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CONTENTS

1. Introduction

Elastic dielectrics are an important and interesting class of solid materials. The photoelastic and piezoelectric effects are but two of the physical phenomena associated with elastic dielectrics which have found applications in engineering and in the laboratory. F. E. Neumann was the first to treat systematically the photoelastic effect in isotropic materials [1, 2].[1] Pockels [3] extended Neumann's theory of the photoelastic effect to the case of crystalline media. What might be called the classical linear theory of piezoelectricity is generally attributed to Voigt [4].[2]

The theories of Neumann, Pockels, and Voigt are restricted to the case of infinitesimal motions (strains and rotations). Also, certain limitations must be placed on the magnitude of the electric field to insure a consistent scheme of approximation. In the case of Neumann's photoelastic theory, some attempt has been made to treat large deformations [2, pp. 192–194]. The results which we obtain here are not inconsistent with the basic hypotheses of this extended Neumann theory of the photoelastic effect in *isotropic* materials. Although we have not made a detailed and specific study of the photoelastic effect in this paper, some of our results may be useful in the analysis of the photoelasticity of crystals. There have been attempts also to generalize Voigt's linear piezoelectric relations to account for finite strain and higher order electrical effects [6]. Our results differ strikingly from the results of others, particularly for the class of elastic dielectrics whose degree of symmetry does not prohibit the piezoelectric effect. For example, if the general theory of elastic dielectrics developed here is specialized to a "polynomial approximation," we find that a "first order" theory results which is consistent with Voigt's classical linear theory of the piezoelectric effect. However, if higher order "photoelastic" terms involving products of a displacement gradient and the electric field or polarization are retained in a consistent manner, the stress tensor does *not* reduce to a polynomial in the symmetric part of the displacement gradients only. The symmetric part of the displacement gradients is the customary measure of infinitesimal strain. It has been a common element of endeavors to generalize Voigt's linear theory to assume that the stress tensor is a polynomial in the electric field (or polarization) and the elements of the infinitesimal strain measure. It can be demonstrated that this assumption violates the invariance of the stored energy to rigid rotations *even to the neglect of terms of first degree in the displacement gradients.*

The theory of stress and the mechanical equilibrium of dielectric media

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[1] The historical development of theory and experiment on the photoelastic effect is traced in [2].

[2] Voigt's work on the piezoelectric effect was originally reported in a series of papers dating from 1890. An excellent bibliography of this subject is given by Cady [5].

presents new problems not encountered in elasticity theory. The material contained in MAXWELL's original treatise cannot be considered definitive. This is according to his own admission [7, Vol. I, §111, p. 166]. LARMOR [8] considered some of the questions pertinent to the mechanics of continuous media which were raised by MAXWELL's theory of electricity and magnetism. New insight was gained with the advent of LORENTZ's particle model of continuous dielectric media [9], but the problem is yet unsettled. BORN's theory of polarizable lattices of particles [10] has done much to clarify the conceptual arrangement and the nature of the forces which determine the electromechanical behavior of crystalline dielectrics. The greatest progress has been made in the study of these particle models. However, the role played by the local field and a theory of stress in dielectrics based on the methods of continuum mechanics and MAXWELL's continuum theory of electricity and magnetism has received too little attention. To illustrate some of the novel questions which arise, we may cite the following: If the electric field and polarization are not parallel to each other, the Maxwell stress tensor in a dielectric medium is not symmetric. Hence, if the medium is to be in static equilibrium there must be an additional stress system whose antisymmetric part is equal and opposite to the antisymmetric part of the Maxwell stress tensor. For our immediate purposes, we may loosely refer to this additional stress system as the "elastic" or "local" stress tensor. Now the Maxwell tensor has the same form in all dielectric materials, whether they be elastic solids, fluids, or other forms of continuous media. The local stress is to be compared with the stress tensor of elasticity theory. There it is assumed that the stress is determined by the deformation and local thermodynamic state of the elastic medium through constitutive relations (stress-strain-temperature relations) whose form varies from one material to another. Is this the part of the stress then to which VOIGT refers in his theory of the piezoelectric effect? Of course, the Maxwell tensor is always quadratic in the electric field and polarization (the antisymmetric part is just $E^iP^j - P^iE^j$) and so is a quantity neglected within the context of the linear theory of piezoelectricity. But we must reckon with such problems in a general theory of the stress in dielectrics. That is, we must entertain the need for constitutive relations which yield an asymmetric stress tensor. It has been claimed elsewhere [11] that such constitutive relations can never be obtained from an energy principle.[3] In §10 of this paper we present an energy principle from which we obtain constitutive relations for an asymmetric local stress tensor. Furthermore, its antisymmetric part has just the required value.

As we have indicated above, there are many new problems not encountered in elasticity theory which must be considered when the combined effects of electrical and elastic properties of dielectrics are considered. For this reason,

[3] The proof is based on a formula relating the stress to derivatives of an energy function which is valid only in an approximation which neglects the antisymmetric part of the stress tensor.

we have reviewed in some detail the essential preliminary topics of kinematics of continuous media, the theory of stress, and the Maxwell-Faraday electrostatics. The material in these sections is conventional for the most part. Some relatively new and interesting material is introduced in the section on Euclidean tensors. The notion of two-point tensor fields and the application of these tensors in the kinematics of continuous media may be of interest even to those not interested in dielectrics. To illustrate the general theory, we have dealt at some length with the isotropic dielectric. Solutions for the homogeneous deformation of an isotropic dielectric ellipsoid and of an infinite slab placed in a uniform external electric field are given as illustrations of the qualitative features of the behavior of elastic dielectrics predicted by the theory. Finally, the theory has been specialized by assuming a special form for the stored energy function. We have done this so that a comparison with the existing approximate theories of the stress and field relations in dielectrics could be made. As far as possible we shall adhere to the notation and shall make free use of the formalism and results of continuum mechanics as presented in the comprehensive review article by C. TRUESDELL [12].

2. Static Equilibrium States of a Continuous Medium

Each point of a continuous medium may be assigned certain physical properties such as position, temperature, density, crystallographic directions, chemical constitution, polarization, stress, *etc.* What constitutes an independent set of such variables will depend on the nature of the substance and on the range of physical phenomena encompassed by the particular theory under consideration. The *local state* of a point in the medium is known if, at the point, the values of an independent set of state variables are specified. A function of the independent variables will be called a *state function.* If the local state of *every* point in the medium is known, we shall say that the *global state* of the medium is known. The distinction which has been made between the concepts of local state and global state will be found useful. For example, consider the gravitational self field of a continuous medium. Let the density of mass ρ be chosen as one of the independent state variables. Then the gravitational potential φ satisfies the Poisson equation $\nabla^2\varphi = -G\rho$. Hence, the potential is not a state function; however, its value at any point may be determined if the global state of the medium is specified.[4] It is, in a broad sense, a functional on the global state of the medium. The environment of a continuous medium consists of the alterable external influences to which we may subject the medium. We shall assume that the medium responds to changes in its environment and that when these changes have ceased the local state of each point in the medium assumes essentially unique equilibrium values commensurate with the static environment.

For the purpose of this work we shall define an elastic dielectric as a continuous medium whose local state is determined by the local deformation of the medium relative to some natural state and by the electric polarization density. It is clear that we are ignoring many interesting physical phenomena by this limited choice of independent state variables. For example, temperature effects will not be considered. Variables which provide a quantitative measure of the deformation and polarization will be defined and their properties will be discussed at the appropriate time. The environment of the elastic dielectric will be a prescribed set of mechanical surface tractions and an externally applied electric field. Hence, we set ourselves the problem: Can a determinate theory based on the principles of mechanics and electrostatics be established which will fix the global state of an elastic dielectric as defined here if the mechanical surface tractions and external electric field are prescribed data?

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[4] Certain continuity and boundary conditions must also be supplied in order that the Poisson equation admit a unique solution.

3. Coordinate Systems and Two-Point Tensor Fields in Euclidean Space

The motion of a continuous medium is conveniently described in terms of two coordinate systems which simultaneously span Euclidean space [12, 13]. The two coordinate systems are introduced in a description of the motion in the following way. A reference configuration of the material particles P of the medium is prescribed by giving the coordinate values $X^A(P)$ of the positions of the particles in one freely chosen coordinate system. If the medium is deformed or displaced rigidly, the material particles move to new positions which may be specified by giving their coordinate values $x^i(P)$ in a second freely chosen coordinate system. For example, the reference configuration of a rectangular block of material may be conveniently described in a rectangular Cartesian coordinate system. If the block is deformed into a spherical cap, it may prove convenient to describe this deformed configuration in a spherical polar coordinate system. Some workers in elasticity theory prefer the use of the so-called convected coordinate systems. Here, the choice of the second coordinate system is made in such a way that the coordinate values of the position of a given particle are the same in the reference and deformed configurations. Thus, each motion or deformed configuration implies a new choice of the second coordinate system. We mention this only as an example since we do not restrict ourselves to this convention. New problems arise in the mathematical formalism of continuum mechanics because of this use of two simultaneous coordinate systems. For this reason we shall review and extend some of the fundamental notions of the tensor analysis.

Since the space is Euclidean, it can be spanned by a rectangular Cartesian net Z^α. The metric tensor in this coordinate system is just the Kronecker delta, $\delta_{\alpha\beta}$. Let V_1^α be the rectangular Cartesian components of a vector at the point Z_1^α. The vector $\mathbf{V}_1$ can be translated by parallel displacement to a second point Z_2^α. The law of parallel displacement is particularly simple in Euclidean space if the coordinate system is rectangular Cartesian. For in this case, the components V_2^α of the displaced vector are numerically equal to the given components V_1^α. Let $V_1^{\alpha\beta\cdots}$ be the components of a tensor of arbitrary rank defined at the point Z_1^α. The components $V_2^{\alpha\beta\cdots}$ of the parallel displaced tensor at an arbitrary point Z_2^α are then given by

$$V_2^{\alpha\beta\cdots} = \delta_\gamma^\alpha \delta_\lambda^\beta \cdots V_1^{\gamma\lambda\cdots}. \tag{3.1}$$

The simplicity of this formula, indeed, its triviality, we owe to the choice of coordinate systems in which we have expressed the components of $\mathbf{V}_1$ and $\mathbf{V}_2$. We wish to obtain the formula analogous to (3.1) for the case when the coordinate system is not rectangular Cartesian. In fact, we wish to allow that the components of $\mathbf{V}_1$ and $\mathbf{V}_2$ might be referred to two different curvilinear coordinate

systems. This is done easily and the results are quite useful. Let two coordinate transformations be defined on the Z^α,

$$X^A = X^A(Z^\alpha), \qquad Z^\alpha = Z_1^\alpha(X^A); \qquad x^i = x^i(Z^\alpha), \qquad Z^\alpha = Z_2^\alpha(x^i). \tag{3.2}$$

The metric tensor in these two coordinate systems has components given by

$$g_{AB} = \delta_{\alpha\beta} \frac{\partial Z_1^\alpha}{\partial X^A} \frac{\partial Z_1^\beta}{\partial X^B}, \qquad g_{ij} = \delta_{\alpha\beta} \frac{\partial Z_2^\alpha}{\partial x^i} \frac{\partial Z_2^\beta}{\partial x^j}. \tag{3.3}$$

Now define the parallel displacement two-point tensor field $g_i^A(\mathbf{X}, \mathbf{x})$ by

$$g_i^A(\mathbf{X}, \mathbf{x}) \equiv \delta_\beta^\alpha \frac{\partial X^A}{\partial Z^\alpha} \frac{\partial Z_2^\beta}{\partial x^i}. \tag{3.4}$$

Note that g_i^A is not an ordinary tensor field. In order to fix the values of its components two points must be specified. That is, *the g_i^A are functions of six independent variables.* Ordinary single-point tensor fields are ordered sets of functions of only *three* independent variables—namely, the coordinates of a single point in the region of space where the tensor field is defined.[5] Simple examples of the displacement tensor are obtained when both coordinate transformations (3.2) reduce to the identity transformation or when both transformations (3.2) yield new coordinate systems which are again rectangular Cartesian. In the first instance, the displacement tensor is just δ_β^α, while in the second instance, the displacement tensor is an orthogonal matrix which would carry the two sets of transformed coordinate axes into parallel sets. In a more general situation when either or both of the coordinate systems are curvilinear, the components of the displacement tensor will be non-constant functions of the six variables, X^A and x^i. If we are given the components of the displacement tensor in one set of coordinate systems (X^A, x^i), the values of the components in a second set (X^{*A}, x^{*i}), where $X^{*A} = X^{*A}(X)$, $x^{*i} = x^{*i}(x)$ are independent coordinate transformations on the two argument points, the new values of the components of the displacement tensor are given by

$$g_i^{*A} = g_i^B \frac{\partial X^{*A}}{\partial X^B} \frac{\partial x^j}{\partial x^{*i}}. \tag{3.5}$$

The formula which generalizes (3.1) and is invariant to the choice of coordinate systems in which we wish to express the tensor $V_1^{AB\cdots}$ or its translated counterpart $V_2^{ij\cdots}$ is

$$V_2^{ij\cdots} = g_A^i g_B^j \cdots V_1^{AB\cdots}. \tag{3.6}$$

That is, for fixed x^i, the $V_2^{ij\cdots}$ are the components in the coordinate system x^i of the tensor obtained from $V_1^{AB\cdots}$ by parallel displacement of this tensor from

[5] We restrict our attention to three-dimensional Euclidean space. Many of our statements are trivially generalized to flat spaces of arbitrary dimension.

the point X^A to the point x^i. A useful application of the displacement tensor is illustrated by the following example. Let $f^i(\mathbf{x})$ be the components of the force per unit of volume on a continuous medium. The *resultant* or *total* force on the medium is an important quantity in Newtonian mechanics. If the components f^i of the force density are given in a curvilinear coordinate system, the integrals $\int f^i \, dV$ taken over the body are *not* the components of the resultant force. In fact, the three integrals so obtained do not transform as the components of a vector under general coordinate transformations. Only if the components f^i refer to the rectangular Cartesian components[6] of the force density will these integrals yield the components of the resultant force. In order to calculate the resultant force in a manner which is invariant to the choice of coordinate system, we must translate the force on each infinitesimal region of the body to a fixed common point by parallel displacement. The resultant force can then be obtained by summing or integrating *at that fixed point.* That is,

$$F^A(\mathbf{X}) = \int g^A_i(\mathbf{X}, \mathbf{x}) f^i(\mathbf{x}) \, dV \tag{3.7}$$

are the components of the resultant force as they appear at the point $\mathbf{X}$. The point $\mathbf{X}$ may be chosen freely. Note that the components F^A are the components of the resultant force $\mathbf{F}$, not in the coordinate system in which the f^i were given, but in the coordinate system X^A. The form of integral expressions of non-scalar tensor quantities such as resultant force, resultant moment, resultant angular momentum, *etc.*, which is invariant to choice of coordinate system will have the structure indicated by this example. Unless the factor g^A_i is included in the integrand, such integrals of non-scalar quantities in curvilinear coordinate systems have no particular transformation properties or physical significance.

Let $T^{AB\cdots ij}_{CD\cdots kl}(\mathbf{X}, \mathbf{x})$ be the components of a two-point tensor field of arbitrary rank. Corresponding to (3.5) we have the general transformation law,

$$T^{*AB\cdots ij}_{*CD\cdots kl} = (x/x^*)^w (X/X^*)^W T^{EF\cdots mn}_{GH\cdots pq} \frac{\partial X^{*A}}{\partial X^E} \frac{\partial X^{*B}}{\partial X^F} \frac{\partial X^G}{\partial X^{*C}} \cdots \frac{\partial x^{*i}}{\partial x^n} \frac{\partial x^q}{\partial x^{*1}}, \tag{3.8}$$

for a two-point relative tensor of weight w with respect to $\mathbf{x}$ transformations and weight W with respect to $\mathbf{X}$ transformations. (x/x^*) and (X/X^*) denote the Jacobians of the coordinate transformations. An *absolute* two-point tensor field is a relative two-point tensor field with both weights zero.[7]

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[6] More precisely, the components f^i may be any set obtained from a rectangular Cartesian set by an affine transformation.

[7] MICHAL [14] emphasized the use of two-point tensor fields in the kinematics of continuous media. ERICKSEN & DOYLE [15] further developed the formalism and applications. TRUESDELL suggested the apt notation g^A_i for the displacement tensor and suggests calling it the "shifter." The tensor formalism which we use here is a natural extension of TRUESDELL's [12]. Many of the formulæ of the kinematics and mechanics of continuous media which have previously defied efforts to place them in coordinate invariant form can now be so written with the help of two-point tensor fields, in particular, with the aid of the displacement tensor or shifter.

We shall denote *partial covariant derivatives* with respect to either type of index by a comma followed by the appropriate index. The partial covariant derivative of a two-point tensor field is defined as follows:

$$\begin{aligned} T^{A\cdots}_{i\cdots,C} &= \frac{\partial T^{A\cdots}_{i\cdots}}{\partial X^C} + T^{D\cdots}_{i\cdots}\Gamma^A_{DC} + \cdots \\ T^{A\cdots}_{i\cdots,j} &= \frac{\partial T^{A\cdots}_{i\cdots}}{\partial x^k} - T^A_k\Gamma^k_{ij} + \cdots . \end{aligned} \tag{3.9}$$

That is, the partial covariant derivative is defined as for single-point tensor fields if we regard the upper case (lower case) indices as mere labels when differentiating covariantly with respect to a lower case (upper case) index. A further word of caution—the two-point tensor field is a function of six independent variables; therefore, the partial derivative indicated in the formulæ (3.9) means a partial derivative holding the remaining *five* independent variables fixed. The Christoffel symbols Γ^C_{AB} and Γ^i_{jk} are given by

$$\Gamma^C_{AB} = \frac{\partial^2 Z^\alpha_1}{\partial X^A\,\partial X^B}\frac{\partial X^C}{\partial Z^\alpha}, \qquad \Gamma^k_{ij} = \frac{\partial^2 Z^\alpha_2}{\partial x^i\,\partial x^j}\frac{\partial x^k}{\partial Z^\alpha}. \tag{3.10}$$

As a special case of (3.9) we have

$$g^A_{i,j} = \frac{\partial g^A_i}{\partial x^j} - g^A_k\Gamma^k_{ij} . \tag{3.11}$$

From the definitions of the parallel displacement tensor and the Christoffel symbols we have

$$\frac{\partial g^A_i}{\partial x^j} = \frac{\partial X^A}{\partial Z^\alpha}\frac{\partial^2 Z^\alpha_2}{\partial x^i\,\partial x^j} = g^A_k\Gamma^k_{ij} . \tag{3.12}$$

Substituting (3.12) into (3.11) it follows that the partial covariant derivative of the displacement tensor vanishes,

$$g^A_{i,j} = 0. \tag{3.13}$$

By similar argument we can show that $g^A_{i,B} = 0$ and as usual that

$$g_{AB,C} = 0, \qquad g_{ij,k} = 0. \tag{3.14}$$

One may readily verify that

$$g_{AB} = g_{ij}g^i_A g^j_B . \tag{3.15}$$

The identification of the particles P of a continuous medium with their coordinates $X^A(P)$ in a reference configuration and a second identification of the particles with their coordinates $x^i(P)$ in a deformed configuration leads to the existence of a one-to-one mapping of the points $X^A \,\varepsilon\, V_0$ onto the points $x^i \,\varepsilon\, V$,

$$x^i = x^i(X^A), \qquad X^A = X^A(x^i), \tag{3.16}$$

where V_0 is the region of space occupied by the body in the reference configuration and V is the region of space occupied by the body in the deformed configuration. We shall assume for our purposes here that the mapping (3.16) is differentiable as many times as may be desired. In the mechanics of continuous media we are concerned with two-point tensor fields $T_i^{A\cdots}{}_{\cdots}(\mathbf{X}, \mathbf{x})$ which are defined for values of the arguments X^A and x^i which range over the regions V_0 and V respectively. However, we wish to consider the case when the argument points X^A and x^i are not independent but are functionally related by the mapping (3.16). Under these circumstances, we shall define the *total covariant derivative* of a two-point tensor field which we denote by the semicolon followed by the appropriate index,

$$\begin{aligned} T_{i\cdots;B}^{A\cdots} &\equiv T_{i\cdots,B}^{A\cdots} + T_{i\cdots,j}^{A\cdots} x^j{}_{;B} \\ T_{i\cdots;j}^{A\cdots} &\equiv T_{i\cdots,j}^{A\cdots} + T_{i\cdots,B}^{A\cdots} X^B{}_{;j} \end{aligned} \tag{3.17}$$

where we have set $x^i{}_{;A} \equiv \partial x^i/\partial X^A$, $X^A{}_{;i} \equiv \partial X^A/\partial x^i$. It follows from (3.17) and the relations $x^i{}_{;A} X^A{}_{;j} = \delta^i{}_j$, $x^i{}_{;A} X^B{}_{;i} = \delta^B_A$, that

$$\begin{aligned} T^A_{i;B} &= T^A_{i;j} x^j{}_{;B} \\ T^A_{i;j} &= T^A_{i;B} X^B{}_{;j} . \end{aligned} \tag{3.18}$$

It should be noted that if the argument points of a two-point tensor field are not functionally independent but are related by a mapping, then either of the ordinary covariant derivatives (3.9) is ambiguous; but the total covariant derivative is not.

