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Abstract

In this work, we focus on on the approach by noncommutative formal power series to study the combinatorial aspects of the renormalization at the singularities in $\{0, 1, +\infty\}$ of the solutions of nonlinear differential equations involved in quantum electrodynamics.

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1 Introduction

During the last century, the functional expansions were common in physics as well as in engineering and have been developped, by Tomonaga, Schwinger and Feynman [16], to represent the nonlinear dynamical systems in quantum electrodynanics. The main difficulty of this approach is the divergence of these expansions at the singularity 0 or at $+\infty$ (see [2]) and leads to the problems of *regularization* and *renormalization* which can be solved by combinatorial technics : Feynman diagrams [18] and their derivatives [14, 23], noncommutative formal power series [19, 20], trees [10], ...

Recently, following the previous vein, and basing on

- the shuffle and quasi-shuffle Hopf algebras [5], the combinatorics on noncommutative formal power series were intensively amplified for the asymptotic analysis of nonlinear dynamical systems with three regular singularities in[2] $\{0, 1, +\infty\}$ [39, 40],
- the monodromy and the Galois differential groups of the Knizhnik-Zamolodchikov differential equation KZ_3 [38, 40], *i.e.* the following noncommutative evolution equation[3]

$$\frac{dG(z)}{dz} = \left(\frac{x_0}{z} + \frac{x_1}{1-z}\right)G(z), \tag{1}$$

  the Schützenberger's monoidal factorization facilitates mainly the renormalization and the computation of the associators[4] via the universal one, *i.e.* the associator Φ_{KZ} of Drinfel'd.

[2] Any differential equation with three singularities in $\{a, b, c\}$, via homographic transformation $(z-a)(c-b)(z-b)^{-1}(c-a)^{-1}$ can be changed into a differential equation with three singularities in $\{0, 1, +\infty\}$.

[3] Here, x_0 and x_1 are noncommuative variables. More precisely (with $\mathrm{i}^2 = -1$),

$$x_0 := \frac{t_{1,2}}{2\mathrm{i}\pi} \quad \text{and} \quad x_1 := -\frac{t_{2,3}}{2\mathrm{i}\pi},$$

where $t_{1,2}$ and $t_{2,3}$ bebong to $\mathcal{T}_3 = \{t_{1,2}, t_{1,3}, t_{2,3}\}$ satisfying the infinitesimal 3-braid relations

$$[t_{1,3}, t_{1,2} + t_{2,3}] = [t_{2,3}, t_{1,2} + t_{1,3}] = 0.$$

[4] They were introduced in quantum field theory by Drinfel'd and it plays an important role for the still open problem of the effective determination of the polynomial invariants of knots and links via Kontsevich's integral (see [6, 23]) and Φ_{KZ}, was obtained firstly, in [23], with explicit coefficients which are polyzetas and regularized polyzetas (see [40, 41] for the computation of the other associators involving *only* convergent polyzetas as local coordinates, and for three algorithmical process to regularize the divergent polyzetas).

In fact, these associators are noncommutative formal power series on two variables and regularize the Chen generating series of the differential forms admitting singularities at 0 or at 1 along the integration paths on the universal covering $\widetilde{\mathbb{C}-\{0,1\}}$ of $\mathbb{C}$ without points 0 and 1. Their coefficients which are, up to multiple of power of $2i\pi$, polynomial on polyzetas, *i.e.* the following real numbers[5] (for $r \geq 1, s_1 \geq 2$ and $s_2, \ldots, s_r \geq 1$) [42, 23, 51]

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}} \tag{2}$$

and it is reasonable to think that these numbers admit a structure of $\mathbb{N}$-graded, by weight, algebra over the rational numbers deduced from the combinatorial aspects of the shuffle and quasi-shuffle Hopf algebras [40, 41].

More precisely, for $s_1 \geq 2, s_2, \ldots, s_r \geq 1$, the polyzeta $\zeta(s_1, \ldots, s_r)$ can be obtained as the limit of

- the polylogarithm $\mathrm{Li}_{s_1,\ldots,s_r}(z)$, for $z \to 1$ (recalled in Section 2.3.1)

$$\mathrm{Li}_{s_1,\ldots,s_r}(z) = \sum_{n_1 > \ldots > n_r > 0} \frac{z^{n_1}}{n_1^{s_1} \ldots n_r^{s_r}} \tag{3}$$

- the harmonic sum $\mathrm{H}_{s_1,\ldots,s_r}(N)$, for $N \to +\infty$ (recalled in Section 2.3.1)

$$\mathrm{H}_{s_1,\ldots,s_r}(N) = \sum_{n_1 > \ldots > n_r > 0}^{N} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}. \tag{4}$$

Then, by a theorem of Abel, one has

$$\zeta(s_1, \ldots, s_r) = \lim_{z \to 1} \mathrm{Li}_{s_1,\ldots,s_r}(z) = \lim_{N \to +\infty} \mathrm{H}_{s_1,\ldots,s_r}(N). \tag{5}$$

Since the algebras of polylogarithms and of harmonic sums are isomorphic respectively to the shuffle algebra $(\mathbb{Q}\langle X\rangle, ⧢)$ and the quasi-shuffle algebra $(\mathbb{Q}\langle Y\rangle, \uplus)$ both admitting the Lyndon words $\mathcal{L}ynX$ over $X = \{x_0, x_1\}$ and $\mathcal{L}ynY$ over $Y = \{y_i\}_{i\geq 1}$, as transcendence bases (recalled in Section 2.1) then, by using

- the correspondence between the composition in $\mathbb{N}_+$ with the words in Y^* and in X^*x_1, *i.e.*

$$(s_1, \ldots, s_r) \longleftrightarrow y_{s_1} \ldots y_{s_r} \longleftrightarrow x_0^{s_1-1}x_1 \ldots x_0^{s_r-1}x_1, \tag{6}$$

- the non commutative generating series in the factorized forms, *i.e.*

$$\mathrm{L} = \prod_{l \in \mathcal{L}ynX} \exp(\mathrm{Li}_{S_l} P_l) \quad \text{and} \quad \mathrm{H} = \prod_{l \in \mathcal{L}ynY} \exp(\mathrm{H}_{\Sigma_l} \Pi_l), \tag{7}$$

$$Z_{⧢} = \prod_{\substack{l \in \mathcal{L}ynX \\ l \neq x_0, x_1}} \exp(\zeta(S_l) P_l) \quad \text{and} \quad Z_{\uplus} = \prod_{\substack{l \in \mathcal{L}ynY \\ l \neq y_1}} \exp(\zeta(\Sigma_l) \Pi_l), \tag{8}$$

[5] The sum $s_1 + \ldots + s_r$ is the *weight* of $\zeta(s_1, \ldots, s_r)$ which equals to the weight of the composition $(s_1, \ldots, s_r)$.

where $\{S_l\}_{l\in\mathcal{L}ynX}$ and $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$ are transcendence bases of $(\mathbb{Q}\langle X\rangle, ⧢)$ and $(\mathbb{Q}\langle Y\rangle, \uplus)$, $\{P_l\}_{l\in\mathcal{L}ynX}$ and $\{\Pi_l\}_{l\in\mathcal{L}ynY}$ are bases of the Lie algebras of the primitive elements of the bi-algebras $\mathcal{H}_{⧢} = (\mathbb{Q}\langle X\rangle, \mathtt{conc}, \Delta_{⧢}, 1_{X^*})$ and $\mathcal{H}_{\uplus} = (\mathbb{Q}\langle Y\rangle, \mathtt{conc}, \Delta_{\uplus}, 1_{Y^*})$ in duality respectively with $\{S_l\}_{l\in\mathcal{L}ynX}$ and $\{\Sigma_l\}_{l\in\mathcal{L}ynY}$,

we obtain the Abel like theorem [39, 40, 41], *i.e.* there exist two noncommutative formal series over Y, Z_1 and Z_2 with constant terms, such that :

$$\lim_{z\to 1}\exp\left(y_1\log\frac{1}{1-z}\right)\pi_Y\mathrm{L}(z) = Z_1, \tag{9}$$

$$\lim_{N\to\infty}\exp\left(\sum_{k\geq 1}\mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\mathrm{H}(N) = Z_2. \tag{10}$$

Moreover, Z_1, Z_2 are equal and stand for the noncommutative generating series of the convergent polyzetas $\{\zeta(w)\}_{w\in Y^*-y_1Y^*}$, or equivalently $\{\zeta(w)\}_{w\in x_0X^*x_1}$:

$$Z_1 = Z_2 = \pi_Y Z_{⧢}, \tag{11}$$

where π_Y denotes the projection over Y^*.

This Abel theorem allows to explicit the counter-terms eliminating the divergence of

- the polylogarithms $\{\mathrm{Li}_w(z)\}_{w\in x_1X^*}$ for $z\to 1$,
- the harmonic sums $\{\mathrm{H}_w(N)\}_{w\in y_1Y^*}$ for $N\to\infty$

and leads naturally to an equation connecting these two algebraic structures

$$\prod_{\substack{l\in\mathcal{L}ynY\\ l\neq y_1}}^{\searrow}\exp(\zeta(\Sigma_l)\Pi_l) = \exp\left(\sum_{k\geq 2}\zeta(k)\frac{(-y_1)^k}{k}\right)\pi_Y\prod_{\substack{l\in\mathcal{L}ynX\\ l\neq x_0,x_1}}^{\searrow}\exp(\zeta(S_l)P_l). \tag{12}$$

This identity (12) enables, in particular,

- to calculate the Euler-Mac Laurin constants and the Hadamard finite parties associated to the divergent polyzetas $\{\zeta(w)\}_{w\in y_1Y^*}$,
- to describe *completely*, via identification of local coordinates in infinite dimension, the kernel of ζ by its *algebraic* generators, and thus this kernel is generated by homogenous polynomials.

In this paper, we will focus on the approach by noncommutative formal power series, adapted from [19, 20], and explaining the results of [39, 40, 41] allowing to study the combinatorial aspects of the renormalization at the singularities in $\{0, 1, +\infty\}$ of the solutions of linear differential equations (see Example 1 bellows) as well as the solutions of nonlinear differential equations (see Examples 2 and 3 bellow) described in Section 3.2 and involved in quantum electrodynamics.

Example 1 (Hypergeometric equation)**.** *Let t_0, t_1, t_2 be parameters and*

$$z(1-z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.$$

Let $q_1(z) = y(z)$ and $q_2(z) = z(1-z)\dot{y}(z)$. One has

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \left(\frac{M_0}{z} + \frac{M_1}{1-z} \right) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

where M_0 and M_1 are the following matrices

$$M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} \quad \text{and} \quad M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}.$$

Or equivalently,

$$\begin{aligned} \dot{q}(z) &= A_0(q)\frac{1}{z} + A_1(q)\frac{1}{1-z}, \\ y(z) &= q(z), \end{aligned}$$

where A_0 and A_1 are the following pamaetrized linear vectors fields

$$A_0(q) = -(t_0 t_1 q_1 + t_2 q_2)\frac{\partial}{\partial q_2} \quad \text{and} \quad A_1(q) = -q_1 \frac{\partial}{\partial q_1} - (t_2 - t_0 - t_1) q_2 \frac{\partial}{\partial q_2}.$$

Example 2 (Harmonic oscillator)**.** *Let k_1, k_2 be parameters and*

$$\dot{y}(z) + k_1 y(z) + k_2 y^2(z) = u_1(z).$$

which can be represented by the following state equations

$$\begin{aligned} \dot{q}(z) &= A_0(q) + A_1(q) u_1(z), \\ y(z) &= q(z), \end{aligned}$$

where A_0 and A_1 are the following vectors fields

$$A_0(q) = -(k_1 q + k_2 q^2)\frac{\partial}{\partial q} \quad \text{and} \quad A_1(q) = \frac{\partial}{\partial q}.$$

Example 3 (Duffing's equation)**.** *Let a, b, c be parameters and*

$$\ddot{y}(z) + a\dot{y}(z) + by(z) + cy^3(z) = u_1(z).$$

which can be represented by the following state equations

$$\begin{aligned} \dot{q}(z) &= A_0(q) + A_1(q) u_1(z), \\ y(z) &= q(z), \end{aligned}$$

where A_0 and A_1 are the following vectors fields

$$A_0(q) = -(a q_2 + b^2 q_1 + c q_1^3)\frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_1} \quad \text{and} \quad A_1(q) = \frac{\partial}{\partial q_2}.$$

This approach by noncommutative formal power series is adequate for studying the algebraic combinatorial aspects of the asymptotic analysis, at the singularities in $\{0, 1, +\infty\}$, of the nonlinear dynamical systems described in Section 3.2 because

- the polylogarithms form the universal Picard-Vessiot of these nonlinear differential equations [38] and their algebra is isomorphic to the shuffle algebra admitting $\{\mathrm{Li}_l\}_{l\in\mathcal{L}ynX}$ as transcendence basis,
- the harmonic sums generate the coefficients of the ordinary Taylor expansion of their solutions (when these expansion exist) [39] and their algebra is isomorphic to the shuffle algebra admitting $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$ as transcendence basis,
- the polyzetas do appear as the fondamental arithmetical constants involved in the computations of the monomdomies [35, 32], the Kummer type functional equations [36, 32], the asymptotic expansions of the solutions of these equations [39, 40] and their algebra is freely generated by the polyzetas encoded by irreducible Lyndon words [40].

