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Mathematical renormalization in quantum electrodynamics via noncommutative generating series

G. H. E. Duchamp - Hoang Ngoc Minh - Q. H. Ngo - K. Penson - P. Simonnet

Abstract In order to push the study of solutions of nonlinear differential equations involved in quantum electrodynamics¹, we focus here on combinatorial aspects of their renormalization at $\{0, 1, +\infty\}$.

1 Introduction

During the last century, the functional expansions were common in physics as well as in engineering and have been developed, by Tomonaga, Schwinger and Feynman [18], to represent the dynamical systems in quantum electrodynamics. The main difficulty of this approach is the divergence of these expansions at the singularity 0 or at $+\infty$ (see [2]) and leads to the problems of *regularization* and *renormalization*

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¹ The present work is part of a series of papers devoted to the study of the renormalization of divergent polyzetas (at positive and at negative indices) via the factorization of the non commutative generating series of polylogarithms and of harmonic sums and via the effective construction of pairs of bases in duality in φ -deformed shuffle algebras. It is a sequel of [6] and its content was presented in several seminars and meetings, including the 66th and 74th Séminaire Lotharingien de Combinatoire.

which can be solved by combinatorial technics : Feynman diagrams [21] and their siblings [15, 26], noncommutative formal power series [23], trees [11].

Recently, in the same vein, and based, on the one hand, on the shuffle and quasi-shuffle bialgebras [6], the combinatorics of noncommutative formal power series was intensively amplified for the asymptotic analysis of dynamical systems with three regular singularities in $\{0, 1, +\infty\}$; and, on the other hand with the monodromy and the Galois differential group of the Knizhnik-Zamolodchikov equation KZ_3 [42, 44] *i.e.*, the following noncommutative evolution equation³

$$\frac{dG(z)}{dz} = \left(\frac{x_0}{z} + \frac{x_1}{1-z} \right) G(z),$$

the monoidal factorization facilitates mainly the renormalization and the computation of the associators⁴ via the universal one, *i.e.* Φ_{KZ} of Drinfel'd [44].

In fact, these associators are noncommutative formal power series on two variables and regularize the Chen generating series of the differential forms admitting singularities at 0 or at 1 along the integration paths on the universal covering of \mathbb{C} without points 0 and 1 (*i.e.* $\mathbb{C} \setminus \widetilde{\{0, 1\}}$). Their coefficients are, up to a multiple of powers of $2i\pi$, polynomial on polyzetas, *i.e.* the following real numbers⁵ [26, 46, 55]

$$\zeta(s_1, \dots, s_r) = \sum_{n_1 > \dots > n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}, \text{ for } r \geq 1, s_1 \geq 2, s_2, \dots, s_r \geq 1,$$

and these numbers admit a natural structure of algebra over the rational numbers deduced from the combinatorial aspects of the shuffle and quasi-shuffle Hopf algebras. It is conjectured that this algebra is \mathbb{N} -graded⁶. More precisely, for $s_1 \geq 2, s_2, \dots, s_r \geq 1$, the polyzeta $\zeta(s_1, \dots, s_r)$ can be obtained as the limit of the polylogarithm $\text{Li}_{s_1, \dots, s_r}(z)$, for $z \rightarrow 1$, and of the harmonic sum $H_{s_1, \dots, s_r}(N)$, for $N \rightarrow +\infty$:

$$\text{Li}_{s_1, \dots, s_r}(z) = \sum_{n_1 > \dots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \dots n_r^{s_r}} \text{ and } H_{s_1, \dots, s_r}(N) = \sum_{n_1 > \dots > n_r > 0}^N \frac{1}{n_1^{s_1} \dots n_r^{s_r}}.$$

Then, by a theorem of Abel, one has

$$\zeta(s_1, \dots, s_r) = \lim_{z \rightarrow 1} \text{Li}_{s_1, \dots, s_r}(z) = \lim_{N \rightarrow +\infty} H_{s_1, \dots, s_r}(N).$$

² Any differential equation with singularities in $\{a, b, c\}$ can be changed into a differential equation with singularities in $\{0, 1, +\infty\}$ via an homographic transformation.

³ $x_0 := t_{1,2}/2i\pi$ and $x_1 := -t_{2,3}/2i\pi$ are noncommutative variables and $t_{1,2}, t_{2,3}$ belong to $\mathcal{F}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ satisfying the infinitesimal 3-braid relations, *i.e.* $[t_{1,3}, t_{1,2} + t_{2,3}] = [t_{2,3}, t_{1,2} + t_{1,3}] = 0$.

⁴ They were introduced in QFT by Drinfel'd and it plays an important role for the still open problem of the effective determination of the polynomial invariants of knots and links via Kontsevich's integral (see [7, 26]) and Φ_{KZ} , was obtained firstly, in [26], with explicit coefficients which are polyzetas and regularized polyzetas (see [44, 45] for the computation of the other involving *only* convergent polyzetas as local coordinates, and for algorithms regularizing divergent polyzetas).

⁵ $s_1 + \dots + s_r$ is the *weight* of $\zeta(s_1, \dots, s_r)$, *i.e.* the weight of the composition (s_1, \dots, s_r) .

⁶ One of us proposed a proof in [44, 45].

Since the algebras of polylogarithms and of harmonic sums are isomorphic to the shuffle algebra $(\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$ and quasi-shuffle algebra $(\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*})$ respectively both admitting the Lyndon words $\mathcal{L}ynX$ over $X = \{x_0, x_1\}$ and $\mathcal{L}ynY$ over $Y = \{y_i\}_{i \geq 1}$, as transcendence bases (recalled in Section 2.1) then, by using

- The (one-to-one) correspondence between the combinatorial compositions, the words⁷ in Y^* and the words in $X^*x_1 + 1_{X^*}$, *i.e.*⁸

$$(\{1\}^k, s_{k+1}, \dots, s_r) \leftrightarrow y_1^k y_{s_{k+1}} \dots y_{s_r} \stackrel{\pi_X}{\longleftarrow} x_1^k x_0^{s_{k+1}-1} x_1 \dots x_0^{s_r-1} x_1. \quad (1)$$

- The ordering $x_1 \succ x_0$ and $y_1 \succ y_2 \succ \dots$ over X and Y respectively.
- The transcendence base $\{S_l\}_{l \in \mathcal{L}ynX}$ (resp. $\{\Sigma_l\}_{l \in \mathcal{L}ynY}$) of $(\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$ (resp. $(\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*})$) in duality⁹ with $\{P_l\}_{l \in \mathcal{L}ynX}$ (resp. $\{\Pi_l\}_{l \in \mathcal{L}ynY}$), a base of the Lie algebra of primitive elements of the bialgebra¹⁰ $\mathcal{H}_{\sqcup} = (\mathbb{Q}\langle X \rangle, \text{conc}, 1_{X^*}, \Delta_{\sqcup}, \varepsilon)$ (resp. $\mathcal{H}_{\sqcup} = (\mathbb{Q}\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\sqcup}, \varepsilon)$) to factorize the following noncommutative generating series of polylogarithms, hormanic sums and polyzetas

$$\begin{aligned} L &= \prod_{l \in \mathcal{L}ynX} \exp(\text{Li}_{S_l} P_l) \text{ and } H = \prod_{l \in \mathcal{L}ynY} \exp(H_{\Sigma_l} \Pi_l), \\ Z_{\sqcup} &= \prod_{l \in \mathcal{L}ynX, l \neq x_0, x_1} \exp(\zeta(S_l) P_l) \text{ and } Z_{\sqcup} = \prod_{l \in \mathcal{L}ynY, l \neq y_1} \exp(\zeta(\Sigma_l) \Pi_l), \end{aligned}$$

we then obtain two formal power series over Y , Z_1 and Z_2 , such that

$$\lim_{z \rightarrow 1} \exp \left[y_1 \log \frac{1}{1-z} \right] \pi_Y L(z) = Z_1, \quad \lim_{N \rightarrow \infty} \exp \left[\sum_{k \geq 1} H_{y_k}(N) \frac{(-y_1)^k}{k} \right] H(N) = Z_2.$$

Moreover, Z_1, Z_2 are equal and stand for the noncommutative generating series of $\{\zeta(w)\}_{w \in Y^* - y_1 Y^*}$, or $\{\zeta(w)\}_{w \in x_0 X^* x_1}$, as one has $Z_1 = Z_2 = \pi_Y Z_{\sqcup}$ [43, 44, 45]. This allows, by extracting the coefficients of the noncommutative generating series, to explicit the counter-terms eliminating the divergence of $\{\text{Li}_w\}_{w \in x_1 X^*}$ and of $\{H_w\}_{w \in y_1 Y^*}$ and this leads naturally to an equation connecting algebraic structures

$$\prod_{l \in \mathcal{L}ynY, l \neq y_1} \exp(\zeta(\Sigma_l) \Pi_l) = \exp \left[\sum_{k \geq 2} -\zeta(k) \frac{(-y_1)^k}{k} \right] \pi_Y \prod_{l \in \mathcal{L}ynX, l \neq x_0, x_1} \exp(\zeta(S_l) P_l) \quad (2)$$

Identity (2) allows to compute the Euler-MacLaurin constants and the Hadamard finite parts associated to divergent polyzetas $\{\zeta(w)\}_{w \in y_1 Y^*}$ and, by identifying local coordinates, to describe the graded core of $\ker \zeta$ by its *algebraic* generators.

⁷ Here, X^* (resp. Y^*) is the monoid generated by X (resp. Y) and its neutral element of is denoted by 1_{X^*} (resp. 1_{Y^*}).

⁸ Here, π_Y is the adjoint of π_X for the canonical scalar products where π_X is the morphism of AAU $k\langle Y \rangle \rightarrow k\langle X \rangle$ defined by $\pi_X(y_k) = x_0^{k-1} x_1$.

⁹ in a more precise way the S and Σ are the ‘‘Lyndon part’’ of the dual bases of the PBW expansions of the P and the Π respectively.

¹⁰ ε is the ‘‘constant term’’ character.

In this paper, we will focus on the approach by noncommutative formal power series, adapted from [22, 23], and explain how the results of [43, 44, 45], allow to study the combinatorial aspects of the renormalization at the singularities in $\{0, 1, +\infty\}$ of the solutions of linear differential equations (see Example 1 below) as well as the solutions of nonlinear differential equations (see Examples 2 and 3 below) described in Section 3.2 and involved in quantum electrodynamics.

Example 1 (Hypergeometric equation). Let t_0, t_1, t_2 be parameters and

$$z(1-z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.$$

Let $q_1(z) = -y(z)$ and $q_2(z) = (1-z)\dot{y}(z)$. One has

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \left(\frac{M_0}{z} + \frac{M_1}{1-z} \right) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

where M_0 and M_1 are the following matrices

$$M_0 = - \begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} \text{ and } M_1 = - \begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}.$$

Or equivalently,

$$\dot{q}(z) = A_0(q) \frac{1}{z} + A_1(q) \frac{1}{1-z} \text{ and } y(z) = -q_1(z),$$

where A_0 and A_1 are the following parametrized linear vector fields

$$A_0 = -(t_0 t_1 q_1 + t_2 q_2) \frac{\partial}{\partial q_2} \text{ and } A_1 = -q_1 \frac{\partial}{\partial q_1} - (t_2 - t_0 - t_1) q_2 \frac{\partial}{\partial q_2}.$$

acting by

$$\frac{\partial}{\partial q_1}(q) = \frac{\partial}{\partial q_1} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \frac{\partial}{\partial q_2}(q) = \frac{\partial}{\partial q_2} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Example 2 (Harmonic oscillator). Let k_1, k_2 be parameters and

$$\ddot{y}(z) + k_1 \dot{y}(z) + k_2 y^2(z) = u_1(z).$$

which can be represented, with the same formalism as above, by the following state equations

$$\dot{q}(z) = A_0(q) + A_1(q) u_1(z) \text{ and } y(z) = q(z),$$

where A_0 and A_1 are the following vector fields

$$A_0 = -(k_1 q + k_2 q^2) \frac{\partial}{\partial q} \text{ and } A_1 = \frac{\partial}{\partial q}.$$

Example 3 (Duffing's equation). Let a, b, c be parameters and

$$\ddot{y}(z) + a\dot{y}(z) + by(z) + cy^3(z) = u_1(z).$$

which can be represented by the following state equations

$$\dot{q}(z) = A_0(q) + A_1(q) u_1(z), \text{ and } y(z) = q_1(z),$$

where A_0 and A_1 are the following vector fields

$$A_0 = -(aq_2 + b^2q_1 + cq_1^3) \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_1} \text{ and } A_1 = \frac{\partial}{\partial q_2}.$$

Example 4 (Van der Pol oscillator). Let γ, g be parameters and

$$\partial_z^2 x(z) - \gamma[1 + x(z)^2] \partial_z x(z) + x(z) = g \cos(\omega z)$$

which can be transformed into (with C is some constant of integration)

$$\partial_z x(z) = \gamma[1 + x(z)^2/3]x(z) - \int_{z_0}^z x(s)ds + \frac{g}{\omega} \sin(\omega z) + C.$$

Setting $y = \int_{z_0}^z x(s)ds$ and $u_1(z) = g \sin(\omega z)/\omega + C$, it leads then to

$$\partial_z^2 y(z) = \gamma[\partial_z y(z) + (\partial_z y(z))^3/3] - y(z) + u_1(z)$$

which can be represented by the following state equations (with $n = 2$)

$$\partial_z q(z) = [A_0 u_0(z) + A_1 u_1(z)](q) \text{ and } y(z) = q_1(z)$$

where A_0 and A_1 are the following vector fields

$$A_1 = \frac{\partial}{\partial q_2} \text{ and } A_0 = [\gamma(q_2 + q_2^3/3) - q_1] \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_1}.$$

This approach by noncommutative formal power series is adequate for studying the algebraic combinatorial aspects of the asymptotic analysis at the singularities in $\{0, 1, +\infty\}$, of the nonlinear dynamical systems described in Section 3.2 because

- The polylogarithms form a basis of an infinite dimensional universal Picard-Vessiot extension by means of these differential equations [42, 12] and their algebra, isomorphic to the shuffle algebra, admits $\{L_i\}_{i \in \mathcal{L}_{ynX}}$ as a transcendence basis,
- The harmonic sums generate the coefficients of the ordinary Taylor expansions of their solutions (when these expansions exist) [43] and their algebra is isomorphic to the quasi-shuffle algebra admitting $\{H_i\}_{i \in \mathcal{L}_{ynY}}$ as a transcendence basis,
- The polyzetas do appear as the fundamental arithmetical constants involved in the computations of the monodromies [39, 36], the Kummer type functional equations [40, 36], the asymptotic expansions of solutions [43, 44] and their algebra is freely generated by the polyzetas encoded by irreducible Lyndon words [44].

Hence, a lot of algorithms can be deduced from these facts and more general studies will be proceeded in [6, 12]. The organisation of this paper is the following

- In Section 2, we will give algebraic and analytic foundations, *i.e.* the combinatorial Hopf algebra of shuffles and the indiscernability respectively, for polyzetas.
- These will be exploited, in Section 3, to expand solutions, of nonlinear dynamical systems with singular inputs and their ordinary and *functional* differentiations.

2 Fundation of the present framework

2.1 Background about combinatorics of shuffle and stuffle bialgebras

2.1.1 Schützenberger's monoidal factorization

Let $\mathbb{Q}\langle X \rangle$ be equipped by the concatenation and the shuffle defined by

$$\begin{aligned} \forall w \in X^*, \quad w \sqcup 1_{X^*} = 1_{X^*} \sqcup w = w, \\ \forall x, y \in X, \forall u, v \in X^*, \quad xu \sqcup yv = x(u \sqcup yv) + y(xu \sqcup v), \end{aligned}$$

or by their dual co-products, Δ_{conc} and Δ_{\sqcup} , defined by, for $w \in X^*$ and $x \in X$,

$$\Delta_{\text{conc}}(w) = \sum_{u, v \in X^*, uv=w} u \otimes v \quad \text{and} \quad \Delta_{\sqcup}(x) = x \otimes 1 + 1 \otimes x,$$

Δ_{\sqcup} is then extended to a conc-morphism $\mathbb{Q}\langle X \rangle \rightarrow \mathbb{Q}\langle X \rangle \otimes \mathbb{Q}\langle X \rangle$. These two comultiplications satisfy, for any $u, v, w \in X^*$,

$$\langle \Delta_{\text{conc}}(w) \mid u \otimes v \rangle = \langle w \mid uv \rangle \quad \text{and} \quad \langle \Delta_{\sqcup}(w) \mid u \otimes v \rangle = \langle w \mid u \sqcup v \rangle.$$

One gets two mutually dual bialgebras

$$\mathcal{H}_{\sqcup} = (\mathbb{Q}\langle X \rangle, \text{conc}, 1_{X^*}, \Delta_{\sqcup}, \varepsilon), \quad \mathcal{H}_{\sqcup}^{\vee} = (\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*}, \Delta_{\text{conc}}, \varepsilon).$$

After a theorem by Radford [49], $\mathcal{L}ynX$ forms a transcendence basis of $(\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$ and it can be completed then to the linear basis $\{w\}_{w \in X^*}$ which is auto-dual :

$$\forall u, v \in X^*, \quad \langle u \mid v \rangle = \delta_{u,v}. \quad (3)$$

But the elements $l \in \mathcal{L}ynX - X$ are not primitive, for Δ_{\sqcup} , and then $\mathcal{L}ynX$ does not constitute a basis for $\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$. Chen, Fox and Lyndon [9] constructed $\{P_w\}_{w \in X^*}$, so-called the Poincaré-Birkhoff-Witt-Lyndon basis, for $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X \rangle)$ as follows

$$P_x = x \quad \text{for } x \in X, \quad (4)$$

$$P_l = [P_s, P_r] \quad \text{for } l \in \mathcal{L}ynX, \text{ standard factorization of } l = (s, r), \quad (5)$$

$$P_w = P_{l_1}^{i_1} \dots P_{l_k}^{i_k} \quad \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 \succ \dots \succ l_k, l_1, \dots, l_k \in \mathcal{L}ynX. \quad (6)$$

where here \succ stands for the lexicographic (strict) ordering defined¹¹ by $x_0 \prec x_1$. Schützenberger constructed bases for $(\mathbb{Q}\langle X \rangle, \sqcup)$ defined by duality as follows :

$$\forall u, v \in X^*, \quad \langle S_u \mid P_v \rangle = \delta_{u,v}$$

¹¹ In here, the order relation \succ on X^* is defined by, for any $u, v \in X^*$, $u \succ v$ iff $u = vw$ with $w \in X^+$ else there are $w, w_1, w_2 \in X^*$ and $a \succ b \in X$ such that $u = waw_1$ and $v = bbw_2$.

and obtained the transcendence and linear bases, $\{S_l\}_{l \in \mathcal{L}ynX}$, $\{S_w\}_{w \in X^*}$, as follows

$$S_l = xS_u, \quad \text{for } l = xu \in \mathcal{L}ynX,$$

$$S_w = \frac{S_{l_1}^{i_1} \sqcup \dots \sqcup S_{l_k}^{i_k}}{i_1! \dots i_k!} \quad \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 \succ \dots \succ l_k.$$

After that, Mélançon and Reutenauer [51] proved that, for any $w \in X^*$,

$$P_w = w + \sum_{v \succ w, |v|_X = |w|_X} c_v v \quad \text{and} \quad S_w = w + \sum_{v \prec w, |v|_X = |w|_X} d_v v. \quad (7)$$

where $|w|_X = (|w|_x)_{x \in X}$ is the family of all partial degrees (number of times a letter occurs in a word). In other words, the elements of the bases $\{S_w\}_{w \in X^*}$ and $\{P_w\}_{w \in X^*}$ are lower and upper triangular respectively and they are of multihomogeneous (all the monomials have the same partial degrees).

