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POLYGRAPHS OF FINITE DERIVATION TYPE

YVES GUIRAUD PHILIPPE MALBOS

Abstract – In 1987, Craig Squier proved that, if a monoid can be presented by a finite convergent string rewriting system, then it satisfies the homological finiteness condition left-FP_3 . Using this result, he has constructed finitely presented decidable monoids that cannot be presented by finite convergent rewriting systems. In 1994, he introduced the condition of finite derivation type, which is a homotopical finiteness property on the presentation complex associated to a monoid presentation. He showed that this condition is an invariant of finite presentations and he gave a constructive way to prove this finiteness property based on the computation of the critical branchings: being of finite derivation type is a necessary condition for a finitely presented monoid to admit a finite convergent presentation. This self-contained survey presents those results in the contemporary language of polygraphs and higher-dimensional categories, providing new proofs and relations between them.

Keywords – higher-dimensional categories, higher-dimensional rewriting, finite derivation type, low-dimensional homotopy.

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1. INTRODUCTION

Given a monoid \mathbf{M} , a generating set Σ_1 for \mathbf{M} provides a way to represent the elements of \mathbf{M} in the free monoid Σ_1^* , *i.e.*, as finite words written with the elements of Σ_1 . But, when the monoid \mathbf{M} is not free, there is no reason for an element of \mathbf{M} to have a single representative in the free monoid Σ_1^* . The *word problem* for \mathbf{M} consists in finding a generating set Σ_1 and a procedure that can determine whether or not any two elements of Σ_1^* represent the same element in the monoid \mathbf{M} .

The word problem and convergent presentations. One way to solve the word problem is to exhibit a finite presentation $\Sigma = (\Sigma_1, \Sigma_2)$ of \mathbf{M} , made of a generating set Σ_1 and a set Σ_2 of directed relations with a good computational property: *convergence*. Indeed, in rewriting theory, one studies presentations where the relations in Σ_2 are not seen as equalities between the words in Σ_1^* , such as $u = v$, but, instead, as *rewriting rules* that can only be applied in one direction, like

$$u \Rightarrow v,$$

thus simulating a non-reversible computational process *reducing* the word u into the word v . In rewriting theory, such a presentation Σ of a monoid is called a *string rewriting system* or, historically, a *semi-Thue system*; in that case, the set Σ_1 of generators is called an *alphabet* and the directed relations of Σ_2 are called *rewriting rules*. A presentation Σ is *convergent* if it has the following two properties:

- *termination*, *i.e.*, all the computations end eventually,
- *confluence*, *i.e.*, different computations on the same input lead to the same result.

The *monoid presented by* Σ is defined as the quotient, denoted by $\bar{\Sigma}$, of the free monoid Σ_1^* over Σ_1 by the congruence generated by Σ_2 . By extension, we also say that Σ presents every monoid that is isomorphic to $\bar{\Sigma}$.

A finite and convergent presentation Σ of a monoid \mathbf{M} gives a solution to the word problem, called the *normal-form procedure* and defined as follows. Given an element u of the free monoid Σ_1^* , convergence ensures that all the applications of (directed) relations to u , in every possible manner, will eventually produce a unique result: an element \hat{u} of Σ_1^* where no relation applies anymore. The word \hat{u} called the *normal form of* u . By construction, two elements u and v of Σ_1^* represent the same element of \mathbf{M} if, and only if, their normal forms are equal in Σ_1^* . Finiteness ensures that one can determine whether an element of Σ_1^* is a normal form or not, by examining all the relations.

Polygraphs. The notion of string rewriting system comes from combinatorial algebra. It was introduced by Axel Thue in 1914 in order to solve the word problem for finitely presented semigroups [Thu14]. It is only in 1947 that the problem was shown to be undecidable, independently by Emil Post [Pos47] and Andrei Markov [Mar47a, Mar47b]. Then in 1943, Maxwell Newman gave a general setting, the abstract rewriting theory, to describe the properties of termination and confluence, and to show the first fundamental result of rewriting: Newman's lemma [New42]. Since then, rewriting theory has been mainly developed in theoretical computer science, producing numerous variants corresponding to different syntaxes of the formulas being transformed: string, terms, terms modulo, λ -terms, term-graphs, etc. Rewriting is also present in other computational formalisms such as Petri nets or logical systems.

More recently, higher-dimensional rewriting has unified several paradigms of rewriting. This approach is based on presentations by generators and relations of higher-dimensional categories, independently introduced by Albert Burroni and Ross Street under the respective names of *polygraphs* in [Bur93] and *computads* in [Str76, Str87]. Those algebraic objects have been subsequently developed in rewriting theory, fixing the terminology to polygraph in that field [Mét03, Gui06, Laf07, Mét08, GM09, Mim10, GM11, GM12a, GM12b, GMM13, GGM13].

The main useful property of polygraph is to encapsulate, in the same globular object, an algebraic structure corresponding to the syntax of the terms and to the computations on the terms, together with a homotopical structure describing the properties of the computations. As a consequence, polygraphs provide a natural setting to formulate Squier's theory, based on the discovery of deep relations between the computational, the homological and the homotopical properties of presentations by generators and relations.

From computational to homological properties. The normal-form procedure proves that, if a monoid admits a finite convergent presentation, then it has a decidable word problem. The converse implication was still an open problem in the middle of the eighties:

Question. [Jan82, Jan85] *Does every finitely presented monoid with a decidable word problem admit a finite convergent presentation?*

In [KN85], Deepak Kapur and Paliath Narendran consider Artin's presentation of the monoid \mathbf{B}_3^+ of positive braids on three strands:

$$\Sigma^{\text{Art}} = \langle s, t \mid sts \Rightarrow tst \rangle.$$

Kapur and Narendran proved that \mathbf{B}_3^+ admits no finite convergent presentation on the two generators s and t . However, they also proved one gets a finite convergent presentation of \mathbf{B}_3^+ by adjunction of a new generator a standing for the product st :

$$\Sigma^{\text{KN}} = \langle s, t, a \mid ta \xrightarrow{\alpha} as, st \xrightarrow{\beta} a, sas \xrightarrow{\gamma} aa, saa \xrightarrow{\delta} aat \rangle.$$

As a consequence, the word problem for \mathbf{B}_3^+ is solvable. The result of Kapur and Narendran shows that the existence of a finite convergent presentation depends on the chosen generators. Thus, to provide the awaited negative answer to the open question, one would have to exhibit a monoid with a decidable word problem but with no finite convergent presentation *on any possible set of generators*: new methods had to be introduced.

And, indeed, Craig Squier answered the question by linking the existence of a finite convergent presentation for a given monoid \mathbf{M} to an invariant of the monoid: the *homological type left-FP₃*. Here, invariant is to be taken in the sense that it is independent of the choice of a presentation of \mathbf{M} and, in particular, of a generating set. A monoid \mathbf{M} is of homological type left-FP₃ if there exists an exact sequence

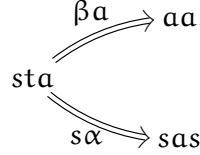
$$P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

of projective and finitely generated (left) $\mathbb{Z}\mathbf{M}$ -modules, where \mathbb{Z} denotes the trivial $\mathbb{Z}\mathbf{M}$ -module. From a presentation Σ of a monoid \mathbf{M} , one can build an exact sequence of free $\mathbb{Z}\mathbf{M}$ -modules

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \quad (1)$$

1. Introduction

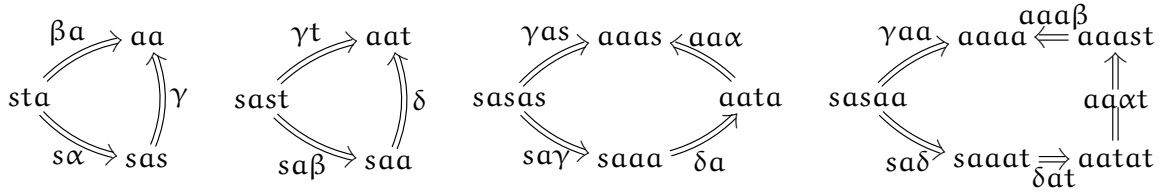
where $\mathbb{Z}\mathbf{M}[\Sigma_k]$ is the free $\mathbb{Z}\mathbf{M}$ -module over Σ_k . In [Squ87], Squier proved that, when Σ is convergent, its *critical branchings* form a generating set of the kernel of d_2 , where a critical branching of Σ is a minimal overlapping application of two relations on the same word of Σ_1^* . For example, the relations $\alpha : ta \Rightarrow as$ and $\beta : st \Rightarrow a$ generate a critical branching $(\beta\alpha, s\alpha)$ on sta :



The convergence of Σ ensures that every critical branching (f, g) is confluent, that is, it can be completed by rewriting sequences f' and g' as in the following diagram:



For example, the presentation Σ^{KN} of \mathbf{B}_3^+ has five critical branchings, and all of them are confluent:



Squier proved that the set Σ_3 of critical branchings of a convergent presentation Σ extends the exact sequence (1) by one step:

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \quad (3)$$

where the boundary map d_3 is defined on the generic branching (2) by:

$$d_3(f, g) = [f] - [g] + [f'] - [g'].$$

Moreover, when the presentation Σ is finite, then its set of critical branchings is finite, yielding *Squier's homological theorem*.

5.3.6. Theorem [Squ87, Theorem 4.1] *If a monoid admits a finite convergent presentation, then it is of homological type left-FP₃.*

Finally, Squier considers in [Squ87] the family \mathbf{S}_k of monoids, indexed by an integer $k \geq 2$, presented by

$$\Sigma^{\text{Sq}_k} = \langle a, b, t, (x_i)_{1 \leq i \leq k}, (y_i)_{1 \leq i \leq k} \mid \alpha, (\beta_i)_{1 \leq i \leq k}, (\gamma_i)_{1 \leq i \leq k}, (\delta_i)_{1 \leq i \leq k}, (\varepsilon_i)_{1 \leq i \leq k} \rangle$$

with

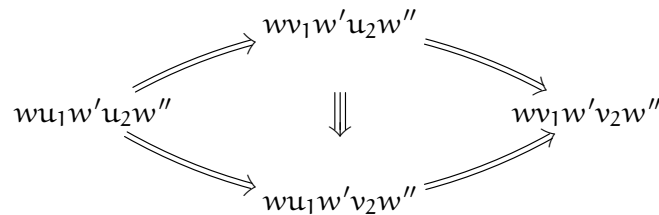
$$ab \xrightarrow{\alpha} 1 \quad x_i a \xrightarrow{\beta_i} atx_i \quad x_i t \xrightarrow{\gamma_i} tx_i \quad x_i b \xrightarrow{\delta_i} bx_i \quad x_i y_i \xrightarrow{\varepsilon_i} 1.$$

Each S_k is a finitely generated monoid with a decidable word problem. But, for $k \geq 2$, the monoid S_k is not of homological type left-FP₃ and, as a consequence, it does not admit a finite convergent presentation. Thus, Squier gave a negative answer to the open question: there exist finitely generated monoids with a decidable word problem that do not admit a finite convergent presentation (for any possible finite set of generators).

Finite derivation type. Given a monoid M with a presentation Σ , Squier considers in [SOK94] the presentation complex of Σ , that is a cellular complex with one 0-cell, whose 1-cells are the elements of the free monoid Σ_1^* and whose 2-cells are generated by the relations of Σ_2 . More precisely, the 2-cells of the presentation complex are constructed as follows. We denote by Σ_2^- the set obtained by reversing the relations of Σ_2 :

$$\Sigma_2^- = \{ v \xrightarrow{\alpha^-} u \text{ such that } u \xrightarrow{\alpha} v \text{ is in } \Sigma_2 \}.$$

There is a 2-cell in the presentation complex between each pair of words with shape $wu w'$ and $wv w'$ such that $\Sigma_2 \sqcup \Sigma_2^-$ contains a relation $u \Rightarrow v$. Then, Squier extends this 2-dimension complex with 3-cells filling all the squares formed by independent applications of relations, such as the following one, where $u_1 \Rightarrow v_1$ and $u_2 \Rightarrow v_2$ are in $\Sigma_2 \sqcup \Sigma_2^-$:



A *homotopy basis* of Σ is a set Σ_3 of additional 3-cells that makes Squier's complex aspherical, *i.e.*, such that every 2-dimensional sphere can be "filled" by the 3-cells of Σ_3 . The presentation Σ is of *finite derivation type* if it admits a finite homotopy basis. Squier proved that the finite derivation type property is an intrinsic property of the presented monoid:

4.2.3. Theorem [SOK94, Theorem 4.3] *Let Σ and Υ be two finite presentations of the same monoid. Then Σ has finite derivation type if and only if Υ has finite derivation type.*

The proof given by Squier is based on Tietze transformations. In these notes, we give another proof based on a homotopy bases transfer theorem, Theorem 4.2.2. As a consequence of Theorem 4.2.3, we can define the condition of finite derivation type for monoids independently of a considered presentation: a monoid is of *finite derivation type* if its finite presentations are of finite derivation type.

From computational to homotopical properties. In [SOK94], Squier links the existence of a finite convergent presentation to the homotopical property of finite derivation type. He proves that, given a convergent presentation Σ , it is sufficient to consider one 3-cell filling the diagram (2) for each critical branching to get a homotopy basis of Σ . Such a set of 3-cells is called a *family of generating confluences* of Σ .

1. Introduction

4.3.2. Theorem [SOK94, Theorem 5.2] *Let Σ be a presentation of a monoid. Every family of generating confluences of Σ is a homotopy basis.*

Moreover, if Σ is finite, the presentation Σ has finitely many critical branchings.

4.3.3. Theorem [SOK94, Theorem 5.3] *If a monoid admits a finite convergent presentation, then it is of finite derivation type.*

In [SOK94], Squier used Theorem 4.3.3 to give another proof that there exist finitely generated monoids with a decidable word problem that do not admit a finite convergent presentation. Moreover, he showed that the homological finiteness condition left-FP_3 is not sufficient for a finitely presented decidable monoid to admit a finite convergent presentation. Indeed, the monoid \mathbf{S}_1 given by the presentation

$$\Sigma^{\mathbf{S}_1} = \langle a, b, t, x, y \mid at^nb \Rightarrow 1, xa \Rightarrow atx, xt \Rightarrow tx, xb \Rightarrow bx, xy \Rightarrow 1 \rangle$$

has a decidable word problem, admits a finite presentation and is of homological type left-FP_3 , yet it is not of a finite derivation type (and, thus, it does not admit a finite convergent presentation). This example is entirely developed in Section 6.1.

Refinements of Squier's conditions. By his results, Squier has opened two different directions to explore, one homological and one homotopical, in the quest for a complete characterisation of the existence of finite convergent presentations of monoids.

Homological and homotopical finiteness conditions are related: finite derivation type implies homological type left-FP_3 , as proved by several authors [CO94, Pri95, Laf95]. The converse implication is false in general, as already noted by Squier in [SOK94]. Yet it is true in the special case of groups [CO96]. However, the invariants homological type left-FP_3 and finite derivation type are not complete characterisations of the property to admit a finite convergent presentation: they are necessary, but not sufficient conditions, as proved by Squier in [SOK94]. Following this observation, various refinements of both invariants have been explored.

In the homological direction, thanks to the notion of abelian resolution, one defines the more restrictive conditions homological type left-FP_n , for every natural number $n > 3$, and homological type left-FP_∞ : a monoid \mathbf{M} has homological type left-FP_∞ if there exists a resolution of the trivial $\mathbb{Z}\mathbf{M}$ -module by finitely generated and projective $\mathbb{Z}\mathbf{M}$ -modules. In [Kob90], a notion of n -fold critical branching is used to complete the exact sequence (3) into a resolution, obtaining the following implication: if a monoid admits a finite convergent presentation, then it is of homological type left-FP_∞ , the converse implication still being false in general. The same results are also known for associative algebras presented by a finite Gröbner basis [Ani86] and for groups [Coh92, Bro92, Gro90]. One can obtain similar implications with the properties right-FP_∞ and bi-FP_∞ , defined with resolutions by right-modules and bimodules, respectively. In [GM12b], the authors give a construction of a resolution involving n -fold critical branchings and based on the notion of normalisation strategy.

In the homotopical direction, the condition finite derivation type has been refined into FDT_2 , a property about the existence of a finite presentation with a finite homotopy basis, itself satisfying a homotopical finiteness property [MPP05]. The condition FDT_2 is also necessary for a monoid to admit a finite convergent presentation and it is sufficient, but not necessary, for having the conditions homological type $\text{left/right/bi-FP}_4$. Higher-dimensional finite derivation type properties, called FDT_n , are defined in [GM12b]. They generalise in any dimension the finite derivation type given by Squier. For any n , the property FDT_n implies the homological type FP_n [GM12b].

Organisation of the paper and prerequisites. In Section 2, we consider presentations of monoids (and, more generally, of categories) as 2-polygraphs, and we explain their main rewriting properties in Section 3. In Section 4, we introduce the property of finite derivation type for categories using the structure of 3-polygraph and, in Section 4.3, we relate convergence and finite derivation type. This survey is self-contained, but wider categorical notions are covered in more detail by Mac Lane in [ML98] and by Barr and Wells in [BW90]. For notions of rewriting theory, one can refer to Baader and Nipkow [BN98], Terese [Ter03] and Book and Otto [BO93] for the special case of string rewriting. For extension of the finite derivation type property to higher-dimensional rewriting systems, we refer the reader to [GM09].