4. Measures of Deformation and Rotation

Let C_0 denote the reference configuration of the material particles of a continuous medium and let C denote any other configuration. Let (3.16) be the mapping which relates the coordinates of the particles in the configurations C_0 and C.

Consider now the two points X^A and $X^A + dX^A$ of the medium in the configuration C_0 . The same two particles in the configuration C will have coordinates x^i and $x^i + dx^i$, and since we refer to the same two particles, their respective coordinates will be related by the mapping (3.16). In particular, we have

$$dx^i = dX^A \, x^i{}_{;A} \, . \tag{4.1}$$

The square of the distance between the two particles in the configuration C_0 is given by

$$dS^2 = g_{AB} \, dX^A \, dX^B = c_{ij} \, dx^i \, dx^j , \tag{4.2}$$

where

$$c_{ij} = g_{AB} X^A{}_{;i} X^B{}_{;j} \, . \tag{4.3}$$

The square of the distance between the particles in the configuration C is given by

$$ds^2 = g_{ij} \, dx^i \, dx^j = C_{AB} \, dX^A \, dX^B \tag{4.4}$$

with

$$C_{AB} = g_{ij} x^i{}_{;A} x^j{}_{;B} \, . \tag{4.5}$$

Now consider the sphere at X^A swept out by the vectors dX^A which satisfy the condition $k^2 = G_{AB} \, dX^A \, dX^B$. The set of points on this sphere is carried by the mapping (3.16) into the quadric surface $k^2 = c_{ij} \, dx^i \, dx^j$ at the point x^i. From the non-singular character of the mapping and the positive definiteness of g_{AB} , it follows that c_{ij} is a positive definite matrix; hence, the quadric at x^i is an ellipsoid. Similarly, the points which satisfy the condition $l^2 = g_{ij} \, dx^i \, dx^j$ and which constitute a sphere at x^i are carried by the inverse mapping into the ellipsoid $l^2 = C_{AB} \, dX^A \, dX^B$ at the point X^A. The two ellipsoids that we have introduced above are called the *spatial* and *material strain ellipsoids*. Let n^i_Γ ($\Gamma = 1, 2, 3$) be an orthogonal triplet of unit vectors at the point x^i,

$$g_{ij} n^i_\Gamma n^j_\Delta = \delta_{\Gamma\Delta} \, ,$$

and N^A_Γ a similar triplet at the point X^A,

$$g_{AB} N^A_\Gamma N^B_\Delta = \delta_{\Gamma\Delta} \, .$$

We can always orient these orthonormal triplets so that they satisfy the equations

$$c_{ij}n_\Gamma^i = c_\Gamma n_{\Gamma j} , \qquad C_{AB}N_\Gamma^A = C_\Gamma N_{\Gamma B} , \tag{4.6}$$

where the c_Γ and C_Γ are the eigenvalues of the matrices c_i^j and C_B^A and satisfy the cubic characteristic equations

$$\det ||c^i{}_j - c_\Gamma \delta^i{}_j|| = 0, \qquad \det ||C^A{}_B - C_\Gamma \delta^A{}_B|| = 0. \tag{4.7}$$

If c_Γ , n_Γ^i is a solution of $(4.6)_1$, $(4.7)_1$, it can be shown that C_Γ and N_Γ^A given by

$$\begin{aligned} C_\Gamma &= c_\Gamma^{-1} \\ N_\Gamma^A &= X^A{}_{,k}n_\Gamma^k/\sqrt{c_\Gamma} \end{aligned} \tag{4.8}$$

constitute a solution of $(4.6)_2$, $(4.7)_2$. If the eigenvalues $c_\Gamma(C_\Gamma)$ are distinct, then the orthonormal triplets $\pm n_\Gamma^i(\pm N_\Gamma^A)$ which satisfy (4.6) are uniquely determined. In this case the eigenvalues c_Γ and C_Γ may be ordered $c_1 > c_2 > c_3$, $C_1 < C_2 < C_3$ and (4.8) constitutes a unique pairing of the eigenvalues and eigenvectors of **c** and **C**. The sign of a normalized eigenvector always remains arbitrary; however, in (4.8) we have made a choice of sign for the square root of c_Γ which fixes the signs of the N_Γ^A in terms of the signs of the n_Γ^k . We shall adhere to this convention in what follows. Now consider the vector $L_\Gamma^A = L_\Gamma N_\Gamma^A$ of length $L_\Gamma > 0$ and parallel to an eigenvector of C_{AB} . It is carried by the mapping into the vector $L_\Gamma^i = (L_\Gamma + \Delta L_\Gamma)x^i{}_{,A}N^A$ of length $(L_\Gamma + \Delta L_\Gamma) > 0$. The vector L^A is carried by the mapping in the sense that a curve of points $X^A(t)$, passing through the point X^A and whose tangent $\dot{X}^A(t)$ at that point is parallel to N^A, is carried by the mapping into a curve $x^i(t)$, passing through the point x^i and whose tangent $\dot{x}^i(t)$ at x^i is locally parallel to n_Γ^i . The ratio of lengths, $L_\Gamma/(L_\Gamma + \Delta L_\Gamma)$ coincides with the ratio of lengths of tangents, $\sqrt{G_{AB}\dot{X}^A\dot{X}^B}/\sqrt{g_{ij}\dot{x}^i\dot{x}^j}$. From $(4.8)_2$ it follows easily that $(L_\Gamma + \Delta L_\Gamma)/L_\Gamma = \sqrt{C_\Gamma}$. Similarly, the vector $l_\Gamma^i = (l_\Gamma + \Delta l_\Gamma)n_\Gamma^i$ is carried by the inverse mapping into the vector $l_\Gamma^A = l_\Gamma N_\Gamma^A$ where $(l_\Gamma + \Delta l_\Gamma)/l_\Gamma = 1/\sqrt{C_\Gamma}$. Hence, by $(4.8)_1$, the ratios $\Delta L_\Gamma/L_\Gamma$ and $\Delta l_\Gamma/l_\Gamma$ have a common value, say δ_Γ . These are the *principal extensions*. The quantities $(1 + \delta_\Gamma)$ are called *principal extension ratios*. The tensors c_{ij} and C_{AB} are called the *Cauchy-Green deformation tensors* and their eigenvectors n_Γ^i and N_Γ^A determine the *principal spatial* and *material axes of strain*.

Let X^A and x^i be the coordinates of the same material particle in the configurations C_0 and C and let the principal extensions for this particle be distinct. Translate the orthonormal triplet n_Γ^i by parallel displacement from the point x^i to the point X^A. We can then write

$$N_\Gamma^A = R^A{}_B n_\Gamma^B , \qquad n_\Gamma^B = g^B{}_i n_\Gamma^i , \tag{4.9}$$

whereby a unique rotation matrix $R^A{}_B$, $g_{AB}R^A{}_C R^A{}_C R^B{}_D = g_{CD}$, is determined. The matrix $R^A{}_B$ has a single real vector invariant which we denote by k^A, $R^A{}_B k^B = k^A$, $k^A k_A = 1$, and a single independent scalar invariant, say R_A^A . The rotation

of the orthogonal triplet n_Γ^A into the orthogonal triplet N_Γ^A is the result of a rotation through an angle θ about an axis parallel to k^A. The cosine of θ is fixed by the relation $R_A^A = 1 + 2\cos\theta$. We may take the scalar function, $W = \frac{1}{4}(3 - R_A^A)$, $0 \leqq W \leqq 1$, and the vector invariant k^A as a measure of the local rotation of the configuration C with respect to the configuration C_0. If $W \ll 1$, the local rotation is said to be small. It should be noted that at a point where $W = 0$, k^A is undefined.

If

$$W_{;A} = 0, \qquad k^A{}_{;B} = 0, \qquad C_{AB;C} = 0, \tag{4.10}$$

the deformation is *homogeneous*. If

$$W = 0, \qquad \mathbf{C} \text{ arbitrary} \tag{4.11}$$

we have a *pure deformation*, which is a special case of

$$W = \text{constant}, \qquad k^A{}_{;B} = 0, \qquad \mathbf{C} \text{ arbitrary} \tag{4.12}$$

which differs from a pure deformation by a gross rigid motion of the configuration C with respect to the configuration C_0.

The components of the orthonormal triplets N_Γ^A and n_Γ^i satisfy the relations,

$$\sum_\Gamma N_\Gamma^A N_\Gamma^B = g^{AB}, \qquad \sum_\Gamma n_\Gamma^i n_\Gamma^j = g^{ij}. \tag{4.13}$$

And the following relations hold between the components of the tensors C^{AB}, c^{ij}, their eigenvalues C_Γ, c_Γ, and the N_Γ^A, n_Γ^i:

$$c^{ij} = \sum_\Gamma c_\Gamma n_\Gamma^i n_\Gamma^j, \qquad C^{AB} = \sum_\Gamma C_\Gamma N_\Gamma^A N_\Gamma^B. \tag{4.14}$$

We also have the more general relations

$$(c^n)^{ij} = \sum_\Gamma c_\Gamma^n n_\Gamma^i n_\Gamma^j, \qquad (C^n)^{AB} = \sum_\Gamma C_\Gamma^n N_\Gamma^A N_\Gamma^B, \tag{4.15}$$

where n may be a fractional exponent. Using (4.8–9) and (4.13–15), one can readily verify the important relations,

$$(C^n)^{AB} = R^A{}_C R^B{}_D g^C{}_i g^D{}_j (c^{-n})^{ij}, \tag{4.16}$$

$$(c^n)_{ij} = R^A{}_C R^B{}_D g^C{}_i g^D{}_j (C^{-n})_{AB}, \tag{4.17}$$

$$x^i{}_{,A} = (c^{-\frac{1}{2}})^i{}_j g^j{}_B R_A{}^B = (C^{\frac{1}{2}})^B{}_A g^i{}_C R_B{}^C, \tag{4.18}$$

$$X^A{}_{,i} = (C^{-\frac{1}{2}})^A{}_C g^B{}_i R^C{}_B = (c^{\frac{1}{2}})^j{}_i g^B{}_j R^A{}_B. \tag{4.19}$$

Consider the vector dX^A at the point X^A. It is carried by the mapping into the vector $dx^i = x^i{}_{;A}\, dX^A$. Using (4.18), the vector dx^i can be written in the form $dx^i = (c^{-\frac{1}{2}})^i{}_j g^j{}_B R_A{}^B\, dX^A$. This latter form of dx^i can be read as follows: The vector dX^A is first rotated rigidly into the vector $dX^{*B} = R_A{}^B\, dX^A$, then translated by parallel displacement to the point x^i which gives us the components $dx^{*i} = g^i{}_B\, dX^{*B}$ of the vector at that point. Finally, the vector dx^{*i} is stretched

into the vector $dx^i = (c^{-\frac{1}{2}})^i{}_j\, dx^{*j}$. Hence, (4.18) constitutes a local decomposition of the motion of a continuous medium into a rotation followed by a displacement and a final "stretching."[8]

The tensors $c^i{}_j$, $C^A{}_B$ and $R^A{}_B$ measure the finite relative deformation and rotation of two configurations of a continuous medium in the way we have described. In order to summarize the description of the motion in terms of these tensors we may state: The mapping $x^i = x^i(X^A)$ carries a sphere of points in the neighborhood of a given point X^A in the configuration C_0 into an ellipsoid of points in the neighborhood of x^i in the configuration C. Conversely, a sphere of points in the neighborhood of x^i in the configuration C is carried into an ellipsoid of points in the neighborhood of X^A by the inverse mapping. The principal axes of the ellipsoids are eigenvectors of the symmetric tensors c^{ij} and C_{AB} . If the eigenvalues of c^{ij} and C_{AB} are distinct, there is a unique set of three mutually orthogonal directions at X^A which are carried by the mapping into three corresponding mutually orthogonal directions at x^i. These directions are the three uniquely determined eigenvectors of C_{AB} and of c_{ij} , respectively. The principal extension ratios $(1 + \delta_{(\Gamma)})$ are related to the eigenvalues of c_{ij} and C_{AB} by $(1 + \delta_\Gamma) = \sqrt{C_\Gamma} = 1/\sqrt{c_\Gamma}$. The rotation tensor $R^A{}_B$ rotates the translated orthogonal triplet of eigenvectors of c_{ij} into coincidence with the orthogonal triplet of eigenvectors of C_{AB} .

In order to reduce the general theory of the elastic dielectric to a linear approximation which can be compared with the classical linear theory of VOIGT and others, we shall have to show how the tensor measures, **c**, **C**, and **R**, of *finite* strain and rotation are related to the corresponding measures of *infinitesimal* strain and rotation. Let R^A and r^i be the position vectors of the material particle P in the configurations C_0 and C. The components of the position vectors in the arbitrary coordinate systems X^A and x^i are related to the rectangular Cartesian coordinates Z^α_1 of P in C_0 and Z^α_2 of P in C by the formulæ,

$$R^A = Z^\alpha_1 \frac{\partial X^A}{\partial Z^\alpha}, \qquad r^i = Z^\alpha_2 \frac{\partial x^i}{\partial Z^\alpha}, \tag{4.20}$$

where $X^A(Z)$ and $x^i(Z)$ are the coordinate transformations (3.2). The total covariant derivatives of the position vectors reduce to

$$R^A{}_{;B} = \delta^A{}_B, \qquad r^i{}_{;j} = \delta^i{}_j. \tag{4.21}$$

The displacement vector of the particle X^A is defined by

$$U^A \equiv g^A{}_i r^i - R^A. \tag{4.22}$$

From (3.13), (3.18), and (4.21), it follows that

$$x^i{}_{,A} = g^i{}_B(U^B{}_{;A} + \delta^B{}_A) \tag{4.23}$$

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[8] Note that if and only if dX^A is an eigenvector of C_{AB} will this final stretching not involve a further rotation of the vector dx^{*i}.

$$X^A{}_{;i} = g^A{}_i(\delta^i{}_i - u^i{}_{;i}) \tag{4.24}$$

where $U^A{}_{;B}$ and $u^i{}_{;j} \equiv U^A{}_{;i}g^i{}_A$ are called *displacement gradients*. Note that these are distinct tensors and, in general, we do *not* have $u^i{}_{;j} = g^i{}_A g^B{}_j U^A{}_{;B}$. Eliminating $x^i{}_{;A}$ and $X^A{}_{;i}$ from (4.23–24) we get

$$U^B{}_{;A} = g^B{}_i g^j{}_C(U^C{}_{;A} + \delta^C{}_A)u^i{}_{;j} . \tag{4.25}$$

It follows from (4.25) that if the physical components [21] of either set of displacement gradients are small, then the physical components of the other set are correspondingly small. To first order terms in the displacement gradients, (4.25) reduces to

$$U_{B;A} \cong g^i{}_B g^j{}_A u_{i;j} . \tag{4.26}$$

Substituting (4.23) and (4.24) into the defining relations, (4.3) and (4.5), of the Cauchy-Green deformation tensors, we obtain expressions for these tensors in terms of the displacement gradients,

$$C_{AB} = g_{AB} + (U_{A;B} + U_{B;A}) + g_{CD}U^C{}_{;A}U^D{}_{;B} , \tag{4.27}$$

$$c_{ij} = g_{ij} - (u_{i;j} + u_{j;i}) + g_{kl}u^k{}_{;i}u^l{}_{;j} . \tag{4.28}$$

Eliminating $x^i{}_{;A}$ from (4.18) and (4.23), we get

$$R_A{}^B = g^B{}_i g^C{}_j (c^{\frac{1}{2}})^{ij}(U_{C;A} + g_{CA}). \tag{4.29}$$

Retaining only first and zero order terms in the displacement gradients, we obtain the following approximate relations from (4.27–29):

$$C_{AB} - g_{AB} \cong (U_{A;B} + U_{B;A}) = 2\tilde{E}_{AB} , \tag{4.30}$$

$$g_{ij} - c_{ij} \cong (u_{i;j} + u_{j;i}) = 2\tilde{e}_{ij} , \tag{4.31}$$

$$g_{ij} - (c^{\frac{1}{2}})_{ij} \cong \tilde{e}_{ij} , \tag{4.32}$$

$$R_{AB} - G_{AB} \cong \tfrac{1}{2}(U_{B;A} - U_{A;B}) = \Omega_{BA} \tag{4.33}$$

$$\Omega_{AB} \cong \tfrac{1}{2}g^i{}_A g^j{}_B(u_{j;i} - u_{i;j}) = g^i{}_A g^j{}_B \omega_{ij} , \tag{4.34}$$

where we have introduced the notation $\tilde{E}_{AB}$ and $\Omega_{AB}(\tilde{e}_{ij}$ and $\omega_{ij})$ for the symmetric and antisymmetric parts of the displacement gradients $U_{A;B}(u_{i;j})$. The symmetric and antisymmetric parts of the displacement gradients are the tensors used to measure infinitesimal strain and rotation in the classical linear theory of elasticity and in VOIGT's theory of the elastic dielectric. The approximate relations (4.30–34) exhibit clearly the relations between these tensor measures of infinitesimal strain and rotation and the tensor measures of finite strain and rotation. In addition, it is seen that no distinction need be made between the displacement gradients $U_{A;B}$ and $u_{i;j}$ in any description of the deformation of a continuous medium which discards all nonlinear terms in either set of displacement gradients.

5. Static Mechanical Equilibrium of Continuous Media

Let the dielectric medium occupy a regular region V with boundary B. We assume that the medium is in static equilibrium with a set of mechanical surface tractions T^i and an external electric field E_0^i . If a particle in the medium is polarized there will be an interaction of the particle with the external field. This interaction gives rise to an extrinsic body force density f^i and an extrinsic body moment density m^{ij} which act on the particle. The magnitude and direction of this force and moment will depend on the degree of polarization as well as the strength and direction of the external electric field. We defer giving explicit forms for the force and moment until later.

The surface tractions and external field exert a resultant force F^A_{ext} on the dielectric. The components of this resultant force are given by the integrals

$$F^A_{\text{ext}} = \int_V g^A{}_i f^i \, dV + \int_B g^A{}_i T^i \, dS. \tag{5.1}$$

The particles of the medium also exert forces on each other. For example, the cohesive forces which bind the medium into an elastic solid are of this interparticle type. Also, if the medium is polarized, the electric self field of this polarized matter interacts with a given polarized particle of the medium. These interparticle forces are sometimes classified according to their "range" of interaction. Thus we hear of "short-range" and of "long-range" interactions. As we shall see later, there is some advantage to introducing these concepts in a continuum theory as well as in the theory of particle interactions where they usually occur. Whatever way the forces of mutual interactions of the particles of the dielectric may be classified, we shall make the *stress hypothesis*. Thus we assume the following: Let v be an arbitrary regular region of space. This region may be either entirely or partially contained in V, or its intersection with V may be zero. The forces of interparticle interaction between particles contained in v and the particles in $V - v$ are *equipollent* to a system of *stress vectors* distributed over the surface of the region v. Let b denote the surface of v. The stress vector field is a function only of position on the surface b and on the direction of the normal to b.[9] Let t^i denote the field of stress vectors. We have then $t^i = t^i(x^i, n^i)$ where n_i are the components of the *unit outward normal* to the surface b at the point x^i. Thus, the stress vector field is not an ordinary vector field whose components are functions only of position. The resultant force exerted by the particles in the region $V - v$ on the particles in the region v is given by

[9] Thus we exclude phenomenon such as surface tension. To include such effects, we would have to assume that the stress vector might also depend on local properties of the surface other than the direction of its normal, *e.g.*, its curvature.

$$F^A_{\text{int}} = \int_b g^A{}_i t^i(\mathbf{x}, \mathbf{n})\, dS. \tag{5.2}$$

The total force on the particles contained in the region v is the sum of the resultant extrinsic force and the resultant interparticle or intrinsic force (5.2). This total force on the arbitrary region v has the form,

$$F^A = \int_b g^A{}_i t^i\, dS + \int_{v\cap B} g^A{}_i T^i\, dS + \int_{v\cap V} g^A{}_i f^i\, dV, \tag{5.3}$$

where $v \cap B$ denotes the set of points common to the region v and the boundary of the dielectric. If the medium is in static mechanical equilibrium, *this total force vanishes for an arbitrarily chosen region v.* Applying this condition of equilibrium to a tetrahedron with a vertex which is not a point of B and passing to the limit of vanishing dimensions of the tetrahedron, we can demonstrate the existence of a *stress tensor* field $t^{ij}(x)$ such that

$$t^i(\mathbf{x}, \mathbf{n}) = t^{ij}(\mathbf{x}) n_j\ . \tag{5.4}$$

Thus, the stress vector is the contracted product of an ordinary tensor field of second rank and the unit normal $\mathbf{n}$. The proof of this standard result depends on the assumption that the stress vector is bounded and continuous throughout space (except at the surface B) and that the extrinsic body force is everywhere bounded. We may also apply the condition of vanishing total force to a pill-box region which contains points of the surface B. On taking the appropriate order of limits as the dimensions of the pill-box approach zero, we obtain the important boundary condition,

$$[t^{ij}] n_j + T^i = 0. \tag{5.5}$$

In (5.5), $[t^{ij}] \equiv t^{+ij} - t^{-ij}$, where t^{+ij} and t^{-ij} are the limiting values of the stress tensor as the surface B is approached from the exterior and interior of the dielectric respectively.[10]

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[10] There are alternative methods of formulating the stress hypothesis and its consequences. For example, in elasticity theory, many authors prefer to identify what we have called "surface tractions" with $t^{+ij}n_j$. The boundary condition corresponding to (5.5) then reads $[t^{ij}]n_j = 0$, and $t^{+ij}n_j$ is regarded as prescribed data. With the arrangement of definitions we have used, the stress tensor in ordinary elastic media vanishes outside the medium; hence, $t^{+ij}n_j$ would be zero in (5.5). If the medium is polarized, however, this is no longer true. What we have done here is to make the stress hypothesis only for the forces of interaction of a limited part of the overall mechanical system—that is, for the forces of interaction between the particles of the dielectric itself. Thus we have treated the extrinsic and intrinsic force systems quite differently. The action of outside agencies (the environment) is to be described by a body force and surface traction—the interaction of the particles of the dielectric itself by a system of stress vectors. The two methods of description of force systems are not entirely equivalent. The use of ex-

The resultant moment exerted on the dielectric by the surface tractions and external electric field is given by

$$M^{AB}_{\text{ext}} = \int_V g^A{}_i g^B{}_j m^{ij}\, dV + \int_B g^A{}_i g^B{}_j (r^i T^j - r^j T^i)\, dS + \int_V g^A{}_i g^B{}_j (r^i f^j - r^j f^i)\, dV, \tag{5.6}$$

where the r^i are the components of the position vectors to the points of application of the forces.

