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THE AFFINE LIE ALGEBRA $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ AND A CONDITIONED SPACE-TIME BROWNIAN MOTION

MANON DEFOSSEUX

ABSTRACT. We construct a sequence of Markov processes on the set of dominant weights of the Affine Lie algebra $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ which involves tensor product of irreducible highest weight modules of $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ and show that it converges towards a Doob's space-time harmonic transformation of a space-time Brownian motion.

1. INTRODUCTION

In [2], Ph. Biane, Ph. Bougerol and N. O'Connell establish a wide extension of Pitman's theorem on Brownian motion and three dimensional Bessel process, in the framework of representation theory of semi-simple complex Lie algebras. In this framework the representation of the Bessel process by a functional of a standard Brownian motion $(B_t)_{t \geq 0}$ on \mathbb{R} ,

$$(B_t - 2 \inf_{0 \leq s \leq t} B_s, t \geq 0),$$

appears to be the continuous counterpart of a similar result which holds for a random walk on the set of integral weights of $\mathfrak{sl}_2(\mathbb{C})$ and a path transformation connected with the Littelmann paths model for semi-simple complex Lie algebras (see for instance [7] for a description of this model).

In [6], C. Lecouvey, E. Lesigne and M. Peigné consider the case when \mathfrak{g} is a Kac-Moody algebra and develop some aspects of [2] in that framework. In particular, they focus on some Markov chains on the Weyl chamber of a Kac-Moody algebra, which are obtained in a similar way as in [2], except that the reference measure can't be the uniform measure when the dimension of the Kac-Moody algebra is infinite. Let us say briefly how the Markov chains are obtained for a Kac-Moody algebra \mathfrak{g} . As in the finite dimensional case, for a dominant weight λ of \mathfrak{g} one defines the character of an irreducible highest-weight representation $V(\lambda)$ of \mathfrak{g} with highest weight λ , as a function defined on a Cartan subalgebra \mathfrak{h} of \mathfrak{g} by

$$\text{ch}_\lambda(h) = \sum_{\mu} \dim(V(\lambda)_{\mu}) e^{\langle \mu, h \rangle}, \quad h \in \mathfrak{h},$$

where $V(\lambda)_{\mu}$ is the weight space of $V(\lambda)$ corresponding to the weight μ . This formal series converges for every h in a subset of the Cartan subalgebra which doesn't depend on λ . For two dominant weights ω and λ , the following decomposition

$$\text{ch}_{\omega} \text{ch}_{\lambda} = \sum_{\beta \in P_+} m_{\lambda}(\beta) \text{ch}_{\beta},$$

where $m_{\lambda}(\beta)$ is the multiplicity of the module with highest weight β in the decomposition of $V(\omega) \otimes V(\lambda)$, allows to define a transition probability q_{ω} on the set of

dominant weights, letting for β and λ two dominant weights of \mathfrak{g} ,

$$q_\omega(\lambda, \beta) = \frac{\text{ch}_\beta(h)}{\text{ch}_\lambda(h)\text{ch}_\omega(h)} m_\lambda(\beta),$$

where h is chosen in the region of convergence of the characters. It is a natural question to ask if there exists a sequence $(h_n)_{n \geq 0}$ of elements of \mathfrak{h} such that the corresponding sequence of Markov chains converges towards a continuous process and what the limit is.

In this paper, we consider the case when \mathfrak{g} is the Kac-Moody algebra of type $A_1^{(1)}$ and ω is its fundamental weight Λ_0 . There is no reason to think that the results are not true in a more general context but this case presents the advantage that explicit computations can easily be done. We show that the sequence of Markov chains, with a proper normalization, converges, for a particular sequence of $(h_n)_{n \geq 0}$, towards a Doob's space-time harmonic transformation of a space-time Brownian motion killed on the boundary of a time-dependent domain. This process is related to the heat equation

$$\frac{1}{2}\Delta + \frac{\partial}{\partial t} = 0,$$

in a time-dependent domain, with Dirichlet boundary conditions and the theta functions play a crucial role in the construction. One can find an extensive literature devoted to the relationship between Brownian motion and the heat equation. One can see for instance [4] for an introduction and [3] for a review of various problems specifically related to time-dependent boundaries.

The paper is organized as follows. Basic definitions and notations related to representation theory of the affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$ are given in section 2. We define in section 3 random walks on the set of integral weights of $\hat{\mathfrak{sl}}_2(\mathbb{C})$ and Markov chains on the set of its dominant weights, considering tensor products of irreducible highest weight representations of $\hat{\mathfrak{sl}}_2(\mathbb{C})$. In section 4, for any positive real numbers x and u such that $x < u$, we define a space-time Brownian motion $(t + u, B_t)_{t \geq 0}$ starting from (u, x) , conditioned to remain in the domain

$$D = \{(r, z) \in \mathbb{R} \times \mathbb{R} : 0 \leq z \leq r\}.$$

For this we introduce a space-time harmonic function remaining positive on D which appears naturally considering the limit of a sequence of characters of $\hat{\mathfrak{sl}}_2(\mathbb{C})$. We prove in section 5 that this conditioned space-time Brownian motion is the limit of a sequence of Markov processes constructed in section 3. We show in section 6 how it is related to a Brownian motion conditioned - in Doob's sense - to remain in an interval.