Hence, a lot of algorithms can be deduced from these facts and the more general studies will be proceed (see [5, 11] for example).

The organisation of this paper is following

- In Section 2, we will give algbraic and analytic foundations, *i.e.* the combinatorial Hopf algebra of shuffles and the indiscernability respectively, for polylogarithms and harmonic sums.
- These will be exploited systematically, in Section 3, to expand the solutions of nonlinear dynamical systems with singular inputs.

2 Background

2.1 Combinatorics on shuffle and stuffle Hopf algebras

2.1.1 Schützenberger's monoidal factorization

Let X be a finite totally ordered alphabet. The free monoid and the set of Lyndon words, over X, are denoted respectively by X^* and $\mathcal{L}ynX$. The neutral element of X^* is denoted by 1_{X^*}. Let $\mathbb{Q}\langle X\rangle$ be equipped by the concatenation and the shuffle defined by

$$\forall w \in X^*, \qquad w ⧢ 1_{X^*} = 1_{X^*} ⧢ w = w, \tag{13}$$

$$\forall x, y \in X, \forall u, v \in X^*, \qquad xu ⧢ yv = x(u ⧢ yv) + y(xu ⧢ v), \tag{14}$$

or by their dual co-products, $\Delta_{\mathtt{conc}}, \Delta_{⧢}$, defined by

$$\forall w \in X^*, \quad \Delta_{\mathtt{conc}}(w) \quad = \sum_{u,v\in X^*, uv=w} u \otimes v, \tag{15}$$

$$\forall x \in X, \quad \Delta_{⧢}(x) \quad = \quad x \otimes 1 + 1 \otimes x, \tag{16}$$

satisfying, for any $u, v, w \in X^*$,

$$\begin{aligned}\langle \Delta_{\texttt{conc}}(w) \mid u \otimes v\rangle &= \langle w \mid uv\rangle, & (17)\\ \langle \Delta_{\sqcup\!\sqcup}(w) \mid u \otimes v\rangle &= \langle w \mid u \sqcup\!\sqcup v\rangle. & (18)\end{aligned}$$

With the co-unit and the antipode defined by

$$\forall P \in \mathbb{Q}\langle X\rangle, \qquad \epsilon(P) = \langle P \mid 1_{X^*}\rangle, \tag{19}$$

$$\forall w = x_{i_1} \ldots x_{i_r} \in X^*, \qquad a_{\sqcup\!\sqcup}(w) = (-1)^{-|w|} x_{i_r} \ldots x_{i_1}, \tag{20}$$

one gets two mutually dual Hopf algebras

$$\begin{aligned}\mathcal{H}_{\sqcup\!\sqcup} &= (\mathbb{Q}\langle X\rangle, \texttt{conc}, 1_{X^*}, \Delta_{\sqcup\!\sqcup}, \epsilon, a_{\sqcup\!\sqcup}), & (21)\\ \mathcal{H}_{\sqcup\!\sqcup}^{\vee} &= (\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{\texttt{conc}}, \epsilon, a_{\sqcup\!\sqcup}). & (22)\end{aligned}$$

By the Cartier-Quillen-Milnor-Moore theorem, the connected graded positively, co-commutative Hopf algebra $\mathcal{H}$ is isomorphic to the envelopping algebra of the Lie algebra of its primitive elements which is equal to $\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$. Hence, any graded[6] suited[7] basis of $\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$, can be completed, by the Poincaré-Birkhoff-Witt theorem, as a linear (graded) basis $\{b_w\}_{w\in X^*}$ for $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$. One can then[8] construct, by duality, a basis $\{\check{b}_w\}_{w\in X^*}$ of $\mathcal{H}_{\sqcup\!\sqcup}$ such that :

$$\forall u, v \in X^*, \quad \langle \check{b}_u \mid b_v\rangle = \delta_{u,v}, \tag{23}$$

and, for any $x \in X$, $\check{b}_x = x$ and for any $w = l_1^{i_1} \ldots l_k^{i_k}$ with $l_1, \ldots, l_k \in \mathcal{L}ynX$ and $l_1 > \ldots > l_k$, one has

$$\check{b}_w = \frac{\check{b}_{l_1}^{i_1} \times \ldots \times \check{b}_{l_k}^{i_k}}{i_1! \ldots i_k!}. \tag{24}$$

For that, one can use a theorem of Viennot [50] to construct, by bijection with $\mathcal{L}ynX$, a basis for $\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$ and, by the PBW theorem, one completes then to a linear basis for $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$.

On an other way, by a Radford's theorem [45], the elements of $\mathcal{L}ynX$ constitute also a transcendence basis for $(\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup)$ and it can be completed then to the linear basis $\{w\}_{w\in X^*}$ which is auto-dual :

$$\forall v, v \in X^*, \quad \langle u \mid v\rangle = \delta_{u,v}. \tag{25}$$

But the elements $l \in \mathcal{L}ynX - X$ are not primitive, for $\Delta_{\sqcup\!\sqcup}$, and then $\mathcal{L}ynX$ does not constitute a basis for $\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$. Denoting $\mathcal{D}_X$ the diagonal series over X, one has

$$\mathcal{D}_X = \sum_{w\in X^*} w \otimes w = \sum_{l\in\mathcal{L}ynX} l \otimes l + \sum_{w\notin\mathcal{L}ynX} w \otimes w \tag{26}$$

[6]for the multidegree.
[7]i.e. a basis of polynomials indexed by their initial words which is a Lyndon word.
[8]because $\mathcal{H}$ is graded in finite dimensions by the multidegree.

and $\mathcal{D}_X$ can be also factorized as follows

$$\mathcal{D}_X = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(l \otimes l). \tag{27}$$

Using the transcendence basis $\{l\}_{l\in\mathcal{L}ynX}$, Mélançon and Reutenauer give an other completion by putting [47]

$$\begin{aligned}
S'_l &= l, && \text{for } l \in \mathcal{L}ynX, && (28)\\
S'_w &= \frac{l_1^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup\, l_k^{\sqcup\!\sqcup\, i_k}}{i_1!\ldots i_k!}, && \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1 > \ldots > l_k, && (29)\\
&= w + \sum_{v<w,|v|=|w|} c'_v v. && && (30)
\end{aligned}$$

The linear basis $\{S'_w\}_{w\in X^*}$ admits the dual basis $\{P'_w\}_{w\in X^*}$ but it is not of PBW type although the elements of $\{P'_l\}_{l\in\mathcal{L}ynX}$ are primitive, for $\Delta_{\sqcup\!\sqcup}$ [12].

Chen, Fox and Lyndon [8] constructed $\{P_w\}_{w\in X^*}$, so-called the PBW-Lyndon basis, for $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle X\rangle)$ as follows

$$\begin{aligned}
P_x &= x && \text{for } x \in X, && (31)\\
P_l &= [P_s, P_r] && \text{for } l \in \mathcal{L}ynX, \text{ standard factorization of } l = (s, r), && (32)\\
P_w &= P_{l_1}^{i_1}\ldots P_{l_k}^{i_k} && \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1 > \ldots > l_k, l_1\ldots, l_k \in \mathcal{L}ynX. && (33)
\end{aligned}$$

Schützenberger constructed bases for $(\mathbb{Q}\langle X\rangle, \sqcup\!\sqcup)$ defined by duality as follows :

$$\forall u, v \in X^*, \quad \langle S_u \mid P_v\rangle \quad = \quad \delta_{u,v}, \tag{34}$$

and obtained the transcendence basis $\{S_l\}_{l\in\in\mathcal{L}ynX}$ and the linear basis $\{S_w\}_{w\in X^*}$ as follows

$$\begin{aligned}
S_l &= xS_u, && \text{for } l = xu \in \mathcal{L}ynX, && (35)\\
S_w &= \frac{S_{l_1}^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup\, S_{l_k}^{\sqcup\!\sqcup\, i_k}}{i_1!\ldots i_k!} && \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1 > \ldots > l_k. && (36)
\end{aligned}$$

After that, Mélançon and Reutenauer [47] proved that[9], for any $w \in X^*$,

$$P_w = w + \sum_{v>w,|v|=|w|} c_v v \quad \text{and} \quad S_w = w + \sum_{v<w,|v|=|w|} c_v v. \tag{37}$$

[9] Recall that the duality preserves the (multi)-homogeneous degree and interchanges the triangularity of polynomials [47]. For that, one can construct the triangular matrices M and N admitting coeffients as the coefficients of the homogeneous of degree k triangular polynomials, $\{\mathrm{P}_w\}_{w\in X^*}$ and $\{S_w\}_{w\in X^*}$ in the basis $\{w\}_{w\in X^*}$ respectively :

$$M_{u,v} = \langle P_u \mid v\rangle \quad \text{and} \quad N_{v,u} = \langle S_u \mid v\rangle.$$

The triangular matrices M and N are unipotent and satisfy the identity $N = ({}^tM)^{-1}$.

In other words, the elements of the bases $\{S_w\}_{w\in X^*}$ and $\{P_w\}_{w\in X^*}$ are lower and upper triangular respectively and they are of homogeneous by degree.

Thanks to the duality of the bases $\{P_w\}_{w\in X^*}$ and $\{S_w\}_{w\in X^*}$ and the basis $\{P_w\}_{w\in X^*}$ is of PBW type, one also has [47]

$$\mathcal{D}_X = \sum_{w\in X^*} S_w \otimes P_w \tag{38}$$

$$= \sum_{l\in\mathcal{L}ynX} S_l \otimes P_l + \sum_{w\notin\mathcal{L}ynX} S_w \otimes P_w \tag{39}$$

$$= \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l). \tag{40}$$

Example 4 ([27])**.** *Let* $X = \{x_0, x_1\}$ *with* $x_0 < x_1$*.*

l	P_l	S_l
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0^2x_1$	$[x_0, [x_0, x_1]]$	$x_0^2x_1$
$x_0x_1^2$	$[[x_0, x_1], x_1]$	$x_0x_1^2$
$x_0^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3x_1$
$x_0^2x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2x_1^2$
$x_0x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0x_1^3$
$x_0^4x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4x_1$
$x_0^3x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3x_1^2$
$x_0^2x_1x_0x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3x_1^2 + x_0^2x_1x_0x_1$
$x_0^2x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2x_1^3$
$x_0x_1x_0x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2x_1^3 + x_0x_1x_0x_1^2$
$x_0x_1^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0x_1^4$
$x_0^5x_1$	$[x_0, [x_0, [x_0, [x_0, [x_0, x_1]]]]]$	$x_0^5x_1$
$x_0^4x_1^2$	$[x_0, [x_0, [x_0, [[x_0, x_1], x_1]]]]$	$x_0^4x_1^2$
$x_0^3x_1x_0x_1$	$[x_0, [[x_0, [x_0, x_1]], [x_0, x_1]]]$	$2x_0^4x_1^2 + x_0^3x_1x_0x_1$
$x_0^3x_1^3$	$[x_0, [x_0, [[[x_0, x_1], x_1], x_1]]]$	$x_0^3x_1^3$
$x_0^2x_1x_0x_1^2$	$[x_0, [[x_0, x_1], [[x_0, x_1], x_1]]]$	$3x_0^3x_1^3 + x_0^2x_1x_0x_1^2$
$x_0^2x_1^2x_0x_1$	$[[x_0, [[x_0, x_1], x_1]], [x_0, x_1]]$	$6x_0^3x_1^3 + 3x_0^2x_1x_0x_1^2 + x_0^2x_1^2x_0x_1$
$x_0^2x_1^4$	$[x_0, [[[[x_0, x_1], x_1], x_1], x_1]]$	$x_0^2x_1^4$
$x_0x_1x_0x_1^3$	$[[x_0, x_1], [[[x_0, x_1], x_1], x_1]]$	$4x_0^2x_1^4 + x_0x_1x_0x_1^3$
$x_0x_1^5$	$[[[[[x_0, x_1], x_1], x_1], x_1], x_1]$	$x_0x_1^5$

2.1.2 Extended Schützenberger's monoidal factorization

Now, let $Y = \{y_i\}_{i\geq 1}$ be an infinite totally ordered alphabet[10]. The free monoid and the set of Lyndon words, over Y, are denoted respectively by Y^* and $\mathcal{L}ynY$. The neutral element of Y^* is denoted by 1_{Y^*}.

[10]by $y_1 > y_2 > y_3 > \ldots$.

Let $u = y_{i_1} \ldots y_{i_k} \in Y^*$, the *length* and the *weight* of u are defined respectively as the numbers $|u| = k$ and $(u) = i_1 + \ldots + i_k$.