The interparticle interaction gives rise to a resultant moment on the region v which is given by

$$M^{AB}_{\text{int}} = \int_b g^A{}_i g^B{}_j (r^i t^j - r^j t^i)\, dS. \tag{5.7}$$

The total moment on the region v is given by

$$M^{AB} = \int_{v\cap B} g^A{}_i g^B{}_j (r^i T^j - r^j T^i)\, dS + \int_{v\cap V} g^A{}_i g^B{}_j (r^i f^j - r^j f^i)\, dV + \int_{v\cap V} g^A{}_i g^B{}_j m^{ij}\, dV + \int_b g^A{}_i g^B{}_j (r^i t^j - r^j t^i)\, dS. \tag{5.8}$$

If the medium is in static mechanical equilibrium this total moment on an arbitrary region v must vanish. From these integral forms of the conditions of equilibrium it follows that if the stress tensor is continuously differentiable except perhaps at points of the boundary of the dielectric then at equilibrium we have

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trinsic body forces and surface tractions in continuum mechanics arises from a desire to focus attention on a limited part of the physical universe. For example, the effect of the earth's gravitational field on the mechanical behavior of continuous media on or near the earth's surface is usually accounted for by a body force type of description. The self gravitational field of the material and the counter effect of the body upon the earth is neglected in many applications. However, for bodies of large size such as the earth itself, this cannot be done and a classification of the forces of interparticle interaction into a "body force" and a "stress tensor" is not particularly appealing or profitable. It may still be desirable in a theory of the mechanics of the planet Earth, to introduce the gravitational force of the sun and other planets as an extrinsic body force. In a universal or cosmological theory, a unified treatment of the entire system of forces by means of a stress hypothesis would probably be a more fundamental formulation of the basic equations of continuum mechanics. However, unless we set ourselves the problem of the mechanics of a completely self contained mechanical system, the device of describing the environment of a limited system which is in interaction with outside agencies by means of a body force and surface traction seems particularly useful. It allows us to simplify the problem by not having to give a detailed account of the mechanics of the external agencies and of the counter effect of the system under consideration upon its environment.

$$\begin{aligned} t^{ji}{}_{,j} + f^i &= 0 \qquad x \,\varepsilon\, V - B \\ t^{ij}{}_{,j} &= 0 \qquad x \,\varepsilon\, E - V - B \end{aligned} \tag{5.9}$$

$$\begin{aligned} t^{ji} - t^{ij} + m^{ij} &= 0 \qquad x \,\varepsilon\, V - B \\ t^{ji} - t^{ij} &= 0 \qquad x \,\varepsilon\, E - V - B \end{aligned} \tag{5.10}$$

where E denotes all of Euclidean space. If we formally define f^i and m^{ij} to have the value zero outside the dielectric we avoid the need for writing $(5.9)_2$ and $(5.10)_2$.

CAUCHY's equations of force balance (5.9) and the COSSERATS' moment equations (5.10) constitute the local conditions of static mechanical equilibrium of a continuous medium. If the extrinsic body moment vanishes, as it usually does in pure elasticity theory, (5.10) reduces to the usual rule that the stress tensor be symmetric. In a theory of the elastic dielectric, greater care and attention must be given to the moment equations (5.10) than is necessary in elasticity theory.

6. The Maxwell-Faraday Theory of the Electric Field in Dielectrics

According to the Maxwell-Faraday electrostatic theory of dielectrics, the electric field is determined by the two conditions,

$$\int_C E_{\mathsf{M}}^i \, dx_i = 0, \tag{6.1}$$

$$\int_S D^i n_i \, dS = Q, \tag{6.2}$$

and a constitutive relation between the components of the *displacement vector* D^i and the Maxwell-Faraday electric field E_{M}^i . The form of this constitutive relation between $\mathbf{D}$ and $\mathbf{E}_{\mathsf{M}}$ may depend on any of the variables which describe the local state of the medium. In a vacuum, the constitutive relation reduces to $D^i = e_0 E_{\mathsf{M}}^i$ where e_0 is a dimensional constant. The surface S in (6.2) is the boundary of an arbitrary regular region R, and Q is the total *free charge* contained in R. By (6.1), the line integral of the electric field around an arbitrary space curve C vanishes. For our purposes here, a sufficiently general form for the total charge Q will be

$$Q = \int_R \sigma \, dV + \sum_k \int_{B_k \cap R} \omega \, dS \tag{6.3}$$

where σ is the volume density of free charge and ω is the surface density of free charge defined over a set of closed surfaces B_k . These surfaces are normally the surfaces of electrical conductors. The electric field and displacement vanish inside a conductor. Let the constitutive relation between $\mathbf{D}$ and $\mathbf{E}_{\mathsf{M}}$ be written in the form

$$D^i - e_0 E_{\mathsf{M}}^i = P^i(E_{\mathsf{M}}^i). \tag{6.4}$$

We shall call P^i the polarization density. The polarization density vanishes in vacuum and in electrical conductors. The local state of a dielectric medium may be characterized in part by the value of the polarization density. That is, we may choose the polarization density as one of the independent variables of state for an elastic dielectric. Let V again denote the region of space occupied by the dielectric and let B denote the boundary of V. Let P^i be zero everywhere except in V. Let V_k denote the regions enclosed by the charge bearing surfaces B_k and let V_0 denote the remainder of space. We assume that the electric field and displacement are continuously differentiable functions of position in each

of the regions, V, V_k, and V_0. It then follows from the law (6.1) that in each of these regions we may represent the electric field as the gradient of a scalar field,

$$E_{\mathsf{M}}^{i} = -\varphi_{,}{}^{i}. \tag{6.5}$$

We also assume that the electric field suffers at most a finite discontinuity at the surfaces B and B_k. We can then use (6.1) to prove that the jump $[\varphi]$ in the scalar field φ is at most a constant over each of these surfaces. Since E_{M} determines only the gradient of φ, we may thus assume without loss in generality that φ is continuous throughout space. The scalar field so defined is called the *electrostatic potential.* From (6.2) it follows that in each of the regions V, V_k, and V_0 we have

$$e_0\varphi_{,}{}^{i}{}_{i} = -\sigma + P^{i}{}_{,i}\,. \tag{6.6}$$

It also follows from (6.2) that at the boundary of the dielectric where the polarization is discontinuous the discontinuity in the normal derivative of the electrostatic potential is given by

$$e_0 n^i[\varphi_{,i}] = n^i[P_i], \tag{6.7}$$

where n^i is the unit normal to the surface of the dielectric. At the surface of a conductor we have

$$e_0 n^i[\varphi_{,i}] = \omega. \tag{6.8}$$

If one now adds the boundary condition that φ vanish at infinity the Poisson equation (6.6) and the boundary conditions (6.7) and (6.8) determine the electrostatic potential uniquely throughout space. The solution can be put in the form

$$\begin{aligned} e_0\varphi(X) = \int \sigma(1/r)\,dV + \sum_k \int_{B_k} \omega(1/r)\,dS \\ - \int_V P^{i}{}_{,i}(1/r)\,dV + \int_B P^i n_i(1/r)\,dS \end{aligned} \tag{6.9}$$

where $r = \sqrt{(r^i - g^i{}_A R^A)(r_i - g^B{}_i R_B)}$ and $(r^i - g^i{}_A R^A)$ are the components of the position vector of the point of integration x^i relative to the point X^A at which the potential is evaluated. Let us put $\varphi = \varphi_0 + \varphi_{\mathsf{MS}}$ where $e_0\varphi_{\mathsf{MS}}$ is the sum of the last two integrals on the right-hand side of (6.9) and $e_0\varphi_0$ is the sum of the two remaining integrals. The *extrinsic field* $\mathbf{E}_0$ will be given by the negative gradient of φ_0. The negative gradient of φ_{MS} will be called the *Maxwell self electric field* of the dielectric and will be denoted by $\mathbf{E}_{\mathsf{MS}}$. The potential of the self field is seen to be equivalent to the potential of a volume distribution of free charge $(-P^{i}{}_{,i})$ plus a surface distribution of free charge $(P^i n_i)$ over the

boundary of the dielectric. These are called the *Poisson-Kelvin equivalent charge distributions* of a polarized dielectric. If the electrostatic potential is calculated from (6.9) for a point where the free charge or equivalent charge distributions do not vanish, the integrand of one or more of the integrals will be infinite at the point owing to the factor $(1/r)$. In such a case, the value of the integral is defined by a limit process. A detailed discussion of the convergence of improper integrals of this general type can be found in [16]. Omitting details, we outline here the definition of such improper integrals. If $F(\mathbf{x})$ is singular at the single point $x_0^i \in V$, then the integral of F over the region V is defined by

$$\int_V F\, dV \equiv \lim_{d\to 0} \int_{V-v} F\, dV \tag{6.10}$$

where v is a regular region containing the point $\mathbf{x}_0$ and whose maximum chord is d. If the limit exists and is independent of the shape of the region v, the integral is said to converge or to exist and its value is defined as the limit of the sequence on the right-hand side of (6.10).

Accordingly, each of the integrals in (6.9) can be shown to converge for every point X^A. If the point X^A lies outside the dielectric we may differentiate the integral expressions for φ_{MS} under the integral sign to obtain

$$e_0 E^A_{\mathsf{MS}} = \int_V P^i{}_{,i}(1/r)_{,}{}^{A}\, dV - \int_B P^i n_i (1/r)_{,}{}^{A}\, dS. \tag{6.11}$$

We now make the observation that $(1/r)_{,}{}^{A} = -(1/r)_{,}{}^{i} g^A{}_i$—a result which may be readily verified by direct calculation of the two different derivatives of the expression which defines the function $r(X^A, x^i)$. If this substitution be made for the factor $(1/r)_{,}{}^{A}$ in (6.11), and if Green's theorem be used to transform the resulting surface integral into a volume integral, certain terms will cancel and the final result can be put in the form,

$$e_0 E^A_{\mathsf{MS}} = \int_V g^A{}_i P^i (1/r)_{,}{}^{i}\, dV = -\int_V P^i (1/r)_{,i}{}^{A}\, dV. \tag{6.12}$$

Thus, the self field at a point outside the dielectric is given by a generalized "Coulomb's law." It is for this reason that $\mathbf{P}$ is called a polarization density. We cannot use (6.12) to calculate the electric self field at a point inside the dielectric. This is so because the integral expressions for the potential of the self field inside the dielectric have singular integrands and the order of integration and the subsequent differentiation of the potential cannot be interchanged. That is, if we made the formal attempt to evaluate the self field inside the dielectric using (6.12), the integrals would not converge. This does not constitute a flaw in the Maxwell theory. The potential (6.9) which converges everywhere

must *first* be obtained, then its gradient may be computed without ambiguity. Within the context of Maxwell-Faraday electrostatic theory, the issue of convergence of (6.12) for points inside a dielectric medium is not relevant. It does give rise to the so-called *Kelvin cavity definitions* of the electrostatic field within a polarized dielectric. A region of definite shape may be excluded from the region of integration in (6.12). Taking the limit as the maximum chord of these differently shaped excluded regions tends to zero, one obtains values of the electric self field which can be varied continuously between certain finite limits. Without further hypotheses of a physical nature, no one of these cavity fields suggests itself as having superior physical significance in the electrostatic theory of dielectrics. As we have seen, the two Maxwell-Faraday laws of electrostatics (6.1) and (6.2) are sufficient to determine a unique self field inside and outside a dielectric which is polarized a given amount. For example, if a sphere of dielectric material is homogeneously polarized to an amount P^i, the Maxwell self field is a constant field inside the dielectric and has a value given by $E^i_{\text{MS}} = -(4\pi/3e_0)P^i$. In the next section we take up certain modifications and extensions of the continuum theory of the electrostatics of dielectrics which were discovered by LORENTZ. The self field of a polarized dielectric will be our major concern.

7. The Lorentz Theory of the Electric Field in Dielectrics

Lorentz has calculated the electrostatic field of an array of point dipoles arranged on a uniform space lattice [9, pp. 305–308]. If the distance between neighboring particles is small compared to the overall dimensions of the lattice, a correspondence may be set up between the electric field of the set of particles and the electric field of a polarized continuum. Of course, since the electric field of an array of point dipoles varies rapidly in the neighborhood of each particle and contains a singularity at the position of each particle, a correspondence between the field of a set of particles and the smooth, differentiable Maxwell field must involve some degree of approximation or some averaging process. We shall review the Lorentz theory here and use his result to motivate an independent hypothesis concerning the electric self field of a continuous elastic dielectric medium. The Lorentz theory of the self field will be presented in a manner which illustrates the physical point of view which we wish to carry over into the continuum theory and which places our hypothesis concerning the electric self field in elastic dielectrics in the most favorable light of known results on particle models of dielectric media.

Let $\underset{\alpha}{r^i}$ denote the components of the position vector to the α^{th} lattice site of a uniform lattice. Let h_Γ^i ($\Gamma = 1, 2, 3$) be three vectors whose directions coincide with the crystal axes and whose lengths are proportional to the lattice spacing. We can choose the constant of proportionality so that the parallelepiped formed by the h_Γ^i will have unit volume, *i.e.*, $\sqrt{g}$ det $h_\Gamma^i = 1$. The lattice vectors can be expressed in terms of the h_Γ^i as follows:

$$\underset{\alpha}{r^i} = \epsilon(\underset{\alpha}{n_1}h_1^i + \underset{\alpha}{n_2}h_2^i + \underset{\alpha}{n_3}h_3^i) = \epsilon\sum_\Gamma \underset{\alpha}{n_\Gamma}h_\Gamma^i \tag{7.1}$$

where the $\underset{\alpha}{n_\Gamma}$ are integers which pick out the lattice vector $\underset{\alpha}{r^i}$. Let $v_a = \epsilon^3$ denote the volume of the unit cell of the lattice. We shall consider here the case of a uniformly polarized lattice with a single particle in each cell. Each lattice site is occupied by a point dipole $\underset{\alpha}{p^i}$ and by uniform polarization we mean that $\underset{\alpha}{p^i}$ is independent of the index α. Let p^i denote the common value of the $\underset{\alpha}{p^i}$. Let $\underset{\alpha}{r} = \sqrt{(g^A{}_i\underset{\alpha}{r^i} - R^A)(g^i{}_A\underset{\alpha}{r}_i - R_A)}$ denote the distance between the lattice site $\underset{\alpha}{r^i}$ and the point whose position vector is **R**. If **R** does not coincide with a lattice site, the electrostatic field of the array of dipoles has a value at X^A which is given by the sum,

$$e_0 E_S^A(\mathbf{X}) = \sum_\alpha p^i (1/r)_{,B}{}^A g^B{}_i$$
$$= \sum_\alpha p^i \left\{ \frac{3(r^j g_i{}^A - R^A)(r_k g^k{}_B - R_B) g^B{}_i}{r^5} - \frac{g^A{}_i}{r^3} \right\}_\alpha . \tag{7.2}$$

If the point X^A coincides with a lattice site, the field is given by the sum (7.2) with the infinite term omitted. If such be the case, we will indicate the modified summation by placing a prime on the summation sign. Now let v be an arbitrarily small but finite regular region which contains the lattice point r^i_0 and let $V - v$ be a regular region which contains the lattice points not in v. Divide the sum (7.2) into two partial sums, $E_S^A|_v$ and $E_S^A|_{V-v}$, where the former is the sum over all $r^i_\alpha \,\varepsilon\, v$ and the latter over all $r^i_\alpha \,\varepsilon\, V - v$. Let $r = \sqrt{(g^A{}_i r^i - R^A)(g^k{}_A r_k - R_A)}$ be the distance from the lattice site r^i_0 to an arbitrary point r^i in the region $V - v$. The function $(1/r)$ is analytic throughout $V - v$, and the function $(1/r)_{,i}{}^A g^A{}_i$ is Riemann integrable over this same region. From the definition of the Riemann integral we have

$$-\int_{V-v} P^i (1/r)_{,i}{}^A \, dV \equiv -\lim_{\Delta V_n \to 0} \sum_n P^i (1/r)_{,i}{}^A \Big|_{r=r_n} \Delta V_n \tag{7.3}$$

where r_n is any point which lies in the three-dimensional interval ΔV_n. P_i is a constant vector whose value we will assign in a moment. The limit is independent of the manner in which the region $V - v$ is subdivided into intervals ΔV_n and independent of the manner in which one chooses the points r_n within the intervals. Hence, we may choose the intervals ΔV_n so that they are the cells of a regular lattice whose corner points are given by (7.1). At the boundary of the region, the intervals may consist of partial lattice cells. We may choose the points r_n so that they coincide with the lattice points (7.1). Let us now identify P^i as the ratio, $p^i/v_a = p^i/\epsilon^3$. Let the limit $\Delta V_n \to 0$ be identified as the limit $\epsilon^3 \to 0$. In this way we see that the integral (7.3) is the limiting value of the sum $E_S^A|_{V-v}$ as the dimensions of the lattice cell approach zero and the dipole moment of each particle approaches zero in a manner which maintains the ratio p^i/ϵ^3 finite and equal to P^i. Now if the lattice has cubic or higher symmetry, the primed sum $E_S^A|_v$, representing the field at r^i_0 of the dipoles within any *sphere* with center at r^i_0, vanishes for every finite value of ϵ. Hence, the limit of this sum as $\epsilon \to 0$ exists and has the value zero. Thus, if the dimensions of the unit cell and the dipole moment on each lattice site are allowed to approach zero, maintaining the ratio $p^i/v_a = P^i$, then the self field of a lattice having cubic symmetry when evaluated at a lattice site is given by

$$e_0 E_{\mathsf{S}}^A = -\int_{V-v} P^i (1/r)_{,i}{}^{,A}\, dV \tag{7.4}$$

where the region v is a sphere whose center is at the lattice site and P^i is the constant density of dipole moment.

The volume integral (7.4) may be transformed into the sum of two surface integrals, one over the boundary of the sphere v and one over the boundary of the dielectric. Since $P^i{}_{,i} = 0$, we have

$$e_0 E_{\mathsf{S}}^A = -\int_b P^i n_i (1/r)_{,}{}^{,A}\, dS + \int_B P^i n_i (1/r)_{,}{}^{,A}\, dS = (4\pi/3)P^A + e_0 E_{\mathsf{MS}}^A\,, \tag{7.5}$$

where we have evaluated the first surface integral explicitly and have identified the last integral as the Maxwell self electric field at an *interior* point of a uniformly polarized continuous medium with polarization density P^i. If the lattice has symmetry lower than cubic, then in general a different value of the excess of E_{S}^i over the corresponding Maxwell-Faraday field E_{MS}^i is obtained. Whatever the lattice symmetry may be, we shall write E_{S}^i in the form

$$E_{\mathsf{L}}^i = E_{\mathsf{L}}^i + E_{\mathsf{MS}}^i \tag{7.6}$$

where E_{L} is called the *Lorentz local field.* If a cubic lattice is deformed homogeneously and the deformed lattice has lower symmetry than cubic, then the local field will have a value which differs from $(4\pi/3e_0)P^i$. Hence, *the local field is a function of the lattice deformation.* The deformation of a lattice may be described quantitatively in the following way. The three linearly independent vectors h_Γ^i which we now let represent the crystal axes of the deformed lattice are the result of rotating and elongating the vectors H_Γ^A which define the crystal axes of the undeformed lattice. If one puts $h_\Gamma^i = S^i{}_A H_\Gamma^A$, then $S^i{}_A$ is determined uniquely in terms of the h_Γ^i and H_Γ^A. It is assumed, of course, that neither det h_Γ^i nor det H_Γ^A is zero. If we introduce the set of three vectors B_A^Γ which are reciprocal to the H_Γ^A, *i.e.*,

$$B_A^\Gamma = \tfrac{1}{2}\sqrt{g}\,\epsilon^{\Gamma\Delta\Upsilon}\epsilon_{ABC}H_\Delta^B H_\Upsilon^C\,, \qquad B_A^\Gamma H_\Delta^A = \delta_\Delta^\Gamma\,, \qquad \sum_\Gamma B_A^\Gamma H_\Gamma^B = \delta^B{}_A\,,$$

then

$$S^i{}_A = \sum_\Gamma h_\Gamma^i B_A^\Gamma\,. \tag{7.7}$$

The nine parameters $S^i{}_A$ afford a quantitative measure of the deformation of the lattice cell. If more than one particle occupies each cell of the lattice, further parameters may be introduced to describe changes in the internal configuration of the particles in a given cell. If the lattice deformation is not homogeneous, then $S^i{}_A$ will vary from cell to cell and we must label the parameters $S^i{}_A$ by the cell index α. If there are N cells in the specimen, a general inhomogeneous

deformation of the specimen is described by the $9N$ parameters $\underset{\alpha}{S^i}_A$. Recall that in the section on kinematics of *continuous* media, it was shown that the vector dX^A at X^A is carried by the mapping $x^i(X^A)$, which describes the relative configurations of the particles of a continuous medium, into the vector $dx^i = x^i{}_{;A}\, dX^A$ at the point x^i. If this be compared with $h^i_\Gamma = S^i{}_A H^A_\Gamma$, we see that the continuum analogues of the $\underset{\alpha}{S^i}_A$ are the displacement gradients $x^i{}_{;A}(X^A)$. The displacement gradients are continuous functions of the material coordinates X^A which replace the discrete index α of the $\underset{\alpha}{S^i}_A$. In the lattice theory, the local field at a lattice point $\underset{\alpha}{r^i}$ depends on the $\underset{\alpha}{S^i}_A$ and $\underset{\alpha}{p^i}$ for the cells in the immediate neighborhood of the point $\underset{\alpha}{r^i}$. In the continuum theory, we shall set down as a primitive assumption that

$$E^i_S = E^i_{MS} + E^i_L(x^i{}_{;A}\ , P^i). \tag{7.8}$$

That is, *the electrostatic self field of a polarized and deformed continuous elastic dielectric is the sum of the Maxwell electrostatic self field and a local field which is a state function of the displacement gradients and polarization density.* We assume that the relation (7.8) holds not only in dielectrics having a crystal structure but also in elastic dielectrics such as rubber or plastic.