2. THE AFFINE LIE ALGEBRA $\hat{\mathfrak{sl}}_2(\mathbb{C})$

We consider the affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$ associated to the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The reader is invited to refer to [5] for a detailed description of this object. Let \mathfrak{h} be a Cartan subalgebra of $\hat{\mathfrak{sl}}_2(\mathbb{C})$. We denote by $S = \{\alpha_0, \alpha_1\}$ the set of simple roots and by $\{\alpha_0^\vee, \alpha_1^\vee\}$ the set of simple coroots. Let Λ_0 be a fundamental weight

such that $\langle \Lambda_0, \alpha_i^\vee \rangle = \delta_{i0}$, $i \in \{0, 1\}$, and $\{\alpha_0, \alpha_1, \Lambda_0\}$ is a basis of \mathfrak{h}^* . We denote by $\mathfrak{h}_{\mathbb{R}}$ the subset of \mathfrak{h} defined by

$$\mathfrak{h}_{\mathbb{R}} = \{x \in \mathfrak{h} : \langle \Lambda_0, x \rangle \in \mathbb{R}, \text{ and } \langle \alpha_i, x \rangle \in \mathbb{R}, i \in \{0, 1\}\}.$$

Let $\delta = \alpha_0 + \alpha_1$ be the so-called null root. We denote by P (resp. P_+) the set of integral (resp. dominant) weights defined by

$$P = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, i = 0, 1\},$$

$$(\text{resp. } P_+ = \{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \geq 0, i = 0, 1\}).$$

The Cartan subalgebra \mathfrak{h} is equipped with a non degenerate symmetric bilinear form (\cdot, \cdot) defined below, which identifies \mathfrak{h} and \mathfrak{h}^* , through the linear isomorphism

$$\begin{aligned} \nu : \mathfrak{h} &\rightarrow \mathfrak{h}^*, \\ h &\mapsto (h, \cdot). \end{aligned}$$

We still denote by (\cdot, \cdot) the induced non degenerate symmetric bilinear form on \mathfrak{h}^* . It is defined on \mathfrak{h}^* by

$$\left\{ \begin{array}{l} (\Lambda_0, \alpha_1) = 0 \\ (\Lambda_0, \Lambda_0) = 0 \\ (\delta, \alpha_1) = 0 \\ (\Lambda_0, \delta) = 1 \\ (\alpha_1, \alpha_1) = 2. \end{array} \right.$$

The level of an integral weight $\lambda \in P$, is defined as the integer (δ, λ) . For $k \in \mathbb{N}$, we denote by P_k the set integral weights of level k . It is defined by

$$P_k = \{\lambda \in P : (\delta, \lambda) = k\}.$$

That is, an integral weight of level k can be written

$$k\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where $x \in \mathbb{Z}$, $y \in \mathbb{C}$, and a dominant weight of level k can be written

$$k\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where $x \in \{0, \dots, k\}$, $y \in \mathbb{C}$. Recall the following important property : all weights of an highest weight irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ have the same level.

Notation. For $\lambda \in \mathfrak{h}^*$, the projection of λ on $\text{vect}\{\Lambda_0, \alpha_1\}$, denoted $\bar{\lambda}$, is defined by $\bar{\lambda} = x\Lambda_0 + y\alpha_1$, when $\lambda = x\Lambda_0 + y\alpha_1 + z\delta$, $x, y, z \in \mathbb{C}$.

Characters. For $\lambda \in P_+$, we denote by ch_λ the character of the irreducible highest-weight module $V(\lambda)$ of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight λ . That is

$$\text{ch}_\lambda(h) = \sum_{\mu \in P} \dim(V(\lambda)_\mu) e^{\langle \mu, h \rangle}, \quad h \in \mathfrak{h},$$

where $V(\lambda)_\mu$ is the weight space of $V(\lambda)$ corresponding to the weight μ . The above series converges absolutely for every $h \in \mathfrak{h}$ such that $\text{Re}\langle \delta, h \rangle > 0$ (see chapter 11 of [5]). For $\beta \in \mathfrak{h}^*$, we write $\text{ch}_\lambda(\beta)$ for $\text{ch}_\lambda(\nu^{-1}(\beta))$. We have

$$\text{ch}_\lambda(\beta) = \sum_{\mu \in P} \dim(V(\lambda)_\mu) e^{\langle \mu, \beta \rangle}, \quad \beta \in \mathfrak{h}^*.$$

The Weyl character's formula states that

$$\text{ch}_\lambda(\cdot) = \frac{\sum_{w \in W} \det(w) e^{(w(\lambda+\rho), \cdot)}}{\sum_{w \in W} \det(w) e^{(w(\rho), \cdot)}},$$

where $\rho = 2\Lambda_0 + \frac{1}{2}\alpha_1$ and W is the group of linear transformations of \mathfrak{h}^* generated by the reflections s_{α_0} and s_{α_1} defined by

$$s_{\alpha_i}(x) = x - 2 \frac{(\alpha_i, x)}{(\alpha_i, \alpha_i)} \alpha_i, \quad x \in \mathfrak{h}^*, \quad i \in \{0, 1\}.$$

