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# THE AFFINE LIE ALGEBRA $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ AND A CONDITIONED SPACE-TIME BROWNIAN MOTION

MANON DEFOSSEUX

ABSTRACT. We construct a sequence of Markov processes on the set of dominant weights of the Affine Lie algebra  $\widehat{\mathfrak{sl}}_2(\mathbb{C})$  which involves tensor product of irreducible highest weight modules of  $\widehat{\mathfrak{sl}}_2(\mathbb{C})$  and show that it converges towards a Doob's space-time harmonic transformation of a space-time Brownian motion.

## 1. INTRODUCTION

In [1], Ph. Biane, Ph. Bougerol and N. O'Connell establish a wide extension of Pitman's theorem on Brownian motion and three dimensional Bessel process, in the framework of representation theory of semi-simple complex Lie algebras. In this framework the representation of the Bessel process by a functional of a standard Brownian motion  $(B_t)_{t \geq 0}$  on  $\mathbb{R}$ ,

$$(B_t - 2 \inf_{0 \leq s \leq t} B_s, t \geq 0),$$

appears to be the continuous counterpart of a similar result which holds for a random walk on the set of integral weights of  $\mathfrak{sl}_2(\mathbb{C})$  and a path transformation connected with the Littelmann paths model for semi-simple complex Lie algebras (see for instance [6] for a description of this model).

In [5], C. Lecouvey, E. Lesigne and M. Peigné consider the case when  $\mathfrak{g}$  is a Kac-Moody algebra and develop some aspects of [1] in that framework. In particular, they focus on some Markov chains on the Weyl chamber of a Kac-Moody algebra, which are obtained in a similar way as in [1], except that the reference measure can't be the uniform measure when the dimension of the Kac-Moody algebra is infinite. Let us say briefly how the Markov chains are obtained for a Kac-Moody algebra  $\mathfrak{g}$ . As in the finite dimensional case, for a dominant weight  $\lambda$  of  $\mathfrak{g}$  one defines the character of an irreducible highest-weight representation  $V(\lambda)$  of  $\mathfrak{g}$  with highest weight  $\lambda$ , as a function defined on a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  by

$$\text{ch}_\lambda(h) = \sum_{\mu} \dim(V(\lambda)_\mu) e^{\langle \mu, h \rangle}, \quad h \in \mathfrak{h},$$

where  $V(\lambda)_\mu$  is the weight space of  $V(\lambda)$  corresponding to the weight  $\mu$ . This formal series converges for every  $h$  in a subset of the Cartan subalgebra which doesn't depend on  $\lambda$ . For two dominant weights  $\omega$  and  $\lambda$ , the following decomposition

$$\text{ch}_\omega \text{ch}_\lambda = \sum_{\beta \in P_+} m_\lambda(\beta) \text{ch}_\beta,$$

where  $m_\lambda(\beta)$  is the multiplicity of the module with highest weight  $\beta$  in the decomposition of  $V(\omega) \otimes V(\lambda)$ , allows to define a transition probability  $q_\omega$  on the set of

dominant weights, letting for  $\beta$  and  $\lambda$  two dominant weights of  $\mathfrak{g}$ ,

$$q_\omega(\lambda, \beta) = \frac{\text{ch}_\beta(h)}{\text{ch}_\lambda(h)\text{ch}_\omega(h)} m_\lambda(\beta),$$

where  $h$  is chosen in the region of convergence of the characters. It is a natural question to ask if there exists a sequence  $(h_n)_{n \geq 0}$  of elements of  $\mathfrak{h}$  such that the corresponding sequence of Markov chains converges towards a continuous process and what the limit is.

In this paper, we consider the case when  $\mathfrak{g}$  is the Kac-Moody algebra of type  $A_1^{(1)}$  and  $\omega$  is its fundamental weight  $\Lambda_0$ . There is no reason to think that the results are not true in a more general context but this case presents the advantage that explicit computations can easily be done. We show that the sequence of Markov chains, with a proper normalization, converges, for a particular sequence of  $(h_n)_{n \geq 0}$ , towards a Doob's space-time harmonic transformation of a space-time Brownian motion killed on the boundary of a time-dependent domain. This process is related to the heat equation

$$\frac{1}{2}\Delta + \frac{\partial}{\partial t} = 0,$$

in a time-dependent domain, with Dirichlet boundary conditions. One can find an extensive literature devoted to the relationship between Brownian motion and the heat equation. One can see for instance [3] for an introduction and [2] for a review of various problems specifically related to time-dependent boundaries.