Let us define the commutative product over $\mathbb{Q}\langle Y\rangle$, denoted by μ, as follows

$$\forall y_n, y_m \in Y, \qquad \mu(y_n, y_m) = y_{n+m}, \tag{41}$$

or by its associated coproduct, Δ_+, defined by

$$\forall y_n \in Y, \qquad \Delta_+ y_n = \sum_{i=1}^{n-1} y_i \otimes y_{n-i} \tag{42}$$

satisfying,

$$\forall x, y, z \in Y, \quad \langle \Delta_+ x \mid y \otimes z\rangle \quad = \quad \langle x \mid \mu(y, z)\rangle. \tag{43}$$

Let $\mathbb{Q}\langle Y\rangle$ be equipped by

1. The concatenation (or by its associated coproduct, $\Delta_{\mathtt{conc}}$).
2. The *shuffle* product, *i.e.* the commutative product defined by [47]

$$\begin{aligned} \forall w \in Y^*, &\qquad w ⧢ 1_{Y^*} = 1_{Y^*} ⧢ w = w, \\ \forall x, y \in Y, \forall u, v \in Y^*, &\qquad xu ⧢ yv = x(u ⧢ yv) + y(xu ⧢ v) \end{aligned}$$

   or by its associated coproduct, $\Delta_{⧢}$, defined, on the letters, by

$$\forall y_k \in Y, \quad \Delta_{⧢} y_k \quad = \quad y_k \otimes 1 + 1 \otimes y_k$$

   and extended by morphism. It satisfies

$$\forall u, v, w \in Y^*, \quad \langle \Delta_{⧢} w \mid u \otimes v\rangle \quad = \quad \langle w \mid u ⧢ v\rangle.$$

3. The *quasi-shuffle* product, *i.e.* the commutative product defined by [43], for any $w \in Y^*$,

$$w \,\text{⧢}\!\!\!\!-\, 1_{Y^*} = 1_{Y^*} \,\text{⧢}\!\!\!\!-\, w = w,$$

   and, for any $y_i, y_j \in Y, \forall u, v \in Y^*$,

$$\begin{aligned} y_i u \,\text{⧢}\!\!\!\!-\, y_j v &= y_j(y_i u \,\text{⧢}\!\!\!\!-\, v) + y_i(u \,\text{⧢}\!\!\!\!-\, y_j v) + \mu(y_i, y_j)(u \,\text{⧢}\!\!\!\!-\, v), \\ &= y_j(y_i u \,\text{⧢}\!\!\!\!-\, v) + y_i(u \,\text{⧢}\!\!\!\!-\, y_j v) + y_{i+j}(u \,\text{⧢}\!\!\!\!-\, v) \end{aligned}$$

   or by its associated coproduct, $\Delta_{\text{⧢}\!\!\!-}$, defined, on the letters, by

$$\forall y_k \in Y, \quad \Delta_{\text{⧢}\!\!\!-} y_k \quad = \quad \Delta_{⧢} y_k + \Delta_+ y_k$$

   and extended by morphism. It satisfies

$$\forall u, v, w \in Y^*, \quad \langle \Delta_{\text{⧢}\!\!\!-} w \mid u \otimes v\rangle \quad = \quad \langle w \mid u \,\text{⧢}\!\!\!\!-\, v\rangle.$$

   Note that $\Delta_{\text{⧢}\!\!\!-}$ and $\Delta_{⧢}$ are morphisms for the concatenation (by definition) whereas Δ_+ is not a morphism for the product of $\mathbf{k}Y$ (for example $\Delta_+(y_1^2) = y_1 \otimes y_1$, whereas $\Delta_+(y_1)^2 = 0$).

Hence, with the counit $\mathtt{e}$ defined by

$$\forall P \in \mathbb{Q}\langle Y\rangle, \qquad \mathtt{e}(P) = \langle P \mid 1_{Y^*}\rangle, \tag{44}$$

one gets two pairs of mutually dual bialgebras

$$\mathcal{H}_{\sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup} = (\mathbb{Q}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\sqcup\mkern-7mu\sqcup}, \mathtt{e}) \quad \text{and} \quad \mathcal{H}^{\vee}_{\sqcup\mkern-7mu\sqcup} = (\mathbb{Q}\langle Y\rangle, \sqcup\mkern-7mu\sqcup, 1_{Y^*}, \Delta_{\mathtt{conc}}, \mathtt{e}), \tag{45}$$

$$\mathcal{H}_{\uplus} = (\mathbb{Q}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{\uplus}, \mathtt{e}) \quad \text{and} \quad \mathcal{H}^{\vee}_{\uplus} = (\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*}, \Delta_{\mathtt{conc}}, \mathtt{e}). \tag{46}$$

As in (26-27), one has

$$\begin{aligned}
\mathcal{D}_Y &= \sum_{w\in Y^*} w\otimes w &(47)\\
&= \sum_{l\in\mathcal{L}ynY} l\otimes l + \sum_{w\notin\mathcal{L}ynY} w\otimes w &(48)\\
&= \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(l\otimes l). &(49)
\end{aligned}$$

By the CQMM theorem (see [5]), the connected $\mathbb{N}$-graded, co-commutative Hopf algebra $\mathcal{H}_{\sqcup\mkern-7mu\sqcup}$ is isomorphic to the enveloping algebra of the Lie algebra of its primitive elements which is equal to $\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle$:

$$\mathcal{H}_{\sqcup\mkern-7mu\sqcup} \cong \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle) \quad \text{and} \quad \mathcal{H}^{\vee}_{\sqcup\mkern-7mu\sqcup} \cong \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle)^{\vee}. \tag{50}$$

Hence, let us consider [8]

1. the PBW-Lyndon basis $\{p_w\}_{w\in Y^*}$ for $\mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle Y\rangle)$ constructed recursively

$$\begin{cases}
p_y = y & \text{for } y\in Y,\\
p_l = [p_s, p_r] & \text{for } l\in\mathcal{L}ynY, \text{ standard factorization of } l=(s,r),\\
p_w = p_{l_1}^{i_1}\ldots p_{l_k}^{i_k} & \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1>\ldots>l_k, l_1\ldots,l_k\in\mathcal{L}ynY,
\end{cases}$$

2. and, by duality[11], the linear basis $\{s_w\}_{w\in Y^*}$ for $(\mathbb{Q}\langle Y\rangle, \sqcup\mkern-7mu\sqcup, 1_{Y^*})$, *i.e.*

$$\forall u,v\in Y^*, \qquad \langle p_u \mid s_v\rangle = \delta_{u,v}.$$

This basis can be computed recursively as follows [47]

$$\begin{cases}
s_y = y, & \text{for } y\in Y,\\
s_l = ys_u, & \text{for } l = yu\in\mathcal{L}ynY,\\
s_w = \dfrac{s_{l_1}^{\sqcup\mkern-7mu\sqcup\, i_1}\sqcup\mkern-7mu\sqcup\ldots\sqcup\mkern-7mu\sqcup s_{l_k}^{\sqcup\mkern-7mu\sqcup\, i_k}}{i_1!\ldots i_k!} & \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1>\ldots>l_k.
\end{cases}$$

[11]The dual family (i.e. the set of coordinate forms) of a basis lies in the algebraic dual which is here the space of noncommutative series, but as the enveloping algebra under consideration is graded in finite dimensions (here by the multidegree), these series are in fact (multihomogeneous) polynomials.

Hence, since $\{p_w\}_{w\in Y^*}$ is of PBW type then, as in (38)-(40), we get also

$$\begin{aligned}
\mathcal{D}_Y &= \sum_{w\in Y^*} s_w \otimes p_w & (51)\\
&= \sum_{l\in \mathcal{L}ynY} s_l \otimes p_l + \sum_{w\notin \mathcal{L}ynY} s_w \otimes p_w & (52)\\
&= \prod_{l\in \mathcal{L}ynY}^{\searrow} \exp(s_l \otimes p_l). & (53)
\end{aligned}$$

Similarly, by the CQMM theorem, the connected $\mathbb{N}$-graded, co-commutative Hopf algebra $\mathcal{H}_{⧢}$ is isomorphic to the enveloping algebra of

$$\mathrm{Prim}(\mathcal{H}_{⧢}) = \mathrm{Im}(\pi_1) = \mathrm{span}_{\mathbb{Q}}\{\pi_1(w) | w \in Y^*\}, \quad (54)$$

where, for any $w \in Y^*, \pi_1(w)$ is obtained as follows [40]

$$\pi_1(w) = w + \sum_{k\geq 2} \frac{(-1)^{k-1}}{k} \sum_{u_1,\ldots,u_k\in Y^+} \langle w \mid u_1 ⧢ \ldots ⧢ u_k\rangle\, u_1 \ldots u_k. \quad (55)$$

Note that Equation (55) is equivalent to the following identity [40, 41]

$$w = \sum_{k\geq 0} \frac{1}{k!} \sum_{u_1,\ldots,u_k\in Y^*} \langle w \mid u_1 ⧢ \ldots ⧢ u_k\rangle\, \pi_1(u_1)\ldots\pi_1(u_k). \quad (56)$$

In particular, for any $y_k \in Y$, we have successively [40, 41]

$$\begin{aligned}
\pi_1(y_k) &= y_k + \sum_{l\geq 2} \frac{(-1)^{l-1}}{l} \sum_{\substack{j_1,\ldots,j_l\geq 1\\ j_1+\ldots+j_l=k}} y_{j_1}\ldots y_{j_l}, & (57)\\
y_n &= \sum_{k\geq 1} \frac{1}{k!} \sum_{s'_1+\cdots+s'_k=n} \pi_1(y_{s'_1})\ldots\pi_1(y_{s'_k}). & (58)
\end{aligned}$$

Hence, by introducing the new alphabet $\bar{Y} = \{\bar{y}\}_{y\in Y} = \{\pi_1(y)\}_{y\in Y}$, one has

$$(\mathbb{Q}\langle\bar{Y}\rangle, \mathtt{conc}, 1_{\bar{Y}^*}, \Delta_{⧢}) \cong (\mathbb{Q}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}, \Delta_{⧢}) \quad (59)$$

and

$$\begin{aligned}
\mathcal{H}_{⧢} &\cong \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle\bar{Y}\rangle) \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{⧢})), & (60)\\
\mathcal{H}_{⧢}^{\vee} &\cong \mathcal{U}(\mathcal{L}ie_{\mathbb{Q}}\langle\bar{Y}\rangle)^{\vee} \cong \mathcal{U}(\mathrm{Prim}(\mathcal{H}_{⧢}))^{\vee}. & (61)
\end{aligned}$$

By considering

1. the PBW-Lyndon basis $\{\Pi_w\}_{w\in Y^*}$ for $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{⧢}))$ constructed recursively as follows [40]

$$\begin{cases}
\Pi_y = \pi_1(y) & \text{for } y \in Y,\\
\Pi_l = [\Pi_s, \Pi_r] & \text{for } l \in \mathcal{L}ynY, \text{ standard factorization of } l = (s,r),\\
\Pi_w = \Pi_{l_1}^{i_1}\ldots\Pi_{l_k}^{i_k} & \text{for } w = l_1^{i_1}\ldots l_k^{i_k}, l_1 > \ldots > l_k, l_1\ldots,l_k \in \mathcal{L}ynY,
\end{cases}$$

2. and, by duality, the linear basis $\{\Sigma_w\}_{w\in Y^*}$ for $(\mathbb{Q}\langle Y\rangle, \uplus, 1_{Y^*})$, *i.e.*

$$\forall u,v\in Y^*,\qquad \langle \Pi_u \mid \Sigma_v\rangle = \delta_{u,v}.$$

This basis can be computed recursively as follows [4, 40]

$$\begin{cases} \Sigma_y &= y & \text{for } y\in Y,\\ \Sigma_l &= \displaystyle\sum_{\substack{\{s'_1,\cdots,s'_i\}\subset\{s_1,\cdots,s_k\}, l_1\ge\cdots\ge l_n\in\mathcal{L}ynY\\ (y_{s_1}\cdots y_{s_k})\overset{*}{\Leftarrow}(y_{s'_1},\cdots,y_{s'_n},l_1,\cdots,l_n)}} \frac{1}{i!} y_{s'_1+\cdots+s'_i}\Sigma_{l_1\cdots l_n} & \text{for } \substack{l\in\mathcal{L}ynY\\ l=y_{s_1}\cdots y_{s_k}},\\ \Sigma_w &= \dfrac{1}{i_1!\ldots i_k!}\Sigma_{l_1}^{\uplus i_1}\uplus\ldots\uplus\Sigma_{l_k}^{\uplus i_k} & \text{for } \substack{l_1>\ldots>l_k\\ w=l_1^{i_1}\ldots l_k^{i_k}}, \end{cases}$$

Hence, we get the following extended Schützenberger's factorization of $\mathcal{D}_Y$ [40, 41]

$$\mathcal{D}_Y = \prod_{l\in\mathcal{L}ynY}^{\searrow} \exp(\Sigma_l\otimes\Pi_l). \tag{62}$$

Proposition 1 ([40]). *Let $\pi_Y : (\mathbb{Q}\oplus\mathbb{Q}\langle X\rangle x_1, .) \longrightarrow (\mathbb{Q}\langle Y\rangle, .)$ be the morphism mapping $x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1$ to $y_{s_1}\ldots y_{s_r}$ and π_X be its inverse. Its extension over $\mathbb{Q}\langle X\rangle$ verifying $\pi_Y(p)=0$, for $p\in\mathbb{Q}\langle X\rangle x_0$, is still denoted by π_Y. Then*

1. *The polynomials $\{\pi_Y P_{\pi_X l}\}_{l\in\mathcal{L}ynY}$ are homogeneous in weight, linearly independent and upper triangular :*

$$\pi_Y P_{\pi_X l} \quad=\quad \Pi_l + \sum_{v>l,(v)=(l)} p_v v.$$

2. *For any $w\in Y^*$, the following polynomial, homogeneous in weight,*

$$\pi_Y P_{\pi_X w} \quad=\quad \Pi_w + \sum_{v>w,(v)=(w)} c_v v$$

is of multi-degree (w) and $\{\pi_Y P_{\pi_X w}\}_{w\in Y^*}$ forms a basis for $\mathbb{Q}\langle Y\rangle$.