Example 5 (of $\{P_w\}_{w \in X^}$ and $\{S_w\}_{w \in X^*}$, [31]).* Let $X = \{x_0, x_1\}$ with $x_0 \prec x_1$.

l	P_l	S_l
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0^2x_1$	$[x_0, [x_0, x_1]]$	$x_0^2x_1$
$x_0x_1^2$	$[[x_0, x_1], x_1]$	$x_0x_1^2$
$x_0^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3x_1$
$x_0^2x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2x_1^2$
$x_0x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0x_1^3$
$x_0^4x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4x_1$
$x_0^3x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3x_1^2$
$x_0^2x_1x_0x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3x_1^2 + x_0^2x_1x_0x_1$
$x_0^2x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2x_1^3$
$x_0x_1x_0x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2x_1^3 + x_0x_1x_0x_1^2$
$x_0x_1^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0x_1^4$
$x_0^5x_1$	$[x_0, [x_0, [x_0, [x_0, [x_0, x_1]]]]]$	$x_0^5x_1$
$x_0^4x_1^2$	$[x_0, [x_0, [x_0, [[x_0, x_1], x_1]]]]$	$x_0^4x_1^2$
$x_0^3x_1x_0x_1$	$[x_0, [[x_0, [x_0, x_1]], [x_0, x_1]]]$	$2x_0^4x_1^2 + x_0^3x_1x_0x_1$
$x_0^3x_1^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0^3x_1^3$
$x_0^2x_1x_0x_1^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0^3x_1^3 + x_0^2x_1x_0x_1^2$
$x_0^2x_1^2x_0x_1$	$[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$	$6x_0^3x_1^3 + 3x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1$
$x_0^2x_1^4$	$[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]]$	$x_0^2x_1^4$
$x_0x_1x_0x_1^3$	$[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]]$	$4x_0^2x_1^4 + x_0x_1x_0x_1^3$
$x_0x_1^5$	$[[[[[x_0, x_1], x_1], x_1], x_1], x_1]$	$x_0x_1^5$

Then, Schützenberger's factorization of the diagonal series \mathcal{D}_X follows [51]

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynX} \exp(S_l \otimes P_l). \quad (8)$$

2.1.2 Extended Schützenberger's monoidal factorization

Let us define the commutative product over $\mathbb{Q}\langle Y \rangle$, denoted by μ , as follows

$$\forall y_n, y_m \in Y, \mu(y_n, y_m) = y_{n+m},$$

or by its associated coproduct, Δ_μ , defined by

$$\forall y_n \in Y, \Delta_\mu(y_n) = \sum_{i=1}^{n-1} y_i \otimes y_{n-i}$$

satisfying,

$$\forall x, y, z \in Y, \langle \Delta_\mu | y \otimes z \rangle = \langle x | \mu(y, z) \rangle.$$

Let $\mathbb{Q}\langle Y \rangle$ be equipped by

1. The concatenation (or by its associated coproduct, Δ_{conc}).
2. The *shuffle* product, *i.e.* the commutative product defined by [23]

$$\begin{aligned} \forall w \in Y^*, w \sqcup 1_{Y^*} &= 1_{Y^*} \sqcup w = w, \\ \forall x, y \in Y, u, v \in Y^*, xu \sqcup yv &= x(u \sqcup yv) + y(xu \sqcup v) \end{aligned}$$

or with its associated coproduct, Δ_{\sqcup} , defined, on the letters, by

$$\forall y_k \in Y, \Delta_{\sqcup} y_k = y_k \otimes 1 + 1 \otimes y_k$$

and extended by morphism. It satisfies

$$\forall u, v, w \in Y^*, \langle \Delta_{\sqcup} w | u \otimes v \rangle = \langle w | u \sqcup v \rangle.$$

3. The *quasi-shuffle* product, *i.e.* the commutative product defined by [47]

$$\begin{aligned} \forall w \in Y^*, w \sqcup\!\!\!\sqcup 1_{Y^*} &= 1_{Y^*} \sqcup\!\!\!\sqcup w = w, \\ \forall x, y \in Y, u, v \in Y^*, y_i u \sqcup\!\!\!\sqcup y_j v &= y_j (y_i u \sqcup\!\!\!\sqcup v) + y_i (u \sqcup\!\!\!\sqcup y_j v) \\ &\quad + \mu(y_i, y_j)(u \sqcup\!\!\!\sqcup v) \end{aligned}$$

or with its associated coproduct, $\Delta_{\sqcup\!\!\!\sqcup}$, defined, on the letters, by

$$\forall y_k \in Y, \Delta_{\sqcup\!\!\!\sqcup} y_k = \Delta_{\sqcup} y_k + \Delta_\mu y_k$$

and extended by morphism. It satisfies

$$\forall u, v, w \in Y^*, \langle \Delta_{\sqcup\!\!\!\sqcup} w | u \otimes v \rangle = \langle w | u \sqcup\!\!\!\sqcup v \rangle.$$

Hence, with the counit \mathbf{e} defined by, for any $P \in \mathbb{Q}\langle Y \rangle$, $\mathbf{e}(P) = \langle P | 1_{Y^*} \rangle$, one gets two pairs of mutually dual bialgebras

$$\begin{aligned}\mathcal{H}_{\sqcup} &= (\mathbb{Q}\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\sqcup}, \mathbf{e}) \text{ and } \mathcal{H}_{\sqcup}^{\vee} = (\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*}, \Delta_{\text{conc}}, \mathbf{e}), \\ \mathcal{H}_{\boxplus} &= (\mathbb{Q}\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\boxplus}, \mathbf{e}) \text{ and } \mathcal{H}_{\boxplus}^{\vee} = (\mathbb{Q}\langle Y \rangle, \boxplus, 1_{Y^*}, \Delta_{\text{conc}}, \mathbf{e}).\end{aligned}$$

By the CQMM theorem (see [6]), the connected \mathbb{N} -graded, co-commutative Hopf algebra \mathcal{H}_{\sqcup} is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements which is equal to $\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle$:

$$\mathcal{H}_{\sqcup} \cong \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle) \text{ and } \mathcal{H}_{\sqcup}^{\vee} \cong \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle)^{\vee}.$$

Hence, let us consider [9]

1. The PBW-Lyndon basis $\{p_w\}_{w \in Y^*}$ for $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y \rangle)$ constructed recursively

$$\begin{cases} p_y = y & \text{for } y \in Y, \\ p_l = [p_s, p_r] & \text{for } l \in \mathcal{L}ynY, \text{ standard factorization of } l = (s, r), \\ p_w = p_{l_1}^{i_1} \dots p_{l_k}^{i_k} & \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 \succ \dots \succ l_k, l_1, \dots, l_k \in \mathcal{L}ynY, \end{cases}$$

2. And, by duality¹², the linear basis $\{s_w\}_{w \in Y^*}$ for $(\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*})$, i.e.

$$\forall u, v \in Y^*, \langle p_u | s_v \rangle = \delta_{u,v}.$$

This basis can be computed recursively as follows [51]

$$\begin{cases} s_y = y, & \text{for } y \in Y, \\ s_l = ys_u, & \text{for } l = yu \in \mathcal{L}ynY, \\ s_w = \frac{s_{l_1}^{i_1} \sqcup \dots \sqcup s_{l_k}^{i_k}}{i_1! \dots i_k!} & \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 \succ \dots \succ l_k \in \mathcal{L}ynY. \end{cases}$$

As in (8), one also has Schützenberger's factorization for the diagonal series \mathcal{D}_Y

$$\mathcal{D}_Y := \sum_{w \in Y^*} w \otimes w = \sum_{w \in Y^*} s_w \otimes p_w = \prod_{l \in \mathcal{L}ynY} \exp(s_l \otimes p_l).$$

Similarly, by the CQMM theorem, the connected \mathbb{N} -graded, co-commutative Hopf algebra \mathcal{H}_{\boxplus} is isomorphic to the enveloping algebra of

$$\text{Prim}(\mathcal{H}_{\boxplus}) = \text{Im}(\pi_1) = \text{span}_{\mathbb{Q}}\{\pi_1(w) | w \in Y^*\},$$

where, for any $w \in Y^*$, $\pi_1(w)$ is obtained as follows [6, 44]

$$\pi_1(w) = w + \sum_{k \geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in Y^+} \langle w | u_1 \boxplus \dots \boxplus u_k \rangle u_1 \dots u_k. \quad (9)$$

¹² The dual family of a basis lies in the algebraic dual which is here the space of noncommutative series, but as the enveloping algebra under consideration is graded in finite dimensions (here by the multidegree), these series are in fact (multihomogeneous) polynomials.

Note that Equation (9) is equivalent to the following identity [6, 44, 45]

$$w = \sum_{k \geq 0} \frac{1}{k!} \sum_{u_1, \dots, u_k \in Y^*} \langle w | u_1 \sqcup \dots \sqcup u_k \rangle \pi_1(u_1) \dots \pi_1(u_k). \quad (10)$$

In particular, for any $y_k \in Y$, we have successively [6, 44, 45]

$$\pi_1(y_k) = y_k + \sum_{l \geq 2} \frac{(-1)^{l-1}}{l} \sum_{\substack{j_1, \dots, j_l \geq 1 \\ j_1 + \dots + j_l = k}} y_{j_1} \dots y_{j_l}, \quad (11)$$

$$y_n = \sum_{k \geq 1} \frac{1}{k!} \sum_{s'_1 + \dots + s'_k = n} \pi_1(y_{s'_1}) \dots \pi_1(y_{s'_k}) \quad (12)$$

Hence, by introducing the new alphabet $\bar{Y} = \{\bar{y}\}_{y \in Y} = \{\pi_1(y)\}_{y \in Y}$, one has

$$(\mathbb{Q}\langle \bar{Y} \rangle, \text{conc}, 1_{\bar{Y}^*}, \Delta_{\sqcup}) \cong (\mathbb{Q}\langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\sqcup})$$

as one can prove through (12) that the endomorphism $y \mapsto \bar{y}$ is, in fact, an isomorphism

$$\begin{aligned} \mathcal{H}_{\sqcup} &\cong \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle \bar{Y} \rangle) \cong \mathcal{U}(\text{Prim}(\mathcal{H}_{\sqcup})), \\ \mathcal{H}_{\sqcup}^{\vee} &\cong \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle \bar{Y} \rangle)^{\vee} \cong \mathcal{U}(\text{Prim}(\mathcal{H}_{\sqcup}))^{\vee}. \end{aligned}$$

By considering

1. The PBW-Lyndon basis $\{\Pi_w\}_{w \in Y^*}$ for $\mathcal{U}(\text{Prim}(\mathcal{H}_{\sqcup}))$ constructed recursively as follows [44]

$$\begin{cases} \Pi_y = \pi_1(y) & \text{for } y \in Y, \\ \Pi_l = [\Pi_s, \Pi_r] & \text{for } l \in \mathcal{L}ynY, \text{ standard factorization of } l = (s, r), \\ \Pi_w = \Pi_{l_1}^{i_1} \dots \Pi_{l_k}^{i_k} & \text{for } w = l_1^{i_1} \dots l_k^{i_k}, l_1 \succ \dots \succ l_k, l_1, \dots, l_k \in \mathcal{L}ynY, \end{cases}$$

2. And, by duality, the linear basis $\{\Sigma_w\}_{w \in Y^*}$ for $(\mathbb{Q}\langle Y \rangle, \sqcup, 1_{Y^*})$, i.e.

$$\forall u, v \in Y^*, \langle \Pi_u | \Sigma_v \rangle = \delta_{u,v}.$$

This basis can be computed recursively as follows [5, 44]

$$\begin{cases} \Sigma_y = y, & \text{for } y \in Y, \\ \Sigma_l = \sum_{\substack{(!) \\ i!}} \frac{y_{s_{k_1} + \dots + s_{k_i}}}{i!} \Sigma_{l_1 \dots l_n}, & \text{for } l = y_{s_1} \dots y_{s_w} \in \mathcal{L}ynY, \\ \Sigma_w = \frac{\sum_{l_1 \sqcup \dots \sqcup l_k} \Sigma_{l_1}^{i_1} \dots \Sigma_{l_k}^{i_k}}{i_1! \dots i_k!}, & \text{for } w = l_1^{i_1} \dots l_k^{i_k}, \text{ with } l_1 \succ \dots \succ l_k \in \mathcal{L}ynY. \end{cases}$$

In (!), the sum is taken over all subsequences $\{k_1, \dots, k_i\} \subset \{1, \dots, k\}$ and all Lyndon words $l_1 \succeq \dots \succeq l_n$ such that $(y_{s_1}, \dots, y_{s_k}) \stackrel{*}{\leftarrow} (y_{s_{k_1}}, \dots, y_{s_{k_i}}, l_1, \dots, l_n)$,

where $\stackrel{*}{\leftarrow}$ denotes the transitive closure of the relation on standard sequences, denoted by \leftarrow (see [5]).

We also proved that, for any $w \in Y^*$, [6, 44, 45]

$$\Pi_w = w + \sum_{v \succ w, (v)=(w)} e_v v \text{ and } \Sigma_w = w + \sum_{v \prec w, (v)=(w)} f_v v. \quad (13)$$

In other words, the elements of the bases $\{\Sigma_w\}_{w \in Y^*}$ and $\{\Pi_w\}_{w \in Y^*}$ are lower and upper triangular respectively and they are of homogeneous in weight.

We also get the extended Schützenberger's factorization of \mathcal{B}_Y [6, 44, 45]

$$\mathcal{B}_Y = \sum_{w \in Y^*} \Sigma_w \otimes \Pi_w = \prod_{l \in \mathcal{L}_{yn} Y} \exp(\Sigma_l \otimes \Pi_l).$$

Example 6 (of $\{\Pi_w\}_{w \in Y^}$ and $\{\Sigma_w\}_{w \in Y^*}$, [5]).*

l	Π_l	Σ_l
y_2	$y_2 - \frac{1}{2}y_1^2$	y_2
y_1^2	y_1^2	$\frac{1}{2}y_2 + y_1^2$
y_3	$y_3 - \frac{1}{2}y_1y_2 - \frac{1}{2}y_2y_1 + \frac{1}{3}y_1^3$	y_3
y_2y_1	$y_2y_1 - y_2y_1$	$\frac{1}{2}y_3 + y_2y_1$
y_1y_2	$y_2y_1 - \frac{1}{2}y_1^3$	y_1y_2
y_1^3	y_1^3	$\frac{1}{6}y_3 + \frac{1}{2}y_2y_1 + \frac{1}{2}y_1y_2 + y_1^3$
y_4	$y_4 - \frac{1}{2}y_1y_3 - \frac{1}{2}y_2^2 - \frac{1}{2}y_3y_1$ $+ \frac{1}{3}y_1^2y_2 + \frac{1}{3}y_1y_2y_1 + \frac{1}{3}y_2y_1^2 - \frac{1}{4}y_1^4$	y_4
y_3y_1	$y_3y_1 - \frac{1}{2}y_2y_1^2 - y_1y_3 + \frac{1}{2}y_1^2y_2$	$\frac{1}{2}y_4 + y_3y_1$
y_2^2	$y_2^2 - \frac{1}{2}y_2y_1^2 - \frac{1}{2}y_1^2y_2 + \frac{1}{4}y_1^4$	$\frac{1}{2}y_4 + y_2^2$
$y_2y_1^2$	$y_2y_1^2 - 2y_1y_2y_1 + y_1^2y_2$	$\frac{1}{6}y_4 + \frac{1}{2}y_3y_1 + \frac{1}{2}y_2^2 + y_2y_1^2$
y_1y_3	$y_1y_3 - \frac{1}{2}y_1^2y_2 - \frac{1}{2}y_1y_2y_1 + \frac{1}{3}y_1^4$	$y_4 + y_3y_1 + y_1y_3$
$y_1y_2y_1$	$y_1y_2y_1 - y_1^2y_2$	$\frac{1}{2}y_4 + \frac{1}{2}y_3y_1 + y_2^2$ $+ y_2y_1^2 + \frac{1}{2}y_1y_3 + y_1y_2y_1$
$y_1^2y_2$	$y_1^2y_2 - \frac{1}{2}y_1^4$	$\frac{1}{2}y_4 + y_3y_1 + y_2^2 + y_2y_1^2$ $+ y_1y_3 + y_1y_2y_1 + y_1^2y_2$
y_1^4	y_1^4	$\frac{1}{24}y_4 + \frac{1}{6}y_3y_1 + \frac{1}{4}y_2^2 + \frac{1}{2}y_2y_1^2$ $+ \frac{1}{6}y_1y_3 + \frac{1}{2}y_1y_2y_1 + \frac{1}{2}y_1^2y_2 + y_1^4$

2.2 Indiscernability over a class of formal power series

2.2.1 Residual calculus and representative series

Definition 1. Let $S \in \mathbb{Q}\langle\langle X \rangle\rangle$ (resp. $\mathbb{Q}\langle X \rangle$) and let $P \in \mathbb{Q}\langle X \rangle$ (resp. $\mathbb{Q}\langle\langle X \rangle\rangle$). The left and right residual of S by P are respectively the formal power series $P \triangleleft S$ and $S \triangleright P$ in $\mathbb{Q}\langle\langle X \rangle\rangle$ defined by $\langle P \triangleleft S \mid w \rangle = \langle S \mid wP \rangle$ (resp. $\langle S \triangleright P \mid w \rangle = \langle S \mid Pw \rangle$).