8. The Equations of Intramolecular Force Balance

It may be said that in adding the polarization vector to the list of independent state variables of an elastic medium we have ascribed an internal structure to the continuum "particle." That is, independent of the values of the displacement gradients which provide a quantitative description of the relative configuration of the particles in the neighborhood of a given one, the magnitude and direction of the polarization vector at a point describes the internal structure of the continuum particle. The forces which maintain this internal configuration must be a static equilibrium system of forces as well as the system of forces which maintain the relative positions of neighboring particles at stationary values. For our purposes here, it proves convenient to refer to a dumbbell model of a single "particle" in an elastic dielectric. According to this model, a polarized particle consists of two equal electric charges of opposite sign separated along a line parallel to the polarization vector. If this particle is in static mechanical equilibrium, the forces which act on either charge must have a zero resultant. The electrical force which acts on a charge q placed in an electrostatic field $\mathbf{E}$ is just $q\mathbf{E}$. The electrostatic field which acts on the charge in an elastic dielectric has three distinct components—they are (1) the Maxwell electric self field $\mathbf{E}_{\mathsf{MS}}$, (2) the Lorentz local field $\mathbf{E}_{\mathsf{L}}$, (3) the external or extrinsic field $\mathbf{E}_0$. In addition to the resultant electrostatic force due to these three components of the electrostatic field, other forces act on either charge of the polarized particle. These are the molecular forces which are made up of the Coulomb attraction between the charges of the particle, dynamical forces and other non-classical or quantum forces. Let $q\mathbf{F}$ denote the resultant of all these molecular forces. Then at static equilibrium we must have

$$q(F^i + E^i_{\mathsf{L}} + E^i_{\mathsf{MS}} + E^i_0) = 0 \tag{8.1}$$

which is just the Newtonian law of force balance applied to either charge of the polarized particle. We shall set $F^i + E_{\mathsf{L}} = \overline{E}^i_{\mathsf{L}}$ and call the sum of these two terms the *effective local field.* We shall assume that *the effective local field in an elastic dielectric is a state function of the displacement gradients and polarization density.* Thus we have

$$\overline{E}^i_{\mathsf{L}}(x^i{}_{;A}\ , P^i) + E^i_0 + E^i_{\mathsf{MS}} = 0. \tag{8.2}$$

We call (8.2) *the equation of intramolecular force balance.* The total Maxwell field is just $\mathbf{E}_{\mathsf{M}} = \mathbf{E}_{\mathsf{MS}} + \mathbf{E}_0$; so that we can write (8.2) in the form[11]

$$\overline{E}^i_{\mathsf{L}}(x^i{}_{;A}\ , P^i) + E^i_{\mathsf{M}} = 0. \tag{8.3}$$

[11] Equation (8.3) could be written in the form $D^i = e_0 E^i_{\mathsf{M}} + P^i = D^i(x^i_{;A}\ , E_{\mathsf{M}})$ which is formally identical to a constitutive relation between the displacement vector, the Maxwell electric field, and the displacement gradients. A constitutive relation of this type is always assumed to exist in Maxwell-Faraday electrostatic theory. We have arrived at a relation of this type using notions of mechanical equilibrium. The two points of view are quite different, however. Equation (8.3) is a condition of static equilibrium, not a constitutive relation. Its form would change if we passed to the dynamical case. As we shall see in §10, the *equilibrium condition* (8.2) results quite naturally from a principle of virtual work.

9. The Form of the Stress Tensor, Extrinsic Body Force and Extrinsic Body Moment in an Elastic Dielectric

In elasticity theory it is assumed that the stress tensor is a state function of the displacement gradients. That is, the displacement gradients are the only state variables and a constitutive relation between the components of the stress tensor and the displacement gradients which is characteristic of the particular elastic material is assumed to exist. To assume a constitutive relation between the stress and displacement gradients implies the physical notion that the stress tensor at a point in the material is determined solely in terms of the local state of the medium. The state of the medium at distant points may be altered without changing the values of the stress tensor components at a given point. The existence of a constitutive relation between stress and local deformation also implies the physical notion of "short-range" forces. Thus, in pure elasticity theory it may be said that the elastic response of the medium is due solely to "short-range" forces. This assumption is in need of modification for the elastic dielectric. Here, in addition to the "short-range" elastic forces which are determined by the local state of the medium, the polarized dielectric interacts with the self field. We have made the stress hypothesis and have assumed that this interaction together with the "short-range" interaction will be described by a system of stress. Now it is known from Maxwell's work that the resultant electrostatic force on a region containing polarized matter is given by $\int_v g^A_i t^{ij}_{MS} n_j \, dS$ where t^{ij}_{MS} is the Maxwell stress tensor given by

$$t^{ij}_{MS} = e_0 E^i_{MS} E^j_{MS} + E^i_{MS} P^j - \tfrac{1}{2} e_0 E^2_{MS} g^{ij}. \tag{9.1}$$

It is clear that the Maxwell stress tensor is not a state function if the displacement gradients and polarization density are the independent state variables. It has the same form in all materials. For a given global state of polarization, we can determine the value of the Maxwell stress using the laws of Maxwell-Faraday electrostatics.

We shall assume that the stress tensor in a polarized elastic dielectric has the form

$$t^{ij} = t^{ij}_L(x^i{}_{;A}, P^i) + t^{ij}_{MS} \tag{9.2}$$

where t^{ij}_L is a state function called the local stress. Note that the Maxwell stress tensor does not vanish at points outside the dielectric. The resultant electrostatic force on any region which lies entirely outside the dielectric is zero. The local stress will be assigned the value zero outside the dielectric.

The interaction of the polarized dielectric with the extrinsic field E^i_0 is given by the following expressions for the extrinsic body force and extrinsic body moment:

$$f^i = E^i_{0,j} P^j, \tag{9.3}$$

$$m^{ij} = P^i E^j_0 - P^j E^i_0 . \tag{9.4}$$

Since $e_0 E^i_{MS,i} = -P^i{}_{,i}$, it follows that $t^{ij}_{MS,j} = E^i_{MS,j} P^j$ at every point of con-

tinuity of $\mathbf{E}_{MS}$ and $\mathbf{P}$. However, at the boundary of the dielectric where $\mathbf{P}$ is discontinuous so also is the self field. From the boundary condition $[e_0 E^i_{MS} + P^i]n_i = 0$, we deduce that the discontinuity in the stress vector of the Maxwell tensor at the boundary of the dielectric is given by

$$[t^{ij}_{MS}]n_i = (\tfrac{1}{2}e_0)(P^i n_i)^2 n^i. \tag{9.5}$$

Summarizing these results and substituting the special forms for the stress tensor, body force, and body moment into the force and moment equations (5.9) and (5.10) we get the following system of equations:

$$t^{ij}_{L,j} + t^{ij}_{MS,j} + E^i_{0,j}P^j = 0 \tag{9.6}$$

$$t^{ji}_{L} - t^{ij}_{L} + t^{ji}_{MS} - t^{ij}_{MS} + E^j_0 P^i - E^i_0 P^j = 0 \tag{9.7}$$

$$\overline{E}^i_L + E^i_0 + E^i_{MS} = 0 \tag{9.8}$$

which may also be written in the form

$$t^{ij}_{L} + E^i_{M,j}P^j = 0, \tag{9.9}$$

$$t^{ji}_{L} - t^{ij}_{L} + E^j_M P^i - E^i_M P^j = 0 \tag{9.10}$$

$$\overline{E}^i_L + E^i_M = 0. \tag{9.11}$$

Substituting (9.5) into the boundary condition (5.5), we obtain

$$-t^{-ij}_{L}n_i + (\tfrac{1}{2}e_0)(P^i n_i)^2 n^i + T^i = 0. \tag{9.12}$$

Recall that t^{-ij}_{L} denotes the limiting values of the local stress as the boundary of the dielectric is approached from the interior.

In order to complete the summary of equations and boundary conditions which will determine the behavior of elastic dielectrics we list the following results from Maxwell-Faraday electrostatic theory:

$$e_0 \varphi_{MS,}{}^{i}{}_{i} = P^i{}_{,i}\,, \tag{9.13}$$

$$e_0 n^i [\varphi_{MS,i}] = n^i [P_i]. \tag{9.14}$$

Finally, in addition to the above equations and equations (9.9–12), two sets of constitutive relations characteristic of the material must be given. These are

$$t^{ij}_{L} = t^{ij}_{L}(x^i{}_{;A}\,, P^i) \tag{9.15}$$

and

$$\overline{E}^i_L = \overline{E}^i_L(x^i{}_{;A}\,, P^i). \tag{9.16}$$

The form of these constitutive relations for the local stress and the effective local field is restricted by certain symmetry properties of the material. Further restrictions are also imposed by the manner in which the dependent variables must transform as the deformed and polarized body is rotated rigidly in space. We take up these questions regarding the form of the constitutive relations in greater detail later.

10. A Principle of Virtual Work for the Elastic Dielectric

In elasticity theory there are two methods which have been used to arrive at stress-strain relations [12, p. 173]. The method used by Cauchy was to assume that the components of the stress tensor were functions of strain. Green's method assumes the existence of a stored energy function which is a function of strain. An energy or work principle is then used to establish formulæ for the components of the stress tensor in terms of certain combinations of the partial derivatives of the stored energy with respect to the variables used to measure the strain. The stress-strain relations obtained by these two different procedures are not always identical. By Cauchy's method, we obtain stress-strain relations which contain those obtained by Green's method as a special case. To this point, we have not used the mechanical concept of work nor have we made any use of the concept of stored energy in formulating the equations of an elastic dielectric summarized at the end of the preceding section. By assuming that the local stress and effective local field were functions of displacement gradients and polarization we have followed a procedure analogous to Cauchy's method in elasticity theory. In this section, we wish to present a natural generalization of Green's method which yields equations and boundary conditions equivalent to those already proposed. As a generalization of the result in elasticity theory, we shall show that the constitutive relations for the local stress and effective local field will follow from a single stored energy function which is characteristic of the material.

The *natural state* of an elastic dielectric is the equilibrium state which the material assumes in the absence of applied surface tractions and an external electric field. Let the material particles of the elastic dielectric be identified by their coordinates X^A in this natural state configuration. As before, let x^i denote the coordinates of the particles in the deformed and polarized equilibrium configuration when surface tractions T^i and an electric field E_0^i are applied. If X^A and x^i are the coordinates of the same material particle in the natural and deformed configurations, then

$$x^i = x^i(X^A). \tag{10.1}$$

As X^A ranges over the region V_0 occupied by the body in its natural state, the correspondence (10.1) constitutes a continuous mapping of the region V_0 onto the region V occupied by the body in its deformed and polarized state. Let B_0 and B denote the boundary of the dielectric in the natural and deformed states respectively. The *total mass* of the dielectric body is given by

$$M = \int_V \rho \, dV \tag{10.2}$$

where ρ is the mass density in the deformed state. Let J denote the absolute scalar given by

$$J = +\sqrt{\frac{\det g_{ij}}{\det g_{AB}}}\,|\det x^{i}{}_{;A}| = \det g^{A}{}_{i}x^{i}{}_{;A}\,. \tag{10.3}$$

The law of conservation of mass may be stated in the form

$$\rho_0(X^A) = J\rho(x^i(X^A)), \tag{10.4}$$

where ρ_0 is the density of mass in the natural state. If the body is homogeneous in the natural state, ρ_0 is a constant independent of X^A. We have found it convenient to introduce the polarization per unit of mass as an independent variable of state instead of the polarization per unit of volume. Let π^i denote the polarization per unit of mass. We have then

$$P^i = \rho\pi^i. \tag{10.5}$$

If the polarization per unit of mass is defined for each point in the deformed and polarized body, we have a vector field $\pi^i(\mathbf{x})$ defined on the region V. There is a corresponding vector field defined on the region V_0 by the following process: Let the vector $\pi^i(\mathbf{x})$ at the point $\mathbf{x}$ be translated by parallel displacement to the point $\mathbf{X}$ which is the position in the natural state of the material particle now at $\mathbf{x}$. Thus we have

$$\pi^A(\mathbf{X}) = g^A{}_i(\mathbf{X}, \mathbf{x})\pi^i(\mathbf{x}); \tag{10.6}$$

whereby a vector field $\pi^A(\mathbf{X})$ is defined over the region V_0 occupied by the body in its natural state. Conversely, if we are given the field π^A we may generate the field π^i by the inverse process,

$$\pi^i = g^i{}_A\pi^A. \tag{10.7}$$

Since the mapping (10.1) has been defined only over the limited regions of space that are occupied by the natural and deformed states of the dielectric medium, the correspondence between tensor fields defined by the process just described can only be extended over these same two limited portions of space. We could regard all of space as being filled with a continuous material medium. A correspondence between material particles would then give us a mapping like (10.1) between every pair of positions in space. Suppose, however, that a portion of space is devoid of material matter. What physical significance could then be attached to a mapping of such a region upon another? A question of this nature arises in our work here in connection with the electrostatic potential of the self electric field of the polarized dielectric. We have met the problem in the following way which appears to be logically sound and physically correct. Let us formally extend the mapping (10.1) throughout all space in an arbitrary fashion. We require only that it join smoothly with the mapping of V onto

V_0 to which we have ascribed physical significance and that it have as many derivatives as we shall need for convenience. A correspondence between tensor fields can now be set up throughout space with formulæ analogous to (10.6) and (10.7). In particular, if $\varphi(\mathbf{x})$ is the electrostatic potential of the self electric field of the polarized and deformed dielectric, it has a value at every point $\mathbf{x}$ and we can define the function $\varphi(\mathbf{X})$ for every point $\mathbf{X}$ by setting

$$\varphi(\mathbf{X}) = \varphi(\mathbf{x}(\mathbf{X})). \tag{10.8}$$

Let Σ denote the *stored energy function of deformation and polarization.* That is, the stored energy Σ is a state function and we have

$$\Sigma = \Sigma(x^i{}_{;A}, \pi^i). \tag{10.9}$$

The principle of virtual work for the elastic dielectric is as follows:

$$\begin{aligned}\delta\Big[-\int_V \rho\Sigma(x^i{}_{;A}, \pi^i)\, dV + \tfrac{1}{2}e_0 \int_E \varphi^{;i}\varphi_{;i}\, dV + \int_V \varphi^{;i}P_i\, dV\Big] \\ + \int_B T_i\delta' x^i\, dS + \int_V f_i\delta' x^i\, dV + \int_V \rho E_{0i}\delta''\pi^i\, dV = 0.\end{aligned} \tag{10.10}$$

The last three integrals in this variational expression represent, respectively, the work done by the applied surface tractions if the boundary of the dielectric is displaced from equilibrium by a small amount $\delta' x^i$, the work done by the body force if any point in the dielectric is displaced from its equilibrium position, and the work done by the external field in changing the polarization a small amount from its equilibrium value. The sum of these three virtual work terms is set equal to the variation in potential energy of the elastic dielectric. This potential energy is written as the sum of three parts which are enclosed in the large brackets in (10.10). The first of these terms represents the variation in the stored energy of deformation and polarization. This term is quite analogous to the stored elastic energy of elasticity theory. The second term is the variation in the potential energy of the self electric field. The third term in the bracket represents an interaction energy between the self field and a polarized particle of the dielectric.

The independent variations of the field variables are now listed:

$$x^i(\mathbf{X}) \to x^i(\mathbf{X}) + \delta' x^i(\mathbf{X}) \tag{10.11}$$

$$\pi^A(\mathbf{X}) \to \pi^A(\mathbf{X}) + \delta''\pi^A(\mathbf{X}) \tag{10.12}$$

$$\varphi(\mathbf{X}) \to \varphi(\mathbf{X}) + \delta'''\varphi(\mathbf{X}). \tag{10.13}$$

The total variation δ of the terms in large brackets in (10.10) means the resultant first order change in the value of these integrals under the replacements

(10.11–13). Note that the total covariant derivative of the electrostatic potential is used in (10.10). Thus we have $\varphi_{;i} = \varphi_{;A}X^{A}{}_{;i}$. Some useful preliminary results are now listed:

$$\delta(x^{i}{}_{;A}) = (\delta' x^{i})_{;A} \tag{10.14}$$

$$\delta(X^{A}{}_{;i}) = -X^{A}{}_{;j}X^{B}{}_{;i}(\delta' x^{j})_{;B} \tag{10.15}$$

$$\delta J = JX^{A}{}_{;i}(\delta' x^{i})_{;A} \tag{10.16}$$

$$\begin{aligned}\delta(\varphi_{;i}) &= (\delta'''\varphi)_{;A}X^{A}{}_{;i} + \varphi_{;A}\delta'(X^{A}{}_{;i}) \\ &= (\delta'''\varphi)_{;i} - \varphi_{;j}(\delta' x^{j})_{;i} .\end{aligned} \tag{10.17}$$

We shall also use the Euler-C. Neumann identities [12, p. 140],

$$(X^{A}{}_{;i}J)_{;A} = 0, \qquad (J^{-1}x^{i}{}_{;A})_{;i} = 0. \tag{10.18}$$

Since the boundary of the region V is subject to variation, it is convenient to transform all the integrals in (10.10) into integrals over the undeformed body V_0 or over the region $E - V_0$ outside the undeformed body. These transformed integrals will then have fixed limits and we may commute the operations of integration and variation. If this be done we find that (10.10) can be put in the form

$$\begin{aligned}\int_{B_0} \Bigg\{ \Bigg[&-\rho_0 \frac{\partial \Sigma}{\partial x^{i}{}_{;A}} + J(t^{+}_{\mathrm{MS}i}{}^{j} - t^{-}_{\mathrm{MS}i}{}^{j})X^{A}{}_{;j} + T_i N^{A}\left(\frac{dS}{dS_0}\right)\Bigg]\delta' x^{i} \\ &+ [e_0(E^{+}_{\mathrm{MS}}{}^{j} - E^{-}_{\mathrm{MS}}{}^{j} - e_0^{-1}P^{j})X^{A}{}_{;j}]J\delta'''\varphi \Bigg\} N_A \, dS_0 \\ &+ \int_{V_0} \Bigg\{ \rho_0 \Bigg[-\frac{\partial \Sigma}{\partial \pi^{j}} + E_{\mathrm{MS}j} + E_{0j} \Bigg] g^{j}{}_{A}\delta''\pi^{A} \\ &+ \Bigg[\left(\rho_0 \frac{\partial \Sigma}{\partial x^{i}{}_{;A}}\right)_{;A} + J t_{\mathrm{MS}i}{}^{j}{}_{;j} + J f_i \Bigg]\delta' x^{i} \\ &+ J[-e_0 V^2\varphi + V\cdot P]\delta'''\varphi \, dV_0 \Bigg\} \\ &+ \int_{E-V_0} \{e_0 V^2\varphi\delta'''\varphi + t_{\mathrm{MS}i}{}^{j}{}_{;j}\delta' x^{i}\} J \, dV_0 = 0.\end{aligned} \tag{10.19}$$

In writing the above result we have grouped certain of the terms using the definition of the Maxwell stress tensor t^{ij}_{MS}. We have also set $\mathbf{E}_{\mathrm{MS}} = -\nabla\varphi$. The quantities N^{A} are the components of the outward unit normal to the surface of the undeformed dielectric. The ratio of the magnitudes of the surface elements (dS/dS_0) is given by

$$dS/dS_0 = J\sqrt{(C^{-1})^{AB}N_A N_B} = J/\sqrt{(c^{-1})^{ij}n_i n_j} . \tag{10.20}$$

We thus obtain the following field equations which must be satisfied at every point inside the dielectric:

$$J^{-1}\left(\rho_0 \frac{\partial \Sigma}{\partial x^i{}_{;A}}\right)_{;A} + t_{\mathrm{MS}i}{}^{i}{}_{;i} + E_{0i;i}P^i = 0 \tag{10.21}$$

$$-\frac{\partial \Sigma}{\partial \pi^i} + E_{\mathrm{MS}i} + E_{0i} = 0 \tag{10.22}$$

$$-e_0\nabla^2\varphi + \nabla\cdot\mathbf{P} = 0. \tag{10.23}$$

At every point outside the dielectric we must have

$$\nabla^2\varphi = 0 \tag{10.24}$$

$$t^{ij}_{\mathrm{MS};j} = 0. \tag{10.25}$$

Equation (10.24) follows from the principle of virtual work by the requirement that the variation of the electrostatic potential of the self field at a point outside the dielectric will give no first order change in the left-hand side of (10.1). Equation (10.25) follows from the same principle applied to a variation of the mapping $x^i(X)$ as extended to points outside the dielectric. But this extended mapping was assigned no physical significance; therefore, it is a happy circumstance that the field equation (10.25) is satisfied identically if the field equation (10.24) is satisfied.

