



HAL
open science

Combining statistical and expert evidence using belief functions: Application to centennial sea level estimation taking into account climate change

Nadia Ben Abdallah, Nassima Mouhous-Voyneau, Thierry Denoeux

► To cite this version:

Nadia Ben Abdallah, Nassima Mouhous-Voyneau, Thierry Denoeux. Combining statistical and expert evidence using belief functions: Application to centennial sea level estimation taking into account climate change. *International Journal of Approximate Reasoning*, 2014, 55 (1), pp.341-354. 10.1016/j.ijar.2013.03.008 . hal-00932879

HAL Id: hal-00932879

<https://hal.science/hal-00932879>

Submitted on 18 Jan 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Combining statistical and expert evidence using belief functions: Application to centennial sea level estimation taking into account climate change

N. Ben Abdallah^a, N. Mouhous-Voyneau^b, T. Dencœur^a

^a*Université de Technologie de Compiègne, CNRS, UMR 7253 Heudiasyc*

^b*Université de Technologie de Compiègne, EA 7284, Avenues-GSU*

Abstract

Estimation of extreme sea levels for high return periods is of prime importance in hydrological design and flood risk assessment. Common practice consists of inferring design levels from historical observations and assuming the distribution of extreme values to be stationary. However, in recent years, there has been a growing awareness of the necessity to integrate the effects of climate change in environmental analysis. In this paper, we present a methodology based on belief functions to combine statistical judgements with expert evidence in order to predict the future centennial sea level at a particular location, taking into account climate change. Likelihood-based belief functions derived from statistical observations are combined with random intervals encoding expert assessments of the 21st century sea level rise. Monte Carlo simulations allow us to compute belief and plausibility degrees for various hypotheses about the design parameter.

Keywords: Sea level rise, structural design, coastal defense, Dempster-Shafer theory, Evidence theory, random intervals, experts opinions, statistical inference, likelihood-based belief functions.

1. Introduction

In design and safety assessment procedures, comprehensive uncertainty analysis is essential to reach reliable results and rational decisions. In the hydrological field, communicating uncertainty about future flood risk to the decision makers has become common practice [2, 27]. Whereas there is a general consensus about the main sources of uncertainty in environmental risk analysis, there is an increasing debate among risk analysts about

the framework to use for quantifying it. The commonly used probabilistic framework has been strongly criticized for treating in the same way *aleatory* uncertainty, which arises from variability of phenomena and *epistemic* uncertainty resulting from lack of knowledge [2, 4]. As both sources of uncertainty coexist in environmental risk analysis, the need for alternative frameworks for representing and propagating uncertainty has emerged. In the past two decades, intensive research work has been devoted to the study of new approaches to uncertainty modeling such as Possibility [45, 18, 15], Imprecise Probability [41, 42] or Dempster-Shafer (DS) [8, 32] theories and their application to reliability and risk analysis.

In recent years, the need to model explicitly epistemic uncertainty in environmental risk analyses has become even more important due to a growing awareness of climate change [24]. Whereas traditional engineering design processes and standards are mainly based on the analysis of historical climate data using, e.g., Extreme Value Theory [21], the underlying assumption of a stable climate is no longer valid in the 21st century [24]. We thus have to develop new design methodologies for combining statistical analysis of past data with expert assessments of expected changes in climate conditions for the next decades.

In this paper, we consider the particular case of coastal defense structural design in the context of expected sea level rise in the next century¹. Our approach will be based on the DS theory of belief functions, which constitutes a unified framework allowing for the representation and combination of expert judgements and statistical evidence. Previous work using DS theory to model uncertainty on climate change has been reported in [26] and [22] focussing, respectively, on uncertainty propagation in numerical models and on expert opinion elicitation. In [26], the authors study the influence of parameter uncertainty in energy balance and radiative forcing models on the estimation of global mean temperature increase. Knowledge about each parameter is represented by a pair of cumulative probability distributions called p-box, which is shown to be equivalent to a special kind of belief functions. These belief functions are then propagated through dynamical models using different dependence assumptions. In [22], Ha-Duong models expert opinions on climate sensitivity, a key parameter in climate change predictions, using belief functions. He shows that expert opinions can be clustered according to “schools of thought”. He then proposes to use different combination rules for within-group and between-group combination.

¹A preliminary version of this work appeared in [5].

In contrast to the two previous pieces of work, the study reported here focuses on the representation of statistical evidence in the belief function framework, and its combination with expert opinion. This approach is considered as a preliminary step towards an integrated approach to uncertainty modeling in environmental risk analysis, taking into account climate change. The rest of the paper is organized as follows. In Sections 2 and 3 we lay down the theoretical foundations of this work by recalling basic definitions of DS theory and discussing in some detail its application to statistical inference using the notion of likelihood-based belief functions. In Section 4, we show how this theoretical approach can be applied to the estimation of centennial sea level at a particular location taking into account sea level rise due to climate change. Numerical results are reported in Section 5 and Section 6 concludes the paper.

2. Belief functions

This section recalls the necessary background notions related to DS theory. Belief functions on finite domains and Dempster’s rule of combination are first presented in Subsections 2.1 and 2.2, respectively. Some notions about random intervals are then recalled in Subsection 2.3.

2.1. Belief functions on finite domains

Let θ be a variable² taking values in a finite domain Θ , called the *frame of discernment*. Uncertain evidence about θ may be represented by a (normalized) *mass function* m on Θ , defined as a function from the powerset of Θ , denoted as 2^Θ , to the interval $[0, 1]$, such that $m(\emptyset) = 0$ and

$$\sum_{A \subseteq \Theta} m(A) = 1. \quad (1)$$

Any subset A of Θ such that $m(A) > 0$ is called a *focal set* of m . A logical mass function has only one focal set (it is thus equivalent to a set), while a Bayesian mass function has only focal sets of cardinality one and is thus equivalent to a probability distribution. The mass function m such that $m(\Theta) = 1$ is said to be *vacuous*.

Each number $m(A)$ is interpreted as a *degree of belief* attached to the proposition $\theta \in A$ and to *no more specific proposition*, based on some evidence. As argued by Shafer [34], the meaning of such degrees of belief can

²In the rest of this paper, we use the notation θ to denote the variable of interest and θ to denote an arbitrary value in Θ .

be better understood by assuming that we have compared our evidence to a canonical chance set-up. The set-up proposed by Shafer consists of an encoded message and a set of codes $\Omega = \{\omega_1, \dots, \omega_n\}$, exactly one of which is selected at random. We know the list of codes as well as the chance p_i of each code ω_i being selected. Decoding the encoded message using code ω_i produces a message of the form “ $\theta \in A_i$ ” for some $A_i \subseteq \Theta$. Then

$$m(A) = \sum_{\{1 \leq i \leq n: A_i = A\}} p_i \quad (2)$$

is the chance that the original message was “ $\theta \in A$ ”. Stated differently, it is the probability of knowing that $\theta \in A$. In particular, $m(\Theta)$ is, in this setting, the probability that the original message was vacuous, i.e., the probability of knowing nothing.

The above setting thus consists of a set Ω , a probability measure P on Ω and a multi-valued mapping $\Gamma : \Omega \rightarrow 2^\Theta \setminus \{\emptyset\}$ such that $A_i = \Gamma(\omega_i)$ for each $\omega_i \in \Omega$. This is the framework initially considered by Dempster in [8]. The triple (Ω, P, Γ) formally defines a finite *random set* [28]: mass functions are thus exactly equivalent to random sets from a mathematical point of view. However, the meaning of mass functions differs from the usual interpretation of a random set as the outcome of a random experiment: here, $m(A)$ is *not* the chance that A was selected, but it can be viewed as the chance of the evidence meaning that θ is in A [34].