For any $S \in \mathbb{Q}\langle\langle X \rangle\rangle$ (resp. $\mathbb{Q}\langle X \rangle$) and $P, Q \in \mathbb{Q}\langle X \rangle$ (resp. $\mathbb{Q}\langle\langle X \rangle\rangle$), we straightforwardly get $P \triangleleft (Q \triangleleft S) = PQ \triangleleft S$, $(S \triangleright P) \triangleright Q = S \triangleright PQ$ and $(P \triangleleft S) \triangleright Q = P \triangleleft (S \triangleright Q)$.

In case $x, y \in X$ and $w \in X^*$, we get¹³ $x \triangleleft (wy) = \delta_x^y w$ and $xw \triangleright y = \delta_x^y w$.

Lemma 1. (Reconstruction lemma) *Let $S \in \mathbb{Q}\langle\langle X \rangle\rangle$. Then*

$$S = \langle S \mid 1_{X^*} \rangle + \sum_{x \in X} x(S \triangleright x) = \langle S \mid 1_{X^*} \rangle + \sum_{x \in X} (x \triangleleft S)x.$$

Theorem 1. *Let $\delta \in \mathfrak{Der}(\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$. Moreover, we suppose that δ is locally nilpotent¹⁴. Then the family $(t\delta)^n/n!$ is summable and its sum, denoted $\exp(t\delta)$, is a one-parameter group of automorphisms of $(\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$.*

Theorem 2. *Let L be a Lie series, i.e. $\Delta_{\sqcup}(L) = L \hat{\otimes} 1 + 1 \hat{\otimes} L$. Let δ_L^r, δ_L^l be defined respectively by $\delta_L^r(P) := P \triangleleft L$, $\delta_L^l(P) := L \triangleright P$. Then δ_L^r, δ_L^l are locally nilpotent derivations of $(\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$. Hence, $\exp(t\delta_L^r), \exp(t\delta_L^l)$ are one-parameter groups of $\text{Aut}(\mathbb{Q}\langle X \rangle, \sqcup, 1_{X^*})$ and $\exp(t\delta_L^r)P = P \triangleleft \exp(tL)$, $\exp(t\delta_L^l)P = \exp(tL) \triangleright P$.*

Example 7. Since $x_1 \triangleleft$ and $\triangleright x_0$ are derivations and the polynomials $\{\Sigma_l\}_{l \in \mathcal{L}_{ynX-X}}$ belong to $x_0 \mathbb{Q}\langle X \rangle x_1$ then $x_1 \triangleleft l = l \triangleright x_0 = 0$ and $x_1 \triangleleft \check{S}_l = \check{S}_l \triangleright x_0 = 0$.

Theorem 3. *Let $S \in \mathbb{Q}\langle\langle X \rangle\rangle$. The following properties are equivalent:*

1. *The left \mathbb{C} -module $\text{Res}_g(S) = \text{span}\{w \triangleleft S \mid w \in X^*\}$ is finite dimensional.*
2. *The right \mathbb{C} -module $\text{Res}_d(S) = \text{span}\{S \triangleright w \mid w \in X^*\}$ is finite dimensional.*
3. *There are matrices $\lambda \in \mathcal{M}_{1,n}(\mathbb{Q})$, $\eta \in \mathcal{M}_{n,1}(\mathbb{Q})$ and $\mu : X^* \rightarrow \mathcal{M}_{n,n}$, such that*

$$S = \sum_{w \in X^*} [\lambda \mu(w) \eta] w = \lambda \left(\prod_{l \in \mathcal{L}_{ynX}} e^{\mu(S_l) P_l} \right) \eta.$$

A series that satisfies the items of Theorem 3 will be called *representative series*. This concept can be found in [1, 14, 48, 17]. The two first items are in [22, 28]. The third can be deduced from [8, 14] for example and it was used to factorize, for the first time, by Lyndon words, the output of bilinear and analytical dynamical systems respectively in [30, 31] and to study polylogarithms, hypergeometric functions and associated functions in [33, 35, 42]. The dimension of $\text{Res}_g(S)$ is equal to that of $\text{Res}_d(S)$, and to the minimal dimension of a representation satisfying the third point of Theorem 3. This rank is then equal to the rank of the Hankel matrix of S , i.e. the infinite matrix $(\langle S \mid uv \rangle)_{u,v \in X}$ indexed by $X^* \times X^*$ so called *Hankel rank*¹⁵ of S [22, 28]. The triplet (λ, μ, η) is called a *linear representation* of S ¹⁶. S is called *rational* if it belongs to the closure by $+$, conc and star operation of proper elements¹⁷. Any noncommutative power series is representative if and only if it is rational [3, 53]. These rationality properties can be expressed in terms of differential operators in noncommutative geometry [14].

¹³ For any words $u, v \in X^*$, if $u = v$ then $\delta_u^v = 1$ else 0.

¹⁴ $\phi \in \text{End}(V)$ is said to be locally nilpotent iff, for any $v \in V$, there exists $N \in \mathbb{N}$ s.t. $\phi^N(v) = 0$.

¹⁵ i.e. the dimension of $\text{span}\{S \triangleright \Pi \mid \Pi \in \mathbb{C}\langle X \rangle\}$ (resp. $\text{span}\{\Pi \triangleleft S \mid \Pi \in \mathbb{C}\langle X \rangle\}$).

¹⁶ The minimal representation of S as being a representation of S of minimal dimension. It can be shown that all minimal representations are isomorphic (see [3]).

¹⁷ For any *proper* series S , i.e. $\langle S \mid 1_{X^*} \rangle = 0$, the series $S^* = 1 + S + S^2 + \dots$ is called “star of S ”

2.2.2 Background on continuity and indiscernability

Definition 2. ([29, 43]) Let \mathcal{H} be a class of $\mathbb{C}\langle\langle X \rangle\rangle$ and $S \in \mathbb{C}\langle\langle X \rangle\rangle$.

1. S is said to be *continuous* over \mathcal{H} if for any $\Phi \in \mathcal{H}$, the following sum, denoted by $\langle S \parallel \Phi \rangle$, is absolutely convergent $\sum_{w \in X^*} \langle S \mid w \rangle \langle \Phi \mid w \rangle$.
The set of continuous power series over \mathcal{H} will be denoted by $\mathbb{C}^{\text{cont}}\langle\langle X \rangle\rangle$.
2. S is said to be *indiscernable* over \mathcal{H} if and only if, for any $\Phi \in \mathcal{H}$, $\langle S \parallel \Phi \rangle = 0$.

Proposition 1. Let $S \in \mathbb{C}^{\text{cont}}\langle\langle X \rangle\rangle$. \mathcal{H} is a monoid containing X and $\{e^{t^x}\}_{x \in X}^{t \in \mathbb{C}}$.

1. If S is indiscernable over \mathcal{H} then for any $x \in X$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbb{C}^{\text{cont}}\langle\langle X \rangle\rangle$ and they are indiscernable over \mathcal{H} .
2. S is indiscernable over \mathcal{H} if and only if $S = 0$.

Proof. 1. Of course, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbb{C}^{\text{cont}}\langle\langle X \rangle\rangle$. Let us calculate $\langle x \triangleleft S \parallel \Phi \rangle = \langle S \parallel \Phi x \rangle$ and $\langle S \triangleright x \parallel \Phi \rangle = \langle S \parallel x \Phi \rangle$. Since S is indiscernable over \mathcal{H} and note that $x\Phi, \Phi x \in \mathcal{H}$ for every $x \in X$; $\Phi \in \mathcal{H}$, then

$$\langle S \parallel \Phi x \rangle = 0, \text{ and } \langle S \parallel x \Phi \rangle = 0.$$

Hence $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbb{C}^{\text{cont}}\langle\langle X \rangle\rangle$ are indiscernable over \mathcal{H} .

2. $S = 0$ is indiscernable over \mathcal{H} . Conversely, if S is indiscernable over \mathcal{H} then by the previous point and by induction on the length of $w \in X^*$, $w \triangleleft S$ is indiscernable over \mathcal{H} . In particular, $\langle w \triangleleft S \mid \text{Id}_{\mathcal{H}} \rangle = \langle S \mid w \rangle = 0$. In other words, $S = 0$.

2.3 Polylogarithms and harmonic sums

2.3.1 Structure of polylogarithms and of harmonic sums

Let $\Omega := \mathbb{C} - (]-\infty, 0] \cup [1, +\infty[)$ and let $\mathcal{C} := \mathbb{C}[z, 1/z, 1/(1-z)]$. Note that the neutral element of \mathcal{C} , for the pointwise product, is $1_\Omega : \Omega \rightarrow \mathbb{C}$ such that $z \mapsto 1$.

One can check that $\text{Li}_{s_1, \dots, s_r}$ is obtained as the iterated integral over the differential forms $\omega_0(z) = dz/z$ and $\omega_1(z) = dz/(1-z)$ and along the path $0 \rightsquigarrow z$ [32]:

$$\text{Li}_{s_1, \dots, s_r}(z) = \alpha_0^z(x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1) = \sum_{n_1 > \dots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \dots n_r^{s_r}}. \quad (14)$$

By (1), $\text{Li}_{s_1, \dots, s_r}$ is then denoted also by $\text{Li}_{x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1}$ or $\text{Li}_{y_{s_1} \dots y_{s_r-1}}$ [33, 35, 36].

Example 8 (of $\text{Li}_2 = \text{Li}_{x_0 x_1}$).

$$\alpha_0^z(x_0 x_1) = \int_0^z \frac{ds}{s} \int_0^s \frac{dt}{1-t} = \int_0^z \frac{ds}{s} \int_0^s dt \sum_{k \geq 0} t^k = \sum_{k \geq 1} \int_0^z ds \frac{s^{k-1}}{k} = \sum_{k \geq 1} \frac{z^k}{k^2}.$$

The definition of polylogarithms is extended over the words $w \in X^*$ by putting $\text{Li}_{x_0}(z) := \log(z)$. The $\{\text{Li}_w\}_{w \in X^*}$ are \mathcal{C} -linearly independent [12, 39, 36] and then the following function, for $v = y_{s_1} \dots y_{s_r} \in Y^*$, are also \mathbb{C} -linearly independent [12, 41]

$$P_v(z) := \frac{\text{Li}_v(z)}{1-z} = \sum_{N \geq 0} H_v(N) z^N, \text{ where } H_v(N) := \sum_{N \geq n_1 > \dots > n_r > 0} \frac{1}{n_1^{s_1} \dots n_r^{s_r}}.$$

Proposition 2. ([41]) *By linearity, the following maps are isomorphisms of algebras*

$$\begin{aligned} \mathbf{P}_\bullet : (\mathbb{C}\langle Y \rangle, \boxplus) &\longrightarrow (\mathbb{C}\{\mathbf{P}_w\}_{w \in Y^*}, \odot), \quad u \longmapsto \mathbf{P}_u, \\ \mathbf{H}_\bullet : (\mathbb{C}\langle Y \rangle, \boxplus) &\longrightarrow (\mathbb{C}\{\mathbf{H}_w\}_{w \in Y^*}, \cdot), \quad u \longmapsto \mathbf{H}_u = \{\mathbf{H}_u(N)\}_{N \geq 0}. \end{aligned}$$

Theorem 4. ([43]) *The Hadamard \mathcal{C} -algebra of $\{\mathbf{P}_w\}_{w \in Y^*}$ can be identified with that of $\{\mathbf{P}_l\}_{l \in \mathcal{L}_{\text{ym}} Y}$. In the same way, the algebra of harmonic sums $\{\mathbf{H}_w\}_{w \in Y^*}$ with polynomial coefficients can be identified with that of $\{\mathbf{H}_l\}_{l \in \mathcal{L}_{\text{ym}} Y}$.*

Let \mathbf{L}, \mathbf{P} and \mathbf{H} be the noncommutative generating series of respectively $\{\text{Li}_w\}_{w \in X^*}$, $\{\mathbf{P}_w\}_{w \in X^*}$ and $\{\mathbf{H}_w(N)\}_{w \in Y^*}$, for $|z| < 1$ and $N > 1$ [39, 41]:

$$\mathbf{L}(z) = \sum_{w \in X^*} \text{Li}_w(z) w; \quad \mathbf{P}(z) = \frac{\mathbf{L}(z)}{1-z}; \quad \mathbf{H}(N) = \sum_{w \in Y^*} \mathbf{H}_w(N) w. \quad (15)$$

Definition 3 (Polylogarithms and harmonic sums at negative multi-indices). For any $s_1, \dots, s_r \in (\mathbb{N})^r$, let us define [16], for $|z| < 1$ and $N > 0$,

$$\text{Li}_{-s_1, \dots, -s_r}(z) := \sum_{n_1 > \dots > n_r > 0} n_1^{s_1} \dots n_r^{s_r} z^{n_1} \text{ and } \mathbf{H}_{-s_1, \dots, -s_r}(N) := \sum_{N \geq n_1 > \dots > n_r > 0} n_1^{s_1} \dots n_r^{s_r}.$$

The ordinary generating series, $\mathbf{P}_{-s_1, \dots, -s_r}(z)$, of $\{\mathbf{H}_{-s_1, \dots, -s_r}(N)\}_{N \geq 0}$ is

$$\mathbf{P}_{-s_1, \dots, -s_r}(z) := \sum_{N \geq 0} \mathbf{H}_{-s_1, \dots, -s_r}(N) z^N = \frac{1}{1-z} \text{Li}_{-s_1, \dots, -s_r}(z).$$

Now, let¹⁸ $Y_0 = Y \cup \{y_0\}$ and let Y_0^* denotes the free monoid generated by Y_0 admitting $1_{Y_0^*}$ as neutral element. As in (1), let us introduce another correspondence

$$(s_1, \dots, s_r) \in \mathbb{N}^r \leftrightarrow y_{s_1} \dots y_{s_r} \in Y_0^*.$$

In all the sequel, for some convenience, we will also adopt the following notations, for any $w = y_{s_1} \dots y_{s_r} \in Y_0^*$,

$$\text{Li}_w^- = \text{Li}_{-s_1, \dots, -s_r}; \quad \mathbf{P}_w^- = \mathbf{P}_{-s_1, \dots, -s_r} \quad \text{and} \quad \mathbf{H}_w^- = \mathbf{H}_{-s_1, \dots, -s_r}.$$

Example 9 ($\text{Li}_{y_0}^-$ and $\mathbf{H}_{y_0}^-$). By Proposition (5), we have $\text{Li}_{y_0}^- = \lambda^r$. Hence,

¹⁸ with $y_0 \succ y_1$.

$$\frac{\text{Li}_{y_0}^-(z)}{1-z} = \frac{z^r}{(1-z)^{r+1}} = \sum_{N \geq 0} \binom{N}{r} z^N \text{ and then } \text{H}_{y_0}^-(N) = \binom{N}{r}.$$

Definition 4. With the convention $\text{H}_{1_{Y_0^*}}^- = 1$, we put

$$\text{L}^-(z) := \sum_{w \in Y_0^*} \text{Li}_w^-(z)w; \quad \text{P}^-(z) := \frac{\text{L}^-(z)}{1-z}; \quad \text{H}^-(N) := \sum_{w \in Y_0^*} \text{H}_w^-(N)w.$$

Since, for $y_k \in Y, u \in Y^*$ (resp. $y_k \in Y_0, u \in Y_0^*$) and $N \geq 1$, one has $\text{H}_{y_k u}(N) - \text{H}_{y_k u}(N-1) = N^{-k} \text{H}_u(N-1)$ (resp. $\text{H}_{y_k u}^-(N) - \text{H}_{y_k u}^-(N-1) = N^k \text{H}_u^-(N-1)$). Then

Proposition 3. H and H^- satisfy the following difference equations

$$\begin{aligned} \text{H}(N) &= \left(1_{Y^*} + \sum_{k \geq 1} \frac{y_k}{N^k}\right) \text{H}(N-1) = \prod_{n=1}^N \left(1_{Y^*} + \sum_{k \geq 1} \frac{y_k}{n^k}\right) = 1_{Y^*} + \sum_{w \in Y^*, |w| \geq N} \text{H}_w(N) w, \\ \text{H}^-(N) &= \left(1_{Y_0^*} + \sum_{k \geq 0} y_k N^k\right) \text{H}^-(N-1) = \prod_{n=1}^N \left(1_{Y_0^*} + \sum_{k \geq 0} y_k n^k\right) = 1_{Y_0^*} + \sum_{w \in Y_0^*, |w| \geq N} \text{H}_w^-(N) w. \end{aligned}$$

Hence, for any $w \in Y^*$ (resp. $w \in Y_0^*$), $\text{H}_w(N)$ (resp. $\text{H}_w^-(N)$) is of valuation N .