As proved for instance in chapter 6 of [5], the affine Weyl group W is the semi-direct product $T \ltimes W_0$ where W_0 is the Weyl group generated by s_{α_1} and T is the group of transformations t_k , $k \in \mathbb{Z}$, defined by

$$t_k(\lambda) = \lambda + k(\lambda, \delta)\alpha_1 - (k(\lambda, \alpha_1) + k^2(\lambda, \delta))\delta, \quad \lambda \in \mathfrak{h}^*.$$

Thus for $a \in \mathbb{R}^*$, $y \in \mathbb{R}_+^*$, and a dominant weight λ of level $n \in \mathbb{N}^*$, such that $\lambda = n\Lambda_0 + \frac{1}{2}x\alpha_1$, the Weyl character formula becomes

$$(1) \quad \text{ch}_\lambda(ia\alpha_1 + y\Lambda_0) = \frac{\sum_{k \in \mathbb{Z}} \sin(a(x+1) + 2ak(n+2)) e^{-y(k(x+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} \sin(a + 8ak) e^{-y(k+4k^2)}}.$$

Letting a goes to zero in the previous identity, one also obtains that

$$(2) \quad \text{ch}_\lambda(y\Lambda_0) = \frac{\sum_{k \in \mathbb{Z}} (x+1 + 2k(n+2)) e^{-y(k(x+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} (1 + 8k) e^{-y(k+4k^2)}},$$

for every $y \in \mathbb{R}_+^*$.

3. MARKOV CHAINS ON THE SETS OF INTEGRAL OR DOMINANT WEIGHTS

Let us choose for this section a dominant weight $\omega \in P_+$ and $h \in \mathfrak{h}_\mathbb{R}$ such that $\langle \delta, h \rangle > 0$.

Random walks on P . We define a probability measure μ_ω on P letting

$$(3) \quad \mu_\omega(\beta) = \frac{\dim(V(\omega)_\beta)}{\text{ch}_\omega(h)} e^{\langle \beta, h \rangle}, \quad \beta \in P.$$

If $(X(n), n \geq 0)$ is a random walk on P whose increments are distributed according to μ_ω , it is important for our purpose to keep in mind that the function

$$x \in \mathbb{R} \mapsto \left[\frac{\text{ch}_\omega(i\frac{x}{2}\alpha_1 + h)}{\text{ch}_\omega(h)} \right]^n,$$

is the Fourier transform of the projection of $X(n)$ on $\mathbb{R}\alpha_1$.

Markov chains on P_+ . Let us consider for $\lambda \in P_+$ the following decomposition

$$\text{ch}_\omega \text{ch}_\lambda = \sum_{\beta \in P_+} m_\lambda(\beta) \text{ch}_\beta,$$

where $m_\lambda(\beta)$ is the multiplicity of the module with highest weight β in the decomposition of $V(\omega) \otimes V(\lambda)$, leads to the definition a transition probability q_ω on P_+ given by

$$(4) \quad q_\omega(\lambda, \beta) = \frac{\text{ch}_\beta(h)}{\text{ch}_\lambda(h) \text{ch}_\omega(h)} m_\lambda(\beta), \quad \beta \in P_+.$$

Let us notice that if $(\Lambda(n), n \geq 0)$ is a Markov process starting from $\lambda_0 \in P_+$, with transition probabilities q_ω then

$$\mathbb{E}\left(\frac{\text{ch}_{\Lambda(n)}(ix+h)}{\text{ch}_{\Lambda(n)}(h)}\right) = \frac{\text{ch}_{\lambda_0}(ix+h)}{\text{ch}_{\lambda_0}(h)} \left[\frac{\text{ch}_\omega(ix\alpha_1+h)}{\text{ch}_\omega(h)}\right]^n,$$

for every $x \in \mathbb{R}$. If λ_1 and λ_2 are two dominant weights such that $\lambda_1 = \lambda_2 \pmod{\delta}$ then the irreducible modules $V(\lambda_1)$ and $V(\lambda_2)$ are isomorphic. Thus if we consider the random process $(\bar{\Lambda}(n), n \geq 0)$, where $\bar{\Lambda}(n)$ is the projection of $\Lambda(n)$ on $\text{vect}\{\Lambda_0, \alpha_1\}$, then $(\bar{\Lambda}(n), n \geq 1)$ is a Markov process satisfying

$$(5) \quad \mathbb{E}\left(\frac{\text{ch}_{\bar{\Lambda}(n)}(ix\alpha_1+h)}{\text{ch}_{\bar{\Lambda}(n)}(h)}\right) = \frac{\text{ch}_{\bar{\lambda}_0}(ix\alpha_1+h)}{\text{ch}_{\bar{\lambda}_0}(h)} \left[\frac{\text{ch}_\omega(ix\alpha_1+h)}{\text{ch}_\omega(h)}\right]^n,$$

for every $x \in \mathbb{R}$, where $\bar{\lambda}_0$ is the projection of λ_0 on $\text{vect}\{\Lambda_0, \alpha_1\}$. More generally, for $n, m \in \mathbb{N}$, one gets

$$(6) \quad \mathbb{E}\left(\frac{\text{ch}_{\bar{\Lambda}(n+m)}(ix\alpha_1+h)}{\text{ch}_{\bar{\Lambda}(n+m)}(h)} \mid \bar{\Lambda}(k), 0 \leq k \leq m\right) = \frac{\text{ch}_{\bar{\Lambda}(m)}(ix\alpha_1+h)}{\text{ch}_{\bar{\Lambda}(m)}(h)} \left[\frac{\text{ch}_\omega(ix\alpha_1+h)}{\text{ch}_\omega(h)}\right]^n,$$

for every $x \in \mathbb{R}$. Let us notice that if ω is a dominant weight of level k , and λ_0 a dominant weight of level k_0 , then $\bar{\Lambda}(n)$ and $\Lambda(n)$ are dominant weights of level $nk + k_0$, for every $n \in \mathbb{N}$.