To each normalized mass function m , we may associate belief and plausibility functions from 2^Θ to $[0, 1]$ defined as follows:

$$Bel(A) = P(\{\omega \in \Omega | \Gamma(\omega) \subseteq A\}) = \sum_{B \subseteq A} m(B) \quad (3a)$$

$$Pl(A) = P(\{\omega \in \Omega | \Gamma(\omega) \cap A \neq \emptyset\}) = \sum_{B \cap A \neq \emptyset} m(B), \quad (3b)$$

for all $A \subseteq \Theta$. These two functions are linked by the relation $Pl(A) = 1 - Bel(\bar{A})$, for all $A \subseteq \Theta$. Each quantity $Bel(A)$ may be interpreted as the degree to which the evidence *supports* A , while $Pl(A)$ can be interpreted as the degree to which the evidence *is not contradictory* to A . The following inequalities always hold: $Bel(A) \leq Pl(A)$, for all $A \subseteq \Theta$. The function $pl : \Theta \rightarrow [0, 1]$ such that $pl(\theta) = Pl(\{\theta\})$ for all $\theta \in \Theta$ is called the *contour function* associated to m .

If m is Bayesian, then function Bel is equal to Pl and it is a probability measure; pl is then the corresponding probability mass function. Another special case of interest is that where m is *consonant*, i.e., its focal elements

are nested. The plausibility function is then a *possibility measure* [45, 18] with possibility distribution pl , i.e., the plausibility function can be recovered from the contour function as follows [32]:

$$Pl(A) = \max_{\theta \in A} pl(\theta), \quad (4)$$

for all $A \subseteq \Theta$.

Given two mass functions m_1 and m_2 , m_1 is said to be *less specific* than m_2 if it can be obtained from m_2 by transferring belief masses $m_2(A)$ to supersets $B \supseteq A$ [44, 17]. In this case, m_1 can be considered as less informative, or less committed³ than m_2 . The *Least Commitment Principle* (LCP) [37] states that, given some constraints on an unknown mass function, the least committed should be selected. This principle provides a justification of consonant mass functions: given a function $\pi : \Theta \rightarrow [0, 1]$ such that $\max \pi = 1$, the least specific mass function m with contour function pl , such that $pl = \pi$, is consonant.

2.2. Dempster's rule

A key idea in DS theory is that beliefs are elaborated by aggregating different items of evidence. The basic mechanism for evidence combination is Dempster's rule of combination, which can be naturally derived using the random code metaphor as follows.

Let m_1 and m_2 be two mass functions induced by triples $(\Omega_1, P_1, \Gamma_1)$ and $(\Omega_2, P_2, \Gamma_2)$ interpreted under the random code framework as before. Let us further assume that the codes are selected independently. For any two codes $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the probability that they both are selected is then $P_1(\{\omega_1\})P_2(\{\omega_2\})$, in which case we can conclude that $\theta \in \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2)$. If $\Gamma_1(\omega_1) \cap \Gamma_2(\omega_2) = \emptyset$, we know that the pair of codes (ω_1, ω_2) could not have been selected: consequently, the joint probability distribution on $\Omega_1 \times \Omega_2$ must be conditioned, eliminating such pairs [34]. This line of reasoning yields the following combination rule, referred to as Dempster's rule [32]:

$$(m_1 \oplus m_2)(A) = \frac{1}{1 - \kappa} \sum_{B \cap C = A} m_1(B)m_2(C) \quad (5)$$

for all $A \subseteq \Theta$, $A \neq \emptyset$ and $(m_1 \oplus m_2)(\emptyset) = 0$, where

$$\kappa = \sum_{B \cap C = \emptyset} m_1(B)m_2(C) \quad (6)$$

³Alternative comparative orderings between belief functions have been proposed, see, e.g., [17].

is the *degree of conflict* between m_1 and m_2 . If $\kappa = 1$, there is a logical contradiction between the two pieces of evidence and they cannot be combined. Dempster's rule is commutative, associative, and it admits as neutral element the *vacuous* mass function defined as $m(\Theta) = 1$.

2.3. Random real intervals

The definition of belief functions and random sets in infinite spaces implies greater mathematical sophistication than it does in finite spaces [33, 28]. Here, we will restrict our discussion to random closed intervals on the real line (see, e.g., [10, 38, 12]), which constitute a simple yet sufficiently general framework for expressing beliefs on a real variable.

Let (Ω, \mathcal{A}, P) be a probability space and $(U, V) : \Omega \rightarrow \mathbb{R}^2$ a two-dimensional real random vector such that $P(\{\omega \in \Omega | U(\omega) \leq V(\omega)\}) = 1$. Let Γ be the multi-valued mapping that maps each $\omega \in \Omega$ to the closed interval $[U(\omega), V(\omega)]$. This setting defines a random interval, as well as belief and plausibility functions on \mathbb{R} defined, respectively, by

$$Bel(A) = P(\{\omega \in \Omega | [U(\omega), V(\omega)] \subseteq A\}) \quad (7)$$

$$Pl(A) = P(\{\omega \in \Omega | [U(\omega), V(\omega)] \cap A \neq \emptyset\}) \quad (8)$$

for all elements A of the Borel sigma-algebra $\mathcal{B}(\mathbb{R})$ on the real line [10]. The intervals $[U(\omega), V(\omega)]$ are referred to as the focal intervals of $[U, V]$. We note that, when U and V are continuous, the notion of mass function should be replaced by that of mass density function defined by $m([u, v]) = p(u, v)$, where $p(u, v)$ denotes the joint probability density function (pdf) of (U, V) .

A special case of interest is that of consonant random closed intervals defined as follows. Let $\Omega = [0, 1]$, $\pi : \mathbb{R} \rightarrow [0, 1]$ a function such that for each $\omega \in \Omega$,

$$\Gamma(\omega) = \{x \in \mathbb{R} | \pi(x) \geq \omega\}$$

is a closed interval $[U(\omega), V(\omega)]$, called the ω -level cut of π (see Figure 1(a)) and let P denote the Lebesgue measure on Ω . Then, $[U, V]$ is a random closed interval and π is its contour function, i.e., $pl(x) = Pl(\{x\}) = \pi(x)$ for all $x \in \mathbb{R}$. Such a random interval is said to be consonant because its focal intervals $\Gamma(\omega)$ are nested. The intervals $\Gamma(1)$ and $\{x \in \mathbb{R} | \pi(x) > 0\}$ are called, respectively, the core and the support of π .

P-boxes are another special class of closed random intervals that has proved quite useful in practice [20, 26, 14]. Let (F_*, F^*) be a pair of functions from \mathbb{R} to $[0, 1]$ such that

1. F_* is nondecreasing and right-continuous;

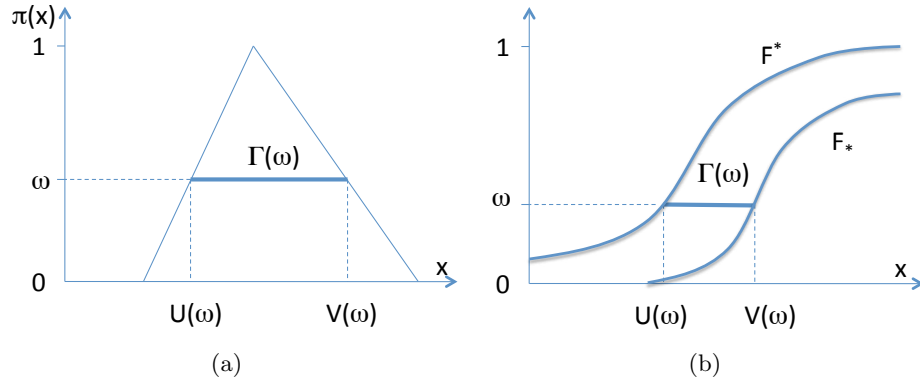


Figure 1: Random closed intervals induced by a possibility distribution (a) and a p-box (b).

2. F^* is nondecreasing and left-continuous;
3. $F_* \leq F^*$.

For each $\omega \in \Omega = [0, 1]$, the set

$$\Gamma(\omega) = \{x \in \mathbb{R} | F^*(x) \geq \omega \text{ and } F_*(x) \leq \omega\}$$

is a closed interval $[U(\omega), V(\omega)]$ (see Figure 1(b)). If, as before, we consider the Lebesgue measure on Ω , we get a random interval $[U, V]$ called a p-box. Functions F_* and F^* can be seen as lower and upper envelopes of a family of cumulative distribution functions [20, 26].