2. LOW-DIMENSIONAL CATEGORIES AND POLYGRAPHS

2.1. Categories and functors

2.1.1. Categories. A *category* (or 1-category) is a data \mathbf{C} made of a set \mathbf{C}_0 , whose elements are called the *0-cells of \mathbf{C}* , and, for every 0-cells x and y of \mathbf{C} , of a set $\mathbf{C}(x, y)$, whose elements are called the *1-cells from x to y of \mathbf{C}* . Those sets are equipped with the following algebraic structure:

- for every 0-cells x, y and z of \mathbf{C} , a map

$$\gamma_{x,y,z} : \mathbf{C}(x, y) \times \mathbf{C}(y, z) \rightarrow \mathbf{C}(x, z),$$

called the *composition* (or 0-composition) of \mathbf{C} ,

- for every 0-cell x , a specified element 1_x of $\mathbf{C}(x, x)$, called the *identity of x* .

The following relations are required to hold, where we write $u : x \rightarrow y$ to mean $u \in \mathbf{C}(x, y)$:

- the composition is associative, *i.e.*, for every 0-cells x, y, z and t and for every 1-cells $u : x \rightarrow y$, $v : y \rightarrow z$ and $w : z \rightarrow t$,

$$\gamma_{x,z,t}(\gamma_{x,y,z}(u, v), w) = \gamma_{x,y,t}(u, \gamma_{y,z,t}(v, w)),$$

- the identities are local units for the composition, *i.e.*, for every 0-cells x and y and for every 1-cell $u : x \rightarrow y$,

$$\gamma_{x,x,y}(1_x, u) = u = \gamma_{x,y,y}(u, 1_y).$$

A *groupoid* is a category where every 1-cell has an inverse, that is, for every 1-cell $u : x \rightarrow y$, there exists a (necessarily unique) 1-cell $u^- : y \rightarrow x$ such that

$$\gamma_{x,y,x}(u, u^-) = 1_x \quad \text{and} \quad \gamma_{y,x,y}(u^-, u) = 1_y.$$

Monoids and groups correspond exactly to categories and groupoids with only one 0-cell, respectively.

2. Low-dimensional categories and polygraphs

2.1.2. A globular point of view. A category \mathbf{C} can also be seen as a graph

$$\mathbf{C}_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \mathbf{C}_1$$

where \mathbf{C}_1 is the disjoint union of all the “hom-sets” $\mathbf{C}(x, y)$, and the maps s_0 and t_0 send a 1-cell $u : x \rightarrow y$ to its *source* x and its *target* y , respectively. We usually simply write $s(u)$ and $t(u)$ instead of $s_0(u)$ and $t_0(u)$. The composition of \mathbf{C} equips this graph with a partial function

$$\star_0 : \mathbf{C}_1 \times \mathbf{C}_1 \rightarrow \mathbf{C}_1$$

mapping a pair (u, v) of *composable* 1-cells (*i.e.*, such that $t(u) = s(v)$) to $u \star_0 v$ (we often simply write uv). By definition, the source and target of a composite 1-cell are given by

$$s(u \star_0 v) = s(u) \quad \text{and} \quad t(u \star_0 v) = t(v).$$

Moreover, the associativity axiom is written as

$$(u \star_0 v) \star_0 w = u \star_0 (v \star_0 w).$$

The identities define an inclusion

$$\mathbf{C}_0 \hookrightarrow \mathbf{C}_1$$

that maps each 0-cell x to the 1-cell 1_x . By definition, the source and target of an identity 1-cell are

$$s(1_x) = x \quad \text{and} \quad t(1_x) = x.$$

Finally, the unit axioms become

$$1_x \star_0 u = u = u \star_0 1_y.$$

This “globular” definition of categories is equivalent to the original definition given in 2.1.1.

2.1.3. Functors. Let \mathbf{C} and \mathbf{D} be categories. A *functor* $F : \mathbf{C} \rightarrow \mathbf{D}$ is a data made of a map $F_0 : \mathbf{C}_0 \rightarrow \mathbf{D}_0$ and, for every 0-cells x and y of \mathbf{C} , a map

$$F_{x,y} : \mathbf{C}(x, y) \rightarrow \mathbf{D}(F(x), F(y)),$$

such that the following relations are satisfied:

- for every 0-cells x, y and z and every 1-cells $u : x \rightarrow y$ and $v : y \rightarrow z$ of \mathbf{C} ,

$$F_{x,z}(u \star_0 v) = F_{x,y}(u) \star_0 F_{y,z}(v),$$

- for every 0-cell x of \mathbf{C} ,

$$F_{x,x}(1_x) = 1_{F(x)}.$$

We often just write $F(x)$ for $F_0(x)$ and $F(u)$ for $F_{x,y}(u)$. A functor F is a *monomorphism* (resp. an *epimorphism*, resp. an *isomorphism*) if the map F_0 and each map $F_{x,y}$ is an injection (resp. a surjection, resp. a bijection). Morphisms of monoids are exactly the functors between the corresponding categories with one 0-cell.

2.1.4. Functors as morphisms of graphs. A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ can be seen as a morphism of graphs

$$\begin{array}{ccc} \mathbf{C}_0 & \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} & \mathbf{C}_1 \\ F_0 \downarrow & & \downarrow F_1 \\ \mathbf{D}_0 & \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} & \mathbf{D}_1 \end{array}$$

where, for every 1-cell $u : x \rightarrow y$ of \mathbf{C} , the 1-cell $F_1(u)$ is defined as $F_{x,y}(u)$.

2.2. The word problem

2.2.1. 1-polygraphs. A 1-polygraph is a graph Σ , *i.e.*, a diagram of sets and maps

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \Sigma_1.$$

The elements of Σ_0 and Σ_1 are called the 0-cells and the 1-cells of Σ , respectively. If there is no confusion, we just write $\Sigma = (\Sigma_0, \Sigma_1)$. A 1-polygraph is *finite* if it has finitely many 0-cells and 1-cells.

2.2.2. Free categories. If Σ is a 1-polygraph, the *free category over Σ* is the category denoted by Σ^* and defined as follows:

- the 0-cells of Σ^* are the ones of Σ ,
- the 1-cells of Σ^* from x to y are the finite paths of Σ , *i.e.*, the finite sequences

$$x \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} \dots \xrightarrow{u_{n-1}} x_{n-1} \xrightarrow{u_n} y$$

of 1-cells of Σ ,

- the composition is given by concatenation,
- the identities are the empty paths.

If Σ has only one 0-cell, then the 1-cells of the free category Σ^* form the free monoid over the set Σ_1 .

2.2.3. Generating 1-polygraph. Let \mathbf{C} be a category. A 1-polygraph Σ *generates* \mathbf{C} if there exists an epimorphism

$$\pi : \Sigma^* \rightarrow \mathbf{C}$$

that is the identity on 0-cells. In that case, the 1-polygraph Σ has the same 0-cells as \mathbf{C} and, for every 0-cells x and y of \mathbf{C} , the map

$$\pi_{x,y} : \Sigma^*(x, y) \rightarrow \mathbf{C}(x, y)$$

is surjective. We usually consider that π is implicitly specified for a given generating 1-polygraph Σ and, if u is a 1-cell of Σ^* , we just write \bar{u} instead of $\pi(u)$. A category is *finitely generated* if it admits a finite generating 1-polygraph (in particular, the category must have finitely many 0-cells).

2. Low-dimensional categories and polygraphs

2.2.4. The word problem for categories. Let \mathbf{C} be a category. The *word problem for \mathbf{C}* is the problem of finding a generating 1-polygraph Σ for \mathbf{C} together with an algorithm that decides, for any two 1-cells u and v of Σ^* , whether or not $\bar{u} = \bar{v}$ holds in \mathbf{C} (that is, whether or not the 1-cells u and v represent the same 1-cell of \mathbf{C}). We note that, to have $\bar{u} = \bar{v}$, it is necessary for u and v to be *parallel*, i.e., they must have the same source and the same target. The word problem is undecidable in general for a given category \mathbf{C} , even if it is finitely generated. However, a finite convergent presentation of \mathbf{C} , see 3.1.7, provides a solution to the word problem of \mathbf{C} .

2.3. Presentations of categories

2.3.1. Spheres and cellular extensions of categories. A *sphere* of a category \mathbf{C} is a pair $\gamma = (u, v)$ of parallel 1-cells of \mathbf{C} , that is, with the same source and the same target; such a sphere is denoted by $\gamma : u \Rightarrow v$. The 1-cell u is the *source of γ* and v is its *target*. A *cellular extension* of \mathbf{C} is a set Γ equipped with a map from Γ to the set of spheres of \mathbf{C} .

2.3.2. Congruences. A *congruence* on a category \mathbf{C} is an equivalence relation \equiv on the parallel 1-cells of \mathbf{C} that is compatible with the composition of \mathbf{C} , that is, for every 1-cells

$$x \xrightarrow{w} y \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} z \xrightarrow{w'} t$$

of \mathbf{C} such that $u \equiv v$, we have $wuw' \equiv wv'w'$. If Γ is a cellular extension of \mathbf{C} , the *congruence generated by Γ* is denoted by \equiv_{Γ} and defined as the smallest congruence relation such that, if $\gamma : u \Rightarrow v$ is in Γ , then $u \equiv_{\Gamma} v$. In the literature, the congruence \equiv_{Γ} is also called the *Thue congruence* generated by Γ .

2.3.3. Quotient categories. If \mathbf{C} is a category and Γ is a cellular extension of \mathbf{C} , the *quotient of \mathbf{C} by Γ* is the category denoted by \mathbf{C}/Γ and defined as follows:

- the 0-cells of \mathbf{C}/Γ are the ones of \mathbf{C} ,
- for every 0-cells x and y of \mathbf{C} , the hom-set $\mathbf{C}/\Gamma(x, y)$ is the quotient of $\mathbf{C}(x, y)$ by the restriction of \equiv_{Γ} .

We denote by $\pi_{\Gamma} : \mathbf{C} \rightarrow \mathbf{C}/\Gamma$ the canonical projection. When the context is clear, we just write π for π_{Γ} and \bar{u} for the image through π of a 1-cell u .

2.3.4. 2-polygraphs. A *2-polygraph* is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ made of a 1-polygraph (Σ_0, Σ_1) , often simply denoted by Σ_1 , and a cellular extension Σ_2 of the free category Σ_1^* . In other terms, a 2-polygraph Σ is a 2-graph

$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \Sigma_1^* \begin{array}{c} \xleftarrow{s_1} \\ \xrightarrow{t_1} \end{array} \Sigma_2$$

whose 0-cells and 1-cells form a free category. The elements of Σ_k are called the *k-cells of Σ* and Σ is *finite* if it has finitely many cells in every dimension.

2.3.5. Presentations of categories. If Σ is a 2-polygraph, the *category presented by Σ* is the category denoted by $\bar{\Sigma}$ and defined by

$$\bar{\Sigma} = \Sigma_1^* / \Sigma_2.$$

If \mathbf{C} is a category, a *presentation of \mathbf{C}* is a 2-polygraph Σ such that \mathbf{C} is isomorphic to $\bar{\Sigma}$. In that case, the 1-cells of Σ are called *the generating 1-cells of \mathbf{C}* , or *the generators of \mathbf{C}* , and the 2-cells of Σ are called *the generating 2-cells of \mathbf{C}* , or *the relations of \mathbf{C}* .

2.3.6. Tietze equivalence. Two 2-polygraphs are *Tietze-equivalent* if they present the same category. It is a standard result that two (finite) 2-polygraphs are Tietze-equivalent if, and only if, they are related by a (finite) sequence of *elementary Tietze transformations*:

- adjunction or elimination of a 1-cell x and of a 2-cell $\alpha : u \Rightarrow x$, where u is a 1-cell of the free category over $\Sigma_1 \setminus \{x\}$,
- adjunction or elimination of a 2-cell $\alpha : u \Rightarrow v$ such that $u \equiv_{\Sigma_2 \setminus \{\alpha\}} v$.

2.4. Two-dimensional categories

2.4.1. 2-categories. A *2-category* is a data \mathcal{C} made of a set \mathcal{C}_0 , whose elements are called the *0-cells of \mathcal{C}* , and, for every 0-cells x and y of \mathcal{C} , a category $\mathcal{C}(x, y)$, whose 0-cells and 1-cells are respectively called the *1-cells* and the *2-cells from x to y of \mathcal{C}* . This data is equipped with the following algebraic structure:

- for every 0-cells x, y and z of \mathcal{C} , a functor

$$\gamma_{x,y,z} : \mathcal{C}(x, y) \times \mathcal{C}(y, z) \rightarrow \mathcal{C}(x, z),$$

- for every 0-cell x , a specified 0-cell 1_x of the category $\mathcal{C}(x, x)$.

The following relations are required to hold:

- the composition is associative, *i.e.*, for every 0-cells x, y, z and t ,

$$\gamma_{x,z,t} \circ (\gamma_{x,y,z} \times \text{Id}_{\mathcal{C}(z,t)}) = \gamma_{x,y,t} \circ (\text{Id}_{\mathcal{C}(x,y)} \times \gamma_{y,z,t}),$$

- the identities are local units for the composition, *i.e.*, for every 0-cells x and y ,

$$\gamma_{x,x,y} \circ (1_x \times \text{Id}_{\mathcal{C}(x,y)}) = \text{Id}_{\mathcal{C}(x,y)} = \gamma_{x,y,y} \circ (\text{Id}_{\mathcal{C}(x,y)}, 1_y).$$

This definition of 2-categories is usually stated as follows: a 2-category is a category enriched in categories. A *(2, 1)-category* is a 2-category whose 2-cells are invertible for the 1-composition: in other terms, it is a 2-category whose hom-categories are groupoids.

2. Low-dimensional categories and polygraphs

2.4.2. The globular point of view. A 2-category can, equivalently, be defined as a 2-graph

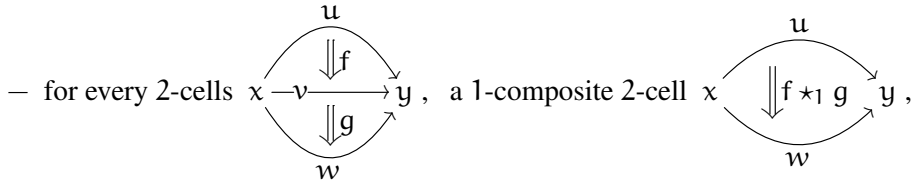
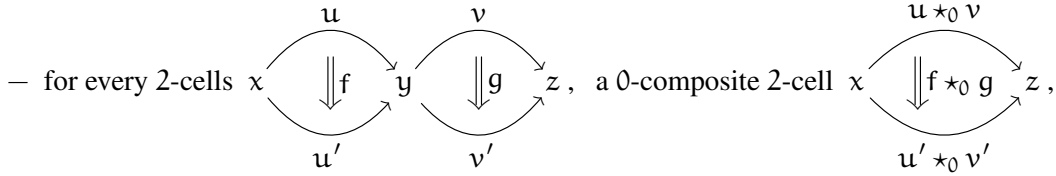
$$\mathcal{C}_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \mathcal{C}_1 \begin{array}{c} \xleftarrow{s_1} \\ \xrightarrow{t_1} \end{array} \mathcal{C}_2$$

equipped with an additional algebraic structure. The definition of 2-graph requires that the source and target maps satisfy the globular relations:

$$s_0 \circ s_1 = s_0 \circ t_1 \quad \text{and} \quad t_0 \circ s_1 = t_0 \circ t_1.$$

The 2-graph is equipped with two compositions, the *0-composition* \star_0 and the *1-composition* \star_1 , respectively defined on 0-composable 1-cells and 2-cells, and on 1-composable 2-cells. We also have an inclusion of \mathcal{C}_0 into \mathcal{C}_1 given by the identities of the 2-category, and an inclusion of \mathcal{C}_1 into \mathcal{C}_2 induced by the identities of the hom-categories. In details, we have the following operations:

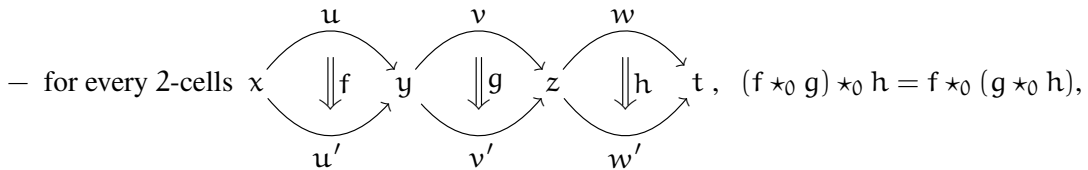
- for every 1-cells $x \xrightarrow{u} y \xrightarrow{v} z$, a 0-composite 1-cell $x \xrightarrow{u \star_0 v} z$,



- for every 0-cell x , an identity 1-cell $x \xrightarrow{1_x} x$,
- for every 1-cell $x \xrightarrow{u} y$, an identity 2-cell $u \xrightarrow{1_u} u$.