The paper is organized as follows. Basic definitions and notations related to representation theory of the affine Lie algebra  $\mathfrak{sl}_2(\mathbb{C})$  are given in section 2. We define in section 3 random walks on the set of integral weights of  $\mathfrak{sl}_2(\mathbb{C})$  and Markov chains on the set of its dominant weights, considering tensor products of irreducible highest weight representations of  $\mathfrak{sl}_2(\mathbb{C})$ . In section 4, for any positive real numbers  $x$  and  $u$  such that  $x < u$ , we define a space-time Brownian motion  $(t + u, B_t)_{t \geq 0}$  starting from  $(u, x)$ , conditioned to remain in the domain

$$D = \{(r, z) \in \mathbb{R} \times \mathbb{R} : 0 \leq z \leq r\}.$$

For this we introduce a space-time harmonic function remaining positive on  $D$  which appears naturally considering the limit of a sequence of characters of  $\mathfrak{sl}_2(\mathbb{C})$ . We prove in section 5 that this conditioned space-time Brownian motion is the limit of a sequence of Markov processes constructed in section 3.

## 2. THE AFFINE LIE ALGEBRA $\mathfrak{sl}_2(\mathbb{C})$

We consider the affine Lie algebra  $\widehat{\mathfrak{sl}}_2(\mathbb{C})$  associated to the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The reader is invited to refer to [4] for a detailed description of this object. Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ . We denote by  $S = \{\alpha_0, \alpha_1\}$  the set of simple roots and by  $\{\alpha_0^\vee, \alpha_1^\vee\}$  the set of simple coroots. Let  $\Lambda_0$  be a fundamental weight such that  $\langle \Lambda_0, \alpha_i^\vee \rangle = \delta_{i0}$ ,  $i \in \{0, 1\}$ , and  $\{\alpha_0, \alpha_1, \Lambda_0\}$  is a basis of  $\mathfrak{h}^*$ . We denote by  $\mathfrak{h}_\mathbb{R}$  the subset of  $\mathfrak{h}$  defined by

$$\mathfrak{h}_\mathbb{R} = \{x \in \mathfrak{h} : \langle \Lambda_0, x \rangle \in \mathbb{R}, \text{ and } \langle \alpha_i, x \rangle \in \mathbb{R}, i \in \{0, 1\}\}.$$

Let  $\delta = \alpha_0 + \alpha_1$  be the so-called null root. We denote by  $P$  (resp.  $P_+$ ) the set of integral (resp. dominant) weights defined by

$$P = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, i = 0, 1\},$$

$$\text{(resp. } P_+ = \{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \geq 0, i = 0, 1\}).$$

The Cartan subalgebra  $\mathfrak{h}$  is equipped with a non degenerate symmetric bilinear form  $(\cdot, \cdot)$  defined below, which identifies  $\mathfrak{h}$  and  $\mathfrak{h}^*$ , through the linear isomorphism

$$\nu : \mathfrak{h} \rightarrow \mathfrak{h}^*,$$

$$h \mapsto (h, \cdot).$$

We still denote by  $(\cdot, \cdot)$  the induced non degenerate symmetric bilinear form on  $\mathfrak{h}^*$ . It is defined on  $\mathfrak{h}^*$  by

$$\begin{cases} (\Lambda_0, \alpha_1) = 0 \\ (\Lambda_0, \Lambda_0) = 0 \\ (\delta, \alpha_1) = 0 \\ (\Lambda_0, \delta) = 1 \\ (\alpha_1, \alpha_1) = 2. \end{cases}$$

The level of an integral weight  $\lambda \in P$ , is defined as the integer  $(\delta, \lambda)$ . For  $k \in \mathbb{N}$ , we denote by  $P_k$  the set integral weights of level  $k$ . It is defined by

$$P_k = \{\lambda \in P : (\delta, \lambda) = k\}.$$

That is, an integral weight of level  $k$  can be written

$$k\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where  $x \in \mathbb{Z}$ ,  $y \in \mathbb{C}$ , and a dominant weight of level  $k$  can be written

$$k\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where  $x \in \{0, \dots, k\}$ ,  $y \in \mathbb{C}$ . Recall the following important property : all weights of an highest weight irreducible representation of  $\mathfrak{sl}_2(\mathbb{C})$  have the same level.

**Notation.** For  $\lambda \in \mathfrak{h}^*$ , the projection of  $\lambda$  on  $\text{vect}\{\Lambda_0, \alpha_1\}$ , denoted  $\bar{\lambda}$ , is defined by  $\bar{\lambda} = x\Lambda_0 + y\alpha_1$ , when  $\lambda = x\Lambda_0 + y\alpha_1 + z\delta$ ,  $x, y, z \in \mathbb{C}$ .