Dempster's rule can be defined for random intervals as follows. Let us assume that we have two random intervals $(\Omega_i, \mathcal{A}_i, P_i, \Gamma_i)$ with $i = 1, 2$ and $[U_i(\omega), V_i(\omega)] = \Gamma_i(\omega)$. Let Γ_{12} be the mapping from $\Omega_1 \times \Omega_2$ to the set of closed real intervals defined by

$$\Gamma_{12}(\omega_1, \omega_2) = \Gamma_1(\omega_1) \cap \Gamma_2(\omega_2), \quad \forall (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$$

and let P_{12} be the product measure $P_1 \times P_2$ conditioned on the set $\{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 | \Gamma_{12}(\omega_1, \omega_2) \neq \emptyset\}$. Then, $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2, P_{12}, \Gamma_{12})$ define a random interval $[U_{12}, V_{12}] = [U_1, V_1] \oplus [U_2, V_2]$.

3. Likelihood-based belief functions

Let us now turn our attention to the representation of statistical evidence. Assume that we have observed a realization \mathbf{x} of a random vector \mathbf{X} with pdf $p(\mathbf{x}; \boldsymbol{\theta})$, where $\boldsymbol{\theta} \in \Theta$ is an unknown parameter. What does

this item of evidence tell us about θ ? This problem was initially studied by Dempster [7, 8, 11], who proposed an approach based on a pivotal quantity, which induces a multi-valued mapping from a probability space to the parameter space once observations have been collected. An alternative approach, proposed by Shafer [32], can be derived from the Likelihood and Least Commitment principles. In this paper, we will focus on the latter approach, which is practically much simpler to implement while having interesting connections with Bayesian and likelihood-based approaches to statistical inference. This approach will first be recalled in Subsection 3.1. Arguments for and against this solution will then be discussed in Subsection 3.2.

3.1. Least committed belief function based on likelihoods

In the standard statistical framework, information about θ is typically assumed to be represented by the likelihood function, defined by $L(\theta; \mathbf{x}) = p(\mathbf{x}; \theta)$ for all $\theta \in \Theta$. More precisely, the likelihood principle [3] [6] [19, chapter 3] states that “Within the framework of a statistical model, *all* the information which the data provide concerning the relative merits of two hypotheses is contained in the likelihood ratio of these hypotheses on the data”. In statistical parlance, the likelihood ratio is often referred to as the “relative plausibility”, which suggests translating the likelihood ratio in the belief function framework as follows:

$$\frac{pl(\theta_1; \mathbf{x})}{pl(\theta_2; \mathbf{x})} = \frac{L(\theta_1; \mathbf{x})}{L(\theta_2; \mathbf{x})},$$

for all $(\theta_1, \theta_2) \in \Theta^2$ or, equivalently,

$$pl(\theta; \mathbf{x}) = cL(\theta; \mathbf{x})$$

for all $\theta \in \Theta$ and some positive constant c . The LCP then leads us to giving the highest possible value to constant c , i.e., defining pl as the relative likelihood :

$$pl(\theta; \mathbf{x}) = \frac{L(\theta; \mathbf{x})}{\sup_{\theta \in \Theta} L(\theta; \mathbf{x})} \quad (9)$$

and representing evidence about θ by the least committed plausibility function induced by pl , i.e.,

$$Pl(A; \mathbf{x}) = \sup_{\theta \in A} pl(\theta; \mathbf{x}) = \frac{\sup_{\theta \in A} L(\theta; \mathbf{x})}{\sup_{\theta \in \Theta} L(\theta; \mathbf{x})}, \quad (10)$$

for all $A \subseteq \Theta$. The corresponding belief function is usually referred to as the likelihood-based belief function [43].

It must be noted that, in (9) and (10), the likelihood function is assumed to be bounded, which is the case for most parametric models considered in practice. For a discussion on this and other technical issues related to the use of the likelihood function, see [19, Chapter 8].

3.2. Discussion

Equation (10) was first proposed by Shafer in [32, chapter 11] who, however, did not justify it by the LCP, but by the more questionable requirement that the belief function on Θ has to be consonant. In the special case where $\Theta = \{\theta_1, \theta_2\}$ has only two points, Wasserman [43] showed that the plausibility function (10) corresponds to the unique belief function $Bel(\cdot; \mathbf{x})$ verifying the following requirements:

1. If $L(\theta_1; \mathbf{x}) = L(\theta_2; \mathbf{x})$, then $Bel(\cdot; \mathbf{x})$ should be vacuous;
2. $Bel(\{\theta\}; \mathbf{x})$ should be nondecreasing in $L(\theta; \mathbf{x})$;
3. If $Bel = Bel(\cdot; \mathbf{x}) \oplus P_0$ and P_0 is a probability measure, then Bel should be equal to the Bayesian posterior.

This argument can be extended to the case where Θ is a complete, separable metric space [43].

In [13], it was shown that (10) can be derived from three basic principles: the likelihood principle, compatibility with Bayes' rule (requirement 3 above) and the LCP. From a non Bayesian perspective, the likelihood principle was placed on firm ground by Birnbaum [6], who showed that it can be derived from the intuitively appealing principles of sufficiency and conditionality (see [13] for further discussion on this topic).

One of the main criticisms against the use of the likelihood-based plausibility function (10) for represented statistical evidence is its incompatibility with Dempster's rule in the case of independent observations [35]. More precisely, assume that \mathbf{X} is an independent sample (X_1, \dots, X_n) and each observation X_i has a marginal pdf $p(x_i; \boldsymbol{\theta})$ depending on $\boldsymbol{\theta}$. We could combine the n observations at the "aleatory level" by computing $Pl(\cdot; \mathbf{x})$ using (10), or we could combine them at the "epistemic level" by first computing the consonant plausibility functions $Pl(\cdot; x_i)$ induced by each of the independent observations and applying Dempster's rule. Obviously, these two procedures yield different results in general, as consonance is not preserved by Dempster's rule.

In [35], Shafer regards the above argument as strong enough to reject (10) as a reasonable method to represent statistical evidence. However, Aickin

[1] proposed to keep (10) but questioned Dempster’s rule as a mechanism for combining statistical evidence, based on the notion of *commitment to the model*. Let (Ω, P, Γ) be random set corresponding to Pl . As stated above, each $\omega \in \Omega$ can be regarded as an interpretation of a given piece of evidence. These interpretations are consistent, or “committed to the model”, if

$$\bigcap_{\omega \in \Omega} \Gamma(\omega) \neq \emptyset,$$

which is equivalent to the condition that $pl(\theta_0) = 1$ for some $\theta_0 \in \Theta$. Aickin [1] argued that, in the context of statistical inference, one should be fully committed to the idea that the model actually generated the observations, which entails that plausibility functions used to represent statistical evidence should be generated by a random set that is committed to the model.

Aickin went on by defining the notion of commitment to a submodel as follows. Let $A \subset \Theta$ be a “submodel”. A random set (Ω, P, Γ) is committed to A if all those $\Gamma(\omega)$ that intersect A have a nonvoid intersection. This property means that, after combining the random set with information stating that $\theta \in A$ for sure using Dempster’s rule, our random set should be committed to the new model $A \subset \Omega$ now considered as certain.

Let Pl be an arbitrary plausibility function on Θ , pl the corresponding contour function, $c = \sup pl$, and Pl^* the consonant plausibility function defined by $Pl^*(A) = c^{-1} \sup_{\theta \in A} pl(\theta)$. Aickin [1, Proposition 1] showed that, if Pl is committed to A , then $Pl(A)/c = Pl^*(A)$, and the converse is true if A is compact and Pl is upper semicontinuous. A consequence of this result is that, assuming commitment to the model, i.e., $c = 1$, the two plausibility functions Pl and Pl^* coincide on all “interesting” submodels A , and nothing is lost by replacing the former by the latter.