The following relations hold:

- for every 1-cells $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} t$, $(u \star_0 v) \star_0 w = u \star_0 (v \star_0 w)$,
- for every 1-cell $x \xrightarrow{u} y$, $1_x \star_0 u = u = u \star_0 1_y$,
- for every 1-cells $x \xrightarrow{u} y \xrightarrow{v} z$, $1_{u \star_0 v} = 1_u \star_0 1_v$,
- for every 2-cells $u \xrightarrow{f} v \xrightarrow{g} w \xrightarrow{h} x$, $(f \star_1 g) \star_1 h = f \star_1 (g \star_1 h)$,



– for every 2-cell $x \begin{array}{c} \xrightarrow{u} \\ \Downarrow f \\ \xrightarrow{v} \end{array} y$, $1_x \star_0 f = f = f \star_0 1_y$,

– for every 2-cell $u \xRightarrow{f} v$, $1_u \star_1 f = f = f \star_1 1_v$,

– for every 2-cells $x \begin{array}{c} \xrightarrow{u} \\ \Downarrow f \\ \xrightarrow{u'} \\ \Downarrow f' \\ \xrightarrow{u''} \end{array} y \begin{array}{c} \xrightarrow{v} \\ \Downarrow g \\ \xrightarrow{v'} \\ \Downarrow g' \\ \xrightarrow{v''} \end{array} z$, $(f \star_1 f') \star_0 (g \star_1 g') = (f \star_0 g) \star_1 (f' \star_0 g')$.

The last relation is usually called the *exchange relation* or the *interchange law* for the compositions \star_0 and \star_1 . This globular definition of 2-categories is equivalent to the enriched one.

The 0-composition of 2-cells with identity 1-cells defines the *whiskering* operations:

– for every $x \xrightarrow{w} y \begin{array}{c} \xrightarrow{u} \\ \Downarrow f \\ \xrightarrow{v} \end{array} z$, the *left whiskering* is $x \begin{array}{c} \xrightarrow{w \star_0 u} \\ \Downarrow w \star_0 f \\ \xrightarrow{w \star_0 v} \end{array} z$,

– for every $x \begin{array}{c} \xrightarrow{u} \\ \Downarrow f \\ \xrightarrow{v} \end{array} y \xrightarrow{w} z$, the *right whiskering* is $x \begin{array}{c} \xrightarrow{u \star_0 w} \\ \Downarrow f \star_0 w \\ \xrightarrow{v \star_0 w} \end{array} z$.

The left and right whiskering operations satisfy the following relations:

– for every $x \xrightarrow{u} y \begin{array}{c} \xrightarrow{v} \\ \Downarrow f \\ \xrightarrow{v'} \\ \Downarrow f' \\ \xrightarrow{v''} \end{array} z$, $u \star_0 (f \star_1 f') = (u \star_0 f) \star_1 (u \star_0 f')$,

– for every $x \begin{array}{c} \xrightarrow{u} \\ \Downarrow f \\ \xrightarrow{u'} \\ \Downarrow f' \\ \xrightarrow{u''} \end{array} y \xrightarrow{v} z$, $(f \star_1 f') \star_0 v = (f \star_0 v) \star_1 (f' \star_0 v)$,

2. Low-dimensional categories and polygraphs

– for every $x \xrightarrow{u} y \xrightarrow{v} z \begin{array}{c} \xrightarrow{w} \\ \Downarrow f \\ \xrightarrow{w'} \end{array} t$, $(u \star_0 v) \star_0 f = u \star_0 (v \star_0 f)$,

– for every $x \xrightarrow{u} y \begin{array}{c} \xrightarrow{v} \\ \Downarrow f \\ \xrightarrow{v'} \end{array} z \xrightarrow{w} t$, $(u \star_0 f) \star_0 w = u \star_0 (f \star_0 w)$,

– for every $x \begin{array}{c} \xrightarrow{u} \\ \Downarrow f \\ \xrightarrow{u'} \end{array} y \xrightarrow{v} z \xrightarrow{w} t$, $(f \star_0 v) \star_0 w = f \star_0 (v \star_0 w)$,

As for categories, we usually write uv and fg instead of $u \star_0 v$ and $f \star_0 g$.

2.4.3. Free 2-categories. Let Σ be a 2-polygraph. The *free 2-category over Σ* is denoted by Σ^* and defined as follows:

- the 0-cells of Σ^* are the ones of Σ ,
- for every 0-cells x and y of Σ , the hom-category $\Sigma^*(x, y)$ is defined as
 - the free category over the 1-polygraph
 - * whose 0-cells are the 1-cells from x to y of Σ_1^* , that is, the finite sequences $a_1 \cdots a_n$ of composable 1-cells of Σ ,
 - * whose 1-cells are the

$$x \xrightarrow{w} y \begin{array}{c} \xrightarrow{u} \\ \Downarrow \alpha \\ \xrightarrow{v} \end{array} z \xrightarrow{w'} t$$

with $\alpha : u \Rightarrow v$ in Σ_2 and w and w' in Σ_1^* ,

- quotiented by the congruence generated by the cellular extension made of all the possible

$$\alpha w v \star_1 u' w \beta \equiv u w \beta \star_1 \alpha w v',$$

for $\alpha : u \Rightarrow u'$ and $\beta : v \Rightarrow v'$ in Σ_2 and w in Σ_1^* ,

- for every 0-cells x, y and z of Σ the composition functor is given by the concatenation on 1-cells and, on 2-cells, as follows:

$$\begin{aligned} & (u_1 \alpha_1 u'_1 \star_1 \cdots \star_1 u_m \alpha_m u'_m) \star_0 (v_1 \beta_1 v'_1 \star_1 \cdots \star_1 v_n \beta_n v'_n) \\ &= u_1 \alpha_1 u'_1 v_1 s(\beta_1) v'_1 \star_1 \cdots \star_1 u_m \alpha_m u'_m v_1 s(\beta_1) v'_1 \\ & \quad \star_1 u_m t(\alpha_m) u'_m v_1 \beta_1 v'_1 \star_1 \cdots \star_1 u_m t(\alpha_m) u'_m v_n \beta_n v'_n \end{aligned}$$

- for every 0-cell x , the identity 1-cell 1_x is the one of Σ_1^* .

By definition of the 2-category Σ^* , for every 1-cells u and v of Σ^* , we have $\bar{u} = \bar{v}$ in the quotient category $\bar{\Sigma}$ if, and only if, there exists a “zig-zag” sequence of 2-cells of Σ^* between them:

$$u \xRightarrow{f_1} u_1 \xleftarrow{g_1} v_1 \xRightarrow{f_2} u_2 \xleftarrow{\dots} \dots \xRightarrow{f_{n-1}} u_{n-1} \xleftarrow{g_{n-1}} v_{n-1} \xRightarrow{f_n} u_n \xleftarrow{g_n} v.$$

2.4.4. Free (2, 1)-categories. If Σ is a 2-polygraph, the *free (2, 1)-category over Σ* is denoted by Σ^\top and is defined as the 2-category whose 0-cells are the ones of Σ and, for every 0-cells x and y , the hom-category $\Sigma^\top(x, y)$ is given as the quotient

$$\Sigma^\top(x, y) = (\Sigma \sqcup \Sigma^-)^*(x, y) / \text{Inv}(\Sigma_2),$$

where:

- the 2-polygraph Σ^- is obtained from Σ by reversing the 2-cells,
- the cellular extension $\text{Inv}(\Sigma_2)$ contains the following two relations for every 2-cell α of Σ and every possible 1-cells u and v of Σ^* such that $s(u) = x$ and $t(v) = y$:

$$u\alpha v \star_1 u\alpha^-v \equiv 1_{us(\alpha)v} \quad \text{and} \quad u\alpha^-v \star_1 u\alpha v \equiv 1_{ut(\alpha)v}.$$

By definition of the (2, 1)-category Σ^\top , for every 1-cells u and v of Σ^* , we have $\bar{u} = \bar{v}$ in the quotient category $\bar{\Sigma}$ if, and only if, there exists a 2-cell $f : u \Rightarrow v$ in the (2, 1)-category Σ^\top .

3. REWRITING PROPERTIES OF 2-POLYGRAPHS

3.1. Convergent presentations of categories

Let us fix a 2-polygraph Σ .

3.1.1. Rewriting and normal forms. A *rewriting step* of Σ is a 2-cell of the free 2-category Σ^* with shape

$$x \xrightarrow{w} y \begin{array}{c} \xrightarrow{u} \\ \Downarrow \varphi \\ \xrightarrow{v} \end{array} z \xrightarrow{w'} t$$

where $\varphi : u \Rightarrow v$ is a generating 2-cell in Σ and w and w' are 1-cells of Σ^* . A *rewriting sequence* of Σ is a finite or infinite sequence

$$u_1 \xRightarrow{f_1} u_2 \xRightarrow{f_2} (\dots) \xRightarrow{f_{n-1}} u_n \xRightarrow{f_n} (\dots)$$

of rewriting steps. If Σ has a non-empty rewriting sequence from u to v , we say that u *rewrites into* v . Let us note that every 2-cell f of Σ^* decomposes into a finite rewriting sequence of Σ , this decomposition being unique up to exchange relations. A 1-cell u of Σ^* is a *normal form* if Σ has no rewriting step with source u . A *normal form* of u is a 1-cell v that is a normal form and such that u rewrites into v .

3. Rewriting properties of 2-polygraphs

3.1.2. Termination. We say that Σ *terminates* if it has no infinite rewriting sequence. In that case, every 1-cell has at least one normal form and *noetherian induction* allows definitions and proofs of properties of 1-cells by induction on the maximum size of the 2-cells leading to normal forms. A *termination order* on Σ is an order relation \leq on parallel 1-cells of Σ^* such that the following properties are satisfied:

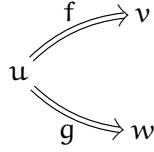
- the composition of 1-cells of Σ^* is strictly monotone in both arguments,
- every decreasing family $(u_n)_{n \in \mathbb{N}}$ of parallel 1-cells of Σ^* is stationary,
- for every 2-cell α of Σ , the strict inequality $s(\alpha) > t(\alpha)$ holds.

As a direct consequence of the definition, if Σ admits a termination order, then Σ terminates. A useful example of termination order is the *left degree-wise lexicographic order* (or *deglex* for short) generated by a given order on the 1-cells of Σ . It is defined by the following strict inequalities, where each x_i and y_j is a 1-cell of Σ :

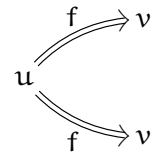
$$\begin{aligned} x_1 \cdots x_p &< y_1 \cdots y_q, & \text{if } p < q, \\ x_1 \cdots x_{k-1} x_k \cdots x_p &< x_1 \cdots x_{k-1} y_k \cdots y_p, & \text{if } x_k < y_k. \end{aligned}$$

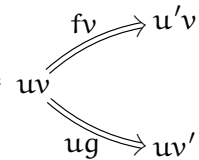
The deglex order is total if, and only if, the original order on 1-cells of Σ is total.

3.1.3. Branchings. A *branching* of Σ is a pair (f, g) of 2-cells of Σ_2^* with a common source, as in the following diagram



The 1-cell u is the *source* of this branching and the pair (v, w) is its *target*. We do not distinguish the branchings (f, g) and (g, f) . A branching (f, g) is *local* if f and g are rewriting steps. Local branchings belong to one of the three following families:

– *aspherical* branchings have shape  where $u \xrightarrow{f} v$ is a rewriting step,

– *Peiffer* branchings have shape  where $u \xrightarrow{f} u'$ and $v \xrightarrow{g} v'$ are rewriting steps,

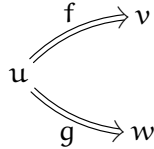
– *overlapping* branchings are the remaining local branchings.

Local branchings are compared by “inclusion”, *i.e.*, by the order \preceq generated by the relations

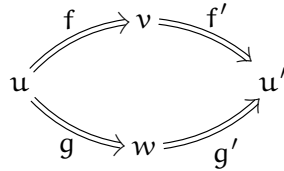
$$(f, g) \preceq (ufv, ugv)$$

given for any local branching (f, g) and any possible 1-cells u and v of Σ_1^* . An overlapping local branching that is minimal for the order \preceq is called a *critical branching* (or a *critical pair*). The terms “aspherical” and “Peiffer” come from the corresponding notions for spherical diagrams in Cayley complexes associated to presentations of groups, [LS01], while “critical” is used in rewriting theory, [BO93, BN98].

3.1.4. Confluence. A branching



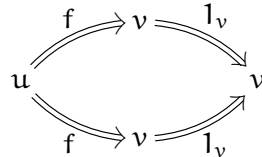
is *confluent* if there exist 2-cells f' and g' in Σ_2^* , as in the following diagram:



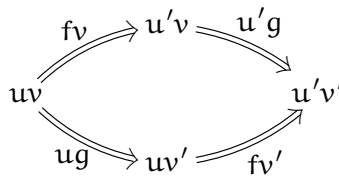
We say that Σ is *confluent* (resp. *locally confluent*) if all of its branchings (resp. local branchings) are confluent. In a confluent 2-polygraph, every 1-cell has at most one normal form.

3.1.5. Theorem. *A 2-polygraph is locally confluent if, and only if, all its critical branchings are confluent.*

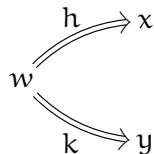
Proof. Every aspherical branching is confluent:



We also have confluence of every Peiffer local branching:

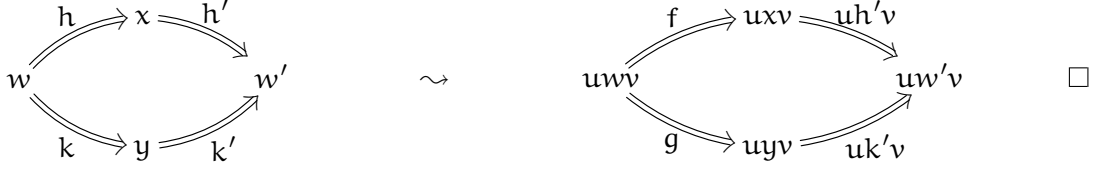


We note that, in the aspherical and Peiffer cases, the 2-cells f' and g' can be chosen in such a way that $f \star_1 f' = g \star_1 g'$ holds. Finally, in the case of an overlapping but not minimal local branching (f, g) , there exist factorisations $f = uhv$ and $g = ukv$ with



3. Rewriting properties of 2-polygraphs

a critical branching of Σ . Moreover, if the branching (h, k) is confluent, then so is (f, g) :



The proof of the following result, also called the diamond lemma, is contained in the one of Theorem 4.3.2.

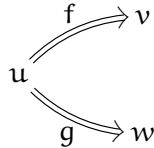
3.1.6. Theorem (Newman's lemma [New42, Theorem 3]). *For terminating 2-polygraphs, local confluence and confluence are equivalent properties.*

3.1.7. Convergent polygraphs. We say that Σ is *convergent* if it terminates and it is confluent. Such a Σ is called a *convergent presentation of $\bar{\Sigma}$* , and of any category that is isomorphic to $\bar{\Sigma}$. In that case, every 1-cell u of Σ_1^* has a unique normal form, denoted by \hat{u} , so that we have $\bar{u} = \bar{v}$ in $\bar{\Sigma}$ if, and only if, $\hat{u} = \hat{v}$ holds in Σ_1^* . This defines a section $\bar{\Sigma} \rightarrow \Sigma_1^*$ of the canonical projection $\Sigma_1^* \rightarrow \bar{\Sigma}$, mapping a 1-cell u of $\bar{\Sigma}$ to the unique normal form of its representative 1-cells in Σ^* , still denoted by \hat{u} .

As a consequence, a finite and convergent 2-polygraph Σ yields a decision procedure for the word problem of the category $\bar{\Sigma}$ it presents: the *normal-form procedure*, which takes, as input, two 1-cells u and v of Σ^* , and decides whether $\bar{u} = \bar{v}$ holds in $\bar{\Sigma}$ or not. For that, the procedure computes the respective normal forms \hat{u} and \hat{v} of u and v . Finiteness is used to test whether a given 1-cell u is a normal form or not, by examination of all the relations and their possible applications on u . Then, the equality $\bar{u} = \bar{v}$ holds in $\bar{\Sigma}$ if, and only if, the equality $\hat{u} = \hat{v}$ holds in Σ^* .

