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# Energy fluctuations in simple conduction models

François Huveneers\*

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## Abstract

We introduce a class of stochastic weakly coupled map lattices, as models for studying heat conduction in solids. Each particle on the lattice evolves according to an internal dynamics that depends on its energy, and exchanges energy with its neighbors at a rate that depends on its internal state. We study energy fluctuations at equilibrium in a diffusive scaling. In some cases, we derive the hydrodynamic limit of the fluctuation field.

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# 1 Introduction

It is generally expected that the temperature profile of thermally isolated solids relaxes according to some non-linear heat equation

$$\partial_t T = \nabla(D(T)\nabla T).$$

The derivation of this equation, starting from a microscopic hamiltonian dynamics, is one of the major open problems in statistical mechanics out of equilibrium [3][16]. Very idealized models of solid constitute a possible starting point to develop our understanding on the problem, and in fact, at the present time, even such cases can be challenging to analyze (see [7] for a good example of such a philosophy).

Lattice gases furnish the first examples where the heat equation can be recovered through a diffusive rescaling of space and time. Each atom reduces here to a point on a lattice, is characterized by its energy only, and interacts stochastically with its neighbors. The stochastic interaction is a reasonable approximation for particles that should evolve according to some chaotic internal dynamics. A set of independent random walks (IRW), the simple symmetric exclusion process (SSEP) [11], and the Kipnis Marchioro Presutti model (KMP) [12], constitute three cases where the heat equation can be easily derived. These systems may be closer to physics than what could be thought at the first glance: in [13], an example of noisy hamiltonian system is found, that gives rise to the SSEP in a weak coupling limit.

This said, true atoms should clearly be described by position and momentum, and not only energy. To move towards a more realistic situation, let us thus consider a lattice where particles at the nodes now have some internal degree of freedom. We assume the local dynamics of particles to have good ergodic properties, and the interaction between them to be weak and controlled by a parameter  $\epsilon > 0$ . So, individual atoms will reach their own equilibrium at a smaller time scale than energy is exchanged.

Much progress has been recently accomplished in the understanding of these systems, by means of a two-stage analysis, as outlined in [9]. First, letting  $\epsilon \rightarrow 0$  and rescaling time properly, an autonomous stochastic process for the energy alone is obtained. Second, rescaling space and time diffusively, the heat equation is derived starting from this mesoscopic stochastic process. The first part of this program has now been rigorously accomplished for a purely hamiltonian dynamics [5]. Though in a rather different way, the distinction of two different time-scales also has allowed to better understand the conductivity of weakly anharmonic solids [1][15].

The introduction of two successive limits is however somehow artificial. One eventually would like to fix some, possibly small,  $\epsilon > 0$ , and directly consider the diffusive limit. Results in that direction are few. Let us quote three of them. First, in [2], the heat equation is recovered starting from a noisy harmonic chain (see also [8] for fluctuations at equilibrium). The proof rests on the fact that an exact fluctuation-dissipation decomposition is available there. Second, a deterministic coupled map lattice is analyzed in [4], and diffusion of an initially localized energy packet is shown. The dynamics of particles is there assumed to be independent of energy. Third, fluctuations of energy for a noisy hamiltonian system

at equilibrium are considered in [17] (see also [10] for a simpler model) ; the fluctuation field is shown to converge to a generalized Ornstein-Uhlenbeck process. Some level of symmetry of the generator, the sector condition, is needed to derive this result.

In this paper, we introduce models of heat conduction that are conceptually very simple. They maybe will even look a bit cheap compared to the aforementioned examples. Though, the analysis reveals that nowadays techniques do not directly apply to describe how energy diffuses in the cases we consider. We will in fact need to look at particular situations in order to derive rigorous results. To make this more concrete, let us now briefly outline the content.

In Section 2, we define three models of conduction. In a sense made precise by (2.3) below, they constitute respective versions of the IRW, SSEP and KMP models, where, now, atoms have an internal degree of freedom. The interaction between near particles is controlled by a parameter  $\epsilon > 0$ . The models so obtained are non-gradient and do not satisfy the sector condition.

Like in [17] and [10], we study the fluctuations of energy in equilibrium, as this might be the easiest problem to look at. We introduce the fluctuation field in Section 3, and we state there Proposition 1, that will allow us to derive the hydrodynamic limit of this field in some cases. While it is essentially a variation on a classical method [11], we here assume that exists an approximative fluctuation-dissipation decomposition of the microscopic current in a particular sense. We hope that this kind of result can be of some use elsewhere. The proof is postponed to Section 5.

Explicit results are gathered in Section 4. The hydrodynamic limit of the field in the first order in  $\epsilon$  is derived in Theorem 1. As a corollary, a weak coupling limit is shown, where  $\epsilon$  is sent to 0 after a diffusive rescaling of space and time. This kind of limit is more satisfactory than the one in [5][13][16], where  $\epsilon$  was sent to 0 for a fixed number of particles without any rescaling of space. It also avoids the two-stage procedure of [9]. One then approximates, like in [4], the local dynamics by a process independent of energy. In Theorem 2, the hydrodynamic limit of the fluctuation field is obtained under that hypothesis for any small enough  $\epsilon > 0$ . While, with respect to [4], we deal with a rather particular case, our proof is much shorter, and we are not limited to an initially localized energy packet.

## 2 Models

Throughout all the paper,  $\epsilon > 0$  represents a strictly positive number. Let us consider a set of  $N \geq 3$  identical particles. The coordinates of the particles are indexed by a point in the periodic one-dimensional lattice  $\mathbb{Z}_N$  of integers modulo  $N$ . They form thus a one-dimensional chain with periodic boundary conditions. Each particle is characterized by a local degree of freedom, and a non negative energy, so that the phase space  $\mathsf{X}$  is given by

$$\mathsf{X} = \mathsf{M}^N \times \mathbb{R}_+^N.$$

where

$$\mathbf{M} = \mathbb{T} \times \{-1, 1\} = (\mathbb{R}/\mathbb{Z}) \times \{-1, 1\}.$$

A point  $r \in \mathbf{M}$  is written as  $r = (q, \varsigma) \in \mathbb{T} \times \{-1, 1\}$ ; a point  $\mathbf{x} \in \mathbf{X}$  is written as

$$\mathbf{x} = (\mathbf{r}, \mathbf{e}) = (r_1, \dots, r_N, e_1, \dots, e_N) = (\mathbf{q}, \boldsymbol{\varsigma}, \mathbf{e}) = (q_1, \dots, q_N, \varsigma_1, \dots, \varsigma_N, e_1, \dots, e_N).$$

We set also  $x_k = (r_k, e_k) = (q_k, \varsigma_k, e_k)$  for  $1 \leq k \leq N$ .

We see  $q_k$  as the position of particle  $k$ , and  $\varsigma_k$  as the sign of its velocity. It could have seemed more natural to first define the usual hamiltonian variables  $q_k$  and  $p_k$ , and then express the energy in function of them. However, for our purposes, the distinction between the local variable  $r_k$  and the energy  $e_k$  is the most important. In fact, we almost never will make use of the variables  $q_k$  and  $\varsigma_k$ . We assume that each particle have a ground state energy  $\mathbf{e}_0 > 0$ , and we think of  $e_k$  as being the extra amount of the energy of particle  $k$  with respect to  $\mathbf{e}_0$ .