3. *Let $\{\Theta_w\}_{w\in Y^*}$ be the family of polynomials, homogeneous in weight, in duality with the family $\{\pi_Y P_{\pi_X w}\}_{w\in Y^*}$:*

$$\forall u,v\in Y^*,\quad \langle \pi_Y P_{\pi_X u} \mid \Theta_u\rangle \quad=\quad \delta_{u,v}.$$

Then, the family $\{\Theta_w\}_{w\in Y^*}$ generates freely $(\mathbb{Q}\langle Y\rangle, \uplus)$ and

$$\forall w\in Y^*,\qquad \Theta_w = \Sigma_w + \sum_{v<w,(v)=(w)} d_v v.$$

Remark 1. *The family $\{\Theta_l\}_{l\in\mathcal{L}ynY}$ is not a transcendence basis of $(\mathbb{Q}\langle Y\rangle, \uplus)$ because if $\{\Theta_l\}_{l\in\mathcal{L}ynY}$ constitutes a transcendence basis of $(\mathbb{Q}\langle Y\rangle, \uplus)$ then, for any $l\in\mathcal{L}ynY, \pi_Y P_{\pi_X l}$ is primitive but it is false in general.*

Example 5 ([41]).

$$
\begin{aligned}
\Pi_{y_4} &= y_4 - \frac{1}{2}y_1y_3 - \frac{1}{2}y_2y_2 - \frac{1}{2}y_3y_1 + \frac{1}{3}y_1^2y_2 + \frac{1}{3}y_1y_2y_1 + \frac{1}{3}y_2y_1^2 - \frac{1}{4}y_1^4,\\
\Pi_{y_3y_1} &= y_3y_1 - \frac{1}{2}y_2y_1^2 - y_1y_3 + \frac{1}{2}y_1^2y_2,\\
\Pi_{y_2y_2} &= y_2y_2 - \frac{1}{2}y_2y_1^2 - \frac{1}{2}y_1^2y_2 + \frac{1}{4}y_1^4,\\
\Pi_{y_2y_1^2} &= y_2y_1^2 - 2\,y_1y_2y_1 + y_1^2y_2,\\
\Pi_{y_1y_3} &= y_1y_3 - \frac{1}{2}y_1^2y_2 - \frac{1}{2}y_1y_2y_1 + \frac{1}{3}y_1^4,\\
\Pi_{y_1y_2y_1} &= y_1y_2y_1 - y_1^2y_2,\\
\Pi_{y_1^2y_2} &= y_1^2y_2 - \frac{1}{2}y_1^4,\\
\Pi_{y_1^4} &= y_1^4.\\
\Sigma_{y_4} &= y_4,\\
\Sigma_{y_3y_1} &= \frac{1}{2}y_4 + y_3y_1,\\
\Sigma_{y_2^2} &= \frac{1}{2}y_4 + y_2^2,\\
\Sigma_{y_2y_1^2} &= \frac{1}{6}y_4 + \frac{1}{2}y_3y_1 + \frac{1}{2}y_2y_2 + y_2y_1^2,\\
\Sigma_{y_1y_3} &= y_4 + y_3y_1 + y_1y_3,\\
\Sigma_{y_1y_2y_1} &= \frac{1}{2}y_4 + \frac{1}{2}y_3y_1 + y_2^2 + +y_2y_1^2 + \frac{1}{2}y_1y_3 + y_1y_2y_1,\\
\Sigma_{y_1^2y_2} &= \frac{1}{2}y_4 + y_3y_1 + y_2^2 + y_2y_1^2 + y_1y_3 + y_1y_2y_1 + y_1^2y_2,\\
\Sigma_{y_1^4} &= \frac{1}{24}y_4 + \frac{1}{6}y_3y_1 + \frac{1}{4}y_2^2 + \frac{1}{2}y_2y_1^2 + \frac{1}{6}y_1y_3 + \frac{1}{2}y_1y_2y_1 + \frac{1}{2}y_1^2y_2 + y_1^4.
\end{aligned}
$$

Example 6 ([41]). *Let us express* $\{\pi_Y P_l\}_{l\in\mathcal{L}yn_{\leq 4}Y}$ *in the basis* $\{\Pi_w\}_{w\in Y^*}$ *:*

l	$\pi_Y P_l$
x_0	0
x_1	Π_{y_1}
x_0x_1	$\Pi_{y_2} + \frac{\Pi_{y_1}^2}{2}$
$x_0^2x_1$	$\Pi_{y_3} + \frac{\Pi_{y_2y_1}}{2} + \Pi_{y_1y_2} + \frac{\Pi_{y_1^3}}{6}$
$x_0x_1^2$	$\Pi_{y_2y_1} - 2\Pi_{y_1y_2} - \frac{\Pi_{y_1^3}}{2}$
$x_0^3x_1$	$\Pi_{y_4} + \frac{\Pi_{y_3y_1}}{2} + \Pi_{y_1y_3} + \frac{\Pi_{y_2^2}}{2} + \frac{\Pi_{y_1^2y_2} + \Pi_{y_1}\Pi_{y_2}\Pi_{y_1} + \Pi_{y_2}\Pi_{y_1^2}}{6} + \frac{\Pi_{y_1^4}}{24}$
$x_0^2x_1^2$	$\Pi_{y_3}\Pi_{y_1} - \frac{\Pi_{y_2}\Pi_{y_1^2}}{2} + \frac{\Pi_{y_1}\Pi_{y_2}\Pi_{y_1}}{2} - \Pi_{y_1^2}\Pi_{y_2} - 2\Pi_{y_2^2} - \frac{\Pi_{y_1^4}}{3}$
$x_0x_1^3$	$\Pi_{y_2}\Pi_{y_1^2} - 3\Pi_{y_1}\Pi_{y_2}\Pi_{y_1} + 3\Pi_{y_1^2}\Pi_{y_2} + \frac{\Pi_{y_1^4}}{2}$

2.2 Indiscernability over a class of formal power series

2.2.1 Residual calculus and representative series

Definition 1. *Let $S \in \mathbb{Q}\langle\langle X\rangle\rangle$ and let $P \in \mathbb{Q}\langle X\rangle$. The left (resp. right)* residual *of S by P is the formal power series $P \triangleleft S$ (resp. $S \triangleright P$) in $\mathbb{Q}\langle\langle X\rangle\rangle$ defined by:*

$$\langle P \triangleleft S \mid w\rangle = \langle S \mid wP\rangle \qquad (\textit{resp.} \quad \langle S \triangleright P \mid w\rangle = \langle S \mid Pw\rangle).$$

We straightforwardly get, for any $P, Q \in \mathbb{Q}\langle X\rangle$:

$$P \triangleleft (Q \triangleleft S) = PQ \triangleleft S,\ (S \triangleright P) \triangleright Q = S \triangleright PQ,\ (P \triangleleft S) \triangleright Q = P \triangleleft (S \triangleright Q). \quad (63)$$

In case $x, y \in X$ and $w \in X^*$, we get[12]:

$$x \triangleleft (wy) = \delta_{x,y} w \qquad \text{and} \qquad xw \triangleright y = \delta_{x,y} w. \quad (64)$$

Lemma 1. (Reconstruction lemma) *Let $S \in \mathbb{Q}\langle\langle X\rangle\rangle$. Then*

$$S = \langle S \mid 1_{X^*}\rangle + \sum_{x \in X} x(S \triangleright x) = \langle S \mid 1_{X^*}\rangle + \sum_{x \in X} (x \triangleleft S)x.$$

Lemma 2. *The residuals by a letter x is a derivation of $(\mathbb{Q}\langle\langle X\rangle\rangle, ⧢)$:*

$$x \triangleleft (u ⧢ v) = (x \triangleleft u) ⧢ v + u ⧢ (x \triangleleft v), \qquad (u ⧢ v) \triangleright x = (u \triangleright x) ⧢ v + u ⧢ (v \triangleright x).$$

Proof. Use the recursive definitions of the shuffle product. □

Consequently,

Lemma 3. *For any Lie polynomial $Q \in \mathcal{L}ie_{\mathbb{Q}}\langle X\rangle$, the linear maps "$Q\triangleleft$" and "$\triangleright Q$" are derivations on $(\mathbb{Q}[\mathcal{L}ynX], ⧢)$.*

Proof. For any $l, l_1, l_2 \in \mathcal{L}ynX$, we have

$$\hat{l} \triangleleft (l_1 ⧢ l_2) = l_1 ⧢ (\hat{l} \triangleleft l_2) + (\hat{l} \triangleleft l_1) ⧢ l_2 = l_1 \delta_{l_2,\hat{l}} + \delta_{l_1,\hat{l}} l_2,$$
$$(l_1 ⧢ l_2) \triangleright \hat{l} = l_1 ⧢ (l_2 \triangleright \hat{l}) + (l_1 \triangleright \hat{l}) ⧢ l_2 = l_1 \delta_{l_2,\hat{l}} + \delta_{l_1,\hat{l}} l_2.$$

□

Lemma 4. *For any Lyndon word $l \in \mathcal{L}ynX$ and $\check{S}_l$ defined as in (**??**), one has*

$$x_1 \triangleleft l = l \triangleright x_0 = 0 \qquad \textit{and} \qquad x_1 \triangleleft \check{S}_l = \check{S}_l \triangleright x_0 = 0.$$

Proof. Since $x_1\triangleleft$ and $\triangleright x_0$ are derivations and for any $l \in \mathcal{L}ynX - X$, the polynomial $\check{S}_l$ belongs to $x_0\mathbb{Q}\langle X\rangle x_1$ then the expected results follow. □

Theorem 1. (On representative series) *The following properties are equivalent for any series $S \in \mathbb{Q}\langle\langle X\rangle\rangle$:*

[12] For any words u and $v \in X^*$, if $u = v$ then $\delta_{u,v} = 1$ else 0.

(1) *The left* $\mathbb{C}$*-module* $Res_g(S) = span\{w \triangleleft S \mid w \in X^*\}$ *is finite dimensional.*

(2) *The right* $\mathbb{C}$*-module* $Res_d(S) = span\{S \triangleright w \mid w \in X^*\}$ *is finite dimensional.*

(3) *There are matrices* $\lambda \in \mathcal{M}_{1,n}(\mathbb{Q})$, $\eta \in \mathcal{M}_{n,1}(\mathbb{Q})$ *and a representation of* X^* *in* $\mathcal{M}_{n,n}$*, such that*

$$\begin{aligned} S &= \sum_{w \in X^*} [\lambda\mu(w)\eta]\, w, \\ &= \lambda\Big(\prod_{l \in \mathcal{L}ynX}^{\searrow} e^{\mu(S_l)\ P_l}\Big)\eta. \end{aligned}$$

A series that satisfies the items of this theorem will be called *representative series*. This concept can be found in [1, 44, 15]. The two first items are in [19, 24]. The third item can be deduced from [7, 13] for example and it was used to factorize first time, by Lyndon words, the output of bilinear and analytical dynamical systems respectively in [26, 27] and to study polylogarithms, hypergeometric functions and associated functions in [29, 31, 38]. The dimension of $Res_g(S)$ is equal to that of $Res_d(S)$, and to the minimal dimension of a representation satisfying the third point of Theorem 1. This rank is then equal to the rank of the Hankel matrix of S, that is the infinite matrix $(\langle S \mid uv\rangle)_{u,v \in X}$ indexed by $X^* \times X^*$ and is also called *Hankel rank* of S [19, 24]:

Definition 2. ([19, 24]) *The* Hankel rank *of a formal power series* $S \in \mathbb{C}\langle\langle X\rangle\rangle$ *is the dimension of the vector space*

$$\{S \triangleright \Pi \mid \Pi \in \mathbb{C}\langle X\rangle\}, \qquad (resp. \qquad \{\Pi \triangleleft S \mid \Pi \in \mathbb{C}\langle X\rangle\}.$$

The triplet (λ, μ, η) is called a *linear representation* of S. We define the minimal representation[13] of S as being a representation of S of minimal dimension.