3.2. Transformations of 2-polygraphs

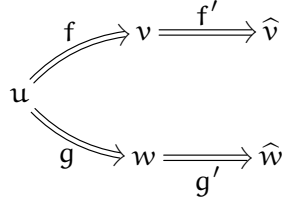
3.2.1. Knuth-Bendix's completion. Let Σ be a terminating 2-polygraph, equipped with a total termination order \leq . The *Knuth-Bendix's completion of Σ* is the 2-polygraph $\check{\Sigma}$ obtained by the following process. We start with $\check{\Sigma}$ equal to Σ and with B equal to the set of critical branchings of Σ . If B is empty, then the procedure stops. Otherwise, it picks a branching



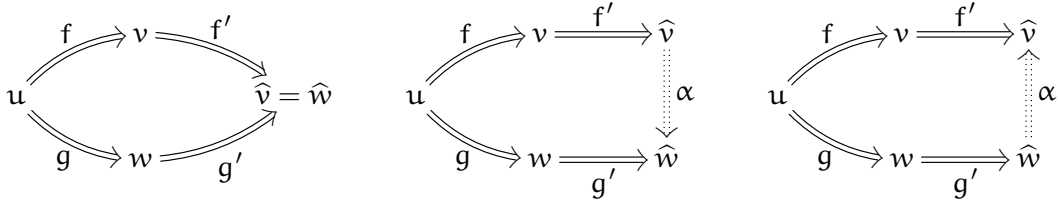
in B and it performs the following operations:

1. It computes 2-cells $f' : v \Rightarrow \hat{v}$ and $g' : w \Rightarrow \hat{w}$ of $\check{\Sigma}^*$, where \hat{v} and \hat{w} are normal forms for v

and w , respectively, as in the following diagram:



2. It tests which (in)equality $\hat{v} = \hat{w}$ or $\hat{v} > \hat{w}$ or $\hat{v} < \hat{w}$ holds, corresponding to the following three situations, respectively:



If $\hat{v} \neq \hat{w}$, the procedure adds the dotted 2-cell α of the corresponding situation to $\check{\Sigma}$ and all the new critical branchings created by α to B .

3. It removes (f, g) from B and restarts from the beginning.

If the procedure stops, it returns the 2-polygraph $\check{\Sigma}$. Otherwise, it builds an increasing sequence of 2-polygraphs, whose limit is denoted by $\check{\Sigma}$. Note that, if the starting 2-polygraph Σ is already convergent, then the Knuth-Bendix's completion of Σ is Σ .

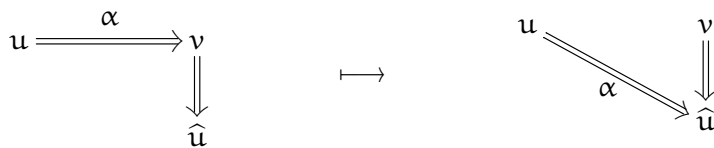
3.2.2. Theorem ([KB70]). *The Knuth-Bendix's completion $\check{\Sigma}$ of a 2-polygraph Σ is a convergent presentation of the category $\bar{\Sigma}$. Moreover, the 2-polygraph $\check{\Sigma}$ is finite if, and only if, the 2-polygraph Σ is finite and if the Knuth-Bendix's completion procedure halts.*

Indeed, by construction, the 2-polygraph $\check{\Sigma}$ is convergent and, since all the operations performed by the procedure are Tietze transformations, it is Tietze-equivalent to Σ .

3.2.3. Métivier-Squier's reduction. A 2-polygraph Σ is *reduced* if, for every 2-cell $\alpha : u \Rightarrow v$ of Σ , we have that u is a normal form for $\Sigma_2 \setminus \{\alpha\}$ and that v is a normal form for Σ_2 .

Given a 2-polygraph Σ , the *Métivier-Squier's reduction* of Σ is the 2-polygraph obtained by the procedure that successively performs the following operations:

1. The procedure replaces every 2-cell $\alpha : u \Rightarrow v$ by $\alpha : u \Rightarrow \hat{u}$:



3. Rewriting properties of 2-polygraphs

2. If the resulting 2-polygraph contains parallel 2-cells, the procedure removes all but one:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & \alpha_1 & \\
 \curvearrowright & & \curvearrowleft \\
 u & & \hat{u} \\
 \curvearrowleft & & \curvearrowright \\
 & \alpha_n & \\
 & \vdots & \\
 & &
 \end{array}
 & \mapsto &
 \begin{array}{ccc}
 & \alpha & \\
 \Longrightarrow & & \Longrightarrow \\
 u & & \hat{u}
 \end{array}
 \end{array}$$

3. Finally, the procedure removes, in the resulting 2-polygraph, every 2-cell whose source contains the source of another 2-cell:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 v w v' & \xrightarrow{\alpha} & \widehat{v w v'} \\
 & \searrow v \beta v' & \uparrow \\
 & & v \widehat{w} v' \\
 & & \uparrow \\
 & & v \widehat{w} v'
 \end{array}
 & \mapsto &
 \begin{array}{ccc}
 v w v' & & \widehat{v w v'} \\
 & \searrow v \beta v' & \uparrow \\
 & & v \widehat{w} v'
 \end{array}
 \end{array}$$

By construction, we get the following result, originally obtained by Métivier for term rewriting and by Squier for string rewriting:

3.2.4. Theorem ([Mét83], [Squ87, Theorem 2.4]). *Every (finite) convergent 2-polygraph is Tietze-equivalent to a (finite) reduced convergent 2-polygraph.*

3.3. Normalisation strategies

3.3.1. Normalisation strategies. Let Σ be a 2-polygraph and let \mathbf{C} denote the category presented by Σ . We consider a section

$$\mathbf{C} \rightarrow \Sigma_1^*$$

of the canonical projection $\pi : \Sigma_1^* \rightarrow \mathbf{C}$, *i.e.*, we choose, for every 1-cell u of \mathbf{C} , a 1-cell \hat{u} that represents u , *i.e.*, such that $\pi(\hat{u}) = u$. In general, we cannot assume that the chosen section is functorial, *i.e.*, that $\widehat{u v} = \widehat{u} \widehat{v}$ holds. However, we assume that $\widehat{1_x} = 1_x$ holds for every 0-cell x of \mathbf{C} . Given a 1-cell u of Σ^* , we simply write \hat{u} for \widehat{u} .

Such a section being fixed, a *normalisation strategy* for Σ is a map

$$\sigma : \Sigma_1^* \rightarrow \Sigma_2^*$$

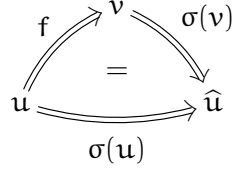
that sends every 1-cell u of Σ^* to a 2-cell

$$u \xrightarrow{\sigma(u)} \hat{u}$$

of Σ^* , such that $\sigma(\hat{u}) = 1_{\hat{u}}$ holds for every 1-cell u of Σ_1^* .

3.3.2. Example. We assume that Σ is a terminating 2-polygraph. We use noetherian induction to define a section and a corresponding normalisation strategy. If u is a normal form, then we define \hat{u} as u and $\sigma(u)$ as the identity. Let us assume that u is not a normal form and that, for every v such that u

rewrites into v , we have built \hat{v} and $\sigma(v)$. In that case, we choose a rewriting step $f : u \Rightarrow v$ and we define \hat{u} to be \hat{v} and $\sigma(u)$ as the following composite:



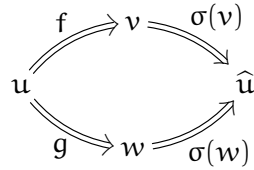
3.3.3. The leftmost and rightmost normalisation strategies. If Σ is a 2-polygraph, then, for every 1-cell u of Σ^* , the set of rewriting steps with source u can be ordered from left to right: for two rewriting steps $f = v\alpha v'$ and $g = w\beta w'$ with source u , we have $f \prec g$ if the length of v is strictly smaller than the length of w . If Σ is finite, then the order \prec is total and the set of rewriting steps of source u is finite. Hence, this set contains a smallest element $\lambda(u)$ and a greatest element $\rho(u)$, respectively called the *leftmost* and the *rightmost rewriting steps on u* . If, moreover, the 2-polygraph Σ terminates, the iteration of λ and ρ yields normalisation strategies respectively called the *leftmost* and the *rightmost normalisation strategies of Σ* . For example, the rightmost normalisation strategy σ is defined, on a 1-cell u of Σ^* by:

$$\sigma(u) = \rho(u) \star_1 \sigma(t(\rho(u))).$$

Moreover, we prove by noetherian induction that, for every composable 1-cells u and v , the rightmost normalisation strategy satisfies the relation

$$\sigma(uv) = u\sigma(v) \star_1 \sigma(u\hat{v}).$$

The leftmost and rightmost normalisation strategies give a way to make constructive most of the results we present here. For example, they provide a deterministic choice of a confluence diagram



for every branching (f, g) .

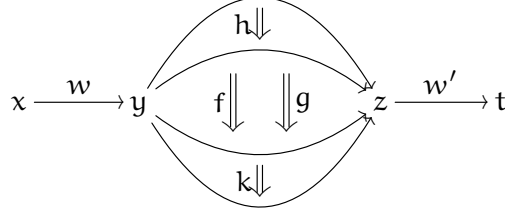
4. FINITE DERIVATION TYPE

4.1. Coherent presentations of categories

4.1.1. Cellular extensions and homotopy bases of 2-categories. Let \mathcal{C} be a 2-category. A *2-sphere of \mathcal{C}* is a pair $\gamma = (f, g)$ of parallel 2-cells of \mathcal{C} , i.e., with $s(f) = s(g)$ and $t(f) = t(g)$; we call f the *source* of γ and g its *target* and we denote such a 2-sphere by $\gamma : f \Rrightarrow g$. A *cellular extension* of the 2-category \mathcal{C} is a set Γ equipped with a map from Γ to the set of 2-spheres of \mathcal{C} .

4. Finite derivation type

A *congruence* on a 2-category \mathcal{C} is an equivalence relation \equiv on the parallel 2-cells of \mathcal{C} such that, for every cells



of \mathcal{C} , if $f \equiv g$, then

$$w \star_0 (h \star_1 f \star_1 k) \star_0 w' \equiv w \star_0 (h \star_1 g \star_1 k) \star_0 w'.$$

If Γ is a cellular extension of \mathcal{C} , the *congruence generated by Γ* is denoted by \equiv_Γ and defined as the smallest congruence such that, if Γ contains a 3-cell $\gamma : f \Rightarrow g$, then $f \equiv_\Gamma g$. The *quotient 2-category* of a 2-category \mathcal{C} by a congruence relation \equiv is the 2-category, denoted by \mathcal{C}/\equiv , whose 0-cells and 1-cells are those of \mathcal{C} and the 2-cells are the equivalence classes of 2-cells of \mathcal{C} modulo the congruence \equiv .

A *homotopy basis of \mathcal{C}* is a cellular extension Γ of \mathcal{C} such that, for every parallel 2-cells f and g of \mathcal{C} , we have $f \equiv_\Gamma g$, that is, the equality $\bar{f} = \bar{g}$ holds in the quotient 2-category $\mathcal{C}/\equiv_\Gamma$. For instance, the set of 2-spheres of \mathcal{C} forms a homotopy basis.

4.1.2. (3, 1)-polygraphs and coherent presentations. A *(3, 1)-polygraph* is a pair $\Sigma = (\Sigma_2, \Sigma_3)$ made of a 2-polygraph Σ_2 and a cellular extension Σ_3 of the free $(2, 1)$ -category Σ_2^\top over Σ_2 , as summarised in the following diagram:

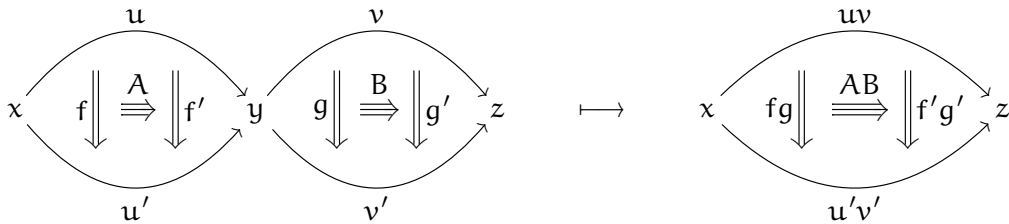
$$\Sigma_0 \begin{array}{c} \xleftarrow{s_0} \\ \xrightarrow{t_0} \end{array} \Sigma_1^* \begin{array}{c} \xleftarrow{s_1} \\ \xrightarrow{t_1} \end{array} \Sigma_2^\top \begin{array}{c} \xleftarrow{s_2} \\ \xrightarrow{t_2} \end{array} \Sigma_3$$

If \mathbf{C} is a category, a *coherent presentation of \mathbf{C}* is a $(3, 1)$ -polygraph $\Sigma = (\Sigma_2, \Sigma_3)$ such that Σ_2 is a presentation of \mathbf{C} and Σ_3 is a homotopy basis of Σ_2^\top .

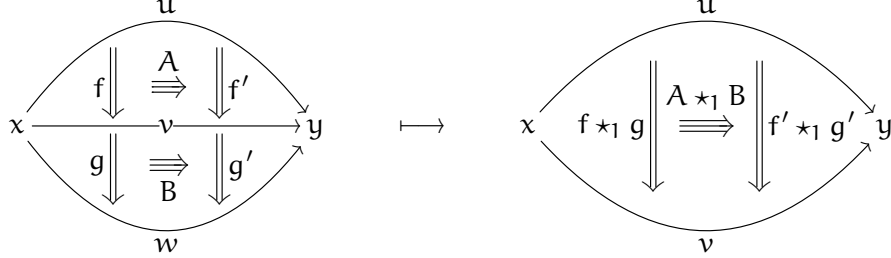
4.1.3. Finite derivation type. A 2-polygraph Σ is of *finite derivation type* if it is finite and if the $(2, 1)$ -category Σ^\top admits a finite homotopy basis. A category \mathbf{C} is of *finite derivation type* if it admits a finite coherent presentation.

4.1.4. 3-categories. The definition of 3-category is adapted from the one of 2-category by replacement of the hom-categories and the composition functors by hom-2-categories and composition 2-functors. In particular, in a 3-category, the 3-cells can be composed in three different ways:

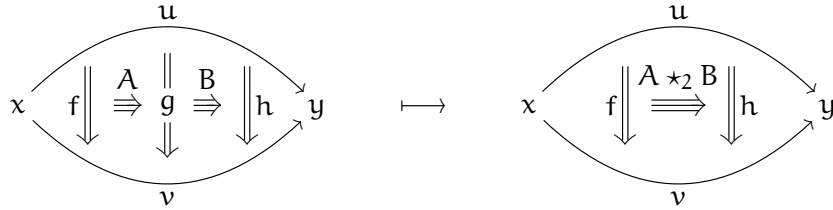
– by \star_0 , along their 0-dimensional boundary:



– by \star_1 , along their 1-dimensional boundary:



– by \star_2 , along their 2-dimensional boundary:



A $(3, 1)$ -category is a 3-category whose 2-cells are invertible for the composition \star_1 and whose 3-cells are invertible for the composition \star_2 . This implies that 3-cells are also invertible for the composition \star_1 .

4.1.5. Free $(3, 1)$ -categories. Given a $(3, 1)$ -polygraph Σ , the *free $(3, 1)$ -category over Σ* is denoted by Σ^\top and defined as follows:

- its underlying 2-category is the free $(2, 1)$ -category Σ_2^\top over Σ_2 ,
- its 3-cells are all the formal compositions by \star_0 , \star_1 and \star_2 of 3-cells of Σ_3 , of their inverses and of identities of 2-cells.

In particular, we get that Σ_3 is a homotopy basis of Σ_2^\top if, and only if, for every pair (f, g) of parallel 2-cells of Σ_2^\top , there exists a 3-cell $A : f \Rightarrow g$ in Σ^\top .

4.2. The homotopy bases transfer theorem

The objective of this section is to prove Theorem 4.2.3: given two finite presentations of the same category, both are of finite derivation type or neither is. Towards this goal, we prove Theorem 4.2.2, that allows transfers of homotopy bases between presentations of the same category.

4.2.1. Lemma. *Let \mathbf{C} be a category and let Σ and Υ be presentations of \mathbf{C} . There exist 2-functors*

$$F : \Sigma^\top \rightarrow \Upsilon^\top \quad \text{and} \quad G : \Upsilon^\top \rightarrow \Sigma^\top$$

and, for every 1-cells u of Σ^\top and v of Υ^\top , there exist 2-cells

$$\sigma_u : GF(u) \Rightarrow u \quad \text{and} \quad \tau_v : FG(v) \Rightarrow v$$

in Σ^\top and Υ^\top , such that the following conditions are satisfied:

4. Finite derivation type

– the 2-functors F and G induce the identity through the canonical projections onto \mathbf{C} :

$$\begin{array}{ccc} \Sigma^\top & \xrightarrow{\pi_\Sigma} & \mathbf{C} \\ F \downarrow & = & \downarrow \text{Id}_{\mathbf{C}} \\ \Upsilon^\top & \xrightarrow{\pi_\Upsilon} & \mathbf{C} \end{array} \qquad \begin{array}{ccc} \Sigma^\top & \xrightarrow{\pi_\Sigma} & \mathbf{C} \\ G \uparrow & = & \uparrow \text{Id}_{\mathbf{C}} \\ \Upsilon^\top & \xrightarrow{\pi_\Upsilon} & \mathbf{C} \end{array}$$

– the 2-cells σ_u and τ_v are functorial in u and v :

$$\sigma_{1_x} = 1_{1_x}, \quad \tau_{1_y} = 1_{1_y}, \quad \sigma_{uu'} = \sigma_u \sigma_{u'} \quad \text{and} \quad \tau_{vv'} = \tau_v \tau_{v'}.$$

Proof. Let us define F , the case of G being symmetric. On a 0-cell x , we take $F(x) = x$. If $a : x \rightarrow y$ is a 1-cell of Σ , we choose, in an arbitrary way, a 1-cell $F(a) : x \rightarrow y$ in Υ^\top such that $\pi_\Upsilon F(a) = \pi_\Sigma(a)$. Then, we extend F to every 1-cell of Σ^\top by functoriality. Let $\alpha : u \Rightarrow u'$ be a 2-cell of Σ . Since Σ is a presentation of \mathbf{C} , we have $\pi_\Sigma(u) = \pi_\Sigma(u')$, so that $\pi_\Upsilon F(u) = \pi_\Upsilon F(u')$ holds. Using the fact that Υ is a presentation of \mathbf{C} , we arbitrarily choose a 2-cell $F(\alpha) : F(u) \Rightarrow F(u')$ in Υ^\top . Then, we extend F to every 2-cell of Σ^\top by functoriality.