We consider a stochastic dynamics defined through the generator

$$L = L' + \epsilon L'' = \sum_{k \in \mathbb{Z}_N} (L'_k + \epsilon L''_k)$$

acting on functions on the phase space  $\mathbf{X}$ . Here  $L'_k$  governs the internal dynamics of particle  $k$ , and  $L''_k$  the exchanges of energy between particles  $k$  and  $k + 1$ . We denote by  $X_t$  the process generated by  $L$  at time  $t \geq 0$ . Let us now concretely define  $L'$  and  $L''$ .

## 2.1 Local dynamics

Let  $1 \leq k \leq N$ . One defines

$$L'_k u = A'_k u + S'_k u = \varsigma_k \sqrt{\mathbf{e}_0 + e_k} \cdot \partial_{q_k} u + \sqrt{\mathbf{e}_0 + e_k} \cdot (u(\dots, -\varsigma_k, \dots) - u(\dots, \varsigma_k, \dots)).$$

Here  $A'_k$  and  $S'_k$  denote respectively the symmetric and antisymmetric part of  $L'_k$ , with respect to the uniform probability measure  $\mu_{\mathbb{U}}$  on  $\mathbf{M}$ . So particle  $k$  moves freely on the circle at speed  $\sqrt{\mathbf{e}_0 + e_k}$  and, at random times, the sign of its velocity gets flipped. We will make use of the following notation: given a function  $u$  on  $\mathbf{X}$ , one writes

$$\langle u | \mathbf{e} \rangle = \int_{\mathbf{M}^N} u(\mathbf{r}, \mathbf{e}) \mu_{\mathbb{U}}(d\mathbf{r}).$$

No sector condition holds. Moreover, the dynamics generated by the symmetric part  $S'_k$  alone is very degenerated since it does not affect the physical location  $q_k$ . The dynamics generated by  $L'$  quickly relaxes to equilibrium: there exists a constant  $c > 0$ , independent of  $N$  such that, given a function  $u$  satisfying  $u(\cdot, \mathbf{e}) \in L^2(\mathbf{M}^N, \mu_{\mathbb{U}})$  and  $\langle u | \mathbf{e} \rangle = 0$  for any  $\mathbf{e} \in \mathbb{R}_+^N$ , it holds that

$$\|e^{L't} u(\cdot, \mathbf{e})\|_2 \leq e^{-ct} \|u(\cdot, \mathbf{e})\|_2.$$

This follows from an explicit diagonalisation of  $L'_k$  (see also [13] or [6] for two different approaches). Given  $e_k \in \mathbb{R}_+$ , the functions

$$\varphi_{l,\tau}(q_k, s_k) = \frac{1}{Z(l,\tau)} (2i\pi l \sqrt{e_0 + e_k} + \delta_0(\lambda_{l,\tau}) + \lambda_{l,\tau} s_k) e^{2i\pi l q_k},$$

with  $Z(l,\tau)$  a normalization factor, and with

$$\lambda_{l,\tau} = -\sqrt{e_0 + e_k} \left( 1 + i\tau \sqrt{(2\pi l)^2 - 1} \right), \quad l \in \mathbb{Z}, \quad \tau \in \{-1, +1\},$$

form an orthonormal basis of  $L^2(\mathbf{M})$  such that

$$L_k \varphi_{l,\tau}(q_k, s_k) = \lambda_{l,\tau} \varphi_{l,\tau}(q_k, s_k).$$

It is seen that  $\lambda_{l,\tau} = 0$  if and only if  $l = 0$  and  $\tau = 1$  (with the convention  $\sqrt{-1} = +i$ ), and that otherwise  $\Re \lambda_{l,\tau} \leq -\sqrt{e_0}$ . Our claim then follows by tensorization since all particles evolve independently under the dynamics generated by  $L'$ .

We will use this information in the following form. Let  $f$  be a function on  $\mathbf{X}$  such that  $f(\cdot, \mathbf{e}) \in L^2(\mathbf{M}^N, \mu_{\mathbf{U}})$  and that  $\langle f | \mathbf{e} \rangle = 0$  for all  $\mathbf{e}$ . Then there exists a unique function  $u$  on  $\mathbf{X}$  such that  $u(\cdot, \mathbf{e}) \in L^2(\mathbf{M}^N, \mu_{\mathbf{U}})$  and  $\langle u | \mathbf{e} \rangle = 0$  for all  $\mathbf{e}$ , that solves the Poisson equation

$$-L' u = f. \tag{2.1}$$

We call this function  $u$  the fundamental solution to this Poisson equation. Moreover there exists a constant  $C < +\infty$  independent of  $N$  and  $f$  such that

$$\|u(\cdot, \mathbf{e})\|_2 \leq C \|f(\cdot, \mathbf{e})\|_2 \tag{2.2}$$

for every  $\mathbf{e} \in \mathbb{R}^N$ . This follows from the representation

$$u = \int_0^\infty e^{L't} f dt,$$

or directly from the explicit diagonalisation of  $L'$ .

## 2.2 Interactions

In a one-dimensional chain of oscillators evolving according to the laws of classical mechanics, the strength of the force between two atoms depends on their relative physical location, and the instantaneous current between them is a function their positions in phase space. To mimic this feature in our model, let us introduce a nonzero uniformly bounded symmetric function  $\chi \geq 0$  on  $\mathbf{M}^2$ . The function  $\chi$  will make the rate of exchange of energy depend on the locations of particles. We set also

$$\bar{\chi} = \int_{\mathbf{M}^2} \chi(r_0, r_1) \mu_{\mathbf{U}}(d\mathbf{r}).$$

As an example, one can take

$$\chi(r_0, r_1) = \chi(q_0, q_1) = \chi_{[a,b]^2}(q_0, q_1),$$

where  $\chi_{[a,b]}$  denotes the characteristic function of some interval  $[a, b] \subset \mathbb{T}$ . Then, the interaction between two particles will only be possible if they both are in the interval  $[a, b]$ .

We will define three possible ways of interacting, and so three different models. In the three cases, the interaction will be purely stochastic, meaning that the operator  $L''$  is symmetric with respect to the canonical invariant measures defined in Section 3. This is admittedly in sharp contrast with hamiltonian interactions. We may view our models as perturbations of the IRW, SSEP or KMP dynamics respectively. Indeed, the operator  $\mathcal{L}$  acting on functions  $u$  that depend only on the energies  $\mathbf{e}$ , and defined by

$$\mathcal{L}u = \frac{1}{\bar{\chi}} \langle L''u | \mathbf{e} \rangle, \quad (2.3)$$

will be nothing else than the generator of these dynamics.