**Characters.** For  $\lambda \in P_+$ , we denote by  $\text{ch}_\lambda$  the character of the irreducible highest-weight module  $V(\lambda)$  of  $\mathfrak{sl}_2(\mathbb{C})$  with highest weight  $\lambda$ . That is

$$\text{ch}_\lambda(h) = \sum_{\mu \in P} \dim(V(\lambda)_\mu) e^{\langle \mu, h \rangle}, \quad h \in \mathfrak{h},$$

where  $V(\lambda)_\mu$  is the weight space of  $V(\lambda)$  corresponding to the weight  $\mu$ . The above series converges absolutely for every  $h \in \mathfrak{h}$  such that  $\text{Re}\langle \delta, h \rangle > 0$  (see chapter 11 of [4]). For  $\beta \in \mathfrak{h}^*$ , we write  $\text{ch}_\lambda(\beta)$  for  $\text{ch}_\lambda(\nu^{-1}(\beta))$ . We have

$$\text{ch}_\lambda(\beta) = \sum_{\mu \in P} \dim(V(\lambda)_\mu) e^{\langle \mu, \beta \rangle}, \quad \beta \in \mathfrak{h}^*.$$

The Weyl character's formula states that

$$\text{ch}_\lambda(\cdot) = \frac{\sum_{w \in W} \det(w) e^{(w(\lambda + \rho), \cdot)}}{\sum_{w \in W} \det(w) e^{(w(\rho), \cdot)}},$$

where  $\rho = 2\Lambda_0 + \frac{1}{2}\alpha_1$  and  $W$  is the group of linear transformations of  $\mathfrak{h}^*$  generated by the reflections  $s_{\alpha_0}$  and  $s_{\alpha_1}$  defined by

$$s_{\alpha_i}(x) = x - 2 \frac{(\alpha_i, x)}{(\alpha_i, \alpha_i)} \alpha_i, \quad x \in \mathfrak{h}^*, \quad i \in \{0, 1\}.$$

As proved for instance in chapter 6 of [4], the affine Weyl group  $W$  is the semi-direct product  $T \ltimes W_0$  where  $W_0$  is the Weyl group generated by  $s_{\alpha_1}$  and  $T$  is the group of transformations  $t_k$ ,  $k \in \mathbb{Z}$ , defined by

$$t_k(\lambda) = \lambda + k(\lambda, \delta)\alpha_1 - (k(\lambda, \alpha_1) + k^2(\lambda, \delta))\delta, \quad \lambda \in \mathfrak{h}^*.$$

Thus for  $a \in \mathbb{R}^*$ ,  $y \in \mathbb{R}_+^*$ , and a dominant weight  $\lambda$  of level  $n \in \mathbb{N}^*$ , such that  $\lambda = n\Lambda_0 + \frac{1}{2}x\alpha_1$ , the Weyl character formula becomes

$$(1) \quad \text{ch}_\lambda(ia\alpha_1 + y\Lambda_0) = \frac{\sum_{k \in \mathbb{Z}} \sin(a(x+1) + 2ak(n+2)) e^{-y(k(x+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} \sin(a + 8ak) e^{-y(k+4k^2)}}.$$

Letting  $a$  goes to zero in the previous identity, one also obtains that

$$(2) \quad \text{ch}_\lambda(y\Lambda_0) = \frac{\sum_{k \in \mathbb{Z}} (x+1 + 2k(n+2)) e^{-y(k(x+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} (1 + 8k) e^{-y(k+4k^2)}},$$

for every  $y \in \mathbb{R}_+^*$ .

### 3. MARKOV CHAINS ON THE SETS OF INTEGRAL OR DOMINANT WEIGHTS

Let us choose for this section a dominant weight  $\omega \in P_+$  and  $h \in \mathfrak{h}_\mathbb{R}$  such that  $\langle \delta, h \rangle > 0$ .

**Random walks on  $P$ .** We define a probability measure  $\mu_\omega$  on  $P$  letting

$$(3) \quad \mu_\omega(\beta) = \frac{\dim(V(\omega)_\beta)}{\text{ch}_\omega(h)} e^{\langle \beta, h \rangle}, \quad \beta \in P.$$

If  $(X(n), n \geq 0)$  is a random walk on  $P$  whose increments are distributed according to  $\mu_\omega$ , it is important for our purpose to keep in mind that the function

$$x \in \mathbb{R} \mapsto \left[ \frac{\text{ch}_\omega(i\frac{x}{2}\alpha_1 + h)}{\text{ch}_\omega(h)} \right]^n,$$

is the Fourier transform of the projection of  $X(n)$  on  $\mathbb{R}\alpha_1$ .