Now, let us define σ , the case of τ being symmetric. Let a be a 1-cell of Σ . By construction of F and G , we have:

$$\pi_\Sigma GF(a) = \pi_\Upsilon F(a) = \pi_\Sigma(a).$$

Since Σ is a presentation of \mathbf{C} , there exists a 2-cell $\sigma_a : GF(a) \Rightarrow a$ in Σ^\top . We extend σ to every 1-cell u of Σ^\top by functoriality. \square

4.2.2. Theorem. *Let \mathbf{C} be a category, let Σ and Υ be two presentations of \mathbf{C} and let F , G and τ be chosen as in Lemma 4.2.1. If Γ is a homotopy basis of Σ^\top , then*

$$\Delta = F(\Gamma) \sqcup \tau_\Upsilon$$

is a homotopy basis of Υ^\top , where:

– the cellular extension $F(\Gamma)$ contains one 3-cell

$$\begin{array}{ccc} & F(f) & \\ & \curvearrowright & \\ F(u) & & F(v) \\ & \Downarrow F(\gamma) & \\ & \curvearrowleft & \\ & F(g) & \end{array}$$

for every 3-cell $\gamma : f \Rightarrow g$ of Γ ,

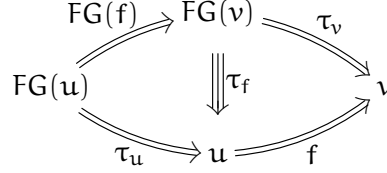
– the cellular extension τ_Υ contains one 3-cell

$$\begin{array}{ccccc} & FG(\alpha) & \rightarrow & FG(v) & \xrightarrow{\tau_v} & v \\ & \curvearrowright & & \Downarrow \tau_\alpha & & \\ FG(u) & & & & & \\ & \xrightarrow{\tau_u} & u & \xrightarrow{\alpha} & & \end{array}$$

for every 2-cell $\alpha : u \Rightarrow v$ of Υ .

4.2. The homotopy bases transfer theorem

Proof. Let us define, for every 2-cell f of Υ^\top , a 3-cell τ_f of Δ^\top with the following shape:



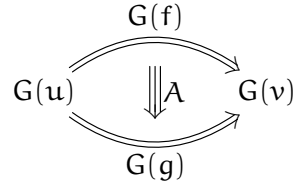
We extend the notation τ_α in a functorial way, according to the following formulas:

$$\begin{aligned} \tau_{1_u} &= 1_{\tau_u}, & \tau_{fg} &= \tau_f \tau_g, & \tau_{f^-} &= FG(f)^- \star_1 \tau_f^- \star_1 f^-, \\ \tau_{f \star_1 g} &= (FG(f) \star_1 \tau_g) \star_2 (\tau_f \star_1 g). \end{aligned}$$

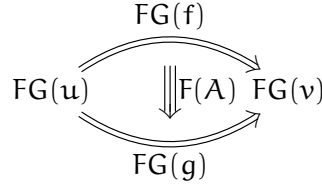
One checks that the 3-cells τ_f are well-defined, *i.e.*, that their definition is compatible with the relations on 2-cells, such as the exchange relation:

$$\tau_{fg \star_1 hk} = \tau_{(f \star_1 h)(g \star_1 k)}.$$

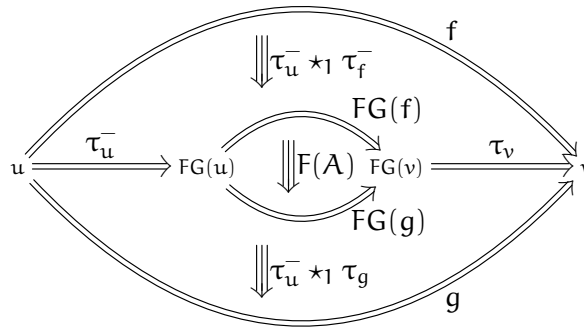
Now, let us consider parallel 2-cells $f, g : u \Rightarrow v$ of Υ^\top . The 2-cells $G(f)$ and $G(g)$ are parallel in Σ^\top so that, since Γ is a homotopy basis of Σ^\top , there exists a 3-cell



in Γ^\top . An application of F to A gives the 3-cell



which, by definition of Δ and functoriality of F , is in Δ^\top . Using the 3-cells $F(A)$, τ_f and τ_g , we get the following 3-cell from f to g in Δ^\top :



This concludes the proof that $\Delta = F(\Gamma) \amalg \tau_\Upsilon$ is a homotopy basis of the $(2, 1)$ -category Υ^\top . □

4. Finite derivation type

We deduce from Theorem 4.2.2 the following result.

4.2.3. Theorem ([SOK94, Theorem 4.3]). *Let Σ and Υ be finite presentations of the same category. Then Σ is of finite derivation type if, and only if, Υ is of finite derivation type.*

The following proposition is useful to prove that a presentation admits no finite homotopy basis.

4.2.4. Proposition. *Let Σ be a 2-polygraph and let Γ be a homotopy basis of Σ^\top . If Σ admits a finite homotopy basis, then there exists a finite subset of Γ that is a homotopy basis of Σ^\top .*

Proof. Let Δ be a finite homotopy basis of Σ and let δ be a 3-cell of Δ . Since Γ is a homotopy basis of Σ , there exists a 3-cell A_δ in Γ^\top with boundary $(s(\delta), t(\delta))$. This induces a 3-functor

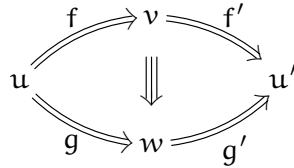
$$F : \Delta^\top \rightarrow \Gamma^\top$$

that is the identity on Σ and such that $F(\delta) = A_\delta$ for every 3-cell δ of Δ . Let Γ_Δ be the subset of Γ that contains all the generating 3-cells that appear in some 3-cell A_δ , for δ in Δ . Since Δ is finite and each A_δ contains finitely many 3-cells of Γ , we have that Γ_Δ is finite. Finally, let us consider a 2-sphere (f, g) of Σ^\top . By hypothesis, there exists a 3-cell A in Δ^\top with boundary (f, g) . By application of F , one gets a 3-cell $F(A)$ in Γ^\top whose boundary is (f, g) . Moreover, the 3-cell $F(A)$ is a composite of cells A_δ : hence, the 3-cell $F(A)$ is in Γ_Δ^\top . As a consequence, one gets $f \equiv_{\Gamma_\Delta} g$, so that Γ_Δ is a finite homotopy basis of Σ^\top , which concludes the proof. \square

4.3. Squier's completion for convergent presentations

Squier's completion provides a way to extend a convergent presentation of a category \mathbf{C} into a coherent presentation of \mathbf{C} . We fix a convergent 2-polygraph Σ .

4.3.1. Squier's completion. A family of generating confluences of Σ is a cellular extension of Σ^\top that contains exactly one 3-cell



for every critical branching (f, g) of Σ . We note that, if Σ is confluent, it always admits a family of generating confluences. However, such a family is not necessarily unique, since the 3-cell can be directed in the reverse way and, for a given branching (f, g) , we can have several possible 2-cells f' and g' with the required shape. Normalisation strategies provide a deterministic way to construct a family of generating confluences, see [GM12b, 4.3.2].

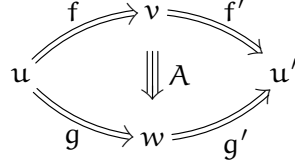
Squier's completion of Σ is the $(3, 1)$ -polygraph denoted by $\mathcal{S}(\Sigma)$ and defined by $\mathcal{S}(\Sigma) = (\Sigma, \Gamma)$, where Γ is a chosen family of generating confluences of Σ .

4.3.2. Theorem ([SOK94, Theorem 5.2]). *Let \mathbf{C} be a category and let Σ be a presentation of \mathbf{C} . Squier's completion $\mathcal{S}(\Sigma)$ of Σ is a coherent presentation of \mathbf{C} .*

4.3. Squier's completion for convergent presentations

Proof. We proceed in three steps.

Step 1. We prove that, for every local branching $(f, g) : u \Rightarrow (v, w)$ of Σ , there exist 2-cells $f' : v \Rightarrow u'$ and $g' : w \Rightarrow u'$ in Σ^* and a 3-cell $A : f \star_1 f' \Rightarrow g \star_1 g'$ in $\mathcal{S}(\Sigma)^\top$, as in the following diagram:

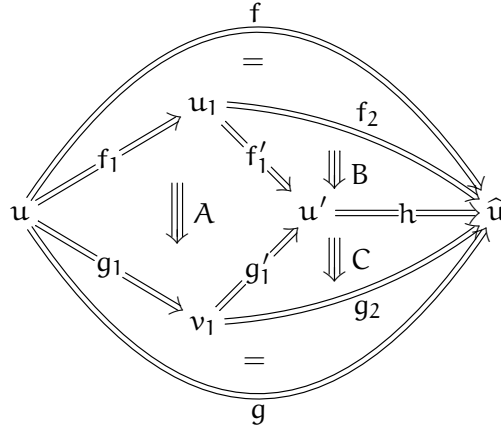


As we have seen in the study of confluence of local branchings, in the case of an aspherical or Peiffer branching, we can choose f' and g' such that $f \star_1 f' = g \star_1 g'$: an identity 3-cell is enough to link them. Moreover, if we have an overlapping branching (f, g) that is not critical, we have $(f, g) = (uhv, ukv)$ with (h, k) critical. We consider the 3-cell $\alpha : h \star_1 h' \Rightarrow k \star_1 k'$ of $\mathcal{S}(\Sigma)$ corresponding to the critical branching (h, k) and we conclude that the following 2-cells f' and g' and 3-cell A satisfy the required conditions:

$$f' = uh'v \quad g' = uk'v \quad A = u\alpha v.$$

Step 2. We prove that, for every parallel 2-cells f and g of Σ^* whose common target is a normal form, there exists a 3-cell from f to g in $\mathcal{S}(\Sigma)^\top$. We proceed by noetherian induction on the common source u of f and g , using the termination of Σ . Let us assume that u is a normal form: then, by definition, both 2-cells f and g must be equal to the identity of u , so that $1_u : 1_u \Rightarrow 1_u$ is a 3-cell of $\mathcal{S}(\Sigma)^\top$ from f to g .

Now, let us fix a 1-cell u with the following property: for any 1-cell v such that u rewrites into v and for any parallel 2-cells $f, g : v \Rightarrow \hat{v} = \hat{u}$ of Σ^* , there exists a 3-cell from f to g in $\mathcal{S}(\Sigma)^\top$. Let us consider parallel 2-cells $f, g : u \Rightarrow \hat{u}$ and let us prove the result by progressively constructing the following composite 3-cell from f to g in $\mathcal{S}(\Sigma)^\top$:



Since u is not a normal form, we can decompose $f = f_1 \star_1 f_2$ and $g = g_1 \star_1 g_2$ so that f_1 and g_1 are rewriting steps. They form a local branching (f_1, g_1) and we build the 2-cells f'_1 and g'_1 , together with the 3-cell A as in the first part of the proof. Then, we consider a 2-cell h from u' to \hat{u} in Σ^* , that must exist by confluence of Σ and since \hat{u} is a normal form. We apply the induction hypothesis to the parallel 2-cells f_2 and $f'_1 \star_1 h$ in order to get B and, symmetrically, to the parallel 2-cells $g'_1 \star_1 h$ and g_2 to get C .

4. Finite derivation type

Step 3. We prove that every 2-sphere of Σ^\top is the boundary of a 3-cell of $\mathcal{S}(\Sigma)^\top$. First, let us consider a 2-cell $f : u \Rightarrow v$ in Σ^* . Using the confluence of Σ , we choose 2-cells

$$\sigma_u : u \Rightarrow \hat{u} \quad \text{and} \quad \sigma_v : v \Rightarrow \hat{v} = \hat{u}$$

in Σ^* . By construction, the 2-cells $f \star_1 \sigma_v$ and σ_u are parallel and their common target \hat{u} is a normal form. Thus, there exists a 3-cell in $\mathcal{S}(\Sigma)^\top$ from $f \star_1 \sigma_v$ to σ_u or, equivalently, a 3-cell σ_f from f to $\sigma_u \star_1 \sigma_v^-$ in $\mathcal{S}(\Sigma)^\top$, as in the following diagram:

$$\begin{array}{ccc} & f & \\ & \curvearrowright & \\ u & & v \\ & \Downarrow \sigma_f & \\ & \curvearrowleft & \\ \sigma_u & \hat{u} & \sigma_v^- \end{array}$$

Moreover, the $(3, 1)$ -category $\mathcal{S}(\Sigma)^\top$ contains a 3-cell σ_{f^-} from f^- to $\sigma_v \star_1 \sigma_u^-$, given as the following composite:

$$\begin{array}{ccccccc} & & \hat{u} & & & & \\ & \sigma_u & \curvearrowright & \sigma_v^- & & & \\ v & \xrightarrow{f^-} & u & & v & \xrightarrow{\sigma_v} & \hat{u} & \xrightarrow{\sigma_u^-} & u \\ & & \Downarrow \sigma_{f^-} & & & & \\ & & \curvearrowleft & & & & \\ & & f & & & & \end{array}$$

Now, let us consider a general 2-cell $f : u \Rightarrow v$ of Σ^\top . By construction of Σ^\top , the 2-cell f can be decomposed (in general in a non-unique way) into a “zig-zag”

$$u \xrightarrow{f_1} v_1 \xrightarrow{g_1^-} u_2 \xrightarrow{f_2} (\dots) \xrightarrow{g_{n-1}^-} u_n \xrightarrow{f_n} v_n \xrightarrow{g_n^-} v$$

where each f_i and g_i is a 2-cell of Σ^* . We define σ_f as the following composite 3-cell of $\mathcal{S}(\Sigma)^\top$, with source f and target $\sigma_u \star_1 \sigma_v^-$:

$$\begin{array}{ccccccc} u & \xrightarrow{f_1} & v_1 & \xrightarrow{g_1^-} & (\dots) & \xrightarrow{f_n} & v_n & \xrightarrow{g_n^-} & v \\ & \Downarrow \sigma_{f_1} & \sigma_{v_1}^- & = & \sigma_{v_1} & \Downarrow \sigma_{g_1^-} & \sigma_{u_2}^- & = & \sigma_{u_n} & \Downarrow \sigma_{f_n} & \sigma_{v_n}^- & = & \sigma_{v_n} & \Downarrow \sigma_{g_n^-} & \sigma_v^- \\ \sigma_u & \hat{u} & \hat{u} & \hat{u} & (\dots) & \hat{u} & \hat{u} & \hat{u} & \hat{u} & \hat{u} & \hat{u} & \hat{u} & \hat{u} & \hat{u} & \hat{u} \end{array}$$

We proceed similarly for any other 2-cell $g : u \Rightarrow v$ of Σ^\top , to get a 3-cell σ_g from g to $\sigma_u \star_1 \sigma_v^-$ in $\mathcal{S}(\Sigma)^\top$. Thus, the composite $\sigma_f \star_2 \sigma_g^-$ is a 3-cell of $\mathcal{S}(\Sigma)^\top$ from f to g , concluding the proof. \square

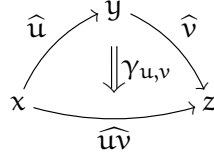
Theorem 4.3.2 is extended to higher-dimensional polygraphs in [GM09, Proposition 4.3.4]. In the special case of presentations of monoids, we recover the original result of Squier.