**Model 1: Exchanging bits of energy.** Let us restrict the set of reachable energies to  $\mathbb{N} = \{0, 1, 2, \dots\}$ , so that the phase space writes  $\mathbf{X} = \mathbb{M}^N \times \mathbb{N}^N$ . Let  $1 \leq k \leq N$ . One defines

$$L_k''u = \chi(r_k, r_{k+1}) \cdot (e_k(u \circ \sigma_k^+ - u) + e_{k+1}(u \circ \sigma_{k+1}^- - u))$$

with

$$\begin{aligned} \sigma_k^-(\mathbf{x}) &= (\mathbf{r}, e_1, \dots, e_{k-1} + 1, e_k - 1, e_{k+1}, \dots, e_N), \\ \sigma_k^+(\mathbf{x}) &= (\mathbf{r}, e_1, \dots, e_{k-1}, e_k - 1, e_{k+1} + 1, \dots, e_N). \end{aligned}$$

So, with such an interaction, particles tend to give part of their energy to their neighbors at a rate proportional to the value of their energy times the value of  $\chi$ . The instantaneous current is defined through the relation

$$Le_k = \epsilon L''e_k = \epsilon(j_{k-1,k} - j_{k,k+1})$$

with

$$j_{k,k+1} = \chi(r_k, r_{k+1})(e_k - e_{k+1}). \quad (2.4)$$

**Model 2: Exchanging the whole energy.** Let us further restrict the set of reachable energies to  $\{0, 1\}$ , so that the phase space now writes  $\mathbf{X} = \mathbb{M}^N \times \{0, 1\}^N$ . Let  $1 \leq k \leq N$ . One defines

$$L_k''u = \chi(r_k, r_{k+1})(u \circ \sigma_k - u)$$

with

$$\sigma_k \mathbf{x} = (\mathbf{r}, e_1, \dots, e_{k+1}, e_k, \dots, e_N).$$

So here, near particles may exchange their energy. It is checked that the instantaneous current is the same as in the first model.

Let us stress at this point that the dynamics still should perfectly make sense if energy was allowed to take more than two values. However, since energy is only exchanged between particles, there should be  $n - 1$  conserved quantities if energy was allowed to take  $n$  values. It is not at all clear in that case that an initial density of energy should evolve autonomously in the hydrodynamic regime, since an initial macroscopic state should in fact be specified by the value of each conserved quantity. We have thus limited ourselves to the case of two reachable energies only, in order to avoid extra complications.

**Model 3: Sharing energy.** Energy is here allowed to take any non negative value, so that the phase space is given by  $\mathsf{X} = \mathsf{M}^N \times \mathbb{R}_+^N$ . Let  $1 \leq k \leq N$ . One defines

$$L_k'' u = 2\chi(r_k, r_{k+1})(T_k - Id)u$$

with

$$T_k u = \int_0^1 u(\mathbf{r}, e_1, \dots, p(e_k + e_{k+1}), (1-p)(e_k + e_{k+1}), \dots, e_N) dp.$$

The instantaneous current is the same as in the two previous models. A similar dynamics is considered in [7].

### 3 Fluctuation field

The derivation of the heat equation requires to consider an energy profile initially out of equilibrium. In this work, like in [17][10], we will however only look at small perturbations out of equilibrium. To see what small means, let us assume that our system is in equilibrium at an inverse temperature  $0 < \beta < +\infty$  (see below), and let us denote by  $\langle e \rangle_\beta$  the average energy of a particle. Let  $I \subset \mathbb{T}$  be an interval. For a typical event in the invariant measure, one has

$$\sum_{k:k/N \in I} (e_k - \langle e \rangle_\beta) \sim \sqrt{N}.$$

So let

$$\mathcal{E}_t(I) = \frac{1}{\sqrt{N}} \sum_{k:k/N \in I} (e_k(t) - \langle e \rangle_\beta)$$

be the macroscopic observable we look at. The random variable  $\mathcal{E}_t(I)$  is expected to fluctuate in a diffusive time-scale. So a typical kind of probability that one would like to estimate in the limit  $N \rightarrow \infty$  is

$$\mathbf{P}(\mathcal{E}_{N^2 t}(I) \simeq b \mid \mathcal{E}_0(I) \simeq a),$$

where  $a, b \in \mathbb{R}$ , and where  $x \simeq a$  means  $x \in [a - \delta, a + \delta]$  for some small  $\delta > 0$ . So in fact, we are considering initial profiles that have an excess of energy of order  $\sqrt{N}$  in subintervals of  $\mathbb{T}$ , rather than of order  $N$  for a profile truly out of equilibrium

We now need some definitions. Let us first define canonical equilibrium measures. For our three models

$$\langle u \rangle_\beta = \int_{\mathbf{X}} u(\mathbf{r}, \mathbf{e}) \mu_{\text{U}}(d\mathbf{r}) \mu_\beta(d\mathbf{e}),$$

where, for  $u = u(\mathbf{e})$ , one has

$$\begin{aligned} \mu_\beta(u) &= \frac{1}{Z(\beta)} \sum_{\mathbf{e} \in \mathbb{N}^N} \frac{(1/\beta)^{e_1 + \dots + e_N}}{e_1! \dots e_N!} u(\mathbf{e}) && \text{for Model 1,} \\ \mu_\beta(u) &= \frac{1}{Z(\beta)} \sum_{\mathbf{e} \in \{0,1\}^N} e^{-\beta(e_1 + \dots + e_N)} u(\mathbf{e}) && \text{for Model 2,} \\ \mu_\beta(u) &= \frac{1}{Z(\beta)} \int_{\mathbb{R}_+^N} e^{-\beta(e_1 + \dots + e_N)} u(\mathbf{e}) d\mathbf{e} && \text{for Model 3,} \end{aligned}$$

with  $Z(\beta)$  a normalization factor.

Let us next define the fluctuation field  $(Y_{N^2t}^{(N)})_{t \geq 0}$ . For any smooth test function  $H$  on  $\mathbb{T}$ , we define

$$Y_{N^2t}^{(N)}(H) = \frac{1}{\sqrt{N}} \sum_{k=1}^N H(k/N) (e_k(N^2t) - \langle e \rangle_\beta).$$

Most of the time we will just write  $Y$  instead of  $Y^{(N)}$ . Formally,  $(Y_{N^2t})_t$  is as a random process with values in  $\mathcal{D}(\mathbb{R}_+, \mathcal{H}_{-3})$ , where  $\mathcal{D}$  denotes the set of cad-lag functions, and where  $\mathcal{H}_{-3}$  denotes the Sobolev space of distributions  $Z$  on  $\mathbb{T}$  with norm  $\|Z\|_{-3}^2 = \sum_{k \in \mathbb{Z}} k^{-6} |Z(\mathbf{f}_k)|^2$ , with  $\mathbf{f}_k(x) = e^{2i\pi kx}$ .