Let us now assume that $Pl(\cdot; \mathbf{x}_1)$ and $Pl(\cdot; \mathbf{x}_2)$ are plausibility functions on Θ induced by two independent observations \mathbf{x}_1 and \mathbf{x}_2 . After combination by Dempster’s rule, the resulting plausibility function $Pl(\cdot; \mathbf{x}_1) \oplus Pl(\cdot; \mathbf{x}_2)$ need not be committed to the model even though $Pl(\cdot; \mathbf{x}_1)$ and $Pl(\cdot; \mathbf{x}_2)$ are, which can be considered as an argument against the use of Dempster’s rule for combining evidence from independent observations in the context of parametric statistical model. However, commitment to the model may be restored by considering the contour function $pl(\cdot; \mathbf{x}_1; \mathbf{x}_2) = pl(\cdot; \mathbf{x}_1)pl(\cdot; \mathbf{x}_2)$ corresponding to $Pl(\cdot; \mathbf{x}_1) \oplus Pl(\cdot; \mathbf{x}_2)$, rescaling it so that its supremum equals 1, and computing the consonant plausibility function. We thus get

$$pl(\theta; \mathbf{x}_1, \mathbf{x}_2) = \frac{pl(\theta; \mathbf{x}_1)pl(\theta; \mathbf{x}_2)}{\sup_{\theta \in \Theta} pl(\theta; \mathbf{x}_1)pl(\theta; \mathbf{x}_2)} = \frac{L(\theta; \mathbf{x}_1, \mathbf{x}_2)}{\sup_{\theta \in \Theta} L(\theta; \mathbf{x}_1, \mathbf{x}_2)}$$

for all $\theta \in \Theta$ and

$$Pl(A; \mathbf{x}_1, \mathbf{x}_2) = \sup_{\theta \in A} pl(\theta; \mathbf{x}_1, \mathbf{x}_2) = \frac{\sup_{\theta \in A} L(\theta; \mathbf{x}_1, \mathbf{x}_2)}{\sup_{\theta \in \Theta} L(\theta; \mathbf{x}_1, \mathbf{x}_2)}$$

for all $A \subseteq \Theta$. This way of combining independent statistical evidence, which Aickin called the DS* rule, restores the equivalent between combination at the aleatory and epistemic levels and reconciles likelihood inference with DS theory. Further arguments against the use of Dempster's rule for combining evidence from independent observations can be found in [40].

Based on the above discussion, we propose to adopt (9) and (10), together as the DS* rule, as models of statistical evidence. Further arguments in favor of this approach are summarized below:

1. This method of inference is considerably simpler than other methods such as Dempster's initial proposal [9] and other methods discussed in [35], while being more widely applicable than Smets' Generalized Bayesian Theorem [36, 16].
2. Combining $Pl(\cdot; \mathbf{x})$ given by (10) with a Bayesian prior P_0 on Θ using Dempster's rule yields a Bayesian plausibility function $Pl(\cdot; \mathbf{x}) \oplus P_0$ which is identical to the posterior probability obtained using Bayes' rule: consequently, the proposed method of inference boils down to Bayesian inference when a Bayesian prior is available.
3. Finally, viewing the relative likelihood function as a possibility distribution seems to be consistent with statistical practice, although this point of view has not been adopted explicitly in the statistical literature. For instance, likelihood intervals [23, 39] are focal intervals of the relative likelihood viewed as a possibility distribution. In the case where $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \Theta_1 \times \Theta_2$ and θ_2 is considered as a nuisance parameter, the relative profile likelihood function can be written

$$pl(\theta_1; \mathbf{x}) = \sup_{\theta_2 \in \Theta_2} pl(\theta_1, \theta_2; \mathbf{x}), \quad (11)$$

which is the marginal possibility distribution on Θ_1 . Eventually, we can remark that the usual likelihood ratio statistics $\Lambda(\mathbf{x})$ for a composite hypothesis $H_0 \subset \Theta$ can be seen as the plausibility of H_0 , as

$$\Lambda(\mathbf{x}) = \frac{\sup_{\theta \in H_0} L(\theta; \mathbf{x})}{\sup_{\theta \in \Theta} L(\theta; \mathbf{x})} = \sup_{\theta \in H_0} pl(\theta; \mathbf{x}) = Pl(H_0; \mathbf{x}).$$

4. Application to centennial sea level estimation

4.1. Model and assumptions

Flood structures have to withstand exceptional sea events and their design has thus to be based on extreme sea level and waves. The main tool for modeling extreme events in environmental applications such as floods, droughts or rainfalls is Extreme Value Theory (EVT), which has emerged giving the limit of the conventional frequency analysis in fitting the tails of probability distributions. The block maxima approach is the original and best known method in EVT. It is based on the assumption that the maximum Z of an independent and identically distributed (i.i.d.) sample has asymptotically a generalized extreme value (GEV) distribution [25] with cumulative distribution function given by:

$$F_Z(z, \mu, \sigma, \xi) = \begin{cases} \exp\left(-\left[1 - \xi \frac{z-\mu}{\sigma}\right]^{\frac{1}{\xi}}\right) & \text{if } \xi \neq 0 \\ \exp\left(-\exp\left[-\frac{z-\mu}{\sigma}\right]\right) & \text{if } \xi = 0, \end{cases} \quad (12)$$

where $\mu, \sigma > 0$ and ξ are, respectively, location, scale and shape parameters. According to the sign of ξ , the distribution is called Fréchet ($\xi > 0$), Weibull ($\xi < 0$) or Gumbel ($\xi = 0$). In the following, Z will be defined as the maximum annual sea level at a particular location.

A key parameter in engineering design procedure is the return period, defined as the average number of years between two consecutive events of a given intensity. Here, the return period T is defined as the average number of years between two successive exceedances of a corresponding return level z_T or, equivalently, the inverse of the probability that Z exceeds z_T . The return level z_T is thus the quantile of the distribution of Z , at level $1 - 1/T$. If Z has the GEV distribution (12), z_T has the following expression:

$$z_T = \begin{cases} \mu - \frac{\sigma}{\xi} \left[1 - \left(-\log\left(1 - \frac{1}{T}\right)\right)^{-\xi}\right] & \text{if } \xi \neq 0 \\ \mu - \sigma \log\left(-\log\left(1 - \frac{1}{T}\right)\right) & \text{if } \xi = 0. \end{cases} \quad (13)$$

Commonly, flood defense structures in coastal areas are designed to withstand events with a return period of at least 100 years. However, due to climate change, they will be subject during their life time to higher loads than predicted by design estimations. The main impact is related to the increase of the mean sea level which affects the frequency and intensity of surges. For adaptation purposes, the current statistics of extreme sea levels derived from statistical observations should be combined with predictions of sea level rise (SLR).

Here, it will be assumed that, because of climate change, the distribution of annual maximum sea level at a given time in the future will be *shifted to the right*, with shift equal to the SLR. We thus assume the future T -return level z'_T to be related to the current return level z_T and the sea level rise SLR by the following equation:

$$z'_T = z_T + SLR. \quad (14)$$

Evidence on z_T comes from past sea level measurements, while evidence on SLR can be obtained from expert judgements expressed in the recent literature on climate change. Our approach will thus be based on (1) the representation of evidence on z_T by a likelihood-based belief function; (2) the representation of evidence on SLR by a belief function summarizing expert opinions and (3) the combination of these two items of evidence to get a belief function on z'_T . These steps are described in the following subsections.