**Markov chains on  $P_+$ .** Let us consider for  $\lambda \in P_+$  the following decomposition

$$\text{ch}_\omega \text{ch}_\lambda = \sum_{\beta \in P_+} m_\lambda(\beta) \text{ch}_\beta,$$

where  $m_\lambda(\beta)$  is the multiplicity of the module with highest weight  $\beta$  in the decomposition of  $V(\omega) \otimes V(\lambda)$ , leads to the definition a transition probability  $q_\omega$  on  $P_+$  given by

$$(4) \quad q_\omega(\lambda, \beta) = \frac{\text{ch}_\beta(h)}{\text{ch}_\lambda(h) \text{ch}_\omega(h)} m_\lambda(\beta), \quad \beta \in P_+.$$

Let us notice that if  $(\Lambda(n), n \geq 0)$  is a Markov process starting from  $\lambda_0 \in P_+$ , with transition probabilities  $q_\omega$  then

$$\mathbb{E} \left( \frac{\text{ch}_{\Lambda(n)}(ix+h)}{\text{ch}_{\Lambda(n)}(h)} \right) = \frac{\text{ch}_{\lambda_0}(ix+h)}{\text{ch}_{\lambda_0}(h)} \left[ \frac{\text{ch}_\omega(ix\alpha_1+h)}{\text{ch}_\omega(h)} \right]^n,$$

for every  $x \in \mathbb{R}$ . If  $\lambda_1$  and  $\lambda_2$  are two dominant weights such that  $\lambda_1 = \lambda_2 \pmod{\delta}$  then the irreducible modules  $V(\lambda_1)$  and  $V(\lambda_2)$  are isomorphic. Thus if we consider the random process  $(\bar{\Lambda}(n), n \geq 0)$ , where  $\bar{\Lambda}(n)$  is the projection of  $\Lambda(n)$  on  $\text{vect}\{\Lambda_0, \alpha_1\}$ , then  $(\bar{\Lambda}(n), n \geq 1)$  is a Markov process satisfying

$$(5) \quad \mathbb{E}\left(\frac{\text{ch}_{\bar{\Lambda}(n)}(ix\alpha_1 + h)}{\text{ch}_{\bar{\Lambda}(n)}(h)}\right) = \frac{\text{ch}_{\bar{\lambda}_0}(ix\alpha_1 + h)}{\text{ch}_{\bar{\lambda}_0}(h)} \left[\frac{\text{ch}_{\omega}(ix\alpha_1 + h)}{\text{ch}_{\omega}(h)}\right]^n,$$

for every  $x \in \mathbb{R}$ , where  $\bar{\lambda}_0$  is the projection of  $\lambda_0$  on  $\text{vect}\{\Lambda_0, \alpha_1\}$ . More generally, for  $n, m \in \mathbb{N}$ , one gets

$$(6) \quad \mathbb{E}\left(\frac{\text{ch}_{\bar{\Lambda}(n+m)}(ix\alpha_1 + h)}{\text{ch}_{\bar{\Lambda}(n+m)}(h)} \mid \bar{\Lambda}(k), 0 \leq k \leq m\right) = \frac{\text{ch}_{\bar{\Lambda}(m)}(ix\alpha_1 + h)}{\text{ch}_{\bar{\Lambda}(m)}(h)} \left[\frac{\text{ch}_{\omega}(ix\alpha_1 + h)}{\text{ch}_{\omega}(h)}\right]^n,$$

for every  $x \in \mathbb{R}$ . Let us notice that if  $\omega$  is a dominant weight of level  $k$ , and  $\lambda_0$  a dominant weight of level  $k_0$ , then  $\bar{\Lambda}(n)$  and  $\Lambda(n)$  are dominant weights of level  $nk + k_0$ , for every  $n \in \mathbb{N}$ .

#### 4. A CONDITIONED SPACE-TIME BROWNIAN MOTION

**A class of space-time harmonic functions.** Considering the asymptotic of the previous characters, one obtains an interesting class of space-time harmonic functions. For  $a \in \mathbb{R}^*$ ,  $x \in [0, t]$ , if  $(\lambda_n)_n$  is a sequence of dominant weights such that

$$\begin{aligned} \lambda_n &= [nt]\Lambda_0 + \frac{1}{2}(\lambda_n, \alpha_1)\alpha_1 \\ &\underset{n \rightarrow \infty}{\sim} nt\Lambda_0 + nx\frac{1}{2}\alpha_1, \end{aligned}$$

then the sum

$$\sum_{k \in \mathbb{Z}} \sin\left(\frac{a}{n}((\lambda_n, \alpha_1) + 1) + 2\frac{a}{n}k([nt] + 2)\right) e^{-\frac{1}{2n}(k((\lambda_n, \alpha_1) + 1) + k^2([nt] + 2))},$$

which is the numerator of  $\text{ch}_{\lambda_n}\left(\frac{a}{n}\alpha_1 + \frac{1}{2n}\Lambda_0\right)$  in the right-hand side of identity (1), converges, when  $n$  goes to infinity, towards

$$\sum_{k \in \mathbb{Z}} \sin(ax + 2kat) e^{-\frac{1}{2}(kx + k^2t)}.$$

**Definition 4.1.** For  $a \in \mathbb{R}^*$ , we define a function  $\phi_a$  on  $\mathbb{R} \times \mathbb{R}_+^*$  letting

$$\phi_a(x, t) = \frac{1}{a} \sum_{k \in \mathbb{Z}} \sin(ax + 2kat) e^{-\frac{1}{2}(kx + k^2t)}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Similarly, considering the asymptotic of the numerator of  $\text{ch}_{\lambda_n}\left(\frac{1}{2n}\Lambda_0\right)$  in (2) leads naturally to the following definition.