Let us finally define the stationary generalized Ornstein-Uhlenbeck process  $(Y_t)_{t \geq 0} \in \mathcal{C}(\mathbb{R}_+, \mathcal{H}_{-3})$ , where  $\mathcal{C}$  denotes the set of continuous functions, for some diffusion constant  $\epsilon D > 0$  and thermal capacity  $\sigma^2 > 0$ . We here have anticipated the fact that, in our applications, the diffusion constant will be of order  $\epsilon$ . Given a smooth test function  $H$  on  $\mathbb{T}$ , and for  $t \geq 0$ , we let  $H_t$  be the solution to the heat equation

$$H_0 = H, \quad \partial_t H_t = \epsilon D \cdot \partial_x^2 H_t. \quad (3.1)$$

Our definition of  $(Y_t)_{0 \leq t \leq T}$  is then as follows. First,  $Y_0(H)$  is a centered Gaussian random variable with variance

$$\sigma^2 \int_{\mathbb{T}} H_0^2(x) dx. \quad (3.2)$$

Next, for  $t \geq 0$ , we have the decomposition

$$Y_t(H) = Y_0(H_t) + M_t(H),$$

where  $M_t(H)$  is a Gaussian centered martingale, independent of  $Y_0(H_t)$ , and with variance

$$2\epsilon D \sigma^2 \int_0^t \left( \int_{\mathbb{T}} (\partial_x H_s(x))^2 dx \right) ds.$$

Since  $Y_t$  is a distribution, it holds that  $Y_t(aG + bH) = aY_t(G) + bY_t(H)$  for any smooth  $G$  and  $H$  on  $\mathbb{T}$ , and  $a, b \in \mathbb{R}$ . Using this property, it is direct to recover the more classical definition of the

stationary Ornstein-Uhlenbeck process:  $(Y_t)_{t \geq 0}$  is the only stationary Gaussian process with zero mean and covariance

$$\mathbb{E}(Y_t(H)Y_0(G)) = \sigma^2 \int_{\mathbb{R}^2} \frac{e^{-\frac{(x-y)^2}{4\epsilon t D}}}{\sqrt{4\pi\epsilon t D}} H(x)G(y) dx dy.$$

Our aim is to show that  $(Y_{N^2 t})_t$  converges in some sense to  $(Y_t)_t$  as  $N \rightarrow \infty$ , with

$$D = D(\beta) > 0 \quad \text{and} \quad \sigma^2 = \sigma^2(\beta) = \langle e^2 \rangle_\beta - \langle e \rangle_\beta^2.$$

We say that a function  $D \cdot \nabla e$  on  $\mathsf{X}$  is gradient if it is of the form

$$D \cdot \nabla e = \sum_{k \in \mathbb{Z}_N} D_k \cdot (e_k - e_{k+1}). \quad (3.3)$$

We then also write  $D = \sum_{k \in \mathbb{Z}_N} D_k$ . This quantity depends on the representation (3.3) of the gradient function  $D \cdot \nabla e$ , which is not unique due to periodic boundary conditions. This however will create no trouble in the sequel. We will use the following proposition, shown in Section 5, to derive our results. The hypotheses are close to optimal for the proof, but could be quite simplified for our applications.

**Proposition 1.** *Let  $\epsilon' \geq 0$ . Let  $\beta > 0$ . Let  $\sigma^2(\beta) = \langle e^2 \rangle_\beta - \langle e \rangle_\beta^2$ . Assume that there are sequences of functions  $((D \cdot \nabla e)_N)_N$ ,  $(u_N)_N$  and  $(g_N)_N$  in  $L^2(\mathsf{X}, \langle \cdot \rangle_\beta)$ , that may depend on  $\beta$ , such that*

$$-Lu_N - \epsilon(j_{0,1} - (D \cdot \nabla e)_N) = \epsilon' L'' g_N,$$

and that

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{k \in \mathbb{Z}_N} |\langle \tau_k u_N, u_N \rangle_\beta| = 0.$$

We moreover assume that, for any  $k \in \mathbb{Z}_N$ ,  $(D_k)_N$  converges to a finite value as  $N \rightarrow \infty$ , that the following quantities converge to a finite value as  $N \rightarrow \infty$ :

$$\sum_{k \in \mathbb{Z}_N} |(D_k)_N|, \quad \sum_{k \in \mathbb{Z}_N} |\langle g_N, \tau_k L'' g_N \rangle_\beta|, \quad \sum_{k \in \mathbb{Z}_N} |\langle u_N, j_{k,k+1} \rangle_\beta|, \quad \sum_{k \in \mathbb{Z}_N} |\langle u_N, \tau_k L'' g_N \rangle_\beta|,$$

and that

$$D := D(\beta) := \lim_{N \rightarrow \infty} \sum_{j \in \mathbb{Z}_N} (D_j)_N > 0.$$

Then, assuming that  $X_0$  is distributed according to the equilibrium measure  $\langle \cdot \rangle_\beta$ , the finite dimensional distributions of the process  $(Y_{N^2 t})_t$  converge weakly as  $N \rightarrow \infty$  to the finite dimensional distributions of a process  $(Y_t)_t$  such that, for any smooth test function  $H$ ,

$$Y_t(H) = Y_0(H_t) + M_t(H) + R_t(H),$$

where  $H_t$  is given by (3.1). Here  $Y_0(H_t)$  is a Gaussian random variable with variance given by (3.2) with  $H_t$  in place of  $H_0$ ,  $M_t(H)$  is a centered Gaussian martingale, independent of  $Y_0(H_t)$ , which variance at time  $t$  is given by

$$2\epsilon \int_0^t \left( \int_{\mathbb{T}} (\partial_x H_s)^2 dx \right) ds \cdot \left( \bar{\chi} \sigma^2(\beta) - \lim_{N \rightarrow \infty} \sum_{k \in \mathbb{Z}_N} \langle u_N, j_{k,k+1} \rangle_\beta + \frac{\epsilon'}{\epsilon} \lim_{N \rightarrow \infty} \sum_{k \in \mathbb{Z}_N} \langle u_N, \tau_k L'' g_N \rangle_\beta \right), \quad (3.4)$$

and  $R_t(H)$  is centered with variance at time  $t$  bounded by a constant times

$$(\epsilon')^2 \epsilon^{-1} \cdot (1 + t^2) \cdot \lim_{k \in \mathbb{Z}_N} \sum_{k \in \mathbb{Z}_N} |\langle g_N, \tau_k L'' g_N \rangle_\beta|.$$

## 4 Diffusion of energy

The results below hold for our three models. We first obtain the hydrodynamic limit of the field  $Y_{N^2 t}$  to the first order in  $\epsilon$ , with a bound on the variance of the rest term. As a direct corollary, we obtain a weak coupling limit, where  $\epsilon$  is sent to zero after that the diffusive limit is taken.

**Theorem 1.** *Let  $D = \bar{\chi}$ . Let us assume that  $X_0$  is distributed according to the equilibrium measure  $\langle \cdot \rangle_\beta$ . The finite dimensional distributions of the process  $(Y_{N^2 t})_t$  converge weakly as  $N \rightarrow \infty$  to the finite dimensional distributions of a process  $(Y_t)_t$  such that*

$$Y_t(H) = Y_0(H_t) + M_t(H) + R_t(H),$$

where  $H_t$  is given by (3.1), where  $Y_0(H_t)$  is a Gaussian random variable with variance given by (3.2) with  $H_t$  in place of  $H_0$ , where  $M_t(H)$  is a Gaussian martingale, independent of  $Y_0(H_t)$ , with variance

$$2\epsilon D \sigma^2(\beta) \int_0^t \left( \int_{\mathbb{T}} (\partial_x H_s)^2 dx \right) ds + \mathcal{O}(\epsilon^2 t) \quad (\text{as } \epsilon \rightarrow 0, t \text{ fixed}),$$

and where  $R_t(H)$  is centered with a variance that is  $\mathcal{O}(\epsilon^3(1 + t^2))$  (as  $\epsilon \rightarrow 0, t$  fixed).