For any proper series S, the following power series is called "star of S"

$$S^* = 1 + S + S^2 + \cdots + S^n + \ldots. \tag{65}$$

Definition 3. ([3, 49]) *A series* S *is called* rational *if it belongs to the closure in* $\mathbb{Q}\langle\langle X\rangle\rangle$ *of the noncommutative polynomial algebra by sum, product, and star operation of* proper[14] *elements. The set of rational power series will be denoted by* $\mathbb{Q}^{rat}\langle\langle X\rangle\rangle$*.*

Lemma 5. *For any noncommutative rational series (resp. polynomial)* R *and for any polynomial* P*, the left and right residuals of* R *by* P *are rational (resp. polynomial).*

Theorem 2. (Schützenberger, [3, 49]) *Any noncommutative power series is representative if and only if it is rational.*

[13] It can be shown that all minimal representations are isomorphic (see [3]).

[14] A series S is said to be proper if $\langle S \mid \epsilon\rangle = 0$.

Example 7. *Let $S = (x_0x_1)^*$ be the rational power sries over $X = \{x_0, x_1\}$. Since*

$$S = (x_0x_1)^* = 1_{X^*} + x_0x_1 + (x_0x_1)^2 + \dots$$

then

$$\begin{array}{rcl} S \triangleright x_0 &=& x_1 + x_1x_0x_1 + \dots, \\ S \triangleright x_1 &=& 0, \\ S \triangleright x_0^2 &=& 0, \\ S \triangleright x_0x_1 &=& S. \end{array}$$

The Hankel rank of S is then 3.

2.2.2 Continuity and indiscernability

Definition 4. ([25, 39]) *Let $\mathcal{H}$ be a class of formal power series over X and let $S \in \mathbb{C}\langle\langle X\rangle\rangle$.*

(1) *S is said to be* continuous[15] *over $\mathcal{H}$ if for any $\Phi \in \mathcal{H}$, the following sum, denoted by $\langle S \parallel \Phi\rangle$, is convergent in norm*

$$\sum_{w \in X^*} \langle S \mid w\rangle\langle \Phi \mid w\rangle.$$

The set of continuous power series over $\mathcal{H}$ will be denoted by $\mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$.

(2) *S is said to be* indiscernable[16] *over $\mathcal{H}$ if and only if*

$$\forall \Phi \in \mathcal{H}, \qquad \langle S \parallel \Phi\rangle = 0.$$

Lemma 6. *Let $\mathcal{H}$ be a monoid containing $\{e^{tx}\}_{x \in X}^{t \in \mathbb{C}}$. Let $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ be indiscernable over $\mathcal{H}$. Then for any $x \in X$, $x \triangleleft S$ and $S \triangleright x$ belong to $\mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ and they are indiscernable over $\mathcal{H}$.*

Proof. Let us calculate $\langle x \triangleleft S \parallel \Phi\rangle = \langle S \parallel \Phi x\rangle$ and $\langle S \triangleright x \parallel \Phi\rangle = \langle S \parallel x\Phi\rangle$. Since

$$\lim_{t \to 0} \frac{e^{tx} - 1}{t} = x \qquad \text{and} \qquad \lim_{t \to 0} \frac{e^{tx} - 1}{t} = x$$

then, for any $\Phi \in \mathcal{H}$, by uniform convergence, one has

$$\begin{aligned} \langle S \parallel \Phi x\rangle &= \langle S \parallel \lim_{t \to 0} \Phi \frac{e^{tx} - 1}{t}\rangle = \lim_{t \to 0} \langle S \parallel \Phi \frac{e^{tx} - 1}{t}\rangle, \\ \langle S \parallel x\Phi\rangle &= \langle S \parallel \lim_{t \to 0} \frac{e^{tx} - 1}{t}\Phi\rangle = \lim_{t \to 0} \langle S \parallel \frac{e^{tx} - 1}{t}\Phi\rangle. \end{aligned}$$

[15] See [25, 39] for a convergence criterion and an example of continuous generating series.
[16] Here, we adapt this notion developped in [25] via the residual calculus.

Since S is indiscernable over $\mathcal{H}$ then

$$\langle S \parallel \Phi x\rangle = \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi e^{tx}\rangle - \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi\rangle = 0,$$
$$\langle S \parallel x\Phi\rangle = \lim_{t\to 0}\frac{1}{t}\langle S \parallel e^{tx}\Phi\rangle - \lim_{t\to 0}\frac{1}{t}\langle S \parallel \Phi\rangle = 0.$$

□

Proposition 2. *Let $\mathcal{H}$ be a monoid containing $\{e^{tx}\}_{x\in X}^{t\in\mathbb{C}}$. The formal power series $S \in \mathbb{C}^{\mathrm{cont}}\langle\langle X\rangle\rangle$ is indiscernable over $\mathcal{H}$ if and only if $S = 0$.*

Proof. If $S = 0$ then it is immediate that S is indiscernable over $\mathcal{H}$. Conversely, if S is indiscernable over $\mathcal{H}$ then by Lemma 6, for any word $w \in X^*$ and, by induction on the length of w, $w \triangleleft S$ is indiscernable over $\mathcal{H}$. In particular,

$$\langle w \triangleleft S \parallel \mathrm{Id}_{\mathcal{H}}\rangle = \langle S \mid w\rangle = 0.$$

In other words, $S = 0$. □

2.3 Polylogarithms and harmonic sums

2.3.1 Structures of polylogarithms and harmonic sums

Let Y be the infinite alphabet $\{y_i\}_{i\geq 1}$ equipped with the order $y_1 > y_2 > y_3 > \ldots$ and let $\mathcal{L}ynY$ be the set of Lyndon words over Y. The length of $w = y_{s_1}\ldots y_{s_r} \in Y^*$ is denoted by $\mid w \mid$ and its degree equals $s_1 + \cdots + s_r$.

Let X be the finite alphabet $\{x_0, x_1\}$ equipped with the order $x_0 < x_1$ and

$$\mathcal{C} := \mathbb{C}\left[z, \frac{1}{z}, \frac{1}{1-z}\right] \qquad \text{and} \qquad \mathcal{G} := \left\{z, \frac{1}{z}, \frac{z-1}{z}, \frac{z}{z-1}, \frac{1}{1-z}, 1-z\right\} \tag{66}$$

This ring $\mathcal{C}$ is invariant under differentiation and under the homographic transformations belonging to the group $\mathcal{G}$ whose elements commute the singularities $\{0, 1, +\infty\}$. The iterated integral over ω_0, ω_1 associated to the word $w = x_{i_1}\cdots x_{i_k}$ over X^* (the monoid generated by X) and along the integration path $z_0 \rightsquigarrow z$ is the following multiple integral defined by

$$\int_{z_0\rightsquigarrow z} \omega_{i_1}\cdots\omega_{i_k} = \int_{z_0}^{z}\omega_{i_1}(t_1)\int_{z_0}^{t_1}\omega_{i_2}(t_2)\ldots\int_{z_0}^{t_{r-1}}\omega_{i_r}(t_r), \tag{67}$$

where $t_1\cdots t_{r-1}$ is a subdivision of the path $z_0 \rightsquigarrow z$. In a shortened notation, we denote this integral by $\alpha_{z_0}^{z}(w)$ and[17] $\alpha_{z_0}^{z}(1_{X^*}) = 1$. One can check that the polylogarithm $\mathrm{Li}_{s_1,\ldots,s_r}$ is also the value of the iterated integral over ω_0, ω_1 and along the integration path $0 \rightsquigarrow z$ [29, 31]:

$$\mathrm{Li}_w(z) \quad = \quad \alpha_0^z(x_0^{s_1-1}x_1\ldots x_0^{s_r-1}x_1). \tag{68}$$

[17]Here, 1_{X^*} stands for the empty word over X.

Example 8.

$$\begin{aligned}
\alpha_0^z(x_0x_1) &= \int_0^z \frac{ds}{s}\int_0^s \frac{dt}{1-t} \\
&= \int_0^z \frac{ds}{s}\int_0^s dt\sum_{k\geq 0} t^k \\
&= \sum_{k\geq 1}\int_0^z ds\frac{s^{k-1}}{k} \\
&= \sum_{k\geq 1}\frac{z^k}{k^2} \\
&= \mathrm{Li}_2(z).
\end{aligned}$$

The definition of polylogarithms is extended over the words $w \in X^*$ by putting $\mathrm{Li}_{x_0}(z) := \log(z)$. Thus, $\{\mathrm{Li}_w\}_{w\in X^*}$, and then $\{\mathrm{P}_w(z) := (1-z)^{-1}\mathrm{Li}_w(z)\}_{w\in Y^*}$, are $\mathcal{C}$-linearly independent [35, 32]. Since, for any $v = y_{s_1}\dots y_{s_r} \in Y^*, \mathrm{P}_v$ is the ordinary generating function of the sequence $\{\mathrm{H}_v(N)\}_{N\geq 0}$ [37]:

$$\mathrm{P}_v(z) = \sum_{N\geq 0}\mathrm{H}_v(N)\, z^N, \quad \text{with} \quad \mathrm{H}_v(N) = \sum_{N\geq n_1>\dots>n_r>0}\frac{1}{n_1^{s_1}\dots n_r^{s_r}} \tag{69}$$

then, as a consequence of the classical isomorphism between convergent Taylor series and their associated sums, the harmonic sums $\{\mathrm{H}_w\}_{w\in Y^*}$ are also $\mathbb{C}$-linearly independent. Firstly, $\ker\mathrm{P} = \{0\}$ and $\ker\mathrm{H} = \{0\}$, and secondly, P is a morphism for the Hadamard product:

$$\mathrm{P}_u(z)\odot\mathrm{P}_v(z) = \sum_{N\geq 0}\mathrm{H}_u(N)\mathrm{H}_v(N)z^N = \sum_{N\geq 0}\mathrm{H}_{u \,⧢\, v}(N)z^N = \mathrm{P}_{u \,⧢\, v}(z). \tag{70}$$

Proposition 3. ([37]) *Extended by linearity, the map*

$$\begin{aligned}
\mathrm{P} : (\mathbb{C}\langle Y\rangle, ⧢) &\longrightarrow (\mathbb{C}\{\mathrm{P}_w\}_{w\in Y^*}, \odot), \\
u &\longmapsto \mathrm{P}_u
\end{aligned}$$

is an isomorphism of algebras. Moreover, the map

$$\begin{aligned}
\mathrm{H} : (\mathbb{C}\langle Y\rangle, ⧢) &\longrightarrow (\mathbb{C}\{\mathrm{H}_w\}_{w\in Y^*}, .), \\
u &\longmapsto \mathrm{H}_u = \{\mathrm{H}_u(N)\}_{N\geq 0}
\end{aligned}$$

is an isomorphism of algebras.

Studying the equivalence between action of $\{(1-z)^l\}_{l\in\mathbb{Z}}$ over $\{\mathrm{P}_w(z)\}_{w\in Y^*}$ and that of $\{N^k\}_{k\in\mathbb{Z}}$ over $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ (see [17]), we have

Theorem 3. ([39]) *The Hadamard $\mathcal{C}$-algebra of $\{\mathrm{P}_w\}_{w\in Y^*}$ can be identified with that of $\{\mathrm{P}_l\}_{l\in\mathcal{L}ynY}$. In the same way, the algebra of harmonic sums $\{\mathrm{H}_w\}_{w\in Y^*}$ with polynomial coefficients can be identified with that of $\{\mathrm{H}_l\}_{l\in\mathcal{L}ynY}$.*

2.3.2 Results *à la Abel* for noncommutative generating and global regularizations

Let L and H be the noncommutative generating series of respectively $\{\mathrm{Li}_w\}_{w\in X^*}$ and $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ [35, 37]:

$$\mathrm{L} = \sum_{w\in X^*} \mathrm{Li}_w\ w \quad \text{and} \quad \mathrm{H} = \sum_{w\in Y^*} \mathrm{H}_w\ w. \tag{71}$$

Hence, L and H are image, by the tensor products $\mathrm{Li}\otimes\mathrm{Id}$ and $\mathrm{H}\otimes\mathrm{Id}$, of the diagonal series $\mathcal{D}_X$ and $\mathcal{D}_Y$ respectively. We get

Theorem 4 (Factorization of L and of H, [35, 32, 40])**.** *Let*

$$\mathrm{L}_{\mathrm{reg}} = \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{\mathrm{Li}_{S_l}\ P_l} \quad \textit{and} \quad \mathrm{H}_{\mathrm{reg}}(N) = \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} e^{\mathrm{H}_{\Sigma_l}(N)\ \Sigma_l}.$$

Then $\mathrm{L}(z) = e^{-x_1\log(1-z)}\mathrm{L}_{\mathrm{reg}}(z)e^{x_0\log z}$ *and* $\mathrm{H}(N) = e^{\mathrm{H}_{y_1}(N)\ y_1}\mathrm{H}_{\mathrm{reg}}(N)$.