4. A CONDITIONED SPACE-TIME BROWNIAN MOTION

A class of space-time harmonic functions. Considering the asymptotic of the previous characters, one obtains an interesting class of space-time harmonic functions involving the Jacobi's theta function θ defined by

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z},$$

for z and τ two complex numbers, τ being in the upper half-plane. This is not surprising as the characters of affine Lie algebras are themselves a linear combination of theta functions (see [5]). For $a \in \mathbb{R}^*$, $x \in [0, t]$, if $(\lambda_n)_n$ is a sequence of dominant weights such that

$$\begin{aligned} \lambda_n &= [nt]\Lambda_0 + \frac{1}{2}(\lambda_n, \alpha_1)\alpha_1 \\ &\underset{n \rightarrow \infty}{\sim} nt\Lambda_0 + nx\frac{1}{2}\alpha_1, \end{aligned}$$

then the sum

$$\sum_{k \in \mathbb{Z}} \sin\left(\frac{a}{n}((\lambda_n, \alpha_1) + 1) + 2\frac{a}{n}k([nt] + 2)\right) e^{-\frac{2}{n}(k((\lambda_n, \alpha_1) + 1) + k^2([nt] + 2))},$$

which is the numerator of $\text{ch}_{\lambda_n}(i\frac{a}{n}\alpha_1 + \frac{2}{n}\Lambda_0)$ in the right-hand side of identity (1), converges, when n goes to infinity, towards

$$\sum_{k \in \mathbb{Z}} \sin(ax + 2kat) e^{-2(kx + k^2t)}.$$

Definition 4.1. For $a \in \mathbb{R}^*$, we define a function ϕ_a on $\mathbb{R} \times \mathbb{R}_+^*$ letting

$$\phi_a(x, t) = \frac{1}{a} \sum_{k \in \mathbb{Z}} \sin(ax + 2kat) e^{-2(kx+k^2t)}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Similarly, considering the asymptotic of the numerator of $\text{ch}_{\lambda_n}(\frac{2}{n}\Lambda_0)$ in (2) leads naturally to the following definition.

Definition 4.2. We define a function ϕ_0 on $\mathbb{R} \times \mathbb{R}_+^*$ letting

$$\phi_0(x, t) = \sum_{k \in \mathbb{Z}} (x + 2kt) e^{-2(kx+k^2t)}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Let us notice that $\lim_{a \rightarrow 0} \phi_a = \phi_0$.

Proposition 4.3. For $a \in \mathbb{R}$, the function

$$(x, t) \in \mathbb{R} \times \mathbb{R}_+^* \mapsto e^{\frac{a^2}{2}t} \phi_a(x, t),$$

is a space-time harmonic function, i.e. it satisfies

$$\left(\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) \phi_a = 0.$$

Moreover it satisfies the following boundary conditions

$$\forall t \in \mathbb{R}_+^*, \quad \begin{cases} \phi_a(0, t) &= 0 \\ \phi_a(t, t) &= 0. \end{cases}$$

Proof. Actually, each summand of the sum in the definition of $e^{\frac{a^2}{2}t} \phi_a$ is a space-time harmonic function because for any $k \in \mathbb{Z}$, one has

$$e^{i(ax+2kat)-2(kx+k^2t)+\frac{a^2}{2}t} = e^{(ia-2k)x - \frac{1}{2}(ia-2k)^2t}.$$

The first boundary condition follows from the change of variable $k \mapsto -k$, whereas the last one follows from the change of variable $k \mapsto -1 - k$. \square

Some properties of the functions ϕ_a , $a \in \mathbb{R}$.

Lemma 4.4. Let $t \in \mathbb{R}_+^*$, and $x \in]0, t[$. If $(\lambda_n)_n$ is a sequence of dominant weights such that

$$\lambda_n \sim nt\Lambda_0 + nx\frac{1}{2}\alpha,$$

then

$$\lim_{n \rightarrow \infty} \frac{\text{ch}_{\lambda_n}(\frac{ia}{n}\alpha_1 + \frac{2}{n}\Lambda_0)}{\text{ch}_{\lambda_n}(\frac{2}{n}\Lambda_0)} = \frac{\phi_a(x, t)}{\phi_0(x, t)}.$$

Proof. A Taylor expansion of the sinus implies easily that

$$\lim_{n \rightarrow \infty} \frac{\sum_{k \in \mathbb{Z}} \sin(\frac{a}{n}(1+8k)) e^{-\frac{2}{n}(k+4k^2)}}{\frac{1}{n} \sum_{k \in \mathbb{Z}} (1+8k) e^{-\frac{2}{n}(k+4k^2)}} = a.$$