*Proof.* Thanks to (2.1), there exists a function  $v \in L^2(\mathbb{X}, \langle \cdot \rangle_\beta)$ , depending only on  $x_0$  and  $x_1$ , that solves the Poisson equation

$$-L'v = j_{0,1} - D \cdot (e_0 - e_1) = (\chi - \bar{\chi})(r_0, r_1) \cdot (e_0 - e_1),$$

and such that  $\langle v | \mathbf{e} \rangle = 0$  for every  $\mathbf{e}$ . Defining  $\epsilon' = \epsilon^2$ ,  $u_N = \epsilon v$  and  $g_N = -v$ , one sees that the hypotheses of Proposition 1 are satisfied, from which our claim follows.  $\square$

**Corollary 1.** *Let  $D = \bar{\chi}$ . Let us assume that  $X_0$  is distributed according to the equilibrium measure  $\langle \cdot \rangle_\beta$ . Letting first  $N \rightarrow \infty$  and then  $\epsilon \rightarrow 0$ , the finite dimensional distributions of the process  $(Y_{N^2 \epsilon^{-1} t})_t$  converge weakly to the finite dimensional distributions of the stationary Ornstein-Uhlenbeck process with diffusion constant  $D$  and thermal capacity  $\sigma^2 = \sigma^2(\beta)$ .*

*Proof.* This follows directly from Theorem 1 thanks to the bound in  $\mathcal{O}(\epsilon^3(1 + t^2))$  on the variance of the rest term  $R_t$ .  $\square$

Like in [4], we next approximate the internal dynamics of particles by a process independent of energy. So we now assume that, for  $k \in \mathbb{Z}_N$ ,

$$L'_k u = \varsigma_k \sqrt{\mathbf{e}_0} \cdot \partial_{q_k} u + \sqrt{\mathbf{e}_0} \cdot (u(\dots, -\varsigma_k, \dots) - u(\dots, \varsigma_k, \dots)). \quad (4.1)$$

With such a definition of  $L'$ , particles evolve now completely independently from each others. This in fact is not needed ; any operator  $L'$ , not involving energy and generating a dynamics that quickly relaxes to equilibrium independently of the system size  $N$ , should be fine for our purposes.

Let us see when the diffusion constant obtained with this new definition of  $L'$  can be expected to be close to that of our original models. For this, let us introduce an extra small parameter  $\delta > 0$  in the original models, by replacing everywhere the energy  $\mathbf{e}_0 + e_k$  by a new, smaller energy,  $\mathbf{e}_0 + e'_k = \mathbf{e}_0 + \delta e_k$ . Doing so, the energy of a typical atom of the chain in the Gibbs state will be very close to the positive ground state energy  $\mathbf{e}_0$ , if  $\delta$  is very small. So, for fixed  $\epsilon > 0$ , we then expect the diffusion constant  $\epsilon D = \epsilon D(\delta)$  to converge to the value given in theorem 2 below, valid for any small enough  $\epsilon > 0$ , as  $\delta \rightarrow 0$ .

**Theorem 2.** *Assume that  $L'_k$  is given by (4.1) for  $k \in \mathbb{Z}_N$ , and that  $\epsilon > 0$  is small enough. Assume also that  $X_0$  is distributed according to the equilibrium measure  $\langle \cdot \rangle_\beta$ . As  $N \rightarrow \infty$ , the finite dimensional distributions of the field  $(Y_{N^2 t})_t$  converge weakly to the finite dimensional distributions of a generalized Ornstein-Uhlenbeck process with some diffusion constant  $\epsilon D > 0$  independent of the temperature, and with the thermal capacity  $\sigma^2 = \sigma^2(\beta)$ .*

*Proof.* The proof is made of two steps: we first show that Proposition 1 can be applied with  $\epsilon' = 0$ , and we next identify the variance of  $M_t(H)$ .

**Step 1.** Let  $n \geq 1$ . We show that the hypotheses of Proposition 1 are satisfied with  $\epsilon' = \epsilon^{n+1}$ , and that the limit as  $N \rightarrow \infty$  of all the quantities appearing in the hypotheses of Proposition 1, are bounded by a constant that does not depend on  $n$ . Let us introduce an extra definition. We say that a function  $u$  on  $\mathbf{X}$  is pre-gradient if it is of the form

$$u(\mathbf{r}, \mathbf{e}) = \sum_{k \in \mathbb{Z}_N} \hat{u}(\mathbf{r}, k)(e_k - e_{k+1}).$$

Let us first make two observations. Given a pre-gradient function  $f$ , the fundamental solution  $u$  to the equation  $-L'u = f$  is itself pregradient. Indeed the coefficients  $\hat{u}(\mathbf{r}, k)$  are the fundamental solutions to the equation  $-L'\hat{u}(\mathbf{r}, k) = \hat{f}(\mathbf{r}, k)$ . Second, if a function  $u$  is pregradient, so is  $L''u$ . Indeed, for the three models, one has

$$L''u(\mathbf{r}, \mathbf{e}) = \sum_{k \in \mathbb{Z}_N} (\hat{u}(\mathbf{r}, k-1) - 2\hat{u}(\mathbf{r}, k) + \hat{u}(\mathbf{r}, k+1)) \chi(r_k, r_{k+1})(e_k - e_{k+1}). \quad (4.2)$$

So moreover, if  $u$  depends on variables  $x_a, \dots, x_b$  with  $a \leq b$ , then  $L''u$  depends on variables  $x_{a-1}, \dots, x_{b+1}$ .

Now, for  $N$  large enough for given  $n$ , the functions  $u_N$ ,  $(D \cdot \nabla e)_N$  and  $g_N$  that we will construct will actually not depend on  $N$ , so that we drop the index  $N$ . We set

$$u = \sum_{k=1}^n \epsilon^k u^{(k)}, \quad (D \cdot \nabla e) = \sum_{k=1}^n \epsilon^{k-1} D^{(k)} \cdot \nabla e \quad \text{and} \quad g = -u^{(n)}, \quad (4.3)$$

with the following recursive definitions of  $u^{(k)}$  and  $D^{(k)}$  for every  $k \geq 1$ . First

$$D^{(1)} \cdot \nabla e = \langle j_{0,1} | \mathbf{e} \rangle = \bar{\chi} (e_0 - e_1),$$

and  $u^{(1)}$  is the fundamental solution of

$$-L'u^{(1)} = j_{0,1} - \langle j_{0,1} | \mathbf{e} \rangle.$$

The function  $u^{(1)}$  is pregradient. Next, given the pregradient function  $u^{(k)}$  for  $k \geq 1$ , one takes

$$D^{(k+1)} \cdot \nabla e = \langle L''u^{(k)} | \mathbf{e} \rangle_\beta$$

This indeed defines a linear gradient since  $L''u^{(k)}$  is pregradient. Then one defines  $u^{(k+1)}$  to be the fundamental solution to

$$-L'u^{(k+1)} = L''u^{(k)} - \langle L''u^{(k)} | \mathbf{e} \rangle_\beta.$$

The function  $u^{(k+1)}$  is pregradient.

