psi]^* \gamma_0 \widetilde{\gamma}_a [\psi]$

$\overline{Tr [\psi]^* \gamma_0 [\psi] [\theta_a]} = Tr \left([\psi]^* \gamma_0 [\psi] [\theta_a] \right)^* = Tr [\theta_a]^* [\psi]^* [\gamma_0] [\psi]$

$= -Tr [\theta_a] [\psi]^* [\gamma_0] [\psi] = -Tr [\psi]^* [\gamma_0] [\psi] [\theta_a]$

$\langle \psi, \psi \rangle = \langle \psi_0, \psi_0 \rangle \Rightarrow \langle \psi, \partial_\alpha \psi \rangle + \langle \partial_\alpha \psi, \psi \rangle = 0$ ■

So :

$\mathrm{Im} \langle \psi, \nabla_\alpha \psi \rangle$

$= \frac{1}{i} Tr [\psi]^* \gamma_0 [\partial_\alpha \psi] + \frac{1}{i} Tr [\psi]^* \gamma_0 [\psi] \left[\grave{A}_a \right] + \frac{1}{2} \sum_{a=1}^{3} G^a_{w\alpha} Tr [\psi]^* \gamma_a [\psi] - G_{r\alpha} Tr [\psi]^* \gamma_0 \widetilde{\gamma}_a [\psi]$

$$\mathrm{Im} \langle \psi, \nabla_\alpha \psi \rangle = \frac{1}{i} \left(\langle \psi, \partial_\alpha \psi \rangle + \left\langle \psi_0, [\psi_0] \left[\grave{A}_a \right] \right\rangle \right) - \frac{1}{2} \sum_{a=1}^{3} \left(G^a_{r\alpha} \langle \psi, \widetilde{\gamma}_a \psi \rangle + G^a_{w\alpha} \langle \psi, \gamma_a \gamma_0 \psi \rangle \right) \tag{49}$$

with $\left\langle \psi, [\psi] \left[\grave{A}_a \right] \right\rangle = \left\langle \vartheta (\sigma, \Theta) \psi_0, \vartheta (\sigma, \Theta) \psi_0 \left[\grave{A}_a \right] \right\rangle = \left\langle \psi_0, \psi_0 \left[\grave{A}_a \right] \right\rangle$

In a given system, each type of particle is assumed to be associated with a unique admissible vector field u_p such that the world line of a particle of this type is defined by the flow of u_p and its origin x. If we assume that the particles stay of the same type all over their trajectory, then we can take the covariant derivatives along this vector field. And to account for the trajectory as measured by the observers, we have to take $V_p (m) = f_p (m) u_p (m)$:

The action is then for a particle of type p :

$\int_0^T \mathrm{Im} \langle \psi_p, \nabla_{V_p} \psi_p \rangle dt = \int_0^T \mathrm{Im} \langle \psi_p, \nabla_{f u_p} \psi_p \rangle dt = \int_0^T f \left(\Phi_{V_p} (t, x) \right) \mathrm{Im} \langle \psi_p, \nabla_{u_p} \psi_p \rangle dt$

$\mathrm{Im} \langle \psi_p, \nabla_{u_p} \psi_p \rangle$

$= \frac{1}{i} \left(\left\langle \psi_p, \frac{d\psi_p}{d\tau_p} \right\rangle + \left\langle \psi_{0p}, [\psi_{0p}] \left[\widehat{\grave{A}} \right] \right\rangle \right) - \frac{1}{2} \sum_{a=1}^{3} \left(\widehat{G}^a_r \langle \psi_p, \widetilde{\gamma}_a \psi_p \rangle + \widehat{G}^a_w \langle \psi_p, \gamma_a \gamma_0 \psi_p \rangle \right)$

with :

$\widehat{\grave{A}}^a = \sum_\alpha u^\alpha_p \grave{A}^a_\alpha$

$\widehat{G}^a_w = \sum_\alpha u^\alpha_p G^a_{w\alpha}$

$\widehat{G}^a_r = \sum_\alpha u^\alpha_p G^a_{r\alpha}$

Moreover we have, in a given system with fixed initial conditions : $\psi_p (m) = \vartheta (\sigma_p, \Theta_p) \psi_{0p}$ and $\sigma_p = \sigma_{wp} \cdot \sigma_{rp}$ so it is logical to single out $\psi_{rp} = \vartheta (\sigma_{rp}, \Theta_p) \psi_{0p}$ that is :

$$\psi_p = \vartheta (\sigma_{wp}, 1) \psi_{rp} = \gamma C (\sigma_{wp}) \psi_{rp} \tag{50}$$

We can then give a more precise formulation of $\mathrm{Im} \langle \psi, \nabla_\alpha \psi \rangle$.

i) $\nabla_\alpha \psi_p = \sum_{ij} \left(\partial_\alpha \psi^{ij} + \sum_k \sum_{a=1}^{6} \left[\gamma C\left(G_\alpha^a\right)\right]_k^i [\psi]_j^k + [\psi]_k^i \left[\grave{A}_a\right]_j^k \right) e_i \otimes f_j$
$= \sum_{ij} \left(\sum_{kl} \left[\gamma C\left(\sigma_{wp}\right)\right]_k^i \left[\gamma C\left(\nabla_\alpha^G \sigma_{wp}\right)\right]_l^k [\psi_{rp}]_j^k + \left[\gamma C\left(\sigma_{wp}\right)\right]_k^i \left[\left(\nabla_\alpha^A \psi_r\right)\right]_j^k \right) e_i \otimes f_j$

$$\nabla_\alpha \psi_p = \gamma C\left(\sigma_{wp}\right)\left(\gamma C\left(\nabla_\alpha^G \sigma_{wp}\right)\psi_{rp} + \nabla_\alpha^A \psi_{rp}\right) \tag{51}$$

ii) $\nabla_\alpha^G \sigma_{wp}$ has been computed previously : $\nabla_\alpha^G \sigma_{wp} = \upsilon\left(X_\alpha, Y_\alpha\right)$ with

Notation 20 $X_\alpha = -\frac{1}{2} j\left(w\right)\partial_\alpha w + \left(a_w^2 - \frac{1}{2} j\left(w\right) j\left(w\right)\right) G_{r\alpha} + a_w j\left(w\right) G_{w\alpha}$

Notation 21 $Y_\alpha = -w\partial_\alpha a_w + a_w \partial_\alpha w + \left[a_w^2 - 1 - a_w j\left(w\right) + \frac{1}{4} j\left(w\right) j\left(w\right)\right] G_{r\alpha} + \left[1 - \frac{3}{4} j\left(w\right) j\left(w\right)\right] G_{w\alpha}$

$\gamma C\left(\nabla_\alpha^G \sigma_{wp}\right) = -i\frac{1}{2}\sum_{a=1}^{3}\left(\left[Y_\alpha\right]^a \gamma_a \gamma_0 + \left[X_\alpha\right]^a \widetilde{\gamma}_a\right)$
$\gamma C\left(\nabla_\alpha^G \sigma_{wp}\right)\psi_{rp} = -i\frac{1}{2}\sum_{a=1}^{3}\left(\left[Y_\alpha\right]^a \gamma_a \gamma_0 \psi_{rp} + \left[X_\alpha\right]^a \widetilde{\gamma}_a \psi_{rp}\right)$
$\nabla_\alpha \psi_p = \left[\gamma C\left(\sigma_{wp}\right)\right]\left(\nabla_\alpha^A \psi_{rp} - i\frac{1}{2}\sum_{a=1}^{3}\left(\left[Y_\alpha\right]^a \gamma_a \gamma_0 \psi_{rp} + \left[X_\alpha\right]^a \widetilde{\gamma}_a \psi_{rp}\right)\right)$
$\langle \psi_p, \nabla_\alpha \psi_p \rangle$
$= \left\langle \gamma C\left(\sigma_{wp}\right)\psi_{rp}, \gamma C\left(\sigma_{wp}\right)\left(\gamma C\left(\nabla_\alpha^G \sigma_{wp}\right)\psi_{rp} + \nabla_\alpha^A \psi_{rp}\right)\right\rangle$
$= \left\langle \psi_{rp}, \left(\gamma C\left(\nabla_\alpha^G \sigma_{wp}\right)\psi_{rp} + \nabla_\alpha^A \psi_{rp}\right)\right\rangle$
$= \left\langle \psi_{rp}, \gamma C\left(\nabla_\alpha^G \sigma_{wp}\right)\psi_{rp}\right\rangle + \left\langle \psi_{rp}, \nabla_\alpha^A \psi_{rp}\right\rangle$
iii) $\left\langle \psi_{rp}, \gamma C\left(\nabla_\alpha^G \sigma_{wp}\right)\psi_{rp}\right\rangle$
$= -i\frac{1}{2}\sum_{a=1}^{3}\left[Y_\alpha\right]^a Tr\left(\left[\psi_{rp}\right]^* \left[\gamma_0\right]\left[\gamma_a\right]\left[\gamma_0\right]\left[\psi_{rp}\right]\right) + \left[X_\alpha\right]^a Tr\left(\left[\psi_{rp}\right]^* \left[\gamma_0\right]\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)$
$= -i\frac{1}{2}\sum_{a=1}^{3}\left[Y_\alpha\right]^a Tr\left(\left[\psi_{rp}\right]^* \left[\gamma_a\right]\left[\psi_{rp}\right]\right) + \left[X_\alpha\right]^a Tr\left(\left[\psi_{rp}\right]^* \left[\gamma_0\right]\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)$
Because $\psi_{rp} \in E_0 \otimes F$:
$\gamma_0 \psi_{rp} = \psi_{rp}; \psi_{rp}^* \gamma_0 = \psi_{rp}^*$
$\left[\psi_{rp}\right]^* \left[\gamma_a\right]\left[\psi_{rp}\right] = 0$
$\left\langle \psi_{rp}, \gamma C\left(\nabla_\alpha^G \sigma_{wp}\right)\psi_{rp}\right\rangle = -i\frac{1}{2}\sum_{a=1}^{3}\left[X_\alpha\right]^a Tr\left(\left[\psi_{rp}\right]^* \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)$
$Tr\left[\psi_{rp}\right]^* \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]$ is real:
$Tr\left[\psi_{rp}\right]^* \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right] = \overline{\left(Tr\left[\psi_{rp}\right]^* \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)^t}$
$= Tr\left[\psi_{rp}\right]^t \left[\widetilde{\gamma}_a\right]^t \overline{\left[\psi_{rp}\right]} = \overline{Tr\left[\psi_{rp}\right]^* \left[\widetilde{\gamma}_a\right]^* \left[\psi_{rp}\right]} = Tr\left[\psi_{rp}\right]^* \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]$

Notation 22 $\varkappa_p = \left(Tr\left[\psi_{rp}\right]^* \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)_{a=1}^{3} = \left\langle \psi_{rp}, \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right\rangle \in \mathbb{R}^3$

$\left\langle \psi_{rp}, \gamma C\left(\nabla_\alpha^G \sigma_{wp}\right)\psi_{rp}\right\rangle = -i\frac{1}{2}\varkappa_p^t X_\alpha$
iv) $\left\langle \psi_{rp}, \nabla_\alpha^A \psi_{rp}\right\rangle = Tr\left[\psi_{rp}\right]^* \gamma_0 \left[\nabla_\alpha^A \psi_{rp}\right]$
$= Tr\left[\psi_{rp}\right]^* \left[\nabla_\alpha^A \psi_{rp}\right] = Tr\left[\psi_{rp}\right]^* \left[\partial_\alpha \psi_{rp}\right] + \sum_a \grave{A}_\alpha^a Tr\left[\psi_{rp}\right]^* \left[\psi_{rp}\right]\left[\theta_0\right]$
$Tr\left[\psi_{rp}^*\right]\left[\psi_{rp}\right]\left[\theta_a\right]$ is purely imaginary :
$Tr\left[\psi_{rp}^*\right]\left[\psi_{rp}\right]\left[\theta_a\right] = \overline{\left(Tr\left[\psi_{rp}^*\right]\left[\psi_{rp}\right]\left[\theta_a\right]\right)^t} = Tr\left[\theta_a\right]^t \left[\psi_{rp}\right]^t \overline{\left[\psi_{rp}\right]}$
$= Tr\overline{\left[\theta_a\right]^* \left[\psi_{rp}\right]^* \left[\psi_{rp}\right]} = -Tr\overline{\left[\theta_a\right]\left[\psi_{rp}\right]^* \left[\psi_{rp}\right]} = -Tr\overline{\left[\psi_{rp}\right]^* \left[\psi_{rp}\right]\left[\theta_a\right]}$
so is $Tr\left[\psi_{rp}\right]^* \left[\partial_\alpha \psi_{rp}\right] : Tr\left[\psi_{rp}\right]^* \left[\psi_{rp}\right] = Ct \Rightarrow Tr\left[\psi_{rp}\right]^* \left[\partial_\alpha \psi_{rp}\right] = -Tr\left[\partial_\alpha \psi_{rp}\right]^* \left[\psi_{rp}\right]$
and $Tr\left[\psi_{rp}\right]^* \left[\nabla_\alpha^A \psi_{rp}\right]$ is purely imaginary :
$\operatorname{Im} Tr\left[\psi_{rp}\right]^* \left[\nabla_\alpha^A \psi_{rp}\right] = \frac{1}{i}\left(Tr\left[\psi_{rp}\right]^* \left[\partial_\alpha \psi_{rp}\right] + \sum_a \grave{A}_\alpha^a Tr\left[\psi_{rp}\right]^* \left[\psi_{rp}\right]\left[\theta\right]\right)$
v) We get :

$$\operatorname{Im}\left\langle \psi_p, \nabla_\alpha \psi_p\right\rangle = \frac{1}{i}\left\langle \psi_{rp}, \nabla_\alpha^A \psi_{rp}\right\rangle - \frac{1}{2}\varkappa_p^t X_\alpha \tag{52}$$

vi) We have the identity,which will be used later on :

$$\operatorname{Im}\langle\psi_p,\partial_\alpha\psi_p\rangle=\frac{1}{i}\langle\psi_{rp},\partial_\alpha\psi_{rp}\rangle+\frac{1}{4}\varkappa_p^t j\left(w\right)\partial_\alpha w \tag{53}$$

The part of the action for the interactions is then :
for a single particle : $C_I\int_0^T f\left(\Phi_V\left(t,x\right)\right)\operatorname{Im}\langle\psi,\nabla_u\psi\rangle\,dt)$
$=C_I\int_0^T f\left(\Phi_V\left(t,x\right)\right)\left(\frac{1}{i}\left\langle\psi_r,\nabla_u^A\psi_r\right\rangle-\frac{1}{2}\sum_\alpha u^\alpha\varkappa^t X_\alpha\right)dt$
for each particle of type p : $C_I\int_0^T f_p\left(\Phi_{V_p}\left(t,x\right)\right)\operatorname{Im}\left\langle\psi_p,\nabla_{u_p}\psi_p\right\rangle dt$
$=C_I\int_0^T f_p\left(\Phi_{V_p}\left(t,x\right)\right)\left(\frac{1}{i}\left\langle\psi_r,\nabla_{u_p}^A\psi_r\right\rangle-\frac{1}{2}\sum_\alpha u_p^\alpha\varkappa_p^t X_{p\alpha}\right)dt$.
with a real constant C_I

11.2.4 Total Action

The total action with the three parts above is expressed in a geometric form in the fiber bundle variables, it is obviously equivariant and covariant. It differs from the general lagrangian by the introduction of the vector field $u_p$which is defined through w :

$$u_p=c\left(\left(2a_w^2-1\right)\varepsilon_0\left(m\right)+a_w\sum_{j=1}^{3}w_{pj}\varepsilon_j\left(m\right)\right) \tag{54}$$

$w_p:\Omega\to\mathbb{R}^3$ is not constrained because $w_p^t w_p$ is not bounded and does not change in a change of chart, which is convenient in GR.

Single particle model
The action is then :

$$\begin{array}{l}\int_\Omega\sum_{\alpha\beta}\left(C_G\left\langle\mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle+C_A\left\langle\mathcal{F}_{A\alpha\beta},\mathcal{F}_A^{\alpha\beta}\right\rangle\right)\varpi_4+\\ \int_0^T f\left(\Phi_V\left(t,x\right)\right)\left(C_P\left\langle\psi_0,\psi_0\right\rangle+C_I\left(\frac{1}{i}\left\langle\psi_r,\nabla_u^A\psi_r\right\rangle-\frac{1}{2}\sum_\alpha u^\alpha\varkappa^t X_\alpha\right)\right)dt\end{array} \tag{55}$$

Distribution of particles
We need to account for the distribution of particles. Uusally one follows a large population of particles, and measures the density of particles, or type of particles, on cross sections along the trajectories.

The volume form $\varpi_4\left(m\right)=\det\left[P'\right]d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\xi^3$ induces on each hypersurface $\Omega_3\left(t\right)=\left\{m=\varphi_M\left(t,x\right),x\in\Omega_3\left(0\right)\right\}$ a volume form :
$\varpi_3\left(m\right)=i_n\varpi_4\Leftrightarrow\varpi_4\left(m\right)=n\wedge\varpi_3\left(m\right)$ where n is the unitary normal future oriented to $\Omega_3\left(t\right)$.
The hypersurfaces are defined by the equation f(m) = t and n is the normalized gradiant :
$grad\left(f'\left(m\right)\right)=\sum_\alpha\left(\sum_\beta g^{\alpha\beta}\partial_\beta f\right)\partial\xi_\alpha=\sum_\alpha\frac{1}{c}g^{\alpha0}\partial\xi_\alpha=\sum_\alpha\frac{1}{c}\sum_{ij}\eta^{ij}P_i^\alpha P_j^0\partial\xi_\alpha$
$=\sum_\alpha\frac{1}{c}\sum_i\eta^{i0}P_i^\alpha\partial\xi_\alpha=-\sum_\alpha\frac{1}{c}P_0^\alpha\partial\xi_\alpha=-\frac{1}{c}\varepsilon_0\left(m\right)$
$n=\varepsilon_0\left(m\right)$
$\varpi_3\left(m\right)=\det\left[P'\right]d\xi^1\wedge d\xi^2\wedge d\xi^3$

The flows $\Phi_V(t,x)$ are complete on Ω so

$$\varpi_4((\Phi_V(t,x))) = \sqrt{-\langle V,V\rangle}dt\varpi_3(x) \tag{56}$$

We assume that :

i) the initial distribution of particles on $\Omega_3(0)$ is given by a function $N_p(x)$ which is locally integrable. It does not need to be smooth, but the measure $N_p(x)\varpi_3(x)$ must be absolutely continuous.

ii) the type of the particles does not change over their trajectories

The action related to the interactions read :

$C_I\int_{\Omega_3(0)} N_p(x)\left(\int_0^T f(\Phi_V(t,x))\left(\frac{1}{i}\langle\psi_r,\nabla^A_\alpha\psi_r\rangle - \frac{1}{2}\varkappa^t X_\alpha\right)dt\right)\varpi_3(x)$

$= C_I\int_{\Omega_3(0)} N_p(x)\left(\int_0^T\left(\frac{1}{i}\langle\psi_r,\nabla^A_\alpha\psi_r\rangle - \frac{1}{2}\varkappa^t X_\alpha\right)\sqrt{-\langle V,V\rangle}dt\right)\varpi_3(x)$

$= C_I\int_{\Omega_3(0)}\int_0^T N_p(x)\left(\left(\frac{1}{i}\langle\psi_r,\nabla^A_\alpha\psi_r\rangle - \frac{1}{2}\varkappa^t X_\alpha\right)\right)\sqrt{-\langle V,V\rangle}dt\varpi_3(x)$

$= C_I\int_\Omega N_p(m)\left(\left(\frac{1}{i}\langle\psi_r,\nabla^A_\alpha\psi_r\rangle - \frac{1}{2}\varkappa^t X_\alpha\right)\right)\varpi_4(m)$

using the fact that ψ are sections defined on M and the prolongation of $N_p(\varphi_M(t,x)) = N_p(x)$. Formally it sums up to replace the chart φ_M by Φ_{V_p} for each type of particles.

The action is then:

$$\begin{aligned}&\int_\Omega\left(C_G\left\langle\mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle + C_A\left\langle\mathcal{F}_{A\alpha\beta},\mathcal{F}_A^{\alpha\beta}\right\rangle + C_I N_p\left(\left(\frac{1}{i}\langle\psi_r,\nabla^A_\alpha\psi_r\rangle - \frac{1}{2}\varkappa^t X_\alpha\right)\right)\right)\varpi_4\\ &+ C_P\int_{\Omega_3(0)} N_p\langle\psi_{0p},\psi_{0p}\rangle\left(\int_0^T f(\Phi_V(t,x))\,dt\right)\varpi_3(x)\end{aligned} \tag{57}$$

11.3 Equations

Find a stationary solution in the single particle model and distribution of particles model requires different tools, but which are similar in many ways. We start with the distributional model.

11.3.1 Equations for a distribution of particles

Proper time

In the case of a population of particles, the function f is involved only with the last term :

$C_P\int_{\Omega_3(0)} N_p(x)\langle\psi_{0p},\psi_{0p}\rangle\left(\int_0^T f(\Phi_V(t,x))\,dt\right)\varpi_3(x)$

which is stationary only if $\int_0^T f(\Phi_V(t,x))\,dt = T_p(x)$ is constant. It is usual to read this term as : $\int_0^T\sqrt{-\langle V,V\rangle}dt$ which leads, for free particles, to the conclusion that the trajectories must be of extremal lengths, that is geodesics. But geodesics depend on the metric. Here, in a general model of interacting particles and fields, the world lines are defined through the other parameters and $T_p(x)$ is determined for each particle. So we are lead to a more significant result : each type of particle has its own proper time. This is consistent with the assumption that the world lines are defined by a common vector field, which depends on the initial conditions, and cannot be arbitrary. This proper time is defined anywhere in M, as well as the sections ψ_p, and we can assume that it does not depend on the system.

In the usual relativist point of view the proper time is linked to the time as measured on the clock of an observer. Here we see that it has a deeper, intrinsic, meaning, as it is linked to the fundamental constituent of material bodies : it is observer independant. The constant c acquires a different status : it is just the ratio of the units used to measure, through different instruments, lengths and times. From this point of view the "speed of propagation of fields", which, as we have seen, is not a simple topic, should be revisited.

Moreover this part of the action is no longer involved, and we can use the usual Lagrange equations to compute the other variables.

Euler Lagrange equations

For a population of particles the action reads :

$\int_\Omega \left(C_G \left\langle \mathcal{F}_{G\alpha\beta}, \mathcal{F}_G^{\alpha\beta} \right\rangle + C_A \left\langle \mathcal{F}_{A\alpha\beta}, \mathcal{F}_A^{\alpha\beta} \right\rangle + C_I N_p(m) \left(\left(\frac{1}{i} \left\langle \psi_r, \nabla_\alpha^A \psi_r \right\rangle - \frac{1}{2} \varkappa^t X_\alpha \right) \right) \right) \varpi_4$

The solutions of the variational problem are given by the Lagrange equations :

$\frac{dL \det P'}{dz^i} - \sum_\beta \frac{d}{d\xi^\beta} \frac{dL \det P'}{dz^i_\beta} = 0$

Whenever z^i is complex, we have two families of real valued equations :

$\frac{\partial L \det P'}{\partial \operatorname{Re} z^i} - \sum_\beta \frac{d}{d\xi^\beta} \frac{\partial L \det P'}{\partial \operatorname{Re} z^i_\beta} = 0$

$\frac{\partial L \det P'}{\partial \operatorname{Im} z^i} - \sum_\beta \frac{d}{d\xi^\beta} \frac{\partial L \det P'}{\partial \operatorname{Im} z^i_\beta} = 0$

and by defining the holomorphic complex valued functions :

$\frac{\partial L \det P'}{\partial z^i} = \frac{\partial L \det P'}{\partial \operatorname{Re} z^i} + \frac{1}{i} \frac{\partial L \det P'}{\partial \operatorname{Im} z^i}$

$\frac{\partial L \det P'}{\partial \overline{z^i}} = \frac{\partial L \det P'}{\partial \operatorname{Re} z^i} - \frac{1}{i} \frac{\partial L \det P'}{\partial \operatorname{Im} z^i}$

the equations read :

$\frac{\partial L \det P'}{\partial z^i} = \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{\partial L \det P'}{\partial \operatorname{Re} z^i_\beta} + \frac{1}{i} \frac{\partial L \det P'}{\partial \operatorname{Im} z^i_\beta} \right) = \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{\partial L \det P'}{\partial z^i_\beta} \right)$

$\frac{\partial L \det P'}{\partial \overline{z^i}} = \overline{\frac{\partial L \det P'}{\partial z^i}} = \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{\partial L \det P'}{\partial \operatorname{Re} z^i_\beta} - \frac{1}{i} \frac{\partial L \det P'}{\partial \operatorname{Im} z^i_\beta} \right)$

$= \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{\partial L \det P'}{\partial \overline{z^i_\beta}} \right) = \overline{\sum_\beta \frac{d}{d\xi^\beta} \left(\frac{\partial L \det P'}{\partial z^i_\beta} \right)}$

and we are left with the unique equation :

$\frac{\partial L \det P'}{\partial z^i} = \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{\partial L \det P'}{\partial z^i_\beta} \right)$

On the manifold M endowed with the volume form ϖ_4 the divergence of a vector field $X = \sum_\alpha X^\alpha \partial \xi_\alpha$ is the function div(X) : $\pounds_X \varpi_4 = div(X) \varpi_4$ and its expression in coordinates is :

$div(X) = \frac{1}{\det P'} \sum_\alpha \partial_\alpha \left(X^\alpha \det P' \right)$ which reads in the SR approximation : $div(X) = \sum_\alpha \partial_\alpha \left(X^\alpha \right)$

$\frac{dL}{dz^i_\beta}$ is a vector : $Z_i = \frac{dL}{dz^i_\beta} \partial \xi_b$ and $\frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{dL}{dz^i_\beta} \det P' \right) = div\left(Z_i \right)$

Equation for ψ_{rp}

We have two families of equations, for each variable ψ_{rp} :

$\forall i, j, p : \frac{dL \det P'}{d \operatorname{Re} \psi^{ij}_{rp}} = \sum_\alpha \frac{d}{d\xi^\alpha} \left(\frac{dL}{d\partial_\alpha \operatorname{Re} \psi^{ij}_{rp}} \det P' \right) ; \frac{dL \det P'}{d \operatorname{Im} \psi^{ij}_{rp}} = \sum_\alpha \frac{d}{d\xi^\alpha} \left(\frac{dL}{d\partial_\alpha \operatorname{Im} \psi^{ij}_{rp}} \det P' \right)$

which combine in :

$\frac{dL}{d\psi^{ij}_{rp}} = \frac{1}{\det P'} \sum_\alpha \frac{d}{d\xi^\alpha} \left(\frac{dL}{d\partial_\alpha \psi^{ij}_{rp}} \det P' \right)$

Partial derivatives :

i) $\frac{\partial \mathcal{L}}{\partial \psi^{ij}_{rp}} = N_p C_I \left(\frac{\partial}{\partial \psi^{ij}_{rp}} \frac{1}{i} Tr \left[\psi_{rp} \right]^* \left[\nabla^A_{u_p} \psi_{rp} \right] - \frac{1}{2} \frac{\partial}{\partial \psi^{ij}_{rp}} \sum_{a=1}^3 \sum_\alpha u^\alpha_p X^a_\alpha Tr \left[\psi_{rp} \right]^* \left[\widetilde{\gamma}_a \right] \left[\psi_{rp} \right] \right)$

with : $X_a^a = \left[-\frac{1}{2}j\left(w\right)\partial_\alpha w + \left(a_w^2 - \frac{1}{2}j\left(w\right)j\left(w\right)\right)G_{r\alpha} + a_w j\left(w\right)G_{w\alpha}\right]^a$
$\frac{1}{i}Tr\left[\psi_{rp}\right]^*\left[\nabla_{u_p}^A\psi_{rp}\right]$
$= \frac{1}{i}\sum_\beta u_p^\beta\left(Tr\left[\psi_{rp}\right]^*\left[\partial_\beta\psi_{rp}\right] + Tr\left[\psi_{rp}\right]^*\left[\psi_{rp}\right]\left[\grave{A}_\beta\right]\right)$
$= \frac{1}{i}\sum_\beta u_p^\beta\left(\sum_{kl}\left[\psi_{rp}^*\right]_l^k\left[\partial_\beta\psi_{rp}\right]_k^l + \left[\grave{A}_\beta\right]_k^q\left[\psi_{rp}^*\right]_l^k\left[\psi_{rp}\right]_q^l\right)$
with $Tr\left[\grave{A}_\beta\right]\left[\psi_{rp}\right]^*\left[\psi_{rp}\right] = Tr\left[\psi_{rp}\right]^*\left[\psi_{rp}\right]\left[\grave{A}_\beta\right]$
$\frac{\partial}{\partial\psi_{rp}^{ij}}\frac{1}{i}Tr\left[\psi_{rp}\right]^*\left[\nabla_{u_p}^A\psi_{rp}\right] = \frac{1}{i}\sum_\beta u_p^\beta\left(\left[\grave{A}_\beta\right]\left[\psi_{rp}^*\right]\right)_i^j$
$\frac{\partial}{\partial\psi_{rp}^{ij}}\sum_{a=1}^3\sum_\alpha u_p^\alpha X_a^a Tr\left[\psi_{rp}\right]^*\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right] = \sum_{a=1}^3\sum_\alpha u_p^\alpha X_a^a\left(\left[\psi_{rp}^*\right]\left[\widetilde{\gamma}_a\right]\right)_i^j$
$\frac{d\mathcal{L}}{d\psi_{rp}^{ij}} = N_pC_I\sum_\alpha u_p^\alpha\left(\frac{1}{i}\sum_\beta u_p^\beta\left(\left[\grave{A}_\beta\right]\left[\psi_{rp}^*\right]\right)_i^j - \frac{1}{2}\sum_{a=1}^3 X_a^a\left(\left[\psi_{rp}^*\right]\left[\widetilde{\gamma}_a\right]\right)\right)_i^j$
ii) $\frac{d\mathcal{L}}{d\partial_\alpha\psi_{rp}^{ij}} = N_pC_I\frac{\partial}{\partial\partial_\alpha\psi_{rp}^{ij}}\frac{1}{i}\sum_\beta u_p^\beta\left(\sum_{kq}\left[\psi_{rp}^*\right]_q^k\left[\partial_\beta\psi_{rp}\right]_k^q + \left[\psi_{rp}^*\right]_q^k\left[\psi_{rp}\right]_l^q\left[\grave{A}_\beta\right]_k^l\right)$

$$\frac{d\mathcal{L}}{d\partial_\alpha\psi_{rp}^{ij}} = N_pC_I\frac{1}{i}u_p^\alpha\left[\psi_{rp}^*\right]_i^j \tag{58}$$

The equation is :
$N_pC_I\sum_{a\alpha}u_p^\alpha\left(\frac{1}{i}\left[\grave{A}_\alpha\right]\left[\psi_{rp}^*\right] - \frac{1}{2}X_a^a\left[\psi_{rp}^*\right]\left[\widetilde{\gamma}_a\right]\right)_i^j$
$= \frac{1}{\det P'}\sum_\alpha\frac{d}{d\xi^\alpha}\left(N_pC_I\frac{1}{i}u_p^\alpha\left[\psi_{rp}^*\right]_i^j\det P'\right)$
and because N_p is constant :
$\sum_{a\alpha}u_p^\alpha\left(\frac{1}{i}\left[\grave{A}_\alpha\right]\left[\psi_{rp}^*\right] - \frac{1}{2}X_a^a\left[\psi_{rp}^*\right]\left[\widetilde{\gamma}_a\right]\right) = \frac{1}{\det P'}\sum_\alpha\frac{d}{d\xi^\alpha}\left(\frac{1}{i}u_p^\alpha\left[\psi_{rp}\right]^*\det P'\right)$
$\sum_{a\alpha}u_p^\alpha\left(-\frac{1}{i}\left[\psi_{rp}\right]\left[\grave{A}_\alpha\right] - \frac{1}{2}X_a^a\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right) = \frac{1}{\det P'}\sum_\alpha\frac{d}{d\xi^\alpha}\left(-\frac{1}{i}u_p^\alpha\left[\psi_{rp}\right]\det P'\right)$
$\sum_{a\alpha}u_p^\alpha\left(\left[\psi_{rp}\right]\left[\grave{A}_\alpha\right] + i\frac{1}{2}X_a^a\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right) = \frac{1}{\det P'}\sum_\alpha\frac{d}{d\xi^\alpha}\left(u_p^\alpha\left[\psi_{rp}\right]\det P'\right)$
$= \sum_\alpha u_p^\alpha\frac{d}{d\xi^\alpha}\left(\left[\psi_{rp}\right]\right) + \left[\psi_{rp}\right]\frac{1}{\det P'}\sum_\alpha\frac{d}{d\xi^\alpha}\left(u_p^\alpha\det P'\right)$
$= \sum_\alpha u_p^\alpha\frac{d}{d\xi^\alpha}\left(\left[\psi_{rp}\right]\right) + \left[\psi_{rp}\right]div\left(u_p\right)i\frac{1}{2}\sum_{a=1}^3\sum_\alpha u_p^\alpha X_a^a\left(\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)$
$= \sum_{a\alpha}u_p^\alpha\left(-\left[\partial_\alpha\psi_{rp}\right] + \left[\psi_{rp}\right]\left[\grave{A}_\alpha\right]\right) + \left[\psi_{rp}\right]div\left(u_p\right)$
By left multiplication with $\left[\psi_{rp}\right]^*$ and taking the trace :
$i\frac{1}{2}\sum_{a=1}^3\sum_\alpha u_p^\alpha X_a^a Tr\left[\psi_{rp}\right]^*\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]$
$= \sum_\alpha u_p^\alpha\left(-Tr\left[\psi_{rp}\right]^*\left[\partial_\alpha\psi_{rp}\right] + Tr\left[\psi_{rp}\right]^*\left[\psi_{rp}\right]\left[\grave{A}_\alpha\right]\right) + Tr\left[\psi_{rp}\right]^*\left[\psi_{rp}\right]div\left(u_p\right)$
Which implies for the complex and real parts :
$Tr\left[\psi_{rp}\right]^*\left[\psi_{rp}\right], Tr\left(\left[\psi_{rp}\right]^*\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right) \in \mathbb{R}$,
$Tr\left[\psi_{rp}\right]^*\left[\partial_\beta\psi_{rp}\right], Tr\left[\psi_{rp}\right]^*\left[\psi_{rp}\right]\left[\grave{A}_\alpha\right] \in i\mathbb{R}$
$i\frac{1}{2}\sum_{a=1}^3\sum_\alpha u_p^\alpha X_a^a Tr\left(\left[\psi_{rp}\right]^*\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)$
$= \sum_\alpha u_p^\alpha\left(-Tr\left[\psi_{rp}\right]^*\left[\partial_\alpha\psi_{rp}\right] + Tr\left[\psi_{rp}\right]^*\left[\psi_{rp}\right]\left[\grave{A}_\alpha\right]\right)$
$div\left(u_p\right)Tr\left[\psi_{rp}\right]^*\left[\psi_{rp}\right] = 0$
So we have the equations :
$\frac{1}{2}\sum_{a=1}^3\sum_\alpha u_p^\alpha X_a^a Tr\left(\left[\psi_{rp}\right]^*\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)$