For any $l\in\mathcal{L}ynX-X$ (resp. $\mathcal{L}ynY-\{y_1\}$), the polynomial S_l (resp. Σ_l) is a finite combination of words in $x_0X^*x_1$ (resp. $Y^*-y_1Y^*$). Then we can state the following

Proposition 4 ([40])**.** *Let* $Z_{⧢} := \mathrm{L}_{\mathrm{reg}}(1)$ *and* $Z_{\text{⧢}\!-} := \mathrm{H}_{\mathrm{reg}}(\infty)$.
Then $Z_{⧢}$ *and* $Z_{\text{⧢}\!-}$ *are group-like, for* $\Delta_{⧢}$ *and* $\Delta_{\text{⧢}\!-}$ *respectively.*

Proposition 5 (Successive differentiations of L, [38])**.** *Let* $\partial_z = d/dz$ *and let* $\theta_0 = zd/dz$. *For any* $n\in\mathbb{N}$, *we have*

$$\partial_z^n\mathrm{L}(z) = D_n(z)\mathrm{L}(z) \quad \textit{and} \quad \theta_0^n\mathrm{L}(z) = E_n(z)\mathrm{L}(z),$$

where[18]

- *the polynomials* $D_n(z)$ *and* $E_n(z)$ *in* $\mathcal{C}\langle X\rangle$ *are defined as follows*

$$\begin{aligned} D_n(z) &= \sum_{\mathrm{wgt}(\mathbf{r})=n}\ \sum_{w\in X^{\deg(\mathbf{r})}}\ \prod_{i=1}^{\deg(\mathbf{r})} \binom{\sum_{j=1}^{i} r_i + j - 1}{r_i}\tau_{\mathbf{r}}(w), \\ E_n(z) &= \sum_{\mathrm{wgt}(\mathbf{r})=n}\ \sum_{w\in X^{\deg(\mathbf{r})}}\ \prod_{i=1}^{\deg(\mathbf{r})} \binom{\sum_{j=1}^{i} r_i + j - 1}{r_i}\rho_{\mathbf{r}}(w), \end{aligned}$$

- *for any* $w = x_{i_1}\cdots x_{i_k}$ *and* $\mathbf{r} = (r_1,\ldots,r_k)$ *of degree* $\deg(\mathbf{r}) = k$ *and of weight* $\mathrm{wgt}(\mathbf{r}) = k+r_1+\cdots+r_k$, *the polynomials* $\tau_{\mathbf{r}}(w) = \tau_{r_1}(x_{i_1})\cdots\tau_{r_k}(x_{i_k})$ *and* $\rho_{\mathbf{r}}(w) = \rho_{r_1}(x_{i_1})\cdots\rho_{r_k}(x_{i_k})$ *are defined respectively by, for any* $r\in\mathbb{N}$,

$$\tau_r(x_0) = \partial_z^r\frac{x_0}{z} = \frac{-r!x_0}{(-z)^{r+1}} \quad \textit{and} \quad \tau_r(x_1) = \partial_z^r\frac{x_1}{1-z} = \frac{r!x_1}{(1-z)^{r+1}},$$

$$\rho_r(x_0) = \theta_0^r\frac{x_0}{z} = 0 \quad \textit{and} \quad \rho_r(x_1) = \theta_0^r\frac{zx_1}{1-z} = \mathrm{Li}_{-r}(z)x_1.$$

[18] Consider also $\theta_1 = (1-z)d/dz$. But $\theta_0+\theta_1 = \partial_z$ and then $\theta_1^n\mathrm{L}(z) = [D_n(z)-E_n(z)]\mathrm{L}(z)$.

Example 9 (Coefficients of $\theta_0^n \mathrm{L}$)**.** *Since, for any $w \in X^+$, one has*

$$\theta_0 \operatorname{Li}_w = \begin{cases} \operatorname{Li}_u & \text{if} \quad w = x_0 u, \\ \operatorname{Li}_0 \operatorname{Li}_u & \text{if} \quad w = x_1 u. \end{cases}$$

then one obtains, for example

- *For any $n \geq 1$ and for any $w \in X^*$, one has*

  $$\theta_0^n \operatorname{Li}_{x_0^n w}(z) = \operatorname{Li}_w(z).$$

  In particular,

  $$\theta_0^n \operatorname{Li}_{x_0^n}(z) = 1.$$

- *Letting $E(n,m)$ denote the Eulerian numbers :*

  $$E(n,m) = \sum_{j=0}^{m} (-1)^j \binom{n+1}{j} (m+1-j)^n,$$

  one has, for any $n \geq 0$,

  $$\theta_0^{n+1} \operatorname{Li}_{x_1}(z) = \frac{z}{(1-z)^{n+1}} \sum_{m=0}^{n} E(n,m) x^m =: \operatorname{Li}_{-n}(z).$$

  In particular,

  $$\theta_0 \operatorname{Li}_{x_1}(z) = \frac{z}{(1-z)} = \operatorname{Li}_0(z).$$

- *For any $k \in \mathbb{N}^*$, one has*

  $$\theta_0^k \operatorname{Li}_{x_1^2}(z) = \sum_{j=0}^{k-1} \binom{k-1}{j} \operatorname{Li}_{-j}(z) \operatorname{Li}_{2+j-k}(z).$$

  For example

  $$\begin{aligned} \theta_0 \operatorname{Li}_{x_1^2}(z) &= \operatorname{Li}_0(z) \operatorname{Li}_1(z), \\ \theta_0^2 \operatorname{Li}_{x_1^2}(z) &= \operatorname{Li}_{-1}(z) \operatorname{Li}_1(z) + \operatorname{Li}_0^2(z), \\ \theta_0^3 \operatorname{Li}_{x_1^2}(z) &= \operatorname{Li}_{-2}(z) \operatorname{Li}_1(z) + 3 \operatorname{Li}_{-1}(z) \operatorname{Li}_0(z), \\ \theta_0^4 \operatorname{Li}_{x_1^2}(z) &= \operatorname{Li}_{-3}(z) \operatorname{Li}_1(z) + 4 \operatorname{Li}_{-2}(z) \operatorname{Li}_0(z) + 3 \operatorname{Li}_{-1}^2(z), \\ &\vdots \end{aligned}$$

The noncommutative generating series of polylogarithms [35, 32] satisfies the differential equation

$$dL \quad = \quad (x_0\omega_0 + x_1\omega_1)L \tag{72}$$

with boundary condition

$$L(z) \underset{z\to 0}{\widetilde{\quad}} \exp(x_0 \log z) \quad \text{and} \quad L(z) \underset{z\to 1}{\widetilde{\quad}} \exp(-x_1\log(1-z))\, Z_{\sqcup\!\sqcup}. \tag{73}$$

This implies L is the exponential of a Lie series [35, 32]. Hence [38],

$$\begin{aligned}\log L(z) &= \sum_{k\geq 1}\frac{(-1)^{k-1}}{k}\sum_{u_1,\ldots,u_k\in X^+}\mathrm{Li}_{u_1\sqcup\!\sqcup\ldots\sqcup\!\sqcup u_k}(z)\, u_1\ldots u_k &(74)\\ &= \sum_{w\in X^*}\mathrm{Li}_w(z)\,\pi_1(w). &(75)\end{aligned}$$

Proposition 6 ([38])**.** *Let $G(z), H(z)$ be exponential solutions of (72). Then there exists a constant Lie series C such that $G(z) = H(z)e^C$.*

Theorem 5 ([38])**.** *Let $\mathrm{Gal}_{\mathbb{C}}(DE)$ be the differential Galois group associated to the Drinfel'd equation. Then $\mathrm{Gal}_{\mathbb{C}}(DE) = \{e^C \mid C \in \mathcal{L}ie_{\mathbb{C}}\langle\langle X\rangle\rangle\}$ and it constains the monodromy group defined as follows*

$$\mathcal{M}_0 L = Le^{2i\pi\mathfrak{m}_0} \quad \textit{and} \quad \mathcal{M}_1 L = LZ_{\sqcup\!\sqcup}^{-1}e^{-2i\pi x_1}Z_{\sqcup\!\sqcup} = Le^{2i\pi\mathfrak{m}_1},$$

$$\textit{where} \quad \mathfrak{m}_0 = x_0 \quad \textit{and} \quad \mathfrak{m}_1 = \prod_{l\in\mathcal{L}ynX-X}^{\searrow} e^{-\zeta(S_l)\,\mathrm{ad}_{P_l}}(-x_1).$$

Let us call $\mathrm{LI}_{\mathcal{C}}$ the smallest algebra containing $\mathcal{C}$, closed under derivation and under integration with respect to ω_0 and ω_1. It is the $\mathcal{C}$-module generated by the polylogarithms $\{\mathrm{Li}_w\}_{w\in X^*}$.

Let $\Lambda(z) := \pi_Y L(z)$. Then [39]

$$\Lambda(z)\underset{z\to 1}{\widetilde{\quad}}\exp\left(y_1\log\frac{1}{1-z}\right)\pi_Y Z_{\sqcup\!\sqcup}. \tag{76}$$

Since the coefficient of z^N in the ordinary Taylor expansion of $\mathrm{P}_{y_1^k}(z)$ is $\mathrm{H}_{y_1^k}(N)$ then let us put [39]

$$\mathrm{Mono}(z) := e^{-(x_1+1)\log(1-z)} = \sum_{k\geq 0}\mathrm{P}_{y_1^k}(z)\, y_1^k \tag{77}$$

$$\mathrm{Const} := \sum_{k\geq 0}\mathrm{H}_{y_1^k}\, y_1^k = \exp\left(-\sum_{k\geq 1}\mathrm{H}_{y_k}\frac{(-y_1)^k}{k}\right). \tag{78}$$

We put also[19] [39]

$$B(y_1) := \exp\left(\sum_{k\geq 1}\zeta(y_k)\frac{(-y_1)^k}{k}\right) \quad \text{and} \quad B'(y_1) := \exp(\gamma y_1)B(y_1). \tag{79}$$

[19] Here, the coefficient $\langle B(y_1) \mid y_1^k\rangle$ corresponds to the Euler–Mac Laurin constant associated to $\langle \mathrm{Const}(N) \mid y_1^k\rangle$, *i.e.* the finite party of its asymptotic expansion in the scale of comparison $\{n^a\log^b(n)\}_{a\in\mathbb{Z},b\in\mathbb{N}}$.

As a consequence of (76)-(78), one gets

Theorem 6 ([39])**.**

$$\pi_Y \mathrm{P}(z) \widetilde{_{z\to 1}} \mathrm{Mono}(z)\pi_Y Z_{\sqcup\!\sqcup} \qquad \textit{and} \qquad \mathrm{H}(N) \widetilde{_{N\to\infty}} \mathrm{Const}(N)\pi_Y Z_{\sqcup\!\sqcup}.$$

Or equivalently,

$$\lim_{z\to 1} \exp\left(y_1 \log\frac{1}{1-z}\right)\Lambda(z) = \lim_{N\to\infty} \exp\left(\sum_{k\geq 1} \mathrm{H}_{y_k}(N)\frac{(-y_1)^k}{k}\right)\mathrm{H}(N) = \pi_Y Z_{\sqcup\!\sqcup}.$$

Theorem 7 ([17])**.** *For any $g \in \mathcal{C}\{\mathrm{P}_w\}_{w\in Y^*}$, there exist algorithmically computable coefficients $c_j \in \mathbb{C}, \alpha_j \in \mathbb{Z}, \beta_j \in \mathbb{N}$ such that*

$$g(z) \quad \sim \quad \sum_{j=0}^{+\infty} c_j(1-z)^{\alpha_j} \log^{\beta_j}(1-z) \quad \textit{for } z \to 1,$$

and the coefficients $b_i \in \mathbb{C}, \eta_i \in \mathbb{Z}, \kappa_i \in \mathbb{N}$ such that

$$\langle g(z) \mid z^n\rangle \quad \sim \quad \sum_{i=0}^{+\infty} b_i n^{\eta_i} \log^{\kappa_i}(n) \quad \textit{for } n \to \infty.$$

Theorem 7 means also that the $\{\mathrm{P}_w\}_{w\in Y^*}$ admit a full singular expansion, at $z = 1$, and then their ordinary Taylor coefficients, $\{\mathrm{H}_w(N)\}_{w\in Y^*}$ admit a full asymptotic expansion, for $N \to +\infty$. More precisely,

Corollary 1. *For any $w \in Y^*$ and for any $k \geq 1$, we have*

$$\mathrm{H}_w(N) \quad = \quad \sum_{i=1}^{|w|} \alpha_i \log^i(N) + \gamma_w + \sum_{j=1}^{k}\sum_{i=0}^{|w|-1} \beta_{i,j}\frac{1}{N^j}\log^i(N) + \mathrm{O}\left(\frac{1}{N^k}\right).$$

where the coefficients γ_w, α_i and $\beta_{i,j}$ belong to the $\mathbb{Q}[\gamma]$-algebra generated by convergent polyzetas.