4.2. Statistical evidence on z_T

Let us assume that we have observed the annual maxima $\mathbf{z} = (z_1, \dots, z_n)$ of sea level at a particular location over n years. These observations will be assumed to be a realization of an i.i.d. random sample Z_1, \dots, Z_n from a Gumbel distribution. From (12), the pdf of each Z_i is

$$f(z_i; \mu, \sigma) = \frac{1}{\sigma} \exp\left(-\frac{z_i - \mu}{\sigma}\right) \exp\left[-\exp\left(-\frac{z_i - \mu}{\sigma}\right)\right]. \quad (15)$$

As the parameter of interest is z_T , the pdf of Z_i can be reparametrized as a function of μ and z_T . Using (13) we get $\sigma = (\mu - z_T)/c$ with

$$c = \log\left[-\log\left(1 - \frac{1}{T}\right)\right], \quad (16)$$

from which we obtain

$$f(z_i; \mu, z_T) = \frac{c}{\mu - z_T} \exp\left(c \frac{z_i - \mu}{z_T - \mu}\right) \exp\left[-\exp\left(c \frac{z_i - \mu}{z_T - \mu}\right)\right]. \quad (17)$$

The likelihood function is thus

$$\begin{aligned} L(\mu, z_T; \mathbf{z}) &= \prod_{i=1}^n f(z_i; \mu, z_T) = \\ &= \left(\frac{c}{\mu - z_T}\right)^n \exp\left(\frac{c}{z_T - \mu} \sum_{i=1}^n (z_i - \mu)\right) \exp\left[-\sum_{i=1}^n \exp\left(c \frac{z_i - \mu}{z_T - \mu}\right)\right]. \end{aligned} \quad (18)$$

Using (9), the corresponding contour function is the normalized likelihood function:

$$pl(\mu, z_T; \mathbf{z}) = \frac{L(\mu, z_T; \mathbf{z})}{\sup_{\mu, z_T} L(\mu, z_T; \mathbf{z})}. \quad (19)$$

Here, the parameter of interest is z_T and μ is a nuisance parameter, which can be marginalized out by considering the normalized profile likelihood (11). We finally get

$$pl(z_T; \mathbf{z}) = \sup_{\mu} pl(\mu, z_T; \mathbf{z}). \quad (20)$$

4.3. Expert evidence on SLR

Future SLR projections as reported by the International Panel on Climate Change Experts (IPCC) in its last Assessment Report [24] are obtained using climate models depending on many ill-known parameters. A principled approach would be to quantify available knowledge about these parameters by belief functions constructed using elicitation methods as described in [22] and to propagate these belief functions in numerical models as done in [26]. An alternative approach is to directly represent final conclusions on the SLR parameter, as reported in the recent literature, in the form of a belief function. Although the former approach is more objective and may be deemed preferable, the latter approach is much simpler and it will be used here as a preliminary step.

Projections provided by the IPCC [24] assess the likely range of values for SLR over the 1990-2095 period as 0.18 to 0.79 meters, not excluding higher values. This range takes into account uncertainties associated to future emissions of greenhouse gases under different scenarios covering a wide range of possible economic, technological and energetic states of the world in the 21st century. They are based on global circulation models as well as impacts models taking into account various phenomena such as melting of the Antarctic and Greenland, ocean expansion, etc.

Since the release of the last IPCC report, other SLR assessments based on semi-empirical models have been undertaken, proposing more pessimistic scenarios for 2100. For example, based on a simple statistical model, Rahmstorf [31] suggests [0.5, 1.4] as a likely range of values (in meters) for SLR at the end of this century. Recent studies suggest that the threshold of 2 meters will not be exceeded by the end of this century due to physical limitations [29].

Current methods for integrating SLR in flood risk or design analysis usually consider a particular deterministic scenario, since there is no information to quantify the probability of any given sea level magnitude within

the IPCC range. However, as shown by Purvis [30], who undertook a flood risk analysis under climate change, using only the most plausible SLR value may significantly underestimate expected consequences and lead to erroneous decisions. Here, we propose to represent the current state of knowledge about SLR during the 21st century, as reflected by the previous studies, by a random interval.

According to the three studies cited above, the interval $[0.5, 0.79] = [0.18, 0.79] \cap [0.5, 1.4]$ seems to be fully supported by the available evidence, as it is considered highly plausible by all three sources, while values outside the interval $[0, 2]$ are considered as impossible. This information may be represented by any belief function verifying $Bel([0, 2]) = 1$ and $Pl([0.5, 0.79]) = 1$. In the absence of more precise information, the most reasonable approach is to carry out some form of sensitivity analysis by considering different belief functions verifying these two constraints. Here, we will consider three kinds of random intervals:

1. Consonant random intervals with core $[0.5, 0.79]$, support $[0, 2]$ and contour function π defined by

$$\pi(x) = \begin{cases} \phi(x/0.5) & 0 < x \leq 0.5 \\ 1 & 0.5 \leq x \leq 0.79 \\ \phi\left(\frac{2-x}{2-0.79}\right) & 0.79 < x \leq 2 \\ 0 & \text{otherwise,} \end{cases} \quad (21)$$

where ϕ is a continuous, non-decreasing function from $[0, 1]$ to $[0, 1]$ such that $\phi(0) = 0$ and $\phi(1) = 1$ (see Figure 2(a));

2. P-boxes with upper and lower bounding functions defined, respectively, as follows:

$$F^*(x) = \begin{cases} 0 & x \leq 0 \\ \phi(x/0.5) & 0 < x \leq 0.5 \\ 1 & x > 0.5, \end{cases} \quad (22a)$$

$$F_*(x) = \begin{cases} 0 & x \leq 0.79 \\ 1 - \phi\left(\frac{2-x}{2-0.79}\right) & 0.79 < x \leq 2 \\ 1 & x > 2, \end{cases} \quad (22b)$$

as shown in Figure 2(b);

3. Random closed intervals $[U, V]$, such that U and V are independent and have cdfs F^* and F_* defined, respectively, by (22a) and (22b).

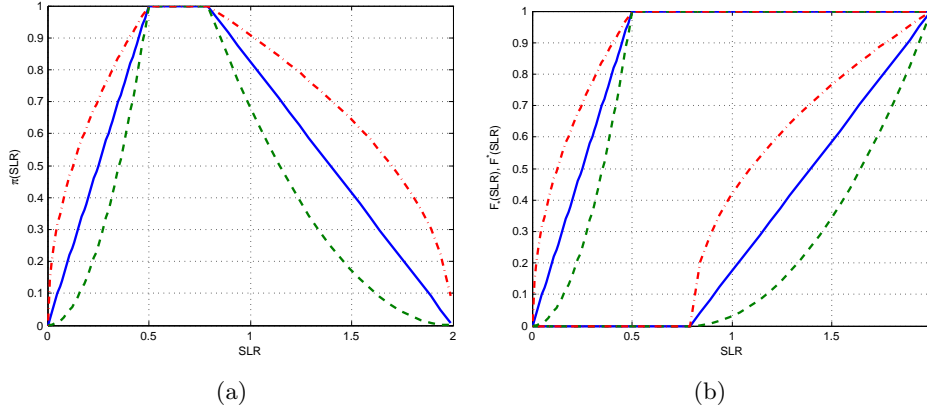


Figure 2: Contour functions (a) and upper and lower cdfs (b) of random sets on SLR for $\phi(x) = x$ (—), $\phi(x) = x^2$ (---) and $\phi(x) = \sqrt{x}$ (-.).

We can remark that the bounds of these three random intervals (for given ϕ) have the same marginal distributions F^* and F_* . Consequently, their plausibility and belief functions coincide on all intervals of the form $(-\infty, x]$, since

$$bel((-\infty, x]) = P(V \leq x) = F_*(x)$$

and

$$pl((-\infty, x]) = P(U \leq x) = F^*(x).$$

More generally, their plausibility functions coincide on all closed intervals since, for all $a \leq b$,

$$Pl([a, b]) = 1 - P(V < a) - P(U > b),$$

which only depends on the marginal distributions of U and V . In particular, they have identical contour function $\pi(x) = Pl([x, x])$.

In the following, we considered three different functions ϕ : linear ($\phi(x) = x$), convex ($\phi(x) = x^2$) and concave ($\phi(x) = \sqrt{x}$). The corresponding contour functions and cumulative distributions are shown, respectively, in Figures 2(a) and 2(b).