**Definition 4.2.** We define a function  $\phi_0$  on  $\mathbb{R} \times \mathbb{R}_+^*$  letting

$$\phi_0(x, t) = \sum_{k \in \mathbb{Z}} (x + 2kt) e^{-\frac{1}{2}(kx + k^2t)}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Let us notice that  $\lim_{a \rightarrow 0} \phi_a = \phi_0$ .

**Proposition 4.3.** For  $a \in \mathbb{R}$ , the function

$$(x, t) \in \mathbb{R} \times \mathbb{R}_+^* \mapsto e^{-\frac{a^2}{2}t} \phi_a(x, t),$$

is a space-time harmonic function, i.e. it satisfies

$$\left(\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) \phi_a = 0.$$

Moreover it satisfies the following boundary conditions

$$\forall t \in \mathbb{R}_+^*, \begin{cases} \phi_a(0, t) = 0 \\ \phi_a(t, t) = 0. \end{cases}$$

*Proof.* Actually, a simple computation shows that each summand of the sum in the definition of  $\phi_a$  is a space-time harmonic function. The first boundary condition follows from the change of variable  $k \mapsto -k$ , whereas the last one follows from the change of variable  $k \mapsto -1 - k$ .  $\square$

**Some properties of the functions  $\phi_a$ ,  $a \in \mathbb{R}$ .**

**Lemma 4.4.** Let  $t \in \mathbb{R}_+^*$ , and  $x \in ]0, t[$ . If  $(\lambda_n)_n$  is a sequence of dominant weights such that

$$\lambda_n \sim nt\Lambda_0 + nx\frac{1}{2}\alpha,$$

then

$$\lim_{n \rightarrow \infty} \frac{ch_{\lambda_n}(\frac{ia}{n}\alpha_1 + \frac{1}{2n}\Lambda_0)}{ch_{\lambda_n}(\frac{1}{2n}\Lambda_0)} = \frac{\phi_a(x, t)}{\phi_0(x, t)}.$$

*Proof.* A Taylor expansion of the sinus implies easily that

$$\lim_{n \rightarrow \infty} \frac{\sum_{k \in \mathbb{Z}} \sin(\frac{a}{n}(1 + 8k)) e^{-\frac{1}{2n}(k+4k^2)}}{\frac{1}{n} \sum_{k \in \mathbb{Z}} (1 + 8k) e^{-\frac{1}{2n}(k+4k^2)}} = a.$$

Thus the lemma follows from identities (1) and (2).  $\square$

**Proposition 4.5.** Let  $a \in \mathbb{R}^*$ , and  $t \in \mathbb{R}_+^*$ . Then

- (1) The function  $\phi_0(\cdot, t)$  is  $C^\infty$  on  $[0, t]$ ,
- (2) the function  $\frac{\phi_a(\cdot, t)}{\phi_0(\cdot, t)}$  is bounded on  $[0, t]$ ,
- (3)  $\forall x \in ]0, t[$ ,  $\phi_0(x, t) \neq 0$ ,
- (4) the function  $\phi_0(\cdot, t)$  doesn't change of sign on  $[0, t]$ .

*Proof.* The first property follows immediately from a dominated convergence theorem. As for any  $r \in \mathbb{R}$  and  $y \in \mathbb{R}_+^*$ ,  $\frac{\chi_\lambda(ir\alpha_1 + y\Lambda_0)}{\chi_\lambda(y\Lambda)}$  is a Fourier transform of a probability measure, it is bounded by 1. Previous lemma implies that

$$\forall x \in ]0, t[, \quad \left| \frac{\phi_a(x, t)}{\phi_0(x, t)} \right| \leq 1.$$

As the function  $\frac{\phi_a(\cdot, t)}{\phi_0(\cdot, t)}$  is easily shown to be continuous on  $[0, t]$ , the second property follows. For the third property, we notice that for  $n \in \mathbb{N}^*$ , the function  $\phi_{\pi \frac{n}{t}}$  is defined by

$$\phi_{\pi \frac{n}{t}}(x, t) = 2 \frac{t}{n} \sin\left(\frac{x}{t} n \pi\right) \sum_{k \in \mathbb{Z}} e^{-\frac{1}{2}(kx + k^2 t)}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Thus for  $x \in \mathbb{R}$ ,

$$\phi_{\pi \frac{n}{t}}(x, t) = 0 \Leftrightarrow \exists p \in \mathbb{Z} : \frac{x}{t} = \frac{p}{n}.$$