In all the sequel, the *length* and the *weight* of $u = y_{i_1} \dots y_{i_k} \in Y^*$ are defined respectively as the numbers $|u| = k$ and $(u) = i_1 + \dots + i_k$.

Definition 5. Let $g, h \in \mathbb{Q} \langle \langle Y_0 \rangle \rangle \llbracket t \rrbracket$ be defined as follows (here, $|1_{Y_0^*}| = (1_{Y_0^*}) = 0$)

$$h(t) := \sum_{w \in Y_0^*} ((w) + |w|)! t^{(w) + |w|} w \text{ and } g(t) := \sum_{w \in Y_0^*} t^{(w) + |w|} w = \left(\sum_{y \in Y_0} t^{(y) + 1} y \right)^*.$$

Remark 1. 1. The generating series h is an extension of the Euler series $\sum_{n \geq 0} n! t^n$ and it can be obtained as Borel-Laplace transform of g .

2. The ordinary generating series $\mathcal{Y}(t) := 1 + \sum_{r \geq 0} y_r t^r$ and its inverse are group-like. The generating series $\Lambda(t) = \sum_{w \in Y_0^*} t^{(w) + |w|} w$ can be obtained from $1/\mathcal{Y}(t)$ by use the following change of alphabet $y_r \leftarrow t y_r$ it can be expressed as

$$g(t) = \left(1 - \sum_{r \geq 0} (-t y_r) t^r\right)^{-1} = \left(\sum_{r \geq 0} (-t y_r) t^r\right)^*.$$

Now, let us consider the following differential and integration operators acting on $\mathbb{C}\{\text{Li}_w\}_{w \in X^*}$ which can be extended over $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$ [42]:

$$\partial_z = d/dz, \quad \theta_0 = z d/dz, \quad \theta_1 = (1-z) d/dz, \quad \iota_0 : \text{Li}_w \mapsto \text{Li}_{x_0 w}, \quad \iota_1 : \text{Li}_w \mapsto \text{Li}_{x_1 w}$$

Let Θ and \mathfrak{S} be monoid morphisms such that $\Theta(1_{X^*}) = \mathfrak{S}(1_{X^*}) = \text{Id}$ and, for $x_i \in X, v \in X^*$, $\Theta(v x_i) = \Theta(v) \theta_i$ and $\mathfrak{S}(v x_i) = \mathfrak{S}(v) \iota_i$. By extension, we obtain $\mathcal{H}_{\text{conc}} \cong (\mathbb{Q} \langle \Theta(X) \rangle, \text{conc}, \text{Id}, \Delta_{\sqcup}, \varepsilon)$ and $\mathcal{H}_{\sqcup} \cong (\mathbb{Q} \langle \mathfrak{S}(X) \rangle, \sqcup, \text{Id}, \Delta_{\text{conc}}, \varepsilon)$. Hence,

Proposition 4. 1. The operators $\{\theta_0, \theta_1, \iota_0, \iota_1\}$ satisfy in particular;

$$\begin{aligned} \theta_1 + \theta_0 &= [\theta_1, \theta_0] = \partial_z \text{ and } \forall k = 0, 1, \theta_k \iota_k = \text{Id}, \\ [\theta_0 \iota_1, \theta_1 \iota_0] &= 0 \text{ and } (\theta_0 \iota_1)(\theta_1 \iota_0) = (\theta_1 \iota_0)(\theta_0 \iota_1) = \text{Id}. \end{aligned}$$

2. For any $w = y_{s_1} \dots y_{s_r} \in Y^*$ ($\pi_X(w) = x_0^{s_1-1} x_1 \dots x_0^{s_r-1} x_1$) and $u = y_{t_1} \dots y_{t_r} \in Y_0^*$, we can rephrase $\text{Li}_w, \text{Li}_u^-$ as follows

$$\begin{aligned} \text{Li}_w &= (\iota_0^{s_1-1} \iota_1 \dots \iota_0^{s_r-1} \iota_1) 1_\Omega \text{ and } \text{Li}_u^- = (\theta_0^{t_1+1} \iota_1 \dots \theta_0^{t_r+1} \iota_1) 1_\Omega, \\ \theta_0 \text{Li}_{x_0 \pi_X(w)} &= \text{Li}_{\pi_X(w)} \text{ and } \theta_1 \text{Li}_{x_1 \pi_X(w)} = \text{Li}_{\pi_X(w)}, \\ \iota_0 \text{Li}_{\pi_X(w)} &= \text{Li}_{x_0 \pi_X(w)} \text{ and } \iota_1 \text{Li}_w = \text{Li}_{x_1 \pi_X(w)}. \end{aligned}$$

3. $\mathcal{C}\{\text{Li}_w\}_{w \in X^*} \cong \mathcal{C} \otimes \mathbb{C}\{\text{Li}_w\}_{w \in X^*}$ is closed under of $\iota_0, \iota_1, \theta_0, \theta_1$.

4. Let $\lambda(z) := z/(1-z) \in \mathcal{C}$. Then λ and $1/\lambda$ are the eigenvalues of $\theta_0 \iota_1$ and $\theta_1 \iota_0$ within $\mathcal{C}\{\text{Li}_w\}_{w \in X^*}$ respectively :

$$\forall f \in \mathcal{C}\{\text{Li}_w\}_{w \in X^*}, (\theta_0 \iota_1)f = \lambda f \text{ and } (\theta_1 \iota_0)f = f/\lambda.$$

5. For any $n \geq 0$ and $w \in X^*$, one has¹⁹

$$\Theta(\tilde{w}) \text{Li}_w = 1_\Omega \text{ and } \partial_z^n = \sum_{w \in X^n} (\Theta \otimes \Theta) \Delta_{\sqcup}(w).$$

6. For any $P, Q \in \mathbb{Q}\langle X \rangle$ and $R \in \mathcal{L}ie_{\mathbb{Q}}\langle X \rangle$, one has

$$\Theta(R) \text{Li}_{P \sqcup Q} = \text{Li}_{(P \sqcup Q) \triangleleft R} = (\Theta(R) \text{Li}_P) \text{Li}_Q + \text{Li}_P (\Theta(R) \text{Li}_Q).$$

Proof. The proofs are immediate.

Proposition 5. [16]

1. For any $w \in Y_0^*$, one has $\text{Li}_w^-(z) = \lambda^{|\mathbf{w}|}(z) A_w^-(z) (1-z)^{-|\mathbf{w}|}$, where A_w^- is the extended Eulerian polynomial defined recursively as follows

$$A_w^-(z) = \begin{cases} \sum_{k=0}^{n-1} A_{n,k} z^k & \text{if } w = y_k \in Y_0, \\ \sum_{i=0}^{s_1} \binom{s_1}{i} A_{y_i} A_{y_{(s_1+s_2-i)} y_{s_3} \dots y_{s_r}}^- & \text{if } w = y_k u \in Y_0 Y_0^*, \end{cases}$$

and $A_{n,k}$ are Eulerian numbers satisfying $A_{n,k} = \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n$.

2. For any $w \in Y^*$, let us define $\{G_w^-(n)\}_{n \in \mathbb{N}}$ by the following generating series

$$\sum_{n \geq |\mathbf{w}|} \frac{(n+1)!}{(n-|\mathbf{w}|)!} G_w^-(n) z^n = \frac{\text{Li}_w^-(z)}{1-z}.$$

Then $H_w^-(N) = (N+1)N(N-1) \dots (N-|\mathbf{w}|+1) G_w^-(N)$.

¹⁹ For any $w = x_{i_1} \dots x_{i_r} \in X^*$, we denote $\tilde{w} = x_{i_r} \dots x_{i_1}$.

3. $\text{Li}_w^-(z) \in \mathbb{Q}[(1-z)^{-1}] \subsetneq \mathcal{C}$ and $H_w^-(N) \in \mathbb{Q}[N]$ of degree $|w| + (w)$.

Example 10. [16][Case of $r = 1$ by Maple]

1. Since $A_n(z)/(1-z)^{n+1} = \sum_{j \geq 0} z^j (j+1)^n$ then $\text{Li}_{y_n}^-(z) = zA_n(z)/(1-z)^{n+1}$ (see [20] for example). For example,

$$\begin{aligned} \text{Li}_{y_1}^-(z) &= z(1-z)^{-2} = -(1-z)^{-1} + (1-z)^{-2}. \\ \text{Li}_{y_2}^-(z) &= z(z+1)(1-z)^{-3} = (1-z)^{-1} - 3(1-z)^{-2} + 2(1-z)^{-3}. \\ \text{Li}_{y_3}^-(z) &= z(z^2+4z+1)(1-z)^{-4} = -(1-z)^{-1} + 7(1-z)^{-2} - 12(1-z)^{-3} + 6(1-z)^{-4}. \end{aligned}$$

2. For any positive integer m , one has

$$H_{y_m}^-(N) = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k (N+1)^{m+1-k} = \frac{1}{m+1} \sum_{k=1}^{m+1} \left[\sum_{l=0}^{m+1-k} \binom{m+1}{l} \binom{m+1-l}{k} B_l \right] N^l,$$

where B_k is the k -th Bernoulli's number given by its exponential generating series

$$\frac{t}{e^t - 1} = \sum_{k \geq 0} B_k \frac{t^k}{k!}.$$

For example, (recall that $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30$),

$$\begin{aligned} H_{y_1}^-(N) &= (N+1)^2/2 - (N+1)/2 = N(N+1)/2, \\ H_{y_2}^-(N) &= (N+1)^3/3 - (N+1)^2/2 + (N+1)/6 = N(2N+1)(N+1)/6, \\ H_{y_3}^-(N) &= (N+1)^4/4 - (N+1)^3/2 + (N+1)^2/4 = (N(N+1)/2)^2. \end{aligned}$$

Example 11. [16][Case of $r = 2$ by Maple]

1. From what precedes, $\text{Li}_{y_m y_n}^- = (\theta_0^{m+1} t_1) \text{Li}_{y_n}^- = \theta_0^m (\theta_0 t_1) \text{Li}_{y_n}^-$. Since, by Example 9, we have $(\theta_0 t_1) \text{Li}_{y_n}^- = \text{Li}_{y_0}^- \text{Li}_{y_n}^-$ then $\text{Li}_{y_m y_n}^- = \theta_0^m [\text{Li}_{y_0}^- \text{Li}_{y_n}^-] = \sum_{l=0}^m \binom{m}{l} \text{Li}_{y_l}^- \text{Li}_{y_{m+n-l}}^-$. For example,

$$\begin{aligned} \text{Li}_{y_2}^-(z) &= \text{Li}_{y_0}^-(z) \text{Li}_{y_2}^-(z) + (\text{Li}_{y_1}^-(z))^2 \\ &= -(1-z)^{-1} + 5(1-z)^{-2} - 7(1-z)^{-3} + 3(1-z)^{-4} \\ \text{Li}_{y_2 y_1}^-(z) &= \text{Li}_{y_0}^-(z) \text{Li}_{y_3}^-(z) + 3 \text{Li}_{y_1}^-(z) \text{Li}_{y_2}^-(z) \\ &= (1-z)^{-1} - 11(1-z)^{-2} + 31(1-z)^{-3} - 33(1-z)^{-4} + 12(1-z)^{-5}, \\ \text{Li}_{y_1 y_2}^-(z) &= \text{Li}_{y_0}^-(z) \text{Li}_{y_3}^-(z) + \text{Li}_{y_1}^-(z) \text{Li}_{y_2}^-(z) \\ &= (1-z)^{-1} - 9(1-z)^{-2} + 23(1-z)^{-3} - 23(1-z)^{-4} + 8(1-z)^{-5}. \end{aligned}$$

2. For any positive integers m, n , one has

$$\begin{aligned} H_{y_m y_n}^-(N) &= \sum_{k_1=0}^n \sum_{k_2=0}^{m+n+1-k_1} \sum_{k_3=0}^{m+n+2-k_1-k_2} \frac{B_{k_1} B_{k_2}}{(n+1)(m+n+2-k_1)} \\ &\quad \binom{n+1}{k_1} \binom{m+n+2-k_1}{k_2} \binom{m+n+2-k_1-k_2}{k_3} N^{k_3}. \end{aligned}$$

For example,

$$\begin{aligned} H_{y_2 y_1}^-(N) &= N(N^2 - 1)(12N^2 + 15N + 2)/120, \\ H_{y_2}^-(N) &= N(N-1)(2N+1)(2N-1)(5N+6)(N+1)/360, \\ H_{y_2 y_3}^-(N) &= N(N-1)(N+1)(30N^4 + 35N^3 - 33N^2 - 35N + 2)/840, \\ H_{y_2 y_4}^-(N) &= N(N-1)(N+1)(63N^5 + 72N^4 - 133N^3 - 138N^2 + 49N + 30)/2520, \end{aligned}$$

$$\begin{aligned} H_{y_2 y_3}^-(N) &= N(N-1)(N+1)(280N^6 + 315N^5 - 920N^4 - 945N^3 + 802N^2 + 630N - 108)/15120, \\ H_{y_3}^-(N) &= N(N-1)(N+1)(21N^5 + 36N^4 - 21N^3 - 48N^2 + 8)/672. \end{aligned}$$

Example 12. [16][General case]

1. One has, for any $y_{s_1} u = y_{s_1} \dots y_{s_r} \in Y_0^*$,

$$\begin{aligned} \text{Li}_{y_{s_1} u}^- &= \theta_0^{s_1}(\theta_0 t_1) \text{Li}_u^- = \theta_0^{s_1}(\lambda \text{Li}_u^-) = \sum_{k_1=0}^{s_1} \binom{s_1}{k_1} (\theta_0^{k_1} \lambda) (\theta_0^{s_1-k_1} \text{Li}_u^-), \\ \text{Li}_{y_{s_1} \dots y_{s_r}}^- &= \sum_{k_1=0}^{s_1} \sum_{k_2=0}^{s_1+s_2-k_1} \dots \sum_{k_r=0}^{\binom{s_1+\dots+s_r}{k_1+\dots+k_{r-1}}} \binom{s_1}{k_1} \binom{s_1+s_2-k_1}{k_2} \dots \\ &\quad \binom{s_1+\dots+s_r-k_1-\dots-k_{r-1}}{k_r} (\theta_0^{k_r} \lambda) (\theta_0^{k_2} \lambda) \dots (\theta_0^{k_r} \lambda). \end{aligned}$$

Denoting $S_2(k_i, j)$ Stirling numbers of the second kind, one has

$$\forall i = 1, \dots, r, \quad \theta_0^{k_i} \lambda(z) = \begin{cases} \lambda(z), & \text{if } k_i = 0, \\ \frac{1}{1-z} \sum_{j=1}^{k_i} S_2(k_i, j) j! \lambda^j(z), & \text{if } k_i > 0. \end{cases}$$

In particular, if $\omega \in Y^*$ then $(1-z)^{|\omega|} \text{Li}_\omega^- (z)$ is polynomial of degree (w) in $\lambda(z)$.

2. We define, firstly, the *polynomials* $\{B_{y_{n_1} \dots y_{n_r}}(z)\}_{n_1, \dots, n_r \in \mathbb{N}}$ by their commutative exponential generating series as follows, for $z \in \mathbb{C}$,

$$\sum_{n_1, \dots, n_r \in \mathbb{N}} B_{y_{n_1} \dots y_{n_r}}(z) \frac{t_1^{n_1} \dots t_r^{n_r}}{n_1! \dots n_r!} = t_1 \dots t_r e^{z(t_1 + \dots + t_r)} \prod_{k=1}^r (e^{t_k + \dots + t_r} - 1)^{-1},$$

or by the difference equation, for $n_1 \in \mathbb{N}_+$,

$$B_{y_{n_1} \dots y_{n_r}}(z+1) = B_{y_{n_1} \dots y_{n_r}}(z) + n_1 z^{n_1-1} B_{y_{n_2} \dots y_{n_r}}(z).$$

For any $w \in y_s Y_0^*$, $s > 1$, we have $B_w(1) = B_w(0)$. Then let also, for any $1 \leq k \leq r$,

$$\begin{aligned} b_w &:= B_w(0) \text{ and } \beta_w(z) := B_w(z) - b_w, \\ b'_{y_k} &:= b_{y_k} \text{ and } b'_{y_{n_k} \dots y_{n_r}} := b_{y_{n_k} \dots y_{n_r}} - \sum_{j=0}^{r-1-k} b_{y_{n_k+j+1} \dots y_{n_r}} b'_{y_{n_k} \dots y_{n_k+j}} \end{aligned}$$

Then we have the extended Faulhaber's identities

$$\begin{aligned} \beta_{y_{n_1} \dots y_{n_r}}(N) &= \sum_{k=1}^r \left(\prod_{i=1}^k n_i \right) b_{y_{n_{k+1}} \dots y_{n_r}} H_{y_{n_1-1} \dots y_{n_k-1}}^-(N-1), \\ H_{y_{n_1} \dots y_{n_r}}^-(N) &= \frac{\beta_{y_{n_1+1} \dots y_{n_r+1}}(N+1) - \sum_{k=1}^{r-1} b'_{y_{n_{k+1}+1} \dots y_{n_r+1}} \beta_{y_{n_1+1} \dots y_{n_k+1}}(N+1)}{\prod_{i=1}^r (n_i + 1)}. \end{aligned}$$

Proposition 6. [16] *The following maps are morphisms of algebras*

$$H^- : (\mathbb{C}\langle Y_0 \rangle, \boxplus) \longrightarrow (\mathbb{C}\{H_w^-\}_{w \in Y_0^*}, \cdot) \text{ and } P^- : (\mathbb{C}\langle Y_0 \rangle, \boxplus) \longrightarrow (\mathbb{C}\{P^- w\}_{w \in Y_0^*}, \odot).$$

Proof. Recall that the quasi-symmetric functions on the variables $\mathbf{t} = \{t_i\}_{N \geq i \geq 1}$, i.e.

$$F_{s_1, \dots, s_r}(\mathbf{t}) = F_{y_{s_1}, \dots, y_{s_r}}(\mathbf{t}) = \sum_{n_1 > \dots > n_r > 0} t_{n_1}^{s_1} \cdots t_{n_r}^{s_r}$$

satisfy the quasi-shuffle relation [51], *i.e.* for any $u, v \in Y_0^*$, $F_{u \sqcup v}(\mathbf{t}) = F_u(\mathbf{t})F_v(\mathbf{t})$.