In addition to the above set of field equations we obtain the boundary conditions which we now list.

$$-\rho \frac{\partial \Sigma}{\partial x^i{}_{;A}} N_A - (t^-_{\mathrm{MS}i}{}^{j} - t^+_{\mathrm{MS}i}{}^{j})X^A{}_{;j}N_A + T_i\sqrt{(C^{-1})^{AB}N_AN_B} = 0, \tag{10.26}$$

$$[e_0(E^+_{\mathrm{MS}}{}^{j} - E^-_{\mathrm{MS}}{}^{j}) - P^j]X^A{}_{;j}N_A = 0. \tag{10.27}$$

In addition we have the continuity of the electrostatic potential which was assumed in the variational principle. The field equations (10.21–24) and the boundary conditions (10.26–27) are the *material form of the equilibrium conditions for an elastic dielectric.* Using the identities (10.18) and the relation $n_i = JN_AX^A{}_{;i}$ between the components of the unit normals to the deformed and undeformed dielectric, the corresponding *spatial* form of the equilibrium conditions can be obtained from the material form. It is as follows:

$$\left(\rho \frac{\partial \Sigma}{\partial x^i{}_{;A}} x^i{}_{;A}\right)_{;i} + t_{\mathrm{MS}i}{}^{i}{}_{;i} + E_{0i;}{}^{i}P_i = 0 \tag{10.28}$$

$$-\frac{\partial \Sigma}{\partial \pi_i} + E^i_{\mathrm{MS}} + E^i_0 = 0 \tag{10.29}$$

$$-e_0\nabla^2\varphi + \nabla\cdot\mathbf{P} = 0 \tag{10.30}$$

$$-\rho\frac{\partial\Sigma}{\partial x^i{}_{;A}}x^i{}_{;A}n_j + [t_{\mathrm{MS}i}{}^j]n_j + T_i = 0 \tag{10.31}$$

$$(e_0[E^i_{\mathrm{MS}}] - P^i)n_i = 0. \tag{10.32}$$

We need only to identify the local stress and effective local field as the expressions,

$$t_{\mathrm{L}i}{}^j = \rho\frac{\partial\Sigma}{\partial x^i{}_{;A}}x^j{}_{;A} \tag{10.33}$$

$$\overline{E}^i_{\mathrm{L}} = -\frac{\partial\Sigma}{\partial\pi_i} \tag{10.34}$$

in order to make the above set of equilibrium equations identical in form to the set of field equations and boundary conditions summarized at the end of the previous section. We do not obtain the moment equation (9.7) as a direct consequence of the principle of virtual work. We can, however, show that *the moment equation is satisfied identically if the stored energy function is invariant under a rigid rotation of the deformed and polarized dielectric.* The proof of this statement is based on the following well known result on invariant functions of several vectors.

Let $F(V^i_1, V^i_2, \cdots, V^i_n)$ be a function of the components V^i_Γ ($\Gamma = 1, 2, \cdots, n$) which is invariant under the substitutions

$$V^i_\Gamma \to S^i{}_j V^j_\Gamma$$

where $S^i{}_j$ is an arbitrary rotation; that is, $g_{ij}S^i{}_kS^j{}_l = g_{kl}$ and $\det S^i{}_j = 1$. An infinitesimal rotation has the form $S^i{}_j = \delta^i{}_j + \epsilon^i{}_j$ where ϵ_{ij} is an arbitrary antisymmetric tensor. Since

$$F(V^i_1, \cdots V^i_n) = F(S^i{}_jV^j_1, \cdots S^i{}_jV^j_n) = F(\overline{V}^i_1, \cdots \overline{V}^i_n)$$

we have as necessary conditions

$$dF = \frac{\partial F}{\partial\overline{V}^i}\frac{\partial\overline{V}^i}{\partial S^k{}_l}dS^k{}_l = 0.$$

Hence, for differentials $S^k{}_l$ about the values $S^k{}_l = \delta^k{}_l$ (the identity transformation) the above condition reads

$$\frac{\partial F}{\partial V^i}V^i\epsilon^i{}_j = 0$$

where ϵ_{ij} is an arbitrary antisymmetric tensor. This condition implies that the coefficients of ϵ_{ij} in this expression are the components of a *symmetric* tensor. We use the notation $T^{[ij\cdots k]}$ to denote the antisymmetric part of a tensor.

For example, $T^{[ij]} = \frac{1}{2}(T^{ij} - T^{ji})$. A necessary and sufficient condition that a tensor of second rank be symmetric is that $T^{[ij]} = 0$. Thus we have the necessary conditions

$$\sum_{\Gamma} \frac{\partial F}{\partial V_{\Gamma[i}} V_{\Gamma}^{j]} = 0 \tag{10.35}$$

if F is a function of n vectors V_{Γ}^{i} which is invariant to a rigid rotation of the vectors. It can also be shown that the conditions (10.35) are sufficient to insure the invariance of F under *finite* rotations.[12]

If the deformed and polarized state of the elastic dielectric is rotated rigidly in space, the displacement gradients and polarization vector change to new values given by

$$x^{i}{}_{;A} \rightarrow S^{i}{}_{j} x^{j}{}_{;A}$$

$$\pi^{i} \rightarrow S^{i}{}_{j} \pi_{j}.$$

Hence, if we assume that the stored energy function of deformation and polarization is invariant under a rigid rotation of the deformed and polarized state it follows from the above theorem that

$$\frac{\partial \Sigma}{\partial x^{[i}{}_{;A}} x_{j]\,;A} + \frac{\partial \Sigma}{\pi^{[i}} \pi_{j]} = 0.$$

Multiplying this equation by ρ and using (10.33–34), it may be put in the form

$$t_{\mathrm{L}}^{[ij]} - \overline{E}_{\mathrm{L}}^{[i} P^{j]} = 0. \tag{10.36}$$

This equation implies the physical notion that the moment exerted on a particle of the dielectric by the system of local stresses is just the moment exerted by the effective local field $\mathbf{E}_{\mathrm{L}}$ acting on the polarized particle. We should comment that (10.36) is identically statisfied whether we are in an equilibrium state or not. It depends only on the assumption that the stored energy function is invariant to rigid rotations. If the result (10.36) is combined with the equilibrium condition (10.29) we find that the moment equation (9.7) is satisfied identically. That is, the equilibrium condition which we have called *intramolecular force balance,* together with the invariance of the stored energy function to rigid rotations, insures that the moment equation will be satisfied. This is to be compared with the corresponding result in elasticity theory where invariance of the stored elastic energy to rigid rotations is sufficient to insure the *symmetry* of the stress tensor in that theory. As is sometimes done in elasticity theory, we could impose the moment equation (symmetry of the stress tensor

[12] The results stated in this theorem are well known consequences of the Lie theory of compact groups.

in the case of elasticity theory) as a side condition. The invariance of the stored energy function to rigid rotations would then follow as a necessary consequence of this side condition instead of by mere assumption. In a theory based on the laws of mechanics, this latter arrangement is probably the preferred order of stating the hypotheses.

We have demonstrated that the variational principle (10.10) yields field equations and boundary conditions in complete agreement with the equilibrium conditions of an elastic dielectric. The main additional result obtained by assuming the existence of a stored energy function of deformation and polarization and the validity of the principle of virtual work lies in the restrictions which are imposed on the constitutive relations for the local stress and effective local field. If the energy principle is used, we see that a single scalar function of the variables of state is sufficient to characterize the mechanical and electrostatic properties of an elastic dielectric completely. Without the energy principle, the same formal set of equations and boundary conditions can be arrived at but the constitutive relations for the local stress and effective local field must be given separately in order to characterize the properties of the material. It is clear that the energy principle leads to constitutive equations for the stress and effective local field which are much less general than they are in the absence of a stored energy function. As in elasticity theory, it is probable that for many elastic dielectrics the restrictions imposed on the constitutive relations by using (10.33) and (10.34) instead of the "Cauchy" forms (9.15) and (9.16) are desirable and are actually borne out by experiment.

11. The Homogeneous Isotropic Elastic Dielectric

There are many important examples of elastic dielectrics which are homogeneous and isotropic. For a study of the effects of large deformations, rubber is the first material which comes to mind. Early experiments on the photoelastic effect were concerned with isotropic media such as glass. Today, the numerous varieties of transparent plastics, which may be regarded as isotropic in their natural state, are widely used in the study of the photoelastic effect. The piezoelectric effect cannot occur, however, in isotropic media.

As we have seen, the properties of a particular elastic dielectric are determined by specifying the form of a single scalar function of the state variables if we adopt the energy principle set forth in §10. Hence, if an elastic dielectric is isotropic or has any other type of material symmetry, this fact must make itself known through the form of the stored energy function of deformation and polarization. It is now our purpose to determine the most general functional form of the energy function which is consistent with the assumption that a particular elastic dielectric is homogeneous and isotropic.

The natural state of a homogeneous isotropic elastic dielectric is a state of zero polarization. The surface tractions and extrinsic electric field vanish. There are no intrinsic directions defined in the material. Without loss in generality we may assume that the stored energy has the value zero in the natural state. We assume that *the stored energy in the deformed and polarized state is a single valued function of the following quantities and following quantities only*:

$$\Sigma = \Sigma(x^i{}_{;A}\ ,\ \pi^i,\ g_{ij}\ ,\ g^i{}_A). \tag{11.1}$$

For emphasis, this may be compared with the case when an intrinsic direction exists in the undeformed and unpolarized dielectric. The intrinsic direction may be characterized by a vector H^A. In this case, the stored energy function could also depend on the components of the vector $\mathbf{H}$. By excluding material descriptors or tensors of this type from the list (11.1), we have given substance to the physical notion of material isotropy. The homogeneity of the material is given quantitative expression by the device of omitting the position vector components R^A from the list (11.1).

Two distinct types of invariance requirements will be made on the stored energy function. These are, *coordinate invariance* and *invariance to rigid rotations.* In many treatments of this and similar problems which arise in the formulation of constitutive relations for continuous media, little or no attempt is made to distinguish clearly between these separate demands. Here, in this work, all of the kinematical and mechanical theory has been developed and presented in a form which is invariant to an arbitrary simultaneous choice of two coordinate systems—one for the description of the natural state configuration, one for the

description of the deformed and polarized state. The transformation law of each of the tensor variables listed in (11.1) is properly indicated by the type and placement of the indices. *Coordinate invariance* requires only that Σ be an absolute scalar function of the list of tensor variables. For example, a term proportional to $g^A{}_i x^i{}_{;A}$ could occur in the stored energy function if coordinate invariance were all that was demanded. That a term of this form cannot occur in the expression for the stored energy function which is also invariant to rigid rotations of the deformed and polarized material will soon be made apparent. In §10, it was shown that the moment equation will not be satisfied unless Σ is invariant to rigid rotations. The transformation law of the variables listed in (11.1) under a rigid rotation of the deformed and polarized body is given by

$$\begin{aligned} x^i{}_{;A} &\to x^j{}_{;A} S^i_j \\ \pi^i &\to \pi^j S^i{}_j \\ g_{ij} &\to g_{ij} \\ g^i{}_A &\to g^i{}_A \end{aligned} \tag{11.2}$$

where $S^i{}_j$ is a rotation tensor and satisfies the equations, $g_{ij} S^i{}_k S^i{}_l = g_{kl}$, $\det S^i{}_j = +1$. Note that the choice of the two coordinate systems is fixed so that the components of the metric and shifter are not altered by this operation. If the natural state of the medium is rotated holding the deformed and polarized state fixed, the transformation law for the variables is

$$\begin{aligned} x^i{}_{;A} &\to x^i{}_{;B} S^B{}_A \\ \pi^i &\to \pi^i \\ g_{ij} &\to g_{ij} \\ g^i{}_A &\to g^i{}_A \end{aligned} \tag{11.3}$$

where $S^A{}_B$ is a rotation tensor satisfying the equations $g_{AB} S^A{}_C S^B{}_D - g_{CD} = 0$, $\det S^A{}_B = +1$. Physically, the two transformations (11.2) and (11.3) represent equivalent operations—namely, a *relative* rotation of the deformed and polarized state and the natural state of the dielectric. If the stored energy function is made insensitive to either type of transformation, it will automatically be insensitive to the other.

We now make use of a theorem on invariant functions of several vectors first given by CAUCHY [17]. If $F(V^i_1, V^i_2, \cdots, V^i_n)$ is a single valued function of the components of n vectors which is invariant to a rigid rotation of the system of vectors, F must reduce to a function of their lengths and scalar products, $I_{\Gamma\Delta} = g_{ij} V^i_\Gamma V^j_\Delta$, and the determinants of their components taken three at a time, $D_{\Gamma\Delta\Upsilon} = \epsilon_{ijk} V^i_\Gamma V^j_\Delta V^k_\Upsilon$.

Let us now impose the condition that Σ be invariant under the substitutions (11.2). By CAUCHY's theorem, Σ must reduce to a function of the following variables:

$$\begin{aligned} C_{AB} &= g_{ij}x^i{}_{;A}x^j{}_{;B}\,, \\ \Pi_A &= g_{ij}x^i{}_{;A}\pi^j, \\ \pi^2 &= g_{ij}\pi^i\pi^j, \\ J &= \tfrac{1}{6}(\det g^A{}_i)\epsilon_{ijk}\epsilon^{ABC}x^i{}_{;A}x^j{}_{;B}x^k{}_{;C}\,, \\ D^A &= \tfrac{1}{2}(\det g^A{}_i)\epsilon_{ijk}\epsilon^{ABC}x^i{}_{;B}x^j{}_{;C}\pi^k, \\ & g_{AB}\,,\ g^i{}_A\,. \end{aligned} \tag{11.4}$$

That is, if the stored energy function is to be invariant to rigid rotations, the original list of variables in (11.1) can occur only in the combinations listed in (11.4). We can now show, however, that the list (11.4) is somewhat redundant. Since $\det C^A{}_B = J^2$ and for real motions J is always positive, we can eliminate J from the list (11.4) since it is determined by the variables C_{AB} and g_{AB}. Also, if we use the fact that $D^A = JX^A{}_{;i}\pi^i$, it is not difficult to show that $C_{AB}D^B = J\Pi_A$, or $D^A = J(C^{-1})^A{}_B\Pi^B$. Since we have already shown that J is expressible in terms of the $C^A{}_B$, we can now eliminate the variables D^A from the list (11.4). In this manner we reduce Σ to a function of the variables now indicated:

$$\Sigma = \Sigma(C_{AB}\,, \Pi_A\,, \pi^2, g_{ij}\,, g^i{}_A). \tag{11.5}$$

The next step is to require coordinate invariance under independent transformations of either set of coordinate systems. With the exception of $g^i{}_A$ and g_{ij} all of the quantities in (11.5) are absolute scalars under coordinate transformations of the x^i. The only absolute scalars which can be formed from the metric and the shifter are the components g_{AB} of the metric in the X^A coordinate system. A well known result which we now use is that a coordinate transformation of the X^A can be made which simultaneously reduces the symmetric tensor C_{AB} and the metric tensor g_{AB} to diagonal form. Let C_1, C_2, and C_3 be the diagonal entries of the matrix C_{AB} in this special coordinate system and let Π_1, Π_2, and Π_3 be the corresponding components of the vector $\mathbf{\Pi}$.[13] The metric has unit entries along the diagonal. Thus, in this special coordinate system, the stored energy function is expressible as a function of the C_Γ and Π_A and we have

$$\Sigma = \Sigma(C_1\,, C_2\,, C_3\,, \Pi_1\,, \Pi_2\,, \Pi_3\,, \pi^2). \tag{11.6}$$

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[13] If the eigenvalues of $\mathbf{C}$ are unique, they can be ordered in such a way that $C_1 > C_2 > C_3$. The coordinate transformation which diagonalizes $\mathbf{C}$ and produces this ordering is unique up to a reversal in the direction of any one of the coordinate axes.

Now under the coordinate transformation from this special coordinate system to one which is obtained by a simple reversal of one of the coordinate directions, the C_Γ are unaltered, π^2 is invariant, but one of the components of the vector $\mathbf{\Pi}$ suffers a change in sign. Since Σ is a single valued function of its arguments and invariant to arbitrary coordinate transformations, it must involve the components of the vector $\mathbf{\Pi}$ only by even powers. By this line of reasoning we see that Σ must reduce to a function of the arguments indicated below:

$$\Sigma = \Sigma(C_1 , C_2 , C_3 , (\Pi_1)^2, (\Pi_2)^2, (\Pi_3)^2, \pi^2). \tag{11.7}$$

It can then be shown that each of the arguments in (11.7) is a *single valued* function of the six independent scalar invariants given by

$$\begin{aligned}
I_1 &= \delta^A{}_B C^B{}_A = \text{trace } \mathbf{C} \\
I_2 &= (\tfrac{1}{2}!)\delta^{AB}_{CD} C^C{}_A C^D{}_B = \text{sum of the principal minors of } \mathbf{C} \\
I_3 &= (1/3!)\delta^{ABC}_{DEF} C^D{}_A C^E{}_B C^F{}_C = \det C^A{}_B \\
I_4 &= \Pi_A \Pi^A = \Pi^2 \\
I_5 &= C^A{}_B \Pi^B \Pi_A = \mathbf{\Pi}\cdot\mathbf{C}\cdot\mathbf{\Pi} \\
I_6 &= \pi^2.
\end{aligned} \tag{11.8}$$

The scalar invariants $I_1 , \cdots , I_6$ can all be written as functions of the strain measure $(c^{-1})^i{}_j$:

$$\begin{aligned}
I_1 &= \text{tr } \mathbf{c}^{-1} \\
I_2 &= \text{sum of the principal minors of } \mathbf{c}^{-1} \\
I_3 &= \det \mathbf{c}^{-1} \\
I_4 &= \boldsymbol{\pi}\cdot\mathbf{c}^{-1}\cdot\boldsymbol{\pi} \\
I_5 &= \boldsymbol{\pi}\cdot\mathbf{c}^{-2}\cdot\boldsymbol{\pi} \\
I_6 &= \pi^2.
\end{aligned} \tag{11.9}$$

This set is entirely equivalent to the set (11.8). Still other choices of the independent variables can be made. For example, in making approximations it is sometimes convenient to take the set obtained from (11.9) by replacing the strain measure $\mathbf{c}^{-1}$ by the strain measure $(\mathbf{c}^{-1} - \mathbf{1})$.