4.4. Combined evidence on z'_T

Let $(\Omega_1, P_1, \Gamma_1)$ and $(\Omega_2, P_2, \Gamma_2)$ denote the random intervals encoding evidence on z_T and SLR , respectively. Assuming independence between the two sources of evidence (a quite natural hypothesis here, as the two pieces of evidence have completely different origins), the combined evidence on

$z'_T = z_T + SLR$ is simply the sum of these two random intervals, which is a random interval (Ω, P, Γ) with $\Omega = \Omega_1 \times \Omega_2$, $P = P_1 \otimes P_2$ and

$$\Gamma(\omega_1, \omega_2) = [U_1(\omega_1) + U_2(\omega_2), V_1(\omega_1) + V_2(\omega_2)], \quad \forall (\omega_1, \omega_2) \in \Omega, \quad (23)$$

where $\Gamma_1(\omega_1) = [U_1(\omega_1), V_1(\omega_1)]$ and $\Gamma_2(\omega_2) = [U_2(\omega_2), V_2(\omega_2)]$.

Let Bel and Pl denote, respectively, the belief and plausibility functions induced by (Ω, P, Γ) . While the analytical expressions of $Bel(I)$ and $Pl(I)$ for an arbitrary interval I may be difficult to derive, these quantities can be easily approximated using Monte Carlo simulation. An i.i.d. random sample $(u_1, v_1), \dots, (u_N, v_N)$ from (U, V) can be generated by sampling N elements $\omega_1, \dots, \omega_N$ from (Ω, P) with replacement and computing $[u_i, v_i] = \Gamma(\omega_i)$, $i = 1 \dots, N$. Quantities $Bel(I)$ and $Pl(I)$ can then be estimated by

$$\widehat{Bel}(I) = \frac{1}{N} \#\{1 \leq i \leq N \mid [u_i, v_i] \subseteq I\} \quad (24)$$

$$\widehat{Pl}(I) = \frac{1}{N} \#\{1 \leq i \leq N \mid [u_i, v_i] \cap I \neq \emptyset\}. \quad (25)$$

5. Results

The above method was applied to a dataset of hourly measurements of sea level in Le Havre, France, recorded during 15 years. Figure 3 shows the empirical cdf of the 15 annual maxima, together with the best fit by a Gumbel distribution.

The joint contour function $pl(\mu, z_T)$ and the marginal contour function $pl(z_T)$ computed from (19) and (20), respectively, are shown in Figure 4.

The consonant random interval with contour function $pl(z_T)$ was combined with random intervals on SLR described in Subsection 4.3 to compute a random interval on z'_T , as explained in Subsection 4.4. The corresponding contour functions and upper and lower cdfs are shown in Figure 5. Again, these functions are the same for the three kinds of random intervals considered in Subsection 4.3 and they only depend on the choice of ϕ . The corresponding functions for the random interval obtained by adding a constant SLR value (the center of the interval $[0.5, 0.79]$) to the random interval of z_T are also shown in Figure 5 for comparison. We can see that the uncertainty on SLR accounts for most of the uncertainty on z'_T , a conclusion that holds for all three choices of ϕ .

Figure 6 shows a contour plot of plausibilities $Pl([x, y])$ for $8.5 \leq x \leq y \leq 12$ and $\phi(x) = x$. Again, these quantities depend only on the marginal distributions of the bounds U and V of the random interval on z'_T . In

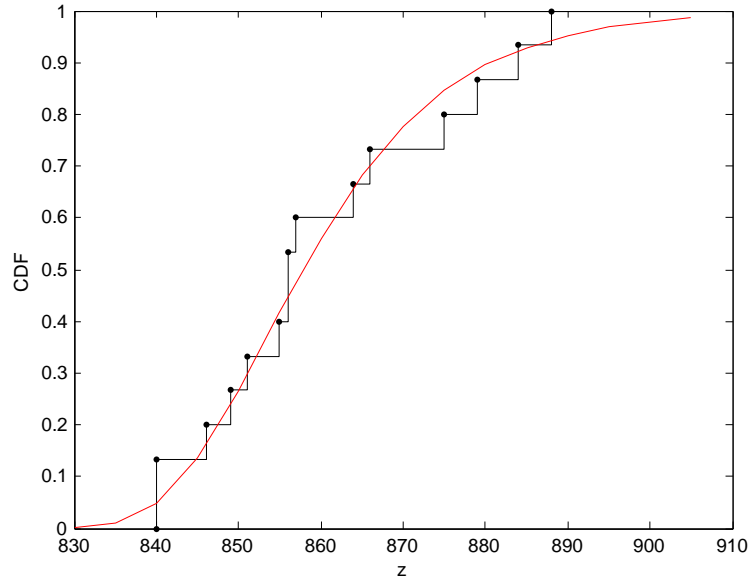
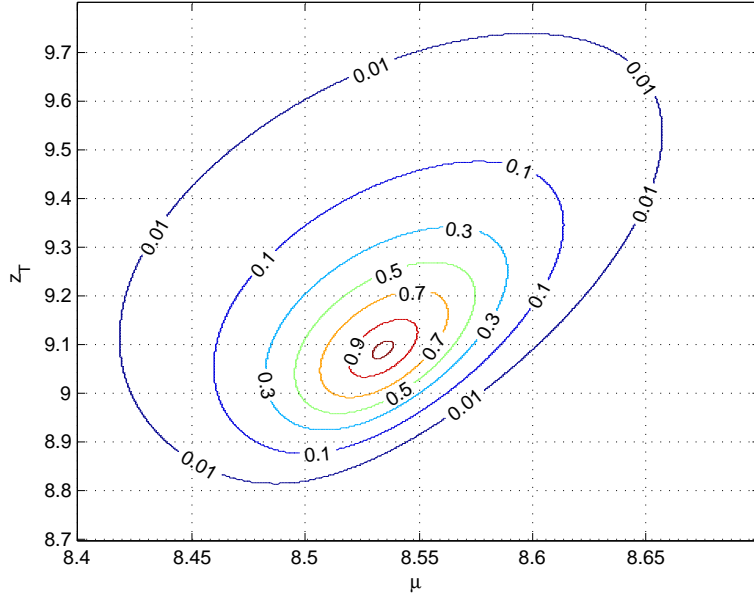
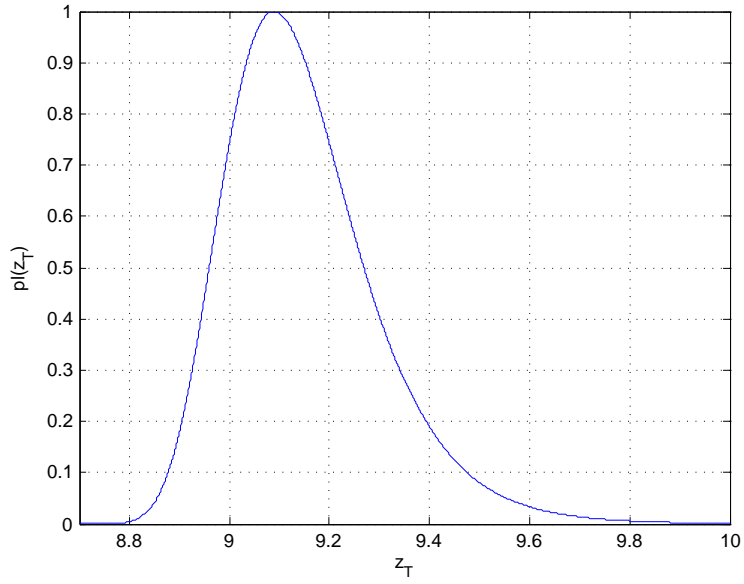


Figure 3: Empirical distribution of annual maxima of sea level recorded in Le Havre, France during 15 years and best fit using a Gumbel Distribution.

contrast, degrees of belief $Bel([x, y])$ depend on the joint distribution of U and V . They are plotted in Figure 7. We can see, however, that the results are very similar for the three kinds of random intervals on SLR . This confirms that the particular choice of random interval on SLR has a minor influence on the results, provided that the core and support are kept constant.