The function  $\frac{\phi_{\pi \frac{n}{t}}(\cdot, t)}{\phi_0(\cdot, t)}$  being bounded on  $[0, t]$ , we get the following inclusion

$$\{x \in ]0, t[ : \phi_0(x, t) = 0\} \subset \{t \frac{p}{n}, p \in \{1, \dots, n-1\}\}.$$

For  $p \in \{1, \dots, n-1\}$ ,  $(n-2)\Lambda_0 + \frac{1}{2}(p-1)\alpha_1$  is a dominant weight and  $\frac{n}{t}\phi_0(t \frac{p}{n}, t)$  is the numerator of  $\text{ch}_{(n-2)\Lambda_0 + \frac{p-1}{2}\alpha_1}(\frac{t}{2n}\Lambda_0)$  given in (2). As a character, evaluated at  $\frac{1}{2n}\Lambda_0$ , is positive, the third property follows. The fourth one is an immediate consequence of the first and the third ones.  $\square$

Let us enounce a classical result on Fourier series that will be used to prove proposition 4.7

**Lemma 4.6.** *Let  $t$  be a positive real number and  $f : [0, t] \rightarrow \mathbb{R}$  be a function such that  $f(0) = f(t) = 0$ , which is  $C^2$  on  $[0, t]$ . Then the series of Fourier coefficients of  $f$  converges absolutely and*

$$f(x) = \sum_{n=1}^{+\infty} \frac{1}{t} \int_0^t \sin\left(\frac{z}{t}n\pi\right) f(z) dz \sin\left(\frac{x}{t}n\pi\right),$$

for every  $x \in [0, t]$ .

**Proposition 4.7.** *Let  $t$  be a positive real number. A probability measure  $\mu$  on  $[0, t]$  is characterized by the quantities*

$$\int_0^t \frac{\phi_{n\pi/t}(x, t)}{\phi_0(x, t)} \mu(dx), \quad n \in \mathbb{N}.$$

*Proof.* For  $t \in \mathbb{R}_+^*$ ,  $x \in \mathbb{R}$ , we let

$$e(x, t) = 2 \frac{t}{n} \sum_{k \in \mathbb{Z}} e^{-\frac{1}{2}(kx+k^2t)}.$$

Let  $u$  be a  $C^2$  function on  $[0, t]$  (a polynomial function for instance). We first notice that the function  $\frac{u(\cdot)\phi_0(\cdot, t)}{e(\cdot, t)}$  satisfies the condition of lemma 4.6. We let for  $n \in \mathbb{N}^*$ ,

$$c_n = \frac{1}{t} \int_0^t \frac{u(x)\phi_0(x, t)}{e(x, t)} \sin\left(\frac{x}{t}n\pi\right) dx.$$

One has

$$\begin{aligned} \int_0^t u(x) \mu(dx) &= \int_0^t u(x) \frac{\phi_0(x, t)}{e(x, t)} \frac{e(x, t)}{\phi_0(x, t)} \mu(dz) \\ &= \int_0^t \sum_{n=1}^{+\infty} c_n \sin\left(\frac{x}{t}n\pi\right) \frac{e(x, t)}{\phi_0(z, t)} \mu(dz) \\ &= \int_0^t \sum_{n=1}^{+\infty} c_n \frac{\phi_{n\pi/t}(x, t)}{\phi_0(x, t)} \mu(dz) \\ &= \sum_{n=1}^{+\infty} c_n \int_0^t \frac{\phi_{n\pi/t}(x, t)}{\phi_0(x, t)} \mu(dz), \end{aligned}$$



where the last identity follows from the fact that the series of  $c_n$  is absolutely convergent and that the function  $\frac{\phi_{n\pi/t}(\cdot, t)}{\phi_0(\cdot, t)}$  is bounded on  $[0, t]$ .  $\square$

**A conditioned space-time Brownian motion.** Let us denote by  $C$  the fundamental Weyl chamber defined by

$$C = \{x \in \mathfrak{h}^* : (x, \alpha_i) \geq 0, i \in \{0, 1\}\}.$$

That is, an element of  $C$  can be written

$$t\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where  $t \in \mathbb{R}_+, x \in [0, t], y \in \mathbb{C}$ . For  $x \in \mathbb{R}$ , we denote by  $\mathbb{W}_x$  the Wiener measure on the set  $C(\mathbb{R}_+)$  of real valued continuous functions on  $\mathbb{R}_+$ , under which the coordinate process  $(X_t, t \geq 0)$  is a Brownian motion starting from  $x$ , and denote the natural filtration of  $(X_t)_{t \geq 0}$  by  $(\mathcal{F}_t)_{t \geq 0}$ . One considers the stopping times  $T_u, u \in \mathbb{R}_+$ , defined by