The functions  $u$ ,  $D \cdot \nabla e$  and  $g$  defined by (4.3) satisfy

$$-Lu - \epsilon(j_{0,1} - D \cdot \nabla e) = \epsilon^{n+1} L''g. \quad (4.4)$$

Thanks to (2.1) and the convergence of geometric series, the quantities appearing in the hypotheses of Proposition 1 are then indeed bounded independently of  $n$  if  $\epsilon > 0$  is taken small enough. Moreover, since  $D^{(1)} = \bar{\chi} > 0$ , it follows from (4.3) that  $D > 0$  if  $\epsilon$  is small enough.

**Step 2.** Let  $u$ ,  $D \cdot \nabla e$  and  $g$  be given by (4.3) Let us first give an expression for  $D$ . Since  $\langle u | \mathbf{e} \rangle = 0$  for every  $\mathbf{e}$ , one has also  $\langle L'u | \mathbf{e} \rangle = 0$  for every  $\mathbf{e}$ , so that, by (4.4),

$$D \cdot \nabla e = \bar{\chi}(e_0 - e_1) + \langle L''u | \mathbf{e} \rangle + \epsilon^n \langle L''g | \mathbf{e} \rangle.$$

The function  $L''u$  is computed by means of (4.2). Writing  $D \cdot \nabla e = \sum_{j \in \mathbb{Z}_N} D_j (e_j - e_{j+1})$ , one has

$$D = \sum_{j \in \mathbb{Z}_N} D_j = \bar{\chi} + \sum_{k \in \mathbb{Z}_N} \langle \chi(\cdot) (\hat{u}(\cdot, k-1) - 2\hat{u}(\cdot, k) + \hat{u}(\cdot, k+1)) \rangle + \mathcal{O}(\epsilon^n).$$

Taking on the other hand the expression (3.4) for the variance of  $M_t(H)$ , and using (4.2) to compute  $\lim_{N \rightarrow \infty} \sum_{k \in \mathbb{Z}_N} \langle u_N, j_{k,k+1} \rangle$ , one gets

$$\sigma^2 D = \sigma^2 \bar{\chi} - \lim_{N \rightarrow \infty} \sum_{k \in \mathbb{Z}_N} \langle u_N, j_{k,k+1} \rangle + \mathcal{O}(\epsilon^n).$$

This shows our claim.  $\square$

## 5 Proof of Proposition 1

We first remind a general formula, that we will use several times. If  $u = u(x, t)$  is a regular enough function on  $\mathbf{X} \times \mathbb{R}_+$ , then the process

$$M_t = \int_0^t Lu(X_s, s) ds + \int_0^t \partial_t u(X_s, s) ds + u(X_0, 0) - u(X_t, t)$$

is a centered martingale, which variance at equilibrium is given by

$$\mathbb{E}_\beta(M_t)^2 = 2 \int_0^t \langle u(\cdot, s), (-L)u(\cdot, s) \rangle_\beta ds.$$

Let us then make some comments on the notations. We will write  $\langle \cdot \rangle$  for  $\langle \cdot \rangle_\beta$ . For  $k \in \mathbb{Z}_N$ , we define the centered variables

$$\eta_k = e_k - \langle e \rangle.$$

We will write gradient functions as  $D \cdot \nabla \eta$  instead of  $D \cdot \nabla e$ . We will not write explicitly the dependence in  $N$  of the functions  $u_N$ ,  $g_N$  and  $(D \cdot \nabla \eta)_N$  appearing in the hypotheses of Proposition 1. We will use the same symbol  $\partial_x f$  for both the true derivative of  $f$  and its approximative derivative  $N(f(x+1/N) - f(x))$ .

*Proof of Proposition 1.* Let  $H = H(x)$  be the smooth test function, and let  $G = G(x, s)$  be defined for  $0 \leq s \leq N^2 t$ , by

$$N^2 \partial_t G + \epsilon D \cdot \partial_x^2 G = 0, \quad G(\cdot, N^2 t) = H(\cdot). \quad (5.1)$$

Since  $L\eta_k = \epsilon(j_{k-1,k} - j_{k,k+1})$  for  $k \in \mathbb{Z}_N$ , one has

$$\int_0^{N^2 t} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} G(k/N, s) L\eta_k ds = \int_0^{N^2 t} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} G(k/N, s) \epsilon(j_{k-1,k} - j_{k,k+1}) ds$$

An integration by parts leads thus to

$$\begin{aligned} Y_{N^2 t}(G(\cdot, N^2 t)) - Y_0(G(\cdot, 0)) &= -M_{N^2 t}^{(0)} + \int_0^{N^2 t} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_t G(k/N, s) \eta_k ds \\ &\quad + \frac{1}{N} \int_0^{N^2 t} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_x G(k/N, s) \epsilon j_{k,k+1} ds, \end{aligned}$$

where  $M_{N^2 t}^{(0)}$  is a martingale. Using then the approximate fluctuation-dissipation decomposition of the current stated in the hypotheses of Proposition 1, one gets

$$Y_{N^2 t}(G(\cdot, N^2 t)) - Y_0(G(\cdot, 0)) = -M_{N^2 t}^{(0)} + \int_0^{N^2 t} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_t G(k/N, s) \eta_k ds \quad (5.2)$$

$$- \frac{1}{N} \int_0^{N^2 t} \frac{L}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_x G(k/N, s) \tau_k u ds \quad (5.3)$$

$$+ \frac{\epsilon}{N} \int_0^{N^2 t} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_x G(k/N, s) \tau_k D \cdot \nabla \eta ds \quad (5.4)$$

$$- \frac{\epsilon'}{N} \int_0^{N^2 t} \frac{L''}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_x G(k/N, s) \tau_k g ds. \quad (5.5)$$

We now proceed to the analysis of the last three terms in the right hand side of this equation. Let us start with (5.3). Since  $\partial_t \partial_x H(k/N, s) = -(\epsilon D/N^2) \partial_x^3 H(k/N, s)$ , there exists a martingale  $M_{N^2 t}^{(1)}$  such

that

$$\begin{aligned}
(5.3) &= -M_{N^2t}^{(1)} - \frac{\epsilon D}{N^3} \int_0^{N^2t} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_x^3 G(k/N, s) \tau_k u \, ds \\
&\quad + \frac{1}{N} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_x G(k/N, N^2t) (\tau_k u(X_0) - \tau_k u(X_{N^2t})) \\
&=: -M_{N^2t}^{(1)} + \rho_{N^2t}^{(0)}
\end{aligned}$$