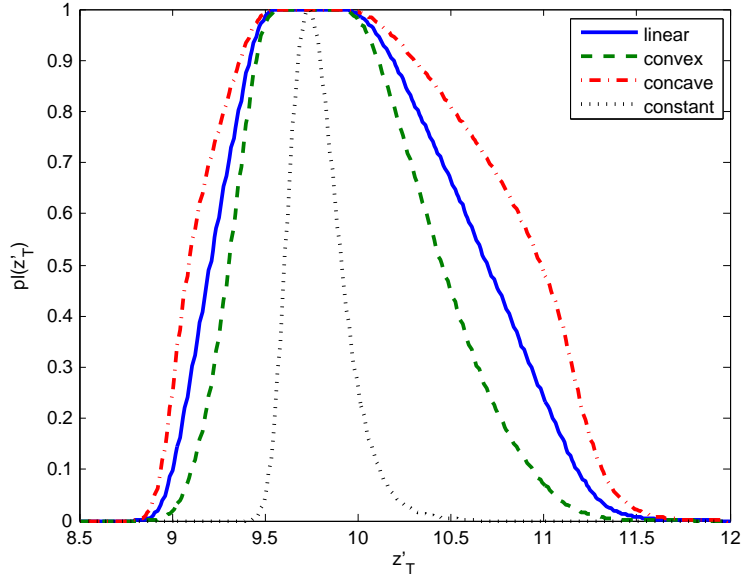


(a)

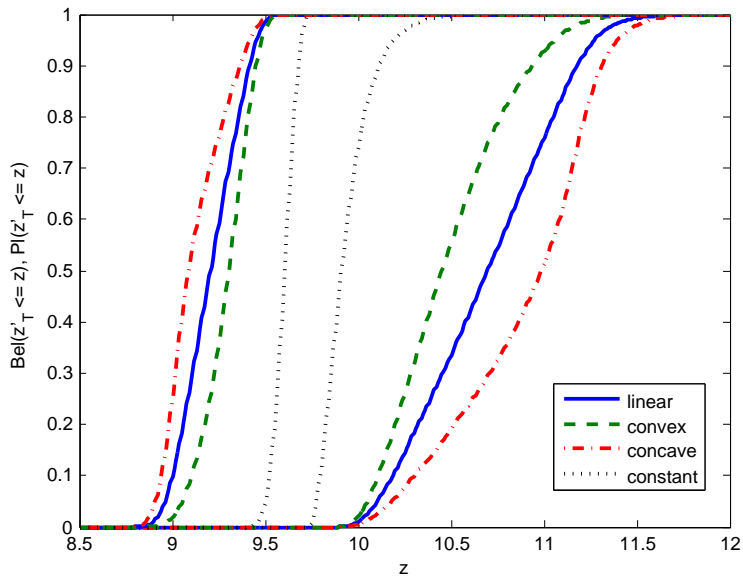


(b)

Figure 4: Joint contour function $pl(\mu, z_T)$ and marginal contour function $pl(z_T)$ for the Le Havre data.



(a)



(b)

Figure 5: Contour functions (a) and upper and lower cdfs (b) on z_T' for the three choices of function ϕ and constant prediction of *SLR*.

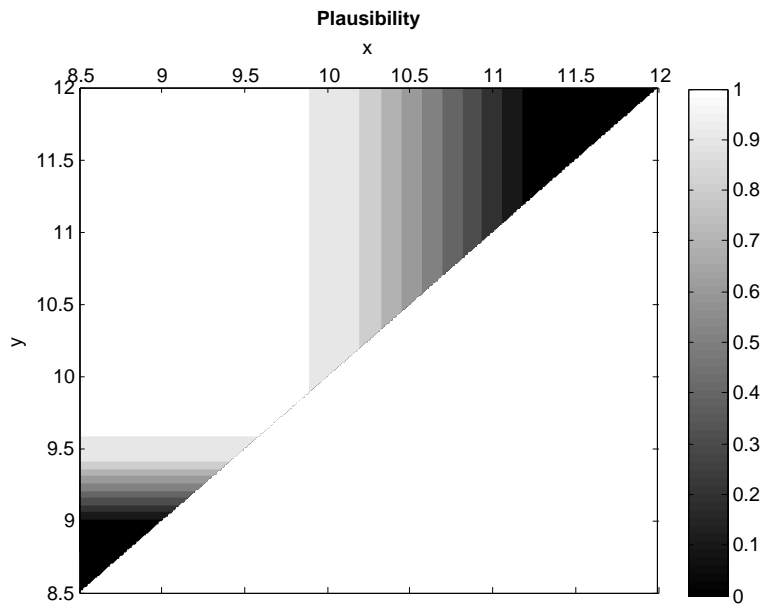


Figure 6: Contour plot of plausibilities $Pl(z'_T \in [x, y])$ for $\phi(x) = x$.

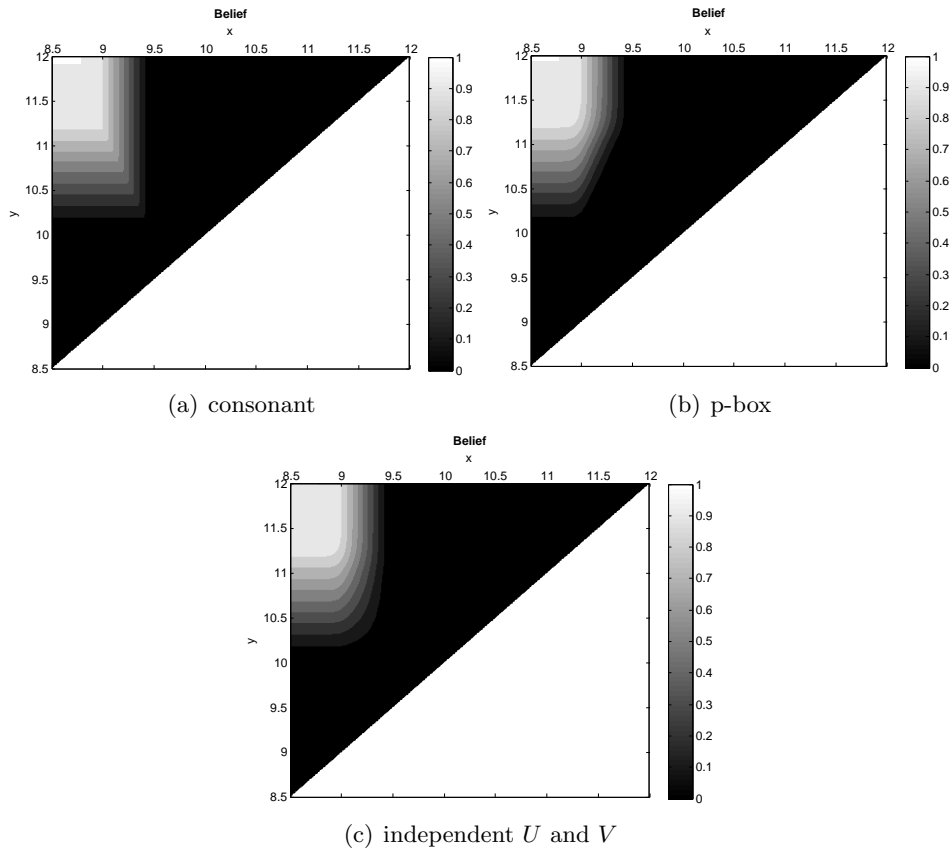


Figure 7: Contour plot of beliefs $Bel(z'_T \in [x, y])$ for the three kinds of random interval on SLR and $\phi(x) = x$.

6. Conclusion

The DS theory of belief functions places emphasis on the representation of evidence for evaluating degrees of belief. The generality and flexibility of this framework makes it suitable for representing and combining expert judgments with statistical evidence. In this paper, this approach has been applied to the estimation of the centennial sea level at a particular location, taking into account historical data and expert assessments of sea level rise during the 21st century. Statistical evidence has been modeled using the likelihood-based approach, which equates the contour function of a consonant belief function with the normalized likelihood. Expert judgments as reported in the last IPCC assessment report and the recent scientific literature have been represented by different random closed intervals. This study shows that the uncertainty on the 21st century sea level rise accounts for most of the uncertainty on the centennial sea level by the end of this century, a parameter that should be taken into account for adapting coastal defenses to climate change. This conclusion holds for different alternative ways of modeling expert opinions.