Thus the lemma follows from identities (1) and (2). \square

Proposition 4.5. Let $a \in \mathbb{R}^*$, and $t \in \mathbb{R}_+^*$. Then

- (1) The function $\phi_0(\cdot, t)$ is C^∞ on $[0, t]$,
- (2) the function $\frac{\phi_a(\cdot, t)}{\phi_0(\cdot, t)}$ is bounded on $[0, t]$,

- (3) $\forall x \in]0, t[, \phi_0(x, t) \neq 0,$
(4) *the function $\phi_0(\cdot, t)$ doesn't change of sign on $[0, t]$.*

Proof. The first property follows immediately from a dominated convergence theorem. As for any $r \in \mathbb{R}$ and $y \in \mathbb{R}_+^*$, $\frac{\text{ch}_\lambda(ir\alpha_1 + y\Lambda_0)}{\text{ch}_\lambda(y\Lambda)}$ is a Fourier transform of a probability measure, it is bounded by 1. Previous lemma implies that

$$\forall x \in]0, t[, \quad \left| \frac{\phi_a(x, t)}{\phi_0(x, t)} \right| \leq 1.$$

As the function $\frac{\phi_a(\cdot, t)}{\phi_0(\cdot, t)}$ is easily shown to be continuous on $[0, t]$, the second property follows. For the third property, we notice that the function $\phi_{\frac{x}{t}}$ is defined by

$$\phi_{\frac{x}{t}}(x, t) = 2t \sin\left(\frac{x}{t}\pi\right) \sum_{k \in \mathbb{Z}} e^{-2(kx + k^2t)}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Thus for $x \in [0, t]$,

$$\phi_{\frac{x}{t}}(x, t) = 0 \Leftrightarrow x \in \{0, t\}.$$

The function $\frac{\phi_{\frac{x}{t}}(\cdot, t)}{\phi_0(\cdot, t)}$ being bounded on $[0, t]$, the third property follows. The fourth one is an immediate consequence of the first and the third ones. \square

Let us enounce a classical result on Fourier series that will be used to prove proposition 4.7

Lemma 4.6. *Let t be a positive real number and $f : [0, t] \rightarrow \mathbb{R}$ be a function such that $f(0) = f(t) = 0$, which is C^2 on $[0, t]$. Then the series of Fourier coefficients of f converges absolutely and*

$$f(x) = \sum_{n=1}^{+\infty} \frac{1}{t} \int_0^t \sin\left(\frac{z}{t}n\pi\right) f(z) dz \sin\left(\frac{x}{t}n\pi\right),$$

for every $x \in [0, t]$.

Proposition 4.7. *Let t be a positive real number. A probability measure μ on $[0, t]$ is characterized by the quantities*

$$\int_0^t \frac{\phi_{n\pi/t}(x, t)}{\phi_0(x, t)} \mu(dx), \quad n \in \mathbb{N}.$$

Proof. For $t \in \mathbb{R}_+^*$, $x \in \mathbb{R}$, we let

$$e(x, t) = 2 \frac{t}{n} \sum_{k \in \mathbb{Z}} e^{-2(kx + k^2t)}.$$

Let u be a C^2 function on $[0, t]$ (a polynomial function for instance). We first notice that the function $\frac{u(\cdot)\phi_0(\cdot, t)}{e(\cdot, t)}$ satisfies the condition of lemma 4.6. We let for $n \in \mathbb{N}^*$,

$$c_n = \frac{1}{t} \int_0^t \frac{u(x)\phi_0(x, t)}{e(x, t)} \sin\left(\frac{x}{t}n\pi\right) dx.$$

One has

$$\begin{aligned}
\int_0^t u(x) \mu(dx) &= \int_0^t u(x) \frac{\phi_0(x, t)}{e(x, t)} \frac{e(x, t)}{\phi_0(x, t)} \mu(dz) \\
&= \int_0^t \sum_{n=1}^{+\infty} c_n \sin\left(\frac{x}{t} n\pi\right) \frac{e(x, t)}{\phi_0(z, t)} \mu(dz) \\
&= \int_0^t \sum_{n=1}^{+\infty} c_n \frac{\phi_{n\pi/t}(x, t)}{\phi_0(x, t)} \mu(dz) \\
&= \sum_{n=1}^{+\infty} c_n \int_0^t \frac{\phi_{n\pi/t}(x, t)}{\phi_0(x, t)} \mu(dz),
\end{aligned}$$

where the last identity follows from the fact that the series of c_n is absolutely convergent and that the function $\frac{\phi_{n\pi/t}(\cdot, t)}{\phi_0(\cdot, t)}$ is bounded on $[0, t]$. \square