4.3.3. Theorem ([SOK94, Theorem 5.3]). *If a monoid admits a finite convergent presentation, then it is of finite derivation type.*

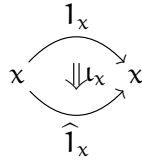
4.3. Squier's completion for convergent presentations

4.3.4. Example. The *standard presentation* of a category \mathbf{C} is the 2-polygraph denoted by $\text{Std}_2(\mathbf{C})$ and defined as follows:

- the 0-cells and 1-cells of $\text{Std}_2(\mathbf{C})$ are the ones of \mathbf{C} , a 1-cell u of \mathbf{C} being denoted by \hat{u} when seen as a generating 1-cell of $\text{Std}_2(\mathbf{C})$,
- for every 1-cells $u : x \rightarrow y$ and $v : y \rightarrow z$ of \mathbf{C} , one 2-cell

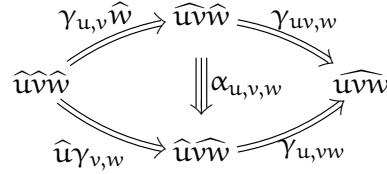


- for every 0-cell x of \mathbf{C} , one 2-cell

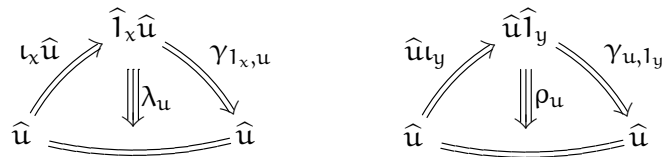


The *standard coherent presentation* of \mathbf{C} is the $(3, 1)$ -polygraph denoted by $\text{Std}_3(\mathbf{C})$ and obtained by extension of $\text{Std}_2(\mathbf{C})$ with the homotopy basis made of the following 3-cells:

- for every 1-cells $u : x \rightarrow y$, $v : y \rightarrow z$ and $w : z \rightarrow t$ of \mathbf{C} , one 3-cell



- for every 1-cell $u : x \rightarrow y$ of \mathbf{C} , two 3-cells



Let us prove that $\text{Std}_+(\mathbf{C})$ is, indeed, a coherent presentation of \mathbf{C} . The standard presentation $\text{Std}(\mathbf{C})$ is not terminating: indeed, for every 0-cell x of \mathbf{C} , the 2-cell l_x creates infinite rewriting sequences

$$l_x \Rightarrow \hat{l}_x \Rightarrow \hat{l}_x \hat{l}_x \Rightarrow \hat{l}_x \hat{l}_x \hat{l}_x \Rightarrow \dots$$

However, we get a convergent presentation of \mathbf{C} by reversing all the 2-cells l_x into l_x^- . Indeed, for termination, we consider the size of the 1-cells (the number of generators they contain) and we check that each 2-cell $\gamma_{u,v}$ has source of size 2 and target of size 1, while each 2-cell l_x^- has source of size 1 and target of size 0. As a consequence, for every non-identity 2-cell $f : u \Rightarrow v$ of the free 2-category, the size of u is strictly greater than the size of v . For confluence, we study the critical branchings, divided into three families:

4. Finite derivation type

- for every 1-cells $u : x \rightarrow y$, $v : y \rightarrow z$ and $w : z \rightarrow t$, one critical branching $(\gamma_{u,v}\widehat{w}, \widehat{u}\gamma_{v,w})$, giving the 3-cell

$$\begin{array}{ccc}
 & \xrightarrow{\gamma_{u,v}\widehat{w}} & \widehat{u}\widehat{v}\widehat{w} & \xrightarrow{\gamma_{uv,w}} & \widehat{u}\widehat{v}\widehat{w} \\
 \widehat{u}\widehat{v}\widehat{w} & & \Downarrow \gamma_{u,v,w} & & \widehat{u}\widehat{v}\widehat{w} \\
 & \xrightarrow{\widehat{u}\gamma_{v,w}} & \widehat{u}\widehat{v}\widehat{w} & \xrightarrow{\gamma_{u,vw}} & \widehat{u}\widehat{v}\widehat{w}
 \end{array}$$

- for every 1-cell $u : x \rightarrow y$ of \mathbf{C} , two critical branchings $(\gamma_{1_x,u}, \iota_x^- \widehat{u})$ and $(\gamma_{u,1_y}, \widehat{u}\iota_y^-)$, producing the 3-cells

$$\begin{array}{ccc}
 & \xrightarrow{\gamma_{1_x,u}} & \widehat{u} & \\
 \widehat{1}_x \widehat{u} & & \Downarrow \lambda_u & & \widehat{u} \\
 & \xrightarrow{\iota_x^- \widehat{u}} & \widehat{u} & & \widehat{u}
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{\gamma_{u,1_y}} & \widehat{u} & \\
 \widehat{u} \widehat{1}_y & & \Downarrow \rho_u & & \widehat{u} \\
 & \xrightarrow{\widehat{u}\iota_y^-} & \widehat{u} & & \widehat{u}
 \end{array}$$

Since considering the 2-cells ι_x or ι_x^- as generators does not change the generated $(2, 1)$ -category, we get that those three families of 3-cells form a homotopy basis for $\text{Std}_2(\mathbf{C})$. We replace λ_u by $\iota_x \widehat{u} \star_1 \lambda_u$ and ρ_u by $\widehat{u}\iota_y \star_1 \rho_u$ to get the result.

One can reduce $\text{Std}_3(\mathbf{C})$ into the smaller *reduced standard coherent presentation* $\text{Std}'_3(\mathbf{C})$ of \mathbf{C} . It is obtained from $\text{Std}_2(\mathbf{C})$ by coherently removing all the cells about units:

- the 3-cells $\gamma_{1_x,u,v}$, $\gamma_{u,1_y,v}$ and $\gamma_{u,v,1_z}$, since they are parallel to composites of λ s and ρ s,
- the 2-cells $\gamma_{1_x,u}$ and the 3-cells λ_u ,
- the 2-cells $\gamma_{u,1_x}$ and the 3-cells ρ_u ,
- the 1-cells $\widehat{1}_x$ and the 2-cells ι_x .

The resulting coherent presentation is detailed in [GM12b, 4.1.6].

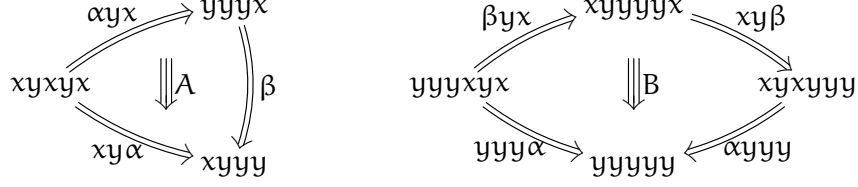
4.3.5. Example. Let us consider the monoid \mathbf{M} presented by the 2-polygraph

$$\Sigma = \langle x, y \mid xyx \xrightarrow{\alpha} yy \rangle.$$

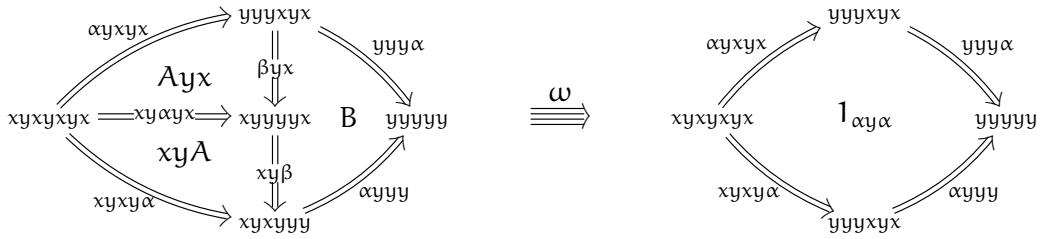
We prove that Σ terminates with the deglex order generated by $x < y$. The 2-polygraph Σ has one, non confluent critical branching $(\alpha y x, x y \alpha)$. Knuth-Bendix's completion $\check{\Sigma}$ of Σ is obtained by adjunction of the following 2-cell $\beta : yy y x \Rightarrow x y y y$:

$$\begin{array}{ccc}
 & \xrightarrow{\alpha y x} & yy y x \\
 xyxyx & & \Downarrow \beta \\
 & \xrightarrow{xy \alpha} & xy y y
 \end{array}$$

Then, Squier's completion $\mathcal{S}(\check{\Sigma})$ extends $\check{\Sigma}$ with the following two 3-cells:



In fact, the 3-cell A is sufficient to get a homotopy basis of $\check{\Sigma}$, as witnessed by the following 3-sphere of the $(3, 1)$ -category $\mathcal{S}(\check{\Sigma})^\top$:



Indeed, the 3-sphere ω proves that the boundary of B is also the boundary of a 3-cell of the $(3, 1)$ -category $(\mathcal{S}(\check{\Sigma}) \setminus \{A\})^\top$. This elimination mechanism, based on the study of the triple critical branchings of $\check{\Sigma}$ is part of the *homotopical reduction procedure* introduced in [GGM13]. This construction coherently eliminates pairs of redundant cells of a coherent presentation. On this particular example, it yields that the $(2, 1)$ -category Σ^\top admits an empty homotopy basis, *i.e.*, that the $(3, 1)$ -polygraph (Σ, \emptyset) is a coherent presentation of the monoid \mathbf{M} .

4.3.6. Polygraphic resolutions from convergent presentations. Polygraphic resolutions have been introduced in [Mét03] as a natural generalisation of coherent presentations in every dimension. In [GM12b, Theorem 4.5.3], Squier's completion is extended to higher dimensions to compute, from a reduced convergent presentation Σ of a category \mathbf{C} , a polygraphic resolution of \mathbf{C} whose n -cells are indexed by the critical $(n - 1)$ -fold branchings of Σ (the minimal nontrivial overlaps of $n - 1$ copies of 2-cells of Σ). In particular, the critical triple branchings induce a homotopy basis of Squier's completion $\mathcal{S}(\Sigma)$: all the parallel 3-cells of $\mathcal{S}(\Sigma)^\top$ are equal up to the 4-cells generated by the triple critical branchings [GM12b, Proposition 4.4.4]. This implies the following result:

4.3.7. Proposition. *Let Σ be a reduced convergent 2-polygraph with no triple critical branching. Then all the parallel 3-cells of the free $(3, 1)$ -category $\mathcal{S}(\Sigma)^\top$ are equal.*

5. A HOMOLOGICAL FINITENESS CONDITION

5.1. Monoids of finite homological type

5.1.1. Resolutions. Let \mathbf{M} be a monoid. We denote by $\mathbb{Z}\mathbf{M}$ the ring generated by \mathbf{M} , that is, the free abelian group over \mathbf{M} , equipped with the canonical extension of the product of \mathbf{M} :

$$\left(\sum_{u \in \mathbf{M}} \lambda_u u\right) \left(\sum_{v \in \mathbf{M}} \lambda_v v\right) = \sum_{u, v \in \mathbf{M}} \lambda_u \lambda_v uv = \sum_{w \in \mathbf{M}} \sum_{uv=w} \lambda_u \lambda_v w.$$

5. A homological finiteness condition

Given a (left) $\mathbb{Z}\mathbf{M}$ -module M , a *resolution* of M is an exact sequence of $\mathbb{Z}\mathbf{M}$ -modules

$$(\dots) \longrightarrow M_n \xrightarrow{d_n} M_{n-1} \longrightarrow (\dots) \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M \longrightarrow 0$$

that is, a sequence $(M_n)_{n \in \mathbb{N}}$ of $\mathbb{Z}\mathbf{M}$ -modules, together with a sequence $(d_n)_{n \in \mathbb{N}}$ of morphisms of $\mathbb{Z}\mathbf{M}$ -modules, called the *boundary maps*, such that d_0 is surjective and

$$\text{Im } d_{n+1} = \text{Ker } d_n$$

holds for every natural number n . In particular, the sequence $(M_n, d_n)_{n \in \mathbb{N}}$ is a (*chain*) *complex of $\mathbb{Z}\mathbf{M}$ -modules*, that is, we have the inclusion $\text{Im } d_{n+1} \subseteq \text{Ker } d_n$ or, equivalently, the relation $d_n d_{n+1} = 0$ holds for every natural number n . Such a resolution is called *projective* (resp. *free*) if all the modules M_n are projective (resp. free).

Given a natural number n , a *partial resolution of length n of M* is defined in a similar way but with a bounded sequence $(M_k)_{0 \leq k \leq n}$ of $\mathbb{Z}\mathbf{M}$ -modules.

5.1.2. Contracting homotopies. Given a complex of $\mathbb{Z}\mathbf{M}$ -modules

$$(\dots) \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow (\dots) \longrightarrow M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M \longrightarrow 0$$

a method to prove that such a complex is a resolution of M is to construct a *contracting homotopy*, that is a sequence of morphisms of \mathbb{Z} -modules

$$(\dots) \longleftarrow M_{n+1} \xleftarrow{s_{n+1}} M_n \xleftarrow{s_n} M_{n-1} \longleftarrow (\dots) \longleftarrow M_1 \xleftarrow{s_1} M_0 \xleftarrow{s_0} M$$

such that $d_0 s_0 = \text{Id}_M$ and, for every $n \geq 1$, we have

$$d_{n+1} s_{n+1} + s_n d_n = \text{Id}_{M_n}.$$

Indeed, in that case, we have that d_0 is surjective. Moreover, for every natural number n and every x in $\text{Ker } d_n$, the equality $d_{n+1} s_{n+1}(x) = x$ holds, proving that x is in $\text{Im } d_{n+1}$, so that $\text{Ker } d_n \subseteq \text{Im } d_{n+1}$ holds. As a consequence, the considered complex is a resolution of M .

5.1.3. Homological type left-FP $_n$. If \mathbf{M} is a monoid, the *trivial $\mathbb{Z}\mathbf{M}$ -module* is the abelian group \mathbb{Z} equipped with the trivial action $un = n$, for every u in \mathbf{M} and n in \mathbb{Z} . A monoid \mathbf{M} is of *homological type left-FP $_n$* , for a natural number n , if there exists a partial resolution of length n of the trivial $\mathbb{Z}\mathbf{M}$ -module \mathbb{Z} by projective, finitely generated $\mathbb{Z}\mathbf{M}$ -modules:

$$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} (\dots) \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0.$$

A monoid \mathbf{M} is of *homological type left-FP $_\infty$* if there exists a resolution of \mathbb{Z} by projective, finitely generated $\mathbb{Z}\mathbf{M}$ -modules.

5.1.4. Lemma. *Let \mathbf{M} be a monoid and let n be a natural number. The following assertions are equivalent:*

- i) The monoid \mathbf{M} is of homological type left-FP_n.
- ii) There exists a free, finitely generated partial resolution of the trivial $\mathbb{Z}\mathbf{M}$ -module \mathbb{Z} of length n

$$F_n \longrightarrow F_{n-1} \longrightarrow (\dots) \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

- iii) For every $0 \leq k < n$ and every projective, finitely generated partial resolution of the trivial $\mathbb{Z}\mathbf{M}$ -module \mathbb{Z} of length k

$$P_k \xrightarrow{d_k} P_{k-1} \xrightarrow{d_{k-1}} (\dots) \longrightarrow P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0,$$

the $\mathbb{Z}\mathbf{M}$ -module $\text{Ker } d_k$ is finitely generated.

This lemma is a consequence of the following generalisation of Schanuel's lemma. Let us consider two exact sequences

$$0 \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow (\dots) \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow P'_{n+1} \longrightarrow P'_n \longrightarrow (\dots) \longrightarrow P'_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

of projective $\mathbb{Z}\mathbf{M}$ -modules, with P_i and P'_i finitely generated for every $0 \leq i \leq n$. Then the $\mathbb{Z}\mathbf{M}$ -module P_{n+1} is finitely generated if, and only if, the $\mathbb{Z}\mathbf{M}$ -module P'_{n+1} is finitely generated.

5.2. Monoids of homological type left-FP₂

5.2.1. Presentations and partial resolutions of length 2. Let \mathbf{M} be a monoid and let Σ be a presentation of \mathbf{M} . Let us define a partial resolution of length 2 of \mathbb{Z} by free $\mathbb{Z}\mathbf{M}$ -modules

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

The $\mathbb{Z}\mathbf{M}$ -modules $\mathbb{Z}\mathbf{M}[\Sigma_1]$ and $\mathbb{Z}\mathbf{M}[\Sigma_2]$ are the free $\mathbb{Z}\mathbf{M}$ -modules over Σ_1 and Σ_2 , respectively: they contain the formal sums of elements denoted by $u[x]$, where u is an element of \mathbf{M} and x is a 1-cell of Σ_1 or a 2-cell of Σ_2 . Let us note that $\mathbb{Z}\mathbf{M}$ is isomorphic to the free $\mathbb{Z}\mathbf{M}$ -module over the singleton Σ_0 . The boundary maps are defined, on generators as follows:

$$\begin{array}{ccc} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} & \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} & \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \\ u \mapsto 1 & [x] \mapsto \bar{x} - 1 & [\alpha] \mapsto [s(\alpha)] - [t(\alpha)] \end{array}$$

The maps ε and d_2 are respectively called the *augmentation map* and the *Reidemester-Fox Jacobian* of Σ . In the definition of d_2 , the bracket $[\cdot]$ is extended to the 1-cells of Σ_1^* thanks to the relation:

$$[uv] = [u] + \bar{u}[v],$$

for any 1-cells u and v in Σ_1 . In particular, this relation implies that $[1] = 0$ holds.

5. A homological finiteness condition

5.2.2. Proposition. *Let \mathbf{M} be a monoid and let Σ be a presentation of \mathbf{M} . The sequence of $\mathbb{Z}\mathbf{M}$ -modules*

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a partial free resolution of length 2 of \mathbb{Z} .