The hypothesis that  $\sum_{k \in \mathbb{Z}_N} |\langle \tau_k u, u \rangle_\beta|$  remains bounded as  $N \rightarrow \infty$ , ensures that  $\mathbb{E}(\rho_{N^2t}^{(0)})^2 \rightarrow 0$  as  $N \rightarrow \infty$ . Let us then look at (5.4). Reminding that  $D = \lim_{N \rightarrow \infty} \sum_{k \in \mathbb{Z}_N} D_k$ , an integration by parts yields

$$\begin{aligned}
(5.4) &= \frac{\epsilon D}{N^2} \int_0^{N^2t} Y_s(\partial_x^2 G(\cdot, s)) \, ds \\
&\quad + \sum_j \frac{\epsilon D_j}{N^2} \int_0^{N^2t} \frac{1}{\sqrt{N}} \sum_k (\partial_x^2 G(k/N, s) - \partial_x^2 G((k+j+1)/N, s)) \eta_{k+j+1} \, ds \\
&=: \frac{\epsilon D}{N^2} \int_0^{N^2t} Y_s(\partial_x^2 H(\cdot, s)) \, ds + \rho_{N^2t}^{(1)}.
\end{aligned}$$

Let us show that  $\mathbb{E}(\rho_{N^2t}^{(1)})^2 \rightarrow 0$  as  $N \rightarrow \infty$ . First, by Jensen's inequality,

$$\begin{aligned}
\mathbb{E}(\rho_{N^2t}^{(1)})^2 &\leq \frac{t^2}{N^2t} \int_0^{N^2t} \mathbb{E} \left( \sum_j \epsilon D_j \frac{1}{\sqrt{N}} \sum_k (\partial_x^2 G(k/N, s) - \partial_x^2 G((k+j+1)/N, s)) \eta_{k+j+1}(s) \right)^2 \, ds \\
&=: \frac{t^2}{N^2t} \int_0^{N^2t} \mathbb{E} \left( \sum_j A_j(s) \right)^2 \, ds.
\end{aligned}$$

We split the sum over  $j$  as

$$\sum_j A_j(s) = \sum_{j: |j| \leq \sqrt{N}} A_j(s) + \sum_{j: |j| > \sqrt{N}} A_j(s),$$

and we separately show that the variance of each of these terms, at a fixed time  $s \in [0, N^2t]$ , converges to 0 uniformly in  $s$ , which will establish that  $\mathbb{E}(\rho_{N^2t}^{(1)})^2 \rightarrow 0$ . First, since by hypothesis  $\sum_{j \in \mathbb{Z}_N} |D_j|$  remains bounded as  $N \rightarrow \infty$ , it holds that

$$\begin{aligned}
\mathbb{E} \left( \sum_{j: |j| \leq \sqrt{N}} A_j(s) \right)^2 &\leq C \sum_{j: |j| \leq \sqrt{N}} |D_j| \mathbb{E} \left( \frac{\epsilon}{\sqrt{N}} \sum_k (\partial_x^2 G(k/N, s) - \partial_x^2 G((k+j+1)/N, s)) \eta_{k+j+1}(s) \right)^2 \\
&\leq C^2 \max_{j: |j| \leq \sqrt{N}} \mathbb{E} \left( \frac{\epsilon}{\sqrt{N}} \sum_k (\partial_x^2 G(k/N, s) - \partial_x^2 G((k+j+1)/N, s)) \eta_{k+j+1}(s) \right)^2 \\
&= C^2 \epsilon^2 \max_{j: |j| \leq \sqrt{N}} \frac{1}{N} \sum_k (\partial_x^2 G(k/N, s) - \partial_x^2 G((k+j+1)/N, s))^2 \tag{5.6}
\end{aligned}$$

for some constant  $C < +\infty$ . Here, to get the last line, we have used the fact that  $\langle \eta_m \eta_n \rangle_\beta = \delta_{m,n}$ . But for  $j$  satisfying  $|j| \leq \sqrt{N}$ , it holds that

$$|\partial_x^2 G(k/N, s) - \partial_x^2 G((k+j+1)/N, s)| = \mathcal{O}(|j+1|/N) = \mathcal{O}(1/\sqrt{N}),$$

so that (5.6) is seen to converge to zero, uniformly in  $s$ , as  $N \rightarrow \infty$ . Next,

$$\begin{aligned} \mathbb{E}\left(\sum_{j:|j|>\sqrt{N}} A_j(s)\right)^2 &\leq \left(\sum_{j:|j|\geq\sqrt{N}} D_j\right)^2 \mathbb{E}\left(\frac{\epsilon}{\sqrt{N}} \sum_k (\partial_x^2 G(k/N, s) - \partial_x^2 G((k+j+1)/N, s)) \eta_{k+j+1}(s)\right)^2 \\ &= \left(\sum_{j:|j|\geq\sqrt{N}} D_j\right)^2 \frac{\epsilon^2}{N} \sum_k (\partial_x^2 G(k/N, s) - \partial_x^2 G((k+j+1)/N, s))^2. \end{aligned}$$

We remind that, in this expression,  $D_j$  is written for  $(D_j)_N$ . Since, by hypothesis, both  $(D_j)_N$  and  $\sum_{l \in \mathbb{Z}_N} |D_l|$  converge as  $N \rightarrow \infty$ , for any given  $j \in \mathbb{Z}_N$ , we conclude that the first factor in the right hand side of this inequality converges to 0 as  $N \rightarrow \infty$ . Since the second factor stays bounded, uniformly in  $s$ , as  $N \rightarrow \infty$ , we deduce that the left hand side converges to 0, uniformly in  $s$ , as  $N \rightarrow \infty$ . We have thus shown that  $\mathbb{E}(\rho_{N^2t}^{(1)})^2 \rightarrow 0$  as  $N \rightarrow \infty$ .

We finally handle the less usual term (5.5), which is a centered process. It is convenient to define

$$F_s = \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_x G(k/N, s) \tau_k g, \quad \tilde{F}_s = \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_x^3 G(k/N, s) \tau_k g$$

For  $z > 0$  one then defines  $V_{z,s}$  and  $\tilde{V}_{z,s}$  as respective unique solutions to the equations

$$(z - L)V_{z,s} = -L''F_s, \quad (z - L)\tilde{V}_{z,s} = -L''\tilde{F}_s. \quad (5.7)$$

So, defining a centered martingale  $M_{N^2t}^{(2)}$ , one has

$$\begin{aligned} (5.5) &= \epsilon' \frac{1}{N} \int_0^{N^2t} (-L'')G_s(X_s) ds \\ &= \frac{\epsilon' z}{N} \int_0^{N^2t} V_{z,s}(X_s) ds - \frac{\epsilon'}{N} \int_0^{N^2t} LV_{z,s}(X_s) ds \\ &= \frac{\epsilon' z}{N} \int_0^{N^2t} V_{z,s}(X_s) ds + \frac{\epsilon'}{N} \int_0^{N^2t} \partial_t V_{z,s}(X_s) ds - \frac{\epsilon'}{N} M_{N^2t}^{(2)} + \frac{\epsilon'}{N} (V_{z,0}(X_0) - V_{z,N^2t}(X_{N^2t})) \end{aligned}$$

Therefore, taking  $z = 1/N^2$ , one finds a constant  $C < +\infty$  such that

$$\begin{aligned} \mathbb{E}_\beta((5.5)^2) &\leq C(\epsilon')^2 \left( t^2 \cdot \frac{1}{N^2t} \int_0^{N^2t} z \langle V_{z,s}^2 \rangle ds + t^2 \cdot \frac{1}{N^2t} \int_0^{N^2t} z \langle \tilde{V}_{z,s}^2 \rangle ds \right. \\ &\quad \left. + t \cdot \frac{1}{N^2t} \int_0^{N^2t} \langle V_{z,s}, (-L)V_{z,s} \rangle ds + z \langle V_{z,0}^2 \rangle + z \langle V_{z,N^2t}^2 \rangle \right). \end{aligned}$$