Since $H_{s_1, \dots, s_r}^- (N)$ can be obtained by specializing, in $F_{s_1, \dots, s_r}(\mathbf{t})$, the variables \mathbf{t} at

$$\forall 1 \leq i \leq N, t_i = i \text{ and } \forall i > N, t_i = 0$$

then H^- is a morphism of algebras. Therefore, P^- is also a morphism of algebras.

2.3.2 Global renormalizations via noncommutative generating series

By (2.3.2), L and H are images, by the tensor products $\text{Li} \otimes \text{Id}$ and $H \otimes \text{Id}$, of the diagonal series \mathcal{D}_X and \mathcal{D}_Y respectively. Then we get

Theorem 5 (Factorization of L and of H , [36, 39, 44]). *Let*

$$L_{\text{reg}} = \prod_{l \in \mathcal{L}_{\text{yn}X - X}} e^{\text{Li}_{s_l} P_l} \text{ and } H_{\text{reg}}(N) = \prod_{l \in \mathcal{L}_{\text{yn}Y - \{y_1\}}} e^{H_{s_l}^{(N)} \Sigma_l}.$$

Then $L(z) = e^{-x_1 \log(1-z)} L_{\text{reg}}(z) e^{x_0 \log z}$ and $H(N) = e^{H_{y_1}^{(N)} y_1} H_{\text{reg}}(N)$.

For any $l \in \mathcal{L}_{\text{yn}X - X}$ (resp. $\mathcal{L}_{\text{yn}Y - \{y_1\}}$), the polynomial S_l (resp. Σ_l) is a finite combination of words in $x_0 X^* x_1$ (resp. $Y^* - y_1 Y^*$). Then we can state

Proposition 7 ([44]). *Let $Z_{\sqcup} := L_{\text{reg}}(1)$ and $Z_{\sqcup} := H_{\text{reg}}(\infty)$. Then Z_{\sqcup} and Z_{\sqcup} are group-like, for Δ_{\sqcup} and Δ_{\sqcup} respectively.*

Proposition 8 (Successive integrations and differentiations of L , [42]). *We have, for any $n \in \mathbb{N}$,*

1. $\iota_0^n L = x_0^n \triangleright L$ and $\iota_1^n L = x_1^n \triangleright L$.
2. $\partial_z^n L = D_n L$ and $\theta_0^n L = E_n L$, where²⁰ the polynomials D_n and E_n in $\mathcal{C}\langle X \rangle$ are

$$D_n = \sum_{\text{wgt}(\mathbf{r})=n} \sum_{w \in X^{\text{deg}(\mathbf{r})}} \prod_{i=1}^{\text{deg}(\mathbf{r})} \binom{\sum_{j=1}^i r_j + j - 1}{r_i} \tau_{\mathbf{r}}(w),$$

$$E_n = \sum_{\text{wgt}(\mathbf{r})=n} \sum_{w \in X^{\text{deg}(\mathbf{r})}} \prod_{i=1}^{\text{deg}(\mathbf{r})} \binom{\sum_{j=1}^i r_j + j - 1}{r_i} \rho_{\mathbf{r}}(w),$$

and for any $w = x_{i_1} \cdots x_{i_k}$ and $\mathbf{r} = (r_1, \dots, r_k)$ of degree $\text{deg}(\mathbf{r}) = k$ and of weight $\text{wgt}(\mathbf{r}) = k + r_1 + \cdots + r_k$, the polynomials $\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1}) \cdots \tau_{r_k}(x_{i_k})$ and $\rho_{\mathbf{r}}(w) = \rho_{r_1}(x_{i_1}) \cdots \rho_{r_k}(x_{i_k})$ are defined respectively by, for any $r \in \mathbb{N}$,

²⁰ Since $\theta_0 + \theta_1 = \partial_z$ then we also have $\theta_1^n L(z) = [D_n(z) - E_n(z)]L(z)$. The more general actions of $\{\Theta(w)\}_{w \in X^*}$ on L are more complicated to be expressed here.

$$\begin{aligned}\tau_r(x_0) &= \partial_z^r \frac{x_0}{z} = \frac{-r!x_0}{(-z)^{r+1}} \text{ and } \tau_r(x_1) = \partial_z^r \frac{x_1}{1-z} = \frac{r!x_1}{(1-z)^{r+1}}, \\ \rho_r(x_0) &= \theta_0^r \frac{(-1)^{-1}x_0}{z} = 0 \text{ and } \rho_r(x_1) = \theta_0^r \frac{zx_1}{1-z} = \text{Li}_{\pi_Y(x_0^{-1}x_1)}^-(z)x_1.\end{aligned}$$

Example 13 (Coefficients of $\theta_0^n L$). Since, for any $u \in X^+$, $\theta_0 \text{Li}_{x_0 u} = \text{Li}_u$ and $\theta_1 \text{Li}_{x_0 u} = \text{Li}_0 \text{Li}_u$, one obtains for example

- For any $n \geq 1$ and $w \in X^*$, one has $\theta_0^n \text{Li}_{x_0^n w} = \text{Li}_w$. Hence,

$$\theta_0 \text{Li}_{x_1} = \text{Li}_0, \theta_0^2 \text{Li}_{x_1} = \text{Li}_{\pi_Y(x_1)}^-, \theta_0^3 \text{Li}_{x_1} = \text{Li}_{\pi_Y(x_0 x_1)}^- \text{ and } \theta_0^4 \text{Li}_{x_1} = \text{Li}_{\pi_Y(x_0^2 x_1)}^-.$$

- $\theta_0 \text{Li}_{x_1^2} = \text{Li}_0 \text{Li}_{x_1}$, $\theta_0^2 \text{Li}_{x_1^2} = \text{Li}_{\pi_Y(x_1)}^- \text{Li}_{x_1} + \text{Li}_0^2$, $\theta_0^3 \text{Li}_{x_1^2} = \text{Li}_{\pi_Y(x_0 x_1)}^- \text{Li}_{x_1} + 3 \text{Li}_{\pi_Y(x_1)}^- \text{Li}_0$ because

$$\forall k > 1, \theta_0^k \text{Li}_{x_1^2} = \sum_{j=0}^{k-1} \binom{k-1}{j} \text{Li}_{-j} \text{Li}_{2+j-k}.$$

The noncommutative generating series L satisfies the differential equation

$$dL = (x_0 \omega_0 + x_1 \omega_1) L \quad (16)$$

with boundary condition

$$L(z) \underset{z \rightarrow 0}{\rightsquigarrow} \exp(x_0 \log z) \text{ and } L(z) \underset{z \rightarrow 1}{\rightsquigarrow} \exp(-x_1 \log(1-z)) Z_{\sqcup}. \quad (17)$$

This implies that L is the exponential of a Lie series [39, 36]. Hence [42],

$$\log L = \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \sum_{u_1, \dots, u_k \in X^+} \text{Li}_{u_1 \sqcup \dots \sqcup u_k} u_1 \dots u_k = \sum_{w \in X^*} \text{Li}_w \pi_1(w).$$

Theorem 6 ([42]).

1. Let G, H be exponential solutions of (16). Then there exists a constant Lie series C such that $G = He^C$.
2. Let $\text{Gal}_{\mathbb{C}}(DE)$ be the differential Galois group associated to the Drinfel'd equation. Then $\text{Gal}_{\mathbb{C}}(DE) = \{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X \rangle\rangle\}$ and it contains the monodromy group defined by $\mathcal{M}_0 L = L \exp(2i\pi m_0)$ and $\mathcal{M}_1 L = LZ_{\sqcup}^{-1} \exp(-2i\pi x_1) Z_{\sqcup} = L \exp(2i\pi m_1)$, where $m_0 = x_0$, $m_1 = \prod_{l \in \mathcal{L}yn X - X} \exp(-\zeta(S_l) \text{ad}_{P_l})(-x_1)$.

Then let us put²¹ $\Lambda := \pi_Y L$ and [43]

²¹ Here, the coefficient $\langle B(y_1) \mid y_1^k \rangle$ corresponds to the Euler–Mac Laurin constant associated to $\langle \text{Const}(N) \mid y_1^k \rangle$, i.e. the finite part of its asymptotic expansion in the scale of comparison $\{n^a \log^b(n)\}_{a \in \mathbb{Z}, b \in \mathbb{N}}$.

$$\text{Mono}(z) := e^{-(x_1+1)\log(1-z)} = \sum_{k \geq 0} P_{y_1^k}(z) y_1^k \quad (18)$$

$$\text{Const} := \sum_{k \geq 0} H_{y_1^k} y_1^k = \exp\left(-\sum_{k \geq 1} H_{y_k} \frac{(-y_1)^k}{k}\right), \quad (19)$$

$$B(y_1) := \exp\left(\sum_{k \geq 1} \zeta(y_k) \frac{(-y_1)^k}{k}\right), \quad (20)$$

and finally, $B'(y_1) := \exp(\gamma y_1) B(y_1)$. Hence, we get $\pi_Y P(z) \underset{z \rightarrow 1}{\sim} \text{Mono}(z) \pi_Y Z_{\sqcup}$ and $H(N) \underset{N \rightarrow +\infty}{\sim} \text{Const}(N) \pi_Y Z_{\sqcup}$ as a consequence of (18)-(19). Or equivalently,

Theorem 7 (First global renormalizations of divergent polyzetas, [43]).

$$\lim_{z \rightarrow 1} \exp\left(-y_1 \log \frac{1}{1-z}\right) \Lambda(z) = \lim_{N \rightarrow +\infty} \exp\left(\sum_{k \geq 1} H_{y_k}(N) \frac{(-y_1)^k}{k}\right) H(N) = \pi_Y Z_{\sqcup}.$$

Theorem 8 ([19]). For any $g \in \mathcal{C}\{P_w\}_{w \in Y^*}$, there exist algorithmically computable coefficients $c_j, b_i \in \mathbb{C}, \alpha_j, \eta_i \in \mathbb{Z}, \beta_j, \kappa_i \in \mathbb{N}$ such that

$$g(z) \underset{z \rightarrow 1}{\sim} \sum_{j=0}^{+\infty} c_j (1-z)^{\alpha_j} \log^{\beta_j}(1-z), \quad \langle g(z) | z^n \rangle \underset{N \rightarrow +\infty}{\sim} \sum_{i=0}^{+\infty} b_i n^{\eta_i} \log^{\kappa_i}(n).$$

Theorem 8 means also that the $\{P_w\}_{w \in Y^*}$ admit a full singular expansion, at 1, and then their ordinary Taylor coefficients, $\{H_w\}_{w \in Y^*}$ admit a full asymptotic expansion, for $+\infty$. More precisely,

Corollary 1. For any $w \in X^*$ and for any $k, i, j \in \mathbb{N}, k \geq 1$, there exists uniquely determined coefficients $a_i, b_{i,j}$ belonging to \mathcal{L} ; $\gamma_{\pi_Y(w)}, \alpha_i$ and $\beta_{i,j}$ belonging to the $\mathbb{Q}[\gamma]$ -algebra generated by convergent polyzetas such that,

$$\text{Li}_w(z) = \sum_{i=1}^{|\mathbf{w}|} a_i \log^i(1-z) + \langle Z_{\sqcup} | w \rangle + \sum_{j=1}^k \sum_{i=0}^{|\mathbf{w}|-1} b_{i,j} \frac{\log^i(1-z)}{(1-z)^{-j}} + o_k^{(1)}((1-z)^k) \quad (21)$$

and, likely

$$H_{\pi_Y(w)}(N) = \sum_{i=1}^{|\mathbf{w}|} \alpha_i \log^i(N) + \gamma_{\pi_Y(w)} + \sum_{j=1}^k \sum_{i=0}^{|\mathbf{w}|-1} \beta_{i,j} \frac{1}{N^j} \log^i(N) + o_k^{(+\infty)}(N^{-k}). \quad (22)$$

Remark 2. i) The two expansions (21) and (22) are asymptotic expansions of Li_w and H_w with respect to the scales $(1-z)^n \log(1-z)^m$; $n, m \geq 0$ and $N^{-k} \log(N)^m$; $k, m \geq 0$ respectively.

ii) In (eq. 21), the error term $o_k^{(1)}((1-z)^k)$ can be put to the form $O_k^{(1)}((1-z)^{k+\varepsilon})$ for any $\varepsilon \in]0, 1[$.

More generally, by Theorem 6, we get

Proposition 9. For any commutative \mathbb{Q} -algebra A and for any Lie series $C \in \mathcal{L}ie_A \langle X \rangle$, we set $\bar{L} = Le^C$, $\bar{\Lambda} = \pi_Y \bar{L}$ and $\bar{P}(z) = (1-z)^{-1} \bar{\Lambda}(z)$, then

1. $\bar{Z}_{\sqcup} = Z_{\sqcup} e^C$ is group-like, for the co-product Δ_{\sqcup} ,
2. $\bar{L}(z) \underset{z \rightarrow 1}{\rightsquigarrow} \exp(-x_1 \log(1-z)) \bar{Z}_{\sqcup}$,
3. $\bar{P}(z) \underset{z \rightarrow 1}{\rightsquigarrow} \text{Mono}(z) \pi_Y \bar{Z}_{\sqcup}$,
4. $\bar{H}(N) \underset{N \rightarrow \infty}{\rightsquigarrow} \text{Const}(N) \pi_Y \bar{Z}_{\sqcup}$,

where, for any $w \in Y^*$ and $N \geq 0$, one defines the coefficient $\langle \bar{H}(N) \mid w \rangle$ of w in the power series $\bar{H}(N)$ as the coefficient $\langle \bar{P}_w(z) \mid z^N \rangle$ of z^N in the ordinary Taylor expansion of the polylogarithmic function $\bar{P}_w(z)$.

By Proposition 9, we get successively

Proposition 10 ([37]). Let $\bar{\zeta}_{\sqcup}$ and $\bar{\zeta}_{\sqcup\sqcup}$ be the characters of respectively $(A \langle X \rangle, \sqcup)$ and $(A \langle Y \rangle, \sqcup\sqcup)$ satisfying $\bar{\zeta}_{\sqcup}(x_0) = \bar{\zeta}_{\sqcup}(x_1) = 0$ and $\bar{\zeta}_{\sqcup\sqcup}(y_1) = 0$. Then

$$\sum_{w \in Y^*} \bar{\zeta}_{\sqcup}(w) w = \bar{Z}_{\sqcup} = \prod_{l \in \mathcal{L}ynX-X} \exp(\bar{\zeta}(S_l) P_l),$$

$$\sum_{w \in Y^*} \bar{\zeta}_{\sqcup\sqcup}(w) w = \bar{Z}_{\sqcup\sqcup} = \prod_{l \in \mathcal{L}ynY-\{y_1\}} \exp(\bar{\zeta}(\Sigma_l) \Pi_l).$$

Proposition 11. Let $\{\bar{\gamma}_w\}_{w \in Y^*}$ be the Euler–Mac Laurin constants associated to $\{\bar{H}_w(N)\}_{w \in Y^*}$. Let \bar{Z}_γ be the noncommutative generating series of these constants. Then,

1. The following map realizes a character :

$$\bar{\gamma}_\bullet : (A \langle Y \rangle, \sqcup\sqcup) \longrightarrow (\mathbb{R}, \cdot), \quad w \longmapsto \langle \bar{\gamma}_\bullet \mid w \rangle = \bar{\gamma}_w.$$

2. The noncommutative power series \bar{Z}_γ is group-like, for $\Delta_{\sqcup\sqcup}$.
3. There exists a group-like element $\bar{Z}_{\sqcup\sqcup}$, for the co-product $\Delta_{\sqcup\sqcup}$, such that

$$\bar{Z}_\gamma = \sum_{w \in Y^*} \bar{\gamma}_w w = \exp(\gamma y_1) \bar{Z}_{\sqcup\sqcup}.$$

By Theorem 7, Propositions 9 and 11, we also get

Proposition 12. For any $C \in \mathcal{L}ie_A \langle X \rangle$ such that $\bar{Z}_{\sqcup} = Z_{\sqcup} e^C$. Then

$$\bar{Z}_\gamma = B(y_1) \pi_Y \bar{Z}_{\sqcup}, \text{ or equivalently by cancellation, } \bar{Z}_{\sqcup\sqcup} = B'(y_1) \pi_Y \bar{Z}_{\sqcup},$$

where $B(y_1)$ and $B'(y_1)$ are given in (20).