Proof. We first note that the sequence is a chain complex. Indeed, the augmentation map is surjective by definition. Moreover, we have

$$\varepsilon d_1[x] = \varepsilon(\bar{x}) - \varepsilon(1) = 1 - 1 = 0,$$

for every 1-cell x of Σ_1 . In order to check that $d_1 d_2 = 0$, we first prove, by induction on the length, that we have $d_1[u] = \bar{u} - 1$ for every 1-cell u of Σ_1^* . For the unit, we have $d_1[1] = d_1(0) = 0$ and $\bar{1} - 1 = 0$. Then, for a composite 1-cell uv such that the result holds for both u and v , we get

$$d_1[uv] = d_1[u] + \bar{u}d_1[v] = \bar{u} - 1 + \bar{u}\bar{v} - \bar{u} = \overline{uv} - 1.$$

As a consequence, we have

$$d_1 d_2[\alpha] = d_1[s(\alpha)] - d_1[t(\alpha)] = \overline{s(\alpha)} - \overline{t(\alpha)} = 0,$$

for every 2-cell α of Σ_2 , where the last equality comes from the fact that $\overline{s(\alpha)} = \overline{t(\alpha)}$ since Σ is a presentation of \mathbf{M} .

The rest of the proof consists in defining contracting homotopies s_0, s_1, s_2 :

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \xleftarrow[s_2]{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xleftarrow[s_1]{d_1} \mathbb{Z}\mathbf{M} \xleftarrow[s_0]{\varepsilon} \mathbb{Z}$$

We choose a representative \hat{u} in Σ_1^* for every element u of \mathbf{M} and we fix a corresponding normalisation strategy σ . Then we define the morphisms of abelian groups s_0, s_1 and s_2 by their values on generic elements

$$s_0(1) = 1, \quad s_1(u) = [\hat{u}], \quad s_2(u[x]) = [\sigma(\hat{u}x)],$$

where the bracket $[\cdot]$ is extended to every 2-cell of Σ^\top thanks to the relations

$$[ufv] = \bar{u}[f] \quad \text{and} \quad [f \star_1 g] = [f] + [g],$$

for any 1-cells u and v and 2-cells f and g such that the composites ufv and $f \star_1 g$ are defined. For every u in \mathbf{M} , we have $s_0\varepsilon(u) = 1$ and

$$d_1 s_1(u) = d_1[\hat{u}] = u - 1.$$

Thus $d_1 s_1 + s_0 \varepsilon = \text{Id}_{\mathbb{Z}\mathbf{M}}$. Then we have, on the one hand,

$$s_1 d_1(u[x]) = s_1(u\bar{x} - u) = [\hat{u}\bar{x}] - [\hat{u}]$$

and, on the other hand,

$$d_2 s_2(u[x]) = d_2[\sigma(\hat{u}x)] = [\hat{u}x] - [\hat{u}\bar{x}] = u[x] + [\hat{u}] - [\hat{u}\bar{x}].$$

For this equality, we check that $d_2[f] = [s(f)] - [t(f)]$ holds for every 2-cell f of Σ^\top by induction on the size of f . Hence $d_2 s_2 + s_1 d_1 = \text{Id}_{\mathbb{Z}\mathbf{M}[\Sigma_1]}$, thus concluding the proof. \square

From Proposition 5.2.2, we deduce the following result:

5.2.3. Theorem. *The following properties hold.*

- i) *Every monoid is of homological type left-FP₀.*
- ii) *Every finitely generated monoid is of homological type left-FP₁.*
- iii) *Every finitely presented monoid is of homological type left-FP₂.*

5.2.4. Remark. The property left-FP₂ implies left-FP₁, yet the converse implication fails to hold. Indeed, let us consider the monoid \mathbf{M} presented by the following 2-polygraph:

$$\Sigma = \langle a, c, t \mid at^{n+1} \xrightarrow{\alpha_n} ct^n, n \in \mathbb{N} \rangle.$$

The monoid \mathbf{M} is finitely generated and, thus, it is of homological type left-FP₁. However, for every natural number n , we have

$$\begin{aligned} d_2[\alpha_{n+1}] &= [at^{n+2}] - [ct^{n+1}], \\ &= [at^{n+1}] + \overline{at^{n+1}}[t] - [ct^n] - \overline{ct^n}[t], \\ &= d_2[\alpha_n] + (\overline{at^{n+1}} - \overline{ct^n})[t]. \end{aligned}$$

The equality $\overline{at^{n+1}} = \overline{ct^n}$ holds in \mathbf{M} by definition, yielding $d_2[\alpha_{n+1}] = d_2[\alpha_n]$. As a consequence, the $\mathbb{Z}\mathbf{M}$ -module $\text{Ker } d_2$ is generated by the elements $[\alpha_{n+1}] - [\alpha_n]$: it is not finitely generated, proving that \mathbf{M} is not of homological type left-FP₂. As a consequence, the monoid \mathbf{M} does not admit a presentation by a finite 2-polygraph.

5.3. Squier's homological theorem

5.3.1. Coherent presentations and partial resolutions of length 3. Let \mathbf{M} be a monoid and let Σ be a coherent presentation of \mathbf{M} . Let us extend the partial resolution of 5.2.2 into the resolution of length 3

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where the $\mathbb{Z}\mathbf{M}$ -module $\mathbb{Z}\mathbf{M}[\Sigma_3]$ is the free $\mathbb{Z}\mathbf{M}$ -module over Σ_3 , formed by the linear combination of elements $u[\gamma]$, with u is in \mathbf{M} and γ is a 3-cell of Σ . The boundary map d_3 is defined, for every 3-cell γ of Σ , by

$$d_3[\gamma] = [s_2(\gamma)] - [t_2(\gamma)].$$

The bracket $[\cdot]$ is extended to 3-cells of Σ^\top thanks to the relations

$$[uAv] = \bar{u}[A], \quad [A \star_1 B] = [A] + [B], \quad [A \star_2 B] = [A] + [B],$$

for any 1-cells u and v and 3-cells A and B such that the composites are defined. We check, by induction on the size, that $d_3[A] = [s_2(A)] - [t_2(A)]$ holds for every 3-cell A of Σ^\top .

5. A homological finiteness condition

5.3.2. Proposition. *Let \mathbf{M} be a monoid and let Σ be a coherent presentation of \mathbf{M} . The sequence of $\mathbb{Z}\mathbf{M}$ -modules*

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a partial free resolution of length 3 of \mathbb{Z} .

Proof. First, we have $d_2 d_3 = 0$ because $s_1 s_2 = s_1 t_2$ and $t_1 s_2 = t_1 t_2$. Then, we define the following morphism of groups

$$\begin{array}{ccc} \mathbb{Z}\mathbf{M}[\Sigma_2] & \xrightarrow{s_3} & \mathbb{Z}\mathbf{M}[\Sigma_3] \\ u[\alpha] & \mapsto & [\sigma(\hat{u}\alpha)] \end{array}$$

where $\sigma(\hat{u}\alpha)$ is a 3-cell of Σ^\top with the following shape, with $v = s(\alpha)$ and $w = t(\alpha)$:

$$\begin{array}{ccc} & \hat{u}w & \\ \hat{u}\alpha \nearrow & & \searrow \sigma(\hat{u}w) \\ & \Downarrow \sigma(\hat{u}\alpha) & \\ \hat{u}v \xrightarrow{\sigma(\hat{u}v)} & & \hat{u}v \end{array}$$

Let us note that such a 3-cell necessarily exists in Σ^\top because Σ_3 is a homotopy basis of Σ^\top . Then we have, on the one hand,

$$s_2 d_2(u[\alpha]) = s_2(u[v] - u[w]) = [\sigma(\hat{u}v)] - [\sigma(\hat{u}w)]$$

and, on the other hand,

$$\begin{aligned} d_3 s_3(u[\alpha]) &= [\hat{u}\alpha \star_1 \sigma(\hat{u}w)] - [\sigma(\hat{u}v)], \\ &= u[\alpha] + [\sigma(\hat{u}w)] - [\sigma(\hat{u}v)]. \end{aligned}$$

Hence $d_3 s_3 + s_2 d_3 = \text{Id}_{\mathbb{Z}\mathbf{M}[\Sigma_2]}$, concluding the proof. \square

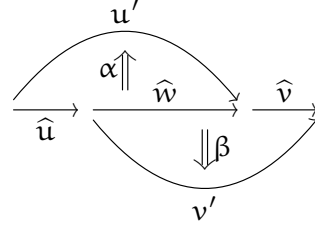
5.3.3. Remark. The proof of Proposition 5.3.2 uses the fact that Σ_3 is a homotopy basis to produce, for every 2-cell α of Σ and every u of \mathbf{M} , a 3-cell $\sigma(\hat{u}\alpha)$ with the required shape. The hypothesis on Σ_3 could thus be modified to only require the existence of such a 3-cell in Σ^\top : however, it is proved in [GM12b] that this implies that Σ_3 is a homotopy basis.

From Proposition 5.3.2, we deduce the following result.

5.3.4. Theorem ([CO94, Theorem 3.2], [Laf95, Theorem 3], [Pri95]). *Let \mathbf{M} be a finitely presented monoid. If \mathbf{M} is of finite derivation type, then it is of homological type left-FP₃.*

5.3.5. The case of a convergent presentation. Let Σ be a reduced convergent 2-polygraph equipped with its leftmost normalisation strategy σ . Since Σ is reduced, every critical branching of Σ has the

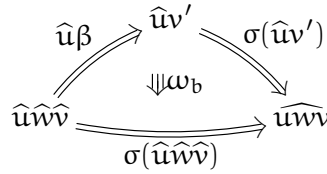
following form



where α and β are 2-cells of Σ and where \hat{u} , \hat{w} and \hat{v} are non-identity normal forms. Let us note that $\alpha\hat{v}$ is the leftmost reduction step of $\hat{u}\hat{w}\hat{v}$ and that $\hat{u}\beta$ is its rightmost reduction step. In particular, we have

$$\sigma(\hat{u}\hat{w}\hat{v}) = \alpha\hat{v} \star_1 \sigma(\hat{u}\hat{v}).$$

We define Σ_3 as the cellular extension of Σ^\top made of one 3-cell with the following shape, for every critical branching $b = (\alpha\hat{v}, \hat{u}\beta)$ of Σ :



Since Σ_3 is a family of generating confluences of Σ , it is a homotopy basis of Σ^\top , so that, by Proposition 5.3.2, the following sequence is a partial free resolution of length 3 of \mathbb{Z} :

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

In particular, if Σ is finite, it has a finite number of critical branchings, so that the monoid \mathbf{M} is of homological type left-FP₃.

5.3.6. Theorem ([Squ87, Theorem 4.1]). *If a monoid admits a finite convergent presentation, then it is of homological type left-FP₃.*

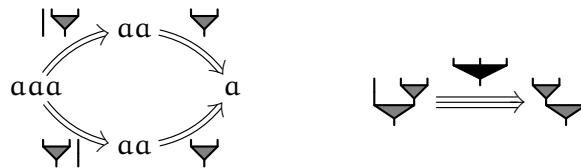
5.3.7. Example. Let us consider the monoid \mathbf{M} with the following convergent presentation:

$$\langle 1, a \mid aa \xrightarrow{\mu} a \rangle.$$

We use “string diagrams” to write ∇ the 2-cell $aa \xrightarrow{\mu} a$. With the leftmost normalisation strategy σ , we get:

$$\sigma(a) = 1_a \quad \sigma(aa) = \nabla \quad \sigma(aaa) = \mu a \star_1 \mu = \nabla \nabla.$$

The presentation has exactly one critical branching, whose corresponding generating confluence can be written in the following two equivalent ways:



5. A homological finiteness condition

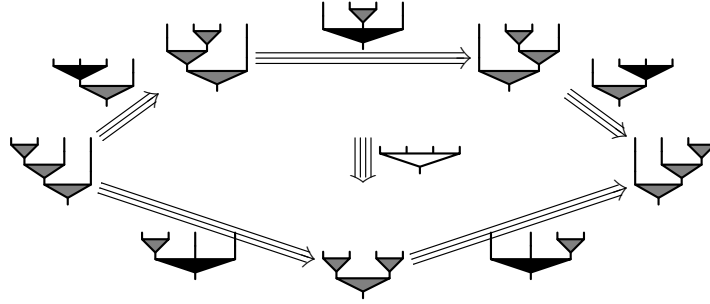
The $\mathbb{Z}\mathbf{M}$ -module $\text{Ker } d_2$ is generated by:

$$\begin{aligned} d_3[\text{triple}] &= [\text{triple}] - [\text{triple}] \\ &= [\text{triple}] + [\text{triple}] - [\text{triple}] - [\text{triple}] \\ &= a[\text{triple}] - [\text{triple}]. \end{aligned}$$

5.3.8. Resolutions from convergent presentations. In [GM12b], the results presented here are generalised to produce a free resolution

$$(\dots) \xrightarrow{d_{n+1}} \mathbb{Z}\mathbf{M}[\Sigma_n] \xrightarrow{d_n} \mathbb{Z}\mathbf{M}[\Sigma_{n-1}] \xrightarrow{d_{n-1}} (\dots) \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M}[\Sigma_0] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of the trivial $\mathbb{Z}\mathbf{M}$ -module \mathbb{Z} from a convergent presentation of a monoid \mathbf{M} . For $k \geq 4$, the $\mathbb{Z}\mathbf{M}$ -module $\mathbb{Z}\mathbf{M}[\Sigma_k]$ is defined as the free $\mathbb{Z}\mathbf{M}$ -module over a family Σ_k of k -cells obtained from the $(k-1)$ -fold critical branchings. For example, the 4-cell



is the only element of Σ_4 in the case of the monoid of Example 5.3.7.

5.3.9. Other homological finiteness conditions. In the definition 5.1.3 of homological type left- FP_n for a monoid \mathbf{M} , the replacement of left-modules by right-modules, bimodules or natural systems gives the definitions of the homological types right- FP_n , bi- FP_n and FP_n , for every $0 \leq n \leq \infty$. We refer the reader to [GM12b, Section 5.2] for the relations between these different finiteness conditions. In particular, for $n = 3$, all of these homotopical conditions are consequences of the finite derivation type property. The proof is similar to the one for left- FP_3 in Section 5.3. For example, in the case of right- FP_3 , we consider right-modules and, to get the contracting homotopy, we construct a left normalisation strategy σ by defining a 3-cell $\sigma(\alpha\hat{u})$ with shape

$$\begin{array}{ccc} & \alpha\hat{u} & \\ & \nearrow & \\ \hat{v}\hat{u} & & \hat{w}\hat{u} \\ & \searrow & \\ & \sigma(\alpha\hat{u}) & \\ & \Downarrow & \\ \hat{v}\hat{u} & & \hat{v}\hat{u} \\ & \nearrow & \\ & \sigma(\hat{v}\hat{u}) & \end{array}$$

for any generating 2-cell $\alpha : v \Rightarrow w$ and u in the monoid.

6. SQUIER'S EXAMPLE AND VARIANT

6.1. Squier's example

In [Squ87], Squier defines, for every $k \geq 1$, the monoid \mathbf{S}_k presented by

$$\Sigma^{\mathbf{S}_k} = \langle a, b, t, x_1, \dots, x_k, y_1, \dots, y_k \mid (\alpha_n)_{n \in \mathbb{N}}, (\beta_i)_{1 \leq i \leq k}, (\gamma_i)_{1 \leq i \leq k}, (\delta_i)_{1 \leq i \leq k}, (\varepsilon_i)_{1 \leq i \leq k} \rangle$$

with

$$at^n b \xrightarrow{\alpha_n} 1, \quad x_i a \xrightarrow{\beta_i} atx_i, \quad x_i t \xrightarrow{\gamma_i} tx_i, \quad x_i b \xrightarrow{\delta_i} bx_i, \quad x_i y_i \xrightarrow{\varepsilon_i} 1.$$

In [SOK94], Squier proves the following properties for \mathbf{S}_1 . With similar arguments, the result extends to every monoid \mathbf{S}_k , for $k \geq 1$.

6.1.1. Theorem ([SOK94, Theorem 6.7, Corollary 6.8]). *For every $k \geq 1$, the monoid \mathbf{S}_k satisfies the following properties:*

- i) *it is finitely presented,*
- ii) *it has a decidable word problem,*
- iii) *it is not of finite derivation type,*
- iv) *it admits no finite convergent presentation.*

In [Squ87], Squier had already proved that the monoids \mathbf{S}_k satisfy the following homological properties.

6.1.2. Proposition ([Squ87, Example 4.5.]). *For $k \geq 2$, the monoid \mathbf{S}_k is not of finite homological type left-FP₃ and, as a consequence, it does not admit a finite convergent presentation.*

Proposition 6.1.2 does not hold for \mathbf{S}_1 . Indeed, this monoid is of homological type left-FP_∞, [Squ87, Example 4.5.]. This proves that, if finite derivation type implies left-FP₃, the reverse implication does not hold for general monoids. However, in the special case of groups, the property of having finite derivation type is equivalent to the homological finiteness condition left-FP₃ [CO96]. The latter result is based on the Brown-Huebschmann isomorphism between homotopical and homological syzygies [BH82].