$= \frac{1}{i}\sum_\alpha u_p^\alpha \left(-Tr\left[\psi_{rp}\right]^* \left[\partial_\alpha \psi_{rp}\right] + Tr\left[\psi_{rp}\right]^* \left[\psi_{rp}\right]\left[\grave{A}_\alpha\right]\right)$

$$div\left(u_p\right) = 0 \tag{59}$$

The vector fields u_p follow the equation :

$$\sum_\alpha \partial_\alpha u_p^\alpha + \frac{1}{2} u_p^\alpha \sum_{\beta\gamma} g^{\beta\gamma}\partial_\alpha g_{\beta\gamma} = 0 \tag{60}$$

which is a single equation for each type of particle, involving w_p and P.
As a consequence the equation becomes :
$i\frac{1}{2}\sum_{a=1}^3 \sum_\alpha u_p^\alpha X_a^a \left(\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right) = \sum_{a\alpha} u_p^\alpha \left(-\left[\partial_\alpha \psi_{rp}\right] + \left[\psi_{rp}\right]\left[\grave{A}_\alpha\right]\right)$
$\sum_\alpha u_p^\alpha \left[\partial_\alpha \psi_{rp}\right] = \sum_\alpha u_p^\alpha \left(\left[\psi_{rp}\right]\left[\grave{A}_\alpha\right] + i\frac{1}{2}\sum_{a=1}^3 X_a^a \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)$

$$\forall p : \frac{d\psi_{rp}}{d\tau_p} = \sum_\alpha u_p^\alpha \left(\left[\psi_{rp}\right]\left[\grave{A}_\alpha\right] + i\frac{1}{2}\sum_{a=1}^3 X_a^a \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right) \tag{61}$$

with : $X_a^a = \left[-\frac{1}{2}j\left(w\right)\partial_\alpha w + \left(a_w^2 - \frac{1}{2}j\left(w\right)j\left(w\right)\right)G_{r\alpha} + a_w j\left(w\right)G_{w\alpha}\right]^a$
The full equation is, in r and w :
$\frac{d\psi_{rp}}{d\tau_p} = \gamma C\left(\frac{d\sigma_{rp}}{d\tau_p}\right)\psi_{0p}$
$\gamma C\left(\sigma_{rp}^{-1}\cdot\frac{d\sigma_{rp}}{d\tau_p}\right)\psi_{0p} = \left[\psi_{0p}\right]\left[\widehat{\grave{A}}\right]$
$+i\frac{1}{2}\sum_{a=1}^3 \left[-\frac{1}{2}j\left(w_p\right)\frac{dw_p}{d\tau_p} + \left(a_w^2 - \frac{1}{2}j\left(w_p\right)j\left(w_p\right)\right)\widehat{G}_r + a_w j\left(w_p\right)\widehat{G}_w\right]^a \gamma C\left(\sigma_{rp}^{-1}\right)\left[\widetilde{\gamma}_a\right]\gamma C\left(\sigma_{rp}\right)\psi_{0p}$
$\gamma C\left(\sigma_{rp}^{-1}\cdot\frac{d\sigma_{rp}}{d\tau_p}\right)\psi_{0p} = -i\frac{1}{2}\sum_{a=1}^3 \left[\left(\frac{1}{a_r} - \frac{1}{2}j\left(r_p\right) + \frac{1}{4a_r}j\left(r_p\right)j\left(r_p\right)\right)\frac{dr_p}{d\tau_p}\right]^a \widetilde{\gamma}_a\psi_{0p}$
$\sum_{a=1}^3 \left(\left[\left(\frac{1}{a_r} - \frac{1}{2}j\left(r_p\right) + \frac{1}{4a_r}j\left(r_p\right)j\left(r_p\right)\right)\frac{dr_p}{d\tau_p}\right]^a \widetilde{\gamma}_a + \left[-\frac{1}{2}j\left(w_p\right)\frac{dw_p}{d\tau_p}\right]^a \gamma C\left(\sigma_{rp}^{-1}\right)\left[\widetilde{\gamma}_a\right]\gamma C\left(\sigma_{rp}\right)\right)\psi_{0p}$
$= 2i\left[\psi_{0p}\right]\left[\widehat{\grave{A}}\right] - \sum_{a=1}^3 \left[\left(a_w^2 - \frac{1}{2}j\left(w_p\right)j\left(w_p\right)\right)\widehat{G}_r + a_w j\left(w_p\right)\widehat{G}_w\right]^a \gamma C\left(\sigma_{rp}^{-1}\right)\left[\widetilde{\gamma}_a\right]\gamma C\left(\sigma_{rp}\right)\psi_{0p}$

As we have :
$\nabla_\alpha \psi_p = \left[\gamma C\left(\sigma_{wp}\right)\right]\left(\nabla_\alpha^A \psi_{rp} - i\frac{1}{2}\sum_{a=1}^3 \left(\left[Y_\alpha\right]^a \gamma_a\gamma_0\psi_{rp} + \left[X_\alpha\right]^a \widetilde{\gamma}_a\psi_{rp}\right)\right)$
and from the previous equation :
$\nabla_{u_p}^A \psi_{rp} = \sum_\alpha u_p^\alpha \left(2\left[\partial_\alpha \psi_{rp}\right] + i\frac{1}{2}\sum_{a=1}^3 X_a^a \left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]\right)$
on shell :

$$\nabla_{u_p}\psi_p = \left[\gamma C\left(\sigma_{wp}\right)\right]\sum_\alpha u_p^\alpha \left(2\left[\partial_\alpha \psi_{rp}\right] - i\frac{1}{2}\sum_{a=1}^3 \left[Y_\alpha\right]^a \gamma_a\gamma_0\psi_{rp}\right) \tag{62}$$

with :
$Y_\alpha = -w\partial_\alpha a_w + a_w\partial_\alpha w + \left[a_w^2 - 1 - a_w j\left(w\right) + \frac{1}{4}j\left(w\right)j\left(w\right)\right]G_{r\alpha} + \left[1 - \frac{3}{4}j\left(w\right)j\left(w\right)\right]G_{w\alpha}$
Moreover:
$\left\langle \psi_p, \nabla_{u_p}\psi_p\right\rangle$
$= \left\langle \gamma C\left(\sigma_{wp}\right)\psi_{rp}, \left[\gamma C\left(\sigma_{wp}\right)\right]\sum_\alpha u_p^\alpha \left(2\left[\partial_\alpha \psi_{rp}\right] - i\frac{1}{2}\sum_{a=1}^3 \left[Y_\alpha\right]^a \gamma_a\gamma_0\psi_{rp}\right)\right\rangle$
$= \sum_\alpha u_p^\alpha \left\langle \psi_{rp}, 2\left[\partial_\alpha \psi_{rp}\right] - i\frac{1}{2}\sum_{a=1}^3 \left[Y_\alpha\right]^a \gamma_a\gamma_0\psi_{rp}\right\rangle$
$= 2\left\langle \psi_{rp}, \frac{d\psi_{rp}}{d\tau_p}\right\rangle - i\frac{1}{2}\sum_\alpha u_p^\alpha \sum_{a=1}^3 \left[Y_\alpha\right]^a Tr\left[\psi_{rp}\right]^* \gamma_0\gamma_a\gamma_0\left[\psi_{rp}\right]$

$= 2\left\langle \psi_{rp}, \frac{d\psi_{rp}}{d\tau_p}\right\rangle + i\frac{1}{2}\sum_\alpha u_p^\alpha \sum_{a=1}^3 \left[Y_\alpha\right]^a Tr\left[\psi_{rp}\right]^* \gamma_a \left[\psi_{rp}\right]$

$$\left\langle \psi_p, \nabla_{u_p}\psi_p\right\rangle = 2\left\langle \psi_{rp}, \frac{d\psi_{rp}}{d\tau_p}\right\rangle \tag{63}$$

with $\psi_{rp} = \gamma C\left(\sigma_{rp}\right)\psi_{0p}$

$\left\langle \psi_{rp}, \frac{d\psi_{rp}}{d\tau_p}\right\rangle$

$= \left\langle \psi_{0p}, \gamma C\left(\sigma_{rp}^{-1}\cdot\frac{d\sigma_{rp}}{d\tau_p}\right)\psi_{0p}\right\rangle$

$= \left\langle \psi_{0p}, -i\frac{1}{2}\sum_{a=1}^3\left[\left(\frac{1}{a_r} - \frac{1}{2}j\left(r_p\right) + \frac{1}{4a_r}j\left(r_p\right)j\left(r_p\right)\right)\frac{dr_p}{d\tau_p}\right]^a \widetilde{\gamma}_a\psi_{0p}\right\rangle$

$= -i\frac{1}{2}\sum_{a=1}^3\left[\left(\frac{1}{a_r} - \frac{1}{2}j\left(r_p\right) + \frac{1}{4a_r}j\left(r_p\right)j\left(r_p\right)\right)\frac{dr_p}{d\tau_p}\right]^a \left\langle \psi_{0p}, \widetilde{\gamma}_a\psi_{0p}\right\rangle$

Equation for w

The variable is w_p (one for each type of particle), which is also involved through u_p.

The equation is :

$a = 1,2,3, p = 1...N : \frac{dL}{dw_p^a} = \frac{1}{\det P'}\sum_\alpha \frac{d}{d\xi^\alpha}\left(\frac{dL}{d\partial_\alpha w_p^a}\det P'\right)$

Partial derivatives :

i) $\frac{\partial\mathcal{L}}{\partial w_p^a} = N_pC_I\left(\frac{\partial}{\partial w_p^a}\operatorname{Im}Tr\left[\psi_{rp}\right]^*\left[\nabla_{u_p}^A\psi_{rp}\right] - \frac{1}{2}\left(\left(\frac{\partial}{\partial w_p^a}u_p^\alpha\right)\varkappa_p^tX_\alpha + u_p^\alpha\varkappa_p^t\left(\frac{\partial}{\partial w_p^a}X_\alpha\right)\right)\right)$

$\frac{\partial\mathcal{L}}{\partial w_p^a} = N_pC_I\left(\sum_\alpha\frac{\partial u_p^\alpha}{\partial w_p^a}\left(\operatorname{Im}Tr\left[\psi_{rp}\right]^*\left[\nabla_\alpha^A\psi_{rp}\right] - \frac{1}{2}\varkappa_p^tX_\alpha\right) - \frac{1}{2}u_p^\alpha\varkappa_p^t\left(\frac{\partial}{\partial w_p^a}X_\alpha\right)\right)$

$\frac{\partial\mathcal{L}}{\partial w_p^a} = N_pC_I\left(\frac{\partial u_p^\alpha}{\partial w_p^a}\operatorname{Im}\left\langle\psi_p, \nabla_\alpha\psi_p\right\rangle - \frac{1}{2}u_p^\alpha\varkappa_p^t\left(\frac{\partial}{\partial w_p^a}X_\alpha\right)\right)$

ii) $\frac{\partial}{\partial w_p^a}X_\alpha$

$= \frac{\partial}{\partial w_p^a}\left[-\frac{1}{2}j\left(w\right)\partial_\alpha w + \left(a_w^2 - \frac{1}{2}j\left(w\right)j\left(w\right)\right)G_{r\alpha} + a_wj\left(w\right)G_{w\alpha}\right]^b$

$= -\frac{1}{2}\left(\epsilon\left(b,a,d\right)\partial_\alpha w_p\right) + \frac{\partial}{\partial w_p^a}\left(\left(3a_w^2 - 2\right)G_{r\alpha}^b - \frac{1}{2}w_p^b\left(w_p^tG_{r\alpha}\right) + a_w\epsilon\left(b,c,d\right)w_p^cG_{w\alpha}^d\right)$

$= -\frac{1}{2}\left(\epsilon\left(a,b,c\right)\partial_\alpha w_p^c\right) + \left(\left(\frac{3}{4}w_p^aG_{r\alpha}^b - \frac{1}{2}\delta_a^b\left(w_p^tG_{r\alpha}\right) - \frac{1}{2}w_p^bG_{r\alpha}^a + a_w\epsilon\left(b,a,c\right)G_{w\alpha}^c\right)\right)$

$\varkappa_p^t\frac{\partial}{\partial w_p^a}X_\alpha$

$= -\frac{1}{2}\epsilon\left(a,b,c\right)\varkappa_p^b\partial_\alpha w_p^c + \frac{3}{4}w_p^a\varkappa_p^bG_{r\alpha}^b - \frac{1}{2}\varkappa_p^b\delta_a^b\left(w_p^tG_{r\alpha}\right) - \frac{1}{2}\varkappa_p^bw_p^bG_{r\alpha}^a - a_w\epsilon\left(a,b,c\right)\varkappa_p^bG_{w\alpha}^c$

$= -\frac{1}{2}\left[j\left(\varkappa_p\right)\partial_\alpha w_p\right]^a + \frac{3}{4}w_p^a\left(\varkappa_p^tG_{r\alpha}\right) - \frac{1}{2}\varkappa_p^a\left(w_p^tG_{r\alpha}\right) - \frac{1}{2}\left(\varkappa_p^tw_p\right)G_{r\alpha}^a - a_w\left[j\left(\varkappa_p\right)G_{w\alpha}\right]^a$

$= \left(-\frac{1}{2}j\left(\varkappa_p\right)\partial_\alpha w_p + \frac{3}{4}w_p\left(\varkappa_p^tG_{r\alpha}\right) - \frac{1}{2}\varkappa_p\left(w_p^tG_{r\alpha}\right) - \frac{1}{2}\left(\varkappa_p^tw_p\right)G_{r\alpha} - a_wj\left(\varkappa_p\right)G_{w\alpha}\right)^a$

$\frac{3}{4}w_p\left(\varkappa_p^tG_{r\alpha}\right) - \frac{1}{2}\varkappa_p\left(w_p^tG_{r\alpha}\right) - \frac{1}{2}\left(w_p^t\varkappa_p\right)G_{r\alpha}$

$= \frac{3}{4}\left(j\left(\varkappa_p\right)j\left(w_p\right)G_{r\alpha} + \left(w_p^t\varkappa_p\right)G_{r\alpha}\right) - \frac{1}{2}\left(j\left(w_p\right)j\left(\varkappa_p\right)G_{r\alpha} + \left(w_p^t\varkappa_p\right)G_{r\alpha}\right) - \frac{1}{2}\left(w_p^t\varkappa_p\right)G_{r\alpha}$

$= \frac{3}{4}j\left(\varkappa_p\right)j\left(w_p\right)G_{r\alpha} - \frac{1}{2}j\left(w_p\right)j\left(\varkappa_p\right)G_{r\alpha} - \frac{1}{4}\left(w_p^t\varkappa_p\right)G_{r\alpha}$

$\varkappa_p^t\frac{\partial}{\partial w_p^a}X_\alpha = \left[-\frac{1}{2}j\left(\varkappa_p\right)\partial_\alpha w_p + \left(\frac{3}{4}j\left(\varkappa_p\right)j\left(w_p\right) - \frac{1}{2}j\left(w_p\right)j\left(\varkappa_p\right) - \frac{1}{4}\left(w_p^t\varkappa_p\right)\right)G_{r\alpha} - a_wj\left(\varkappa_p\right)G_{w\alpha}\right]^a$

iii) $\frac{\partial\mathcal{L}}{\partial w_p^a} = N_pC_I\{\sum_\alpha\operatorname{Im}\left\langle\psi_p, \nabla_\alpha\psi_p\right\rangle\frac{\partial u_p^\alpha}{\partial w_p^a}$

$-\frac{1}{2}u_p^\alpha\left[-\frac{1}{2}j\left(\varkappa_p\right)\partial_\alpha w_p + \left(\frac{3}{4}j\left(\varkappa_p\right)j\left(w_p\right) - \frac{1}{2}j\left(w_p\right)j\left(\varkappa_p\right) - \frac{1}{4}\left(w_p^t\varkappa_p\right)\right)G_{r\alpha} - a_wj\left(\varkappa_p\right)G_{w\alpha}\right]^a\}$

iv) $\frac{dL}{d\partial_\alpha w_p^a} = C_IN_p\frac{\partial}{\partial\partial_\alpha w_p^a}\left(-\frac{1}{2}\sum_\alpha u_p^\alpha\varkappa_p^t\left[-\frac{1}{2}j\left(w_p\right)\partial_\alpha w_p\right]\right)$

$= \frac{1}{4}C_IN_p\frac{\partial}{\partial\partial_\alpha w_p^a}\left(u_p^\alpha\varkappa_p^b\epsilon\left(b,c,d\right)w_p^c\partial_\alpha w_p^d\right)$

$= \frac{1}{4}C_IN_p\left(u_p^\alpha\varkappa_p^b\epsilon\left(b,c,a\right)w_p^c\right)$

$$\frac{dL}{d\partial_\alpha w_p^a} = -\frac{1}{4} C_I N_p u_p^\alpha \left[j\left(w_p\right) \varkappa_p \right]^a \tag{64}$$

The equation is with $N_p = Ct$
$a = 1, 2, 3, p = 1...N$:
$C_I \sum_\alpha \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle \frac{\partial u_p^\alpha}{\partial w_p^a}$
$-\frac{1}{2} u_p^\alpha \left[-\frac{1}{2} j\left(\varkappa_p\right) \partial_\alpha w_p + \left(\frac{3}{4} j\left(\varkappa_p\right) j\left(w_p\right) - \frac{1}{2} j\left(w_p\right) j\left(\varkappa_p\right) - \frac{1}{4}\left(w_p^t \varkappa_p\right)\right) G_{r\alpha} - a_w j\left(\varkappa_p\right) G_{w\alpha} \right]^a$
$= -\frac{1}{4} C_I \frac{1}{\det P'} \sum_\alpha \frac{d}{d\xi^\alpha} \left(u_p^\alpha \left[j\left(w_p\right) \varkappa_p \right]^a \det P' \right)$
$= -\frac{1}{4} C_I \sum_\alpha u_p^\alpha \frac{d}{d\xi^\alpha} \left[j\left(w_p\right) \varkappa_p \right]^a - \frac{1}{4} C_I \left[j\left(w_p\right) \varkappa_p \right]^a \frac{1}{\det P'} \sum_\alpha \frac{d}{d\xi^\alpha} \left(u_p \det P' \right)$
On shell :
$div\left(u_p\right) = 0$
$\sum_\alpha \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle \frac{\partial u_p^\alpha}{\partial w_p^a}$
$-\frac{1}{2} u_p^\alpha \left[-\frac{1}{2} j\left(\varkappa_p\right) \partial_\alpha w_p + \left(\frac{3}{4} j\left(\varkappa_p\right) j\left(w_p\right) - \frac{1}{2} j\left(w_p\right) j\left(\varkappa_p\right) - \frac{1}{4}\left(w_p^t \varkappa_p\right)\right) G_{r\alpha} - a_w j\left(\varkappa_p\right) G_{w\alpha} \right]^a$
$= -\frac{1}{4} \sum_\alpha u_p^\alpha \frac{d}{d\xi^\alpha} \left[j\left(w_p\right) \varkappa_p \right]^a$
The equation is then :

$$\begin{aligned} &\sum_\alpha \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle \frac{\partial u_p^\alpha}{\partial w_p^a} = \\ &\frac{1}{4} \left[\left(-j\left(w_p\right) \frac{d\varkappa_p}{d\tau_p} + \left(\frac{3}{2} j\left(\varkappa_p\right) j\left(w_p\right) - j\left(w_p\right) j\left(\varkappa_p\right) - \frac{1}{2}\left(w_p^t \varkappa_p\right)\right) \widehat{G}_r - 2 a_w j\left(\varkappa_p\right) \widehat{G}_w \right) \right]^a \end{aligned} \tag{65}$$

with :
$u_p^0 = c\left(2a_w^2 - 1\right); i = 1, 2, 3 : u_p^i = c a_w w_p^i$
$u_p^\alpha = \sum_{j=0}^3 P_j^\alpha u_p^j$
$a_w = \sqrt{1 + \frac{1}{4} w^t w} \Rightarrow \frac{da_w}{dw_p^a} = \frac{1}{4a_w} w_p^a$
$\frac{\partial u_p^0}{\partial w_p^a} = c4a_w \frac{da_w}{dw_p^a} = c w_p^a$
$\frac{\partial u_p^i}{\partial w_p^a} = c\left(\frac{1}{4a_w} w_p^a w_p^i + a_w \delta_a^i \right)$
$\frac{\partial u_p^\alpha}{\partial w_p^a} = P_0^\alpha c w_p^a + \sum_{j=0}^3 P_j^\alpha c \left(\frac{1}{4a_w} w_p^a w_p^j + a_w \delta_a^j \right)$
$= c\left(P_0^\alpha w_p^a + a_w P_a^\alpha + \frac{1}{4a_w} w_p^a \sum_{j=1}^3 P_j^\alpha w_p^j \right)$
$= c\left(P_0^\alpha w_p^a + a_w P_a^\alpha + \frac{1}{4a_w} w_p^a \sum_{j=1}^3 \frac{1}{c a_w} P_j^\alpha u_p^j \right)$
$= c\left(P_0^\alpha w_p^a + a_w P_a^\alpha + \frac{1}{4ca_w^2} w_p^a \left(u_p^\alpha - P_0^\alpha u_p^0 \right) \right)$
$= c\left(\frac{1}{2} w_p^a P_0^\alpha \left(1 + \frac{1}{2a_w^2} \right) + a_w P_a^\alpha + \frac{1}{4ca_w^2} w_p^a u_p^\alpha \right)$
The full equation with respect to w is then :
$\left[j\left(w_p\right) \frac{d\varkappa_p}{d\tau_p} \right]^a =$
$\sum_\alpha \{ \left(-2 \sum_{b=1}^3 \left[\left(\frac{1}{a_r} - \frac{1}{2} j\left(r_p\right) + \frac{1}{4a_r} j\left(r_p\right) j\left(r_p\right) \right) \partial_\alpha r \right]^b \langle \psi_{0p}, \widetilde{\gamma}_b \psi_{0p} \rangle \right)$
$+ \frac{1}{i} \left\langle \psi_{0p}, \psi_{0p} \left[\grave{A}_\alpha \right] \right\rangle - \frac{1}{2} \varkappa_p^t X_\alpha \} c \left(\frac{1}{2} w_p^a P_0^\alpha \left(1 + \frac{1}{2a_w^2} \right) + a_w P_a^\alpha \right)$

$-\left[\left(\frac{3}{2}j\left(\varkappa_p\right)j\left(w_p\right)-j\left(w_p\right)j\left(\varkappa_p\right)-\frac{1}{2}\left(w_p^t\varkappa_p\right)\right)\widehat{G}_r-2a_wj\left(\varkappa_p\right)\widehat{G}_w\right]^a$

with $\left\langle\psi_{rp},\frac{d\psi_{rp}}{d\tau_p}\right\rangle=0$ (see below)

Equation for the gravitational field

The equations are :

$\forall a,\alpha:\frac{dL}{dG^a_{r\alpha}}=\frac{1}{\det P'}\sum_\beta\frac{d}{d\xi^\beta}\left(\frac{dL}{d\partial_\beta G^a_{r\alpha}}\det P'\right)$

$\forall a,\alpha:\frac{dL}{dG^a_{w\alpha}}=\frac{1}{\det P'}\sum_\beta\frac{d}{d\xi^\beta}\left(\frac{dL}{d\partial_\beta G^a_{w\alpha}}\det P'\right)$

Partial derivatives related to $C_G\left\langle\mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle$

i) $\mathcal{F}^b_{r\lambda\mu}=\frac{\partial G^b_{r\mu}}{\partial\xi^\lambda}-\frac{\partial G^b_{r\lambda}}{\partial\xi^\mu}+j\left(G_{r\lambda}\right)G_{r\mu}-j\left(G_{w\lambda}\right)G_{w\mu}=\frac{\partial G^b_{r\mu}}{\partial\xi^\lambda}-\frac{\partial G^b_{r\lambda}}{\partial\xi^\mu}+\epsilon\left(b,c,d\right)G^c_{r\lambda}G^d_{r\mu}-\epsilon\left(b,c,d\right)G^c_{w\lambda}G^d_{w\mu}$

$\mathcal{F}^b_{w\lambda\mu}=\frac{\partial G^b_{w\mu}}{\partial\xi^\lambda}-\frac{\partial G^b_{w\lambda}}{\partial\xi^\mu}+j\left(G_{w\alpha}\right)G_{r\beta}+j\left(G_{r\alpha}\right)G_{w\beta}=\frac{\partial G^b_{w\mu}}{\partial\xi^\lambda}-\frac{\partial G^b_{w\lambda}}{\partial\xi^\mu}+\epsilon\left(b,c,d\right)G^c_{w\lambda}G^d_{r\mu}+\epsilon\left(b,c,d\right)G^c_{r\lambda}G^d_{w\mu}$

$\left\langle\mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle=\frac{1}{4}\sum^3_{\zeta\eta\lambda\mu=0}g^{\xi\lambda}g^{\zeta\mu}G_G\frac{1}{4}\frac{d}{dG^a_{r\alpha}}\left(\mathcal{F}^t_{r\xi\zeta}\mathcal{F}_{r\lambda\mu}-\mathcal{F}^t_{w\xi\varepsilon}\mathcal{F}_{w\lambda\mu}\right)$

ii) $\frac{d}{dG^a_{r\alpha}}\sum^3_{\zeta\eta\lambda\mu=0}\sum_b g^{\xi\lambda}g^{\zeta\mu}\left(\mathcal{F}^b_{r\xi\zeta}\mathcal{F}^b_{r\lambda\mu}-\mathcal{F}^b_{w\xi\varepsilon}\mathcal{F}^b_{w\lambda\mu}\right)$

$=\sum_b g^{\xi\lambda}g^{\zeta\mu}\left(\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{r\xi\zeta}\right)\mathcal{F}^b_{r\lambda\mu}+g^{\xi\lambda}g^{\zeta\mu}\mathcal{F}^b_{r\xi\zeta}\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{r\lambda\mu}$

$-g^{\xi\lambda}g^{\zeta\mu}\left(\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{w\xi\zeta}\right)\mathcal{F}_{w\lambda\mu}-g^{\xi\lambda}g^{\zeta\mu}\mathcal{F}^b_{w\xi\zeta}\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{w\lambda\mu}$

$=\sum_b\mathcal{F}^{b\alpha\zeta}_r\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{r\alpha\zeta}+\mathcal{F}^{b\xi\alpha}_r\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{r\xi\alpha}+\mathcal{F}^{b\alpha\mu}_r\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{r\alpha\mu}+\mathcal{F}^{b\lambda\alpha}_r\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{r\lambda\alpha}$

$-\left(\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{w\alpha\zeta}\right)\mathcal{F}^{\alpha\zeta}_w-\left(\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{w\xi\alpha}\right)\mathcal{F}^{\xi\alpha}_w-\mathcal{F}^{b\alpha\mu}_w\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{w\alpha\mu}-\mathcal{F}^{b\lambda\alpha}_w\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{w\lambda\alpha}$

$=4\sum_{b\gamma}\mathcal{F}^{b\alpha\gamma}_r\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{r\alpha\gamma}-\left(\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{w\alpha\gamma}\right)\mathcal{F}^{\alpha\gamma}_w$

$\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{r\alpha\gamma}=\frac{d}{dG^a_{r\alpha}}\left(\epsilon\left(b,c,d\right)G^c_{r\alpha}G^d_{r\gamma}-\epsilon\left(b,c,d\right)\left(G^c_{w\alpha}G^d_{w\gamma}\right)\right)=\epsilon\left(b,a,c\right)G^{cc}_{r\gamma r\gamma}$

$\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{w\alpha\lambda}=\frac{d}{dG^a_{r\alpha}}\epsilon\left(b,c,d\right)\left(G^c_{w\alpha}G^d_{r\gamma}\right)+\epsilon\left(b,c,d\right)\left(G^c_{r\alpha}G^d_{w\gamma}\right)=2\epsilon\left(b,a,c\right)G^c_{w\gamma}$

$\sum^3_{\zeta\eta\lambda\mu=0}g^{\xi\lambda}g^{\zeta\mu}G_G\frac{1}{4}\frac{d}{dG^a_{r\alpha}}\left(\mathcal{F}^t_{r\xi\zeta}\mathcal{F}_{r\lambda\mu}-\mathcal{F}^t_{w\xi\varepsilon}\mathcal{F}_{w\lambda\mu}\right)$

$=G_G\sum_{b\gamma}\epsilon\left(b,a,c\right)\left(\mathcal{F}^{b\alpha\gamma}_rG^c_{r\gamma}-\mathcal{F}^{\alpha\gamma}_wG^c_{w\gamma}\right)$

$=-G_G\sum_{bc}\epsilon\left(a,b,c\right)\left(\mathcal{F}^{b\alpha\gamma}_rG^c_{r\gamma}-\mathcal{F}^{\alpha\gamma}_wG^c_{w\gamma}\right)$

$=-G_G\sum_\gamma\left[j\left(\mathcal{F}^{\alpha\gamma}_r\right)G_{r\gamma}-j\left(\mathcal{F}^{\alpha\gamma}_w\right)G_{w\gamma}\right]^a$

$\frac{d}{dG^a_{r\alpha}}\left\langle\mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle=-G_G\sum_\gamma\left[j\left(\mathcal{F}^{\alpha\gamma}_r\right)G_{r\gamma}-j\left(\mathcal{F}^{\alpha\gamma}_w\right)G_{w\gamma}\right]^a$

iii) $\frac{d}{dG^a_{w\alpha}}\frac{1}{4}\sum^3_{\zeta\eta\lambda\mu=0}\sum_b g^{\xi\lambda}g^{\zeta\mu}\left(\mathcal{F}^b_{r\xi\zeta}\mathcal{F}^b_{r\lambda\mu}-\mathcal{F}^b_{w\xi\varepsilon}\mathcal{F}^b_{r\lambda\mu}\right)$

$=\sum_{b\gamma}\mathcal{F}^{b\alpha\gamma}_r\frac{d}{dG^a_{w\alpha}}\mathcal{F}^b_{r\alpha\gamma}-\left(\frac{d}{dG^a_{w\alpha}}\mathcal{F}^b_{w\alpha\gamma}\right)\mathcal{F}^{\alpha\gamma}_r$

$\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{r\alpha\gamma}=\frac{d}{dG^a_{w\alpha}}\left(\epsilon\left(b,c,d\right)\left(G^c_{r\alpha}G^d_{r\gamma}\right)-\epsilon\left(b,c,d\right)\left(G^c_{w\alpha}G^d_{w\gamma}\right)\right)=-\epsilon\left(b,a,c\right)G^c_{w\gamma}$

$\frac{d}{dG^a_{r\alpha}}\mathcal{F}^b_{w\alpha\lambda}=\frac{d}{dG^a_{w\alpha}}\epsilon\left(b,c,d\right)\left(G^c_{w\alpha}G^d_{r\gamma}\right)+\epsilon\left(b,c,d\right)\left(G^c_{r\alpha}G^d_{w\gamma}\right)=\epsilon\left(b,a,c\right)G^c_{r\gamma}$

$\frac{d}{dG^a_{w\alpha}}\sum^3_{\zeta\eta\lambda\mu=0}\frac{1}{4}\sum_b g^{\xi\lambda}g^{\zeta\mu}\left(\mathcal{F}^b_{r\xi\zeta}\mathcal{F}^b_{r\lambda\mu}-\mathcal{F}^b_{w\xi\varepsilon}\mathcal{F}^b_{r\lambda\mu}\right)$

$=\sum_{bc\gamma}\epsilon\left(b,a,c\right)\left(-\mathcal{F}^{b\alpha\gamma}_rG^c_{w\gamma}-\mathcal{F}^{\alpha\gamma}_wG^c_{r\gamma}\right)$

$=\sum_{bc}\epsilon\left(a,b,c\right)\left(G^c_{w\gamma}\mathcal{F}^{b\alpha\gamma}_r+\mathcal{F}^{b\alpha\gamma}_wG^{cc}_{r\gamma}\right)$

$= \sum_{\gamma} \left[j\left(\mathcal{F}_r^{\alpha\gamma}\right) G_{w\gamma} + j\left(\mathcal{F}_w^{\alpha\gamma}\right) G_{r\gamma} \right]^a$
$\frac{d}{dG_{r\alpha}^a} \left\langle \mathcal{F}_{G\alpha\beta}, \mathcal{F}_G^{\alpha\beta} \right\rangle = C_G \sum_{\gamma} \left[j\left(\mathcal{F}_r^{\alpha\gamma}\right) G_{w\gamma} + j\left(\mathcal{F}_w^{\alpha\gamma}\right) G_{r\gamma} \right]^a$
iv) $\frac{d\mathcal{L}}{d\partial_\beta G_{r\alpha}^a} = \sum_{\zeta\eta\lambda\mu=0}^{3} g^{\xi\lambda} g^{\zeta\mu} C_G \frac{1}{4} \frac{d}{d\partial_\beta G_{r\alpha}^a} \left(\mathcal{F}_{r\xi\zeta}^t \mathcal{F}_{r\lambda\mu} \right)$
$= C_G \frac{1}{4} 4 \sum_{\gamma} \mathcal{F}_r^{a\beta\gamma} \frac{d}{d\partial_\beta G_{r\alpha}^a} \mathcal{F}_{r\beta\gamma}^a$

$$\frac{d\mathcal{L}}{d\partial_\beta G_{w\alpha}^a} = -C_G \mathcal{F}_w^{a\beta\alpha}; \frac{d\mathcal{L}}{d\partial_\beta G_{r\alpha}^a} = C_G \mathcal{F}_r^{a\beta\alpha} \tag{66}$$