By Proposition 9, the noncommutative generating series \bar{Z}_{\sqcup} and $\bar{Z}_{\sqcup\sqcup}$ are group-like, for the co-product Δ_{\sqcup} and $\Delta_{\sqcup\sqcup}$ respectively, and we also have

$$\begin{aligned}\bar{Z}_{\sqcup} &= \sum_{l \in \mathcal{L}_{ynX-X}} \bar{\zeta}(S_l) P_l + \sum_{w \notin \mathcal{L}_{ynX-X}} \bar{\zeta}_{\sqcup}(S_w) P_w, \\ \bar{Z}_{\sqcup} &= \sum_{l \in \mathcal{L}_{ynY-\{y_1\}}} \bar{\zeta}(\Sigma_l) \Pi_l + \sum_{w \notin \mathcal{L}_{ynY-\{y_1\}}} \bar{\zeta}_{\sqcup}(\Sigma_w) \Pi_w.\end{aligned}$$

Hence, by Proposition 12, we deduce in particular,

$$\sum_{l \in \mathcal{L}_{ynY-\{y_1\}}} \bar{\zeta}(\Sigma_l) \Pi_l + \dots = B'(y_1) \left(\sum_{l \in \mathcal{L}_{ynX-X}} \bar{\zeta}(\pi_Y S_l) \pi_Y P_l + \dots \right).$$

The elements of $\{\pi_Y P_l\}_{l \in \mathcal{L}_{ynX}}$ are decomposable in the linear basis $\{\Pi_w\}_{w \in Y^*}$ of $\mathcal{U}(\text{Prim}(\mathcal{H}_{\sqcup}))$. Thus, by identification of local coordinates, *i.e.* the coefficients of $\{\Pi_l\}_{l \in \mathcal{L}_{ynY-\{y_1\}}}$ in the basis $\{\Sigma_l\}_{l \in \mathcal{L}_{ynY-\{y_1\}}}$, we get homogenous polynomial relations on polyzetas encoded by $\{\Sigma_l\}_{l \in \mathcal{L}_{ynY-\{y_1\}}}$ [44].

Proposition 13. *There exist A, B and $C \in \mathbb{Q}\langle Y_0 \rangle$ such that*

$$L^-(z) \underset{z \rightarrow 1}{\rightsquigarrow} A \odot g\left(\frac{1}{1-z}\right), P^-(z) \underset{z \rightarrow 1}{\rightsquigarrow} B \odot \frac{1}{1-z} g\left(\frac{1}{1-z}\right), H^-(N) \underset{N \rightarrow +\infty}{\rightsquigarrow} C \odot g(N).$$

where the series g, h were defined in the definition 5.

Proof. By Propositions 5, for $w = y_{s_1} \dots y_{s_r}$, there exists $a, b, c \in \mathbb{Q}$ such that

$$\text{Li}_w^-(z) \underset{z \rightarrow 1}{\rightsquigarrow} \frac{a}{(1-z)^{|w|+(w)}}, P_w^-(z) \underset{z \rightarrow 1}{\rightsquigarrow} \frac{b}{(1-z)^{|w|+(w)+1}}, H_w^-(N) \underset{N \rightarrow +\infty}{\rightsquigarrow} cN^{|w|+(w)}.$$

Putting $\langle A | w \rangle = (-1)^{|w|} a, \langle B | w \rangle = (-1)^{|w|} b, \langle C | w \rangle = (-1)^{|w|} c$, it follows the expected results.

Proposition 14. [16] *For any $w \in Y_0^*$, there are non-zero constants, namely C_w^- and B_w^- , which only depend on w and r such that*

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{H_w^-(N)}{N^{(w)+|w|} C_w^-} &= 1, \text{ i.e. } H_w^-(N) \underset{N \rightarrow +\infty}{\rightsquigarrow} N^{(w)+|w|} C_w^-, \\ \lim_{z \rightarrow 1^-} \frac{(1-z)^{(w)+|w|} \text{Li}_w^-(z)}{B_w^-} &= 1, \text{ i.e. } \text{Li}_w^-(z) \underset{z \rightarrow 1}{\rightsquigarrow} \frac{N^{(w)+|w|} B_w^-}{(1-z)^{n+1}}.\end{aligned}$$

Moreover, C_w^- and B_w^- are well determined by

$$C_w^- = \prod_{w=uv; v \neq 1_{Y_0^*}} \frac{1}{(v)_+ + |v|} \in \mathbb{Q} \text{ and } B_w^- = ((w)_+ + |w|)! C_w^- \in \mathbb{N}.$$

Example 14. [16][of C_w^- and B_w^-]

w	C_w^-	B_w^-	w	C_w^-	B_w^-
y_0	1	1	$y_1 y_2$	1/15	8
y_1	1/2	1	$y_2 y_3$	1/28	180
y_2	1/3	2	$y_3 y_4$	1/49	8064
y_n	$1/(n+1)$	$n!$	$y_m y_n$	$1/[(n+1)(m+n+2)]$	$n! m! \binom{m+n+1}{n+1}$
y_0^2	1/2	1	$y_2 y_2 y_3$	1/280	12960
y_0^3	$1/(n!)$	1	$y_2 y_{10} y_1^2$	1/2160	9686476800
y_1^2	1/8	3	$y_2^2 y_4 y_3 y_{11}$	1/2612736	4167611825465088000000

Proposition 15. [16] Let $u, v \in Y_0^*$. We get $H_u^- H_v^- = H_{u \boxplus v}^-$.

Proof. Let $w \in Y_0^*$ associated to $s = (s_1, \dots, s_k)$. The quasi-symmetric monomial functions on the commutative alphabet $t = \{t_i\}_{i \geq 1}$ are defined as follows

$$M_{1Y_0^*}(t) = 1 \text{ and } M_w(t) = \sum_{n_1 > \dots > n_k > 0} t_{n_1}^{s_1} \dots t_{n_k}^{s_k},$$

For any $u, v \in Y_0^*$, we have $M_u(t)M_v(t) = M_{u \boxplus v}(t)$. Then, the harmonic sum $H_{s_1, \dots, s_k}^-(N)$ is obtained by specializing the indeterminates $t = \{t_i\}_{i \geq 1}$ from $M_w(t)$ as follows: $t_i = i$ for $1 \leq i \leq N$ and $t_i = 0$ for $N < i$.

Theorem 9 (Second global renormalizations of divergent polyzetas).

1. The generating series H^- is group-like and $\log H^-$ is primitive. Moreover²²,

$$\lim_{N \rightarrow +\infty} g^{\odot -1}(N) \odot H^-(N) = \lim_{z \rightarrow 1} h^{\odot -1}((1-z)^{-1}) \odot L^-(z) = C^-.$$

2. $\ker H_{\bullet}^-$ is a prime ideal of $(\mathbb{Q}\langle Y_0 \rangle, \boxplus)$, i.e. $\mathbb{Q}\langle Y_0 \rangle \setminus \ker H_{\bullet}^-$ is closed by \boxplus .

Proof. The first result is a consequence of the extended Friedrichs criterion [6, 44, 45] and the second is a consequence of Proposition 13.

Definition 6. For any $n \in \mathbb{N}_+$, let $\mathbb{P}_n := \text{span}_{\mathbb{R}_+} \{w \in Y_0^* \mid (w) + |w| = n\} \setminus \{0\}$ be the blunt²³ convex cone generated by the set $\{w \in Y_0^* \mid (w) + |w| = n\}$.

By definition, C_{\bullet}^- is linear on the set \mathbb{P}_n . For any $u, v \in Y_0^*$, one has $u \boxplus v = u \boxdot v + \sum_{\substack{|w| < |u| + |v| \\ (w) = (u) + (v)}} x_w w$ and the x_w 's are positive. Moreover, for any w which belongs to the support of $\sum_{\substack{|w| < |u| + |v| \\ (w) = (u) + (v)}} x_w w$, one has $(w) + |w| < (u) + (v) + |u| + |v|$, thus, by the definition of C_{\bullet}^- , one obtains

Corollary 2. 1. Let $w, v \in Y_0^*$. Then $C_w^- C_v^- = C_{w \boxdot v}^- = C_{w \boxplus v}^-$.

2. For any $P, Q \notin \ker H_{\bullet}^-$, $C_P^- C_Q^- = C_{P \boxplus Q}^-$ and $\mathbb{Q}\langle Y_0 \rangle \setminus \ker H_{\bullet}^-$ is a \boxplus -multiplicative monoid containing Y_0^* .

Now, let us prove that C_{\bullet}^- can be extended as a character, for \boxdot , or equivalently, C^- is group-like (see the Friedrichs' criterion [51]) and then $\log C^-$ is primitive.

²² Here, the Hadamard product is denoted by \odot and its dual law, the diagonal comultiplication is denoted by Δ_{\odot} . The series g, h are defined in Definition 5.

²³ i.e. without zero or see Appendix A.

Lemma 2. Let \mathcal{A} be an unitary \mathbb{R} -associative algebra and $f : \sqcup_{n \geq 0} \mathbb{P}_n \rightarrow \mathcal{A}$ such that

1. For any $u, v \in Y_0^*$, $f(u \sqcup v) = f(u)f(v)$. In particular, $f(1_{Y_0^*}) = 1_{\mathcal{A}}$.
2. On every \mathbb{P}_n , one has $f(\sum_{i \in I} \alpha_i w_i) = \sum_{i \in I} \alpha_i f(w_i)$, where $\alpha_i \in \mathbb{R}_+^*$.

Then f can be extended uniquely as a character, i.e. $S_f = \sum_{w \in Y_0^*} f(w)w$ is group-like for Δ_{\sqcup} .

Proof. By definition of f and S_f , it is immediate $\langle S_f | 1_{Y_0^*} \rangle = 1_{\mathcal{A}}$. One can check easily that $\Delta_{\sqcup}(S_f) = S_f \otimes S_f$. Hence, S_f is group-like, for Δ_{\sqcup} .

Corollary 3. The noncommutative generating series C^- is group-like, for Δ_{\sqcup} .

Proof. It is a consequence of Lemma 2 and Corollary 2.

Example 15. [16][of $C_{u \sqcup v}^-$ and $C_{u \sqcup \sqcup v}^-$] Let $Y_0 = \{y_i\}_{i \geq 0}$ be an infinite alphabet.

u	C_u^-	v	C_v^-	$u \sqcup v$	$C_{u \sqcup v}^-$
y_0	1	y_0	1	$2y_0^2$	1
y_0^2	1/2				
y_1	1/2	y_2	1/3	$y_1y_2 + y_2y_1$	1/6
y_1y_2	1/15	y_2y_1	1/10		
y_m	$(m+1)^{-1}$	y_n	$(n+1)^{-1}$	$y_my_n + y_ny_m$	$[(m+1)(n+1)]^{-1}$
y_my_n	$\frac{(n+1)^{-1}}{(n+m+2)}$	y_ny_m	$\frac{(m+1)^{-1}}{(m+n+2)}$		
y_1	1/2	y_2y_5	1/54	$y_1y_2y_5 + y_2y_1y_5 + y_2y_5y_1$	1/108
$y_1y_2y_5$	1/594	$y_2y_1y_5$	1/528		
$y_2y_5y_1$	1/176				
y_0y_1	1/6	y_2y_3	1/28	$y_0y_1y_2y_3 + y_0y_2y_1y_3 + y_0y_2y_3y_1 + y_2y_3y_0y_1 + y_2y_0y_1y_3 + y_2y_0y_3y_1$	1/168
$y_0y_1y_2y_3$	1/2520	$y_0y_2y_1y_3$	1/2160		
$y_0y_2y_3y_1$	1/1080	$y_2y_3y_0y_1$	1/420		
$y_2y_0y_1y_3$	1/1680	$y_2y_0y_3y_1$	1/840		
$yaYb$	$\frac{(b+1)^{-1}}{(a+b+2)}$	$YcYd$	$\frac{(d+1)^{-1}}{(c+d+2)}$	$YaYbYcYd + YaYcYbYd + YaYcYdYb + YcYdYaYb + YcYaYbYd + YcYaYdYb$	$\frac{(b+1)^{-1}(d+1)^{-1}}{(a+b+2)(c+d+2)}$

u	C_u^-	v	C_v^-	$u \sqcup \sqcup v$	$C_{u \sqcup \sqcup v}^-$
y_0	1	y_0	1	$2y_0^2 + y_0$	1
y_1	1/2	y_2	1/3	$y_1y_2 + y_2y_1 + y_3$	1/6
y_m	$(m+1)^{-1}$	y_n	$(n+1)^{-1}$	$y_my_n + y_ny_m + y_{n+m}$	$[(m+1)(n+1)]^{-1}$
y_1	1/2	y_2y_5	1/54	$y_1y_2y_5 + y_2y_1y_5 + y_2y_5y_1 + y_3y_5 + y_2y_6$	1/108
y_0y_1	1/6	y_2y_3	1/28	$y_0y_1y_2y_3 + y_0y_2y_1y_3 + y_0y_2y_3y_1 + y_2y_3y_0y_1 + y_2y_0y_1y_3 + y_2y_0y_3y_1 + y_0y_2y_4 + y_0y_3^2 + y_2y_3y_1 + y_2y_1y_3 + y_2y_0y_4 + y_2y_3y_1 + y_2y_4$	1/168
$YaYb$	$\frac{(b+1)^{-1}}{(a+b+2)}$	$YcYd$	$\frac{(d+1)^{-1}}{(c+d+2)}$	$YaYbYcYd + YaYcYbYd + YaYcYdYb + YcYdYaYb + YcYaYbYd + YcYaYdYb + YaYcYb+d + YaYb+cYd + YcYaYb+d + YcYa+dYb + Ya+cYbYd + Ya+cYdYb + Ya+cYb+d$	$\frac{(b+1)^{-1}(d+1)^{-1}}{(a+b+2)(c+d+2)}$

In the above tables, it is clearly seen that C_{\bullet}^{-} is linear on \mathbb{P}_n . For example, let $u = y_1$ and $v = y_2 y_5$. Then $u \sqcup v = y_1 y_2 y_5 + y_2 y_1 y_5 + y_2 y_5 y_1$. Hence, we get $C_{y_1 y_2 y_5}^{-} + C_{y_2 y_1 y_5}^{-} + C_{y_2 y_5 y_1}^{-} = \frac{1}{594} + \frac{1}{528} + \frac{1}{176} = \frac{1}{108} = C_{y_1}^{-} C_{y_2 y_5}^{-} = C_{y_1 \sqcup y_2 y_5}^{-}$. Note that $y_1 y_2 y_5, y_2 y_1 y_5, y_2 y_5 y_1 \in \mathbb{P}_{11}$. But we have also $u \sqcup v = y_1 y_2 y_5 + y_2 y_1 y_5 + y_2 y_5 y_1 + y_3 y_5 + y_2 y_6$. Moreover, $C_{y_1 y_2 y_5}^{-} + C_{y_2 y_1 y_5}^{-} + C_{y_2 y_5 y_1}^{-} + C_{y_3 y_5}^{-} + C_{y_2 y_6}^{-} = \frac{1}{108} + \frac{13}{420} \neq \frac{1}{108} = C_{y_1}^{-} C_{y_2 y_5}^{-}$. However, from $y_3 y_5, y_2 y_6 \in \mathbb{P}_{10}$, we can conclude that

$$C_{y_1 \sqcup y_2 y_5}^{-} = C_{y_1 y_2 y_5 + y_2 y_1 y_5 + y_2 y_5 y_1 + y_3 y_5 + y_2 y_6}^{-} = C_{y_1 y_2 y_5 + y_2 y_1 y_5 + y_2 y_5 y_1}^{-} = 1/108 = C_{y_1}^{-} C_{y_2 y_5}^{-}.$$

3 Polysystems and differential realization

3.1 Polysystems and convergence criterion

3.1.1 Estimates (from above) for series

Here, $(\mathbb{K}, \|\cdot\|)$ is a normed space.

Definition 7. ([29, 43]) Let ξ, χ be real positive functions over X^* . Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$.

1. S will be said ξ -exponentially bounded from above if it satisfies

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \|\langle S | w \rangle\| \leq K \xi(w) |w|!.$$

We denote by $\mathbb{K}^{\xi\text{-em}}\langle\langle X \rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X \rangle\rangle$ which are ξ -exponentially bounded from above.

2. S satisfies the χ -growth condition if it satisfies

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \|\langle S | w \rangle\| \leq K \chi(w) |w|!.$$

We denote by $\mathbb{K}^{\chi\text{-gc}}\langle\langle X \rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X \rangle\rangle$ satisfying the χ -growth condition.

Lemma 3. If $R = \sum_{w \in X^*} |w|! w$ then $\langle R^{\sqcup^2} | w \rangle = \sum_{\substack{u, v \in X^* \\ \text{supp}(u \sqcup v) \ni w}} |u|! |v|! \leq 2^{|w|} |w|!$.

Proof. One has

$$\sum_{\substack{u, v \in X^* \\ \text{supp}(u \sqcup v) \ni w}} |u|! |v|! = \sum_{k=0}^{|w|} \sum_{\substack{|u|=k, |v|=|w|-k \\ \text{supp}(u \sqcup v) \ni w}} k! (|w| - k)! = \sum_{k=0}^{|w|} \binom{|w|}{k} k! (|w| - k)! = \sum_{k=0}^{|w|} |w|!.$$

The last sum is equal to $(1 + |w|) |w|!$. By induction on $|w|$, one has $1 + |w| \leq 2^{|w|}$. Then the expected result follows.

Proposition 16. If S_1, S_2 satisfy the growth condition then $S_1 + S_2, S_1 \sqcup S_2$ do also.

Proof. It is immediate for $S_1 + S_2$. Next, since $\|\langle S_i | w \rangle\| \leq K_i \chi_i(w) |w|!$ then

$$\begin{aligned} \langle S_1 \sqcup S_2 | w \rangle &= \sum_{\text{supp}(u \sqcup v) \ni w} \langle S_1 | u \rangle \langle S_2 | v \rangle, \\ \Rightarrow \|\langle S_1 \sqcup S_2 | w \rangle\| &\leq K_1 K_2 \sum_{\substack{u, v \in X^* \\ \text{supp}(u \sqcup v) \ni w}} (\chi_1(u) |u|!) (\chi_2(v) |v|!). \end{aligned}$$

Let $K = K_1 K_2$ and let χ be a real positive function over X^* such that, for any $w \in X^*$

$$\chi(w) = \max\{\chi_1(u) \chi_2(v) \mid u, v \in X^* \text{ and } \text{supp}(u \sqcup v) \ni w\}.$$

With the notations in Lemma 3, we get $\|\langle S_1 \sqcup S_2 | w \rangle\| \leq K \chi(w) \langle S_1 R^{\sqcup 2} | w \rangle$. Hence, $S_1 \sqcup S_2$ satisfies the χ' -growth condition with $\chi'(w) = 2^{|w|} \chi(w)$.