A conditioned space-time Brownian motion. Let us denote by C the fundamental Weyl chamber defined by

$$C = \{x \in \mathfrak{h}^* : (x, \alpha_i) \geq 0, i \in \{0, 1\}\}.$$

That is, an element of C can be written

$$t\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where $t \in \mathbb{R}_+, x \in [0, t], y \in \mathbb{C}$. For $x \in \mathbb{R}$, we denote by \mathbb{W}_x the Wiener measure on the set $C(\mathbb{R}_+)$ of real valued continuous functions on \mathbb{R}_+ , under which the coordinate process $(X_t, t \geq 0)$ is a Brownian motion starting from x , and denote the natural filtration of $(X_t)_{t \geq 0}$ by $(\mathcal{F}_t)_{t \geq 0}$. One considers the stopping times $T_u, u \in \mathbb{R}_+$, defined by

$$T_u = \inf\{t \geq 0 : X_t = 0 \text{ or } X_t = t + u\}.$$

Proposition 4.5 ensures that $\phi_0(X_s, s + u)$ doesn't change of sign whenever $s \in [0, T_u]$. Let x be a positive real number such that $u > x$. One has $\mathbb{W}_x(T_u > 0) = 1$. The function ϕ_0 being space-time harmonic, the process $\phi_0(X_t, t + u)$ is a local martingale. Actually, each summand of the sum is a local martingale, for which the quadratic variation is easily shown to be integrable, so that, each summand is a true Martingale. As their sum converges absolutely in L_2 norm, one obtains that $\phi_0(X_t, t + u), t \geq 0$, is a true martingale. As $\phi(X_{T_u}, T_u + u) = 0$, one defines a measure $\mathbb{Q}_{x, u}$ on $C(\mathbb{R}_+)$ letting

$$\mathbb{Q}_{x, u}(A) = \mathbb{E}_x\left(\frac{\phi_0(X_t, t + u)}{\phi_0(x, u)} \mathbf{1}_{\{T_u > t\} \cap A}\right), A \in \mathcal{F}_t.$$

Let r and s be two positive real number such that $r < t$. Using that

$$T_u = T_{u+r} \circ \theta_r + r \text{ on } \{T_u \geq r\},$$

where θ_r the shift operator defined by

$$\forall t \in \mathbb{R}_+, X_t \circ \theta_r = X_{t+r},$$

one easily proves that $(X_t, t \geq 0)$ is an inhomogeneous Markov process under $\mathbb{Q}_{x,u}$ satisfying

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_{x,u}}(f(X_{t+r})|\mathcal{F}_r) &= \mathbb{E}_{X_r}\left(\frac{\phi_0(X_t, t+r+u)}{\phi_0(X_r, r+u)}f(X_t)1_{T_{u+r}>t}\right) \\ (7) \qquad \qquad \qquad &= \mathbb{E}_{\mathbb{Q}_{X_r, r+u}}(f(X_t)), \end{aligned}$$

for any real valued measurable bounded function f .

Proposition 4.8. *For $r, t, u \in \mathbb{R}_+^*$, $x \in]0, u[$, and $a \in \mathbb{R}$, one has*

$$(8) \qquad \qquad \mathbb{Q}_{x,u}\left(\frac{\phi_a(X_t, t+u)}{\phi_0(X_t, t+u)}\right) = \frac{\phi_a(x, u)}{\phi_0(x, u)}e^{-\frac{a^2}{2}t}.$$

and

$$(9) \qquad \mathbb{E}_{\mathbb{Q}_{x,u}}\left(\frac{\phi_a(X_{t+r}, t+r+u)}{\phi_0(X_{t+r}, t+r+u)}|\mathcal{F}_r\right) = \frac{\phi_a(X_r, r+u)}{\phi_0(X_r, r+u)}e^{-\frac{a^2}{2}t}.$$

Proof. One proves as previously that

$$(e^{\frac{a^2}{2}t}\phi_a(X_t, t+u), t \geq 0)$$

is a true martingale, which implies that

$$\mathbb{W}_x(\phi_a(X_t, t+u)e^{\frac{a^2}{2}t}1_{T_u < t}) = 0,$$

and identity (8). The second identity follows, using (7). \square

5. THE CONDITIONED BROWNIAN MOTION AND THE MARKOV CHAINS ON THE SET OF DOMINANT WEIGHTS

Let us focus on the Markov chains defined in section 3 when $\omega = \Lambda_0$. We recall that the weights occurring in $V(\Lambda_0)$ are

$$\Lambda_0 + k\alpha_1 - (k^2 + s)\delta, \quad k \in \mathbb{Z}, \quad s \in \mathbb{N},$$

with respective multiplicities $p(s)$, the number of partitions of s (see for instance chapter 9 in [8]). If we consider, for $h = \frac{1}{2}(h_1\alpha_1 + h_2\Lambda_0)$, with $h_1 \in \mathbb{R}, h_2 \in \mathbb{R}_+^*$, the associated probability measure μ_{Λ_0} defined by (3) and the associated random walk $(X(n), n \geq 0)$, then its projection on $\mathbb{Z}\alpha_1$ is a random walk with increments distributed according to a probability measure $\bar{\mu}_{\Lambda_0}$ defined by

$$\bar{\mu}_{\Lambda_0}(k) = C_h e^{kh_1 - \frac{h_2}{2}k^2}, \quad k \in \mathbb{Z},$$

where C_h is a normalizing constant depending on h .

The main theorem. For $n \in \mathbb{N}^*$, we consider a random walk $(X_k^n, k \geq 0)$ starting from 0, whose increments are distributed according to probability measure μ_{Λ_0} associated to $h = \frac{2}{n}\Lambda_0$. If we denote by $(\bar{X}_k^n, k \geq 0)$ its projection on $\mathbb{Z}\alpha_1$, standard method shows that the sequence of processes $(\frac{2}{n}\bar{X}_{[nt]}^n, t \geq 0)$ converges towards a standard Brownian motion on \mathbb{R} when n goes to infinity.