Partial derivatives related to $\sum_{p=1}^{N} N_p C_I \operatorname{Im} \sum_{\alpha} u_p^{\alpha} \langle \psi_p, \nabla_\alpha \psi_p \rangle$
$\operatorname{Im} \sum_{\alpha} u_p^{\alpha} \langle \psi_p, \nabla_\alpha \psi_p \rangle = \frac{1}{i} Tr \left[\psi_{rp} \right]^* \left[\nabla_{u_p}^A \psi_{rp} \right] - \frac{1}{2} \sum_{\alpha} u_p^{\alpha} \varkappa_p^t X_\alpha$
with $X_\alpha = -\frac{1}{2} j(w) \partial_\alpha w + \left(a_w^2 - \frac{1}{2} j(w) j(w) \right) G_{r\alpha} + a_w j(w) G_{w\alpha}$
i) $\frac{d}{dG_{r\alpha}^a} \left(\operatorname{Im} \sum_{\alpha} u_p^{\alpha} \langle \psi_p, \nabla_\alpha \psi_p \rangle \right) = -\frac{1}{2} \left(\sum_{\alpha} u_p^{\alpha} \frac{d}{dG_{r\alpha}^a} \left(\varkappa_p^t X_\alpha \right) \right)$
$\frac{d}{dG_{r\alpha}^a} \sum_{b=1}^{3} \varkappa_p^b X_\alpha^b$
$= \frac{d}{dG_{r\alpha}^a} \left[\left(a_{wp}^2 - \frac{1}{2} j(w_p) j(w_p) G_{r\alpha} \right) \right]^b \varkappa_p^b$
$= \sum_{b=1}^{3} \frac{d}{dG_{r\alpha}^a} \left[\left(a_w^2 G_{r\alpha}^b - \frac{1}{2} w_p^b \left(w_p^t G_{r\alpha} \right) + \frac{1}{2} \left(w_p^t w_p \right) G_{r\alpha}^b \right) \right] \varkappa_p^b$
$= \sum_{b=1}^{3} \frac{d}{dG_{r\alpha}^a} \left[\left(\left(3a_w^2 - 2 \right) G_{r\alpha}^b - \frac{1}{2} w_p^b \left(w_p^t G_{r\alpha} \right) \right) \right] \varkappa_p^b$
$= \sum_{b=1}^{3} \left[\left(3a_{wp}^2 - 2 \right) \delta_a^b - \frac{1}{2} w_p^b w_p^a \right] \varkappa_p^b$
$= \left(3a_{wp}^2 - 2 \right) \varkappa_p^a - \frac{1}{2} \left(w_p^t \varkappa_p \right) w_p^a$
$= \left[\left(3a_{wp}^2 - 2 \right) \varkappa_p - \frac{1}{2} \left(w_p^t \varkappa_p \right) w_p \right]^a$
$= \left[\left(3a_{wp}^2 - 2 \right) \varkappa_p - \frac{1}{2} \left(j(w_p) j(w_p) \varkappa_p + 4 \left(a_w^2 - 1 \right) \varkappa_p \right) w_p \right]^a$
$= \left[\left(a_w^2 - \frac{1}{2} j(w_p) j(w_p) \right) \varkappa_p \right]^a$
$\frac{d}{dG_{r\alpha}^a} \left(\operatorname{Im} \sum_{\alpha} u_p^{\alpha} \langle \psi_p, \nabla_\alpha \psi_p \rangle \right) = -\frac{1}{2} \left(\sum_{\alpha} u_p^{\alpha} \left[\left(a_w^2 - \frac{1}{2} j(w_p) j(w_p) \right) \varkappa_p \right]^a \right)$
ii) $\frac{d}{dG_{w\alpha}^a} \left(\operatorname{Im} \sum_{\alpha} u_p^{\alpha} \langle \psi_p, \nabla_\alpha \psi_p \rangle \right) = -\frac{1}{2} \left(\sum_{\alpha} u_p^{\alpha} \frac{d}{dG_{w\alpha}^a} \left(\varkappa_p^t X_\alpha \right) \right)$
$\frac{d}{dG_{w\alpha}^a} \sum_{b=1}^{3} \varkappa_p^b X_\alpha^b$
$= \frac{d}{dG_{w\alpha}^a} \sum_{b=1}^{3} \left[j(w_p) G_{w\alpha} \right]^b \varkappa_p^b$
$= \frac{d}{dG_{w\alpha}^a} \sum_{b=1}^{3} \epsilon(b,c,d) w_p^c G_{w\alpha}^d \varkappa_p^b$
$= \sum_{b=1}^{3} \epsilon(a,b,c) w_p^c \varkappa_p^b$
$= \left[j(\varkappa_p) w_p \right]^a$
$= -\left[j(w_p) \varkappa_p \right]^a$
$\frac{d}{dG_{w\alpha}^a} \left(\operatorname{Im} \sum_{\alpha} u_p^{\alpha} \langle \psi_p, \nabla_\alpha \psi_p \rangle \right) = -\frac{1}{2} \sum_{\alpha} u_p^{\alpha} \left(-\left[j(w_p) \varkappa_p \right]^a \right)$

The equations are:
$\forall a, \alpha :$
$-G_G \sum_{\gamma} \left[j\left(\mathcal{F}_r^{\alpha\gamma}\right) G_{r\gamma} - j\left(\mathcal{F}_w^{\alpha\gamma}\right) G_{w\gamma} \right]^a - \frac{1}{2} C_I \sum_{p=1}^{N} N_p u_p^{\alpha} \left[\left(a_w^2 - \frac{1}{2} j(w_p) j(w_p) \right) \varkappa_p \right]^a$
$= C_G \frac{1}{\det P'} \sum_{\beta} \frac{d}{d\xi^\beta} \left(\mathcal{F}_r^{a\beta\alpha} \det P' \right)$
$G_G \sum_{\gamma} \left[j\left(\mathcal{F}_r^{\alpha\gamma}\right) G_{w\gamma} + j\left(\mathcal{F}_w^{\alpha\gamma}\right) G_{r\gamma} \right]^a - \frac{1}{2} C_I \sum_{p=1}^{N} N_p u_p^{\alpha} a_{wp} \left[-j(w_p) \varkappa_p \right]^a$
$= -C_G \frac{1}{\det P'} \sum_{\beta} \frac{d}{d\xi^\beta} \left(\mathcal{F}_w^{a\beta\alpha} \det P' \right)$
$G_G \sum_{\gamma} \left[j\left(\mathcal{F}_r^{\alpha\gamma}\right) G_{r\gamma} - j\left(\mathcal{F}_w^{\alpha\gamma}\right) G_{w\gamma} \right]^a + \frac{1}{2} C_I \sum_{p=1}^{N} N_p u_p^{\alpha} \left[\left(a_w^2 - \frac{1}{2} j(w_p) j(w_p) \right) \varkappa_p \right]^a$

$= C_G \frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_r^{a\alpha\beta} \det P' \right)$
$G_G \sum_\gamma \left[j\left(\mathcal{F}_r^{\alpha\gamma}\right) G_{w\gamma} + j\left(\mathcal{F}_w^{\alpha\gamma}\right) G_{r\gamma} \right]^a + \frac{1}{2} C_I \sum_{p=1}^N N_p u_p^\alpha a_{wp} \left[j\left(w_p\right) \varkappa_p \right]^a$
$= C_G \frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_w^{a\alpha\beta} \det P' \right)$
In the Clifford algebra :
$\sum_{a=1}^3 \sum_\gamma \left[j\left(\mathcal{F}_r^{\alpha\gamma}\right) G_{r\gamma} - j\left(\mathcal{F}_w^{\alpha\gamma}\right) G_{w\gamma} \right]^a \overrightarrow{\kappa}_a + \sum_\gamma \left[j\left(\mathcal{F}_r^{\alpha\gamma}\right) G_{w\gamma} + j\left(\mathcal{F}_w^{\alpha\gamma}\right) G_{r\gamma} \right]^a \overrightarrow{\kappa}_{a+3}$
$= \upsilon \left(j\left(\mathcal{F}_r^{\alpha\beta}\right) G_{r\beta} - j\left(\mathcal{F}_w^{\alpha\beta}\right) G_{w\beta}, j\left(\mathcal{F}_r^{\alpha\beta}\right) G_{w\beta} + j\left(\mathcal{F}_w^{\alpha\beta}\right) G_{r\beta r\xi} \right)$
$= \left[\upsilon\left(\mathcal{F}_r^{\alpha\beta}, \mathcal{F}_w^{\alpha\beta}\right), \upsilon\left(G_{r\beta}, G_{w\beta}\right) \right]$
$\sum_{a=1}^3 \left[\left(a_w^2 - \frac{1}{2} j\left(w_p\right) j\left(w_p\right) \right) \varkappa_p \right]^a \overrightarrow{\kappa}_a - \left[j\left(w_p\right) \varkappa_p \right]^a \overrightarrow{\kappa}_{a+3} = \upsilon \left(\left(a_w^2 - \frac{1}{2} j\left(w_p\right) j\left(w_p\right) \right) \varkappa_p, a_{wp} j\left(w_p\right) \varkappa_p \right)$
$\sum_{a=1}^3 \mathcal{F}_r^{a\alpha\beta} \overrightarrow{\kappa}_a + \mathcal{F}_w^{a\alpha\beta} \overrightarrow{\kappa}_{a+3} = \upsilon \left(\mathcal{F}_r^{\alpha\beta}, \mathcal{F}_w^{\alpha\beta} \right) = \mathcal{F}_G^{\alpha\beta}$
So the equation reads:
$\forall \alpha :$

$$G_G \left[\mathcal{F}_G^{\alpha\beta}, G_\beta \right] + \frac{1}{2} C_I \sum_{p=1}^N N_p u_p^\alpha \upsilon \left(\left(a_w^2 - \frac{1}{2} j\left(w_p\right) j\left(w_p\right) \right) \varkappa_p, a_{wp} j\left(w_p\right) \varkappa_p \right) = C_G \frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_G^{\alpha\beta} \det P' \right) \tag{67}$$

Remark :
$\operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle = \frac{1}{i} \left\langle \psi_{rp}, \nabla_\alpha^A \psi_{rp} \right\rangle - \frac{1}{2} \varkappa_p^t \left(-\frac{1}{2} j\left(w\right) \partial_\alpha w + \left(a_w^2 - \frac{1}{2} j\left(w\right) j\left(w\right) \right) G_{r\alpha} + a_w j\left(w\right) G_{w\alpha} \right)$
$\operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle = \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle = \frac{1}{i} \langle \psi, \partial_\alpha \psi \rangle + \frac{1}{i} \left\langle \psi_0, \left[\psi_0 \right] \left[\grave{A}_a \right] \right\rangle - \frac{1}{2} \sum_{a=1}^3 G_{r\alpha}^a \langle \psi, \widetilde{\gamma}_a \psi \rangle - \frac{1}{2} \sum_{a=1}^3 G_{w\alpha}^a \langle \psi, \gamma_a \gamma_0 \psi \rangle$
So we have the identities :
$\frac{d}{dG_{r\alpha}^a} \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle = -\frac{1}{2} \langle \psi_p, \widetilde{\gamma}_a \psi_p \rangle = -\frac{1}{2} \frac{d}{dG_{r\alpha}^a} \left(\varkappa_p^t X_\alpha \right) = -\frac{1}{2} \left[\left(a_w^2 - \frac{1}{2} j\left(w_p\right) j\left(w_p\right) \right) \varkappa_p \right]^a$
$\frac{d}{dG_{w\alpha}^a} \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle = -\frac{1}{2} \langle \psi_p, \gamma_a \gamma_0 \psi_p \rangle = -\frac{1}{2} \frac{d}{dG_{w\alpha}^a} \left(\varkappa_p^t X_\alpha \right) = -\frac{1}{2} \left[a_w j\left(w_p\right) \varkappa_p \right]^a$
$Tr \left[\psi_p \right]^* \gamma_0 \widetilde{\gamma}_a \left[\psi_p \right] = \left[\left(a_w^2 - \frac{1}{2} j\left(w_p\right) j\left(w_p\right) \right) \varkappa_p \right]^a = \langle \psi_p, \widetilde{\gamma}_a \psi_p \rangle$
$-Tr \left[\psi_p \right]^* \gamma_a \left[\psi_p \right] = a_w \left[j\left(w_p\right) \varkappa_p \right]^a = \langle \psi_p, \gamma_a \gamma_0 \psi_p \rangle$

Equation for the other fields

The equations are :
$\forall a, \alpha : \frac{d\mathcal{L}}{d\grave{A}_\alpha^a} = \frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\frac{d\mathcal{L} \det P'}{d\partial_\beta \grave{A}_\alpha^a} \right)$

Partial derivatives :
i) $\frac{d\mathcal{L}}{d\grave{A}_\alpha^a} = \sum_{\zeta\eta\lambda\mu=0}^3 g^{\xi\lambda} g^{\zeta\mu} C_A \frac{d}{d\grave{A}_\alpha^a} \left(\mathcal{F}_{A\xi\zeta}^t \mathcal{F}_{A\lambda\mu} \right) + \sum_{p=1}^N N_p C_I \frac{d}{d\grave{A}_\alpha^a} \operatorname{Im} Tr \left[\psi_{rp} \right]^* \left[\nabla_{u_p}^A \psi_{rp} \right]$
$\mathcal{F}_{A\lambda\mu}^b = \partial_\lambda \grave{A}_\mu^b - \partial_\mu \grave{A}_\lambda^b + \sum C_{cd}^b \grave{A}_\lambda^c \grave{A}_\mu^d$
$C_A \sum_{\zeta\eta\lambda\mu=0}^3 g^{\xi\lambda} g^{\zeta\mu} \frac{d}{d\grave{A}_\alpha^a} \left(\mathcal{F}_{A\xi\zeta}^t \mathcal{F}_{A\lambda\mu} \right)$
$= C_A \sum_{\zeta\eta\lambda\mu=0}^3 \sum_{bcd} g^{\alpha\lambda} g^{\zeta\mu} C_{ad}^b \grave{A}_\zeta^d \mathcal{F}_{A\lambda\mu}^b + g^{\xi\lambda} g^{\alpha\mu} C_{ca}^b \grave{A}_\xi^c \mathcal{F}_{A\lambda\mu}^b + g^{\xi\alpha} g^{\zeta\mu} \mathcal{F}_{A\xi\zeta}^b C_{ad}^b \grave{A}_\mu^d + g^{\xi\lambda} g^{\zeta\alpha} \mathcal{F}_{A\xi\zeta}^b C_{ca}^b \grave{A}_\lambda^c$
$= C_A \sum_{\zeta\eta\lambda\mu=0}^3 \sum_{bc} C_{ac}^b \grave{A}_\beta^c \mathcal{F}_A^{b\alpha\beta} - C_{ac}^b \grave{A}_\xi^c \mathcal{F}_A^{b\xi\alpha} + \mathcal{F}_A^{b\alpha\mu} C_{ac}^b \grave{A}_\mu^c - \mathcal{F}_A^{b\lambda\alpha} C_{ac}^b \grave{A}_\lambda^c$
$= 4 C_A \sum_{\beta bc} C_{ac}^b \grave{A}_\beta^c \mathcal{F}_A^{b\alpha\beta} = 4 C_A \sum_{\beta b} \left[\overrightarrow{\theta}_a, \grave{A}_\beta \right]^b \mathcal{F}_A^{b\alpha\beta} = 4 C_A \sum_\beta \left\langle \left[\overrightarrow{\theta}_a, \grave{A}_\beta \right], \mathcal{F}_A^{\alpha\beta} \right\rangle_{T_1 U}$
Using : $\forall X, Y, Z \in T_1 U : \langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle$
$\left\langle \left[\overrightarrow{\theta}_a, \grave{A}_\beta \right], \mathcal{F}_A^{\alpha\beta} \right\rangle_{T_1 U} = \left\langle \overrightarrow{\theta}_a, \left[\grave{A}_\beta, \mathcal{F}_A^{\alpha\beta} \right] \right\rangle_{T_1 U} = \left[\grave{A}_\beta, \mathcal{F}_A^{\alpha\beta} \right]^a$
because the basis is orthonormal.
$\sum_{p=1}^N N_p C_I \frac{d}{d\grave{A}_\alpha^a} \operatorname{Im} Tr \left[\psi_{rp} \right]^* \left[\nabla_{u_p}^A \psi_{rp} \right]$

$\operatorname{Im} Tr\left[\psi_{rp}\right]^{*}\left[\nabla_{u_p}^{A}\psi_{rp}\right]=\frac{1}{i}\sum_{\beta}u_p^{\beta}\left(Tr\left[\psi_{rp}\right]^{*}\left[\partial_{\beta}\psi_{rp}\right]+\sum_{a}\grave{A}_{\beta}^{a}Tr\left[\psi_{rp}\right]^{*}\left[\psi_{rp}\right]\left[\theta\right]\right)$
$\frac{d}{d\grave{A}_{\alpha}^{a}}\operatorname{Im} Tr\left[\psi_{rp}\right]^{*}\left[\nabla_{u_p}^{A}\psi_{rp}\right]=u_p^{\alpha}\frac{1}{i}Tr\left(\left[\theta_a\right]\left[\psi_{rp}^{*}\right]\left[\psi_{rp}\right]\right)$
$\frac{d\mathcal{L}}{d\grave{A}_{\alpha}^{a}}=\left(4C_A\sum_{\beta}\left[\grave{A}_{\beta},\mathcal{F}_A^{\alpha\beta}\right]^{a}+C_I\frac{1}{i}\sum_{p=1}^{N}N_pu_p^{\alpha}Tr\left[\psi_{rp}^{*}\right]\left[\psi_{rp}\right]\left[\theta_a\right]\right)$
ii)

$$\frac{d\mathcal{L}}{d\partial_{\beta}\grave{A}_{\alpha}^{a}}=4C_A\mathcal{F}_A^{a\beta\alpha} \tag{68}$$

The equation is :
$\forall a,\alpha:4C_A\sum_{\beta}\left[\grave{A}_{\beta},\mathcal{F}_A^{\alpha\beta}\right]^{a}+C_I\frac{1}{i}\sum_{p=1}^{N}N_pu_p^{\alpha}Tr\left[\psi_{rp}^{*}\right]\left[\psi_{rp}\right]\left[\theta_a\right]$
$=4C_A\frac{1}{\det P'}\sum_{\beta}\frac{d}{d\xi^{\beta}}\left(\mathcal{F}_A^{a\beta\alpha}\det P'\right)$
$Tr\left[\psi_{rp}^{*}\right]\left[\psi_{rp}\right]\left[\theta_a\right]=\left\langle\psi_{rp},\psi_{rp}\left[\theta_a\right]\right\rangle=\left\langle\vartheta\left(\sigma_{rp},\Theta_p\right)\psi_{0p},\vartheta\left(\sigma_{rp},\Theta_p\right)\psi_{0p}\left[\theta_a\right]\right\rangle=\left\langle\psi_{0p},\psi_{0p}\left[\theta_a\right]\right\rangle$
or in the Lie algebra :

$$\forall\alpha:C_A\sum_{a=1}^{m}\sum_{\beta}\left[\mathcal{F}_A^{\alpha\beta},\grave{A}_{\beta}\right]^{a}\overrightarrow{\theta}-\frac{1}{4}\frac{1}{i}C_I\sum_{p=1}^{N}N_pu_p^{\alpha}\left\langle\psi_{0p},\psi_{0p}\left[\theta_a\right]\right\rangle\overrightarrow{\theta}=C_A\frac{1}{\det P'}\sum_{\beta}\frac{d}{d\xi^{\beta}}\left(\mathcal{F}_A^{\alpha\beta}\det P'\right) \tag{69}$$

Equations for the tetrad P

The equation are :
$\forall\alpha,i:\frac{dL\det P'}{dP_i^{\alpha}}=\sum_{\beta}\frac{d}{d\xi^{\beta}}\left(\frac{\partial L\det P'}{\partial\partial_{\beta}P_i^{\alpha}}\right)$

The derivative of the determinant is :
$\frac{\partial\det P'}{\partial P_i^{\alpha}}=-\left(\frac{1}{\det P}\right)^2\frac{\partial\det P}{\partial P_i^{\alpha}}=-\left(\frac{1}{\det P}\right)^2P_{\alpha}^{\prime i}\det P=-P_{\alpha}^{\prime i}\det P'$
So the equations read :
$\frac{d\mathcal{L}}{dP_i^{\alpha}}\det P'-\mathcal{L}\left(\det P'\right)P_{\alpha}^{\prime i}=0$
By product with P_i^{β} and summation on i :
$\forall\alpha,\beta=0...3:\sum_i\frac{d\mathcal{L}}{dP_i^{\alpha}}P_i^{\beta}-\mathcal{L}\delta_{\beta}^{\alpha}=0$

Partial derivatives:
$\frac{d\mathcal{L}}{dP_i^{\alpha}}=\frac{\partial\mathcal{L}}{\partial u^{\alpha}}\frac{\partial u^{\alpha}}{\partial P_i^{\alpha}}+\sum_{\rho\theta\lambda\mu}\frac{\partial}{\partial P_i^{\alpha}}\left(g^{\lambda\rho}g^{\mu\theta}\right)\left(C_G\left\langle\mathcal{F}_{G\lambda\mu},\mathcal{F}_{G\rho\theta}\right\rangle+C_A\left\langle\mathcal{F}_{A\rho\theta},\mathcal{F}_{A\rho\theta}\right\rangle\right)$
i) $\frac{\partial\mathcal{L}}{\partial u^{\alpha}}\frac{\partial u^{\alpha}}{\partial P_i^{\alpha}}=C_I\sum_{p=1}^{N}\sum_{\alpha}N_p\operatorname{Im}\left\langle\psi_p,\nabla_{\alpha}\psi_p\right\rangle\frac{\partial u^{\alpha}}{\partial P_i^{\alpha}}$
with $u^{\alpha}=c\left(P_0^{\alpha}\left(2a_w^2-1\right)+\sum_{b=1}^{3}a_wP_b^{\alpha}w_b\right)$
$i=0:\frac{\partial\mathcal{L}}{\partial u^{\alpha}}\frac{\partial u^{\alpha}}{\partial P_i^{\alpha}}=cC_I\sum_{p=1}^{N}N_p\left(2a_w^2-1\right)\operatorname{Im}\left\langle\psi_p,\nabla_{\alpha}\psi_p\right\rangle$
$i>0:\frac{\partial\mathcal{L}}{\partial u^{\alpha}}\frac{\partial u^{\alpha}}{\partial P_i^{\alpha}}=cC_I\sum_{p=1}^{N}N_pa_ww^i\operatorname{Im}\left\langle\psi_p,\nabla_{\alpha}\psi_p\right\rangle$
$\sum_i\frac{\partial\mathcal{L}}{\partial u^{\alpha}}\frac{\partial u^{\alpha}}{\partial P_i^{\alpha}}P_i^{\beta}$
$=cC_I\sum_{p=1}^{N}N_p\left(\left(2a_w^2-1\right)P_0^{\beta}+a_w\sum_{i=1}^{3}w^iP_i^{\beta}\right)\operatorname{Im}\left\langle\psi_p,\nabla_{\alpha}\psi_p\right\rangle$
$=C_I\sum_{p=1}^{N}N_pu_p^{\beta}\operatorname{Im}\left\langle\psi_p,\nabla_{\alpha}\psi_p\right\rangle$

ii) $\sum_{\rho\theta\lambda\mu} \frac{\partial}{\partial P_i^\alpha} \left(g^{\lambda\rho} g^{\mu\theta}\right) \langle \mathcal{F}_{G\lambda\mu}, \mathcal{F}_{G\rho\theta} \rangle$
$= \sum_{\lambda\mu\rho\theta} \frac{\partial}{\partial P_i^\alpha} \left(\sum_{jkpq=0}^{3} \eta^{jk} P_j^\lambda P_k^\rho \eta^{pq} P_p^\mu P_q^\theta\right) \langle \mathcal{F}_{G\lambda\mu}, \mathcal{F}_{G\rho\theta} \rangle$
$= 4 \sum_{\lambda\mu\rho} \sum_{j=0}^{3} \eta^{ij} P_j^\lambda g^{\mu\rho} \langle \mathcal{F}_{G\alpha\mu}, \mathcal{F}_{G\lambda\rho} \rangle$
$\sum_{\rho\theta\lambda\mu} \frac{\partial}{\partial P_i^\alpha} \left(g^{\lambda\rho} g^{\mu\theta}\right) \left(C_G \langle \mathcal{F}_{G\lambda\mu}, \mathcal{F}_{G\rho\theta} \rangle + C_A \langle \mathcal{F}_{A\rho\theta}, \mathcal{F}_{A\rho\theta} \rangle\right)$
$= 4 \sum_{\lambda\mu\rho} \sum_{j=0}^{3} \eta^{ij} P_j^\lambda g^{\mu\rho} \left(C_G \langle \mathcal{F}_{G\alpha\mu}, \mathcal{F}_{G\lambda\rho} \rangle + C_A \langle \mathcal{F}_{A\alpha\mu}, \mathcal{F}_{A\lambda\rho} \rangle\right)$
$\sum_i P_i^\beta \sum_{\rho\theta\lambda\mu} \frac{\partial}{\partial P_i^\alpha} \left(g^{\lambda\rho} g^{\mu\theta}\right) \left(C_G \langle \mathcal{F}_{G\lambda\mu}, \mathcal{F}_{G\rho\theta} \rangle + C_A \langle \mathcal{F}_{A\rho\theta}, \mathcal{F}_{A\rho\theta} \rangle\right)$
$= 4 \sum_i P_i^\beta \sum_{\lambda\mu\rho} \sum_{j=0}^{3} \eta^{ij} P_j^\lambda g^{\mu\rho} \left(C_G \langle \mathcal{F}_{G\alpha\mu}, \mathcal{F}_{G\lambda\rho} \rangle + C_A \langle \mathcal{F}_{A\alpha\mu}, \mathcal{F}_{A\lambda\rho} \rangle\right)$
$= 4 \sum_{\lambda\mu\rho} g^{\lambda\beta} g^{\mu\rho} \left(C_G \langle \mathcal{F}_{G\alpha\mu}, \mathcal{F}_{G\lambda\rho} \rangle + C_A \langle \mathcal{F}_{A\alpha\mu}, \mathcal{F}_{A\lambda\rho} \rangle\right)$
$= 4 \left(C_G \left\langle \mathcal{F}_{G\alpha\mu}, \mathcal{F}_G^{\beta\mu} \right\rangle + C_A \left\langle \mathcal{F}_{A\alpha\mu}, \mathcal{F}_A^{\beta\mu} \right\rangle\right)$
iii) So we have the identity :

$$\sum_i \frac{d\mathcal{L}}{dP_i^\alpha} P_i^\beta = C_I \sum_{p=1}^{N} N_p u_p^\beta \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle + 4 \sum_{\gamma=0}^{3} \left(C_G \left\langle \mathcal{F}_{G\alpha\gamma}, \mathcal{F}_G^{\beta\gamma} \right\rangle + C_A \left\langle \mathcal{F}_{A\alpha\gamma}, \mathcal{F}_A^{\beta\gamma} \right\rangle\right) \tag{70}$$

The tetrad equation reads :
$\forall \alpha, \beta = 0...3$:

$$C_I \sum_{p=1}^{N} N_p u_p^\beta \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle + 4 \sum_{\gamma=0}^{3} \left(C_G \left\langle \mathcal{F}_{G\alpha\gamma}, \mathcal{F}_G^{\beta\gamma} \right\rangle + C_A \left\langle \mathcal{F}_{A\alpha\gamma}, \mathcal{F}_A^{\beta\gamma} \right\rangle\right) = \mathcal{L} \delta_\beta^\alpha \tag{71}$$

By taking $\alpha = \beta$ and summing :
$C_I \sum_{p=1}^{N} N_p \operatorname{Im} \left\langle \psi_p, \nabla_{u_p} \psi_p \right\rangle + 4 \sum_{\alpha\gamma=0}^{3} \left(C_G \langle \mathcal{F}_{G\alpha\gamma}, \mathcal{F}_G^{\alpha\gamma} \rangle + C_A \langle \mathcal{F}_{A\alpha\gamma}, \mathcal{F}_A^{\alpha\gamma} \rangle\right) = 4\mathcal{L}$
$= 4 C_G \left\langle \mathcal{F}_{G\alpha\beta}, \mathcal{F}_G^{\alpha\beta} \right\rangle + C_A \left\langle \mathcal{F}_{A\alpha\beta}, \mathcal{F}_A^{\alpha\beta} \right\rangle + 4 C_I \sum_{p=1}^{N} N_p \operatorname{Im} \left\langle \psi_p, \nabla_{u_p} \psi_p \right\rangle$
$\Rightarrow$

$$\sum_{p=1}^{N} N_p \operatorname{Im} \left\langle \psi_p, \nabla_{u_p} \psi_p \right\rangle = 0 \tag{72}$$

11.3.2 Equations of a single particle

We consider a system of a single particle interacting with the fields, represented as above :
- the state of the particle is represented by a vector $\psi \in \widehat{E}_0 \otimes F$
- the fields are represented by their potential $G_\alpha, \grave{A}_\alpha$ and their strengh $\mathcal{F}_{G\alpha\beta}, \mathcal{F}_{A\alpha\beta}$
The action is :
$\int_\Omega \sum_{\alpha\beta} \left(C_G \left\langle \mathcal{F}_{G\alpha\beta}, \mathcal{F}_G^{\alpha\beta} \right\rangle + C_A \left\langle \mathcal{F}_{A\alpha\beta}, \mathcal{F}_A^{\alpha\beta} \right\rangle\right) \varpi_4$
$+ C_P \langle \psi_0, \psi_0 \rangle \int_0^T f\left(\Phi_V(t,x)\right) dt + C_I \int_0^T f\left(\Phi_V(t,x)\right) \left(\frac{1}{i} \left\langle \psi_r, \nabla_u^A \psi_r \right\rangle - \frac{1}{2} \sum_\alpha u^\alpha \varkappa^t X_\alpha\right) dt$

Functional derivatives

We cannot use the Euler-Lagrange equations, so we will use the functional derivatives, using the extension of distributions on vector bundles (see Dutailly 2012).

A functional : $\ell : J^r E \to \mathbb{R}$ defined on a normed subspace of sections $\mathfrak{X}(J^r E)$ of a vector bundle E has a functional derivative $\frac{\delta \ell}{\delta z}(Z_0)$ with respect to a section $Z \in \mathfrak{X}(E)$ in Z_0 if there is a distribution $\frac{\delta \ell}{\delta z}$ such that for any smooth, compactly supported $\delta Z \in \mathfrak{X}_{c\infty}(E)$:

$\lim_{\|\delta Z\| \to 0} \left\| \ell(Z_0 + \delta Z) - \ell(Z_0) - \frac{\delta \ell}{\delta z}(Z_0) Z \right\| = 0$

Because Z and δZ are sections of E their r-jets extensions are computed by taking the partial derivatives. The key point in the definition is that only δZ, and not its derivatives, is involved. It is clear that the functional is stationary in Z_0 if $\frac{\delta \ell}{\delta z}(Z_0) = 0$.

When the functional is given by an integral : $\int_\Omega \lambda(J^r Z)\, d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$ the functional derivative is the distribution :

$\frac{\delta \ell}{\delta z}(\delta Z) = \int_\Omega \sum_{s=0}^{r} \sum_{\alpha_1 \ldots \alpha_s} (-1)^s D_{\alpha_1 \ldots \alpha_s} \frac{\partial \lambda}{\partial Z_{\alpha_1 \ldots \alpha_s}} \delta Z d\xi^0 \wedge d\xi^1 \wedge d\xi^2 \wedge d\xi^3$

so that we get back the Euler Lagrange equations if all the functionals are integral similarly defined.