We have proven that a *single valued* stored energy function for an elastic dielectric is reducible to a *single valued* function of the scalar invariants (11.8) or (11.9). The classical theory of invariant functions of vectors and tensors [18] is concerned primarily with the reduction of tensor invariant *polynomial* functions of the components of a set of tensors to *polynomial* functions in a minimal

set of basic tensor invariants which are the elements of the so-called *integrity basis*. We may state the general problem treated there as follows: Let $F^{AB\cdots}$ be the components of a tensor $\mathbf{F}$ of given rank. It is supposed that the components of $\mathbf{F}$ are polynomial functions of the components of a set of dependent variables which we shall denote by $H_1^{AB\cdots}$, $H_2^{AB\cdots}$, *etc.* This leads to a set of relations having the form

$$F^{AB\cdots} = F^{AB\cdots}(H_1^{AB\cdots}, H_2^{AB\cdots}, \cdots) \tag{11.10}$$

where the right-hand side denotes a polynomial in the variables listed. Let $T^A{}_B$ denote the elements of a "transformation matrix," $\mathbf{T}$. Under the transformation $\mathbf{T}$, the dependent variables $F^{AB\cdots}$ and each of the independent variables are transformed according to a definite law of transformation. For our purposes here, we may assume that this law of transformation is

$$\begin{aligned} F^{AB\cdots} &\to T^A{}_C T^B{}_D \cdots F^{CD\cdots} \\ H_\Gamma^{AB\cdots} &\to T^A{}_C T^B{}_D \cdots H_\Gamma^{CD\cdots}. \end{aligned} \tag{11.11}$$

That is, the law of transformation is the linear homogeneous law of transformation for tensors. The notation of (11.10) and (11.11) may be conveniently shortened by the use of the more abstract symbolic notation $\mathbf{F} = \mathbf{F}(\mathbf{H}_1, \mathbf{H}_2, \cdots)$, $\mathbf{F} \to \mathbf{TF}$, $\mathbf{H}_\Gamma \to \mathbf{TH}_\Gamma$. If $\mathbf{F}$ is an *invariant* tensor function of the $\mathbf{H}_\Gamma$ under the transformation $\mathbf{T}$, then

$$\mathbf{TF} = \mathbf{F}(\mathbf{TH}_1, \mathbf{TH}_2, \cdots). \tag{11.12}$$

More generally, a *set* of elements $\mathbf{T}_\Gamma$ which form a group $\mathcal{G}$ under the multiplication law $T^A_{\Gamma B} T^B_{\Delta C} = T^A_{\Upsilon C}$, are defined and the functional relation (11.12) is required to be satisfied for each element of the group. The group may be finite or continuous. For example, $\mathbf{T}_\Gamma$ may be defined as an arbitrary element of the orthogonal group or T_Γ may be an arbitrary element of a finite subgroup of the orthogonal group. The tensor $\mathbf{F}$ is then said to be an invariant tensor function of the tensors $\mathbf{H}_\Gamma$ under the group of transformations $\mathcal{G}$. A fundamental theorem of the classical invariant theory states that an arbitrary tensor invariant *polynomial* function of a set of tensors $\mathbf{H}_\Gamma$ under any finite or compact group $\mathcal{G}$, is reducible to a *polynomial* in a *finite* set of basic tensor invariants $\mathbf{I}_1, \mathbf{I}_2, \cdots, \mathbf{I}_h$. If the set $\{\mathbf{I}\}$ is minimal, the set constitutes an *integrity basis* [18, p. 274]. The elements of an integrity basis may not be functionally independent. Polynomial relations called syzygies more often exist than not between the elements of an integrity basis. Many important special cases of the above stated problem of finding the integrity basis which is relevant to a given dependent tensor $\mathbf{F}$, a given set of variable tensors $\mathbf{H}_\Gamma$ and a given group $\mathcal{G}$ have been considered by the workers in this field of mathematics. It is sometimes difficult to translate these known results, particularly those results found in the older

literature on invariant theory, into the language we have chosen to state the problem here. However, a little patience is well rewarded and answers to many difficult problems can be found. For example, suppose that the $F^{AB\cdots}$ are constant functions and G is the orthogonal group, then (11.12) reduces to

$$T^{A}{}_{C}T^{B}{}_{D}\cdots F^{CD} = F^{AB},$$

where $T^{A}{}_{B}$ is an arbitrary solution of the equations $g_{AB}T^{A}{}_{C}T^{B}{}_{D} - g_{CD} = 0$. Then it is known [18, p. 144] that the elements g^{AB} of the metric tensor constitute an integrity basis. That is, an arbitrary constant, invariant tensor of the orthogonal group is expressible as a linear combination of outer products of g^{AB} with coefficients which transform as scalars under the group $\mathcal{G}$. An immediate corollary of this theorem is that there exists no constant invariant tensor of the orthogonal group which is of odd rank. If $\mathbf{F}$ is an invariant constant tensor, its rank is even and $\mathbf{F}$ has the general form

$$F^{A_1A_2\cdots A_{2n}} = C_{pq\cdots rs}g^{A_pA_q}\cdots g^{A_rA_s}, \tag{11.13}$$

where $(p, q, \cdots, r, s)$ is a permutation of the numbers $(1, 2, \cdots, 2n)$ and the $C_{pq\cdots rs}$ are scalars. We shall use the result (11.13) in §13 of this paper where we consider polynomial approximations to the stored energy function. We conclude this digression on classical invariant theory by pointing out that the integrity basis for scalar invariant functions of a single symmetric tensor, a single vector under the orthogonal group, the metric tensor g_{ij}, and its inverse g^{ij} under the full linear group (arbitrary coordinate transformations) is a known result of classical invariant theory [19, p. 61]. But this is just the problem which confronts us in the reduction of the stored energy function of an elastic dielectric after reaching the point (11.5) if we demand that Σ be a *polynomial* in the variables listed in (11.5) and we wish to preserve the polynomial character of Σ. The scalar invariants (11.7) and (11.9) also constitute an *integrity basis* for the same variables under the full linear group. That is, *if we assume that the stored energy function is a polynomial function of the components* $(c^{-1})_{ij}$, π^{i}, g_{ij}, g^{ij}, *it is expressible as a polynomial in the basic invariants* $I_1, I_2, \cdots, I_6$. Since the set of basic invariants $I_1, I_2, \cdots, I_6$ are functionally independent, there are no syzygies.

Let us assume that, for some range of values of the invariants I_Γ, Σ is a differentiable function of the I_Γ. Then, from (10.33) and (10.34) we have

$$t_{L}{}^{i}{}_{i}{}^{j} = \rho\sum_{\Gamma}\frac{\partial\Sigma}{\partial I_\Gamma}\frac{\partial I_\Gamma}{\partial x^{i}{}_{;A}}x^{j}{}_{;A} \tag{11.14}$$

$$\overline{E}{}^{i}_{L} = -\sum_{\Gamma}\frac{\partial\Sigma}{\partial I_\Gamma}\frac{\partial I_\Gamma}{\partial\pi_i}. \tag{11.15}$$

Working out the various expressions $(\partial I_{\mathrm{I}}/\partial x^{i}{}_{;A})x^{j}{}_{;A}$ and $\partial I_{\mathrm{I}}/\partial \pi_{i}$, and substituting in (11.14) and (11.15) we obtain the following expressions for the local stress and effective local field in an isotropic elastic dielectric:

$$t_{\mathrm{L}i}{}^{j} = 2\rho\Big\{I_3 \frac{\partial \Sigma}{\partial I_3}\,\delta^{j}{}_{i} + \Big(\frac{\partial \Sigma}{\partial I_1} + I_1 \frac{\partial \Sigma}{\partial I_2}\Big)(c^{-1})^{j}{}_{i} - \frac{\partial \Sigma}{\partial I_2}\,(c^{-2})^{j}{}_{i} + \frac{\partial \Sigma}{\partial I_4}\,(c^{-1})^{j}{}_{k}\pi^{k}\pi_{i} + \frac{\partial \Sigma}{\partial I_5}\,(c^{-2})^{j}{}_{k}\pi^{k}\pi_{i} + \frac{\partial \Sigma}{\partial I_5}\,(c^{-1})^{j}{}_{k}(c^{-1})_{i}{}^{l}\pi_{l}\pi^{k}\Big\} \tag{11.16}$$

$$\overline{E}^{i}_{\mathrm{L}} = -2\Big\{\frac{\partial \Sigma}{\partial I_4}\,(c^{-1})^{i}{}_{j} + \frac{\partial \Sigma}{\partial I_5}\,(c^{-2})^{i}{}_{j} + \frac{\partial \Sigma}{\partial I_6}\,\delta^{i}{}_{j}\Big\}\pi^{j}. \tag{11.17}$$

The above set of constitutive relations between the local stress, effective local field, strain, and polarization are the general form which they take in isotropic homogeneous materials if we assume the validity of the energy principle. In the next section, we shall determine some simple solutions of the equilibrium equations using the stress-strain-field-polarization relations for isotropic materials. These special solutions reveal many interesting physical phenomena predicted by the general non-linear theory.

12. Some Simple Solutions for an Arbitrary Form of the Stored Energy Function of Isotropic Dielectrics

We shall consider, first, the simple shearing of an infinite slab of homogeneous isotropic elastic dielectric whose deformed and undeformed boundaries are the planes $X^1 = 0, a$. We here choose the X^A and x^i coordinate systems to be one and the same rectangular Cartesian system. By simple shearing of the slab, we mean the deformation indicated by the mapping,

$$x^1 = X^1, \qquad x^2 = X^2 + \beta X^1, \qquad x^3 = X^3 \tag{12.1}$$

where the constant β is a measure of the amount of shear. In addition, let the dielectric be polarized in the amount,

$$\pi = (\pi_1 , \pi_2 , 0) \tag{12.2}$$

where π_1 and π_2 are constants throughout the slab. For the deformation (11.1), the $x^i{}_{;A}$ has the matrix of values

$$||x^i{}_{;A}|| = \begin{Vmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{12.3}$$

from which it follows that $J = \det |x^i{}_{;A}| = 1$, whence $\rho = \rho_0$. The deformation tensor $(c^{-1})^i{}_j$ has the form

$$||(c^{-1})^i{}_j|| = \begin{Vmatrix} 1 & \beta & 0 \\ \beta & 1 + \beta^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{12.4}$$

The Maxwell self field of the polarized slab is equivalent to that of a uniform surface charge density of strength $P \cdot n/e_0 = \pm \rho_0 \pi_1 / e_0$ on the faces of the slab, the positive sign holding on the face $x^1 = a$, the negative sign holding on the face $x^1 = 0$. The self field is of the form

$$\begin{aligned} \mathbf{E}_{\mathsf{MS}} &= [-\rho_0 \pi_1/e_0 , 0, 0] \qquad && 0 < x < a \\ \mathbf{E}_{\mathsf{MS}} &= [0, 0, 0] && x < 0 \\ & && x > a. \end{aligned} \tag{12.5}$$

Since the deformation, the polarization, *and the Maxwell self field* are homogeneous, the divergence of the local stress and the Maxwell stress vanish separately, $t^{ij}_{\mathsf{L}}{}_{;j} = t^{ij}_{\mathsf{MS}}{}_{;j} = 0$. Hence, the slab will be in equilibrium in the absence

of an extrinsic body force. The extrinsic body force will vanish provided the external field $\mathbf{E}_0$ is uniform throughout the slab. The external field $\mathbf{E}_0$ plus the Maxwell field $\mathbf{E}_{MS}$ must be equal and opposite to $-\partial\Sigma/\partial\pi^i$ at each interior point of the slab, so that

$$E_0^i = -E_{MS}^i + \frac{\partial\Sigma}{\partial\pi_i} \tag{12.6}$$

is a condition of equilibrium. Working out the consequences of this requirement, we find

$$E_0^1 = e_0^{-1}\rho_0\pi_1 + 2\Big\{(\pi_1 + \beta\pi_2)\frac{\partial\Sigma}{\partial I_4} + [(1+\beta^2)\pi_1 + \beta(2+\beta^2)\pi_2]\frac{\partial\Sigma}{\partial I_5} + \pi_1\frac{\partial\Sigma}{\partial I_6}\Big\} \tag{12.7}$$

$$E_0^2 = 2\Big\{[\beta\pi_1 + (1+\beta^2)\pi_2]\frac{\partial\Sigma}{\partial I_4} + \beta(2+\beta^2)\pi_1 + [(1+3\beta^2+\beta^4)\pi_2]\frac{\partial\Sigma}{\partial I_5} + \pi_2\frac{\partial\Sigma}{\partial I_6}\Big\} \tag{12.8}$$

$$E_0^3 = 0. \tag{12.9}$$

Consider for a moment the expression (12.8) for E_0^2 . It is a rather general function of the parameters π_1 , π_2 , and β,

$$E_0^2 = E_0^2(\pi_1 , \pi_2 , \beta), \tag{12.10}$$

whose form is not known explicitly until one specifies the stored energy function Σ explicitly. Let us regard the parameters β and π_1 as having specified values and attempt to satisfy the condition $E_0^2 = 0$, by suitable choice of π_2 . For given β, there is always the solution obtained by setting $\pi_1 = \pi_2 = 0$. Then, if the derivative $\partial E_0^2/\partial\pi_2$ is non-zero and continuous, we are assured of non-zero solutions $\pi_2 = \pi_2(\pi_1 , \beta)$ in some neighborhood of the zero solution. This follows from the theory of implicit functions [20]. Hence, we may contemplate an equilibrium state of a sheared elastic dielectric slab in the presence of an applied field $\mathbf{E}_0$ which is *normal* to the faces of the slab. In general, the polarization of the slab will have a component π_2 perpendicular to the applied field for non-zero values of the shear measure β. This will, of course, give rise to an extrinsic body moment of the form

$$m^{ij} = \begin{Vmatrix} 0 & -\rho_0\pi^2 E_0^1 & 0 \\ \rho_0\pi^2 E_0^1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{12.11}$$

tending to rotate the slab about the x^3 axis.

Consider, next, the boundary condition (10.31), which implies that to maintain the state of deformation and polarization, mechanical surface tractions T^i must be applied to the faces of the slab. They are given by

$$T^i = t_{\mathrm{L}}^{ij} n_j - [\![t_{\mathrm{MS}}^{ij}]\!] n_j \,. \tag{12.12}$$

We find that

$$[\![t_{\mathrm{MS}}^{ij}]\!] n_j = \pm(\tfrac{1}{2} e_0^{-1} (P_1)^2, 0, 0), \tag{12.13}$$

the positive sign holding at the surface $x^1 = a$, and the negative sign at the surface $x^1 = 0$. Hence, the Maxwell self field gives rise to a stress system which exerts an *apparent* normal surface traction on the faces of the slab which tends to elongate the slab. A normal surface traction over and above that required to balance the local stress $t_{\mathrm{L}}^{ij} n_j$ must be applied to maintain the prescribed state of deformation and polarization. Using the stress-deformation-polarization relations (11.15) for the local stress, we find that the total surface traction which must be applied to the face of the slab, $x^1 = a$, in order to maintain equilibrium has components given by

$$\begin{aligned} T_1 = 2\rho_0 \Big\{ & \Big(I_3 \frac{\partial \Sigma}{\partial I_3} + \frac{\partial \Sigma}{\partial I_1} + I_1 \frac{\partial \Sigma}{\partial I_2} \Big) - \frac{\partial \Sigma}{\partial I_2} (1 + \beta^2) \\ & + \Big(\frac{\partial \Sigma}{\partial I_4} + 2 \frac{\partial \Sigma}{\partial I_5} - \tfrac{1}{4} e_0^{-1} \rho_0 \Big) (\pi_1)^2 \\ & + \beta \Big(\frac{\partial \Sigma}{\partial I_4} + 4 \frac{\partial \Sigma}{\partial I_5} \Big) \pi_1 \pi_2 + \beta^2 \frac{\partial \Sigma}{\partial I_5} [(\pi_1)^2 + (\pi_2)^2] + \beta^3 \frac{\partial \Sigma}{\partial I_5} \pi_1 \pi_2 \Big\} \end{aligned} \tag{12.14}$$

$$\begin{aligned} T_2 = 2\rho_0 \Big\{ & \beta \Big(\frac{\partial \Sigma}{\partial I_1} + I_1 \frac{\partial \Sigma}{\partial I_2} - 2 \frac{\partial \Sigma}{\partial I_2} \Big) - \beta^3 \frac{\partial \Sigma}{\partial I_2} \\ & + \Big(\frac{\partial \Sigma}{\partial I_4} + 2 \frac{\partial \Sigma}{\partial I_5} \Big) \pi_1 \pi_2 + \beta \Big(\frac{\partial \Sigma}{\partial I_4} + 3 \frac{\partial \Sigma}{\partial I_5} \Big) (\pi_1)^2 \\ & + \beta \frac{\partial \Sigma}{\partial I_5} (\pi_2)^2 + \beta^2 \Big(\frac{\partial \Sigma}{\partial I_4} + 5 \frac{\partial \Sigma}{\partial I_5} \Big) \pi_1 \pi_2 \\ & + \beta^3 \frac{\partial \Sigma}{\partial I_5} [(\pi_1)^2 + (\pi_2)^2] + \beta^4 \frac{\partial \Sigma}{\partial I_5} \pi_1 \pi_2 \Big\} \end{aligned} \tag{12.15}$$

$$T_3 = 0. \tag{12.16}$$

At the face $x^1 = 0$, tractions equal in magnitude and opposite in direction to those listed above must be applied.

A number of qualitative features of the theory are apparent upon examination of the above expressions for the surface tractions. Since the theory is equivalent

to the theory of finite deformations of homogeneous isotropic perfectly elastic solids if one neglects the dependence of the stored energy function on the invariants I_4, I_5, and I_6, all of the qualitative features of that theory will occur here as possibilities even in the limit of vanishing polarization. If the various scalar coefficients in the expressions for surface traction are thought of as power series expansions about the undeformed and unpolarized state, $\pi_1 = \pi_2 = \beta = 0$, there occurs a term in the normal component of the surface traction of order zero in the polarization components and of second order in the shear measure β. This is the well known *Poynting effect*, whereby to maintain a state of finite simple shear, tangential surface tractions are insufficient. This is a non-linear effect and does not occur in the classical linear theory of elasticity. There also occur terms in the normal component of the surface traction which are of zero order in the shear parameter β and which involve the components of the polarization to at least the second power. We identify the existence of such terms with the well known electrostrictive effect. It is sometimes argued that the *electrostrictive effect* is due solely to the tendency of the Maxwell field to elongate the slab, but it is apparent here that what might be called the local field electrostrictive effect may either support this tendency to elongate or have an overriding influence in the opposite direction tending to shorten the slab. The ambiguity that remains is similar to that which remains at this stage of development of the theory of non-linear elasticity in any discussion of the sign of the Poynting effect. It is possible that arguments based on stability or thermodynamic inequalities may dictate the positive or negative character of the electrostrictive phenomenon, but we do not enter upon these questions here. It is also clear on examination of the terms in T_1 that there is a modification of the electrostrictive effect due to shearing of the slab. If we assume, as is reasonable, that $\pi_2 \sim \beta$, then this deformation-polarization cross effect is of at least second order in the shear parameter, and hence would not occur in a linear theory of the deformation.

As a second example, we consider the homogeneously deformed and polarized ellipsoid.

KELLOGG [16] exhibits the solution of the following problem in potential theory:

$$\nabla^2 U = -\kappa/e_0 \qquad x \,\varepsilon\, V, \qquad \kappa = \text{constant}, \tag{12.17}$$

$$\nabla^2 U^* = 0 \qquad x \,\varepsilon\, E - V. \tag{12.18}$$

The region V is the interior of an ellipsoid with semi-axes a_1, a_2, and a_3. The solution is rendered unique by the conditions, (1) $U = U^*$ at the boundary B of the ellipsoid, (2) $U_{,i} = U^*_{,i}$ at B, (3) U^* regular at infinity. We now demonstrate that φ, as determined by

$$\varphi = -U_{,i}P^i/\kappa \qquad x \,\varepsilon\, V, \qquad P^i = \text{constant}, \tag{12.19}$$

$$\varphi^* = -U^*_{,i}P^i/\kappa \qquad x \,\varepsilon\, E - V, \tag{12.20}$$

is the potential of a homogeneously polarized ellipsoid. That is, φ is a solution of the potential problem,

$$\nabla^2\varphi = 0 \qquad x \,\varepsilon\, V \tag{12.21}$$

$$\nabla^2\varphi^* = 0 \qquad x \,\varepsilon\, E - V \tag{12.22}$$

$$[e_0\varphi_{,i} - e_0\varphi^*{}_{,i} - P_i]n^i = 0 \qquad x \,\varepsilon\, B. \tag{12.23}$$

It follows from (12.17–18) that φ and φ^* satisfy (12.21–22). From the continuity of the gradient, $U_{,i}$, it follows [16] that the jump $[\![U_{,ij}]\!] \equiv U_{,ij} - U^*{}_{,ij} = K_i n_j$. The symmetry of $[\![U_{,ij}]\!]$ in the indices i and j allows one to set $[\![U_{,ij}]\!] = Kn_in_j$. We can evaluate K by taking the trace of this latter form and employing the field equations (12.17–18) satisfied by U and U^*. In this manner we deduce that

$$[\varphi_{,i}] \equiv \varphi_{,i} - \varphi^*{}_{,i} = -[\![U_{,ij}]\!]P^i/\kappa = e_0^{-1}n_in_jP^j. \tag{12.24}$$

Substituting this result for the jump in $\varphi_{,i}$ into the boundary condition (12.23), we readily verify that it is satisfied by this solution; furthermore, the solution is unique.

The solution for the potential U at interior points of the ellipsoid [16, p. 194] is

$$U = -A_1x^2 - A_2y^2 - A_3z^2 + D \tag{12.25}$$

where the A_Γ and D are positive constants given by

$$\begin{aligned} A_\Gamma &= \tfrac{1}{4}a_1a_2a_3\kappa e_0^1 \int_0^\infty ds/(a_\Gamma^2 + s)\sqrt{\psi(s)} \\ D &= \tfrac{1}{4}a_1a_2a_3\kappa e_0^{-1} \int_0^\infty ds/\sqrt{\psi(s)} \\ \psi(s) &= (a_1^2 + s)(a_2^2 + s)(a_3^2 + s). \end{aligned} \tag{12.26}$$

In the solution (12.25), the center of the ellipsoid is at the origin. The principal axes of the ellipsoid coincide with the coordinate axes of the rectangular Cartesian frame, (x, y, z). In a general coordinate system, U has the form,

$$U = -A_{ij}(r^i - r_0^i)(r^j - r_0^j) \tag{12.27}$$

where the proper values of the symmetric tensor A_{ij} are the A_Γ of (12.26), and the principal directions of A_{ij} are the principal directions of the ellipsoid. The r^i are the components of the position vector of an interior point of an ellipsoid centered at the point r_0^i. Thus, *the Maxwell self field,* $E^i_{\text{MS}} = -\varphi_{,}{}^i$, *in the interior of a homogeneously polarized ellipsoid, is a homogeneous field whose value is given by*

$$E^i_{\text{MS}} = -A^i{}_jP^j. \tag{12.28}$$

This is a known result of electrostatic theory, and is of major importance for obtaining non-trivial inverse solutions for a deformed and polarized elastic dielectric. It is conjectured that the ellipsoid and its various degenerate forms, such as the sphere and the infinite slab, are the only bodies for which a homogeneous polarization field leads to a *homogeneous* self field at interior points.