Let x and u be two positive numbers such that $x < u$. For $n \in \mathbb{N}^*$, we consider a Markov process $(\Lambda_k^n, k \geq 0)$ starting from $[nu]\Lambda_0 + [xn]\frac{1}{2}\alpha_1$, with the transition probability q_ω defined by (4), with $\omega = \Lambda_0$ and $h = \frac{1}{2n}\Lambda_0$. It is important to notice that $(\Lambda_k^n, \delta) = [nu] + k$ for every $k \in \mathbb{N}$. If $\bar{\Lambda}_k^n$ is the projection of Λ_k^n on $\text{vect}\{\Lambda_0, \alpha_1\}$ for every $k \in \mathbb{N}$ and $n \in \mathbb{N}^*$, then the following convergence holds.

Theorem 5.1. *The sequence of processes $(\frac{1}{n}\bar{\Lambda}_{[nt]}^n, t \geq 0)$ converges when n goes to infinity towards the process $((t+u)\Lambda_0 + \frac{X_t}{2}\alpha_1, t \geq 0)$ under $\mathbb{Q}_{x,u}$*

Proof. Let $t \in \mathbb{R}_+^*$. We denote by μ_t^n the law of $\frac{1}{n}(\Lambda_{[nt]}^n, \alpha_1)$, for $n \in \mathbb{N}$. The probability measure μ_t^n is carried by $[0, t+u]$. The interval $[0, t+u]$ being a compact set, the space of probability measures on $[0, t+u]$ endowed with the weak topology is also compact. Suppose that a subsequence of $(\mu_t^n)_n$ converges towards μ_t . For $\lambda \in P_m^+$, one has

$$\frac{\text{ch}_\lambda(\frac{a}{n}\alpha_1 + \frac{2}{n}\Lambda_0)}{\text{ch}_\lambda(\frac{2}{n}\Lambda_0)} = \frac{\phi_a(\frac{1}{n}((\lambda, \alpha_1) + 1), \frac{1}{n}(m+2)) \phi_0(\frac{1}{n}, \frac{4}{n})}{\phi_0(\frac{1}{n}((\lambda, \alpha_1) + 1), \frac{1}{n}(m+2)) \phi_a(\frac{1}{n}, \frac{4}{n})},$$

for any $a \in \mathbb{R}$, and $n \in \mathbb{N}^*$. The function $(x, t) \mapsto \frac{\phi_a(x, t+u)}{\phi_0(x, t)}$ can be shown to be uniformly continuous on $\{(x, t) \in \mathbb{R} \times [0, T] : 0 \leq x \leq u+t\}$ for every $T \in \mathbb{R}_+$. As $\lim_{n \rightarrow \infty} \frac{\phi_0(\frac{1}{n}, \frac{4}{n})}{\phi_a(\frac{1}{n}, \frac{4}{n})} = 1$, and

$$\left[\frac{\text{ch}_{\Lambda_0}(\frac{a}{n}\alpha_1 + \frac{2}{n}\Lambda_0)}{\text{ch}_{\Lambda_0}(\frac{2}{n}\Lambda_0)} \right]^{[nt]} = \mathbb{E}(e^{ia\frac{2}{n}\bar{X}_{[nt]}^{(n)}}),$$

identity (5) implies that μ_t satisfies

$$\int_0^{t+u} \frac{\phi_a(z, t+u)}{\phi_0(z, t+u)} \mu_t(dz) = \frac{\phi_a(x, u)}{\phi_0(x, u)} e^{-\frac{a^2}{2}t}.$$

Proposition 4.7 implies that $(\mu_t^n)_n$ converges towards μ_t and proposition 4.8 implies that μ_t is the distribution of X_t under $\mathbb{Q}_{x,u}$. Convergence of the sequence of random processes $(\frac{1}{n}(\Lambda_{[nt]}^n, \alpha_1), t \geq 0)$ - in the sense of finite dimensional distributions convergence - follows similarly from identity (6) and (9). As one has the inequality

$$\frac{1}{n} |(\Lambda_{[nt]}^n, \alpha_1) - (\Lambda_{[ns]}^n, \alpha_1)| \leq |t-s| + \frac{1}{n},$$

for every $n \in \mathbb{N}^*$ and $s, t \in \mathbb{R}_+^*$, the sequence of random processes $(\frac{1}{n}(\Lambda_{[nt]}^n, \alpha_1), t \geq 0)$ is clearly tight and thus it converges also in $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$ endowed with the topology of uniform convergence on compact sets. \square

6. BROWNIAN MOTION CONDITIONED TO REMAIN IN AN INTERVAL

In this section we discuss the connection between the conditioned Brownian motion constructed in this paper and the Brownian motion conditioned - in the sense of Doob - to remain in an interval. The connection is not surprising when we keep in mind that the dominant term in a character of an affine algebra involves the so-called asymptotic dimensions, which are related to eigenfunctions for the Laplacian on an interval (see chapter 13 of [5]). Let $u \in \mathbb{R}_+^*$. The function h defined on $[0, u]$ by

$$h(x) = \sin(\pi \frac{x}{u}), \quad x \in [0, u],$$

is the Dirichlet eigenfunction on the interval $[0, u]$ corresponding to the eigenvalue $-\frac{\pi^2}{u^2}$ at the bottom of the spectrum. Brownian motion conditioned - in the sense

of Doob - to remain in the interval $[0, u]$, has the Doob-transformed semi-group $(q_t)_{t \geq 0}$ defined for $t \in \mathbb{R}_+^*$ by

$$q_t(x, y) = \frac{h(y)}{h(x)} e^{\frac{\pi^2}{u^2} \frac{t}{2}} p_t^0(x, y), \quad x, y \in]0, u[,$$

where p_t^0 is the semi-group of the standard Brownian motion on \mathbb{R} , killed on the boundary of $[0, u]$.