Equations for ψ and w

The variables are involved in the last integral only, and the equations can be deduced from the Euler-Lagrange equations :

$$div(u) = 0 \tag{73}$$

$$\frac{d\psi_r}{d\tau} = \psi_r \left[\widehat{\grave{A}}\right] - \frac{1}{2} i \sum_{a=1}^{3} \left[-\frac{1}{2} j(w) \frac{dw}{d\tau} + \left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \widehat{G}_r + a_w j(w_p) \widehat{G}_w \right]^a [\widetilde{\gamma}_a] [\psi_r] \tag{74}$$

with :

$\widehat{\grave{A}}^a = \sum_\alpha u_p^\alpha \grave{A}_\alpha^a$

$\widehat{G}_w^a = \sum_\alpha u^\alpha G_{w\alpha}^a$

$\widehat{G}_r^a = \sum_\alpha u^\alpha G_{r\alpha}^a$

$\forall a = 1, 2, 3$:

$$\sum_\alpha \operatorname{Im} \langle \psi, \nabla_\alpha \psi \rangle \frac{\partial u^\alpha}{\partial w^a} = \frac{1}{4} \left[\left(-j(w) \frac{d\varkappa}{d\tau} + \left(\frac{3}{2} j(\varkappa) j(w) - j(w) j(\varkappa) - \frac{1}{2} \left(w^t \varkappa\right) \right) \widehat{G}_r - 2 a_w j(\varkappa) \widehat{G}_w \right) \right]^a \tag{75}$$

$\operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle = \frac{1}{i} \langle \psi_{rp}, \partial_\alpha \psi_{rp} \rangle + \frac{1}{i} \left\langle \psi_{0p}, \psi_{0p} \left[\widehat{\grave{A}}\right] \right\rangle$

$-\frac{1}{2} \varkappa^t \left[-\frac{1}{2} j(w_p) \frac{dw_p}{d\tau_p} + \left(a_w^2 - \frac{1}{2} j(w_p) j(w_p)\right) \widehat{G}_r + a_w j(w_p) \widehat{G}_w \right]$

$\langle \psi_{rp}, \partial_\alpha \psi_{rp} \rangle = \left\langle \psi_{0p}, \gamma C \left(\sigma_{rp}^{-1} \cdot \partial_\alpha \sigma_{rp}\right) \psi_{0p} \right\rangle$

$= -i \frac{1}{2} \sum_{a=1}^{3} \left[\left(\frac{1}{a_r} - \frac{1}{2} j(r) + \frac{1}{4a_r} j(r) j(r) \right) \partial_\alpha r \right]^a \langle \psi_{0p}, \widetilde{\gamma}_a \psi_{0p} \rangle$

$\operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle = \frac{1}{i} \left\langle \psi_{rp}, \nabla_\alpha^A \psi_{rp} \right\rangle - \frac{1}{2} \varkappa_p^t X_\alpha$

Equation for the fields

The functional derivatives for the gravitational field are :

$\frac{\delta}{\delta G^a_{w\alpha}}\left(\int_\Omega C_G\left\langle \mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle \varpi_4\right)(\delta G^a_{w\alpha})$
$= C_G\int_\Omega\left(\frac{\partial}{\partial G^a_{w\alpha}}\left\langle \mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle \det P' - \sum_\beta \frac{d}{d\xi^\beta}\frac{\partial}{\partial\partial_\beta G^a_{w\alpha}}\left(\left\langle \mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle \det P'\right)\right)$
$(\delta G^a_{w\alpha})\, d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\xi^3$
$= C_G\int_\Omega\sum_\beta\left(\left[j\left(\mathcal{F}_r^{\alpha\beta}\right)G_{w\beta}+j\left(\mathcal{F}_w^{\alpha\beta}\right)G_{r\beta}\right]^a\det P' - \frac{d}{d\xi^\beta}\left(\mathcal{F}_w^{a\beta\alpha}\det P'\right)\right)$
$(\delta G^a_{w\alpha})\, d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\xi^3$
$= C_G\int_\Omega\sum_\beta\left[j\left(\mathcal{F}_r^{\alpha\beta}\right)G_{w\beta}+j\left(\mathcal{F}_w^{\alpha\beta}\right)G_{r\beta}\right]^a(\delta G^a_{w\alpha})\,\varpi_4$
$+\int_\Omega\sum_\beta\frac{d}{d\xi^\beta}\left(\mathcal{F}_w^{a\alpha\beta}\det P'\right)(\delta G^a_{w\alpha})\, d\xi^0\wedge d\xi^1\wedge d\xi^2\wedge d\xi^3$
$= C_G\int_\Omega\sum_\beta\left[j\left(\mathcal{F}_r^{\alpha\beta}\right)G_{w\beta}+j\left(\mathcal{F}_w^{\alpha\beta}\right)G_{r\beta}\right]^a(\delta G^a_{w\alpha})\,\varpi_4$
$+C_G\int_\Omega(\delta G^a_{w\alpha})\left(\sum_\beta\frac{d}{d\xi^\beta}\mathcal{F}_w^{a\alpha\beta}\right)\varpi_4 - C_G\int_\Omega(\delta G^a_{w\alpha})\sum_\beta\mathcal{F}_w^{a\alpha\beta}\sum_{\gamma\beta i}P'^i_\gamma\frac{dP^\gamma_i}{d\xi^\beta}\varpi_4$
$= C_G\int_\Omega(\delta G^a_{w\alpha})\left(\sum_\beta\left[j\left(\mathcal{F}_r^{\alpha\beta}\right)G_{w\beta}+j\left(\mathcal{F}_w^{\alpha\beta}\right)G_{r\beta}\right]^a+\frac{d}{d\xi^\beta}\mathcal{F}_w^{a\alpha\beta}-\mathcal{F}_w^{a\alpha\beta}\sum_{\gamma i}P'^i_\gamma\frac{dP^\gamma_i}{d\xi^\beta}\right)\varpi_4$
with
$\frac{d}{dG^a_{r\alpha}}\left\langle \mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle = C_G\sum_\gamma\left[j\left(\mathcal{F}_r^{\alpha\gamma}\right)G_{w\gamma}+j\left(\mathcal{F}_w^{\alpha\gamma}\right)G_{r\gamma}\right]^a$
$\frac{d\mathcal{L}}{d\partial_\beta G^a_{w\alpha}} = -C_G\mathcal{F}_w^{a\beta\alpha}$
$\sum_\beta\frac{d}{d\xi^\beta}\det P' = \sum_{\gamma\beta i}\frac{\partial\det P'}{\partial P^\gamma_i}\frac{dP^\gamma_i}{d\xi^\beta} = -\left(\sum_{\gamma\beta i}P'^i_\gamma\frac{dP^\gamma_i}{d\xi^\beta}\right)\det P'$

For the simple integral a direct computation gives:
$-\frac{1}{2}C_I\sum_\alpha\int_0^T fu^\alpha\varkappa^t X_\alpha\left(G_{w\alpha}+\delta G_{w\alpha}\right)dt - \left(-\frac{1}{2}C_I\sum_\alpha\int_0^T fu^\alpha\varkappa^t X_\alpha\left(G_{w\alpha}\right)dt\right)$
$= -\frac{1}{2}C_I\int_0^T fu^\alpha\sum_{bc}\varkappa^b\left(a_w\epsilon\left(b,c,a\right)w_c\delta G^a_{w\alpha}\right)dt$
$= -\frac{1}{2}C_I\int_0^T fu^\alpha\delta G^a_{w\alpha}\sum_{bc}\left(a_w\epsilon\left(a,b,c\right)\varkappa^b w_c\right)dt$
$= -\frac{1}{2}C_I a_w\int_0^T fu^\alpha\delta G^a_{w\alpha}\left[j\left(\varkappa\right)w\right]^a dt$
with $X_\alpha = -\frac{1}{2}j\left(w\right)\partial_\alpha w+\left(a_w^2-\frac{1}{2}j\left(w\right)j\left(w\right)\right)G_{r\alpha}+a_w j\left(w\right)G_{w\alpha}$

The equation reads :
$\forall\delta G_{w\alpha}$:
$C_G\int_\Omega(\delta G^a_{w\alpha})\left(\sum_\beta\left[j\left(\mathcal{F}_r^{\alpha\beta}\right)G_{w\beta}+j\left(\mathcal{F}_w^{\alpha\beta}\right)G_{r\beta}\right]^a+\frac{d}{d\xi^\beta}\mathcal{F}_w^{a\alpha\beta}-\mathcal{F}_w^{a\alpha\beta}\sum_{\gamma i}P'^i_\gamma\frac{dP^\gamma_i}{d\xi^\beta}\right)\varpi_4$
$+\frac{1}{2}C_I a_w\int_0^T fu^\alpha\left[j\left(w\right)\varkappa\right]^a(\delta G^a_{w\alpha})\,dt = 0$
And similarly :
$\frac{\delta}{\delta G^a_{r\alpha}}\left(\int_\Omega C_G\left\langle \mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle \varpi_4\right)(\delta G^a_{w\alpha})$
$= C_G\int_\Omega(\delta G^a_{w\alpha})\left(\sum_\beta-\left[j\left(\mathcal{F}_r^{\alpha\beta}\right)G_{r\beta}-j\left(\mathcal{F}_w^{\alpha\beta}\right)G_{w\beta}\right]^a-\frac{d}{d\xi^\beta}\mathcal{F}_r^{a\alpha\beta}+\mathcal{F}_r^{a\alpha\beta}\sum_{\gamma i}P'^i_\gamma\frac{dP^\gamma_i}{d\xi^\beta}\right)\varpi_4$
$\forall\delta G^a_{r\alpha}$:
$C_G\int_\Omega(\delta G^a_{w\alpha})\left(\sum_\beta-\left[j\left(\mathcal{F}_r^{\alpha\beta}\right)G_{r\beta}-j\left(\mathcal{F}_w^{\alpha\beta}\right)G_{w\beta}\right]^a-\frac{d}{d\xi^\beta}\mathcal{F}_r^{a\alpha\beta}+\mathcal{F}_r^{a\alpha\beta}\sum_{\gamma i}P'^i_\gamma\frac{dP^\gamma_i}{d\xi^\beta}\right)\varpi_4$
$-\frac{1}{2}C_I\int_0^T fu^\alpha\left[\left(a_w^2-\frac{1}{2}j\left(w\right)j\left(w\right)\right)\varkappa\right]^a\delta G^a_{r\alpha}(\delta G^a_{w\alpha})\,dt = 0$
with
$\frac{d}{dG^a_{r\alpha}}\left\langle \mathcal{F}_{G\alpha\beta},\mathcal{F}_G^{\alpha\beta}\right\rangle = -G_G\sum_\gamma\left[j\left(\mathcal{F}_r^{\alpha\gamma}\right)G_{r\gamma}-j\left(\mathcal{F}_w^{\alpha\gamma}\right)G_{w\gamma}\right]^a$
$\frac{d\mathcal{L}}{d\partial_\beta G^a_{r\alpha}} = C_G\mathcal{F}_r^{a\beta\alpha}$
$X_\alpha = -\frac{1}{2}j\left(w\right)\partial_\alpha w+\left(a_w^2-\frac{1}{2}j\left(w\right)j\left(w\right)\right)G_{r\alpha}+a_w j\left(w\right)G_{w\alpha}$

$\sum_b \varkappa^b \left[\left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \delta G_{r\alpha}\right]^b = a_w^2 \varkappa^a \delta G_{r\alpha}^a - \frac{1}{2} \sum_b \varkappa^b \left[j(w) j(w) \delta G_{r\alpha}\right]^b$
$= a_w^2 \varkappa^a \delta G_{r\alpha}^a - \frac{1}{2} \sum_b \varkappa^b \left(w^b \left(w^t \delta G_{r\alpha}\right) - \left(w^t w\right) \delta G_{r\alpha}^b\right)$
$= a_w^2 \varkappa^a \delta G_{r\alpha}^a - \frac{1}{2} \sum_b \varkappa^b w^b \left(w^a \delta G_{r\alpha}^a\right) - \left(w^t w\right) \varkappa^a \delta G_{r\alpha}^a$
$= \left(a_w^2 + \frac{1}{2}\left(w^t w\right)\right) \varkappa^a \delta G_{r\alpha}^a - \frac{1}{2}\left(w^a \delta G_{r\alpha}^a\right)\left(\varkappa^t w\right)$
$= \left(\left(a_w^2 + \frac{1}{2}\left(w^t w\right)\right) \varkappa^a - \frac{1}{2} w^a \left(\varkappa^t w\right)\right) \delta G_{r\alpha}^a$
$= \left[\left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \varkappa\right]^a \delta G_{r\alpha}^a$
$C_G \int_\Omega \left(\delta G_{w\alpha}^a\right) \left(\sum_\beta \left[j\left(\mathcal{F}_r^{\alpha\beta}\right) G_{w\beta} + j\left(\mathcal{F}_w^{\alpha\beta}\right) G_{r\beta}\right]^a + \frac{d}{d\xi^\beta} \mathcal{F}_w^{a\alpha\beta} - \mathcal{F}_w^{a\alpha\beta} \sum_{\gamma i} P_\gamma'^i \frac{dP_i^\gamma}{d\xi^\beta}\right) \varpi_4$
$+ \frac{1}{2} C_I \int_0^T f u^\alpha a_w \left[j(w) \varkappa\right]^a \left(\delta G_{w\alpha}^a\right) dt = 0$
$C_G \int_\Omega \left(\delta G_{w\alpha}^a\right) \left(\sum_\beta \left[j\left(\mathcal{F}_r^{\alpha\beta}\right) G_{r\beta} - j\left(\mathcal{F}_w^{\alpha\beta}\right) G_{w\beta}\right]^a + \frac{d}{d\xi^\beta} \mathcal{F}_r^{a\alpha\beta} - \mathcal{F}_r^{a\alpha\beta} \sum_{\gamma i} P_\gamma'^i \frac{dP_i^\gamma}{d\xi^\beta}\right) \varpi_4$
$+ \frac{1}{2} C_I a_w \int_0^T f u^\alpha \left[\left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \varkappa\right]^a \delta G_{r\alpha}^a \left(\delta G_{w\alpha}^a\right) dt = 0$

By combination, as done previously :
$C_G \int_\Omega \left\langle \delta G_\alpha, \sum_\beta \left[\mathcal{F}_G^{\alpha\beta}, G_\beta\right] + \frac{d}{d\xi^\beta} \mathcal{F}_G^{\alpha\beta} - \mathcal{F}_G^{\alpha\beta} \sum_{\gamma i} P_\gamma'^i \frac{dP_i^\gamma}{d\xi^\beta} \right\rangle \varpi_4$
$+ \frac{1}{2} C_I \int_0^T f u^\alpha \left\langle \delta G_\alpha, \upsilon \left(\left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \varkappa, a_w j(w) \varkappa\right)\right\rangle dt = 0$

The equation holds for any compactly smooth δG. Take $\delta G = \widehat{\delta G}_\alpha d\xi^\alpha$ with $\widehat{\delta G}$ null outside a small tube ∂C enclosing the world line u.
$C_G \int_\Omega \left\langle \delta G_\alpha, \sum_\beta \left[\mathcal{F}_G^{\alpha\beta}, G_\beta\right] \right\rangle \varpi_4$
$= \int_0^T \left(\int_{\partial C} \left\langle \widehat{\delta G}_\alpha, \sum_\beta \left[\mathcal{F}_G^{\alpha\beta}, G_{r\beta}\right] \right\rangle \varpi_3(x)\right) \sqrt{-\langle V, V\rangle} dt$
$= \int_0^T f \left(\int_{\partial C} \left\langle \widehat{\delta G}_\alpha, \sum_\beta \left[\mathcal{F}_G^{\alpha\beta}, G_{r\beta}\right] \right\rangle \varpi_3(x)\right) dt$
By shrinking ∂C the integral converges to : $\int_0^T f \left\langle \widehat{\delta G}_\alpha, \sum_\beta \left[\mathcal{F}_G^{\alpha\beta}, G_\beta\right] \right\rangle (\Phi_v(t,x)) dt$
and similarly for :
$C_G \int_\Omega \left\langle \delta G_\alpha, \sum_\beta \frac{d}{d\xi^\beta} \mathcal{F}_G^{\alpha\beta} - \mathcal{F}_G^{\alpha\beta} \sum_{\gamma i} P_\gamma'^i \frac{dP_i^\gamma}{d\xi^\beta} \right\rangle \varpi_4$
$\to C_G \int_0^T f \left\langle \widehat{\delta G}_\alpha, \sum_\beta \frac{d}{d\xi^\beta} \mathcal{F}_G^{\alpha\beta} - \mathcal{F}_G^{\alpha\beta} \sum_{\gamma i} P_\gamma'^i \frac{dP_i^\gamma}{d\xi^\beta} \right\rangle (\Phi_v(t,x)) dt$
So that the equation reads for any point $\Phi_V(t,x)$ on the trajectory
$f \sum_\beta \left[\mathcal{F}_G^{\alpha\beta}, G_\beta\right] + f \upsilon \left(\left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \varkappa, a_w j(w) \varkappa\right)$
$+ C_G f \left(\sum_\beta \frac{d}{d\xi^\beta} \mathcal{F}_G^{\alpha\beta} - \mathcal{F}_G^{\alpha\beta} \sum_{\gamma i} P_\gamma'^i \frac{dP_i^\gamma}{d\xi^\beta}\right) = 0$
$\forall \alpha$:

$$\sum_\beta \left[\mathcal{F}_G^{\alpha\beta}, G_\beta\right] + \upsilon \left(\left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \varkappa, a_w j(w) \varkappa\right) + C_G \frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_G^{\alpha\beta} \det P'\right) = 0 \tag{76}$$

So we get back the previous equation, which holds along the trajectory.
By a similar method we can get the equation for the other fields :
$\forall a, \alpha$:

$$C_A \sum_\beta \left[\mathcal{F}_A^{\alpha\beta}, \grave{A}_\beta\right]^a - C_I \frac{1}{4i} \sum_{p=1}^N N_p u_p^\alpha \left\langle \psi_0, \psi_0 [\theta_a]\right\rangle + C_A \frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_A^{\alpha\beta} \det P'\right) = 0 \tag{77}$$

Tetrad equation

The functional derivative reads:

$\frac{\delta}{\delta P_i^\alpha}\left(\int_\Omega \sum_{\lambda\mu}\left(C_G\left\langle \mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle\right)\varpi_4\right)(\delta P_i^\alpha)$
$= 4\sum_{j\gamma\lambda\mu=0}^3 \int_\Omega \eta^{ij}P_j^\lambda g^{\mu\gamma}\left(C_G\left\langle \mathcal{F}_{G\alpha\gamma},\mathcal{F}_{G\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\alpha\gamma},\mathcal{F}_{A\lambda\mu}\right\rangle\right)(\delta P_i^\alpha)\varpi_4$
$-\int_\Omega\left(\sum_{\lambda\mu}\left(C_G\left\langle \mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle\right)\right)\left(P_\alpha'^i\right)(\delta P_i^\alpha)\varpi_4$
$\frac{\delta}{\delta P_i^\alpha}\left(C_I\int_0^T f\left(\Phi_V(t,x)\right)\sum_\beta u^\beta \operatorname{Im}\left\langle \psi_p,\nabla_\beta\psi_p\right\rangle dt\right)(\delta P_i^\alpha)$
$i=0 := C_I\int_0^T f\left(\Phi_V(t,x)\right)c\left(2a_w^2-1\right)\operatorname{Im}\left\langle \psi_p,\nabla_\alpha\psi_p\right\rangle(\delta P_i^\alpha)\,dt$
$i>0 := C_I\int_0^T f\left(\Phi_V(t,x)\right)ca_w w^i \operatorname{Im}\left\langle \psi_p,\nabla_\alpha\psi_p\right\rangle(\delta P_i^\alpha)\,dt$

The equation holds for any compactly smooth δP. Take δP_α^i null outside a small tube ∂C enclosing the world line u. When shrinking $\partial C$the first integral converges to :

$4\sum_{j\gamma\lambda\mu=0}^3\int_0^T f\eta^{ij}g^{\gamma\mu}P_j^\lambda\left(C_G\left\langle \mathcal{F}_{G\alpha\gamma},\mathcal{F}_{G\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\alpha\gamma},\mathcal{F}_{A\lambda\mu}\right\rangle\right)(\delta P_i^\alpha)\,dt$
$-\int_0^T f\left(\sum_{\lambda\mu}\left(C_G\left\langle \mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle\right)\right)\left(P_\alpha'^i\right)(\delta P_i^\alpha)\,dt$

So we have the equations along the trajectory :

i = 0 :
$4\sum_{j\gamma\lambda\mu=0}^3\eta^{ij}g^{\gamma\mu}P_j^\lambda\left(C_G\left\langle \mathcal{F}_{G\alpha\gamma},\mathcal{F}_{G\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\alpha\gamma},\mathcal{F}_{A\lambda\mu}\right\rangle\right)$
$-\sum_{\lambda\mu}\left(C_G\left\langle \mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle\right)\left(P_\alpha'^i\right)$
$+C_I c\left(2a_w^2-1\right)\operatorname{Im}\left\langle \psi,\nabla_\alpha\psi\right\rangle = 0$

i > 0 :
$4\sum_{j\gamma\lambda\mu=0}^3\eta^{ij}g^{\gamma\mu}P_j^\lambda\left(C_G\left\langle \mathcal{F}_{G\alpha\gamma},\mathcal{F}_{G\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\alpha\gamma},\mathcal{F}_{A\lambda\mu}\right\rangle\right)$
$-\sum_{\lambda\mu}\left(C_G\left\langle \mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle\right)\left(P_\alpha'^i\right)$
$+C_I ca_w w^i \operatorname{Im}\left\langle \psi,\nabla_\alpha\psi\right\rangle = 0$

By multiplication with P_i^β an summation over i :
$\sum_i 4\sum_{j\gamma\lambda\mu=0}^3\eta^{ij}g^{\gamma\mu}P_j^\lambda P_i^\beta\left(C_G\left\langle \mathcal{F}_{G\alpha\gamma},\mathcal{F}_{G\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\alpha\gamma},\mathcal{F}_{A\lambda\mu}\right\rangle\right)$
$= 4\sum_{\gamma\lambda\mu=0}^3 g^{\gamma\mu}g^{\lambda\beta}\left(C_G\left\langle \mathcal{F}_{G\alpha\gamma},\mathcal{F}_{G\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\alpha\gamma},\mathcal{F}_{A\lambda\mu}\right\rangle\right)$
$= 4\sum_{\gamma=0}^3 C_G\left\langle \mathcal{F}_{G\alpha\gamma},\mathcal{F}_G^{\beta\gamma}\right\rangle + C_A\left\langle \mathcal{F}_{A\alpha\gamma},\mathcal{F}_A^{\beta\gamma}\right\rangle$
$\sum_i\sum_{\lambda\mu}\left(C_G\left\langle \mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle\right)\left(P_\alpha'^i\right)P_i^\beta$
$= \delta_\beta^\alpha\sum_{\lambda\mu}\left(C_G\left\langle \mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle\right)$
$C_I c\left(\left(2a_w^2-1\right)P_0^\beta + a_w\sum_{i=1}^3 w^i P_i^\beta\right)\operatorname{Im}\left\langle \psi,\nabla_\alpha\psi\right\rangle$
$= C_I u^\beta \operatorname{Im}\left\langle \psi,\nabla_\alpha\psi\right\rangle$

we get the equation on the trajectory :

$\forall\alpha,\beta$:

$$\begin{aligned}&4\sum_\gamma\left(C_G\left\langle \mathcal{F}_{G\alpha\gamma},\mathcal{F}_G^{\beta\gamma}\right\rangle + C_A\left\langle \mathcal{F}_{A\alpha\gamma},\mathcal{F}_A^{\beta\gamma}\right\rangle\right) + C_I u^\beta\operatorname{Im}\left\langle \psi,\nabla_\alpha\psi\right\rangle\\ &= \delta_\beta^\alpha\sum_{\lambda\mu}\left(C_G\left\langle \mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle \mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle\right)\end{aligned} \tag{78}$$

and with $\alpha=\beta$ and summing :

$$\left\langle \psi,\nabla_u\psi\right\rangle = 0 \tag{79}$$

As :
$\langle\psi,\nabla_u\psi\rangle = 2\left\langle\psi_r,\frac{d\psi_r}{d\tau}\right\rangle \in i\mathbb{R}$

$$\left\langle\psi_r,\frac{d\psi_r}{d\tau}\right\rangle = 0 \tag{80}$$

f equation
$\int_0^T f\left(\Phi_V\left(t,x\right)\right)\left(C_P\left\langle\psi_0,\psi_0\right\rangle + C_I\,\mathrm{Im}\left\langle\psi_p,\nabla_u\psi_p\right\rangle\right)dt$
$= C_P\left\langle\psi_0,\psi_0\right\rangle\int_0^T f\left(\Phi_V\left(t,x\right)\right)dt = C_P\left\langle\psi_0,\psi_0\right\rangle T\left(x\right)$
So the proper time of arrival is defined by the world line, depending of all the other variables.

11.3.3 Conclusion

The equations are identical or similar in both cases. This is due to the specification of the lagrangian by scalar products. But notice that in the case of a distribution we have equations which hold all over Ω, meanwhile they hold only on the trajectory for a single particle. However the proximity of the equations show that, by continuity, the results hold for a general distribution of particles, as long as the type of the particles does not change.

We can sum up these results :
i) the vector fields defining the world lines are divergence free : $div\left(u_p\right)=0$
ii) the state of the particles and the world lines are defined, for each type of particle, by the first order ODE in $\frac{dw_p}{d\tau_p},\frac{dr_p}{d\tau_p}$ with parameters the value of the fields along the world line $\widehat{A},\widehat{G}_r,\widehat{G}_w,\widehat{P}$:
$\frac{d\psi_{rp}}{d\tau_p} = \left[\psi_{rp}\right]\left[\widehat{A}\right]$
$+i\frac{1}{2}\sum_{a=1}^3\left[-\frac{1}{2}j\left(w_p\right)\frac{dw_p}{d\tau_p}+\left(a_w^2-\frac{1}{2}j\left(w_p\right)j\left(w_p\right)\right)\widehat{G}_r+a_wj\left(w_p\right)\widehat{G}_w\right]^a\left[\widetilde{\gamma}_a\right]\left[\psi_{rp}\right]$
$\forall a=1,2,3:$
$\sum_\alpha\mathrm{Im}\left\langle\psi_p,\nabla_\alpha\psi_p\right\rangle\frac{\partial u_p^\alpha}{\partial w_p^a}$
$=\frac{1}{4}\left[\left(-j\left(w_p\right)\frac{d\varkappa_p}{d\tau_p}+\left(\frac{3}{2}j\left(\varkappa_p\right)j\left(w_p\right)-j\left(w_p\right)j\left(\varkappa_p\right)-\frac{1}{2}\left(w_p^t\varkappa_p\right)\right)\widehat{G}_r-2a_wj\left(\varkappa_p\right)\widehat{G}_w\right)\right]^a$
$\mathrm{Im}\left\langle\psi_p,\nabla_\alpha\psi_p\right\rangle=\frac{1}{i}\left\langle\psi_{rp},\partial_\alpha\psi_{rp}\right\rangle$
$+\frac{1}{i}\left\langle\psi_{0p},\psi_{0p}\left[\widehat{A}\right]\right\rangle-\frac{1}{2}\varkappa^t\left[-\frac{1}{2}j\left(w_p\right)\frac{dw_p}{d\tau_p}+\left(a_w^2-\frac{1}{2}j\left(w_p\right)j\left(w_p\right)\right)\widehat{G}_r+a_wj\left(w_p\right)\widehat{G}_w\right]$
$\left\langle\psi_p,\nabla_{u_p}\psi_p\right\rangle=2\left\langle\psi_{rp},\frac{d\psi_{rp}}{d\tau_p}\right\rangle$
$=-i\frac{1}{2}\sum_{a=1}^3\left[\left(\frac{1}{a_r}-\frac{1}{2}j\left(r_p\right)+\frac{1}{4a_r}j\left(r_p\right)j\left(r_p\right)\right)\frac{dr_p}{d\tau_p}\right]^a\left\langle\psi_{0p},\widetilde{\gamma}_a\psi_{0p}\right\rangle=0$
iii) the fields are defined by :
$\forall\alpha:$
$G_G\left[\mathcal{F}_G^{\alpha\beta},G_\beta\right]+\frac{1}{2}C_I\sum_{p=1}^N N_pu_p^\alpha\upsilon\left(\left(a_w^2-\frac{1}{2}j\left(w_p\right)j\left(w_p\right)\right)\varkappa_p,a_wj\left(w_p\right)\varkappa_p\right)=C_G\frac{1}{\det P'}\sum_\beta\frac{d}{d\xi^\beta}\left(\mathcal{F}_G^{\alpha\beta}\det P'\right)$
$\forall a,\alpha:$
$C_A\sum_\beta\left[\grave{A}_\beta,\mathcal{F}_A^{\alpha\beta}\right]^a+C_I\frac{1}{4i}\sum_{p=1}^N N_pu_p^\alpha Tr\left[\psi_{rp}^*\right]\left[\psi_{rp}\right]\left[\theta_a\right]=C_A\frac{1}{\det P'}\sum_\beta\frac{d}{d\xi^\beta}\left(\mathcal{F}_A^{a\beta\alpha}\det P'\right)$
iv) the tetrad over the world line is defined by :
$\forall\alpha,\beta,p:$
$4\sum_{\gamma=0}^3\left(C_G\left\langle\mathcal{F}_{G\alpha\gamma},\mathcal{F}_G^{\beta\gamma}\right\rangle+C_A\left\langle\mathcal{F}_{A\alpha\gamma},\mathcal{F}_A^{\beta\gamma}\right\rangle\right)+C_Iu_p^\beta\,\mathrm{Im}\left\langle\psi,\nabla_\alpha\psi\right\rangle$

$= \delta_\beta^\alpha \sum_{\lambda\mu} \left(C_G \left\langle \mathcal{F}_{G\lambda\mu}, \mathcal{F}_G^{\lambda\mu} \right\rangle + C_A \left\langle \mathcal{F}_{A\lambda\mu}, \mathcal{F}_A^{\lambda\mu} \right\rangle \right)$
and over Ω by :
$\forall \alpha, \beta = 0...3$:
$C_I \sum_{p=1}^N N_p u_p^\beta \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle + 4 \sum_{\gamma=0}^3 \left(C_G \left\langle \mathcal{F}_{G\alpha\gamma}, \mathcal{F}_G^{\beta\gamma} \right\rangle + C_A \left\langle \mathcal{F}_{A\alpha\gamma}, \mathcal{F}_A^{\beta\gamma} \right\rangle \right) = \mathcal{L}\delta_\beta^\alpha$
with the identity :
$\sum_i \frac{d\mathcal{L}}{dP_i^\alpha} P_i^\beta = C_I \sum_{p=1}^N N_p u_p^\beta \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle + 4 \sum_{\gamma=0}^3 \left(C_G \left\langle \mathcal{F}_{G\alpha\gamma}, \mathcal{F}_G^{\beta\gamma} \right\rangle + C_A \left\langle \mathcal{F}_{A\alpha\gamma}, \mathcal{F}_A^{\beta\gamma} \right\rangle \right)$
v) the proper time τ_p is specific to each particle

The equations for the fields are not linear in $\mathcal{F}$, if U is not abelian, so that practically they give a true physical meaning to the potentials : there is no possibility to fix a gauge for the fields.

The state of each particle is defined in its proper time from the value of the fields and P at each point of its world line. So actually the position m does not matter, as long as the value of these variables is known.

The proper time itself cannot be computed from the equations. So we have the paradoxical situation that the position of the particle can be forecasted in its proper time, but not in the time of the observer. Meanwhile of course the observable is the trajectory, in the time of the observer.

The map $\psi_r : \mathbb{R} \to E_0 \otimes F$ belongs to the kernel of the operator :
$\psi_r \to$
$\frac{d\psi_r}{d\tau} - [\psi_r] \left[\widehat{\hat{A}} \right] + i \frac{1}{2} \sum_{a=1}^3 \left[-\frac{1}{2} j(w) \frac{dw}{d\tau} + \left(a_w^2 - \frac{1}{2} j(w) j(w) \right) \widehat{G}_r + a_w j(w) \widehat{G}_w \right]^a [\widetilde{\gamma}_a] [\psi_r]$
which is strongly elliptic. As Ω is relatively compact, the operator is Fredholm and its kernel is finite dimensional in the space of sections. So the map : $\psi_{rp} : \mathbb{R} \to E_0 \otimes F$ belongs to a finite dimensional vector space. This extend to r and w moreover we have :
$\sum_{a=1}^3 \left[\left(\frac{1}{a_r} - \frac{1}{2} j(r) + \frac{1}{4a_r} j(r) j(r) \right) \frac{dr}{d\tau} \right]^a \langle \psi_0, \widetilde{\gamma}_a \psi_0 \rangle = 0$

The equations give continuous solutions for $r, w \in C\left(\mathbb{R}; \mathbb{R}^3\right)$ and so continuous solutions for σ_w, σ_r. The domain of these maps is connected, so are their ranges. σ_r stays in the same connected component of Spin(3) : the spatial spin does not change in a continuous process. One can see that the equations for w, ψ_p do not depend on the spin, and if we take the opposite spin ($\sigma_w \to -\sigma_w$) then $\frac{\partial u_p^\alpha}{\partial w_p^a} \to -\frac{\partial u_p^\alpha}{\partial w_p^a}$ and we have the same curve wih the opposite spatial speed. Similarly if ψ_r is solution for w, then $-\psi_r$ is still solution for -w.

11.4 Currents

11.4.1 Geometric form of the fields equations

The Noether currents are usually introduced through the equivariance of the Lagrange equations. But it is easier and more straigthforward to proceed with a direct computation. As they involve the fields equations we will use the distribution model.