Consider the homogeneously deformed and polarized ellipsoid whose interior points satisfy the condition $S_{ij}(r^i - r_0^i)(r^j - r_0^j) \leqq 1$. A general homogeneous deformation may be characterized by the condition,

$$x^i{}_{;A;B} = x^i{}_{;A;j} = 0. \tag{12.29}$$

The equilibrium condition (10.29) requires the applied field E_0^i to be uniform throughout the body. Since the applied field is uniform, the extrinsic body force, which is proportional to the gradient of $\mathbf{E}_0$, vanishes. The equilibrium condition (10.22) reduces to $t_i{}^j{}_{;j} = 0$. Since all of the quantities upon which $t_i{}^j$ depends are constant tensor fields in a homogeneously deformed and polarized ellipsoid, the equilibrium condition (10.22) will be satisfied for homogeneous deformations of an elastic dielectric ellipsoid subjected to a uniform applied electric field.

Using the boundary condition (10.32), it can be shown that the non-local part of the stress, $t_{\mathrm{MS}i}{}^j$, *i.e.*, the Maxwell stress, always yields an *apparent* surface traction $[t_{\mathrm{MS}i}{}^j]n_j = \frac{1}{2}e_0^{-1}P_n^2 n_i$, where P_n is the component of polarization normal to the surface. The surface tractions required to maintain the homogeneously deformed and polarized ellipsoidal dielectric body in equilibrium will be

$$T_i = t_{\mathrm{L}i}{}^j n_j - \tfrac{1}{2}e_0^{-1}P_n^2 n_i \,. \tag{12.30}$$

The local part of the stress, $t_{\mathrm{L}i}{}^j$, will be given by the expression (11.15) with $(c^{-1})_i{}^j$ and π^i restricted to the class of constant tensors. The uniform external field required to maintain equilibrium will be

$$E_0^i = A^i{}_j P^j + \frac{\partial \Sigma}{\partial \pi_i}. \tag{12.31}$$

In principle, the solutions (12.30) and (12.31) for homogeneous deformation and polarization of an ellipsoid would be sufficient to determine the stored energy function $\Sigma(I_\Gamma)$ if sufficient data relating measured values of the $(T_i\,,\ E_0^i)$ to the corresponding values of $((c^{-1})_{ij}\,,\ \pi_i)$ were available.

As an illustration of the solutions (12.30–31), consider the case when the principal directions n_Γ^i of the deformation tensor $\mathbf{c}^{-1}$, and the principal axes of the ellipsoid coincide. Let π^i be in one of these directions, say n_1^i, and let π be its magnitude. For this case, the applied field given by (12.31), is parallel to the polarization and has magnitude E_0 given by

$$E_0 = \left[\rho A_1 + 2\frac{\partial \Sigma}{\partial I_4}(c_1)^{-1} + 2\frac{\partial \Sigma}{\partial I_5}(c_1)^{-2} + 2\frac{\partial \Sigma}{\partial I_6}\right]\pi. \tag{12.32}$$

At the vertices of the ellipsoid, the surface tractions are normal to the surface and have magnitudes $T^{(\Gamma)}$ given by

$$T^{(\Gamma)} = 2\rho\left\{I_3 \frac{\partial\Sigma}{\partial I_3} + \left(\frac{\partial\Sigma}{\partial I_1} + I_1 \frac{\partial\Sigma}{\partial I_2}\right)(c_\Gamma)^{-1} - \frac{\partial\Sigma}{\partial I_2}(c_\Gamma)^{-2}\right\} \qquad \Gamma = 2, 3$$

$$T^{(1)} = 2\rho\left\{I_3 \frac{\partial\Sigma}{\partial I_3} + \left(\frac{\partial\Sigma}{\partial I_1} + I_1 \frac{\partial\Sigma}{\partial I_2}\right)(c_1)^{-1} - \frac{\partial\Sigma}{\partial I_2}(c_1)^{-2}\right. \tag{12.33}$$

$$\left. + \left(\frac{\partial\Sigma}{\partial I_4}(c_1)^{-1} + 2\frac{\partial\Sigma}{\partial I_5}(c_1)^{-2} - \tfrac{1}{4}e_0^{-1}\rho\right)(\pi^2)\right\}.$$

The solution (12.32–33) for this subclass of homogeneous deformations and polarization of an ellipsoid is not sufficient to determine the stored energy function completely. The essential lack of generality in this solution results from the requirement that the polarization be in the direction of a principal axis of the deformation tensor. However, in principle, sufficient information about the form of Σ can be obtained from data relating measured values of the $(T^{(\Gamma)}, E_0)$ to the corresponding values of (c_Γ, π) in this class of solutions to enable one to make quantitative predictions in the general problem of plane strain of an elastic dielectric polarized in a direction normal to the plane of strain.

13. Anisotropic Dielectrics

In many applications of elastic dielectrics, the deformations are extremely small. Voigt's theory of the piezoelectric effect is based on constitutive relations for the stress and electric field which are linear in the displacement gradients and polarization. We wish to show that the linear constitutive relations of this classical theory are contained as a special case of (10.33) and (10.34). However, since the piezoelectric effect can occur only in anisotropic media with exceptional symmetry properties, we shall have to consider first the conditions imposed on the form of the stored energy function by the symmetry of an anisotropic medium.

A fundamental assumption of our energy principle is that the stored energy is a single valued function of the $9 + 3 = 12$ variables $x^i{}_{;A}$ and π^i. In the course of our discussion of isotropic dielectrics, we made the functional character of Σ more explicit by assuming that

$$\Sigma = \Sigma(x^i{}_{;A}, \pi^i, g_{ij}, g^i{}_A). \tag{13.1}$$

That is, the metric tensor and shifter were explicitly listed as variables. This was done so as not to exclude the dependence of Σ on variables such as $\pi_i = g_{ij}\pi^j$ or $\pi_A = g^i{}_A\pi_i$, which represent (measure) the *same* physical quantity. It was stated that, for isotropic materials, the energy function depends *only* on the variables listed in (13.1). This was motivated by the notion of material isotropy of the natural state.

Now in anisotropic dielectrics we shall again single out the natural state as the state of zero polarization.[14] It is the equilibrium state of the dielectric in the absence of applied surface tractions and external field. The local stress and effective local field vanish in the natural state.

The *point symmetry* of a crystalline medium may be fully characterized by a finite subgroup of the orthogonal group. We shall also be interested in anisotropic media which are not crystalline. For example, materials which possess *transverse isotropy* are of some interest in elasticity theory [15, 21, p. 160]. We shall make our discussion of anisotropic media general enough to include the case of *curvilinear anisotropy* [21, p. 164]. An example of a curvilinear anisotropic state of a continuous medium is the case of an elastic medium which is isotropic in some natural state and which has been deformed inhomogeneously. At each point in the deformed medium, the symmetry may be characterized by the group of orthogonal transformations which generate the eight equivalent points of the Cauchy deformation ellipsoid (the quadric of c_{ij}) corresponding to a given point on the ellipsoid. Stated otherwise, it is the subgroup of the orthogonal group which leaves the Cauchy deformation quadric invariant.

Whatever may be the material point symmetry of a particular state of a continuous

[14] There are examples of materials possessing a permanent polarization or electric moment. We do not consider these materials here.

medium, we shall assume that the symmetry is fully characterized by some subgroup $\mathcal{G}$ *of the orthogonal group* $\mathcal{O}$.

This subgroup may be finite as in the case of crystalline media, or may be continuous as in the case of transversely isotropic media. In a given coordinate system X^A with metric g_{AB}, each element of the orthogonal group has a matrix representation $T^A{}_B$ which satisfies the equations

$$g_{AB}T^A{}_C T^B{}_D - g_{CD} = 0. \tag{13.2}$$

Under coordinate transformations, the elements of the matrix $T^A{}_B$ transform as the elements of a mixed tensor of rank two. Hence (13.2) is a coordinate invariant definition of an "orthogonal matrix."

An *invariant tensor* of the group $\mathcal{G}$ is any tensor which satisfies each of the equations,

$$T^{A_1}{}_{B_1} T^{A_2}{}_{B_2} \cdots T^{A_n}{}_{B_n} H^{B_1 B_2 \cdots B_n} = H^{A_1 A_2 \cdots A_n}, \tag{13.3}$$

for every element T of the group. If $H^{AB\cdots}$ is an invariant tensor of the group which characterizes the point symmetry of the *natural* state of a continuous elastic medium, we shall call $H^{AB\cdots}$ a *material descriptor* or simply a *material tensor*. Note that an arbitrary scalar satisfies (13.3).

If the natural state is *homogeneous*, all the material descriptors are *spatially constant*. A spatially constant tensor is one whose covariant derivative vanishes.

The characteristic group for materials whose natural state is isotropic is the complete orthogonal group. According to a previously mentioned result (11.12), the material descriptors of isotropic materials are tensors of even rank which can be constructed by taking linear combinations of products of the metric tensor. The coefficients in these linear combinations are arbitrary scalar material descriptors. If an isotropic material is also homogeneous, it follows that each of these scalar coefficients must be spatially constant.

If the characteristic group of the material symmetry of the natural state of an elastic dielectric is a *proper* subgroup of the orthogonal group we shall call the dielectric *anisotropic*. If the material descriptors are not all spatially constant, we shall say that the material is *inhomogeneous*. If the material symmetry of the natural state is described by one of the thirty-two finite crystal groups, we shall call the material a *crystalline dielectric*.

An assumption which generalizes (13.1) to the case of anisotropic media and which includes (13.1) as a special case is that the stored energy function depends only on the following list of variables:

$$\Sigma = \Sigma(x^i{}_{;A}, \pi^i, H_\Gamma^{AB\cdots}, g_{ij}, g^i{}_A) \tag{13.4}$$

where the set of tensors $H_\Gamma^{AB\cdots}$ ($\Gamma = 1, 2, \cdots$) denotes the set of material descriptors. Let us adopt the convention that given the dependence of Σ on the contravariant components f^i of any tensor that the possible dependence of Σ on the covariant or shifted components, $f_i = g_{ij}f^j$, $f^A = g^A{}_i f^i$ is to be understood. Using this standard convention, (13.4) may be shortened to

$$\Sigma = \Sigma(x^i{}_{;A}, \pi^i, H_\Gamma^{AB\cdots}). \tag{13.5}$$

We have introduced, so far, three distinct types of transformations on the variables which occur in (13.5). These are: (I) independent *coordinate transformations* of the two coordinate systems which simultaneously span the Euclidean space; (II) the two groups of *rotations*—the first of which corresponds to rigid rotations of the deformed and polarized dielectric, the second of which corresponds to a rigid rotation of the natural state of the dielectric; (III) the subgroup $\mathcal{G}$ of the orthogonal group which describes the material point symmetry of the dielectric. The law of transformation of each of the variables under these three groups of transformations is as follows:

(I) Under coordinate transformations, each of the variables transforms according to the general transformation law for two-point tensor fields given in §3. The index notation and the convention regarding the type and position of the indices are sufficient to identify at a glance the law of transformation of any particular set of variables in (13.5).

(II) A. Under the group of rigid rotations of the deformed and polarized dielectric, the variables transform according to

$$\begin{aligned} x^i{}_{;A} &\to S^i{}_j x^j{}_{;A} \\ \pi^i &\to S^i{}_j \pi^j \\ H_\Gamma^{AB\cdots} &\to H_\Gamma^{AB\cdots}. \end{aligned} \tag{13.6}$$

(II) B. Under the group of rigid rotations of the natural state of the dielectric, the variables transform according to

$$\begin{aligned} x^i{}_{;A} &\to S^B{}_A x^i{}_{;B} \\ \pi^i &\to \pi_i \\ H_\Gamma^{AB\cdots} &\to S^A{}_C S^B{}_D \cdots H_\Gamma^{CD\cdots}. \end{aligned} \tag{13.7}$$

Note that material descriptors are invariant under a rigid rotation of the deformed and polarized state and that the polarization is not. The converse is true for rigid rotations of the natural state.

(III) Under the group $\mathcal{G}$, characteristic of the material symmetry, the variables transform according to

$$\begin{aligned} x^i{}_{;A} &\to x^i{}_{;A} \\ \pi^i &\to \pi^i \\ H_\Gamma^{AB\cdots} &\to T^A{}_C T^B{}_D \cdots H_\Gamma^{CD\cdots} \end{aligned} \tag{13.8}$$

where $\mathbf{T}$ is an arbitrary element of the group $\mathcal{G}$. Note that under this group of transformations, the displacement gradients and the polarization are invariants.[15]

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[15] In physical theories, yet another type of transformation of the variables is important. These are the dimensional transformations. TRUESDELL [12] has considered the restrictions placed on the form of some types of constitutive relations by the requirement of dimensional invariance.

A fundamental assumption in this theory of anisotropic elastic dielectrics is that the stored energy function of deformation and polarization is absolutely invariant under each of the three types of transformations (I), (II), *and* (III).

We shall consider the restrictions placed on the form of the energy function by each type of transformation in the order (III), (II), (I). Since we have already assumed that the $H_\Gamma^{AB}\ldots$ are material descriptors, no further conditions on the form of Σ follow from (III). That is, each and every variable listed in (13.5) is invariant under (III), hence an arbitrary function of the variables is invariant under (III). Next, consider the transformations (II). Precisely the same argument which carried us from (11.1) to (11.5) in the case of isotropic materials allows us to conclude that if Σ is invariant to rigid rotations of the deformed and polarized state, it is reducible to a function of the variables indicated now:[16]

$$\Sigma = \Sigma(C_{AB}, \Pi_A, H^{AB\cdots}), \tag{13.9}$$

where the C_{AB} are the components of the Cauchy measure of strain $C_{AB} \equiv g_{ij}x^i{}_{;A}x^j{}_{;B}$, and the Π_A are given by $\Pi_A \equiv x^i{}_{;A}\pi_i$. Note that the Π_A are not the shifted components of π_i. We shall refrain from giving the vector Π_A any physical interpretation.

In the case of isotropic materials, we were able at this point to use the invariance of Σ under the coordinate transformations (I) to demonstrate that a single valued stored energy function of an isotropic dielectric must reduce to a function of only six independent variables—namely, the six scalar invariants (11.7–8). A further result which we were able to establish was that the same six scalar invariants constituted an integrity basis. Unfortunately, we are unable to proceed with such generality here in the case of anisotropic media. This is so for a number of reasons. First of all, the number and rank of the material descriptors is not known until the material symmetry is specified. Each type of material symmetry must therefore be considered separately. Depending on the type of material symmetry, the invariant theoretic problem which must be solved may involve considerable labor. For these reasons and others, we have assumed a special form for the energy function which allows us to proceed with the theoretical development without a specification of the material symmetry. The classical linear theory of the piezoelectric effect will be shown to follow from this special form for the energy function if the special constitutive relations for the local stress and effective local field are linearized.

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[16] We have omitted writing π^2 in the list (13.9) since it is a single valued function of the C_{AB} and Π_A. In discussing the isotropic case, we retained π^2 in the list of variables since we were also interested in the case where Σ was a polynomial in the variables listed. Since π^2 is not a polynomial in the C_{AB} and Π_A, a polynomial in the set C_{AB}, Π_A, π^2 is not reducible in general to a polynomial in the functionally independent set C_{AB} and Π_A. Of course, π^2 can be written as a power series in the remaining variables so that polynomials in the C_{AB} and Π_A can approximate a given analytic function of π^2 as closely as desired. Since we shall later assume that Σ is a polynomial in C_{AB} and Π_A, the effect of eliminating π^2 from the list (13.9) should be understood at this time.

14. A Special Form for the Stored Energy Function—Polynomial Approximations

We have shown that the energy function of an anisotropic (includes isotropic as a special case) dielectric is reducible to the variables listed in (13.9). For purposes of approximation, it is convenient to introduce the tensor measure of strain $E_{AB} = (C_{AB} - g_{AB})$ which vanishes in the natural state. No loss in generality is incurred by writing (13.9) in the form

$$\Sigma = \Sigma(E_{AB}, \Pi_A, H_\Gamma^{AB\cdots}). \tag{14.1}$$

Any single valued absolute scalar function of these variables is a possible energy function for some elastic dielectric. Since each of the variables E_{AB} and Π_A vanishes in the natural state, we have been led to consider the following special form of the function

$$\begin{aligned}\rho_0\Sigma = {} & H_0{}^A\Pi_A + H_1{}^{AB}\Pi_A\Pi_B + H_2{}^{AB}E_{AB} + H_3{}^{ABCD}E_{AB}E_{CD} + H_4{}^{ABC}E_{AB}\Pi_C \\ & + H_5{}^{ABCD}E_{AB}\Pi_C\Pi_D + H_6{}^{ABCDE}E_{AB}E_{CD}\Pi_E \\ & + H_7{}^{ABCDEF}E_{AB}E_{CD}\Pi_E\Pi_F,\end{aligned} \tag{14.2}$$

where the $H_\Gamma^{AB\cdots}$ ($\Gamma = 1, 2, \cdots, 7$) are independent of the E_{AB} and Π_A. It follows that the $H_\Gamma^{AB\cdots}$ are material descriptors of the rank and type as indicated by the number and position of their indices. The invariance of Σ under each type of transformation (I), (II), and (III) is insured. Since this special form of the stored energy function satisfies all of the invariance requirements *without approximation*, we can regard it in either of two ways: (1) the exact form of the energy function of an elastic dielectric which may or may not be found in nature, or (2) the first few terms in a power series expansion about the natural state of an arbitrary elastic dielectric. As a polynomial in the E_{AB} and Π_A we have included all possible terms of order zero, one, and two in the E_{AB} and all possible terms of order zero, one, and two in the polarization. In the sense of (2), it can only be expected that quantitative predictions based on this form of the energy function will be accurate for sufficiently small values of the strains and for weak fields.[17]

Many of the terms in (14.2) may vanish identically owing to the material symmetry. For example, if the natural state is isotropic, there are no material descriptors of odd rank; hence, no terms of odd degree in the polarization can occur in (14.2) if the dielectric is isotropic. This constitutes a formal reason why

[17] From dimensional considerations which we do not give here, it is possible to establish dimensionless criteria for "small strain and weak fields."

isotropic dielectrics do not exhibit the piezoelectric effect. The tensor coefficients in (14.2) will be assumed to have the same symmetry as the tensor of variables. For example, in the term $H_1{}^{AB}\Pi_A\Pi_B$, only the symmetric part of the tensor $H_1{}^{AB}$ contributes to the value of the indicated sum over A and B; therefore, it is assumed that $H_2{}^{AB} = H_2{}^{BA}$.

We have established the general formulæ for the local stress and effective local field when the energy principle is adopted. These are

$$t_{\mathrm{L}i}{}^{j} = \rho \frac{\partial \Sigma}{\partial x^{i}{}_{;A}} x^{j}{}_{;A} \tag{14.3}$$

$$\overline{E}_{\mathrm{L}}{}^{i} = -\frac{\partial \Sigma}{\partial \pi_i}. \tag{14.4}$$

It is a matter of straightforward calculation now to determine the form of the stress-strain-field-polarization relations of an anisotropic media whose stored energy function is given by (14.2) or whose stored energy function is approximated by this form for sufficiently small strain and polarization. To carry out this calculation, it is convenient to introduce the following preliminary formulæ:

$$x^{j}{}_{;C}\frac{\partial E_{AB}}{\partial x_{i;C}} = x^{i}{}_{;B}x^{j}{}_{;A} + x^{i}{}_{;A}x^{j}{}_{;B} \equiv M^{ij}_{AB} ,$$

$$x^{j}{}_{;B}\frac{\partial \Pi_A}{\partial x_{i;B}} = x^{j}{}_{;A}\pi^{i} \equiv N^{ij}_{A}$$

$$\frac{\partial \Pi_A}{\partial \pi_i} = x^{i}{}_{;A} .$$

We find that the local stress and effective local field have the form

$$\begin{aligned}\frac{\rho_0}{\rho} t_{\mathrm{L}}^{ij} &= H_0{}^{A}N^{ij}_{A} + 2H_1{}^{AB}N^{ij}_{A}\Pi_B + H_2{}^{AB}M^{ij}_{AB} + 2H_3{}^{ABCD}M^{ij}_{AB}E_{CD} \\ &+ H_4{}^{ABC}M^{ij}_{AB}\Pi_C + H_4{}^{ABC}E_{AB}N^{ij}_{C} + H_5{}^{ABCD}M^{ij}_{AB}\Pi_C\Pi_D \\ &+ 2H_5{}^{ABCD}E_{AB}\Pi_C N^{ij}_{D} + 2H_6{}^{ABCDE}E_{AB}M^{ij}_{CD}\Pi_E + H_6{}^{ABCDE}E_{AB}E_{CD}N^{ij}_{E} \\ &+ 2H_7{}^{ABCDEF}E_{AB}\Pi_E\Pi_F M^{ij}_{CD} + 2H_7{}^{ABCDEF}E_{AB}E_{CD}\Pi_E N^{ij}_{F} ,\end{aligned} \tag{14.5}$$

$$\begin{aligned}\rho_0\overline{E}_{\mathrm{L}}^{i} = -[&H_0{}^{A}x^{i}{}_{;A} + 2H_1{}^{AB}x^{i}{}_{;A}\Pi_B + H_4{}^{ABC}E_{AB}x^{i}{}_{;C} \\ &+ 2H_5{}^{ABCD}E_{AB}\Pi_C x^{i}{}_{;D} + H_6{}^{ABCDE}E_{AB}E_{CD}x^{i}{}_{;E} \\ &+ 2H_7{}^{ABCDEF}E_{AB}E_{CD}\Pi_E x^{i}{}_{;F}].\end{aligned} \tag{14.6}$$

It follows from (14.5) that, if the local stress is to vanish in the natural state, we must set $H_2{}^{AB} = 0$. Similarly, it follows from (14.6) that, if the effective

local field is to vanish in the natural state, we must set $H_0{}^A = 0$. Since M^{ij}_{AB} is symmetric in i and j, it is a simple matter to write down the antisymmetric part of the local stress. We have

$$\frac{\rho_0}{\rho} t_L^{[ij]} = [2H_1{}^{AB}\Pi_B + H_4{}^{BCA}E_{BC} + 2H_5{}^{DBCA}E_{DB}\Pi_C + H_6{}^{EBCDA}E_{EB}E_{CD} + 2H_7{}^{FBCDEA}E_{FB}E_{CD}\Pi_E]N_A^{[ij]}. \tag{14.7}$$

Multiplying (14.6) by π^j and taking the antisymmetric part of the tensor $(\rho_0/\rho)E_L{}^iP^j$ obtained in this manner, we verify that

$$\frac{\rho_0}{\rho} t_L^{[ij]} = \frac{\rho_0}{\rho} \overline{E}_L^{[i}P^{j]}. \tag{14.8}$$

This result serves to check the general theorem proven in §10, (10.36). That is, since the special form for the energy function (14.2) is invariant to rigid rotations of the deformed and polarized dielectric, we are thereby insured that the moment equation will be satisfied as an algebraic identity if the constitutive relations (14.5) and (14.6) for the local stress and effective local field are adopted.