6.2. Proof of Theorem 6.1.1

Let us prove the result in the case of the monoid \mathbf{S}_1 , with the following infinite presentation:

$$\Sigma^{\mathbf{S}_1} = \langle a, b, t, x, y \mid (\alpha_n)_{n \in \mathbb{N}}, \beta, \gamma, \delta, \varepsilon \rangle$$

with

$$at^n b \xrightarrow{\alpha_n} 1, \quad xa \xrightarrow{\beta} atx, \quad xt \xrightarrow{\gamma} tx, \quad xb \xrightarrow{\delta} bx, \quad xy \xrightarrow{\varepsilon} 1.$$

In what follows, we denote by $\gamma_n : xt^n \Rightarrow t^n x$ the 2-cell of $(\Sigma^{\mathbf{S}_1})^*$ defined by induction on n as follows:

$$\gamma_0 = 1_x \quad \text{and} \quad \gamma_{n+1} = \gamma t^n \star_1 t \gamma_n.$$

6. Squier's example and variant

For every n , we write $f_n : \chi a t^b \Rightarrow a t^{n+1} b \chi$ the 2-cell of $(\Sigma^{Sq_1})^*$ defined as the following composite:

$$\chi a t^n b \xrightarrow{\beta t^n b} a t x t^n b \xrightarrow{a t \gamma_n b} a t^{n+1} \chi b \xrightarrow{a t^{n+1} \delta} a t^{n+1} b \chi.$$

We note that f_n contains no 2-cell α_k .

6.2.1. Proposition. *The monoid \mathbf{S}_1 admits the following finite presentation:*

$$\Upsilon = \langle a, b, t, \chi, y \mid \alpha_0, \beta, \gamma, \delta, \varepsilon \rangle.$$

Proof. For every natural number n , we consider the following 2-sphere of $(\Sigma^{Sq_1})^\top$:

$$\begin{array}{ccccc} & & a t^{n+1} b \chi y & \xrightarrow{a t^{n+1} b \varepsilon} & a t^{n+1} b & & \\ & f_n y \nearrow & & & & \searrow \alpha_{n+1} & \\ \chi a t^n b y & & & & & & 1 \\ & \searrow \chi \alpha_n y & & & & \nearrow \varepsilon & \\ & & \chi y & & & & \end{array} \quad (4)$$

Thus, the 2-cell α_{n+1} is parallel to the composite 2-cell

$$a t^{n+1} b \varepsilon^- \star_1 f_n^- y \star_1 \chi \alpha_n y \star_1 \varepsilon. \quad (5)$$

Since f_n contains no α_k , this proves by induction on n that the monoid \mathbf{S}_1 admits Υ as a presentation. \square

6.2.2. Proposition. *The 2-polygraph Σ^{Sq_1} is convergent and Squier's completion of Σ^{Sq_1} contains a 3-cell A_n for every natural number n with the following shape:*

$$\begin{array}{ccc} & a t^{n+1} b \chi & \\ f_n \nearrow & & \searrow \alpha_{n+1} \chi \\ \chi a t^n b & & \chi \\ \chi \alpha_n \searrow & & \nearrow \end{array} \quad \Downarrow A_n$$

Proof. Let us prove that Σ^{Sq_1} terminates. For that, we build a termination order based on derivations, similar to the method of [GM09, Theorem 4.2.1] for 3-polygraphs. We associate, to every 1-cell u of $(\Sigma^{Sq_1})^*$, two maps

$$u_* : \mathbb{N} \rightarrow \mathbb{N} \quad \text{and} \quad \partial(u) : \mathbb{N} \rightarrow \mathbb{N}$$

as follows. First, we define them on the 1-cells of Σ^{Sq_1} :

$$x_*(n) = n + 1, \quad a_*(n) = b_*(n) = t_*(n) = y_*(n) = n,$$

$$\partial(a)(n) = 3^n, \quad \partial(b)(n) = \partial(y)(n) = 2^n, \quad \partial(t)(n) = \partial(x)(n) = 0.$$

Then, we extend the mappings to every 1-cell of $(\Sigma^{Sq_1})^*$ thanks to the following relations:

$$1_*(n) = n, \quad (uv)_*(n) = v_*(u_*(n)), \quad \partial(1)(n) = 0, \quad \partial(uv)(n) = \partial(u)(n) + \partial(v)(u_*(n)).$$

We compare parallel 1-cells of $(\Sigma^{\text{Sq}_1})^*$ by the order generated by $u < v$ if $u_* \leq v_*$ and $\partial(u) < \partial(v)$. The defining relations of $(\cdot)_*$ and ∂ imply that the composition of 1-cells of $(\Sigma^{\text{Sq}_1})^*$ is strictly monotone in both arguments. The natural order on \mathbb{N} implies that every decreasing family of parallel 1-cells of $(\Sigma^{\text{Sq}_1})^*$ is stationary. To get a termination order, hence the termination of Σ^{Sq_1} , there remains to check that $u > v$ for every 2-cell $u \Rightarrow v$ of Σ^{Sq_1} . Indeed, we check that the following (in)equalities are satisfied:

$$\begin{aligned} (\text{at}^k\text{b})_*(n) &= n = 1_*(n), & (\text{x}\alpha)_*(n) &= n + 1 = (\text{atx})_*(n), & (\text{x}\text{t})_*(n) &= n + 1 = (\text{t}\text{x})_*(n), \\ (\text{x}\text{b})_*(n) &= n + 1 = (\text{b}\text{x})_*(n), & (\text{x}\text{y})_*(n) &= n + 1 > n = 1_*(n), \end{aligned}$$

and

$$\begin{aligned} \partial(\text{at}^k\text{b})(n) &= 3^n + 2^n > 0 = \partial(1)(n), & \partial(\text{x}\alpha)(n) &= 3^{n+1} > 2^n + 3^n = \partial(\text{atx})(n), \\ \partial(\text{x}\text{t})(n) &= 2^{n+1} > 2^n = \partial(\text{t}\text{x})(n), & \partial(\text{x}\text{b})(n) &= 2^{n+1} > 2^n = \partial(\text{b}\text{x})(n), \\ \partial(\text{x}\text{y})(n) &= 2^{n+1} > 0 = \partial(1)(n). \end{aligned}$$

Let us prove that Σ^{Sq_1} is confluent and compute Squier's completion of Σ^{Sq_1} . The 2-polygraph Σ^{Sq_1} has exactly one critical branching $(\beta\text{t}^n\text{b}, \text{x}\alpha_n)$ for every natural number n , and each of those critical branchings is confluent, yielding the 3-cell A_n . We conclude thanks to Theorem 4.3.2. \square

6.2.3. Proposition. *The monoid \mathbf{S}_1 has a decidable word problem.*

Proof. The convergent presentation Σ^{Sq_1} of \mathbf{S}_1 is infinite, so that the normal-form procedure cannot be applied directly. Indeed, one cannot check whether or not a 1-cell u is a normal form by inspecting all the possible applications of the 2-cells of Σ^{Sq_1} . However, the sources of the 2-cells α_n are exactly the elements of the regular language at^*b : thus, a finite automaton can check, in finite time, whether or not a 1-cell of $(\Sigma^{\text{Sq}_1})^*$ can be reduced by any of the 2-cells α_n . As a conclusion, the normal-form procedure can be adapted to decide the word problem with the convergent presentation Σ^{Sq_1} . \square

6.2.4. Proposition. *The monoid \mathbf{S}_1 is not of finite derivation type.*

Proof. By Theorem 4.2.3, it is sufficient to check that the finite presentation Υ of \mathbf{S}_1 given in Proposition 6.2.1 admits no finite homotopy basis. We denote by

$$\pi : (\Sigma^{\text{Sq}_1})^\top \longrightarrow \Upsilon^\top$$

the projection that sends the 2-cells β , γ , δ and ε to themselves and whose value on α_n is given by induction thanks to (5), *i.e.*,

$$\pi(\alpha_0) = \alpha \quad \text{and} \quad \pi(\alpha_{n+1}) = g_n^- \star_1 \text{x}\pi(\alpha_n)\text{y} \star_1 \varepsilon$$

where

$$g_n = f_n\text{y} \star_1 \text{at}^{n+1}\text{b}\varepsilon.$$

6. Squier's example and variant

We apply the homotopy bases transfer theorem 4.2.2 to Σ^{Sq_1} and Υ , with F the canonical inclusion of Υ^\top into $(\Sigma^{\text{Sq}_1})^\top$, with $G = \pi$ and with τ mapping each 1-cell u to 1_u . We obtain that the following family of 3-cells $(\pi(A_n))_{n \in \mathbb{N}}$ forms a homotopy basis of Υ^\top :

$$\begin{array}{ccc}
 & \text{at}^{n+1}\text{bx} & \\
 f_n \nearrow & & \searrow \pi(\alpha_{n+1})x \\
 \text{xat}^n\text{b} & \Downarrow \pi(A_n) & \text{x} \\
 & \text{x}\pi(\alpha_n) &
 \end{array}$$

We note that the 3-cell $A_n y \star_1 \varepsilon$ of $\mathcal{S}(\Sigma^{\text{Sq}_1})^\top$ has the 2-sphere (4) as boundary:

$$\begin{array}{ccccc}
 & \text{at}^{n+1}\text{bxy} & \xrightarrow{\text{at}^{n+1}\text{b}\varepsilon} & \text{at}^{n+1}\text{b} & \\
 f_n y \nearrow & & & & \searrow \alpha_{n+1} \\
 \text{xat}^n\text{by} & \Downarrow A_n y & & & \text{1} \\
 & \alpha_{n+1}xy & = & & \\
 & \searrow & & \nearrow & \\
 & \text{xy} & & & \varepsilon
 \end{array}$$

As a consequence, for every natural number n , the 3-cell $g_n^- \star_1 A_n y \star_1 \varepsilon$ has source α_{n+1} and target $g_n^- \star_1 x\alpha_n y \star_1 \varepsilon$. We define the 3-cell B_n of $\mathcal{S}(\Sigma^{\text{Sq}_1})^\top$ by induction on n as

$$B_0 = 1_{\alpha_0} \quad \text{and} \quad B_{n+1} = (g_n^- \star_1 A_n y \star_1 \varepsilon) \star_2 B_n$$

so that B_n has source α_n and target $\pi(\alpha_n)$, by definition of π . As a consequence, the 3-cell $\pi(A_n)$, when seen as a 3-cell of $\mathcal{S}(\Sigma^{\text{Sq}_1})^\top$ through the canonical inclusion, is parallel to the following composite 3-cell:

$$\begin{array}{ccc}
 & \text{at}^{n+1}\text{bx} & \\
 f_n \nearrow & & \searrow \pi(\alpha_{n+1})x \\
 \text{xat}^n\text{b} & \Downarrow A_n & \text{x} \\
 & \alpha_{n+1}x & \\
 & \Downarrow \text{x}B_n & \\
 & \text{x}\pi(\alpha_n) &
 \end{array}
 \tag{6}$$

We observe that the convergent 3-polygraph is reduced and has no critical triple branching. Thus, as a consequence of Proposition 4.3.7, all the parallel 3-cells of $\mathcal{S}(\Sigma^{\text{Sq}_1})^\top$ are equal. This implies that $\pi(A_n)$ is equal to the composite (6). Expanding the definition of B_{n+1} , we get:

$$\pi(A_n) = (f_n \star_1 B_n^- x) \star_2 (f_n \star_1 g_n^- x \star_1 A_n^- y x \star_1 \varepsilon x) \star_2 A_n \star_2 xB_n. \tag{7}$$

Now, let us assume that Υ admits a finite homotopy basis. By Proposition 4.2.4, there exists a natural number n such that the 3-cells $\pi(A_0), \dots, \pi(A_n)$ form a homotopy basis of Υ^\top . In particular, the 3-cell $\pi(A_{n+1})$ is parallel to a composite W of the 3-cells $\pi(A_0), \dots, \pi(A_n)$, hence it is equal to W in $(\Sigma^{\text{Sq}_1})^\top$.

Thus, on the one hand, by application of (7) to each of $\pi(A_0), \dots, \pi(A_n)$ and by definition of B_0, \dots, B_n , we get that $\pi(A_{n+1})$ is a composite of the 3-cells A_0, \dots, A_n . But, on the other hand, the relation (7) tells us that $\pi(A_{n+1})$ is equal to a composite

$$\pi(A_{n+1}) = C \star_2 (h \star_1 A_{n+1}^- yx \star_1 k) \star_2 A_{n+1} \star_2 D \quad (8)$$

where C and D contain A_0, \dots, A_n only.

To prove that this leads to a contradiction, let us consider the free right- \mathbb{ZS}_1 -module $\mathbb{ZS}_1[\Upsilon_3]$ over the homotopy basis $\Upsilon_3 = (\pi(A_n))_{n \in \mathbb{N}}$. We define a map

$$[\cdot] : \Upsilon_3^\top \longrightarrow \mathbb{ZS}_1[\Upsilon_3]$$

thanks to the relations

$$[uAv] = [A]\bar{v}, \quad [A \star_1 B] = [A] + [B], \quad [A \star_2 B] = [A] + [B],$$

for any 1-cells u and v and 3-cells A and B such that the composites are defined. From (8), we deduce that

$$[\pi(A_{n+1})] = [C] - [A_{n+1}]\bar{y}\bar{x} + [A_{n+1}] + [D] \quad (9)$$

holds in $\mathbb{ZS}_1[\Upsilon_3]$. Since the 3-cell $\pi(A_{n+1})$ is a composite of the 3-cells A_0, \dots, A_n , we have that $[\pi(A_{n+1})]$ is a linear combination of $[A_0], \dots, [A_n]$. Since the right-module $\mathbb{ZS}_1[\Upsilon_3]$ is free, it follows from (9) that

$$\bar{y}\bar{x} = 1$$

holds in \mathbf{S}_1 . However, the 1-cells yx and 1 are distinct normal forms of the convergent presentation Σ^{Sq_1} of \mathbf{S}_1 . This means that $\bar{y}\bar{x}$ is distinct from 1 in \mathbf{S}_1 and, thus, leads to a contradiction, so that we conclude that Υ does not admit a finite homotopy basis. \square

Finally, by Theorem 4.3.3, we get:

6.2.5. Corollary. *The monoid \mathbf{S}_1 admits no finite convergent presentation.*

6.3. A variant of Squier's example

Let us consider the monoid \mathbf{M} presented by the following 2-polygraph from [LP91, Laf95]:

$$\Sigma^{\text{LP}} = \langle a, b, c, d, d' \mid ab \xrightarrow{\alpha_0} a, da \xrightarrow{\beta} ac, d'a \xrightarrow{\gamma} ac \rangle.$$

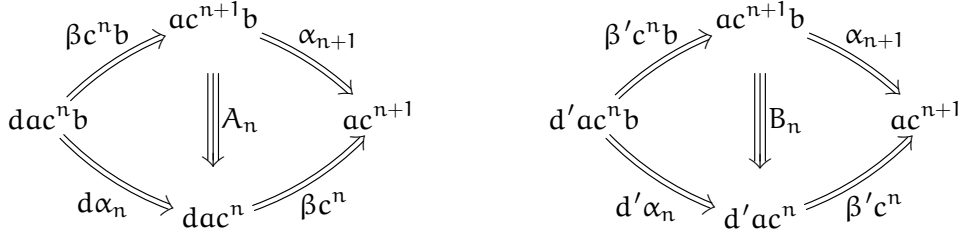
The monoid \mathbf{M} has similar properties to Squier's example: it admits a finite presentation, it has a decidable word problem, yet it is not of finite derivation type and, as a consequence, it does not admit a finite convergent presentation.

To prove these facts, the 2-polygraph Σ^{LP} is completed, by Knuth-Bendix's procedure, into the infinite convergent 2-polygraph

$$\check{\Sigma}^{\text{LP}} = \langle a, b, c, d, d' \mid (ac^n b \xrightarrow{\alpha_n} ac^n)_{n \in \mathbb{N}}, da \xrightarrow{\beta} ac, d'a \xrightarrow{\beta'} ac \rangle.$$

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Squier's completion of $\check{\Sigma}^{\text{LP}}$ has two infinite families of 3-cells:



Moreover, the 2-polygraph $\check{\Sigma}^{\text{LP}}$ has no triple critical branching. In a similar way to the case of Squier's monoid \mathbf{S}_1 , we get that the (decidable and finitely generated) monoid \mathbf{M} is not of finite derivation type: we prove that the 3-cells B_n induce a projection π of $(\check{\Sigma}^{\text{LP}})^\top$ onto $(\Sigma^{\text{LP}})^\top$, so that the family $(\pi(A_n))_{n \in \mathbb{N}}$ is an infinite homotopy basis of $(\Sigma^{\text{LP}})^\top$. Then we prove that no finite subfamily of $(\pi(A_n))_{n \in \mathbb{N}}$ can be a homotopy basis of $(\Sigma^{\text{LP}})^\top$.

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