$$T_u = \inf\{t \geq 0 : X_t = 0 \text{ or } X_t = t + u\}.$$

Proposition 4.5 ensures that  $\phi_0(X_s, s + u)$  doesn't change of sign whenever  $s \in [0, T_u]$ . Let  $x$  be a positive real number such that  $u > x$ . One has  $\mathbb{W}_x(T_u > 0) = 1$ . The function  $\phi_0$  being space-time harmonic, the process  $\phi_0(X_t, t + u)$  is a local martingale. Actually, each summand of the sum is a local martingale, for which the quadratic variation is easily shown to be integrable, so that, each summand is a true Martingale. As their sum converges absolutely in  $L_2$  norm, one obtains that  $\phi_0(X_t, t + u), t \geq 0$ , is a true martingale. As  $\phi_0(X_{T_u}, T_u + u) = 0$ , one defines a measure  $\mathbb{Q}_{x,u}$  on  $C(\mathbb{R}_+)$  letting

$$\mathbb{Q}_{x,u}(A) = \mathbb{E}_x\left(\frac{\phi_0(X_t, t + u)}{\phi_0(x, u)} 1_{\{T_u > t\} \cap A}\right), A \in \mathcal{F}_t.$$

Let  $r$  and  $s$  be two positive real number such that  $r < t$ . Using that

$$T_u = T_{u+r} \circ \theta_r + r \text{ on } \{T_u \geq r\},$$

where  $\theta_r$  the shift operator defined by

$$\forall t \in \mathbb{R}_+^*, X_t \circ \theta_r = X_{t+r},$$

one easily proves that  $(X_t, t \geq 0)$  is an inhomogeneous Markov process under  $\mathbb{Q}_{x,u}$  satisfying

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_{x,u}}(f(X_{t+r}) | \mathcal{F}_r) &= \mathbb{E}_{X_r}\left(\frac{\phi_0(X_t, t + r + u)}{\phi_0(X_r, r + u)} f(X_t) 1_{T_{u+r} > t}\right) \\ (7) \qquad \qquad \qquad &= \mathbb{E}_{\mathbb{Q}_{X_r, r+u}}(f(X_t)), \end{aligned}$$

for any real valued measurable bounded function  $f$ .

**Proposition 4.8.** For  $r, t, u \in \mathbb{R}_+^*, x \in ]0, u[$ , and  $a \in \mathbb{R}$ , one has

$$(8) \qquad \mathbb{Q}_{x,u}\left(\frac{\phi_a(X_t, t + u)}{\phi_0(X_t, t + u)}\right) = \frac{\phi_a(x, u)}{\phi_0(x, u)} e^{\frac{a^2}{2}t}.$$

and

$$(9) \qquad E_{\mathbb{Q}_{x,u}}\left(\frac{\phi_a(X_{t+r}, t + r + u)}{\phi_0(X_{t+r}, t + r + u)} | \mathcal{F}_r\right) = \frac{\phi_a(X_r, r + u)}{\phi_0(X_r, r + u)} e^{\frac{a^2}{2}t}.$$

*Proof.* One proves as previously that

$$(e^{-\frac{a^2}{2}t} \phi_a(X_t, t+u), t \geq 0)$$

is a true martingale, which implies that

$$\mathbb{W}_x(\phi_a(X_t, t+u)e^{-\frac{a^2}{2}t} 1_{T_u < t}) = 0,$$

and identity (8). The second identity follows, using (7).  $\square$

## 5. THE CONDITIONED BROWNIAN MOTION AND THE MARKOV CHAINS ON THE SET OF DOMINANT WEIGHTS

Let us focus on the Markov chains defined in section 3 when  $\omega = \Lambda_0$ . We recall that the weights occurring in  $V(\Lambda_0)$  are

$$\Lambda_0 + k\alpha_1 - (k^2 + s)\delta, \quad k \in \mathbb{Z}, \quad s \in \mathbb{N},$$

with respective multiplicities  $p(s)$ , the number of partitions of  $s$  (see for instance chapter 9 in [7]). If we consider, for  $h = \frac{1}{2}(h_1\alpha_1 + h_2\Lambda_0)$ , with  $h_1 \in \mathbb{R}, h_2 \in \mathbb{R}_+$ , the associated probability measure  $\mu_{\Lambda_0}$  defined by (3) and the associated random walk  $(X(n), n \geq 0)$ , then its projection on  $\mathbb{Z}\alpha_1$  is a random walk with increments distributed according to a probability measure  $\bar{\mu}_{\Lambda_0}$  defined by

$$\bar{\mu}_{\Lambda_0}(k) = C_h e^{kh_1 - \frac{h_2}{2}k^2}, \quad k \in \mathbb{Z},$$

where  $C_h$  is a normalizing constant depending on  $h$ .