But one has the bounds

$$z \langle V_{z,s}^2 \rangle \leq \epsilon^{-1} \langle F_s, (-L'')F_s \rangle, \quad \langle V_{z,s}, (-L)V_{z,s} \rangle \leq \epsilon^{-1} \langle F_s, (-L'')F_s \rangle, \quad (5.8)$$

as well as the same bounds with the tilted quantities. Indeed, (5.7) implies

$$\begin{aligned} z \langle V_{z,s}^2 \rangle + \langle V_{z,s}, (-L)V_{z,s} \rangle &= \langle V_{z,s}, (-L'')F_s \rangle \leq \langle F_s, (-L'')F_s \rangle^{1/2} \langle V_{z,s}, (-L'')V_{z,s} \rangle^{1/2} \\ &\leq \epsilon^{-1/2} \langle F_s, (-L'')F_s \rangle^{1/2} \langle V_{z,s}, (-L'')V_{z,s} \rangle^{1/2} \\ &\leq \epsilon^{-1/2} \langle F_s, (-L'')F_s \rangle^{1/2} \langle V_{z,s}, (-L)V_{z,s} \rangle^{1/2}, \end{aligned}$$

from where (5.8) is derived. So finally we obtain

$$\mathbf{E}_\beta((5.5)^2) \leq C(\epsilon')^2 \epsilon^{-1}(1+t^2) \lim_{N \rightarrow \infty} \sum_{k \in \mathbb{Z}_N} |\langle g, L'' \tau_k g \rangle_\beta|. \quad (5.9)$$

So at this point, let us define

$$R_{N^{2t}} = (5.5) \quad \text{and} \quad M_{N^{2t}} = -M_{N^{2t}}^{(0)} - M_{N^{2t}}^{(1)}.$$

Taking into account that  $G$  satisfies (5.1), one has so far obtained

$$Y_{N^{2t}}(H) = Y_0(G(\cdot, 0)) + M_{N^{2t}} + R_{N^{2t}} + \rho_{N^{2t}}^{(0)} + \rho_{N^{2t}}^{(1)}.$$

It holds that  $G(\cdot, 0) = H_t(\cdot)$ , that  $\mathbf{E}_\beta(R_{N^{2t}}^2)$  is bounded by (5.9) and that both  $\rho_{N^{2t}}^{(0)}$  and  $\rho_{N^{2t}}^{(1)}$  converge to 0 in  $L^2(\mathbf{P}_\beta)$ . Therefore, if we establish that

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbf{E}_\beta(M_{N^{2t}}^2) = \\ 2\epsilon \lim_{N \rightarrow \infty} \frac{1}{N^2} \int_0^{N^{2t}} \left( \int (\partial_x G)^2(x, s) dx \right) ds \cdot \left( \bar{\chi} \sigma^2 - \sum_{k \in \mathbb{Z}_N} \langle u, j_{k, k+1} \rangle + \frac{\epsilon'}{\epsilon} \sum_{k \in \mathbb{Z}_N} \langle u, \tau_k L'' g \rangle \right), \end{aligned} \quad (5.10)$$

then we will be able to conclude that  $M_{N^{2t}}$  converges as a process to a centered Gaussian martingale, independent of  $Y_0(H_t)$ , with the variance given by (3.4) (see [14] or also [10]). It thus remains to show (5.10) to conclude the proof.

Writing  $Y_s$  for  $Y_s(H(\cdot, s))$  and putting

$$U_s = \frac{1}{N} \frac{1}{\sqrt{N}} \sum_{k \in \mathbb{Z}_N} \partial_x H(k/N, s) \tau_k u,$$

the variance of  $M_{N^{2t}}$  is given by

$$\begin{aligned} \mathbf{E}(M_{N^{2t}})^2 &= 2 \int_0^{N^{2t}} \langle Y_s + U_s, (-L)(Y_s + U_s) \rangle ds \\ &= 2\epsilon \int_0^{N^{2t}} \langle Y_s, (-L'')Y_s \rangle ds + 4\epsilon \int_0^{N^{2t}} \langle U_s, (-L'')Y_s \rangle ds + 2 \int_0^{N^{2t}} \langle U_s, (-L)U_s \rangle ds. \end{aligned}$$

One first readily computes that, since  $\sum_{k \in \mathbb{Z}_N} |\langle u, j_{k, k+1} \rangle|$  remains bounded as  $N \rightarrow \infty$ ,

$$\begin{aligned} 2\epsilon \int_0^{N^{2t}} \langle Y_s, (-L'')Y_s \rangle ds &\rightarrow 2\epsilon \bar{\chi} \sigma^2 \cdot \lim_{N \rightarrow \infty} \frac{1}{N^2} \int_0^{N^{2t}} \left( \int (\partial_x G)^2(x, s) dx \right) ds, \\ 4\epsilon \int_0^{N^{2t}} \langle U_s, (-L'')Y_s \rangle ds &\rightarrow -4\epsilon \lim_{N \rightarrow \infty} \frac{1}{N^2} \int_0^{N^{2t}} \left( \int (\partial_x G)^2(x, s) dx \right) ds \cdot \sum_{k \in \mathbb{Z}_N} \langle u, j_{k, k+1} \rangle. \end{aligned}$$

Then, one uses the approximate fluctuation-dissipation decomposition of the current stated as hypothesis in Proposition 1 to obtain

$$-LU_s = \frac{\epsilon}{N^{3/2}} \sum_{k \in \mathbb{Z}_N} \partial_x G(k/N, s) j_{k, k+1} - \frac{\epsilon}{N^{3/2}} \sum_{k \in \mathbb{Z}_N} \partial_x G(k/N, s) \tau_k D \cdot \nabla \eta + \frac{\epsilon'}{N^{3/2}} \sum_{k \in \mathbb{Z}_N} \partial_x G(k/N, s) \tau_k L'' g.$$

The second term in the right hand side of this equation will not contribute since it is a gradient. Taking into account that both  $\sum_{k \in \mathbb{Z}_N} |\langle u, j_{k,k+1} \rangle|$  and  $\sum_{k \in \mathbb{Z}_N} |\langle u, \tau_k L'' g \rangle|$  remain bounded as  $N \rightarrow \infty$ , a computation furnishes

$$2 \int_0^{N^2 t} \langle U_s, (-L)U_s \rangle ds \rightarrow 2 \lim_{N \rightarrow \infty} \frac{1}{N^2} \int_0^{N^2 t} \left( \int (\partial_x G)^2(x, s) dx \right) ds \cdot \left( \epsilon \sum_{k \in \mathbb{Z}_N} \langle u, j_{k,k+1} \rangle + \epsilon' \sum_{k \in \mathbb{Z}_N} \langle u, \tau_k L'' g \rangle \right).$$

So (5.10) is shown.  $\square$

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