More generally, by Theorem 5, we get

Proposition 7. *For any commutative $\mathbb{Q}$-algebra A and for any Lie series $C \in \mathcal{L}ie_A\langle X\rangle$, let*

$$\overline{\mathrm{L}} = \mathrm{L}e^C, \quad \overline{\Lambda} = \pi_Y\overline{\mathrm{L}}, \quad \overline{\mathrm{P}}(z) = \frac{\overline{\Lambda}(z)}{1-z}.$$

Then

1. $\overline{Z}_{\sqcup\!\sqcup} = Z_{\sqcup\!\sqcup} e^C$ *is group-like, for the co-product* $\Delta_{\sqcup\!\sqcup}$,
2. $\overline{\mathrm{L}}(z) \widetilde{_{z\to 1}} \exp(-x_1\log(1-z))\,\overline{Z}_{\sqcup\!\sqcup}$,
3. $\overline{\mathrm{P}}(z) \widetilde{_{z\to 1}} \mathrm{Mono}(z)\pi_Y\overline{Z}_{\sqcup\!\sqcup}$,

4. $\overline{\mathrm{H}}(N) \underset{N\to\infty}{\widetilde{\quad}} \mathrm{Const}(N)\pi_Y \overline{Z}_{\sqcup\!\sqcup}$,

where, for any $w \in Y^*$ and $N \geq 0$, one defines the coefficient $\langle \overline{\mathrm{H}}(N) \mid w\rangle$ of w in the power series $\overline{\mathrm{H}}(N)$ as the coefficient $\langle \overline{\mathrm{P}}_w(z) \mid z^N\rangle$ of z^N in the ordinary Taylor expansion of the polylogarithmic function $\overline{\mathrm{P}}_w(z)$.

By Proposition 7, we get successively

Proposition 8 ([33])**.** *Let $\overline{\zeta}_{\sqcup\!\sqcup}$ and $\overline{\zeta}_{\uplus}$ be the characters of respectively $(A\langle X\rangle, \sqcup\!\sqcup)$ and $(A\langle Y\rangle, \uplus)$ satisfying*

$$\overline{\zeta}_{\sqcup\!\sqcup}(x_0) = \overline{\zeta}_{\sqcup\!\sqcup}(x_1) = 0 \quad \text{and} \quad \overline{\zeta}_{\uplus}(y_1) = 0.$$

Then

$$\sum_{w\in Y^*} \overline{\zeta}_{\sqcup\!\sqcup}(w)\, w = \overline{Z}_{\sqcup\!\sqcup} = \prod_{l\in\mathcal{L}ynX-X}^{\searrow} \exp(\overline{\zeta}(S_l)\; P_l),$$

$$\sum_{w\in Y^*} \overline{\zeta}_{\uplus}(w)\, w = \overline{Z}_{\uplus} = \prod_{l\in\mathcal{L}ynY-\{y_1\}}^{\searrow} \exp(\overline{\zeta}(\Sigma_l)\; \Pi_l).$$

Proposition 9. *Let $\{\overline{\gamma}_w\}_{w\in Y^*}$ be the Euler–Mac Laurin constants associated to $\{\overline{\mathrm{H}}_w(N)\}_{w\in Y^*}$. Let $\overline{Z}_\gamma$ be its noncommutative generating series. Then,*

1. *The following map realizes a character :*

$$\begin{array}{rcccl} \overline{\gamma}_\bullet & : & (A\langle Y\rangle, \uplus) & \longrightarrow & (\mathbb{R}, .), \\ & & w & \longmapsto & \langle \overline{\gamma}_\bullet \mid w\rangle = \overline{\gamma}_w. \end{array}$$

2. *The noncommutative power series $\overline{Z}_\gamma$ is group-like, for $\Delta_{\uplus}$.*
3. *There exists a group-like element $\overline{Z}_{\uplus}$, for the co-product $\Delta_{\uplus}$, such that*

$$\overline{Z}_\gamma = \sum_{w\in Y^*} \overline{\gamma}_w\, w = \exp(\gamma y_1)\overline{Z}_{\uplus}.$$

By Theorem 6, Propositions 7 and 9, we also get

Proposition 10. *For any $C \in \mathcal{L}ie_A\langle X\rangle$ such that $\overline{Z}_{\sqcup\!\sqcup} = Z_{\sqcup\!\sqcup} e^C$. Then*

$$\overline{Z}_\gamma = B(y_1)\pi_Y \overline{Z}_{\sqcup\!\sqcup}, \quad \text{or equivalently by cancellation,} \quad \overline{Z}_{\uplus} = B'(y_1)\pi_Y \overline{Z}_{\sqcup\!\sqcup},$$

where $B(y_1)$ and $B'(y_1)$ are given in (79).

By Propositions 7, the noncommutative generationg series $\overline{Z}_{\sqcup\!\sqcup}$ and $\overline{Z}_{\uplus}$ are group-like, for the co-product $\Delta_{\sqcup\!\sqcup}$ and $\Delta_{\uplus}$ respectively, and we also have

$$\overline{Z}_{\sqcup\!\sqcup} = \sum_{l\in\mathcal{L}ynX-X} \overline{\zeta}(S_l)\; P_l + \sum_{w\notin\mathcal{L}ynX-X} \overline{\zeta}_{\sqcup\!\sqcup}(S_w)\; P_w, \tag{80}$$

$$\overline{Z}_{\uplus} = \sum_{l\in\mathcal{L}ynY-\{y_1\}} \overline{\zeta}(\Sigma_l)\; \Pi_l + \sum_{w\notin\mathcal{L}ynY-\{y_1\}} \overline{\zeta}_{\uplus}(\Sigma_w)\; \Pi_w. \tag{81}$$

Hence, by Proposition 10, we deduce in particular,

$$\sum_{l\in\mathcal{L}ynY-\{y_1\}} \overline{\zeta}(\Sigma_l)\,\Pi_l+\ldots \quad=\quad B'(y_1)\sum_{l\in\mathcal{L}ynX-X} \overline{\zeta}(S_l)\,\pi_Y P_l+\ldots. \qquad (82)$$

By Proposition 1, the elements of the family $\{\pi_Y P_l\}_{l\in\mathcal{L}ynX}$ are decomposable in the linear basis $\{\Pi_w\}_{w\in Y^*}$ of $\mathcal{U}(\mathrm{Prim}(\mathcal{H}_{\text{⧢}}))$. Thus, by identification of locale coordinates, *i.e.* the coefficients of $\{\Pi_l\}_{l\in\mathcal{L}ynY-\{y_1\}}$, we get the homogenous polynomial relations among convergent polyzetas [40].

3 Polysystem and differential realization

3.1 Polysystem and convergence criterion

3.1.1 Serial estimates from above

Here, generalizing a little, $\mathbb{K}$ is supposed a $\mathbb{C}$-algebra and a complete normed vector space equipped with a norm denoted by $\|.\|$.

For any $n\in\mathbb{N}, X^{\geq n}$ denotes the set of words over X of length greater than or equal to n. The set of formal power series (resp. polynomials) on X, is denoted by $\mathbb{K}\langle\langle X\rangle\rangle$ (resp. $\mathbb{K}\langle X\rangle$).

Definition 5. ([25, 39]) *Let ξ,χ be real positive functions over X^*. Let $S\in\mathbb{K}\langle\langle X\rangle\rangle$.*

*(1) S will be said $\xi-$*exponentially bounded from above *if it verifies*

$$\exists K\in\mathbb{R}_+,\exists n\in\mathbb{N},\forall w\in X^{\geq n},\qquad \|\langle S\mid w\rangle\|\leq K\frac{\xi(w)}{|w|!}.$$

We denote by $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X\rangle\rangle$ which are $\xi-$exponentially bounded from above.

*(2) S verifies the $\chi-$*growth condition *if it satisfies*

$$\exists K\in\mathbb{R}_+,\exists n\in\mathbb{N},\forall w\in X^{\geq n},\qquad \|\langle S\mid w\rangle\|\leq K\chi(w)\,|w|!.$$

We denote by $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X\rangle\rangle$ the set of formal power series in $\mathbb{K}\langle\langle X\rangle\rangle$ verifying the $\chi-$growth condition.

Lemma 7. *We have*

$$R=\sum_{w\in X^*}|w|!\,w\quad\Rightarrow\quad\langle R^{\sqcup\!\sqcup 2}\mid w\rangle=\sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u\,\sqcup\!\sqcup\,v)\ni w}}|u|!\,|v|!\leq 2^{|w|}\,|w|!.$$

Proof. One has

$$\begin{aligned}
\sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u \sqcup\!\sqcup\, v)\ni w}} |u|!\,|v|! &= \sum_{k=0}^{|w|}\sum_{\substack{|u|=k,|v|=|w|-k\\ \mathrm{supp}(u \sqcup\!\sqcup\, v)\ni w}} k!(|w|-k)!\\
&= \sum_{k=0}^{|w|}\binom{|w|}{k}k!(|w|-k)!\\
&= \sum_{k=0}^{|w|}|w|! = (1+|w|)\,|w|!.
\end{aligned}$$

By induction on $|w|$, one has $1+|w|\le 2^{|w|}$. The expected result follows. $\square$

Proposition 11. *Let S_1 and S_2 verify the growth condition. Then S_1+S_2 and $S_1 \sqcup\!\sqcup\, S_2$ also verify the growth condition.*

Proof. The proof for S_1+S_2 is immediate.

Next, since $\|\langle S_i \mid w\rangle\| \le K_i\chi_i(w)\,|w|!$, for $i=1$ or 2 and for $w\in X^*$, then[20]

$$\begin{aligned}
\langle S_1 \sqcup\!\sqcup\, S_2 \mid w\rangle &= \sum_{\mathrm{supp}(u \sqcup\!\sqcup\, v)\ni w}\langle S_1\mid u\rangle\langle S_2\mid v\rangle,\\
\Rightarrow\quad \|\langle S_1 \sqcup\!\sqcup\, S_2 \mid w\rangle\| &\le K_1K_2\sum_{\substack{u,v\in X^*\\ \mathrm{supp}(u \sqcup\!\sqcup\, v)\ni w}}(\chi_1(u)\,|u|!)(\chi_2(v)\,|v|!).
\end{aligned}$$

Let $K=K_1K_2$ and let χ be a real positive function over X^* such that

$$\forall w\in X^*,\qquad \chi(w)=\max\{\chi_1(u)\chi_2(v)\mid u,v\in X^* \text{ and } \mathrm{supp}(u \sqcup\!\sqcup\, v)\ni w\}.$$

With the notations in Lemma 7, we get

$$\|\langle S_1 \sqcup\!\sqcup\, S_2 \mid w\rangle\| \quad\le\quad K\chi(w)\langle R^{\sqcup\!\sqcup\, 2}\mid w\rangle.$$

Hence, $S_1 \sqcup\!\sqcup\, S_2$ verifies the χ'-growth condition with $\chi'(w)=2^{|w|}\chi(w)$. $\square$

Definition 6. ([25, 39]) *Let ξ be a real positive function defined over X^*, S will be said ξ-*exponentially continuous *if it is continuous over $\mathbb{K}^{\xi-\mathrm{em}}\langle\!\langle X\rangle\!\rangle$. The set of formal power series which are ξ-exponentially continuous is denoted by $\mathbb{K}^{\xi-ec}\langle\!\langle X\rangle\!\rangle$.*

Lemma 8. [25, 39] *For any real positive function ξ defined over X^*, we have $\mathbb{K}\langle X\rangle\subset\mathbb{K}^{\xi-ec}\langle\!\langle X\rangle\!\rangle$. Otherwise, for $\xi=0$, we get $\mathbb{K}\langle X\rangle=\mathbb{K}^{0-ec}\langle\!\langle X\rangle\!\rangle$. Hence, any polynomial is $0-$exponentially continuous.*

Proposition 12. ([25, 39]) *Let ξ,χ be real positive functions over X^* and let $P\in\mathbb{K}\langle X\rangle$.*

[20] $\langle S_1 \sqcup\!\sqcup\, S_2 \mid w\rangle$ is the coefficient of the word w in the power series $S_1 \sqcup\!\sqcup\, S_2$.

(1) *Let* $S \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X \rangle\rangle$. *The right residual of* S *by* P *belongs to* $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X \rangle\rangle$.

(2) *Let* $R \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X \rangle\rangle$. *The concatenation* SR *belongs to* $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X \rangle\rangle$.