**The main theorem.** For  $n \in \mathbb{N}^*$ , we consider a random walk  $(X_k^n, k \geq 0)$  starting from 0, whose increments are distributed according to probability measure  $\mu_{\Lambda_0}$  associated to  $h = \frac{1}{2n}\Lambda_0$ . If we denote by  $(\bar{X}_k^n, k \geq 0)$  its projection on  $\mathbb{Z}\alpha_1$ , standard method shows that the sequence of processes  $(\frac{1}{n}\bar{X}_{[nt]}^n, t \geq 0)$  converges towards a standard Brownian motion on  $\mathbb{R}$  when  $n$  goes to infinity.

Let  $x$  and  $u$  be two positive numbers such that  $x < u$ . For  $n \in \mathbb{N}^*$ , we consider a Markov process  $(\Lambda_k^n, k \geq 0)$  starting from  $[nu]\Lambda_0 + [xn]\frac{1}{2}\alpha_1$ , with the transition probability  $q_\omega$  defined by (4), with  $\omega = \Lambda_0$  and  $h = \frac{1}{2n}\Lambda_0$ . It is important to notice that  $(\Lambda_k^n, \delta) = [nu] + k$  for every  $k \in \mathbb{N}$ . If  $\bar{\Lambda}_k^n$  is the projection of  $\Lambda_k^n$  on  $\text{vect}\{\Lambda_0, \alpha_1\}$  for every  $k \in \mathbb{N}$  and  $n \in \mathbb{N}^*$ , then the following convergence holds.

**Theorem 5.1.** *The sequence of processes  $(\frac{1}{n}\bar{\Lambda}_{[nt]}^n, t \geq 0)$  converges when  $n$  goes to infinity towards the process  $((t+u)\Lambda_0 + \frac{X_t}{2}\alpha_1, t \geq 0)$  under  $\mathbb{Q}_{x,u}$*

*Proof.* Let  $t \in \mathbb{R}_+$ . We denote by  $\mu_t^n$  the law of  $\frac{1}{2n}(\Lambda_{[nt]}^n, \alpha_1)$ , for  $n \in \mathbb{N}$ . The probability measure  $\mu_t^n$  is carried by  $[0, t+u]$ . The interval  $[0, t+u]$  being a compact set, the space of probability measures on  $[0, t+u]$  endowed with the weak topology is also compact. Suppose that a subsequence of  $(\mu_t^n)_n$  converges towards  $\mu_t$ . For  $\lambda \in P_m^+$ , one has

$$\frac{\text{ch}_\lambda(\frac{a}{n}\alpha_1 + \frac{1}{2n}\Lambda_0)}{\text{ch}_\lambda(\frac{1}{2n}\Lambda_0)} = \frac{\phi_a(\frac{1}{n}((\lambda, \alpha_1) + 1), \frac{1}{n}(m+2)) \phi_0(\frac{1}{n}, \frac{4}{n})}{\phi_0(\frac{1}{n}((\lambda, \alpha_1) + 1), \frac{1}{n}(m+2)) \phi_a(\frac{1}{n}, \frac{4}{n})},$$

for any  $a \in \mathbb{R}$ , and  $n \in \mathbb{N}^*$ . The function  $(x, t) \mapsto \frac{\phi_a(x, t+u)}{\phi_0(x, t)}$  can be shown to be uniformly continuous on  $\{(x, t) \in \mathbb{R} \times [0, T] : 0 \leq x \leq u+t\}$  for every  $T \in \mathbb{R}_+$ . As

$\lim_{n \rightarrow \infty} \frac{\phi_0(\frac{1}{n}, \frac{t}{n})}{\phi_a(\frac{1}{n}, \frac{t}{n})} = 1$ , identity (5) implies that  $\mu_t$  satisfies

$$\int_0^{t+u} \frac{\phi_a(z, t+u)}{\phi_0(z, t+u)} \mu_t(dz) = \frac{\phi_a(x, u)}{\phi_0(x, u)} e^{\frac{a^2}{2}t}.$$

Proposition 4.7 implies that  $(\mu_t^n)_n$  converges towards  $\mu_t$  and proposition 4.8 implies that  $\mu_t$  is the distribution of  $X_t$  under  $\mathbb{Q}_{x,u}$ . Convergence of the sequence of random processes  $(\frac{1}{2n}(\Lambda_{[nt]}^n, \alpha_1), t \geq 0)$  - in the sense of finite dimensional distributions convergence - follows similarly from identity (6) and (9). As one has the inequality

$$\frac{1}{2n} |(\Lambda_{[nt]}^n, \alpha_1) - (\Lambda_{[ns]}^n, \alpha_1)| \leq |t - s| + \frac{1}{n},$$

for every  $n \in \mathbb{N}^*$  and  $s, t \in \mathbb{R}_+$ , the sequence of random processes  $(\frac{1}{2n}(\Lambda_{[nt]}^n, \alpha_1), t \geq 0)$  is clearly tight and thus it converges also in  $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$  endowed with the topology of uniform convergence on compact sets. □

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