Definition 8. ([29, 43]) Let ξ be a real positive function defined over X^* , S will be said ξ -exponentially continuous if it is continuous over $\mathbb{K}^{\xi-\text{em}}\langle\langle X \rangle\rangle$. The set of formal power series which are ξ -exponentially continuous is denoted by $\mathbb{K}^{\xi-\text{ec}}\langle\langle X \rangle\rangle$.

Lemma 4. [29, 43] For any real positive function ξ defined over X^* , we have $\mathbb{K}\langle X \rangle \subset \mathbb{K}^{\xi-\text{ec}}\langle\langle X \rangle\rangle$. Otherwise, for $\xi = 0$, we get $\mathbb{K}\langle X \rangle = \mathbb{K}^{0-\text{ec}}\langle\langle X \rangle\rangle$. Hence, any polynomial is 0-exponentially continuous.

Proposition 17 ([29, 43]). Let ξ, χ be real positive functions over X^* and $P \in \mathbb{K}\langle X \rangle$.

1. Let $S \in \mathbb{K}^{\xi-\text{em}}\langle\langle X \rangle\rangle$. The right residual of S by P belongs to $\mathbb{K}^{\xi-\text{em}}\langle\langle X \rangle\rangle$.
2. Let $R \in \mathbb{K}^{\chi-\text{gc}}\langle\langle X \rangle\rangle$. The concatenation SR belongs to $\mathbb{K}^{\chi-\text{gc}}\langle\langle X \rangle\rangle$.
3. Moreover, if ξ and χ are morphisms over X^* satisfying $\sum_{x \in X} \chi(x) \xi(x) < 1$ then, for any $F \in \mathbb{K}^{\chi-\text{gc}}\langle\langle X \rangle\rangle$, F is continuous over $\mathbb{K}^{\xi-\text{em}}\langle\langle X \rangle\rangle$.

Proof. 1. Since $S \in \mathbb{K}^{\xi-\text{em}}\langle\langle X \rangle\rangle$ then

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \|\langle S | w \rangle\| \leq K \xi(w) / |w|!$$

If $u \in \text{supp}(P)$ then, for any $w \in X^*$, one has $\langle S \triangleright u | w \rangle = \langle S | uw \rangle$ and $S \triangleright u$ belongs to $\mathbb{K}^{\xi-\text{em}}\langle\langle X \rangle\rangle$:

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \|\langle S \triangleright u | w \rangle\| \leq [K \xi(u)] \xi(w) / |w|!$$

It follows that $S \triangleright P$ is $\mathbb{K}^{\xi-\text{em}}\langle\langle X \rangle\rangle$ by taking $K_1 = K \max_{u \in \text{supp}(P)} \xi(u)$.

2. Since $R \in \mathbb{K}^{\chi-\text{gc}}\langle\langle X \rangle\rangle$ then

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \|\langle S | w \rangle\| \leq K \chi(w) |w|!$$

Let $v \in \text{supp}(P)$ such that $v \neq \varepsilon$. Since Rv belongs to $\mathbb{K}^{\chi-\text{gc}}\langle\langle X \rangle\rangle$ and one has, for $w \in X^*$, $\langle Rv | w \rangle = \langle R | v \triangleleft w \rangle$, i.e. there exists $K \in \mathbb{R}_+, n \in \mathbb{N}$ such that

$$\|\langle R | v \triangleleft w \rangle\| \leq K \chi(v \triangleleft w) (|w| - |v|)! \leq K |w| \chi(w) / \chi(v).$$

Note if $v \triangleleft w = 0$ then $\langle Rv | w \rangle = 0$ and the previous conclusion holds. It follows that RP is $\mathbb{K}^{\mathcal{X}\text{-gc}}\langle\langle X \rangle\rangle$ by taking $K_2 = K \min_{v \in \text{supp}(P)} \chi(v)^{-1}$.

3. Let ξ, χ be functions which satisfy the upper bound condition. The following quantity is well defined

$$\sum_{w \in X^*} \chi(w) \xi(w) = \left(\sum_{x \in X} \chi(x) \xi(x) \right)^*.$$

If $F \in \mathbb{K}^{\mathcal{X}\text{-gc}}\langle\langle X \rangle\rangle, C \in \mathbb{K}^{\xi\text{-em}}\langle\langle X \rangle\rangle$ then there exist $K_i \in \mathbb{R}_+, n_i \in \mathbb{N}, i = 1, 2$ such that, for $w \in X^{\geq n_i}, \| \langle F | w \rangle \| \leq K_1 \chi(w) |w|!$ and $\| \langle C | w \rangle \| \leq K_2 \xi(w) |w|!$. Thus,

$$\begin{aligned} \forall w \in X^*, |w| \geq \max\{n_1, n_2\}, \quad \| \langle F | w \rangle \langle C | w \rangle \| &\leq K_1 K_2 \chi(w) \xi(w), \\ \Rightarrow \sum_{w \in X^*} \| \langle F | w \rangle \langle C | w \rangle \| &\leq K_1 K_2 \sum_{w \in X^*} \chi(w) \xi(w) = K_1 K_2 \left(\sum_{x \in X} \chi(x) \xi(x) \right)^*. \end{aligned}$$

3.1.2 Upper bounds à la Cauchy

The algebra of formal power series on commutative indeterminates $\{q_1, \dots, q_n\}$ with coefficients in \mathbb{C} is denoted by $\mathbb{C}[[q_1, \dots, q_n]]$.

Definition 9. ([29, 43]) Let $f = \in \mathbb{C}[[q_1, \dots, q_n]]$. We set

$$\begin{aligned} E(f) &:= \{ \rho \in \mathbb{R}_+^n : \exists C_f \in \mathbb{R}_+ \text{ s.t. } \forall i_1, \dots, i_n \geq 0, |f_{i_1, \dots, i_n}| \rho_1^{i_1} \dots \rho_n^{i_n} \leq C_f \}. \\ \check{E}(f) &:= \text{the interior of } E(f) \text{ in } \mathbb{R}^n. \\ \text{CV}(f) &:= \{ q \in \mathbb{C}^n : (|q_1|, \dots, |q_n|) \in \check{E}(f) \} : \text{the convergence domain of } f. \end{aligned}$$

f is *convergent* if $\text{CV}(f) \neq \emptyset$. Let $\mathcal{U} \subset \mathbb{C}^n$ be an open domain and $q \in \mathbb{C}^n$. f is *convergent on q* (resp. over \mathcal{U}) if $q \in \text{CV}(f)$ (resp. $\mathcal{U} \subset \text{CV}(f)$). We set $\mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]] := \{ f \in \mathbb{C}[[q_1, \dots, q_n]] : \text{CV}(f) \neq \emptyset \}$. Let $q \in \text{CV}(f)$. There exist $C_f \in \mathbb{R}_+, \rho \in E(f), \bar{\rho} \in \check{E}(f)$ such that $|q_1| < \bar{\rho}_1 < \rho_1, \dots, |q_n| < \bar{\rho}_n < \rho_n$ and $|f_{i_1, \dots, i_n}| \rho_1^{i_1} \dots \rho_n^{i_n} \leq C_f$, for $i_1, \dots, i_n \geq 0$.

The *convergence modulus* of f at q is $(C_f, \rho, \bar{\rho})$.

Suppose $\text{CV}(f) \neq \emptyset$ and let $q \in \text{CV}(f)$. If $(C_f, \rho, \bar{\rho})$ is a convergence modulus of f at q then $|f_{i_1, \dots, i_n} q_1^{i_1} \dots q_n^{i_n}| \leq C_f (\bar{\rho}_1 / \rho_1)^{i_1} \dots (\bar{\rho}_1 / \rho_1)^{i_n}$. Hence, at q , f is majored termwise by $C_f \prod_{k=0}^m (1 - \bar{\rho}_k / \rho_k)^{-1}$ and it is uniformly absolutely convergent in $\{ q \in \mathbb{C}^n : |q_1| < \bar{\rho}_1, \dots, |q_n| < \bar{\rho}_n \}$ which is open in \mathbb{C}^n . Thus, $\text{CV}(f)$ is open in \mathbb{C}^n . Since the partial derivation $D_1^{j_1} \dots D_n^{j_n} f$ is estimated by

$$\| D_1^{j_1} \dots D_n^{j_n} f \| \leq C_f \frac{\partial^{j_1 + \dots + j_n}}{\partial^{j_1} \bar{\rho}_1 \dots \partial^{j_n} \bar{\rho}_n} \prod_{k=0}^m \left(1 - \frac{\bar{\rho}_k}{\rho_k} \right)^{-1}.$$

Proposition 18. ([29]) We have $\text{CV}(f) \subset \text{CV}(D_1^{j_1} \dots D_n^{j_n} f)$.

Let $f \in \mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]]$. Let $\{A_i\}_{i=0,1}$ be a polysystem defined as follows

$$A_i = \sum_{j=1}^n A_i^j \frac{\partial}{\partial q_j}, \forall j = 1, \dots, n, A_i^j(q) \in \mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]]. \quad (23)$$

Let $(\rho, \bar{\rho}, C_f), \{(\rho, \bar{\rho}, C_i)\}_{i=0,1}$ be convergence modulus at $q \in \text{CV}(f) \cap_{i=0,1, j=1, \dots, n} \text{CV}(A_i^j)$ of f and $\{A_i^j\}_{j=1, \dots, n}$. Let us consider the following monoid morphisms

$$\mathcal{A}(1_{X^*}) = \text{identity and } C(1_{X^*}) = 1, \quad (24)$$

$$\forall w = vx_i, x_i \in X, v \in X^*, \quad \mathcal{A}(w) = \mathcal{A}(v)A_i \text{ and } C(w) = C(v)C_i. \quad (25)$$

Lemma 5. ([25]) For $i = 0, 1$ and $j = 1, \dots, n$, one has $A_i \circ q_j = A_i^j$. Hence,

$$\forall i = 0, 1, A_i = \sum_{j=1}^n (A_i \circ q_j) \frac{\partial}{\partial q_j}.$$

Lemma 6. ([23]) For any word w , $\mathcal{A}(w)$ is continuous over $\mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]]$ and, for any $f, g \in \mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]]$, one has

$$\mathcal{A}(w) \circ (fg) = \sum_{u, v \in X^*} \langle u \sqcup v \mid w \rangle (\mathcal{A}(u) \circ f) (\mathcal{A}(v) \circ g).$$

These notations are extended, by linearity, to $\mathbb{K}\langle X \rangle$ and we will denote $\mathcal{A}(w) \circ f|_q$ the evaluation of $\mathcal{A}(w) \circ f$ at q .

Definition 10. ([23]) Let $f \in \mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]]$. The generating series of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f is given by

$$\sigma f := \sum_{w \in X^*} \mathcal{A}(w) \circ f w \in \mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]] \langle\langle X \rangle\rangle.$$

Then the following generating series is called *Fliess generating series* of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f at q :

$$\sigma f|_q := \sum_{w \in X^*} \mathcal{A}(w) \circ f|_q w \in \mathbb{C} \langle\langle X \rangle\rangle.$$

Lemma 7. ([23]) The map $\sigma : (\mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]], \cdot) \longrightarrow (\mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]] \langle\langle X \rangle\rangle, \sqcup)$ is an algebra morphism, i.e. for any $f, g \in \mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]]$ and $\mu, \nu \in \mathbb{C}$, one has $\sigma(\nu f + \mu g) = \nu \sigma f + \mu \sigma g$ and $\sigma(fg) = \sigma f \sqcup \sigma g$.

Lemma 8. ([25]) For any $w \in X^*$, $\sigma(\mathcal{A}(w) \circ f) = w \triangleleft \sigma f \in \mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]] \langle\langle X \rangle\rangle$.

Theorem 10. ([29])

1. Let $\tau = \min_{1 \leq k \leq n} \rho_k$ and $r = \max_{1 \leq k \leq n} \bar{\rho}_k / \rho_k$. We have

$$\begin{aligned} \|\mathcal{A}(w) \circ f\| &\leq C_f \frac{(n+1)}{(1-r)^n} \frac{C(w)}{\binom{n+|w|-1}{|w|}} \left[\frac{n}{\tau(1-r)^{n+1}} \right]^{|w|} \\ &\leq C_f \frac{(n+1)}{(1-r)^n} C(w) \left[\frac{n}{\tau(1-r)^{n+1}} \right]^{|w|} |w|!. \end{aligned}$$

2. Let $K = C_f(n+1)(1-r)^{-n}$ and χ be the real positive function defined over X^* :

$$\forall i = 0, 1, \quad \chi(x_i) = C_i n (1-r)^{-(n+1)} / \tau.$$

Then²⁴ the generating series σf of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f satisfies the χ -growth condition.

3.2 Polysystem and nonlinear differential equation

3.2.1 Nonlinear differential equation (with three singularities)

Let us consider the singular inputs²⁵ $u_0(z) := z^{-1}$ and $u_1(z) := (1-z)^{-1}$, and

$$\begin{cases} y(z) = f(q(z)), \\ \dot{q}(z) = A_0(q) u_0(z) + A_1(q) u_1(z), \\ q(z_0) = q_0, \end{cases} \quad (26)$$

where the state $q = (q_1, \dots, q_n)$ belongs to a complex analytic manifold of dimension n , q_0 is the initial state, the observation f belongs to $\mathbb{C}^{\text{cv}}[[q_1, \dots, q_n]]$ and $\{A_i\}_{i=0,1}$ is the polysystem defined on (23).

Definition 11. ([31]) The following power series is called *transport operator*²⁶ of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f

$$\mathcal{T} := \sum_{w \in X^*} \alpha_{z_0}^z(w) \mathcal{A}(w).$$

By the factorization of the monoid by Lyndon words, we have [31]

$$\mathcal{T} = (\alpha_{z_0}^z \otimes \mathcal{A}) \left(\sum_{w \in X^*} w \otimes w \right) = \prod_{l \in \mathcal{L}_{\text{yn}X}} \exp[\alpha_{z_0}^z(S_l) \mathcal{A}(P_l)].$$

The Chen generating series along the path $z_0 \rightsquigarrow z$, associated to ω_0, ω_1 is

²⁴ It is the same for the Fliess generating series $\sigma f|_q$ of $\{A_i\}_{i=0,1}$ and of f at q .

²⁵ These singular inputs are not included in the studies of Fliess motivated, in particular, by the renormalization of y at $+\infty$ [23, 25].

²⁶ It plays the rôle of the resolvent in Mathematics and the evolution operator in Physics.

$$S_{z_0 \rightsquigarrow z} := \sum_{w \in X^*} \langle S | w \rangle w \text{ with } \langle S | w \rangle = \alpha_{z_0}^z(w) \quad (27)$$

which solves the differential equation (16) with the initial condition $S_{z_0 \rightsquigarrow z_0} = 1$. Thus, $S_{z_0 \rightsquigarrow z}$ and $L(z)L(z_0)^{-1}$ satisfy the same differential equation taking the same value at z_0 and $S_{z_0 \rightsquigarrow z} = L(z)L(z_0)^{-1}$. Any Chen generating series $S_{z_0 \rightsquigarrow z}$ is group like [50] and depends only on the homotopy class of $z_0 \rightsquigarrow z$ [10]. The product of $S_{z_1 \rightsquigarrow z_2}$ and $S_{z_0 \rightsquigarrow z_1}$ is $S_{z_0 \rightsquigarrow z_2} = S_{z_1 \rightsquigarrow z_2} S_{z_0 \rightsquigarrow z_1}$. Let $\varepsilon \in]0, 1[$ and $z_i = \varepsilon \exp(i\beta_i)$, for $i = 0, 1$. We set $\beta = \beta_1 - \beta_0$. Let $\Gamma_0(\varepsilon, \beta_0)$ (resp. $\Gamma_1(\varepsilon, \beta_1)$) be the path turning around 0 (resp. 1) in the positive direction from z_0 to z_1 . By induction on the length of w , one has $|\langle S_{\Gamma_i(\varepsilon, \beta)} | w \rangle| = (2\varepsilon)^{|w|_{x_i}} \beta^{|w|} / |w|!$, where $|w|$ denotes the length of w and $|w|_{x_i}$ denotes the number of occurrences of letter x_i in w , for $i = 0$ or 1 . When ε tends to 0^+ , these estimations yield $S_{\Gamma_i(\varepsilon, \beta)} = e^{i\beta x_i} + o(\varepsilon)$. In particular, if $\Gamma_0(\varepsilon)$ (resp. $\Gamma_1(\varepsilon)$) is a circular path of radius ε turning around 0 (resp. 1) in the positive direction, starting at $z = \varepsilon$ (resp. $1 - \varepsilon$), then, by the noncommutative residue theorem [39, 36], we get

$$S_{\Gamma_0(\varepsilon)} = e^{2i\pi x_0} + o(\varepsilon) \text{ and } S_{\Gamma_1(\varepsilon)} = e^{-2i\pi x_1} + o(\varepsilon). \quad (28)$$

Finally, the asymptotic behaviors of L on (17) give [36, 39]

$$S_{\varepsilon \rightsquigarrow 1 - \varepsilon} \underset{\varepsilon \rightarrow 0^+}{\sim} e^{-x_1 \log \varepsilon} Z_{\perp\perp} e^{-x_0 \log \varepsilon}. \quad (29)$$

In other terms, $Z_{\perp\perp}$ is the regularized Chen generating series $S_{\varepsilon \rightsquigarrow 1 - \varepsilon}$ of differential forms ω_0 and ω_1 : $Z_{\perp\perp}$ is the noncommutative generating series of the finite parts of the coefficients of the Chen generating series $e^{x_1 \log \varepsilon} S_{\varepsilon \rightsquigarrow 1 - \varepsilon} e^{x_0 \log \varepsilon}$.