Let us denote the tensors :

Notation 23 $F_G = C_I \frac{1}{2} \sum_{p=1}^N N_p \upsilon \left(\left(a_w^2 - \frac{1}{2} j(w_p) j(w_p) \right) \varkappa_p, a_w j(w_p) \varkappa_p \right) \otimes u_p \in T_1 Spin \otimes TM$

Notation 24 $F_A = -C_I \frac{1}{4i} \sum_{a=1}^m \sum_{p=1}^N N_p \langle \psi_{0p}, \psi_{0p} [\theta_a] \rangle \overrightarrow{\theta}_a \otimes u_p \in T_1 U \otimes TM$

Notation 25 $B_G = G_G \sum_{\alpha\beta} \left[\mathcal{F}_G^{\alpha\beta}, G_\beta\right] \otimes \partial\xi_\alpha \in T_1 Spin \otimes TM$

Notation 26 $B_A = C_A \sum_{\alpha\beta} \left[\mathcal{F}_A^{\alpha\beta}, \grave{A}_\beta\right] \otimes \partial\xi_\alpha \in T_1 U \otimes TM$

Notation 27 $\Phi_G = F_G + B_G \in T_1 Spin \otimes TM$

Notation 28 $\Phi_A = F_A + B_A \in T_1 U \otimes TM$

With :
$\left[\left(a_w^2 - \frac{1}{2} j\left(w_p\right) j\left(w_p\right)\right) \varkappa_p\right]^a = \langle \psi_p, \widetilde{\gamma}_a \psi_p \rangle$
$a_w \left[j\left(w_p\right) \varkappa_p\right]^a = \langle \psi_p, \gamma_a \gamma_0 \psi_p \rangle$
$\upsilon\left(\left(a_w^2 - \frac{1}{2} j\left(w_p\right) j\left(w_p\right)\right) \varkappa_p, a_w j\left(w_p\right) \varkappa_p\right) = \sum_{a=1}^{3} \langle \psi_p, \widetilde{\gamma}_a \psi_p \rangle \overrightarrow{\kappa}_a - \langle \psi_p, \gamma_a \gamma_0 \psi_p \rangle \overrightarrow{\kappa}_{a+3}$

The equations for the potentials read :
$\alpha = 0..3 : \Phi_G^\alpha = C_G \frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_G^{\alpha\beta} \det P'\right)$
$\alpha = 0..3 : \Phi_A^\alpha = C_A \frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_A^{\alpha\beta} \det P'\right)$
The application of the 4 form ϖ_4 gives:
$i_{\Phi_G} \varpi_4 = \varpi_4\left(\Phi_G\right) = \sum_{\alpha=0}^{3} (-1)^{\alpha+1} \Phi_G^\alpha \left(\det P'\right) d\xi^0 \wedge ... \widehat{d\xi^\alpha} ... \wedge d\xi^3$
$i_{\Phi_A} \varpi_4 = \varpi_4\left(\Phi_A\right) = \sum_{\alpha=0}^{3} (-1)^{\alpha+1} \Phi_A^\alpha \left(\det P'\right) d\xi^0 \wedge ... \widehat{d\xi^\alpha} ... \wedge d\xi^3$
For any tensor : $X = \sum_{\{\alpha\beta\}} X^{\alpha\beta} \partial\xi_\alpha \wedge \partial\xi_\beta \in \Lambda_2 TM$, by a straigthforward computation, we get the value of :
$\varpi_4\left(X\right) = 2\left(\det P'\right) \{X^{32} d\xi^0 \wedge d\xi^1 + X^{13} d\xi^0 \wedge d\xi^2 + X^{21} d\xi^0 \wedge d\xi^3$
$+ X^{03} d\xi^2 \wedge d\xi^1 + X^{02} d\xi^1 \wedge d\xi^3 + X^{01} d\xi^3 \wedge d\xi^2\}$
and : $d \circ \varpi_4\left(X\right) = -2 \sum_{\alpha\beta=0}^{3} (-1)^{\alpha+1} \partial_\beta \left(X^{\alpha\beta} \det P'\right) d\xi^0 \wedge ... \widehat{d\xi^\alpha} ... \wedge d\xi^3$
So we have for the tensors :
$d \circ \varpi_4 \left(\sum_{\alpha\beta} \mathcal{F}_G^{\alpha\beta} \partial\xi_\alpha \wedge \partial\xi_\beta\right) = d \circ \varpi_4 \left(\frac{1}{2} \sum_{\{\alpha\beta\}} \mathcal{F}_G^{\alpha\beta} \partial\xi_\alpha \wedge \partial\xi_\beta\right)$
$= -\sum_{\alpha\beta=0}^{3} (-1)^{\alpha+1} \partial_\beta \left(\mathcal{F}_G^{\alpha\beta} \det P'\right) d\xi^0 \wedge ... \widehat{d\xi^\alpha} ... \wedge d\xi^3$
$d \circ \varpi_4 \left(\sum_{\alpha\beta} \mathcal{F}_A^{\alpha\beta} \partial\xi_\alpha \wedge \partial\xi_\beta\right) = d \circ \varpi_4 \left(\frac{1}{2} \sum_{\{\alpha\beta\}} \mathcal{F}_A^{\alpha\beta} \partial\xi_\alpha \wedge \partial\xi_\beta\right)$
$= -\sum_{\alpha\beta=0}^{3} (-1)^{\alpha+1} \partial_\beta \left(\mathcal{F}_A^{\alpha\beta} \det P'\right) d\xi^0 \wedge ... \widehat{d\xi^\alpha} ... \wedge d\xi^3$
Thus we can write the equations in the geometric form :

$$i_{\Phi_G} \varpi_4 = -C_G d \circ \varpi_4 \left(\sum_{\alpha\beta} \mathcal{F}_G^{\alpha\beta} \partial\xi_\alpha \wedge \partial\xi_\beta\right) \tag{81}$$

$$i_{\Phi_A} \varpi_4 = -C_A d \circ \varpi_4 \left(\sum_{\alpha\beta} \mathcal{F}_A^{\alpha\beta} \partial\xi_\alpha \wedge \partial\xi_\beta\right) \tag{82}$$

The use of ϖ_4 is, from a mathematical point of view, necessary to go from a two form on TM to a more usual two form on TM*. But physically it has a clear meaning : it represents a density, accounting for the varying volume measure detP' on M. So in the equations above we have a relation between the density of the currents Φ and the variation of the density of the fields $\mathcal{F}$: the sign -

shows that the fields $\mathcal{F}$ appear to be the "source" of the currents, or conversely, the currents are "stored" in the fields.

The tensors Φ_G^a, Φ_A^a are vector fields. They are divergence free:
$\pounds_{\Phi_G}\varpi_4 = div(\Phi_G)\varpi_4 = i_{\Phi_G} d \circ \varpi_4 + d \circ i_{\Phi_G}\varpi_4 = d \circ i_{\Phi_G}\varpi_4$
$= -C_G d \circ d \circ \varpi_4 \left(\sum_{\alpha\beta} \mathcal{F}_G^{\alpha\beta} \partial\xi_\alpha \wedge \partial\xi_\beta\right) = 0$

They are conserved in the following meaning. The flow of the vector through a hypersurface ∂C can be defined as $\int_{\partial C} i_{\Phi_G}\varpi_4$,and if C is a manifold with boundary : $\int_C d \circ i_{\Phi_G}\varpi_4 = \int_{\partial C} i_{\Phi_G}\varpi_4 = 0$

Two hypersurfaces, defined as subsets of M with $t = t_1, t = t_2$ can be taken as boundaries, so :
$\int_{\partial C(t_1)} i_{\Phi_G}\varpi_4 = \int_{\partial C(t_2)} i_{\Phi_G}\varpi_4$

11.4.2 The meaning of the currents

Currents for the particles

Their meaning is clear : the currents follow the world line (they have the same curves) and are proportional to the density of particles :
$F_G = C_I \frac{1}{2} \sum_{p=1}^N N_p \upsilon \left(\left(a_w^2 - \frac{1}{2} j(w_p)\, j(w_p)\right) \varkappa_p, a_w j(w_p)\, \varkappa_p\right) \otimes u_p \in T_1 Spin \otimes TM$
$F_A = -C_I \frac{1}{4i} \sum_{a=1}^m \sum_{p=1}^N N_p \langle \psi_{0p}, \psi_{0p}[\theta_a]\rangle \overrightarrow{\theta}_a \otimes u_p \in T_1 U \otimes TM$

They are valued in the Lie algebras. And, as it is obvious in the single particle model, their support is the world line : their extension to the whole of Ω is uniquely due to the introduction of N.

$\rho_A^a = \langle \psi_{0p}, \psi_{0p}[\theta_a]\rangle$ can be seen as the charge carried by the particles. There are as many charges as the dimension m of T_1U, and each type of particle carries a specific, fixed charge $-C_I \frac{1}{4i} \langle \psi_{0p}, \psi_{0p}[\theta_a]\rangle$. In a Yang-Mills model this value has a meaning with respect to a standard given by the elementary particles, as basis of F.

Similarly one should expect that $C_I \frac{1}{2} \upsilon \left(\left(a_w^2 - \frac{1}{2} j(w_p)\, j(w_p)\right) \varkappa_p, a_w j(w_p)\, \varkappa_p\right) = C_I \frac{1}{2} \upsilon \left(\langle \psi_p, \widetilde{\gamma}_a \psi_p\rangle, \langle \psi_p, \gamma_a \gamma_0 \psi_p\rangle\right)$ is related to the gravitational charge carried by the particle, that is its mass. Let us write :
$\left(a_w^2 - \frac{1}{2} j(w_p)\, j(w_p)\right) \varkappa_p = \rho_r$
$a_w j(w_p)\, \varkappa_p = \rho_w$
then :
$\left(a_w^2 - \frac{1}{2} j(w_p)\, j(w_p)\right) \varkappa_p = \rho_r \Leftrightarrow \varkappa_p = \left(\frac{1}{a_w^2} + \frac{1}{2a_w^2(3a_w^2-2)} j(w)\, j(w)\right) \rho_r$
$a_w j(w_p) \left(\frac{1}{a_w^2} + \frac{1}{2a_w^2(3a_w^2-2)} j(w)\, j(w)\right) \rho_r = \rho_w \Leftrightarrow j(w_p)\, \rho_r = \frac{3a_w^2-2}{a_w} \rho_w$
and $\rho_w^t \rho_r = 0$

From there if ρ_w, ρ_r were constant, the world line w would be defined, whatever the field, which is not realistic.

Unfortunately we cannot go further with this simple model. It would be nice to express the scalar products $\langle \psi_p, \widetilde{\gamma}_a \psi_p\rangle, \langle \psi_p, \gamma_a \gamma_0 \psi_p\rangle$ from ψ_0 as it is done for A, but this should require a further knowledge of U and, indeed, an integration of the Spin and U groups.

It is clear that, in a Yang-Mills model like this one, the gravitational field and the other fields should be put on the same footing, which requires a true GUT model. And for the same reason it seems difficult to deal correctly with the mass in a model which ignores gravitation, such as the Standard Model.

Currents for the fields

The currents B_G, B_A share some characteristics with the currents carried by the particles : they define vector fields, they are valued in the Lie algebras, and there are as many kinds of currents as the dimensions of these algebras:

$B_G = G_G \sum_{\alpha\beta} \left[\mathcal{F}_G^{\alpha\beta}, G_\beta\right] \otimes \partial\xi_\alpha \in T_1 Spin \otimes TM$

$B_A = C_A \sum_{\alpha\beta} \left[\mathcal{F}_A^{\alpha\beta}, \grave{A}_\beta\right] \otimes \partial\xi_\alpha \in T_1 U \otimes TM$

However there are several important differences.

i) they are defined all over Ω : they do not have a compact support.

ii) the vector fields are usually different for each kind of current

iii) there is no obvious "charge" linked to the currents, except that it is null if the group U is abelian (and indeed the photon does not carry a charge, contrary to the other bosons).

The equation for G reads in the single particle model :

$\forall\alpha : \int_0^T f \left\langle \delta G_\alpha, F_G^\alpha \right\rangle dt = \int_\Omega \left\langle \delta G_\alpha, B_G^\alpha + \frac{1}{\det P'} C_G \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_G^{\alpha\beta} \det P'\right) \right\rangle \varpi_4$

For α fixed $\delta G_\alpha \in C_{\infty c}\left(\Omega; T_1 Spin\right)$, on the left hand side F_G^α is a distribution, with compact support, acting on δG_α. Similarly on the right hand side $B_G^\alpha + \frac{1}{\det P'} C_G \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_G^{\alpha\beta} \det P'\right)$is a distribution, with continuous support in Ω, acting on δG_α.

F is a perturbation, which occurs at the location of the particle. The perturbation entails a variation of the field through the usual derivative

$\sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_G^{\alpha\beta} \det P'\right)$ and an additional effect B_G^α which can be seen as the propagation of the disturbance over Ω, a radiation emitted along a specific vector field. And similarly for the other fields. The radiation (B_G, B_A) is similar to the Bremsstrahlung effect : indeed in the GR context the particle is usually submitted to accelerations. It is continuous, so the spectrum of the energy which is radiated is usually continuous.

For the reader familiar with generalized functions, B_G^α is equivalent to the jump which complements the derivative at a point where a function is not normally differentiable.

This equation shows particles as the sources of the fields, however the radiation which is emitted has a life of its own, and exists even in areas which are void of any particles. This leads to give to B a special status, and identify these currents to the bosons.

The bosons should propagate at the speed of light. We have seen what could be the mathematical formulation of this rule. In particular we should have :

$\forall a : \sum_{\alpha\beta\lambda\mu} g_{\alpha\beta} \left[\mathcal{F}_G^{\alpha\lambda}, G_\lambda\right]^a \left[\mathcal{F}_G^{\beta\mu}, G_\mu\right]^a = 0$

but it is not obious that this is met in the model. Of course such relations can be taken as additional hypothesis and indeed this seems natural with regard to what is said for the particles. However, as well as for the mass and charges, we face the issues raised by the non integration of the gravitational and other fields. It is clear that a "true" model for the boson should give :

$B_A = \left(\sum_a B_A^a \overrightarrow{\theta_a}\right) \otimes W, B_G = \left(\sum_a B_G^a \overrightarrow{\kappa_a}\right) \otimes W$

with a single vector field W for all the currents B_A, B_G and $\langle W, W\rangle = 0$.

As B comes from the partial derivative $\frac{\partial}{\partial \grave{A}_\alpha^a}$ of the part of the action related to the fields with respect to the potential, this action should be something like : $\sum_{\lambda\mu} C_A B_A \grave{A}\left(W\right) + C_G B_G G\left(W\right)$ where B_A, B_G do not depend on the potential.

The key point to see if and how these currents can be assimilated to particles is their behaviour with respect to the momentum and energy.

Electromagnetic field

The second Maxwell equations in the GR context can be written:

$\mu_0 J^\alpha \sqrt{-\det g} = \sum_\beta \partial_\beta \left(\sqrt{-\det g}\mathcal{F}_A^{\alpha\beta}\right)$

where $J = qu$ is the charges current. Because U(1) is abelian $B_A = 0$. In the model the equation reads:

$\alpha = 0..3 : F_A^\alpha = C_A \frac{1}{\det P'} \sum_\beta \frac{d}{d\xi^\beta} \left(\mathcal{F}_A^{\alpha\beta} \det P'\right)$

with $[\theta_a] = i$, this leads to identify $\mu_0 q$ with :

$-C_I \frac{1}{4} \operatorname{Im} i \left[\psi_{rp}^*\right] [\psi_{rp}] = \mu_0 q = -C_I \frac{1}{4} \left(S_{0p}^* S_{0p}\right) \left(\phi_{0p}^* \phi_{0p}\right)$

$= -C_I \frac{1}{2} \left(S_{R0p}^* S_{R0p}\right) \left(\sum_{j=1}^n \left|\phi_{0p}^j\right|^2\right)$

The charge q depends on the physical state $\phi_{0p} = \sum_{j=1}^n \phi_{0p}^j f_j$ and the spin $\left(S_{R0p}^* S_{R0p}\right)$, which are both constant.

11.5 Energy-momentum tensor

11.5.1 Definition

From analytical mechanics, for a scalar lagrangian $L\left(z^i, z_\alpha^i\right)$ it is traditional to call energy-momentum tensor the quantity $T_\beta^\alpha = \sum_i \frac{\partial L}{\partial z_\alpha^i} \partial_\beta z^i - \delta_\beta^\alpha L$ with $z_\alpha^i = \partial_\alpha z^i$. It is a tensor $T = \sum_{\alpha\beta} \left(\sum_i \frac{\partial L}{\partial z_\alpha^i} \partial_\beta z^i - \delta_\beta^\alpha L\right) \partial\xi_\alpha \otimes d\xi^\beta$. The main justification for this definition is that, for β fixed, the vector $\sum_\alpha T_\beta^\alpha \partial\xi_\alpha$ is divergence free,on shell (when the Lagrange equations are satisfied), as it is easy to show.

Proof. Any lagrangian built according to the predeeding rules cannot depend on the derivatives $\partial_\beta P_i^\alpha$ of the tetrad. So let us denote this lagrangian $L\left(z^i, z_\alpha^i, P_i^\alpha\right)$. Some of the indexes i are related to coordinates α but here only the derivatives are involved.

i) proceeding as previously in setting the covariance conditions, it is easy to show that:

$T = \sum_{\alpha\beta} \left(\sum_i \frac{\partial L}{\partial z_\alpha^i} \partial_\beta z^i - \delta_\beta^\alpha L\right) \partial\xi_\alpha \otimes d\xi^\beta$ is a tensor

the covariance equation is : $\forall \alpha, \beta : \sum_i \frac{dL}{dz_\beta^i} z_\alpha^i = \sum_i \frac{\partial L}{\partial P_\beta^i} P_\alpha^i$

ii) So for β fixed T_β is a vector and its divergence is :

$\frac{1}{\det P'} \sum_\alpha \partial_\alpha \left(T_\beta^\alpha \det P'\right)$

$= \frac{1}{\det P'} \sum_{\alpha i} \partial_\alpha \left(\frac{\partial L}{\partial z_\alpha^i} \partial_\beta z^i \det P'\right) - \frac{1}{\det P'} \sum_\alpha \partial_\alpha \left(L \delta_\beta^\alpha \det P'\right)$

$= \sum_i \partial_\beta z^i \frac{1}{\det P'} \sum_\alpha \partial_\alpha \left(\frac{\partial L}{\partial \partial_\alpha z^i} \det P'\right) + \sum_i \frac{\partial L}{\partial \partial_\alpha z^i} \sum_\alpha \partial_\alpha \left(\partial_\beta z^i\right) - \frac{1}{\det P'} \partial_\beta \left(L \det P'\right)$

$= \sum_i \partial_\beta z^i \frac{1}{\det P'} \sum_\alpha \partial_\alpha \left(\frac{\partial L}{\partial \partial_\alpha z^i} \det P'\right) + \sum_i \frac{\partial L}{\partial \partial_\alpha z^i} \sum_\alpha \partial_\beta \left(z_\alpha^i\right)$

$- \sum_{i\alpha} \left(\frac{\partial L}{\partial z^i} \partial_\beta z^i + \frac{\partial L}{\partial z_\alpha^i} \partial_\beta z_\alpha^i + \frac{\partial L}{\partial P_i^\alpha} \partial_\beta P_i^\alpha\right) - \frac{1}{\det P'} L \frac{\partial \det P'}{\partial P_i^\alpha} \partial_\beta P_i^\alpha$

$= \sum_i \partial_\beta z^i \left(\frac{1}{\det P'} \sum_\alpha \partial_\alpha \left(\frac{\partial L}{\partial \partial_\alpha z^i} \det P'\right) - \sum_{i\alpha} \frac{\partial L}{\partial z^i}\right) - \sum_{i\alpha} \left(\frac{\partial L}{\partial P_i^\alpha} - L P_\alpha'^i\right) \partial_\beta P_i^\alpha$

with $\frac{\partial \det P'}{\partial P_i^\alpha} = -\left(\frac{1}{\det P}\right)^2 \frac{\partial \det P}{\partial P_i^\alpha} = -\left(\frac{1}{\det P}\right)^2 P_\alpha'^i \det P = -P_\alpha'^i \det P'$

We have the Lagrange equations for the variables z^i, P_α^i ■

So if we take two hypersurfaces $\partial C_k = \{\varphi_M(t_k, x), x \in \Omega_3(0)\}, k = 1, 2$ then :

$\int_{\partial C_1} i_{T_\beta}\varpi_4 = \int_{\partial C_2} i_{T_\beta}\varpi_4$

The flow of the energy momentum tensor is preserved. And this still holds for the intersection of any open of M which intersects the hypersurfaces : this is a local property.

11.5.2 Energy momentum tensor for a distribution of particles

There are several ways to define the energy-momentum tensor, according to the variables that we involve. As the conservation is one of its key features, and the motion is fully represented in ψ, we define :

$T^\alpha_\beta = \frac{1}{2}\sum_{ijp}\frac{dL}{d\,\mathrm{Re}\,\partial_\alpha\psi^{ij}_p}\partial_\beta\,\mathrm{Re}\,\psi^{ij}_p + \frac{dL}{d\,\mathrm{Im}\,\partial_\alpha\psi^{ij}_p}\partial_\beta\,\mathrm{Im}\,\psi^{ij}_p + \sum_{a\gamma}\frac{dL}{d\partial_\alpha G^a_{r\gamma}}\partial_\beta G^a_{r\gamma} + \sum_{a\gamma}\frac{dL}{d\partial_\alpha G^a_{w\gamma}}\partial_\beta G^a_{w\gamma} + \frac{dL}{d\partial_\alpha \grave{A}^a_\gamma}\partial_\beta \grave{A}^a_\gamma - \delta^\alpha_\beta L$

$$T^\alpha_\beta = \mathrm{Re}\sum_{ijp}\frac{dL}{d\partial_\alpha\psi^{ij}_p}\partial_\beta\psi^{ij}_p + \sum_{a\gamma}\frac{dL}{d\partial_\alpha G^a_{r\gamma}}\partial_\beta G^a_{r\gamma} + \sum_{a\gamma}\frac{dL}{d\partial_\alpha G^a_{w\gamma}}\partial_\beta G^a_{w\gamma} + \frac{dL}{d\partial_\alpha \grave{A}^a_\gamma}\partial_\beta \grave{A}^a_\gamma - \delta^\alpha_\beta L \quad (83)$$

With :

$\frac{d\mathcal{L}}{d\partial_\alpha\psi^{ij}_p} = N_p C_I \frac{1}{i} u^\alpha_p \left[\psi^*_p\right]^j_i$

$\frac{dL}{d\partial_\alpha G^a_{r\gamma}} = C_G \mathcal{F}^{a\alpha\gamma}_r$

$\frac{dL}{d\partial_\alpha G^a_{w\gamma}} = -C_G \mathcal{F}^{a\alpha\gamma}_w$

$\frac{dL}{d\partial_\alpha \grave{A}^a_\gamma} = 4C_A \mathcal{F}^{a\beta\alpha}_A$

$T^\alpha_\beta = C_I\sum_{ijp} N_p u^\alpha_p \,\mathrm{Re}\left(\frac{1}{i}\left(\left[\psi^*_p\right]\gamma_0\right)^j_i\left[\partial_\beta\psi_p\right]^i_j\right) + \sum_{a\gamma} C_G\mathcal{F}^{a\alpha\gamma}_r\partial_\beta G^a_{r\gamma} - \sum_{a\gamma} C_G\mathcal{F}^{a\alpha\gamma}_w\partial_\beta G^a_{w\gamma} + 4C_A\mathcal{F}^{a\beta\alpha}_A\partial_\beta\grave{A}^a_\gamma - \delta^\alpha_\beta L$

$T^\alpha_\beta = C_I\sum_p N_p u^\alpha_p\,\mathrm{Im}\,Tr\left[\psi_p\right]^*\gamma_0\left[\partial_\beta\psi_p\right] + 4\sum_\gamma C_G\left\langle\mathcal{F}^{\alpha\gamma},\partial_\beta G_\gamma\right\rangle + C_A\left\langle\mathcal{F}^{\beta\alpha}_A,\partial_\beta\grave{A}_\gamma\right\rangle - \delta^\alpha_\beta L$

This explains the motivation for the factor 1/2 in the definition : to avoid giving a disproportionate weight to the particles.

The alternate definition

$T^\alpha_\beta = \mathrm{Re}\sum_{ijp}\frac{dL}{d\partial_\alpha\psi^{ij}_{rp}}\partial_\beta\psi^{ij}_{rp} + \sum_a\frac{dL}{d\partial_\alpha w^a_p}\partial_\beta w^a_p + \sum_{a\gamma}\frac{dL}{d\partial_\alpha G^a_{r\gamma}}\partial_\beta G^a_{r\gamma} + \sum_{a\gamma}\frac{dL}{d\partial_\alpha G^a_{w\gamma}}\partial_\beta G^a_{w\gamma} + \frac{dL}{d\partial_\alpha \grave{A}^a_\gamma}\partial_\beta\grave{A}^a_\gamma - \delta^\alpha_\beta L$

gives the same result. The first two terms read :

$C_I\sum_p N_p u^\alpha_p\left(\mathrm{Im}\,Tr\left[\psi_{rp}\right]^*\gamma_0\left[\partial_\beta\psi_{rp}\right] + \frac{1}{4}\varkappa^t_p j\left(w_p\right)\partial_\beta w_{rp}\right) = C_I\sum_p N_p u^\alpha_p\,\mathrm{Im}\,Tr\left[\psi_p\right]^*\gamma_0\left[\partial_\beta\psi_p\right]$

Computation of the momentum-energy tensor

As seen above the trick is to transform the equation above by showing the Lagrange equations.

i) $C_I\sum_p N_p u^\alpha_p\,\mathrm{Im}\,Tr\left[\psi_p\right]^*\gamma_0\left[\partial_\beta\psi_p\right]$

$= C_I\sum_p N_p u^\alpha_p\frac{1}{i}\left\langle\psi_p,\partial_\alpha\psi_p\right\rangle$

$= C_I\sum_p N_p u^\alpha_p\{\mathrm{Im}\left\langle\psi_p,\nabla_\alpha\psi_p\right\rangle - \frac{1}{i}\left\langle\psi_{0p},\left[\psi_{0p}\right]\left[\grave{A}_a\right]\right\rangle$

$+\frac{1}{2}\sum_{a=1}^3\left(G^a_{r\alpha}\left\langle\psi_p,\widetilde{\gamma}_a\psi_p\right\rangle + G^a_{w\alpha}\left\langle\psi_p,\gamma_a\gamma_0\psi_p\right\rangle\right)\}$

$= C_I\sum_p N_p u^\alpha_p\,\mathrm{Im}\left\langle\psi_p,\nabla_\alpha\psi_p\right\rangle + 4\left\langle F^\alpha_G,G_\beta\right\rangle + 4\left\langle F^\alpha_A,\grave{A}_\beta\right\rangle$

ii) $\sum_{\gamma=0}^3 C_G\left\langle\mathcal{F}^{\alpha\gamma}_G,\partial_\beta G^a_\gamma\right\rangle$

$= \sum_{\gamma=0}^{3} C_G \left(\langle \mathcal{F}_{G\beta\gamma}, \mathcal{F}_G^{\alpha\gamma} \rangle + \langle \mathcal{F}^{\alpha\gamma}, \partial_\gamma G_\beta \rangle - \langle \mathcal{F}_G^{\alpha\gamma}, [G_\beta, G_\gamma] \rangle \right)$
$= \sum_{\gamma=0}^{3} C_G \left(\langle \mathcal{F}_{G\beta\gamma}, \mathcal{F}_G^{\alpha\gamma} \rangle + \langle \mathcal{F}^{\alpha\gamma}, \partial_\gamma G_\beta \rangle + \langle \mathcal{F}_G^{\alpha\gamma}, [G_\gamma, G_\beta] \rangle \right)$
$= \sum_{\gamma=0}^{3} C_G \{ \langle \mathcal{F}_{G\beta\gamma}, \mathcal{F}_G^{\alpha\gamma} \rangle + \frac{1}{\det P'} \partial_\gamma \langle \mathcal{F}_G^{\alpha\gamma} \det P', G_\beta \rangle$
$- \frac{1}{\det P'} \langle \partial_\gamma \left(\mathcal{F}_G^{\alpha\gamma} \det P' \right), G_\beta \rangle + \langle [\mathcal{F}_G^{\alpha\gamma}, G_\gamma], G_\beta \rangle \}$
$= \sum_{\gamma=0}^{3} C_G \{ \langle \mathcal{F}_{G\beta\gamma}, \mathcal{F}_G^{\alpha\gamma} \rangle + \frac{1}{\det P'} \partial_\gamma \langle \mathcal{F}_G^{\alpha\gamma} \det P', G_\beta \rangle$
$- \frac{1}{\det P'} \langle \partial_\gamma \left(\mathcal{F}_G^{\alpha\gamma} \det P' \right), G_\beta \rangle \} + \langle B_G^\alpha, G_\beta \rangle$
$= \sum_{\gamma=0}^{3} C_G \left(\langle \mathcal{F}_{G\beta\gamma}, \mathcal{F}_G^{\alpha\gamma} \rangle + \frac{1}{\det P'} \partial_\gamma \langle \mathcal{F}_G^{\alpha\gamma} \det P', G_\beta \rangle \right) + \langle \Phi_G^\alpha, G_\beta \rangle$
$- \frac{1}{\det P'} C_G \left\langle \sum_{\gamma=0}^{3} \partial_\gamma \left(\mathcal{F}_G^{\alpha\gamma} \det P' \right), G_\beta \right\rangle + \langle B_G^\alpha, G_\beta \rangle - \langle \Phi_G^\alpha, G_\beta \rangle \}$
$= \sum_{\gamma=0}^{3} C_G \left(\langle \mathcal{F}_{G\beta\gamma}, \mathcal{F}_G^{\alpha\gamma} \rangle + \frac{1}{\det P'} \partial_\gamma \langle \mathcal{F}_G^{\alpha\gamma} \det P', G_\beta \rangle \right)$
$+ \left\langle \Phi_G^\alpha - \frac{1}{\det P'} C_G \sum_{\gamma=0}^{3} \partial_\gamma \left(\mathcal{F}_G^{\alpha\gamma} \det P' \right), G_\beta \right\rangle - \langle F_G^\alpha, G_\beta \rangle$
and similarly :
$\sum_{\gamma=0}^{3} C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \partial_\beta \grave{A}_\gamma^a \right\rangle =$
$\sum_{\gamma=0}^{3} C_A \left(\langle \mathcal{F}_{A\beta\gamma}, \mathcal{F}_A^{\alpha\gamma} \rangle + \frac{1}{\det P'} \partial_\gamma \left\langle \mathcal{F}_A^{\alpha\gamma} \det P', \grave{A}_\beta \right\rangle \right)$
$+ \left\langle \Phi_A^\alpha - \frac{1}{\det P'} C_A \sum_{\gamma=0}^{3} \partial_\gamma \left(\mathcal{F}_A^{\alpha\gamma} \det P' \right), G_\beta \right\rangle - \left\langle F_A^\alpha, \grave{A}_\beta \right\rangle$
Thus:
$T_\beta^\alpha = 4 \frac{1}{\det P'} \sum_{\gamma=0}^{3} \partial_\gamma \left(\left(G_G \langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \det P' \right)$
$+ C_I \sum_p N_p u_p^\alpha \operatorname{Im} \langle \psi_p, \nabla_\beta \psi_p \rangle + 4 \sum_{\gamma=0}^{3} C_G \langle \mathcal{F}_{G\beta\gamma}, \mathcal{F}_G^{\alpha\gamma} \rangle + C_A \langle \mathcal{F}_{A\beta\gamma}, \mathcal{F}_A^{\alpha\gamma} \rangle - \delta_\beta^\alpha L$
$+ 4 \left\langle \Phi_G^\alpha - \frac{1}{\det P'} C_G \sum_{\gamma=0}^{3} \partial_\gamma \left(\mathcal{F}_G^{\alpha\gamma} \det P' \right), G_\beta \right\rangle + 4 \left\langle \Phi_A^\alpha - \frac{1}{\det P'} C_A \sum_{\gamma=0}^{3} \partial_\gamma \left(\mathcal{F}_A^{\alpha\gamma} \det P' \right), G_\beta \right\rangle$
iii) We have the identity :
$\sum_i \frac{d\mathcal{L}}{dP_i^\alpha} P_i^\beta =$
$C_I \sum_{p=1}^{N} N_p u_p^\beta \operatorname{Im} \langle \psi_p, \nabla_\alpha \psi_p \rangle + 4 \sum_{\gamma=0}^{3} \left(C_G \left\langle \mathcal{F}_{G\alpha\gamma}, \mathcal{F}_G^{\beta\gamma} \right\rangle + C_A \left\langle \mathcal{F}_{A\alpha\gamma}, \mathcal{F}_A^{\beta\gamma} \right\rangle \right)$
$T_\beta^\alpha = 4 \frac{1}{\det P'} \sum_{\gamma=0}^{3} \partial_\gamma \left(\left(G_G \langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \det P' \right)$
$+ \sum_i \frac{d\mathcal{L}}{dP_i^\alpha} P_i^\beta - \delta_\beta^\alpha L + 4 \left\langle \Phi_G^\alpha - \frac{1}{\det P'} C_G \sum_{\gamma=0}^{3} \partial_\gamma \left(\mathcal{F}_G^{\alpha\gamma} \det P' \right), G_\beta \right\rangle$
$+ 4 \left\langle \Phi_A^\alpha - \frac{1}{\det P'} C_A \sum_{\gamma=0}^{3} \partial_\gamma \left(\mathcal{F}_A^{\alpha\gamma} \det P' \right), G_\beta \right\rangle$
iv) On shell, if the equations are met :
$T_\beta^\alpha = 4 \frac{1}{\det P'} \sum_{\gamma=0}^{3} \partial_\gamma \left(\left(G_G \langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \det P' \right)$

$$T = 4 div \left(\sum_{\gamma=0}^{3} \partial_\gamma \left(\left(G_G \langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \partial \xi_\gamma \right) \right) d\xi^\alpha \otimes d\xi^\beta \tag{84}$$

As can be expected usually the tensor is not symmetric : $T_\beta^\alpha \neq T_\alpha^\beta$
And it is easy to check that :

Proposition 29 *For β fixed T_β is a vector field. Its divergence is null.*

Proof. $div(\sum_\alpha T_\beta^\alpha \partial \xi_\alpha) = \frac{1}{\det P'} \sum_\alpha \partial_\alpha \left(T_\beta^\alpha \det P' \right)$
$div \left(\frac{1}{\det P'} \sum_{\gamma=0}^{3} \partial_\gamma \left(\left(G_G \langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \det P' \right) \det P' \right)$
$= \frac{1}{\det P'} \sum_{a\alpha\gamma} \partial_{\alpha\gamma}^2 \left(\left(G_G \langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \det P' \right)$

$= \frac{1}{\det P'} \sum_{a\alpha<\gamma} \partial^2_{\alpha\gamma} \left(\left(G_G \left\langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \right\rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \det P' \right)$
$+ \frac{1}{\det P'} \sum_{a\alpha>\gamma} \partial^2_{\alpha\gamma} \left(\left(G_G \left\langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \right\rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \det P' \right)$
$= \frac{1}{\det P'} \sum_{a\alpha<\gamma} \partial^2_{\alpha\gamma} \left(\left(\left(G_G \left\langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \right\rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \right) \det P' \right)$
$- \frac{1}{\det P'} \sum_{a\gamma>\alpha} \partial^2_{\gamma\alpha} \left(\left(G_G \left\langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \right\rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \det P' \right) = 0$ ■

Alternate expression of T

It would be nice to see the currents in the expression of T. We proceed in several steps.

i) The equation above leads to look at the tensor

$\Theta = \sum_{\alpha\beta\gamma} \left(G_G \left\langle \mathcal{F}_G^{\alpha\gamma}, G_\beta \right\rangle + C_A \left\langle \mathcal{F}_A^{\alpha\gamma}, \grave{A}_\beta \right\rangle \right) \partial\xi_\alpha \wedge \partial\xi_\gamma \otimes d\xi^\beta$

For β fixed we can implement the formula above :

$d \circ \varpi_4 (\Theta_\beta) = - \sum_{\alpha\gamma=0}^{3} (-1)^{\alpha+1} \partial_\gamma \left(\Theta^{\alpha\gamma} \det P' \right) d\xi^0 \wedge ... \widehat{d\xi^\alpha} ... \wedge d\xi^3$

$= - (\det P') \sum_{\alpha=0}^{3} (-1)^{\alpha+1} div \left(\sum_\gamma \Theta^{\alpha\gamma} \partial\xi_\gamma \right) d\xi^0 \wedge ... \widehat{d\xi^\alpha} ... \wedge d\xi^3$

$= - \sum_{\alpha=0}^{3} (-1)^{\alpha+1} T^\alpha_\beta (\det P') d\xi^0 \wedge ... \widehat{d\xi^\alpha} ... \wedge d\xi^3$

and on the other hand :