For $c \in]0, 1[$, one defines a space-time harmonic function $\phi_0^{(c)}$ on $\mathbb{R} \times \mathbb{R}_+^*$ letting

$$\phi_0^{(c)}(x, t) = \phi_0(cx, c^2t),$$

for $x, t \in \mathbb{R} \times \mathbb{R}_+^*$. This function satisfies the following boundary conditions

$$\forall t \in \mathbb{R}_+^*, \quad \begin{cases} \phi_0^{(c)}(0, t) & = 0 \\ \phi_0^{(c)}(ct, t) & = 0. \end{cases}$$

As in section 4, one defines for a real number x satisfying $0 < x < u$, a probability $\mathbb{Q}_{x,u}^{(c)}$ on $C(\mathbb{R}_+)$ letting

$$\mathbb{Q}_{x,u}^{(c)}(A) = \mathbb{E}_x \left(\frac{\phi_0^{(c)}(X_t, t + \frac{u}{c})}{\phi_0^{(c)}(x, \frac{u}{c})} 1_{\{T_u^{(c)} > t\} \cap A} \right), \quad A \in \mathcal{F}_t,$$

where $T_u^{(c)} = \inf\{s \geq 0 : X_s = 0 \text{ or } X_s = cs + u\}$. Thus, under the probability measure $\mathbb{Q}_{x,u}^{(c)}$, $(t + \frac{u}{c}, X_t)_{t \geq 0}$ is a space-time Brownian motion starting from $(\frac{u}{c}, x)$, conditioned to remain in the domain

$$\{(r, z) \in \mathbb{R} \times \mathbb{R} : 0 \leq z \leq cr\}.$$

Theorem 6.1. *The probability measure $\mathbb{Q}_{x,u}^{(c)}$ converges, when c goes to 0, towards the law of a standard Brownian starting from x , conditioned - in the sense of Doob - to remain in $[0, u]$.*

Proof. Let us admit the lemma 6.2 for the proof. It easily implies that

$$\lim_{c \rightarrow 0} \frac{\phi_0^{(c)}(y, t + \frac{u}{c})}{\phi_0^{(c)}(x, \frac{u}{c})} = \frac{\sin(y \frac{\pi}{u})}{\sin(x \frac{\pi}{u})} e^{\frac{\pi^2}{u^2} \frac{t}{2}},$$

for every $y \in [0, u]$, $t > 0$, which implies the theorem, as the quotient inside the limit is uniformly bounded for $y \in [0, ct + u]$ and $c \in]0, 1[$. \square

Lemma 6.2. *For $x \in \mathbb{R}$ and $t \in \mathbb{R}_+^*$, one has the identity*

$$\phi_0(x, t) = \sqrt{\frac{\pi}{2t}} e^{\frac{x^2}{2t}} \sum_{n \in \mathbb{Z}} n \pi \sin(\pi n \frac{x}{t}) e^{-\frac{n^2 \pi^2}{2t}}.$$

Proof. The identity could be obtained directly computing the Fourier coefficients of the $2t$ -periodic function $x \mapsto e^{-\frac{x^2}{2t}} \phi_0(x, t)$. Nevertheless, we prefer to show how it derives from the well known Jacobi's theta function identity

$$(10) \quad \frac{1}{\sqrt{\pi t}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{t}(n+x)^2} = \sum_{n \in \mathbb{Z}} \cos(2n\pi x) e^{-n^2 \pi^2 t},$$

which is valid for $x \in \mathbb{R}$, $t \in \mathbb{R}_+^*$. The Jacobi's theta identity, which is a particular case of the Poisson summation formula, can be proved computing the Fourier coefficients of the left hand expression, considered as 1-periodic function of x (see [1]).

Considering the partial derivative with respect to x of the left and the right hand sides in identity (10) leads to the identity

$$(11) \quad \frac{1}{\sqrt{\pi t}} \sum_{n \in \mathbb{Z}} \frac{1}{t} (n+x) e^{-\frac{1}{t}(n+x)^2} = \sum_{n \in \mathbb{Z}} n \pi \sin(2n\pi x) e^{-n^2 \pi^2 t},$$

for $x \in \mathbb{R}$, $t \in \mathbb{R}_+^*$. As

$$\phi_0(x, t) = \sum_{n \in \mathbb{Z}} 2t \left(\frac{x}{2t} + n \right) e^{-2t \left(n + \frac{x}{2t} \right)^2 + \frac{x^2}{2t}},$$

one obtains the lemma replacing respectively t and x by $\frac{1}{2t}$ and $\frac{x}{2t}$ in (11). \square

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