3.2.2 Asymptotic behavior via extended Fliess fundamental formula

Theorem 11. ([43]) $y(z) = \mathcal{T} \circ f|_{q_0} = \langle \sigma f|_{q_0} | S_{z_0 \rightsquigarrow z} \rangle$.

This extends then Fliess fundamental formula [23]. By Theorem 5, the expansions of the output y of nonlinear dynamical system with singular inputs follow

Corollary 4 (Combinatorics of Dyson series).

$$\begin{aligned} y(z) &= \sum_{w \in X^*} g_w(z) \mathcal{A}(w) \circ f|_{q_0} \\ &= \sum_{k \geq 0} \sum_{n_1, \dots, n_k \geq 0} g_{x_0^{n_1} x_1 \dots x_0^{n_k} x_1}(z) \text{ad}_{A_0}^{n_1} A_1 \dots \text{ad}_{A_0}^{n_k} A_1 e^{\log z A_0} \circ f|_{q_0} \\ &= \prod_{l \in \mathcal{L}_{ynX}} \exp \left(g_{S_l}(z) \mathcal{A}(P_l) \circ f|_{q_0} \right) \\ &= \exp \left(\sum_{w \in X^*} g_w(z) \mathcal{A}(\pi_1(w)) \circ f|_{q_0} \right), \end{aligned}$$

where, for any word w in X^* , g_w belongs to the polylogarithm algebra.

Since $S_{z_0 \rightsquigarrow z} = L(z)L(z_0)^{-1}$ and $\sigma f_{|q_0}, L(z_0)^{-1}$ are invariant by $\partial_z = d/dz$ and $\theta_0 = zd/dz$ then we get the n -th order differentiation of y , with respect to ∂_z and θ_0 :

$$\begin{aligned}\partial_z^n y(z) &= \langle \sigma f_{|q_0} \mid \partial^n S_{z_0 \rightsquigarrow z} \rangle = \langle \sigma f_{|q_0} \mid \partial_z^n L(z)L(z_0)^{-1} \rangle, \\ \theta_0^n y(z) &= \langle \sigma f_{|q_0} \mid \theta_0^n S_{z_0 \rightsquigarrow z} \rangle = \langle \sigma f_{|q_0} \mid \theta_0^n L(z)L(z_0)^{-1} \rangle.\end{aligned}$$

With the notations of Proposition 8, we get respectively

$$\begin{aligned}\partial_z^n y(z) &= \langle \sigma f_{|q_0} \mid [D_n(z)L(z)]L(z_0)^{-1} \rangle = \langle \sigma f_{|q_0} \triangleright D_n(z) \mid L(z)L(z_0)^{-1} \rangle, \\ \theta_0^n y(z) &= \langle \sigma f_{|q_0} \mid E_n(z)L(z)]L(z_0)^{-1} \rangle = \langle \sigma f_{|q_0} \triangleright E_n(z) \mid L(z)L(z_0)^{-1} \rangle.\end{aligned}$$

For $z_0 = \varepsilon \rightarrow 0^+$, the asymptotic behavior and the renormalization at $z = 1$ of $\partial_z^n y$ and $\theta_0^n y$ (or the asymptotic expansion and the renormalization of its Taylor coefficients at $+\infty$) are deduced from (29) and extend a little bit results of [43, 44] :

Corollary 5 (Asymptotic behavior of output).

1. The n -order differentiation of the output y of the system (26) is a \mathcal{C} -combination of the elements g belonging to the polylogarithm algebra and²⁷, for any $n \geq 0$,

$$\begin{aligned}\partial_z^n y(1) &\underset{\varepsilon \rightarrow 0^+}{\widetilde{\sim}} \sum_{w \in X^*} \langle \mathcal{A}(w) \circ f_{|q_0} \mid w \rangle \langle D_n(1 - \varepsilon) e^{-x_1 \log \varepsilon} Z_{\perp\perp} e^{-x_0 \log \varepsilon} \mid w \rangle, \\ \theta_0^n y(1) &\underset{\varepsilon \rightarrow 0^+}{\widetilde{\sim}} \sum_{w \in X^*} \langle \mathcal{A}(w) \circ f_{|q_0} \mid w \rangle \langle E_n(1 - \varepsilon) e^{-x_1 \log \varepsilon} Z_{\perp\perp} e^{-x_0 \log \varepsilon} \mid w \rangle.\end{aligned}$$

2. If the ordinary Taylor expansions of $\partial_z^n y$ and $\theta_0^n y$ exist then the coefficients of these expansions belong to the algebra of harmonic sums and there exist algorithmically computable coefficients $a_i, a'_i \in \mathbb{Z}, b_i, b'_i \in \mathbb{N}, c_i, c'_i \in \mathcal{X}[\gamma]$ such that

$$\begin{aligned}\partial_z^n y(z) &= \sum_{k \geq 0} d_k z^n \text{ and } d_k \underset{k \rightarrow \infty}{\widetilde{\sim}} \sum_{i \geq 0} c_i k^{a_i} \log^{b_i} k, \\ \theta_0^n y(z) &= \sum_{k \geq 0} t_k z^k \text{ and } t_k \underset{k \rightarrow \infty}{\widetilde{\sim}} \sum_{i \geq 0} c'_i k^{a'_i} \log^{b'_i} k.\end{aligned}$$

²⁷ Moreover, we get more out of this i.e. $\theta_1^n y(z) = \langle \sigma f_{|q_0} \parallel \theta_1^n S_{z_0 \rightsquigarrow z} \rangle = \langle \sigma f_{|q_0} \parallel \theta_1^n L(z)L(z_0)^{-1} \rangle$. Therefore, $\theta_1^n y(z) = \langle \sigma f_{|q_0} \mid [D_n(z) - E_n(z)]L(z)L(z_0)^{-1} \rangle = \langle \sigma f_{|q_0} \triangleright [D_n(z) - E_n(z)] \mid L(z)L(z_0)^{-1} \rangle$. Hence, $\theta_1^n y(1) \underset{\varepsilon \rightarrow 0^+}{\widetilde{\sim}} \sum_{w \in X^*} \langle \mathcal{A}(w) \circ f_{|q_0} \mid w \rangle \langle [D_n(1 - \varepsilon) - E_n(1 - \varepsilon)] e^{-x_1 \log \varepsilon} Z_{\perp\perp} e^{-x_0 \log \varepsilon} \mid w \rangle$.

The actions of $\theta_0 = u_0(z)^{-1}d/dz$ and $\theta_1 = u_1(z)^{-1}d/dz$ over y are equivalent to those of the residuals of $\sigma f_{|q_0}$ by respectively x_0 and x_1 . They correspond to *functional* differentiations [24] while $\partial_z = d/dz$ is the ordinary differentiation and is equivalent to the residual by $x_0 + x_1$.

3.3 Differential realization

3.3.1 Differential realization

Definition 12. The *Lie rank* of a formal power series $S \in \mathbb{K}\langle\langle X \rangle\rangle$ is the dimension of the vector space generated by

$$\{S \triangleright \Pi \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X \rangle\}, \text{ or respectively by } \{\Pi \triangleleft S \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X \rangle\}.$$

Definition 13. Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$ and let us put $\text{Ann}(S) := \{\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X \rangle \mid S \triangleright \Pi = 0\}$, and $\text{Ann}^{\perp}(S) := \{Q \in (\mathbb{K}\langle\langle X \rangle\rangle, \sqcup) \mid Q \triangleright \text{Ann}(S) = 0\}$.

It is immediate that $\text{Ann}^{\perp}(S) \ni S$. It follows then(see [25, 52] and Lemma 7),

Lemma 9. Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$. Then

1. If S is of finite Lie rank, d , then the dimension of $\text{Ann}^{\perp}(S)$ is d .
2. For any Q_1 and $Q_2 \in \text{Ann}^{\perp}(S)$, one has $Q_1 \sqcup Q_2 \in \text{Ann}^{\perp}(S)$.
3. For any $P \in \mathbb{K}\langle X \rangle$ and $Q_1 \in \text{Ann}^{\perp}(S)$, one has $P \triangleleft Q_1 \in \text{Ann}^{\perp}(S)$.

Definition 14. ([25]) The formal power series $S \in \mathbb{K}\langle\langle X \rangle\rangle$ is *differentially produced* if there exist an integer d , a power series $f \in \mathbb{K}[[\bar{q}_1, \dots, \bar{q}_d]]$, a homomorphism \mathcal{A} from X^* to the algebra of differential operators generated by

$$\mathcal{A}(x_i) = \sum_{j=1}^d A_i^j(\bar{q}_1, \dots, \bar{q}_d) \frac{\partial}{\partial \bar{q}_j}, \text{ where } \forall j = 1, \dots, d, A_i^j(\bar{q}_1, \dots, \bar{q}_d) \in \mathbb{K}[[\bar{q}_1, \dots, \bar{q}_d]]$$

such that, for any $w \in X^*$, one has $\langle S \mid w \rangle = \mathcal{A}(w) \circ f|_0$.

The pair (\mathcal{A}, f) is called the *differential representation* of S of dimension d .

Proposition 19. ([52]) Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$. If S is differentially produced then it satisfies the growth condition and its Lie rank is finite.

Proof. Let (\mathcal{A}, f) be a differential representation of S of dimension d . Then, by the notations of Definition 10, we get

$$\sigma f|_0 = S = \sum_{w \in X^*} (\mathcal{A}(w) \circ f)|_0 w.$$

We put

$$\forall j = 1, \dots, d, T_j = \sum_{w \in X^*} \frac{\partial(\mathcal{A}(w) \circ f)}{\partial \bar{q}_j} w.$$

Firstly, by Theorem 10, the generating series σf satisfies the growth condition. Secondly, for any $\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X \rangle$ and for any $w \in X^*$, one has

$$\langle \sigma f \triangleright \Pi \mid w \rangle = \langle \sigma f \mid \Pi w \rangle = \mathcal{A}(\Pi w) \circ f = \mathcal{A}(\Pi) \circ (\mathcal{A}(w) \circ f).$$

Since $\mathcal{A}(\Pi)$ is a derivation over $\mathbb{K}[[\bar{q}_1, \dots, \bar{q}_d]]$:

$$\begin{aligned}\mathcal{A}(\Pi) &= \sum_{j=1}^d (\mathcal{A}(\Pi) \circ \bar{q}_j) \frac{\partial}{\partial \bar{q}_j}, \\ \Rightarrow \mathcal{A}(\Pi) \circ (\mathcal{A}(w) \circ f) &= \sum_{j=1}^d (\mathcal{A}(\Pi) \circ \bar{q}_j) \frac{\partial (\mathcal{A}(w) \circ f)}{\partial \bar{q}_j}\end{aligned}$$

then we deduce that

$$\begin{aligned}\forall w \in X^*, \quad \langle \sigma f \triangleright \Pi \mid w \rangle &= \sum_{j=1}^d (\mathcal{A}(\Pi) \circ \bar{q}_j) \langle T_j \mid w \rangle, \\ \Leftrightarrow \sigma f \triangleright \Pi &= \sum_{j=1}^d (\mathcal{A}(\Pi) \circ \bar{q}_j) T_j.\end{aligned}$$

This means that $\sigma f \triangleright \Pi$ is a \mathbb{K} -linear combination of $\{T_j\}_{j=1, \dots, d}$ and the dimension of the vector space $\text{span}\{\sigma f \triangleright \Pi \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X \rangle\}$ is less than or equal to d .

3.3.2 Fliess' local realization theorem

Proposition 20. ([52]) *Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$ with Lie rank d . Then there exists a basis $S_1, \dots, S_d \in \mathbb{K}\langle\langle X \rangle\rangle$ of $(\text{Ann}^\perp(S), \sqcup) \cong (\mathbb{K}[[S_1, \dots, S_d]], \sqcup)$ such that the S_i 's are proper and for any $R \in \text{Ann}^\perp(S)$, one has*

$$R = \sum_{i_1, \dots, i_d \geq 0} \frac{r_{i_1, \dots, i_d}}{i_1! \dots i_d!} S_1^{\sqcup i_1} \sqcup \dots \sqcup S_d^{\sqcup i_d}, \text{ where } r_{0, \dots, 0} = \langle R \mid 1_{X^*} \rangle, r_{i_1, \dots, i_d} \in \mathbb{K}.$$

Proof. By Lemma 9, such a basis exists. More precisely, since the Lie rank of S is d then there exist $P_1, \dots, P_d \in \mathcal{L}ie_{\mathbb{K}}\langle X \rangle$ such that $S \triangleright P_1, \dots, S \triangleright P_d \in (\mathbb{K}\langle\langle X \rangle\rangle, \sqcup)$ are \mathbb{K} -linearly independent. By duality, there exists $S_1, \dots, S_d \in (\mathbb{K}\langle\langle X \rangle\rangle, \sqcup)$ such that

$$\forall i, j = 1, \dots, d, \quad \langle S_i \mid P_j \rangle = \delta_{i,j}, \text{ and } R = \prod_{i=1}^d \exp(S_i P_i).$$

Expanding this product, one obtains, via PBW theorem, the expected expression for the coefficients $\{r_{i_1, \dots, i_d} = \langle R \mid P_1^{i_1} \dots P_d^{i_d} \rangle\}_{i_1, \dots, i_d \geq 0}$. Hence, $(\text{Ann}^\perp(S), \sqcup)$ is generated by S_1, \dots, S_d .

With the notations of Proposition 20, one has

- Corollary 6.** 1. *If $S \in \mathbb{K}[S_1, \dots, S_d]$ then, for any $i = 0, 1$ and for any $j = 1, \dots, d$, one has $x_i \triangleleft S \in \text{Ann}^\perp(S) = \mathbb{K}[S_1, \dots, S_d]$.*
 2. *The power series S satisfies the growth condition if and only if, for any $i = 1, \dots, d$, S_i also satisfies the growth condition.*

Proof. Assume there exists $j \in [1, \dots, d]$ such that S_j does not satisfy the growth condition. Since $S \in \text{Ann}^\perp(S)$ then using the decomposition of S on S_1, \dots, S_d , one obtains a contradiction with the fact that S satisfies the growth condition.

Conversely, using Proposition 16, we get the expected results.

Theorem 12. ([25]) *The formal power series $S \in \mathbb{K}\langle\langle X \rangle\rangle$ is differentially produced if and only if its Lie rank is finite and if it satisfies the χ -growth condition.*

Proof. By Proposition 19, one gets a direct proof. Conversely, since the Lie rank of S equals d then by Proposition 20, setting $\sigma f|_0 = S$ and, for $j = 1, \dots, d$, $\sigma \bar{q}_i = S_i$,

1. We choose the observation f as follows

$$f(\bar{q}_1, \dots, \bar{q}_d) = \sum_{i_1, \dots, i_d \geq 0} \frac{r_{i_1, \dots, i_d}}{i_1! \dots i_d!} \bar{q}_1^{i_1} \dots \bar{q}_d^{i_d} \in \mathbb{K}[[\bar{q}_1, \dots, \bar{q}_d]]$$

such that

$$\sigma f|_0(\bar{q}_1, \dots, \bar{q}_d) = \sum_{i_1, \dots, i_d \geq 0} \frac{r_{i_1, \dots, i_d}}{i_1! \dots i_d!} (\sigma \bar{q}_1)^{\sqcup i_1} \sqcup \dots \sqcup (\sigma \bar{q}_d)^{\sqcup i_d},$$

2. It follows that, for $i = 0, 1$ and for $j = 1, \dots, d$, the residual $x_i \triangleleft \sigma \bar{q}_j$ belongs to $\text{Ann}^\perp(\sigma f|_0)$ (see also Lemma 9),
3. Since σf satisfies the χ -growth condition then, the generating series $\sigma \bar{q}_j$ and $x_i \triangleleft \sigma \bar{q}_j$ (for $i = 0, 1$ and for $j = 1, \dots, d$) verify also the growth condition. We then take (see Lemma 8)

$$\forall i = 0, 1, \forall j = 1, \dots, d, \quad \sigma A_j^i(\bar{q}_1, \dots, \bar{q}_d) = x_i \triangleleft \sigma \bar{q}_j,$$

by expressing σA_j^i on the basis $\{\sigma \bar{q}_i\}_{i=1, \dots, d}$ of $\text{Ann}^\perp(\sigma f|_0)$,

4. The homomorphism \mathcal{A} is then determined as follows

$$\forall i = 0, 1, \quad \mathcal{A}(x_i) = \sum_{j=0}^d A_j^i(\bar{q}_1, \dots, \bar{q}_d) \frac{\partial}{\partial \bar{q}_j},$$

where, by Lemma 5, one has $A_j^i(\bar{q}_1, \dots, \bar{q}_d) = \mathcal{A}(x_i) \circ \bar{q}_j$.

Thus, (\mathcal{A}, f) provides a differential representation²⁸ of dimension d of S .

Moreover, one also has the following

Theorem 13. ([25]) *Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$ be a differentially produced formal power series. Let (\mathcal{A}, f) and (\mathcal{A}', f') be two differential representations of dimension n of S . There exist a continuous and convergent automorphism h of \mathbb{K} such that*

$$\forall w \in X^*, \forall g \in \mathbb{K}, \quad h(\mathcal{A}(w) \circ g) = \mathcal{A}'(w) \circ (h(g)) \text{ and } f' = h(f).$$

²⁸ In [25, 52], the reader can find the discussion on the *minimal* differential representation.

Since any rational power series satisfies the growth condition and its Lie rank is less than or equal to its Hankel rank which is finite [25] then

Corollary 7. *Any rational power series and any polynomial over X with coefficients in \mathbb{K} are differentially produced.*

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