$i_{T_\beta} \varpi_4 = \sum_{\alpha=0}^{3} (-1)^{\alpha+1} T^\alpha_\beta (\det P') d\xi^0 \wedge ... \widehat{d\xi^\alpha} ... \wedge d\xi^3$

So we have the geometric formulation of the energy-momentum tensor :

$$i_{T_\beta} \varpi_4 = -d \circ \varpi_4 (\Theta_\beta) \tag{85}$$

ii) $\Theta_\beta = \sum_{\alpha\gamma} \sum_a \left(G_G G^a_\beta \left\langle \mathcal{F}_G^{\alpha\gamma}, \overrightarrow{\kappa}_a \right\rangle + C_A \grave{A}_\beta \left\langle \mathcal{F}_A^{\alpha\gamma}, \overrightarrow{\theta}_a \right\rangle \right) \partial\xi_\alpha \wedge \partial\xi_\gamma$

$= \sum_{\alpha\gamma} \sum_a \left(G_G \frac{1}{4} \left(G^a_{r\beta} \mathcal{F}_r^{a\alpha\gamma} - G^a_{w\beta} \mathcal{F}_w^{a\alpha\gamma} \right) + C_A \grave{A}^a_\beta \mathcal{F}_A^{a\alpha\gamma} \right) \partial\xi_\alpha \wedge \partial\xi_\gamma$

$\varpi_4 (\Theta_\beta) = \sum_a G_G \frac{1}{4} G^a_{r\beta} \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_r^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right)$

$- G_G \frac{1}{4} G^a_{w\beta} \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_w^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) + C_A \grave{A}^a_\beta \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_A^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right)$

$d \circ \varpi_4 (\Theta_\beta)$

$= \sum_a G_G \frac{1}{4} dG^a_{r\beta} \wedge \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_r^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) - G_G \frac{1}{4} G^a_{r\beta} d\varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_r^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right)$

$- G_G \frac{1}{4} dG^a_{w\beta} \wedge \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_w^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) + G_G \frac{1}{4} G^a_{w\beta} d\varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_w^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right)$

$+ C_A d\grave{A}^a_\beta \wedge \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_A^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) - C_A \grave{A}^a_\beta d\varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_A^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right)$

$= \sum_a G_G \frac{1}{4} dG^a_{r\beta} \wedge \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_r^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) + \frac{1}{4} G^a_{r\beta} i_{\Phi^a_{Gr}} \varpi_4$

$- G_G \frac{1}{4} dG^a_{w\beta} \wedge \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_w^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) - \frac{1}{4} G^a_{w\beta} i_{\Phi^a_{Gw}} \varpi_4$

$+ C_A d\grave{A}^a_\beta \wedge \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_A^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) + \grave{A}^a_\beta i_{\Phi_A} \varpi_4$

$= \sum_a G_G \frac{1}{4} \left(dG^a_{r\beta} \wedge \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_r^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) - dG^a_{w\beta} \wedge \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_w^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) \right)$

$+ C_A d\grave{A}^a_\beta \wedge \varpi_4 \left(\sum_{\alpha\gamma} \mathcal{F}_A^{a\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma \right) + \frac{1}{4} G^a_{r\beta} i_{\Phi^a_{Gr}} \varpi_4 - \frac{1}{4} G^a_{w\beta} i_{\Phi^a_{Gw}} \varpi_4 + \grave{A}^a_\beta i_{\Phi_A} \varpi_4$

iii) For any one form Z : $Z = \sum_\gamma Z_\gamma d\xi^\gamma$ and tensor $X = \sum_{\alpha\gamma} X^{\alpha\gamma} \partial\xi_\alpha \wedge \partial\xi_\gamma$:

$dZ \wedge \varpi_4 (X) = \varpi_4 \left(\sum_{\alpha\gamma} X^{\alpha\gamma} Z_\gamma \partial\xi_\alpha \right)$

(see Annex for the proof)

$d \circ \varpi_4 (\Theta_\beta)$
$= \sum_a G_G \frac{1}{4} \left(\varpi_4 \left(\sum_{\alpha\gamma} \partial_\gamma G^a_{r\beta} \mathcal{F}^{a\alpha\gamma}_r \partial\xi_\alpha \wedge \partial\xi_\gamma \right) - \varpi_4 \left(\sum_{\alpha\gamma} \partial_\gamma G^a_{w\beta} \mathcal{F}^{a\alpha\gamma}_w \partial\xi_\alpha \wedge \partial\xi_\gamma \right) \right)$
$+ C_A \varpi_4 \left(\sum_{\alpha\gamma} \partial_\gamma \grave{A}^a_\beta \mathcal{F}^{a\alpha\gamma}_A \partial\xi_\alpha \wedge \partial\xi_\gamma \right) + \frac{1}{4} \left(G^a_{r\beta} i_{\Phi^a_{Gr}} \varpi_4 - G^a_{w\beta} i_{\Phi^a_{Gw}} \varpi_4 \right) + \grave{A}^a_\beta i_{\Phi_A} \varpi_4$
$= G_G \frac{1}{4} \varpi_4 \left(\sum_{a\alpha\gamma} \left(\partial_\gamma G^a_{r\beta} \mathcal{F}^{a\alpha\gamma}_r - \partial_\gamma G^a_{w\beta} \mathcal{F}^{a\alpha\gamma}_w \right) \partial\xi_\alpha \wedge \partial\xi_\gamma \right)$
$+ C_A \varpi_4 \left(\sum_{a\alpha\gamma} \partial_\gamma \grave{A}^a_\beta \mathcal{F}^{a\alpha\gamma}_A \partial\xi_\alpha \wedge \partial\xi_\gamma \right) + \frac{1}{4} \left(\sum_a G^a_{r\beta} i_{\Phi^a_{Gr}} \varpi_4 - G^a_{w\beta} i_{\Phi^a_{Gw}} \varpi_4 \right) + \sum_a \grave{A}^a_\beta i_{\Phi_A} \varpi_4$
$= G_G \varpi_4 \left(\sum_{\alpha\gamma} \langle \partial_\gamma G_\beta, \mathcal{F}^{\alpha\gamma}_G \rangle \partial\xi_\alpha \wedge \partial\xi_\gamma \right) + C_A \varpi_4 \left(\sum_{\alpha\gamma} \left\langle \partial_\gamma \grave{A}_\beta, \mathcal{F}^{\alpha\gamma}_A \right\rangle \partial\xi_\alpha \wedge \partial\xi_\gamma \right)$
$+ \sum_\alpha \left(\langle \Phi^\alpha_G, G_\beta \rangle + \left\langle \Phi^\alpha_A, \grave{A}_\beta \right\rangle \right) \partial\xi_\alpha$

$$\varpi_4 \left(\sum_\alpha T^\alpha_\beta \partial\xi_\alpha \right) = -\varpi_4 \left(\sum_\alpha \left(\langle \Phi^\alpha_G, G_\beta \rangle + \left\langle \Phi^\alpha_A, \grave{A}_\beta \right\rangle + \sum_\gamma C_G \langle \mathcal{F}^{\alpha\gamma}_G, \partial_\gamma G_\beta \rangle + C_A \left\langle \mathcal{F}^{\alpha\gamma}_A, \partial_\gamma \grave{A}_\beta \right\rangle \right) \partial\xi_\alpha \right) \tag{86}$$

So, from the conservation of the energy momentum tensor, we have :
$\int_{\partial C(t_1)} i_{T_B} \varpi_4 = \int_{\partial C(t_2)} i_{T_B} \varpi_4$
$\Leftrightarrow$
$\int_{\partial C(t_2)} \varpi_4 (\Phi_{Ga}) (G_\beta) + \varpi_4 (\Phi_{Aa}) \left(\grave{A}_\beta \right) - \int_{\partial C(t_1)} \varpi_4 (\Phi_{Ga}) (G_\beta) + \varpi_4 (\Phi_{Aa}) \left(\grave{A}_\beta \right)$
$= \int_{\partial C(t_1)} \varpi_4 \left(\sum_{\alpha\gamma} \left(C_G \langle \mathcal{F}^{\alpha\gamma}_G, \partial_\gamma G_\beta \rangle + C_A \left\langle \mathcal{F}^{\alpha\gamma}_A, \partial_\gamma \grave{A}_\beta \right\rangle \right) \partial\xi_\alpha \right)$
$- \int_{\partial C(t_2)} \varpi_4 \left(\sum_{\alpha\gamma} \left(C_G \langle \mathcal{F}^{\alpha\gamma}_G, \partial_\gamma G_\beta \rangle + C_A \left\langle \mathcal{F}^{\alpha\gamma}_A, \partial_\gamma \grave{A}_\beta \right\rangle \right) \partial\xi_\alpha \right)$
The energy-momentum carried by the currents is stored in the fields through a variation of the potentials.

This formulation is closer to a classic representation of interacting fields and particles, and more familiar to physicists. And it puts the currents linked to particles and fields on the same footing : they interact similarly with the fields by $\langle F_G, G \rangle + \left\langle F_A, \grave{A} \right\rangle$ on one hand, and $\langle B_G, G \rangle + \left\langle B_A, \grave{A} \right\rangle$ on the other.

11.5.3 Energy momentum tensor for a single particle

Computation of T

We must adjust the definition to account for the two parts of the lagrangian. The fourth order integral can be seen as :
$\int_{\Omega_3(0)} \left(\int_0^T f (\Phi_V (t, x)) \sum_{\lambda\mu} \left(C_G \left\langle \mathcal{F}_{G\lambda\mu}, \mathcal{F}^{\lambda\mu}_G \right\rangle + C_A \left\langle \mathcal{F}_{A\lambda\mu}, \mathcal{F}^{\lambda\mu}_A \right\rangle \right) dt \right) \varpi_3 (x)$
which leads to take :
$T^\alpha_\beta = C_I f u^\alpha \operatorname{Im} Tr [\psi]^* \gamma_0 [\partial_\beta \psi] + 4f \sum_\gamma C_G \langle \mathcal{F}^{\alpha\gamma}, \partial_\beta G_\gamma \rangle + C_A \left\langle \mathcal{F}^{\beta\alpha}_A, \partial_\beta \grave{A}_\gamma \right\rangle$
$- \delta^\alpha_\beta f \left(C_P \langle \psi_0, \psi_0 \rangle + C_I \operatorname{Im} \langle \psi, \nabla_u \psi \rangle + \sum_{\lambda\mu} \left(C_G \left\langle \mathcal{F}_{G\lambda\mu}, \mathcal{F}^{\lambda\mu}_G \right\rangle + C_A \left\langle \mathcal{F}_{A\lambda\mu}, \mathcal{F}^{\lambda\mu}_A \right\rangle \right) \right)$

$\frac{1}{f} T^\alpha_\beta$ is defined as above, and the same computation leads to :

$\frac{1}{f}T^{\alpha}_{\beta} = 4\frac{1}{\det P'}\sum_{\gamma=0}^{3}\partial_{\gamma}\left(\left(G_G\left\langle\mathcal{F}_G^{\alpha\gamma},G_{\beta}\right\rangle + C_A\left\langle\mathcal{F}_A^{\alpha\gamma},\grave{A}_{\beta}\right\rangle\right)\det P'\right)$
$+C_I u^{\alpha}\operatorname{Im}\left\langle\psi,\nabla_{\beta}\psi\right\rangle + 4\sum_{\gamma=0}^{3}C_G\left\langle\mathcal{F}_{G\beta\gamma},\mathcal{F}_G^{\alpha\gamma}\right\rangle + C_A\left\langle\mathcal{F}_{A\beta\gamma},\mathcal{F}_A^{\alpha\gamma}\right\rangle$
$-\delta^{\alpha}_{\beta}\left(\sum_{\lambda\mu}\left(C_G\left\langle\mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle\mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle\right) + C_P\left\langle\psi_0,\psi_0\right\rangle + C_I\operatorname{Im}\left\langle\psi,\nabla_u\psi\right\rangle\right)$
$+4\left\langle\Phi^{\alpha}_G - \frac{1}{\det P'}C_G\sum_{\gamma=0}^{3}\partial_{\gamma}\left(\mathcal{F}_G^{\alpha\gamma}\det P'\right),G_{\beta}\right\rangle + 4\left\langle\Phi^{\alpha}_A - \frac{1}{\det P'}C_A\sum_{\gamma=0}^{3}\partial_{\gamma}\left(\mathcal{F}_A^{\alpha\gamma}\det P'\right),G_{\beta}\right\rangle$
On shell and on the world line :
$\Phi^{\alpha}_G - \frac{1}{\det P'}C_G\sum_{\gamma=0}^{3}\partial_{\gamma}\left(\mathcal{F}_G^{\alpha\gamma}\det P'\right) = 0$
$\Phi^{\alpha}_A - \frac{1}{\det P'}C_A\sum_{\gamma=0}^{3}\partial_{\gamma}\left(\mathcal{F}_A^{\alpha\gamma}\det P'\right) = 0$
$4\sum_{\gamma=0}^{3}\left(C_G\left\langle\mathcal{F}_{G\beta\gamma},\mathcal{F}_G^{\alpha\gamma}\right\rangle + C_A\left\langle\mathcal{F}_{A\beta\gamma},\mathcal{F}_A^{\alpha\gamma}\right\rangle\right)$
$-\delta^{\beta}_{\alpha}\sum_{\lambda\mu}\left(C_G\left\langle\mathcal{F}_{G\lambda\mu},\mathcal{F}_G^{\lambda\mu}\right\rangle + C_A\left\langle\mathcal{F}_{A\lambda\mu},\mathcal{F}_A^{\lambda\mu}\right\rangle + C_P\left\langle\psi_0,\psi_0\right\rangle\right) = -C_I u^{\alpha}\operatorname{Im}\left\langle\psi,\nabla_{\beta}\psi\right\rangle$
$\operatorname{Im}\left\langle\psi,\nabla_u\psi\right\rangle = 0$
$\frac{1}{f}T^{\alpha}_{\beta} = 4\frac{1}{\det P'}\sum_{\gamma=0}^{3}\partial_{\gamma}\left(\left(G_G\left\langle\mathcal{F}_G^{\alpha\gamma},G_{\beta}\right\rangle + C_A\left\langle\mathcal{F}_A^{\alpha\gamma},\grave{A}_{\beta}\right\rangle\right)\det P'\right)$

$$T^{\alpha}_{\beta} = 4\frac{1}{\det P'}f\sum_{\gamma=0}^{3}\partial_{\gamma}\left(\left(G_G\left\langle\mathcal{F}_G^{\alpha\gamma},G_{\beta}\right\rangle + C_A\left\langle\mathcal{F}_A^{\alpha\gamma},\grave{A}_{\beta}\right\rangle\right)\det P'\right) \tag{87}$$

Moreover, as we have the same equations for the fields, we still have :

$$\varpi_4\left(\frac{1}{f}\sum_{\alpha}T^{\alpha}_{\beta}\partial\xi_{\alpha}\right) \tag{88}$$

$$= -\varpi_4\left(\sum_{\alpha}\left(\left\langle\Phi^{\alpha}_G,G_{\beta}\right\rangle + \left\langle\Phi^{\alpha}_A,\grave{A}_{\beta}\right\rangle + \sum_{\gamma}C_G\left\langle\mathcal{F}_G^{\alpha\gamma},\partial_{\gamma}G_{\beta}\right\rangle + C_A\left\langle\mathcal{F}_A^{\alpha\gamma},\partial_{\gamma}\grave{A}_{\beta}\right\rangle\right)\partial\xi_{\alpha}\right) \tag{89}$$

Contribution of the particle

It is useful to compute the contribution of the particle :
$\varpi_4\left(\sum_{\alpha}\left(\left\langle F^{\alpha}_G,G_{\beta}\right\rangle + \left\langle F^{\alpha}_A,\grave{A}_{\beta}\right\rangle\right)\partial\xi_{\alpha}\right)$
On shell :
$\left\langle F^{\alpha}_G,G_{\beta}\right\rangle = fC_I u^{\alpha}\frac{1}{2}\left\langle\upsilon\left(\left(a_w^2 - \frac{1}{2}j(w)j(w)\right)\varkappa, a_w j(w)\varkappa\right),G_{\beta}\right\rangle$
$= \frac{1}{4}fC_I u^{\alpha}\frac{1}{2}\left(G^t_{r\beta}\left(\left(a_w^2 - \frac{1}{2}j(w)j(w)\right)\varkappa\right) - a_w G^t_{w\beta}\left[j(w)\varkappa\right]\right)$
$= \frac{1}{4}fC_I u^{\alpha}\frac{1}{2}\sum_a\left(G^a_{r\beta}\left\langle\psi,\widetilde{\gamma}_a\psi\right\rangle + G^a_{w\beta}\left\langle\psi,\gamma_a\gamma_0\psi\right\rangle\right)$
$= \frac{1}{8}fC_I u^{\alpha}\sum_a\left\langle\psi,\sum_a\left(\widetilde{\gamma}_a G^a_{r\beta} + \gamma_a\gamma_0 G^a_{w\beta}\right)\psi\right\rangle$
$= -\frac{1}{4i}fC_I u^{\alpha}\sum_a\left\langle\psi,\gamma C\left(G_{\beta}\right)\psi\right\rangle$
$\left\langle F^{\alpha}_A,\grave{A}_{\beta}\right\rangle$
$= fC_I u^{\alpha}\left(-\frac{1}{4}\grave{A}^a_{\beta}\operatorname{Im}Tr\left[\psi^*_r\right]\left[\psi_r\right]\left[\theta_a\right]\right)$
$= fC_I u^{\alpha}\left(-\frac{1}{4i}\left\langle\psi,\psi\left[\grave{A}_{\beta}\right]\right\rangle\right)$
$\left\langle F^{\alpha}_G,G_{\beta}\right\rangle + \left\langle F^{\alpha}_A,\grave{A}_{\beta}\right\rangle$
$= -\frac{1}{4i}fC_I u^{\alpha}\sum_a\left\langle\psi,\psi\left[\grave{A}_{\beta}\right] + \gamma C\left(G_{\beta}\right)\psi\right\rangle$

$$\left\langle F_G^\alpha, G_\beta \right\rangle + \left\langle F_A^\alpha, \grave{A}_\beta \right\rangle = -\tfrac{1}{4i} C_I f u^\alpha \left\langle \psi, \nabla_\beta \psi - \partial_\beta \psi \right\rangle \tag{90}$$

Energy

The energy is the trace of T : $\sum_\alpha T_\alpha^\alpha$

The density of energy is then :

$\varpi_4 \left(\sum_\alpha T_\alpha^\alpha\right)$

$= -\varpi_4 \left(\sum_\alpha \left\langle \Phi_G^\alpha, G_\alpha \right\rangle + \left\langle \Phi_A^\alpha, \grave{A}_\beta \right\rangle + \sum_{\alpha\beta} C_G \left\langle \mathcal{F}_G^{\alpha\beta}, \partial_\beta G_\alpha \right\rangle + C_A \left\langle \mathcal{F}_A^{\alpha\beta}, \partial_\beta \grave{A}_\alpha \right\rangle\right)$

So for the particles on shell :

$-\sum_\alpha \left\langle F_G^\alpha, G_\alpha \right\rangle + \left\langle F_G^\alpha, G_\alpha \right\rangle$

$= \frac{1}{4i} C_I \sum_{p=1}^N N_p f_p \sum_a u_p^\alpha \left\langle \psi_p, \nabla_\alpha \psi_p - \partial_\alpha \psi_p \right\rangle$

$= \frac{1}{4i} C_I \sum_{p=1}^N N_p f_p \left\langle \psi_p, \nabla_{u_p} \psi_p - \frac{d\psi_p}{d\tau_p} \right\rangle$

In the absence of fields $\nabla_{u_p} \psi_p = \frac{d\psi_p}{d\tau_p}$ the energy is null.

In the general case the contribution a single particle is :

$-\left\langle F_G^\alpha, G_\beta \right\rangle + \left\langle F_A^\alpha, \grave{A}_\beta \right\rangle = \frac{1}{4i} C_I f \left\langle \psi, \nabla_u \psi - \frac{d\psi}{d\tau} \right\rangle$

$= \frac{1}{4} C_I f \left(2 \frac{1}{i} \left\langle \psi_r, \frac{d\psi_r}{d\tau} \right\rangle - \left(\frac{1}{i} \left\langle \psi_r, \frac{d\psi_r}{d\tau} \right\rangle + \frac{1}{4} \varkappa^t j(w) \frac{dw}{d\tau}\right)\right)$

$$-\sum_\alpha \left\langle F_G^\alpha, G_\alpha \right\rangle + \left\langle F_G^\alpha, G_\alpha \right\rangle = -C_I f \tfrac{1}{16} \varkappa^t j(w) \tfrac{dw}{dt} \tag{91}$$

with $\operatorname{Im} \left\langle \psi, \partial_\alpha \psi \right\rangle = \frac{1}{i} \left\langle \psi_r, \partial_\alpha \psi_r \right\rangle + \frac{1}{4} \varkappa^t j(w) \partial_\alpha w$ and $\left\langle \psi_r, \frac{d\psi_r}{d\tau} \right\rangle = 0$

The energy is a combination of rotational kinetic energy ($\varkappa^a = \left\langle \psi_r, \widetilde{\gamma}_a \psi_r \right\rangle$) and translational kinetic energy $\left(j(w) \frac{dw}{dt}\right)$.

From the equation for ψ_r :

$\left\langle \psi_r, \frac{d\psi_r}{d\tau} \right\rangle = \left\langle \psi_r, \psi_r \left[\widehat{\grave{A}}\right] \right\rangle - \frac{1}{2} i \varkappa^t \left[-\frac{1}{2} j(w) \frac{dw}{d\tau} + \left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \widehat{G}_r + a_w j(w_p) \widehat{G}_w\right] = 0$

The energy of the particle is :

$-C_I \frac{1}{16} \varkappa^t j(w) \frac{dw}{dt}$

$= f C_I \frac{1}{4} \left(\frac{1}{i} \left\langle \psi_0, \psi_0 \left[\widehat{\grave{A}}\right] \right\rangle - \frac{1}{2} \varkappa^t \left[\left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \widehat{G}_r + a_w j(w_p) \widehat{G}_w\right]\right)$

$= f C_I \frac{1}{4} \left(\frac{1}{i} \left\langle \psi_0, \psi_0 \left[\widehat{\grave{A}}\right] \right\rangle - \frac{1}{2} \varkappa^t \left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \widehat{G}_r - \frac{1}{2} \varkappa^t a_w j(w_p) \widehat{G}_w\right)$

$= f C_I \frac{1}{4} \left(\frac{1}{i} \left\langle \psi_0, \psi_0 \left[\widehat{\grave{A}}\right] \right\rangle - \frac{1}{2} \widehat{G}_r^t \left(a_w^2 - \frac{1}{2} j(w) j(w)\right) \varkappa + \frac{1}{2} \widehat{G}_w^t a_w j(w_p) \varkappa\right)$

$= f C_I \frac{1}{4} \left(\frac{1}{i} \left\langle \psi_0, \psi_0 \left[\widehat{\grave{A}}\right] \right\rangle - \frac{1}{2} \widehat{G}_r^t \rho_r + \frac{1}{2} \widehat{G}_w^t \rho_w\right)$

$= f C_I \frac{1}{4} \left(\frac{1}{i} \left\langle \psi_0, \psi_0 \left[\widehat{\grave{A}}\right] \right\rangle + \frac{1}{2} \left(\widehat{G}_w^t \frac{a_w}{3a_w^2 - 2} j(w) - \widehat{G}_r^t\right) \rho_r\right)$

with

$\left(a_w^2 - \frac{1}{2} j(w_p) j(w_p)\right) \varkappa_p = \rho_r$

$a_w j(w_p) \varkappa_p = \rho_w$

$j(w_p) \rho_r = \frac{3a_w^2 - 2}{a_w} \rho_w$

There is no "energy at rest". The first term corresponds to the energy linked to the charge $\frac{1}{i} \left\langle \psi_0, \psi_0 [\theta_a] \right\rangle$ and the other parts to the contribution of the gravitational charges. The radiation

does not necessarily involve a loss of energy for the particle : if the motion and the fields are constant, the energy stays the same.

If the trajectory is circular, with a constant rotational speed ω and radius ρin a plane orthogonal to ε_3

$$w = \nu k\left(-\sin\omega t, \cos\omega t; 0\right) \text{ with } \nu = \frac{\rho\varpi}{c}; a = \epsilon\sqrt{\frac{1}{2}\left(\frac{1}{\sqrt{1-\upsilon^2}}+1\right)}; k = \frac{\sqrt{2}}{\sqrt{1+\sqrt{1-\nu^2}}}$$

$$j\left(w_p\right)\frac{dw_p}{dt} = \nu^2k^2\omega \begin{array}{ccccc} 0 & 0 & \cos\omega t & -\cos\omega t & \\ 0 & 0 & \sin\omega t & -\sin\varpi t & = \nu^2k^2\omega \\ -\cos\omega t & -\sin\omega t & 0 & 0 & \end{array} \begin{array}{c} 0 \\ 0 \\ 1 \end{array}$$

$$-\sum_\alpha \left\langle F_G^\alpha, G_\alpha\right\rangle + \left\langle F_G^\alpha, G_\alpha\right\rangle = \frac{1}{8}C_I\varkappa_p\nu^2k^2\omega$$

11.6 Conclusion

We can sum up the main results in our quest for duality.

The concept of particle leads, by quantization, to assume that there is a finite number of types of particles, and to each of them is associated a section of a fiber bundle, defined all over Ω, which can take a spin up or down and has its own proper time. The state of a particle is fully defined along its worldline, in its proper time, by the value of the force fields where it is located.

These matters fields propagate along currents which share the same curves as the world lines, and carry charges (at least for À). So they coincide with real particles wherever they exist. And they can be seen as virtual particles elsewhere.

The particles contribute to the energy momentum, by vectors which are parallel to their trajectories, as we could expect. And this contribution comprises a part related to the spatial rotation and a part related to the translation. The energy-momentum tensor does not depend on the spatial spin (it is the same for spin up or down, and they have the same level of energy). However the picture is a bit more complicated than seen in elementary models.

The energy momentum tensor depends of the trajectory as seen by the observer : the time of the observer t replaces the proper time τ (through the function f). Of course this is a direct consequence of its definition, which is somewhat arbitrary, but starting from the action this result seems unavoidable. Physically this makes sense : the energy-momentum is an additive quantity, so in the relativist context it must be evaluated in a common time frame. Moreover the energy-momentum tensor has a true physical content, as it is through its measures that usually the state of the particle is actually estimated. So it needs to be related to the frame of the observer.

The energy-momentum tensor is proportional to the values of the potential and the charges, so it is null when there is no "force field" present. In particular there is no equivalent of an energy or a "mass at rest". This is is the logical consequence of a Yang-Mills model, where the mass is the gravitational charge. But this raises also intriguing questions about the significance of the cosmological constant, which guarantees that the gravitational field is never null. Indeed the assumption that material bodies travel along world lines, a motion which is linked to the cosmological constant as we noticed before, entails that its kinetic energy is never null. So the "mass at rest" should be understood as the mass of a particle which is submitted only to the unavoidable cosmological motion of all material bodies.

The currents asociated to the force fields can be seen as the propagation of the fields, and of the perturbations generated by the particles, similar to a radiation emitted as the particles interact with the fields. These vector fields are associated to quantities defined in the same vector spaces as the "charges" of the particles. They contribute also to the energy momentum tensor in a way similar to the particles.

However the laws of their propagation, the charge and mass which could be attributed to the bosons cannot be fully explained in the famework of a Yang-Mills model without full integration of the gravitational and the other fields.

But actually the strongest motivation for the introduction of bosons lies with discontinuous processes.

12 DISCONTINUOUS MODELS

Most physical phenomena involve, at some step, processes which can be seen as discontinuous :

- collision of molecules or particles, elastic (without loss of energy) or not
- the photo-electric effect : interaction of an electromagnetic field with electrons of the atoms of a solid material
- desintegration of a nucleus, spontaneous or following collisions
- black body : interaction of the electromagnetic field with the electrons of a body, under thermal agitation

We have seen that, in a continuous model, the spatial spin does not change. So the change of its orientation (up or down) requires a discontinuous process.

So the models used to represent these phenomena feature an explicit discontinuous operation. There are many variants.

The oldest are the kinetic models. They usually derive from a hydrodynamic model (similar to the continuous models), but add a collision operator to represent elastic collisions between particles. They are based upon a distribution function f(m,p) of particles of linear momentum p which shall follow a conservation law, using the collision operator. So the distribution of charges is itself given by a specific equation. Then the 4 dimensional action, with a lagrangian adptated to the fields considered, gives an equation relating the field and the distribution of charges. Usually the particles are assumed to have the same physical characteristic (mass and charge), which imposes an additional condition on the linear momentum : $\langle p,p\rangle = mc^2$. Such models have been extensively studied with gravitational fields only ("Boltzman systems"), notably in astrophysics, and the electromagnetic field for plasmas ("Vlasov-Maxwell systems").

In QTF these discontinuous processes are represented by Feynman diagrams, which desribe all the possible interactions between fermions and bosons, and the probability of occurence of a given interaction is computed by path-integrals, integrals of the action along each possible evolution of the system.

12.1 Micro-system

It we consider a single particle as a system of its own (a microsystem), its state is fully defined by a map $\psi \in C\left(\mathbb{R};\widehat{E}_0 \otimes F\right)$, which, at equilibrium, is solution of first order differential equations, where the parameters are the values G,À,P at each point along the world line, and actually the position m does not matter : all the variables can be defined as functions of τ.

The map ψ belongs to a Hilbert space $\mathcal{F}$, which is a unitary representation of Spin(3,1)$\times U$ and the solutions belong to a finite dimensional vector subspace, which is the kernel of Freholm operators.

If a discontinuity occurs on some point τ_0 of the world line, between two states of equilibrium at τ_1, τ_2, the equations above still holds at these dates τ_1, τ_2. The state jumps from some value $\psi(\tau_1)$ to $\psi(\tau_2)$. The continuous fields $\mathcal{F}$ all over Ω acts as a reservoir : by definition they are smooth, defined all over Ω and can change only continuously as $\frac{1}{\det P'}\sum_\beta \frac{d}{d\xi^\beta}\left(\mathcal{F}_G^{\alpha\beta}\det P'\right)$. In equilibrium the propagation of fields follow the differential equations for the potentials, the equations are smooth and the fields adjust in accordance with the state of the particles. The radiation which is emitted has a continuous spectrum. When a discontinuity occurs, the field cannot locally absorb the adjustment,

so there is a discontinuous radiation ΔB which is emitted, whose value is direcly linked to the jump ΔF. And of course the energy released in the radiation is closely related to the change of the energy of the particle. The effect is all the more important as the range of the fields are small, because the size of the reservoir is smaller. For the weak and strong interactions the phenomenon is essentially the emission of a boson.

Without additional information about the discontinuous process involved it is impossible to tell more about the radiation emitted. Even the Planck's law ($E = \hbar\nu$) does not state a minimum for E. So, as usual in discontinuous process, we have probabilist models.

The theorems of QM do not require any smoothness from the variables, so they hold in this situation.

The map ψ belongs to a closed (because of the continuity at equilibrium) vector subspace $\digamma_0$ of of a Hilbert space, which is also a Hilbert space. In a hilbertian basis of $\digamma_0$: $\psi = \sum_{k=0}^{N} c_k \psi_k$. The space $\digamma_0$ does not change because of the discontinuity, so we have :

$\psi_1 = \sum_{k=0}^{N} c_k \psi_k \rightarrow \psi_2 = \sum_{k=0}^{N} d_k \psi_k$

between the evolutions : $\tau \in [0, \tau_1], \tau \in [\tau_2, +\infty]$ and there is a matrix $\langle \psi_1, \psi_2 \rangle_{\digamma_0}$ called the S-matrix, which represents the "scattering process".

The observed value for ψ is as usual : $\widehat{\psi} = \sum_{k=0}^{N} \widehat{c}_k \psi_k$ with $\widehat{c}_k = 0$ or $= c_k$. And the matrix $\left\langle \widehat{\psi}_1, \widehat{\psi}_2 \right\rangle_{\digamma_0}$ represents the probability of transition between the states ψ_1, ψ_2.

All this holds when one considers a population of particles, interacting together and with the fields.Then the S-matrix is related to the measures of the distributions of the states of particles in a cross-section at different times.

In this very general scheme, the hypothesis of equilibrium at the two different dates is essential. And, except if there is a discernable discontinuity (such as a collision), we cannot say when such a process occurs. This is the topic of models of desequilibrium, which are, so far, essentially phenomenological.

12.2 Gas of interacting particles

12.2.1 Fock spaces

Starting with these microsystems, we can go a step further, on a way similar to the classical kinetic theory of gas. Because QM allows to replace interacting systems by a more global model where the vector spaces are replaced by their tensorial products, we can consider a population of k microsystems, represented each by the variables (ψ_k, X_k) where X_k stands for the value $\left(G, \grave{A}, P\right)$ at τ on the world line of the particle N, and a global system (the reservoir) represented by the fields $(\mathcal{F}_G, \mathcal{F}_A, P)$ sections of the respective fiber bundles. We have also to account for the propagation of the fields, meaning the bosons which are emitted, both in a continuous and discontinuous manner through the interactions with particles, and propagate all over the system along their currents. They are represented as vectors $B = (B_G, B_A)$ of $TM \otimes (T_1 Spin \times T_1 U)$.

The particles can be distinguished along their type (to alleviate the notation we will not do it). At any time their number is fixed, but it is allowed to vary along the evolution of the system, as well as the repartition along the types.

The types of bosons are fixed, but not their number. They must be accounted for, as part of the balance for the local equilibrium of fields and of the energy-momentum tensor, but as they propagate one cannot know their number.

The space of representation of the microsystems is then the tensorial algebra, called a Fock space : $\oplus_{k=0}^{\infty}(\otimes_k H_P)$ where H_P stands for the Hilbert space of the variables (ψ_N, X_N) representing the particles, and H_F for the fields.

Similarly the space of representation of the bosons is $\oplus_{N=1}^{\infty}(\otimes_N H_B)$ with H_B the Hilbert space of the variables representing the bosons, and the space of representation of the fields is H_F.

All together the state of the system is :

$(\oplus_{k=0}^{\infty}(\otimes_k H_P)) \otimes (\oplus_{k=0}^{\infty}(\otimes_k H_B)) \otimes H_F$

Notice that k can be 0 so scalars can be vectors of the Fock spaces. The "ground state" is the vector $(1, 0, 0, ...)$ in the algebra.