We next take up the questions of approximate constitutive relations, linearizations, small rotations, approximate invariance of the stored energy function, *etc.* We shall show that by linearizing the forms (14.5) and (14.6) we obtain VOIGT's piezoelectric constitutive relations.

15. Linearizations of the Constitutive Relations of an Elastic Dielectric

To effect a comparison of the constitutive relations (14.5–6) corresponding to the special form of the stored energy function (14.2) with the stress and field relations of Voigt's linear theory we must first write (14.5–6) in terms of the displacement gradients $u_{i;j}$ or $U_{A;B}$ which were defined in §4. As pointed out in §4, the symmetric part of either of these tensors is the customary measure of infinitesimal strain. To first order terms in the components of either set of these displacement gradients we have $u_{i;j} \approx g^{A}{}_{i} g^{B}{}_{j} U_{A;B}$. That is, according to the convention by which we regard the shifted components of a tensor merely as a different representation of the *same* tensor, no distinction need be made between the two sets of displacement gradients $u_{i;j}$ and $U_{A;B}$; however, if the displacement gradients are not regarded as infinitesimals they are not equivalent tensors.[18] In order to show that Voigt's linear piezoelectric constitutive relations are contained in (14.5–6) as a special case, it is sufficient to linearize these expressions with respect to both the displacement gradients $u_{i;j}$ and the polarization π_i . In order to show that various non-linear generalization of Voigt's linear relations which have been proposed are *not* contained as a special case of (14.5–6), it is sufficient to linearize with respect to the displacement gradients only and to retain all the terms in the polarization. Therefore, for either purpose, we may first examine the approximate form of (14.5–6) obtained by regarding the displacement gradients $u_{i;j}$ (or $U_{A;B}$) as infinitesimals. This linearization process is facilitated by the use of the approximate relations:

$$M^{ij}_{AB} \approx (g^{i}{}_{A} g^{j}{}_{B} + g^{i}{}_{B} g^{j}{}_{A}) + (g^{i}{}_{A} g^{k}{}_{B} + g^{i}{}_{B} g^{k}{}_{A}) u^{j}{}_{;k} + (g^{i}{}_{A} g^{k}{}_{B} + g^{j}{}_{B} g^{k}{}_{A}) u^{i}{}_{;k}$$

$$N^{ij}_{A} \approx g^{j}{}_{A} \pi^{i} + g^{k}{}_{A} u^{j}{}_{;k} \pi^{i}$$

$$x^{i}{}_{;A} \approx g^{i}{}_{A} + g^{k}{}_{A} u^{i}{}_{;k} .$$

Substituting these approximate expressions for M^{ij}_{AB} , N^{ij}_{A} , and $x^{i}{}_{;A}$ into (14.5–6) we have obtained the following approximate relations for the stress and field:

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[18] One often finds non-linear generalizations of Voigt's linear piezoelectric constitutive relations which involve non-linear expressions in *the* displacement gradients. It would seem desirable in such circumstances to have a more explicit definition of what is meant by *the* displacement gradients. The components of these tensors have different transformation laws under rigid rotations of the deformed and natural states. A little thought will reveal that the transformation laws of the $u_{i;j}$ and $U_{A;B}$ are extremely complicated if any non-linear terms in these quantities are retained in a consistent fashion. It is for this reason that we have preferred to use the $x^{i}{}_{;A}$ as independent variables since their transformation law is considerably simpler than the transformation law of the $u_{i;j}$ or $U_{A;B}$ under rotations of the deformed or undeformed material.

$$\begin{aligned}\frac{\rho_0}{\rho}\, t_{\mathrm{L}}^{ij} \approx\ & 2H_1{}^{ik}\pi^i\pi_k + 2H_1{}^{ik}u^l{}_{;k}\pi^i\pi_l + 2H_1{}^{kl}u^j{}_{;k}\pi^i\pi_l + 4H_3{}^{ijkl}\tilde{e}_{kl} \\ & + 2H_4{}^{ijk}\pi_k + 2H_4{}^{ijk}u^l{}_{;k}\pi_l + 2H_4{}^{ikl}u^j{}_{;k}\pi_l + 2H_4{}^{jkl}u^i{}_{;k}\pi_l + H_4{}^{kli}\tilde{e}_{kl}\pi^i \\ & + 2H_5{}^{ijkl}\pi_k\pi_l + 2H_5{}^{imkl}u^j{}_{;m}\pi_k\pi_l + 2H_5{}^{jmkl}u^i{}_{;m}\pi_k\pi_l + 4H_5{}^{ijkl}u^m{}_{;l}\pi_k\pi_m \\ & + 2H_5{}^{klmi}\tilde{e}_{kl}\pi_m\pi^i + 2H_6{}^{klij m}\tilde{e}_{kl}\pi_m + 2H_7{}^{klijmn}\tilde{e}_{kl}\pi_m\pi_n\ ,\end{aligned} \tag{15.1}$$

$$\rho_0\overline{E}_{\mathrm{L}}^i \approx -[2H_1{}^{ik}\pi_k + 2H_1{}^{ik}u^l{}_{;k}\pi_l + 2H_1{}^{kl}u^i{}_{;k}\pi_l + H_4{}^{kli}\tilde{e}_{kl} + 2H_5{}^{klmi}\tilde{e}_{kl}\pi_m]. \tag{15.2}$$

For infinitesimal displacement gradients we also have $(\rho_0/\rho) = 1 + \tilde{e}^k{}_k$; hence, this factor may be cleared from the expression for the stress by multiplying each term on the right which does not already contain a displacement gradient by the factor $(1 - \tilde{e}^k{}_k)$. Note that, even for infinitesimal displacement gradients, the components of the local stress and effective local field do *not* in general reduce to polynomials in the symmetric part of the displacement gradients only, as is sometimes assumed.

If we now completely linearize these constitutive relations by dropping all terms involving squares of the polarization or a product of a polarization component and a displacement gradient, we obtain the linear relations:

$$t_{\mathrm{L}}^{ij} \approx 4H_3{}^{ijkl}\tilde{e}_{kl} + 2H_4{}^{ijk}\pi_k \tag{15.3}$$

$$\overline{E}_{\mathrm{L}}^i \approx -[2H_1{}^{ik}\pi_k + H_4{}^{kli}\tilde{e}_{kl}]. \tag{15.4}$$

Now, at *static* equilibrium we have $\overline{\mathbf{E}}_{\mathrm{L}} + \mathbf{E}_{\mathrm{M}} = 0$, (§8, (8.3)), where $\mathbf{E}_{\mathrm{M}}$ is the total Maxwell field at a point inside the dielectric. This is the field which occurs in VOIGT's relations. Also, the total stress t^{ij} which is always symmetric if we neglect the Maxwell stress tensor (a legitimate approximation in this linearized theory since it always involves the field or polarization squared) must be assumed to be the stress tensor referred to in VOIGT's theory, since the concept of a local stress is not introduced. For the linearized theory, this is not an issue since $t^{ij} - t_{\mathrm{L}}^{ij}$ is negligible; hence, we may set $t^{ij} \approx t_{\mathrm{L}}^{ij}$ in (15.3). Thus we have from (15.3) and (15.4) that at *static* equilibrium

$$t^{ij} \approx c^{ijkl}e_{kl} + q^{ijk}P_k \tag{15.5}$$

$$e_0E_{\mathrm{M}}^i \approx (\chi^{-1})^{ik}P_k + p^{kli}e_{kl} \tag{15.6}$$

with $c^{ijkl} = 4H_3{}^{ijkl}$, $q^{ijk} = (2/\rho_0)H_4{}^{ijk}$, $(\chi^{-1})^{ij} = (2e_0/\rho_0{}^2)H_1{}^{ij}$, and $p^{kli} = (e_0/\rho_0)H_4{}^{kli}$.

The linear relations (15.5) and (15.6) are identical in form to the piezoelectric

relations proposed by VOIGT. The values of the components of the tensors c^{ijkl}, q^{ijk}, p^{ijk}, and χ^{ij} in certain special coordinate systems are called by some authors, elastic constants, piezoelectric constants, and susceptibility constants. Note that within the context of the theory of elastic dielectrics given here, (15.5–6) cannot be strictly regarded as constitutive relations. In order to obtain these particular formulæ we had to assume that the dielectric was in static equilibrium. That is, to eliminate the effective local field from (15.4) and obtain the relations (15.6) involving the Maxwell field, we used the static equilibrium condition $\overline{\mathbf{E}}_{\mathrm{L}} + \mathbf{E}_{\mathrm{M}} = 0$. It is to be expected that in the dynamic case, the right-hand side of this last equation will not be zero. In elasticity theory when acceleration forces are taken into account, the constitutive relations for the stress in terms of the strain are the same as they are in the static case. Thus, by analogy, we can expect that the constitutive relations (15.3–4) would be unaltered if we were to treat dynamic equilibrium of an elastic dielectric. However, it is not expected that VOIGT's linear relations will follow as a necessary consequence of the linear constitutive relations of this theory except in the special case of static equilibrium which has been considered throughout this paper.

16. The Linear Constitutive Relations for Isotropic Elastic Dielectrics

We have already considered the isotropic elastic dielectric and have obtained the form of the constitutive relations for the stress and effective local field corresponding to an arbitrary isotropic stored energy function (§11, (11.16–17)). It is instructive, however, to follow the formalism developed for anisotropic materials in §13, regarding the isotropic dielectric as a special case. The stress and field relations (15.1) and (15.2) which have been linearized with respect to displacement gradients will be specialized to the isotropic case. The work in this section will serve to illustrative the methods we use to treat materials of arbitrary symmetry. Also, the isotropic form of (15.1–2) can be compared with the work of Helmholtz [22, pp. 140–146].

According to the formalism of §13, if the natural state of an elastic dielectric is isotropic, the material descriptors H_Γ^{AB} are invariant tensors of the orthogonal group. Invariant tensors of the orthogonal group are also called *isotropic tensors*. As pointed out before, it is known that the most general form of an isotropic tensor is a linear combination of outer products of the metric tensor.[19] Given this result, we can now reduce the approximate constitutive relations (15.1) and (15.2) to their most general form in isotropic dielectrics. First, the material descriptors $H_4{}^{iik}$ and $H_6{}^{iiklm}$ must vanish identically since there are no isotropic tensors of odd rank. The remaining material descriptors can now be written down in their most general form allowed by isotropic symmetry. These are

$$
\begin{aligned}
2H_1{}^{ii} &= a_1 g^{ii} \\
4H_3{}^{iikl} &= \lambda g^{ii}g^{kl} + \mu(g^{ik}g^{il} + g^{il}g^{ik}) \\
2H_5{}^{iikl} &= a_2 g^{ii}g^{kl} + a_3(g^{ik}g^{il} + g^{il}g^{ik}) \\
2H_7{}^{iiklmn} &= a_4 g^{ii}g^{kl}g^{mn} + a_5(g^{ik}g^{il} + g^{ik}g^{il})g^{mn} \\
&\quad + a_6[(g^{im}g^{in} + g^{im}g^{in})g^{kl} + (g^{km}g^{ln} + g^{lm}g^{kn})g^{ii}] \\
&\quad + a_7[g^{ik}(g^{im}g^{ln} + g^{in}g^{lm}) + g^{ik}(g^{im}g^{ln} + g^{in}g^{lm}) \\
&\quad + g^{il}(g^{im}g^{kn} + g^{in}g^{km}) + g^{il}(g^{im}g^{kn} + g^{in}g^{km})].
\end{aligned}
\tag{16.1}
$$

---

[19] There exists an elegant and simple theorem in the theory of group representations (the theorem on characters) which enables one to calculate the number of linearly independent invariant tensors of given rank for finite or compact groups. This application of the theory of group representations was first made by Racah [23] to calculate the number of linearly independent invariant tensors of given rank for the rotation subgroup of the orthogonal group in three dimensions. Invariant tensors of the rotation group are called hemitropic tensors. Bhagavantam and his coworkers [24] have made many applications of this same theorem on characters in the study of invariant tensors of the crystal groups.

Each of the above material tensors has been symmetrized to have the same symmetry as the tensor of variables in (14.2). For example, since H_7^{iiklmn} is the tensor of coefficients of the variable tensor $E_{ii}E_{kl}\Pi_m\Pi_n$, in writing (16.1) we have made H_7^{iiklmn} symmetric in each of the pairs (ij), (kl), and (mn) and also symmetric under an interchange of the pairs (ij) and (kl). This is important since it will in general reduce the number of scalar material descriptors necessary to define the material tensor descriptor completely. For example, there are three linearly independent isotropic tensors of rank four but only two with the symmetry of either H_3^{iikl} or H_5^{iikl}. We can determine from (16.1) that nine scalar material descriptors are necessary to determine the special form of the stored energy function (14.2) in the isotropic case. If the dielectric is also homogeneous, these nine descriptors will be spatially constant. It is appropriate to call these scalars *material constants*. Substituting (16.1) into (15.1–2), we obtain

$$\begin{aligned} t_{\mathsf{L}}^{ij} \approx\ & [\lambda\tilde{e}^k{}_k + a_2\pi^2 + 2(a_2 + a_6)\boldsymbol{\pi}\cdot\mathbf{e}\cdot\boldsymbol{\pi} + (a_4 - a_2)\tilde{e}^k{}_k\pi^2]g^{ij} \\ & + [2\mu + 2(a_2 + a_5)\pi^2]\tilde{e}^{ij} + [(a_1 + 2a_3) + (a_2 + 2a_6 - a_1 - 2a_3)\tilde{e}^k{}_k]\pi^i\pi^j \\ & + 2(a_1 + 3a_3 + 2a_7)\tilde{e}^i{}_k\pi^i\pi^k + 4(a_3 + a_7)\tilde{e}^i{}_k\pi^i\pi^k, \end{aligned} \tag{16.2}$$

$$-\rho_0\overline{E}_{\mathsf{L}}^i \approx (a_1 + a_2\tilde{e}^k{}_k)\pi^i + 2(a_1 + a_3)\tilde{e}^i{}_k\pi^k. \tag{16.3}$$

From (16.2–3), it follows that the antisymmetric part of the local stress is given by

$$t_{\mathsf{L}}^{[ij]} \approx \overline{E}_{\mathsf{L}}^{[i}P^{j]} = 2(a_1 + a_3)\tilde{e}^{[i}{}_k\pi^{i]}\pi^k. \tag{16.4}$$

We may call the stress-strain-field-polarization relations (15.1–2) the quasilinear constitutive relations of an elastic dielectric since, in obtaining these forms, we have linearized with respect to displacement gradients but have retained nonlinear terms in the polarization and terms which are a product of a displacement gradient and a polarization component. Note that the *isotropic* quasilinear local stress *is* a polynomial in the components of the infinitesimal strain measure $\tilde{e}_{ij}$; whereas, the quasilinear local stress of an anisotropic dielectric does not in general reduce to a polynomial in the symmetric part of the displacement gradients only.

The quasilinear constitutive relations for isotropic materials may be further specialized by dropping all the terms which contain a product of a displacement gradient and a component of polarization. This process yields

$$\begin{aligned} t_{\mathsf{L}}^{ij} &\approx \lambda\tilde{e}^k{}_k g^{ij} + 2\mu\tilde{e}^{ij} + a_2\pi^2 g^{ij} + (a_1 + 2a_3)\pi^i\pi^j \\ \rho_0\overline{E}_{\mathsf{L}}^i &= -a_1\pi^i. \end{aligned} \tag{16.5}$$

Then using the equilibrium condition $E_{\mathsf{M}} + \overline{E}_{\mathsf{L}} = 0$, we can write the local stress in this approximation in the form

$$t_{\mathsf{L}}^{ij} \approx \tilde{e}^k{}_k g^{ij} + 2\mu\tilde{e}^{ij} + A_1E_{\mathsf{M}}^2g^{ij} + A_2E_{\mathsf{M}}^iE_{\mathsf{M}}^j \tag{16.6}$$

where the A's are material constants. This last formula is identical to the stress-strain-field-relation derived by STRATTON [22, pp. 140–146] from an energy principle attributed to HELMHOLTZ and KORTEWEG.

17. Photoelasticity

The photoelastic effect cannot be properly treated within the context of statics. By its very nature, optics is a dynamical phenomenon. We can, however, give a qualitative sketch of the relations between static electro-elastic theory and the classical theories of the photoelastic effect.

Historically, the theories of the piezoelectric and photoelastic effects were developed using quite different physical principles. An account of the development of the theory of photoelasticity is given by Coker & Filon [2]. It appears that Neumann was the first to formulate a definite theory of the photoelastic effect. If we adopt Maxwell's electromagnetic equations and assume that the photoelastic medium is magnetically isotropic in an arbitrary state of deformation, the propagation of electromagnetic waves through such a medium can then be discussed in terms of the *inductive capacity tensor* $\epsilon^i{}_j$. This tensor is defined by writing the Maxwellian constitutive relation between the electric displacement and Maxwell electric field in the form

$$D^i = \epsilon^i{}_j E^j_{\mathsf{M}} . \tag{17.1}$$

We have shown that such a relation between $\mathbf{D}$ and $\mathbf{E}_{\mathsf{M}}$ exists at static equilibrium of an elastic dielectric as defined here. The tensor $\epsilon^i{}_j$ is a function of the state of deformation and polarization. In a magnetically isotropic medium $\mathbf{B} = \mu\mathbf{H}$ where $\mathbf{B}$ is called the magnetic induction, $\mathbf{H}$ is the magnetic field and μ is the magnetic inductive capacity. The quadric surface defined by the reciprocal of the tensor $\mu\epsilon^{ij}$ is called the *Fresnel ellipsoid.* Let F_{ij} denote the tensor which describes the Fresnel ellipsoid at a point in the deformed dielectric. Neumann's theory of the photoelastic effect in isotropic dielectrics was based on the assumption that the Fresnel tensor F^{ij} was a linear isotropic function of the infinitesimal strain measure $\tilde{e}_{ij}$. That is, he assumed

$$F^{ij} = H_0{}^{ij} + H_1{}^{ijkl}\tilde{e}_{kl} \tag{17.2}$$

where the $\mathbf{H}_\Gamma$ are isotropic material descriptors. The most general form of (17.2) is

$$F^{ij} = a_1 g^{ij} + a_2 g^{ik}\tilde{e}^k{}_k + a_3 \tilde{e}^{ij} \tag{17.3}$$

where the a_Γ are scalar material descriptors. If the dielectric is homogeneous, the a_Γ are spatially constant.

According to Coker & Filon, Pockels was the first to recognize that Neumann's photoelastic relations (17.3) should not be applied in the analysis of the photoelastic effect in crystalline media. In effect, Pockels' theory is based on (17.2) also, where the $\mathbf{H}_\Gamma$ are material descriptors of the crystalline medium and not necessarily isotropic tensors.

Both NEUMANN's and POCKELS' theories are restricted to the case of infinitesimal displacement gradients. The quasilinear constitutive relation for the effective local field in isotropic elastic dielectrics (16.3) leads to a Maxwellian constitutive relation (17.1) where $\epsilon^i{}_j$ is a linear isotropic function of the infinitesimal strain measure $\tilde{e}_{ij}$. Hence, if we assume that the medium is magnetically isotropic in any state of deformation, we are led, in this case, to a Fresnel tensor in agreement with NEUMANN's form (17.3). For anisotropic media, we are unable to reproduce POCKELS' generalization of (17.2) since, in general, the effective local field in crystals does not reduce to a polynomial in the infinitesimal strain measure. We are reluctant to consider the topic of photoelasticity beyond these few observations. It is our opinion that the dynamics of elastic dielectrics and the theory of electromagnetic wave propagation through general non-linear media must first be formulated from a unified point of view before the foundations of these approximate theories of NEUMANN and POCKELS will be properly understood.

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