The work reported in this paper is only preliminary. It would be interesting to adopt a more objective approach to SLR assessment by modeling uncertainty on basic physical parameters (such as climate sensitivity) and propagating it in climate models. A deeper study would also benefit from closer interaction with climate experts and should be based on rigorous knowledge elicitation procedures that remain to be developed. Finally, the design of flood defense structures depends not only on extreme sea level, but also on other parameters such as wave height and length. The approach sketched here will have to be extended to represent uncertainty on several design parameters and propagate it in hydraulic and flood models so as to devise optimal decision strategies in response to sea level rise.

References

- [1] M. Aickin. Connecting Dempster-Shafer belief functions with likelihood-based inference. *Synthese*, 123:347–364, 2000.
- [2] H. Apel, A. H. Thikeken, B. Merz, and G. Blöschl. Flood risk assessment and associated uncertainty. *Natural Hazards and Earth System Sciences*, 4:295–308, 2004.
- [3] G. A. Barnard, G. M. Jenkins, and C. B. Winsten. Likelihood inference

- and time series. *Journal of the Royal Statistical Society*, 125(3):321–372, 1962.
- [4] C. Baudrit, I. Couso, and D. Dubois. Joint propagation of probability and possibility in risk analysis: Towards a formal framework. *International Journal of Approximate Reasoning*, 45(1):82–105, 2007.
- [5] N. Ben Abdallah, N. Mouhous Voyneau, and T. Denœux. Combining statistical and expert evidence within the D-S framework: Application to hydrological return level estimation. In T. Denœux and M.-H. Masson, editors, *Belief functions: theory and applications (Proc. of the 2nd Int. Conf. on Belief Functions)*, number AISC 164 in Advances in Intelligent and Soft Computing, pages 393–400, Compiègne, France, 2012. Springer.
- [6] Allan Birnbaum. On the foundations of statistical inference. *Journal of the American Statistical Association*, 57(298):269–306, 1962.
- [7] A. P. Dempster. New methods for reasoning towards posterior distributions based on sample data. *Annals of Mathematical Statistics*, 37:355–374, 1966.
- [8] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Annals of Mathematical Statistics*, 38:325–339, 1967.
- [9] A. P. Dempster. A generalization of Bayesian inference (with discussion). *J. R. Statistical Society B*, 30:205–247, 1968.
- [10] A. P. Dempster. Upper and lower probabilities generated by a random closed interval. *Annals of Mathematical Statistics*, 39(3):957–966, 1968.
- [11] A. P. Dempster. The Dempster-Shafer calculus for statisticians. *International Journal of Approximate Reasoning*, 48(2):365–377, 2008.
- [12] T. Denœux. Extending stochastic ordering to belief functions on the real line. *Information Sciences*, 179:1362–1376, 2009.
- [13] T. Denœux. Likelihood-based belief function: justification and some extensions to low-quality data. *International Journal of Approximate Reasoning (submitted)*, 2013.
- [14] S. Destercke, D. Dubois, and E. Chojnacki. Unifying practical uncertainty representations – I: generalized p-boxes. *International Journal of Approximate Reasoning*, 49(3):649 – 663, 2008.

- [15] D. Dubois. Possibility theory and statistical reasoning. *Computational Statistics and Data Analysis*, 51(1):47–69, 2006.
- [16] D. Dubois and T. Denœux. Statistical inference with belief functions and possibility measures: a discussion of basic assumptions. In C. Borgelt et al., editor, *Combining soft computing and statistical methods in data analysis (Proceedings of SMPS 2010)*, Advances in Intelligent and Soft Computing, pages 217–225, Oviedo, Spain, 2010. Springer.
- [17] D. Dubois and H. Prade. A set-theoretic view of belief functions: logical operations and approximations by fuzzy sets. *International Journal of General Systems*, 12(3):193–226, 1986.
- [18] D. Dubois and H. Prade. *Possibility Theory: An approach to computerized processing of uncertainty*. Plenum Press, New-York, 1988.
- [19] A. W. F. Edwards. *Likelihood (expanded edition)*. The John Hopkins University Press, Baltimore, USA, 1992.
- [20] S. Ferson, V. Kreinovitch, L. Ginzburg, D. S. Myers, and K. Sentz. Constructing probability boxes and Dempster-Shafer structures. Technical Report SAND2002-4015, Sandia National Laboratories, Albuquerque, NM, 2003.
- [21] E. J. Gumbel. *The statistics of extremes*. Columbia University Press, New York, 1958.
- [22] M. Ha-Duong. Hierarchical fusion of expert opinions in the Transferable Belief Model. *International Journal of Approximate Reasoning*, 49(3):555–574, 2008.
- [23] D. J. Hudson. Interval estimation from the likelihood function. *J. R. Statistical Society B*, 33(2):256–262, 1973.
- [24] IPCC. Fourth assessment report. Technical report, Intergovernmental Panel on Climate Change, 2007.
- [25] A. F. Jenkinson. The frequency distribution of the annual maximum (or minimum) values of meteorological elements. *Quarterly Journal of the Royal Meteorological Society*, 81:158–171, 1955.
- [26] E. Kriegler and H. Held. Utilizing belief functions for the estimation of future climate change. *International Journal of Approximate Reasoning*, 39(2–3):185–209, 2005.

- [27] B. Merz and A. H. Thielen. Flood risk curves and uncertainty bounds. *Natural hazards*, 51:437–458, 2009.
- [28] H.T. Nguyen. *An Introduction to Random Sets*. Chapman and Hall/CRC Press, Boca Raton, Florida, 2006.
- [29] W. T. Pfeffer, J. T. Harper, and S. O’Neel. Kinematic constraints on glacier contribution to 21st century sea level rise. *Science*, 321:1340–1343, 2008.
- [30] M. J. Purvis, P. D. Bates, and C. M. Hayes. Probabilistic methodology to estimate future coastal flood risk due to sea level rise. *Coastal Engineering*, 55(12):1062–1073, 2008.
- [31] S. Rahmstorf. A semi-empirical approach to projecting future sea level rise. *Science*, 315:368–370, 2007.
- [32] G. Shafer. *A mathematical theory of evidence*. Princeton University Press, Princeton, N.J., 1976.
- [33] G. Shafer. Allocations of probability. *Annals of Probability*, 7(5):827–839, 1979.
- [34] G. Shafer. Constructive probability. *Synthese*, 48(1):1–60, 1981.
- [35] G. Shafer. Belief functions and parametric models (with discussion). *J. Roy. Statist. Soc. Ser. B*, 44:322–352, 1982.
- [36] Ph. Smets. Belief functions: the disjunctive rule of combination and the generalized Bayesian theorem. *International Journal of Approximate Reasoning*, 9:1–35, 1993.
- [37] Ph. Smets. The Transferable Belief Model for quantified belief representation. In D. M. Gabbay and Ph. Smets, editors, *Handbook of Defeasible reasoning and uncertainty management systems*, volume 1, pages 267–301. Kluwer Academic Publishers, Dordrecht, 1998.
- [38] Ph. Smets. Belief functions on real numbers. *International Journal of Approximate Reasoning*, 40(3):181–223, 2005.
- [39] D. A. Sprott. *Statistical Inference in Science*. Springer-Verlag, Berlin, 2000.
- [40] P. Walley. Belief function representations of statistical evidence. *The Annals of Statistics*, 15(4):1439–1465, 1987.

- [41] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.
- [42] P. Walley. Towards a unified theory of imprecise probability. *International Journal of Approximate Reasoning*, 24(2–3):125–148, 2000.
- [43] L. A. Wasserman. Belief functions and statistical evidence. *The Canadian Journal of Statistics*, 18(3):183–196, 1990.
- [44] R. R. Yager. The entailment principle for Dempster-Shafer granules. *Int. J. of Intelligent Systems*, 1:247–262, 1986.
- [45] L. A. Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.