Actually these intimidating mathematical structures allow some computations.

Any operator on the Hilbert spaces can be extended to a linear continuous operator on the Fock space. So there are operators $\widehat{\psi}, \widehat{G}, ...$ which are related to the estimation which could be made of the values of $(\psi_1, \psi_2, ...\psi_N)$.

For each Fock space $\oplus_{k=1}^{\infty}(\otimes_k H)$ there is a number operator N, whose, dense, domain is :

$D(N) = \left\{\Upsilon^k \in \otimes_k H, \sum_{k\geq 0} k^2 \left\|\Upsilon^k\right\|^2 < \infty\right\}$

$N(\Upsilon) = \left(0, \Upsilon^1, 2\Upsilon^2, ...k\Upsilon^k...\right)$

N is self adjoint.

The annihiliation operator cuts a tensor at its beginning :

$a_k : H \to \mathcal{L}(\otimes_k H; \otimes_{k-1} H) ::$

$a_k(v)(\Upsilon_1 \otimes \Upsilon_2 ... \otimes \Upsilon_k) = \frac{1}{\sqrt{k}} \langle v, \Upsilon_1 \rangle \Upsilon_2 \otimes \Upsilon_2 ... \otimes \Upsilon_k$

The creation operator adds a vector to a tensor at its beginning :

$a_k^* : H \to \mathcal{L}(\otimes_k H; \otimes_{k+1} H) ::$

$a_k^*(v)(\Upsilon_1 \otimes \Upsilon_2 ... \otimes \Upsilon_k) = \sqrt{k+1} v \otimes \Upsilon_1 \otimes \Upsilon_2 \otimes \Upsilon_2 ... \otimes \Upsilon_k$

a_k^* is the adjoint of a_k and a_k, a_k^* can be extended to the Fock space as a, a*.

The physical meaning of these operators is clear from their names. They are the main tool to represent the crucial phenomenon in a disctontinuous process, that is the variation of the number of particles.

12.2.2 The Spin issue

In our assumptions about particles, in a given system, to each type p of particle is associated a section $\Psi_p \in \mathfrak{X}\left(Q\left[\widehat{E}_0 \otimes F, \vartheta\right]\right)$ and, for given initial conditions ψ_{0p} we have unique sections $\sigma \in P_G, \Theta \in P_U$ such that :

$\psi_p(m) = \vartheta(\sigma(m), \Theta(m))\psi_{0p}$

$\sigma(m)$ is projected on the homogeneous space Spin(3,1)/Spin(3) and there are two elements $\sigma_w = \epsilon(a_w + \upsilon(0, w))$ such that

$\sigma(m) = \epsilon\sigma_w \cdot \epsilon\sigma_r$

Indeed, on the same world line the spin can be up or down. The state ψ_p is the same, the spatial state $\psi_{pr} = \gamma C\left(\epsilon\sigma_w^{-1}\right)\psi_p$ takes opposite values. We have seen that the equations for ψ and w hold similarly in both cases, and the continuity of the evolution impose that the spatial state (say the spin) does not change, so it is imposed by the initial conditions. However in a discontinuous model this assumption does not hold any longer and, as ψ is not sufficient to discriminate the spin, we have to introduce and additional variable $s \in \{+1, -1\}$ to represent the spin of the particle in a microsystem.

In a gas of particles it is natural to assume that the particles are undistinguishable, at least among the particles of the same type. The group of permutations acts on the Fock space $\oplus_{k=0}^{\infty}(\otimes_k H_P)$: it exchanges the vectors representing the states of two particles. If the particles are undistinguishable the vectors of the Fock space representing the system belong to an invariant vector subspace : an observer cannot distinguish two states of the same system where the particles have been exchanged.

However we have to account for the spin : even if we cannot tell one particle for another of the same type, the spin is an observable. So actually we have two subpopulations of particles (and further subdivised by types).

The same reasoning holds for the bosons, but here there is no spin involved, and no specific invariant subspace.

The Wiener's theorem applies to the action of the group of permutations on the Fock space : we have a representation of the group, so the vector subspace of the states of particle is isometric to a representation of the group. These representations are well known. They involve classes of conjugations (subsets which are globally invariant by permutations). In the case of bosons no specific subset can be involved, and as a consequence the tensors of the Fock space must be symmetric. For particles we have two distinct subsets, and the tensors of the Fock space must be antisymmetric.

As a consequence the tensors representing the state of particles must be such that their coordinates are antisymmetric : they must be different for each particle, otherwise the value of any observable, which would change according to the signature of the permutation, would be null. So two particles, in such a gas of interacting particles, cannot have the same state. This is the "Pauli's exclusion principle". Notice that the state is related here to the coordinates of maps ψ in a space of maps : so the set of values at a given time is huge.

So far we have not dwelled on the nature of particles, and our basic assumptions hold for a large class of objects. So what is said here stands whenever there are two distinguishable states of spin. "Particles", as assembing of elementary particles having a proper spin and behaving in a coordinated manner, have a spin which is the result of the elementary spins. The same can be said of nuclei or atoms with their electrons. But as we have assumed that particles have no internal structure this fact does not matter, the result stands for any particle in this larger definition.

The spaces of symmetric (called the Bose-Fock space) and antisymmetric (called the Fermi-Fock space) tensors in a Fock space have special properties. They are closed vector subspaces, so are themselves Hilbert spaces, with an adjusted scalar product. Any tensor of the Fock space can be projected on the Bose subspace (by P_+) or the Fermi space (by P_-) by symmetrization and antisymmetrization respectively, and P_+, P_- are orthogonal. The operator $\exp itN$ leaves both subspaces invariant. Any self-adjoint operator on the underlying Hilbert space has an essentially self adjoint prolongation on these subspaces (called its "second quantification"). However the creation and annihiliation operators have extensions with specific commutation rules :

Canonical commutation rules (CCR) in the Bose space:

$[a_+(u), a_+(v)] = \left[a_+^*(u), a_+^*(v)\right] = 0$

$\left[a_+(u), a_+^*(v)\right] = \langle u, v\rangle 1$

Canonical anticommutation rules (CAR) in the Fermi space :

$\{a_+(u), a_+(v)\} = \left\{a_-^*(u), a_-^*(v)\right\} = 0$

$\left\{a_+(u), a_+^*(v)\right\} = \langle u, v\rangle 1$

where

$[X, Y] = X \circ Y - Y \circ X$

$\{X, Y\} = X \circ Y + Y \circ X$

These differences have important mathematical consequences. In the Fermi space the operators a_-, a_-^* have bounded (continuous) extensions. Any configuration of particles can be generated by the product of "creation operators" acting on the ground state. We have nothing equivalent for the bosons.

We will not pursue on these topics, which are exposed at lengths in many books, and imply sophisticated mathematical concepts.

Part IV
CONCLUSION

At the end of this paper, I hope to have brought some clarification on the concept of duality between particles and fields.

The most important result is probably the new formalism to represent the motion of a body. It explains clearly the need for spinors, which are an essential but somewhat obscure part of any Quantum Theory of Fields. Spinors cannot be fully understood and used out of the framework of Clifford algebras. By fully assuming this framework, and the more general representation of General Relativity by fiber bundles, we have been able to develop new tools which are physically meaningful and easy to use in the context of curved space-times. These tools should find some usage in the studies of moving bodies in astrophysics.

The picture of "matter fields" is quite clear. However the concept of mass must be revisited. In any Yang-Mills model in the GR context mass is a gravitational charge, and so is not attached to a specific unit : it is a way to compare the behaviour of particles in a given field . It reveals itself, in the energy-tensor, in the presence of a gravitational field. So the concept of "mass at rest" must be understood in the light of the basic assumption of Relativity, that material body travels along world lines. If we fully accept this tenet, then there is no such thing as a "body at rest", and the source of this universal motion shall be related to cosmologic features, and to see the cosmological constant in a different perspective.

The bosons can be recognized, from the currents, the energy-momentum tensor, and more importantly in all discontinuous processes. However it is difficult to understand the rules of propagation, or the charges carried by bosons, without a better representation of the fields as connections on fiber bundles. This leads to underline the need for a unified representation of the gravitational and other field, a big and so far unsuccesful endeavour. The use of Clifford algebras should be pursued, in the light of the present results. The similarities between Fock spaces (notably in their Bose / Fermi versions) with Clifford algebras are too stricking to be a simple coincidence, even if they have been muddled by the usual quest for "commutation relations".

Eventually it seems to me that the concept of propagation of a field, which is so familiar to physicists and so essential in all theory about causality, desserves a better attention. I have sketched here some basic ideas on this topic, but more has to be done, notably in the framework of differential equations.

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Part V
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Part VI
ANNEX

13 CLIFFORD ALGEBRAS

This annex gives proofs of some results presented in the core of the paper.

13.1 Products in the Clifford algebra

Many results are consequences of the computation of products in the Clifford algebra. The computation are straightforward but the results precious.

Product $\upsilon(r,w)\cdot\upsilon(r',w')$
$\upsilon(r,w)=\frac{1}{2}\left(w^1\varepsilon_0\cdot\varepsilon_1+w^2\varepsilon_0\cdot\varepsilon_2+w^3\varepsilon_0\cdot\varepsilon_3+r^3\varepsilon_2\cdot\varepsilon_1+r^2\varepsilon_1\cdot\varepsilon_3+r^1\varepsilon_3\cdot\varepsilon_2\right)$
$\upsilon(r',w')=\frac{1}{2}\left(w'^1\varepsilon_0\cdot\varepsilon_1+w'^2\varepsilon_0\cdot\varepsilon_2+w'^3\varepsilon_0\cdot\varepsilon_3+r'^3\varepsilon_2\cdot\varepsilon_1+r'^2\varepsilon_1\cdot\varepsilon_3+r'^1\varepsilon_3\cdot\varepsilon_2\right)$

In Cl$(\mathbb{R},3,1)$
$\upsilon(r,w)\cdot\upsilon(r',w')=\frac{1}{4}\left(w^tw'-r^tr'\right)+\frac{1}{2}\upsilon\left(j(r)r'-j(w)w',j(w)r'+j(r)w'\right)-\frac{1}{4}\left(w^tr'+r^tw'\right)\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$
From there the bracket on the Lie algebra :
$[\upsilon(r,w),\upsilon(r',w')]=\upsilon(r,w)\cdot\upsilon(r',w')-\upsilon(r',w')\cdot\upsilon(r,w)$

$$[\upsilon(r,w),\upsilon(r',w')]=\upsilon\left(j(r)r'-j(w)w',j(w)r'+j(r)w'\right) \tag{92}$$

In Cl$(\mathbb{R},1,3)$
$\upsilon(r,w)\cdot\upsilon(r',w')=\frac{1}{4}\left(w^tw'-r^tr'\right)-\frac{1}{2}\upsilon\left(-j(r)r'+j(w)w',j(w)r'+j(r)w'\right)-\frac{1}{4}\left(w^tr'+r^tw'\right)\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$
From there the bracket on the Lie algebra :

$$[\upsilon(r,w),\upsilon(r',w')]=-\upsilon\left(j(r)r'-j(w)w',j(w)r'+j(r)w'\right) \tag{93}$$

Product on Spin$(3,1)$
Because they belong to $Cl_0(\mathbb{R},3,1)$ the elements of Spin(3,1) can be written :
$s=a+\frac{1}{2}\left(w^1\varepsilon_0\cdot\varepsilon_1+w^2\varepsilon_0\cdot\varepsilon_2+w^3\varepsilon_0\cdot\varepsilon_3+r^3\varepsilon_2\cdot\varepsilon_1+r^2\varepsilon_1\cdot\varepsilon_3+r^1\varepsilon_3\cdot\varepsilon_2\right)+b\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$
where $a,(w^j,r^j)_{j=1}^3,b$ are real scalar which are related. That we will write :

$$s=a+\upsilon(r,w)+b\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3=a+\upsilon(r,w)+b\varepsilon_5 \tag{94}$$

And similarly in $Cl(\mathbb{R},1,3)$
$s=a+\upsilon(r,w)+b\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$
The product of two elements of the spin group expressed as :
$s=a+\upsilon(r,w)+b\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$
$s'=a'+\upsilon(r',w')+b'\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$
can be computed with the previous formulas.

i) In $Cl(\mathbb{R},3,1)$:

$\upsilon(r,w)\,\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3=\upsilon(r,-w)$

$\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\upsilon(r',w')=\upsilon(r',-w')$

$\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\cdot\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3=-1$

$s\cdot s'$

$=aa'-b'b+\frac{1}{4}\left(w^tw'-r^tr'\right)+\left(ab'+ba'-\frac{1}{4}\left(w^tr'+r^tw'\right)\right)\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$

$$\frac{1}{2}\begin{pmatrix}\left(b'w^1+bw'^1\right)\varepsilon_2\cdot\varepsilon_3-\left(b'w^2+bw'^2\right)\varepsilon_1\cdot\varepsilon_3+\left(b'w^3+bw'^3\right)\varepsilon_1\cdot\varepsilon_2\\+\left(b'r^3+br'^3\right)\varepsilon_0\cdot\varepsilon_3+\left(b'r^2+br'^2\right)\varepsilon_0\cdot\varepsilon_2+\left(b'r^1+br'^1\right)\varepsilon_0\cdot\varepsilon_1\end{pmatrix}$$

$$+\frac{1}{4}\begin{pmatrix}\left(\left(j(w)r'\right)^1+\left(j(r)w'\right)^1\right)\varepsilon_0\cdot\varepsilon_1+\left(\left(j(w)r'\right)^2+\left(j(r)w'\right)^2\right)\varepsilon_0\cdot\varepsilon_2\\+\left(\left(j(w)r'\right)^3+\left(j(r)w'\right)^1\right)\varepsilon_0\cdot\varepsilon_3\end{pmatrix}$$

$$+\frac{1}{4}\begin{pmatrix}\left(-\left(j(r)r'\right)^3+\left(j(w)w'\right)^3\right)\varepsilon_1\cdot\varepsilon_2+\left(\left(j(r)r'\right)^2-\left(j(w)w'\right)^2\right)\varepsilon_1\cdot\varepsilon_3\\+\left(-\left(j(r)r'\right)^1+\left(j(w)w'\right)^1\right)\varepsilon_2\cdot\varepsilon_3\end{pmatrix}$$

$$+\frac{1}{2}\begin{pmatrix}\left(a'w^1+aw'^1\right)\varepsilon_0\cdot\varepsilon_1+\left(a'w^2+aw'^2\right)\varepsilon_0\cdot\varepsilon_2+\left(a'w^3+aw'^3\right)\varepsilon_0\cdot\varepsilon_3\\+\left(a'r^3+ar'^3\right)\varepsilon_2\cdot\varepsilon_1+\left(a'r^2+ar'^2\right)\varepsilon_1\cdot\varepsilon_3+\left(a'r^1+ar'^1\right)\varepsilon_3\cdot\varepsilon_2\end{pmatrix}$$

Which can be expressed as :

$s\cdot s'=a"+\upsilon(r",w")+b"\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$

with :

$a"=aa'-b'b+\frac{1}{4}\left(w^tw'-r^tr'\right)$

$b"=ab'+ba'-\frac{1}{4}\left(w^tr'+r^tw'\right)$

$r"=\frac{1}{2}\left(j(r)r'-j(w)w'\right)+a'r+ar'-b'w-bw'$

$w"=\frac{1}{2}\left(j(w)r'+j(r)w'\right)+a'w+aw'+b'r+br'$

So we have in particular :

$(a+\upsilon(0,w))\cdot(a'+\upsilon(0,w'))=aa'+\frac{1}{4}w^tw'+\upsilon\left(-\frac{1}{2}\left(j(w)w',a'w+aw'\right)\right)$

$(a+\upsilon(r,0))\cdot(a'+\upsilon(r',0))=aa'-\frac{1}{4}r^tr'+\upsilon\left(\frac{1}{2}j(r)r'+\left(a'r+ar'\right),0\right)$

$(a_w+\upsilon(0,w))\cdot(a_r+\upsilon(r,0))=a_wa_r+\upsilon(a_wr,a_rw)-\frac{1}{2}\left(w^tr\right)\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$

ii) In $Cl(\mathbb{R},1,3)$:

$s=a+\upsilon(r,w)+b\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$

$s'=a'+\upsilon(r',w')+b'\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$

$\upsilon(r,w)\cdot\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3=\upsilon(w,r)$

$\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\upsilon(r',w')=\upsilon(w',r')$

$\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3\cdot\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3=-1$

$s\cdot s'=a"+\upsilon(r",w")+b"\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$

with :

$a"=aa'-b'b+\frac{1}{4}\left(w^tw'-r^tr'\right)$

$b"=ab'+ba'-\frac{1}{4}\left(w^tr'+r^tw'\right)$

$r"=\frac{1}{2}\left(j(r)r'-j(w)w'\right)+a'r+ar'+b'w+bw'$

$w"=-\frac{1}{2}\left(j(w)r'+j(r)w'\right)+a'w+aw'+b'r+br'$

13.2 Characterization of the elements of the Spin group

13.2.1 Inverse

The elements of Spin(3,1) are the product of an even number of vectors of norm ± 1. Consequently we have :

$s \cdot s^t = (v_1 \cdot ... v_{2p}) \cdot (v_{2p} \cdot ... \cdot v_1) = 1$

The transposition is an involution on the Clifford algebra, thus :

$(a + \upsilon(r,w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) \cdot \left(a + \upsilon(r,w)^t + b\varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0\right) = 1$

$(a + \upsilon(r,w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) \cdot (a - \upsilon(r,w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) = 1$

$\Leftrightarrow (a + \upsilon(r,w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3)^{-1} = (a - \upsilon(r,w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3)$

and we have the same result in $\mathrm{Cl}(\mathbb{R},1,3)$

$$(a + \upsilon(r,w) + b\varepsilon_5)^{-1} = a - \upsilon(r,w) + b\varepsilon_5 \tag{95}$$

13.2.2 Relation between a,b, r, w

By a straightforward computation this identity gives the following relation between a,b,r,w :

1. in $Cl(\mathbb{R},3,1)$:

$(a + \upsilon(r,w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) \cdot (a - \upsilon(r,w) + b\varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0) = 1$

$= a" + \upsilon(r", w") + b"\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

with :

$a" = a^2 - b^2 + \frac{1}{4}(-w^t w + r^t r) = 1$

$b" = ab + ba - \frac{1}{4}(-w^t r - r^t w) = 0$

$r" = \frac{1}{2}(-j(r)r + j(w)w) + ar - ar - bw + bw = 0$

$w" = \frac{1}{2}(-j(w)r - j(r)w) + aw - aw + br - br = 0$

$a^2 - b^2 = 1 + \frac{1}{4}(w^t w - r^t r)$

So, for any element : $a + \upsilon(r,w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

we have :

$$a^2 - b^2 = 1 + \frac{1}{4}(w^t w - r^t r) \tag{96}$$

$$ab = -\frac{1}{4} r^t w \tag{97}$$

and we keep only 6 free parameters. a,b are defined from r,w, up to sign, with the conditions:

i) $r^t w \neq 0 : b = -\frac{1}{4a} r^t w$

$a^2 = \frac{1}{2}\left(\left(1 + \frac{1}{4}(w^t w - r^t r)\right) + \sqrt{\left(1 + \frac{1}{4}(w^t w - r^t r)\right)^2 + \frac{1}{4}(r^t w)^2}\right)$

ii) $r^t w = 0$:

$(w^t w - r^t r) \geq -4 : a = \epsilon\sqrt{1 + \frac{1}{4}(w^t w - r^t r)}; b = 0$

$(w^t w - r^t r) \leq -4 : b = \epsilon\sqrt{-\left(1 + \frac{1}{4}(w^t w - r^t r)\right)}; a = 0$

So :

if r=0 then $a = \epsilon\sqrt{1 + \frac{1}{4} w^t w}; b = 0 : s = \epsilon\sqrt{1 + \frac{1}{4} w^t w} + \upsilon(0,w)$

if w=0 then

$r^t r \leq 4 : a = \epsilon\sqrt{1 - \frac{1}{4}r^t r}; b = 0 : s = \epsilon\sqrt{1 - \frac{1}{4}r^t r} + \upsilon(r, 0)$
$r^t r \geq 4 : b = \epsilon\sqrt{-1 + \frac{1}{4}r^t r}; a = 0 : s = \upsilon(r, 0) + \epsilon\sqrt{\frac{1}{4}r^t r - 1}\varepsilon_5$

2. In $Cl(\mathbb{R}, 1, 3)$
$(a - \upsilon(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3) \cdot (a + \upsilon(r, w) + b\varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0) = 1$
$= a" + \upsilon(r", w") + b"\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$
with :
$r" = \frac{1}{2}(-j(r)r + j(w)w) + ar - ar + bw - bw = 0$
$w" = -\frac{1}{2}(-j(w)r - j(r)w) + aw - aw + br - br'$
$a" = a^2 - b^2 + \frac{1}{4}(-w^t w + r^t r)$
$b" = ab + ba - \frac{1}{4}(-w^t r - r^t w)$
we get the same relations.

13.3 Quotient set

13.3.1 Spin(R,3)

Proposition 30 *The subset of $Spin(3,1)$ of the elements which commute with ε_0 is a subgroup of $Spin(3,1)$. They leave ε_0 unchanged :* $\mathbf{Ad}_{s_r}\varepsilon_0 = s_r \cdot \varepsilon_0 \cdot s_r^{-1} = \varepsilon_0 = [g]_0^i \varepsilon_i$. *They read :* $s_r = a + \upsilon(r, 0)$

Proof. $s_r = a + \upsilon(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$
$s_r \cdot \varepsilon_0 = \varepsilon_0 \cdot s_r$
In Cl$(\mathbb{R}, 3, 1)$:
$s \cdot \varepsilon_0 = a\varepsilon_0 + \upsilon(r, w)\varepsilon_0 - b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = \varepsilon_0 \cdot s = a\varepsilon_0 + \varepsilon_0\upsilon(r, w) + b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$
$\upsilon(r, w)\varepsilon_0 =$
$\frac{1}{2}\left(g^4\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_0 + g^5\varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_0 + g^6\varepsilon_0 \cdot \varepsilon_3 \cdot \varepsilon_0 + g^3\varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0 + g^2\varepsilon_1 \cdot \varepsilon_3 \cdot \varepsilon_0 + g^1\varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_0\right)$
$= \frac{1}{2}\left(g^4\varepsilon_1 + g^5\varepsilon_2 + g^6\varepsilon_3 - g^3\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 + g^2\varepsilon_0\varepsilon_1 \cdot \varepsilon_3 - g^1\varepsilon_0\varepsilon_2 \cdot \varepsilon_3\right)$
$\varepsilon_0\upsilon(r, w)$
$= \frac{1}{2}\left(g^4\varepsilon_0\varepsilon_0 \cdot \varepsilon_1 + g^5\varepsilon_0\varepsilon_0 \cdot \varepsilon_2 + g^6\varepsilon_0\varepsilon_0 \cdot \varepsilon_3 + g^3\varepsilon_0\varepsilon_2 \cdot \varepsilon_1 + g^2\varepsilon_0\varepsilon_1 \cdot \varepsilon_3 + g^1\varepsilon_0\varepsilon_3 \cdot \varepsilon_2\right)$
$= \frac{1}{2}\left(-g^4\varepsilon_1 - g^5\varepsilon_2 - g^6\varepsilon_3 - g^3\varepsilon_0\varepsilon_1 \cdot \varepsilon_2 + g^2\varepsilon_0\varepsilon_1 \cdot \varepsilon_3 - g^1\varepsilon_0\varepsilon_2 \cdot \varepsilon_3\right)$
$a\varepsilon_0 + \frac{1}{2}\left(g^4\varepsilon_1 + g^5\varepsilon_2 + g^6\varepsilon_3 - g^3\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 + g^2\varepsilon_0\varepsilon_1 \cdot \varepsilon_3 - g^1\varepsilon_0\varepsilon_2 \cdot \varepsilon_3\right) - b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$
$= a\varepsilon_0 + \frac{1}{2}\left(-g^4\varepsilon_1 - g^5\varepsilon_2 - g^6\varepsilon_3 - g^3\varepsilon_0\varepsilon_1 \cdot \varepsilon_2 + g^2\varepsilon_0\varepsilon_1 \cdot \varepsilon_3 - g^1\varepsilon_0\varepsilon_2 \cdot \varepsilon_3\right) + b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$
$\frac{1}{2}\left(g^4\varepsilon_1 + g^5\varepsilon_2 + g^6\varepsilon_3\right) - b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = \frac{1}{2}\left(-g^4\varepsilon_1 - g^5\varepsilon_2 - g^6\varepsilon_3\right) + b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$
$\Rightarrow w = 0, b = 0$
In Cl$(\mathbb{R}, 1, 3)$:
$s \cdot \varepsilon_0 = a\varepsilon_0 - \upsilon(g)\varepsilon_0 - b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = \varepsilon_0 \cdot s = a\varepsilon_0 - \varepsilon_0\upsilon(g) + b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 \Rightarrow b = 0$
$\upsilon(g)\varepsilon_0 =$
$\frac{1}{2}\left(g^4\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_0 + g^5\varepsilon_0 \cdot \varepsilon_2 \cdot \varepsilon_0 + g^6\varepsilon_0 \cdot \varepsilon_3 \cdot \varepsilon_0 + g^3\varepsilon_2 \cdot \varepsilon_1 \cdot \varepsilon_0 + g^2\varepsilon_1 \cdot \varepsilon_3 \cdot \varepsilon_0 + g^1\varepsilon_3 \cdot \varepsilon_2 \cdot \varepsilon_0\right)$
$= \frac{1}{2}\left(-g^4\varepsilon_1 - g^5\varepsilon_2 - g^6\varepsilon_3 - g^3\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 + g^2\varepsilon_0\varepsilon_1 \cdot \varepsilon_3 - g^1\varepsilon_0\varepsilon_2 \cdot \varepsilon_3\right)$
$\varepsilon_0\upsilon(g) = \frac{1}{2}\left(g^4\varepsilon_0\varepsilon_0 \cdot \varepsilon_1 + g^5\varepsilon_0\varepsilon_0 \cdot \varepsilon_2 + g^6\varepsilon_0\varepsilon_0 \cdot \varepsilon_3 + g^3\varepsilon_0\varepsilon_2 \cdot \varepsilon_1 + g^2\varepsilon_0\varepsilon_1 \cdot \varepsilon_3 + g^1\varepsilon_0\varepsilon_3 \cdot \varepsilon_2\right)$
$= \frac{1}{2}\left(g^4\varepsilon_1 + g^5\varepsilon_2 + g^6\varepsilon_3 - g^3\varepsilon_0\varepsilon_1 \cdot \varepsilon_2 + g^2\varepsilon_0\varepsilon_1 \cdot \varepsilon_3 - g^1\varepsilon_0\varepsilon_2 \cdot \varepsilon_3\right)$
$\Rightarrow \frac{1}{2}\left(g^4\varepsilon_1 + g^5\varepsilon_2 + g^6\varepsilon_3\right) - b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = \frac{1}{2}\left(-g^4\varepsilon_1 - g^5\varepsilon_2 - g^6\varepsilon_3\right) + b\varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$ ■

So the elements such that $s = \upsilon(r,0) + \epsilon\sqrt{\frac{1}{4}r^t r - 1}\varepsilon_5$ are excluded and we are left with $s = \epsilon\left(\sqrt{1-\frac{1}{4}r^t r} + \upsilon(r,0)\right)$. These elements constitute a subgroup, as it can easily be checked with the formula for the product. They are generated by vectors belonging to the subspace spanned by the vectors $(\varepsilon_i)_{i=1}^3$ so they belong to Spin(3) which can be seen as a subgroup of $Spin(3,1)$.

The scalars $\epsilon = \pm 1$ belong to the group. The group is not connected. The elements $s = \sqrt{1-\frac{1}{4}r^t r} + \upsilon(r,0)$ constitute the component of the identity.

13.3.2 Decomposition of the Lie algebra

The Lie algebra $T_1 Spin(3,1)$ is the direct sum of the two vector vector subspaces :

$X \in T_1 Spin(3,1) :: X = \upsilon(r,w) = \upsilon(r,0) + \upsilon(0,w)$

The Lie bracket reads :

$[\upsilon(r,w), \upsilon(r',w')] = \upsilon(j(r)r' - j(w)w', j(w)r' + j(r)w')$

thus :

$[\upsilon(r,0), \upsilon(r',0)] = \upsilon(j(r)r', 0)$

$[\upsilon(0,w), \upsilon(0,w')] = \upsilon(-j(w)w', 0)$

The vector subspace $\upsilon(r,0)$, generated by the vectors $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ is a Lie subalgebra, which can be identified with $T_1 Spin(3)$.

And similarly in $Cl(\mathbb{R},1,3)$.

For any element $\upsilon(r,w)$of $T_1 Spin(3,1)$ we have the identity :

$\upsilon(r,w)\cdot\varepsilon_0 - \varepsilon_0\cdot\upsilon(r,w) = w$

Proof. $\upsilon(r,w)\cdot\varepsilon_0 =$

$\frac{1}{2}\left(w^1\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_0 + w^2\varepsilon_0\cdot\varepsilon_2\cdot\varepsilon_0 + w^3\varepsilon_0\cdot\varepsilon_3\cdot\varepsilon_0 + r^3\varepsilon_2\cdot\varepsilon_1\cdot\varepsilon_0 + r^2\varepsilon_1\cdot\varepsilon_3\cdot\varepsilon_0 + r^1\varepsilon_3\cdot\varepsilon_2\cdot\varepsilon_0\right)$

$= \frac{1}{2}w + \frac{1}{2}\left(-r^3\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2 + r^2\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_3 - r^1\varepsilon_0\cdot\varepsilon_2\cdot\varepsilon_3\right)$

$\varepsilon_0\cdot\upsilon(r,w) =$

$\frac{1}{2}\left(w^1\varepsilon_0\cdot\varepsilon_0\cdot\varepsilon_1 + w^2\varepsilon_0\cdot\varepsilon_0\cdot\varepsilon_2 + w^3\varepsilon_0\cdot\varepsilon_0\cdot\varepsilon_3 + r^3\varepsilon_0\cdot\varepsilon_2\cdot\varepsilon_1 + r^2\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_3 + r^1\varepsilon_0\cdot\varepsilon_3\cdot\varepsilon_2\right)$

$= -\frac{1}{2}w + \frac{1}{2}\left(-r^3\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2 + r^2\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_3 - r^1\varepsilon_0\cdot\varepsilon_2\cdot\varepsilon_3\right)$

$\upsilon(r,w)\cdot\varepsilon_0 - \varepsilon_0\cdot\upsilon(r,w) = w$ ■

Thus $\upsilon(r,w) \in T_1 Spin(3) \Leftrightarrow \upsilon(r,w)\cdot\varepsilon_0 - \varepsilon_0\cdot\upsilon(r,w) = 0 \Leftrightarrow w = 0$

13.3.3 Homogeneous space

The quotient space $Spin(3,1)/Spin(3)$ (called a homogeneous space) is not a group but a 3 dimensional manifold. It is characterized by the equivalence relation :

$s = a + \upsilon(r,w) + b\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3 \sim s' = a' + \upsilon(r',w') + b'\varepsilon_0\cdot\varepsilon_1\cdot\varepsilon_2\cdot\varepsilon_3$

$\Leftrightarrow \exists s_r \in Spin(3) : s' = s\cdot s_r$

As any quotient space its elements are *subsets* of $Spin(3,1)$.

Proposition 31 *In each class of the homogeneous space there are two elements, defined up to sign, which read :* $s_w = \pm(a_w + \upsilon(0,w))$

Proof. Each coset [s] is in bijective correspondance with $Spin(3)$.

So $[s] = \left\{s' = s\cdot\left(\epsilon\sqrt{1-\frac{1}{4}\rho^t\rho} + \upsilon(\rho,0)\right), \rho^t\rho \leq 4\right\}$

In Spin(3,1) :

$s' = a' + \upsilon(r', w') + b'\varepsilon_5$
$a' = a\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho} - \frac{1}{4}r^t\rho$
$b' = b\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho} - \frac{1}{4}w^t\rho$
$r' = \frac{1}{2}j(r)\rho + r\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho} + a\rho$
$w' = \frac{1}{2}j(w)\rho + w\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho} + b\rho$
$a^2 - b^2 = 1 + \frac{1}{4}(w^t w - r^t r)$
$ab = -\frac{1}{4}r^t w$
We can always choose in the class an element s' such that : $r' = 0$. It requires : $\left(\frac{1}{2}j(r) + aI\right)\rho = -r\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho}$
This linear equation in ρ has always a unique solution :
$\rho = -\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho}\frac{1}{a(4a^2+r^tr)}\left(\left(4a^2 + r^tr\right)I - 2aj(r) + j(r)j(r)\right)r$
$= -\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho}\frac{1}{a(4a^2+r^tr)}\left(4a^2 + r^tr\right)r$
$= -\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho}\frac{1}{a}r$
$\rho^t\rho = \left(1 - \frac{1}{4}\rho^t\rho\right)\frac{1}{a^2}(r^tr) \Rightarrow$
$\left(a^2 + \frac{1}{4}(r^tr)\right)\rho^t\rho = (r^tr)$
$\rho^t\rho = \frac{4(r^tr)}{4a^2+(r^tr)} \leq 4$
$\sqrt{1 - \frac{1}{4}\rho^t\rho} = \sqrt{\frac{4a^2}{4a^2+(r^tr)}} = \frac{2a}{\sqrt{4a^2+(r^tr)}}$
$\rho = -\epsilon\frac{2}{\sqrt{4a^2+(r^tr)}}r$
$a' = a\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho} - \frac{1}{4}r^t\rho = \frac{1}{2}\frac{\epsilon}{\sqrt{4a^2+(r^tr)}}\left(4a^2 + r^tr\right) = \frac{1}{2}\epsilon\sqrt{4a^2 + r^tr}$
$w' = \frac{1}{2}j(w)\rho + w\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho} + b\rho = \epsilon\frac{2}{\sqrt{4a^2+(r^tr)}}\left(\frac{1}{2}j(r)w + aw - br\right)$
$b' = b\epsilon\sqrt{1 - \frac{1}{4}\rho^t\rho} - \frac{1}{4}w^t\rho = \epsilon\frac{2}{\sqrt{4a^2+(r^tr)}}\left(ab + \frac{1}{4}w^tr\right) = 0$
$s = \left(\epsilon\sqrt{4a^2 + r^tr}\right)\left(\frac{1}{2} + \upsilon\left(0, \epsilon\frac{2}{4a^2+(r^tr)}\left(\frac{1}{2}j(r)w + aw - br\right)\right)\right) \cdot \left(\epsilon\frac{2}{\sqrt{4a^2+(r^tr)}}\right)(a + \upsilon(r, 0))$
So any element of $\mathrm{Spin}(3,1)$ can be written uniquely (up to sign) :
$s = a + \upsilon(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 = s_w \cdot s_r = (a_w + \upsilon(0, w_w)) \cdot (a_r + \upsilon(0, r_r))$
and :
$(a_w + \upsilon(0, w_w)) \cdot (a_r + \upsilon(0, r_r)) = a_w a_r + \upsilon(a_w r_r, a_r w_w) - \frac{1}{2}(w_w^t r_r)\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$
In $\mathrm{Cl}(\mathbb{R}, 1, 3)$ we have the same decomposition with the same components.
$s = a + \upsilon(r, w) + b\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3 =$
$r" = a_w r_r = \frac{1}{2}\epsilon\sqrt{4a^2 + r^tr}\epsilon\frac{2}{\sqrt{4a^2+(r^tr)}}r = r$
$w" = \frac{1}{2}j(w_w)r_r + a_r w_w$
$= \frac{1}{2}j\left(\left(\epsilon\sqrt{4a^2 + r^tr}\right)\epsilon\frac{2}{4a^2+(r^tr)}\left(\frac{1}{2}j(r)w + aw - br\right)\right)\left(\epsilon\frac{2}{\sqrt{4a^2+(r^tr)}}\right)r$
$+ \left(\epsilon\frac{2}{\sqrt{4a^2+(r^tr)}}\right)a\left(\epsilon\sqrt{4a^2 + r^tr}\right)\epsilon\frac{2}{4a^2+(r^tr)}\left(\frac{1}{2}j(r)w + aw - br\right)$
$= 2j\left(\epsilon\frac{1}{4a^2+(r^tr)}\left(\frac{1}{2}j(r)w + aw - br\right)\right)r + a\epsilon\frac{4}{4a^2+(r^tr)}\left(\frac{1}{2}j(r)w + aw - br\right)$

$= \left(\epsilon\frac{2}{4a^2+(r^tr)}\right)\left(j\left(\left(\frac{1}{2}j(r)w+aw-br\right)\right)r+a2\left(\frac{1}{2}j(r)w+aw-br\right)\right)$
$= \left(\epsilon\frac{2}{4a^2+(r^tr)}\right)\left(\frac{1}{2}j(j(r)w)r-aj(w)r+aj(r)w+2a^2w-2abr\right)$
$= \left(\epsilon\frac{2}{4a^2+(r^tr)}\right)\left(\frac{1}{2}(wr^t-rw^t)r+2a^2w+\frac{1}{2}(r^tw)r\right)$
$= \left(\epsilon\frac{2}{4a^2+(r^tr)}\right)\left(\frac{1}{2}w(r^tr)-\frac{1}{2}r(w^tr)+2a^2w+\frac{1}{2}(r^tw)r\right)$
$= \left(\epsilon\frac{1}{4a^2+(r^tr)}\right)\left(\left(4a^2+(r^tr)\right)w\right)=w$ ■

Remark : the elements $\pm s_w$ are equivalent :
$(a_w+\upsilon(0,w_w))\sim-(a_w+\upsilon(0,w_w))$
Take $s_r=-1\in Spin(3):-s_w=s_w\cdot s_r$

So $\pm s_w$ belong to the same class of equivalence. In the decomposition : $s=\epsilon s_w\cdot\epsilon s_r$, ϵs_w is a specific projection of s on the homoheneous space.

13.4 Adjoint map

The translations on $Spin(3,1)$ are :
$L_gh=g\cdot h, R_gh=h\cdot g$
and their derivatives :
$L'_gh:T_hSpin(3,1)\to T_{g\cdot h}Spin(3,1)::L'_gh(X_h)=g\cdot X_h$
$R'_gh:T_hSpin(3,1)\to T_{h\cdot g}Spin(3,1)::R'_gh(X_h)=X_h\cdot g$
Their inverse are, as in any Lie groups :
$\left(L'_gh\right)^{-1}=L'_{g^{-1}}(g\cdot h);\left(R'_gh\right)^{-1}=R'_{g^{-1}}(h\cdot g)$
$T_gSpin(3,1)\subset Cl(\mathbb{R},3,1)$ and there are two linear maps :
$L'_{g^{-1}}g:T_gSpin(3,1)\to T_1Spin(3,1)::L'_{g^{-1}}g(Z_g)=g^{-1}\cdot Z_g$
$R'_{g^{-1}}g:T_gSpin(3,1)\to T_1Spin(3,1)::R'_{g^{-1}}g(Z_g)=Z_g\cdot g^{-1}$
And the adjoint map:
$Ad_g:T_1Spin(3,1)\to T_1Spin(3,1)::Ad_g=L'_gg^{-1}\circ R'_{g^{-1}}1=R'_{g^{-1}}g\circ L'_g1$
$Ad_gZ=L'_gg^{-1}\circ R'_{g^{-1}}1(Z)=L'_gg^{-1}\left(Z\cdot g^{-1}\right)=g\cdot Z\cdot g^{-1}$
$Ad_gZ=\left(\mathbf{Ad}_gZ\right)'_{x=1}=\mathbf{Ad}_gZ$
With
$g=a+\upsilon(r,w)+b\varepsilon_5$
$Z=\upsilon(x,y)$
$Ad_gX=(a+\upsilon(r,w)+b\varepsilon_5)\cdot\upsilon(x,y)\cdot(a-\upsilon(r,w)+b\varepsilon_5)$
A straightforward computation gives :
$Ad_g\upsilon(x,y)=\upsilon(X,Y)$
with
$X=\left[(a+b)^2+(a+b)j(r)\right]x+[(b-a)j(w)]y$
$-\frac{1}{4}(-[j(w)j(r)+j(r)j(r)-4ab+r^tr]x+[j(w)j(r)-j(r)j(r)-4ab-r^tr]y)$
$-\frac{1}{2}([j(w)j(w)-j(r)j(r)]x+[j(w)j(r)+j(r)j(w)]y)$
$Y=[(a+b)j(w)]x+\left[(a-b)^2+(a-b)j(r)\right]y$
$-\frac{1}{4}(-[j(w)j(w)+j(r)j(w)+w^tw-4ab]x+[j(w)j(w)-j(r)j(w)+w^tw+4ab]y)$
$-\frac{1}{2}(-[j(w)j(r)x+j(r)j(w)]x+[j(w)j(w)-j(r)j(r)]y)$
$X=\left[(a+b)^2-ab+\frac{1}{4}r^tr+(a+b)j(r)+\frac{1}{4}(j(w)j(r)-2j(w)j(w)+3j(r)j(r))\right]x$

$+\left[ab+\frac{1}{4}r^tr+(b-a)\,j\,(w)-\frac{1}{4}\left(3j\,(w)\,j\,(r)+2j\,(r)\,j\,(w)-j\,(r)\,j(r)\right)\right]y$
$Y=\left[\frac{1}{4}w^tw-ab+(a+b)\,j\,(w)+\frac{1}{4}\left(j\,(w)\,j\,(w)+2j\,(w)\,j\,(r)+3j\,(r)\,j\,(w)\right)\right]x$
$+\left[(a-b)^2-\frac{1}{4}w^tw-ab+(a-b)\,j\,(r)-\frac{1}{4}\left(3j\,(w)\,j\,(w)-j\,(r)\,j\,(w)-2j\,(r)\,j\,(r)\right)\right]y$

13.5 Derivatives

Let $g:M\rightarrow Spin\,(3,1))::g\,(m)=a\,(m)+\upsilon\,(r\,(m)\,,w\,(m))+b\,(m)\,\varepsilon_5$
$g'\,(m):T_mM\rightarrow T_gSpin\,(3,1)::$
$g'\,(m)\,u_m=a'\,(m)\,u_m+\upsilon\,(r'\,(m)\,u_m,w'\,(m)\,u_m)+b'\,(m)\,u_m\varepsilon_5$
where $u_m\in T_mM,a'\,(m)\,u_m,b'\,(m)\,u_m\in\mathbb{R},r'\,(m)\,u_m,w'\,(m)\,u_m\in\mathbb{R}^3$
Thus :
$g^{-1}\cdot(a'\,(m)\,u_m+\upsilon\,(r'\,(m)\,u_m,w'\,(m)\,u_m)+b'\,(m)\,u_m\varepsilon_5)\in T_1Spin\,(3,1)$
$(a'\,(m)\,u_m+\upsilon\,(r'\,(m)\,u_m,w'\,(m)\,u_m)+b'\,(m)\,u_m\varepsilon_5)\cdot g^{-1}\in T_1Spin\,(3,1)$
$L'_{g^{-1}}g\,(g'\,(m)\,u_m)=$
$\upsilon\left(-\frac{1}{2}\left(j\,(r)\,r'-j\,(w)\,w'\right)-a'r+ar'+b'w-bw',-\frac{1}{2}\left(j\,(w)\,r'+j\,(r)\,w'\right)-a'w+aw'-b'r+br'\right)u_m$
$R'_{g^{-1}}g\,(g'\,(m)\,u_m)=$
$\upsilon\left(-\frac{1}{2}\left(j\,(r)\,r'-j\,(w)\,w'\right)-a'r+ar'+b'w-bw',-\frac{1}{2}\left(j\,(w)\,r'+j\,(r)\,w'\right)-a'w+aw'-b'r+br'\right)u_m$

Computation of $\sigma_w^{-1}\cdot\partial_\alpha\sigma_w$:
$\sigma_w^{-1}\cdot\partial_\alpha\sigma_w=\upsilon\left(-\frac{1}{2}j\,(w)\,\partial_\alpha\sigma_w,-w\partial_\alpha a_w+a_w\partial_\alpha w\right)$
$2a_w\partial_\alpha a_w=\frac{1}{2}w^t\partial_\alpha w$
$j\,(w)\,j\,(w)\,\partial_\alpha w=ww^t\partial_\alpha w-w^tw\partial_\alpha w$
$-w\partial_\alpha a_w+a_w\partial_\alpha w=-w\frac{1}{4a_w}w^t\partial_\alpha w+a_w\partial_\alpha w$
$=-\frac{1}{4a_w}\left(j\,(w)\,j\,(w)\,\partial_\alpha w+w^tw\partial_\alpha w\right)+a\partial_\alpha w$
$=\left(-\frac{1}{4a_w}j\,(w)\,j\,(w)-\frac{1}{4a_w}w^tw+a_w\right)\partial_\alpha w$
$=\left(-\frac{1}{4a_w}j\,(w)\,j\,(w)+\frac{1}{a_w}\right)\partial_\alpha w$
$\sigma_w^{-1}\cdot\partial_\alpha\sigma_w=\upsilon\left(-\frac{1}{2}j\,(w)\,\partial_\alpha w,\left(-\frac{1}{4a_w}j\,(w)\,j\,(w)+\frac{1}{a_w}\right)\partial_\alpha w\right)$
$\gamma C\left(\sigma_w^{-1}\partial_\alpha\sigma_w\right)=-i\frac{1}{2}\frac{1}{a_w}\sum_{a=1}^3\left(\left[\left(1-\frac{1}{4}j\,(w)\,j\,(w)\right)\partial_\alpha w\right]^a\gamma_a\gamma_0-\frac{1}{2}a_w\left[j\,(w)\,\partial_\alpha w\right]^a\widetilde{\gamma}_a\right)$

Computation of $\partial_\alpha\sigma_w\cdot\sigma_w^{-1}$:
$\partial_\alpha\sigma_w\cdot\sigma_w^{-1}=\upsilon\left(-\frac{1}{2}\left(-j\,(w)\,\partial_\alpha w\right),-w\partial_\alpha a_w+a\partial_\alpha w\right)$
$=\upsilon\left(\frac{1}{2}\,(w)\,\partial_\alpha w,\left(-\frac{1}{4a_w}j\,(w)\,j\,(w)+\frac{1}{a_w}\right)\partial_\alpha w\right)$
$\gamma C\left(\sigma_w^{-1}\partial_\alpha\sigma_w\right)=-i\frac{1}{2}\frac{1}{a_w}\sum_{a=1}^3\left(\left[\left(1-\frac{1}{4}j\,(w)\,j\,(w)\right)\partial_\alpha w\right]^a\gamma_a\gamma_0+\frac{1}{2}a_w\left[j\,(w)\,\partial_\alpha w\right]^a\widetilde{\gamma}_a\right)$

Computation of $\sigma_r^{-1}\cdot\partial_\alpha\sigma_r$:
$\sigma_r^{-1}\cdot\partial_\alpha\sigma_r=\upsilon\left(-\frac{1}{2}j\,(r)\,\partial_\alpha r-r\partial_\alpha r+a_r\partial_\alpha r,0\right)$
$a_r^2=1-\frac{1}{4}r^tr$
$2a_r\partial_\alpha a_r=-\frac{1}{2}r\partial_\alpha r$
$-\frac{1}{2}j\,(r)\,\partial_\alpha r-r\partial_\alpha r+a_r\partial_\alpha r=-\frac{1}{2}j\,(r)\,\partial_\alpha r+\frac{1}{4a_r}\left(r_p^t\partial_\alpha r\right)r_p+a_r\partial_\alpha r$
$=-\frac{1}{2}j\,(r)\,\partial_\alpha r+\frac{1}{4a_r}\left(j\,(r)\,j\,(r)+r^tr\right)\partial_\alpha r+a_r\partial_\alpha r$
$=-\frac{1}{2}j\,(r)\,\partial_\alpha r+\frac{1}{4a_r}\left(j\,(r)\,j\,(r)+4\left(1-a_r^2\right)\right)\partial_\alpha r+a_r\partial_\alpha r$

$$= \left(\frac{1-a_r^2}{a_r} + a_r - \frac{1}{2}j\left(r\right) + \frac{1}{4a_r}j\left(r\right)j\left(r\right)\right)\partial_\alpha r$$
$$= \left(\frac{1}{a_r} - \frac{1}{2}j\left(r\right) + \frac{1}{4a_r}j\left(r\right)j\left(r\right)\right)\partial_\alpha r$$
$$\sigma_r^{-1}\cdot\partial_\alpha\sigma_r = \upsilon\left(\left(\frac{1}{a_r} - \frac{1}{2}j\left(r\right) + \frac{1}{4a_r}j\left(r\right)j\left(r\right)\right)\partial_\alpha r, 0\right)$$
$$\gamma C\left(\sigma_r^{-1}\cdot\partial_\alpha\sigma_r\right) = -i\frac{1}{2}\sum_{a=1}^{3}\left[\left(\frac{1}{a_r} - \frac{1}{2}j\left(r\right) + \frac{1}{4a_r}j\left(r\right)j\left(r\right)\right)\partial_\alpha r\right]^a$$

14 Energy-momentum Tensor

For any tensor : $X=\sum_{\{\alpha\beta\}}X^{\alpha\beta}\partial\xi_\alpha\wedge\partial\xi_\beta\in\Lambda_2TM$, by a straigthforward computation, we get the value of :

$\varpi_4(X)=2(\det P)\{X^{32}d\xi^0\wedge d\xi^1+X^{13}d\xi^0\wedge d\xi^2+X^{21}d\xi^0\wedge d\xi^3$
$+X^{03}d\xi^2\wedge d\xi^1+X^{02}d\xi^1\wedge d\xi^3+X^{01}d\xi^3\wedge d\xi^2\}$
and : $d\varpi_4(X)=-2\sum_{\alpha\beta=0}^{3}(-1)^{\alpha+1}\left(\partial_\beta\left(X^{\alpha\beta}\det P\right)\partial_\alpha\right)d\xi^0\wedge...\widehat{d\xi^\alpha}...\wedge d\xi^3$
For any function Z : $dZ=\sum_\gamma\partial_\gamma Zd\xi^\gamma$ and tensor $X=\sum_{\{\alpha\beta\}}X^{\alpha\beta}\partial\xi_\alpha\wedge\partial\xi_\beta$:
$dZ\wedge\varpi_4(X)$
$=2(\det P)\times$
$\{X^{32}\partial_0Zd\xi^0\wedge d\xi^0\wedge d\xi^1+X^{13}\partial_0Zd\xi^0\wedge d\xi^0\wedge d\xi^2+X^{21}\partial_0Zd\xi^0\wedge d\xi^0\wedge d\xi^3$
$+X^{03}\partial_0Zd\xi^0\wedge d\xi^2\wedge d\xi^1+X^{02}\partial_0Zd\xi^0\wedge d\xi^1\wedge d\xi^3+X^{01}\partial_0Zd\xi^0\wedge d\xi^3\wedge d\xi^2$
$+X^{32}\partial_1Zd\xi^1\wedge d\xi^0\wedge d\xi^1+X^{13}\partial_1Zd\xi^1\wedge d\xi^0\wedge d\xi^2+X^{21}\partial_1Zd\xi^1\wedge d\xi^0\wedge d\xi^3$
$+X^{03}\partial_1Zd\xi^1\wedge d\xi^2\wedge d\xi^1+X^{02}\partial_1Zd\xi^1\wedge d\xi^1\wedge d\xi^3+X^{01}\partial_1Zd\xi^1\wedge d\xi^3\wedge d\xi^2$
$+X^{32}\partial_2Zd\xi^2\wedge d\xi^0\wedge d\xi^1+X^{13}\partial_2Zd\xi^2\wedge d\xi^0\wedge d\xi^2+X^{21}\partial_2Zd\xi^2\wedge d\xi^0\wedge d\xi^3$
$+X^{03}\partial_2Zd\xi^2\wedge d\xi^2\wedge d\xi^1+X^{02}\partial_2Zd\xi^2\wedge d\xi^1\wedge d\xi^3+X^{01}\partial_2Zd\xi^2\wedge d\xi^3\wedge d\xi^2$
$+X^{32}\partial_3Zd\xi^3\wedge d\xi^0\wedge d\xi^1+X^{13}\partial_3Zd\xi^3\wedge d\xi^0\wedge d\xi^2+X^{21}\partial_3Zd\xi^3\wedge d\xi^0\wedge d\xi^3$
$+X^{03}\partial_3Zd\xi^3\wedge d\xi^2\wedge d\xi^1+X^{02}\partial_3Zd\xi^3\wedge d\xi^1\wedge d\xi^3+X^{01}\partial_3Zd\xi^3\wedge d\xi^3\wedge d\xi^2\}$
$=2(\det P)\times$
$\{+X^{03}\partial_0Zd\xi^0\wedge d\xi^2\wedge d\xi^1+X^{02}\partial_0Zd\xi^0\wedge d\xi^1\wedge d\xi^3+X^{01}\partial_0Zd\xi^0\wedge d\xi^3\wedge d\xi^2$
$+X^{13}\partial_1Zd\xi^1\wedge d\xi^0\wedge d\xi^2+X^{21}\partial_1Zd\xi^1\wedge d\xi^0\wedge d\xi^3+X^{01}\partial_1Zd\xi^1\wedge d\xi^3\wedge d\xi^2$
$+X^{32}\partial_2Zd\xi^2\wedge d\xi^0\wedge d\xi^1+X^{21}\partial_2Zd\xi^2\wedge d\xi^0\wedge d\xi^3+X^{02}\partial_2Zd\xi^2\wedge d\xi^1\wedge d\xi^3$
$+X^{32}\partial_3Zd\xi^3\wedge d\xi^0\wedge d\xi^1+X^{13}\partial_3Zd\xi^3\wedge d\xi^0\wedge d\xi^2+X^{03}\partial_3Zd\xi^3\wedge d\xi^2\wedge d\xi^1\}$
$=2(\det P)\times$
$\{+X^{03}\partial_0Zd\xi^0\wedge d\xi^2\wedge d\xi^1+X^{13}\partial_1Zd\xi^1\wedge d\xi^0\wedge d\xi^2+X^{32}\partial_2Zd\xi^2\wedge d\xi^0\wedge d\xi^1$
$+X^{02}\partial_0Zd\xi^0\wedge d\xi^1\wedge d\xi^3+X^{21}\partial_1Zd\xi^1\wedge d\xi^0\wedge d\xi^3+X^{32}\partial_3Zd\xi^3\wedge d\xi^0\wedge d\xi^1$
$+X^{01}\partial_0Zd\xi^0\wedge d\xi^3\wedge d\xi^2+X^{21}\partial_2Zd\xi^2\wedge d\xi^0\wedge d\xi^3+X^{13}\partial_3Zd\xi^3\wedge d\xi^0\wedge d\xi^2$
$+X^{01}\partial_1Zd\xi^1\wedge d\xi^3\wedge d\xi^2+X^{02}\partial_2Zd\xi^2\wedge d\xi^1\wedge d\xi^3+X^{03}\partial_3Zd\xi^3\wedge d\xi^2\wedge d\xi^1\}$
$=2(\det P)\times$
$\{\left(-X^{03}\partial_0Z-X^{13}\partial_1Z+X^{32}\partial_2Z\right)d\xi^0\wedge d\xi^1\wedge d\xi^2$
$+\left(X^{02}\partial_0Z-X^{21}\partial_1Z+X^{32}\partial_3Z\right)d\xi^0\wedge d\xi^1\wedge d\xi^3$
$+\left(-X^{01}\partial_0Z-X^{21}\partial_2Z+X^{13}\partial_3Z\right)d\xi^0\wedge d\xi^2\wedge d\xi^3$
$+\left(-X^{01}\partial_1Z-X^{02}\partial_2Z-X^{03}\partial_3Z\right)\}d\xi^1\wedge d\xi^2\wedge d\xi^3$
$=2(\det P)\times$
$\left(-X^{01}\partial_1Z-X^{02}\partial_2Z-X^{03}\partial_3Z\right)\}d\xi^1\wedge d\xi^2\wedge d\xi^3$
$\left(X^{10}\partial_0Z+X^{12}\partial_2Z+X^{13}\partial_3Z\right)d\xi^0\wedge d\xi^2\wedge d\xi^3$
$\left(-X^{20}\partial_0Z-X^{21}\partial_1Z-X^{23}\partial_3Z\right)d\xi^0\wedge d\xi^1\wedge d\xi^3$
$\left(X^{30}\partial_0Z+X^{31}\partial_1Z+X^{32}\partial_2Z\right)d\xi^0\wedge d\xi^1\wedge d\xi^2$
$=2(\det P)\sum_{\alpha\beta}(-1)^{\alpha+1}X^{\alpha\beta}(\partial_\beta Z)d\xi^0\wedge...\widehat{d\xi^\alpha}...\wedge d\xi^3$
$dZ\wedge\varpi_4(X)=2(\det P)\sum_{\alpha\beta}(-1)^{\alpha+1}X^{\alpha\beta}(\partial_\beta Z)d\xi^0\wedge...\widehat{d\xi^\alpha}...\wedge d\xi^3$
$dZ\wedge\varpi_4(X)=2\varpi_4\left(\sum_{\alpha\beta}X^{\alpha\beta}(\partial_\beta Z)\partial\xi_\alpha\right)$
Of course the result still holds for any one form $Z=\sum_\alpha Z_\alpha d\xi^\alpha$

15 FORMULAS

15.1 Operator j

$$[j(r)]\,w = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} r_2 w_3 - r_3 w_2 \\ -r_1 w_3 + r_3 w_1 \\ r_1 w_2 - r_2 w_1 \end{bmatrix}$$

$[j(r)\,w]^a = \epsilon(a,b,c)\,(r_b w_c - r_c w_b)$
$j(r)^t = -j(r) = j(-r)$
$j(x)y = -j(y)x$
$y^t j(x) = -x^t j(y)$
$j(x)y = 0 \Leftrightarrow \exists k \in R : y = kx$
$j(x)j(y) = yx^t - (y^t x)\,I$
$j(j(x)y) = yx^t - xy^t = j(x)j(y) - j(y)j(x)$
$j(x)j(y)j(x) = -(y^t x)\,j(x)$
$x^t j(r)\,j(s)y = x^t\,(sr^t - r^t sI)\,y = (x^t s)\,(r^t y) - (x^t y)\,(r^t s)$
$\|j(x)y\|^2 = (x^t x)\,(y^t y) - (x^t y)^2$
$M \in L(3) : M^t j(Mx)M = (\det M)\,j(x)$
$M \in O(3) : j(Mx)My = Mj(x)y \Leftrightarrow Mx \times My = M\,(x \times y)$
$k > 0 : j(r)^{2k} = (-r^t r)^{k-1}\,(rr^t - (r^t r)\,I) = (-r^t r)^{k-1}\,j(r)j(r)$
$k \geq 0 : J(r)^{2k+1} = (-r^t r)^k\,j(r)$

15.2 Dirac's matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\sigma_0$
j,k,l=1,2,3: $\sigma_j \sigma_k = \epsilon(j,k,l)\,i\sigma_l$
$(\sum_a U_a \sigma_a)\,(\sum_b V_b \sigma_b)$
$= \sum_{a<b=1}^3 U_a V_b \sigma_a \sigma_b + \sum_{a>b=1}^3 U_a V_b \sigma_a \sigma_b + \sum_{a=1}^3 U_a V_a \sigma_0$
$= \sum_{a<b=1}^3 U_a V_b \sigma_a \sigma_b + \sum_{b>a=1}^3 U_b V_a \sigma_b \sigma_a + \sum_{a=1}^3 U_a V_a \sigma_0$
$= \sum_{a<b=1}^3 (U_a V_b - U_b V_a)\,\sigma_a \sigma_b + \sum_{a=1}^3 U_a V_a \sigma_0$
$= \sum_{a<b=1}^3 (U_a V_b - U_b V_a)\,i\epsilon(c,a,b)\,\sigma_c + \sum_{a=1}^3 U_a V_a \sigma_0$
$= \sum_{a<b=1}^3 (j(U)\,V)^a\,i\sigma_a + \sum_{a=1}^3 U_a V_a \sigma_0$

15.3 γ matrices

$$\gamma_0 = \begin{bmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{bmatrix}; \gamma_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}; \gamma_2 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix}; \gamma_3 = \begin{bmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{bmatrix};$$

$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij} I_4$
$\gamma_i = \gamma_i^* = \gamma_i^{-1}$
$\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \begin{bmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{bmatrix}$
$\gamma_5 \gamma_a = -\gamma_a \gamma_5$
$Cl(\mathbb{R},3,1) : \gamma C(\varepsilon_j) = \gamma_j, j = 1,2,3; \gamma C(\varepsilon_0) = i\gamma_0; \gamma C(\varepsilon_5) = i\gamma_5$
$Cl(\mathbb{R},1,3) : \gamma C'(\varepsilon_j) = i\gamma_j, j = 1,2,3; \gamma C'(\varepsilon_j) = \gamma_0; \gamma C'(\varepsilon_5) = \gamma_5$

j = 1,2,3 : $\widetilde{\gamma}_j = \begin{bmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{bmatrix}$
$j \neq k, l = 1, 2, 3 : \gamma_j \gamma_k = -\gamma_k \gamma_j = i\epsilon (j, k, l) \widetilde{\gamma}_l$
$\widetilde{\gamma}_a \widetilde{\gamma}_a = I_4$
$\widetilde{\gamma}_a \widetilde{\gamma}_b = i\epsilon (a, b, c) \widetilde{\gamma}_c$
$\widetilde{\gamma}_a \widetilde{\gamma}_b + \widetilde{\gamma}_b \widetilde{\gamma}_a = 2\delta_{ab} I_4$
$\gamma_a \widetilde{\gamma}_b = i\epsilon (a, b, c) \gamma_c$
$\gamma_a \widetilde{\gamma}_a = \begin{bmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{bmatrix}$
$\gamma_0 \widetilde{\gamma}_a = \begin{bmatrix} 0 & -i\sigma_a \\ i\sigma_a & 0 \end{bmatrix} = (\gamma_0 \widetilde{\gamma}_a)^* = \begin{bmatrix} 0 & -i\sigma_a^* \\ i\sigma_a^* & 0 \end{bmatrix}$
$j = 1, 2, 3 : \gamma_j \gamma_0 = -\gamma_0 \gamma_j = i \begin{bmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{bmatrix} = i\gamma_5 \widetilde{\gamma}_j$
$\gamma_1 \gamma_2 = -\gamma_2 \gamma_1 = i \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}; \gamma_2 \gamma_3 = -\gamma_3 \gamma_2 = i \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix};$
$\gamma_3 \gamma_1 = -\gamma_1 \gamma_3 = i \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}$
$\gamma_1 \gamma_0 = -\gamma_0 \gamma_1 = i \begin{bmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix}; \gamma_2 \gamma_0 = -\gamma_0 \gamma_2 = i \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix};$
$\gamma_3 \gamma_0 = -\gamma_0 \gamma_3 = i \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}$
$\gamma C (\upsilon (G_{r\alpha}, G_{w\alpha})) = -i\frac{1}{2} \sum_{a=1}^{3} (G_{w\alpha} \gamma_a \gamma_0 + G_{r\alpha} \widetilde{\gamma}_a)$
$\gamma C' (\upsilon (G_{r\alpha}, G_{w\alpha})) = -i\frac{1}{2} \sum_{a=1}^{3} (G_{w\alpha} \gamma_a \gamma_0 - G_{r\alpha} \widetilde{\gamma}_a)$

15.4 Clifford algebra

$\varepsilon_i \cdot \varepsilon_j + \varepsilon_j \cdot \varepsilon_i = 2\eta_{ij}$
$\varepsilon_5 \cdot \varepsilon_5 = -1$

15.4.1 Lie Algebras

$\upsilon (r, w) = \frac{1}{2} \left(w^1 \varepsilon_0 \cdot \varepsilon_1 + w^2 \varepsilon_0 \cdot \varepsilon_2 + w^3 \varepsilon_0 \cdot \varepsilon_3 + r^3 \varepsilon_2 \cdot \varepsilon_1 + r^2 \varepsilon_1 \cdot \varepsilon_3 + r^1 \varepsilon_3 \cdot \varepsilon_2 \right)$
In Cl(3,1):
$\gamma C (\upsilon (r, w)) = -i\frac{1}{2} \sum_{a=1}^{3} (w_a \gamma_a \gamma_0 + r_a \widetilde{\gamma}_a)$
$\upsilon (r, w) \cdot \varepsilon_5 = \varepsilon_5 \cdot \upsilon (r, w) = \upsilon (r, -w)$
$\upsilon (r', w') \cdot \upsilon (r, w)$
$= \frac{1}{4} (w^t w' - r^t r') + \frac{1}{2} \upsilon (-j (r) r' + j (w) w', -j (w) r' - j (r) w') - \frac{1}{4} (w^t r' + r^t w') \varepsilon_5$
$[\upsilon (r, w), \upsilon (r', w')] = \upsilon (j (r) r' - j (w) w', j (w) r' + j (r) w')$
In Cl(1,3):
$\gamma C' (\upsilon (r, w)) = -i\frac{1}{2} \sum_{a=1}^{3} (w_a \gamma_a \gamma_0 - r_a \widetilde{\gamma}_a)$
$\upsilon (r, w) \cdot \varepsilon_5 = \varepsilon_5 \cdot \upsilon (r, w) = \upsilon (w, r)$
$\upsilon (r, w) \cdot \upsilon (r', w')$
$= \frac{1}{4} (w^t w' - r^t r') - \frac{1}{2} \upsilon (-j (r) r' + j (w) w', j (w) r' + j (r) w') - \frac{1}{4} (w^t r' + r^t w') \varepsilon_5$
$[\upsilon (r, w), \upsilon (r', w')] = -\upsilon (j (r) r' - j (w) w', j (w) r' + j (r) w')$

15.4.2 Spin groups

$s = a + \upsilon\,(r, w) + b\varepsilon_5$

$a^2 - b^2 = 1 + \frac{1}{4}\left(w^t w - r^t r\right)$

$ab = -\frac{1}{4} r^t w$

if r=0 then $a = \epsilon\sqrt{1 + \frac{1}{4} w^t w}; b = 0$

if w=0 then

$r^t r \leq 4 : a = \epsilon\sqrt{1 - \frac{1}{4} r^t r}; b = 0$

$r^t r \geq 4 : b = \epsilon\sqrt{-1 + \frac{1}{4} r^t r}; a = 0$

Product :

$s \cdot s' = a" + \upsilon\,(r", w") + b"\varepsilon_0 \cdot \varepsilon_1 \cdot \varepsilon_2 \cdot \varepsilon_3$

with :

$a" = aa' - b'b + \frac{1}{4}\left(w^t w' - r^t r'\right)$

$b" = ab' + ba' - \frac{1}{4}\left(w^t r' + r^t w'\right)$

i) In $Cl(\mathbb{R}, 3, 1)$ **:**

$r" = \frac{1}{2}\left(j\,(r)\,r' - j\,(w)\,w'\right) + a'r + ar' - b'w - bw'$

$w" = \frac{1}{2}\left(j\,(w)\,r' + j\,(r)\,w'\right) + a'w + aw' + b'r + br'$

$(a + \upsilon\,(0, w)) \cdot (a' + \upsilon\,(0, w')) = aa' + \frac{1}{4} w^t w' + \upsilon\left(-\frac{1}{2}\left(j\,(w)\,w', a'w + aw'\right)\right)$

$(a + \upsilon\,(r, 0)) \cdot (a' + \upsilon\,(r', 0)) = aa' - \frac{1}{4} r^t r' + \upsilon\left(\frac{1}{2} j\,(r)\,r' + (a'r + ar'), 0\right)$

ii) In $Cl(\mathbb{R}, 1, 3)$ **:**

$r" = \frac{1}{2}\left(j\,(r)\,r' - j\,(w)\,w'\right) + a'r + ar' + b'w + bw'$

$w" = -\frac{1}{2}\left(j\,(w)\,r' + j\,(r)\,w'\right) + a'w + aw' + b'r + br'$

γ **Matrix**

$\gamma C\,(a + \upsilon\,(r, w) + b\varepsilon_5) = aI + ib\gamma_5 - i\frac{1}{2}\sum_{a=1}^{3}\left(w_a \gamma_a \gamma_0 + r_a \widetilde{\gamma}_a\right)$