Proof. (1) Since $S \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X \rangle\rangle$ then

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\frac{\xi(w)}{|w|!}.$$

If $u \in \mathrm{supp}(P) := \{w \in X^* \mid \langle P \mid w\rangle \neq 0\}$ then, for any $w \in X^*$, one has $\langle S \triangleright u \mid w\rangle = \langle S \mid uw\rangle$ and $S \triangleright u$ belongs to $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X \rangle\rangle$:

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \triangleright u \mid w\rangle\| \leq [K\xi(u)]\frac{\xi(w)}{|w|!}.$$

It follows that $S \triangleright P$ is $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X \rangle\rangle$ by taking $K_1 = K\max_{u \in \mathrm{supp}(P)} \xi(u)$.

(2) Since $R \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X \rangle\rangle$ then

$$\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \qquad \|\langle S \mid w\rangle\| \leq K\chi(w)\,|w|!.$$

Let $v \in \mathrm{supp}(P)$ such that $v \neq \epsilon$. Since, for any $w \in X^*$, Rv belongs to $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X \rangle\rangle$ and one has $\langle Rv \mid w\rangle = \langle R \mid v \triangleleft w\rangle$:

$$\begin{aligned}\exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^{\geq n}, \quad \|\langle R \mid v \triangleleft w\rangle\| &\leq K\chi(v \triangleleft w)(|w| - |v|)! \\ &\leq K\,|w|\,\frac{\chi(w)}{\chi(v)}.\end{aligned}$$

Note if $v \triangleleft w = 0$ then $\langle Rv \mid w\rangle = 0$ and the previous conclusion holds. It follows that RP is $\mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X \rangle\rangle$ by taking $K_2 = K\min_{v \in \mathrm{supp}(P)} \chi(v)^{-1}$. □

Proposition 13. ([25, 39]) *Two real positive morphisms over* X^*, ξ *and* χ *are assumed to verify the condition*

$$\sum_{x \in X} \chi(x)\xi(x) \quad < \quad 1.$$

Then for any $F \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X \rangle\rangle$, F *is continuous over* $\mathbb{K}^{\xi-\mathrm{em}}\langle\langle X \rangle\rangle$.

Proof. If ξ, χ verify the upper bound condition then the following power series

$$\sum_{w \in X^*} \chi(w)\xi(w) \quad = \quad \Big(\sum_{x \in X} \chi(x)\xi(x)\Big)^*$$

is well defined. If $F \in \mathbb{K}^{\chi-\mathrm{gc}}\langle\langle X \rangle\rangle$ and $C \in \mathbb{K}^{\xi-\mathrm{em}}\langle\langle X \rangle\rangle$ then there exist $K_i \in \mathbb{R}_+$ and $n_i \in \mathbb{N}$ such that for any $w \in X^{\geq n_i}, i = 1, 2$, one has

$$\|\langle F \mid w\rangle\| \leq K_1\chi(w)\,|w|! \qquad \text{and} \qquad \|\langle C \mid w\rangle\| \leq K_2\frac{\xi(w)}{|w|!}.$$

Hence,

$$\begin{aligned}
&\forall w\in X^*, |\,w\,|\geq \max\{n_1,n_2\},\quad \|\langle F|w\rangle\langle C|w\rangle\|\leq K_1K_2\chi(w)\xi(w),\\
\Rightarrow\quad &\sum_{w\in X^*}\|\langle F|w\rangle\langle C|w\rangle\|\leq K_1K_2\sum_{w\in X^*}\chi(w)\xi(w)=K_1K_2\Bigg(\sum_{x\in X}\chi(x)\xi(x)\Bigg)^*.
\end{aligned}$$

□

3.1.2 Upper bounds *à la* Cauchy

Let $q_1,\ldots,q_n$ be commutative indeterminates over $\mathbb{C}$. The algebra of formal power series (resp. polynomials) over $\{q_1,\ldots,q_n\}$ with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}[\![q_1,\ldots,q_n]\!]$ (resp. $\mathbb{C}[q_1,\ldots,q_n]$).

Definition 7. ([25, 39]) *Let*

$$f=\sum_{i_1,\ldots,i_n\geq 0} f_{i_1,\ldots,i_n}q_1^{i_1}\ldots q_n^{i_n}\in\mathbb{C}[\![q_1,\ldots,q_n]\!].$$

We set

$$\begin{aligned}
E(f) &:= \{\rho\in\mathbb{R}_+^n:\exists C_f\in\mathbb{R}_+ \text{ s.t. } \forall i_1,\ldots,i_n\geq 0, |\,f_{i_1,\ldots,i_n}|\,\rho_1^{i_1}\ldots\rho_n^{i_n}\leq C_f\}.\\
\breve{E}(f) &: \text{ the interior of } E(f) \text{ in } \mathbb{R}^n.\\
\mathrm{CV}(f) &:= \{q\in\mathbb{C}^n:(|\,q_1\,|,\ldots,|\,q_n\,|)\in\breve{E}(f)\}:\quad \text{the convergence domain of } f.
\end{aligned}$$

The power series f is said to be convergent *if* $\mathrm{CV}(f)\neq\emptyset$*. Let $\mathcal{U}$ be an open domain in $\mathbb{C}^n$ and let $q\in\mathbb{C}^n$. The power series f is said to be convergent on q (resp. over $\mathcal{U}$) if $q\in\mathrm{CV}(f)$ (resp. $\mathcal{U}\subset\mathrm{CV}(f)$). We set*

$$\mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]\quad=\quad\{f\in\mathbb{C}[\![q_1,\ldots,q_n]\!]:\mathrm{CV}(f)\neq\emptyset\}.$$

Let $q\in\mathrm{CV}(f)$. There exist the constants C_f,ρ and $\bar\rho$ such that

$$|\,q_1\,|<\bar\rho<\rho,\ldots,|\,q_n\,|<\bar\rho<\rho\qquad\text{and}\qquad |\,f_{i_1,\ldots,i_n}|\,\rho_1^{i_1}\ldots\rho_n^{i_n}\leq C_f,$$

for $i_1,\ldots,i_n\geq 0$. The convergence modulus *of f at q is $(C_f,\rho,\bar\rho)$.*

Suppose that $\mathrm{CV}(f)\neq\emptyset$ and let $q\in\mathrm{CV}(f)$. If $(C_f,\rho,\bar\rho)$ is a convergence modulus of f at q then $|\,f_{i_1,\ldots,i_n}q_1^{i_1}\ldots q_n^{i_n}\,|\leq C_f(\bar\rho_1/\rho_1)^{i_1}\ldots(\bar\rho_1/\rho_1)^{i_n}$. Hence, at q, the power series f is majored termwise by

$$C_f\prod_{k=0}^m\bigg(1-\frac{\bar\rho_k}{\rho_k}\bigg)^{-1}. \tag{83}$$

Hence, f is uniformly absolutely convergent in $\{q\in\mathbb{C}^n:|\,q_1\,|<\bar\rho,\ldots,|\,q_n\,|<\bar\rho\}$ which is an open domain in $\mathbb{C}^n$. Thus, $\mathrm{CV}(f)$ is an open domain in $\mathbb{C}^n$. Since the partial derivation $D_1^{j_1}\ldots D_n^{j_n}f$ is estimated by

$$\|D_1^{j_1}\ldots D_n^{j_n}f\|\quad\leq\quad C_f\frac{\partial^{j_1+\ldots+j_n}}{\partial\bar\rho^{j_1+\ldots+j_n}}\prod_{k=0}^m\bigg(1-\frac{\bar\rho_k}{\rho_k}\bigg)^{-1}. \tag{84}$$

Proposition 14. ([25]) *We have* $\mathrm{CV}(f) \subset \mathrm{CV}(D_1^{j_1} \ldots D_n^{j_n} f)$.

Let $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$. Let $\{A_i\}_{i=0,1}$ be a polysystem defined as follows

$$A_i(q) \quad = \quad \sum_{j=1}^{n} A_i^j(q) \frac{\partial}{\partial q_j}, \tag{85}$$

where for any $j = 1, \ldots, n$, $A_i^j(q) \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$.

Lemma 9. ([22]) *For* $i = 0, 1$ *and* $j = 1, \ldots, n$, *one has* $A_i \circ q_j = A_i^j(q)$. *Thus,*

$$\forall i = 0, 1, \quad A_i(q) \quad = \quad \sum_{j=1}^{n} (A_i \circ q_j) \frac{\partial}{\partial q_j}.$$

Let $(\rho, \bar{\rho}, C_f), \{(\rho, \bar{\rho}, C_i)\}_{i=0,1}$ be respectively the convergence modulus at

$$q \quad \in \quad \mathrm{CV}(f) \bigcap_{\substack{i=0,1\\ j=1,\ldots,n}} \mathrm{CV}(A_i^j) \tag{86}$$

of f and $\{A_i^j\}_{j=1,\ldots,n}$. Let us consider the following monoid morphisms

$$\mathcal{A}(\epsilon) = \text{identity} \quad \text{and} \quad C(\epsilon) = 1, \tag{87}$$

$$\forall w = v x_i, x_i \in X, v \in X^*, \quad \mathcal{A}(w) = \mathcal{A}(v) A_i \quad \text{and} \quad C(w) = C(v) C_i. \tag{88}$$

Lemma 10. ([20]) *For any word* w, $\mathcal{A}(w)$ *is continuous over* $\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$ *and, for any* $f, g \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$, *one has*

$$\mathcal{A}(w) \circ (fg) \quad = \quad \sum_{u,v \in X^*} \langle u ⧢ v \mid w \rangle (\mathcal{A}(u) \circ f)(\mathcal{A}(v) \circ g).$$

These notations are extended, by linearity, to $\mathbb{K}\langle X \rangle$ and we will denote $\mathcal{A}(w) \circ f_{|q}$ the evaluation of $\mathcal{A}(w) \circ f$ at q.

Definition 8. ([20]) *Let* $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$. *The generating series of the polysystem* $\{A_i\}_{i=0,1}$ *and of the observation* f *is given by*

$$\sigma f \quad := \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f \; w \quad \in \quad \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]\langle\!\langle X \rangle\!\rangle.$$

Then the following generating series is called Fliess generating series *of the polysystem* $\{A_i\}_{i=0,1}$ *and of the observation* f *at* q:

$$\sigma f_{|_q} \quad := \quad \sum_{w \in X^*} \mathcal{A}(w) \circ f_{|q} \; w \quad \in \quad \mathbb{C}\langle\!\langle X \rangle\!\rangle.$$

Lemma 11. ([20]) *Let* $\{A_i\}_{i=0,1}$ *be a polysystem. Then, the map*

$$\sigma : (\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!], .) \quad \longrightarrow \quad (\mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]\langle\!\langle X \rangle\!\rangle, ⧢),$$

is an algebra morphism, i.e. for $f, g \in \mathbb{C}^{\mathrm{cv}}[\![q_1, \ldots, q_n]\!]$ *and* $\mu, \nu \in \mathbb{C}$, *one has:*

$$\sigma(\nu f + \mu h) = \nu \sigma f + \mu \sigma g \qquad \text{and} \qquad \sigma(fg) = \sigma f ⧢ \sigma g.$$

Lemma 12. ([22]) *Let $\{A_i\}_{i=0,1}$ be a polysystem and $f \in \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$. Then*

$$\begin{array}{rcl} \forall x_i \in X, \quad \sigma(A_i \circ f) = x_i \triangleleft \sigma f & \in & \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]\langle\langle X\rangle\rangle \\ \forall w \in X^*, \quad \sigma(\mathcal{A}(w) \circ f) = w \triangleleft \sigma f & \in & \mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]\langle\langle X\rangle\rangle. \end{array}$$

Lemma 13. ([25]) *Let $\tau = \min_{1\le k\le n} \rho_k$ and $r = \max_{1\le k\le n} \bar{\rho}_k/\rho_k$. We have*

$$\begin{array}{rcl} \|\mathcal{A}(w) \circ f\| & \le & C_f \dfrac{(n+1)}{(1-r)^n} \dfrac{C(w)\ |w|!}{\binom{n+|w|-1}{|w|}} \left[\dfrac{n}{\tau(1-r)^{n+1}}\right]^{|w|} \\ & \le & C_f \dfrac{(n+1)}{(1-r)^n} C(w) \left[\dfrac{n}{\tau(1-r)^{n+1}}\right]^{|w|} |w|!. \end{array}$$

Theorem 8. ([25]) *Let $K = C_f(n+1)(1-r)^{-n}$ and let χ be the real positive function defined over X^* by*

$$\forall i = 0, 1, \quad \chi(x_i) \quad = \quad \frac{C_i n}{\tau(1-r)^{(n+1)}}.$$

Then the generating series σf of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f satisfies the $\chi-$growth condition.

It is the same for the Fliess generating series $\sigma f_{|q}$ of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f at q.

3.2 Polysystem and nonlinear differential equation

3.2.1 Nonlinear differential equation (with three singularities)

Let us consider the singular inputs[21] $u_0(z) := z^{-1}$ and $u_1(z) := (1-z)^{-1}$, and the following nonlinear dynamical system

$$\left\{ \begin{array}{rcl} y(z) & = & f(q(z)), \\ \dot{q}(z) & = & A_0(q)\ u_0(z) + A_1(q)\ u_1(z), \\ q(z_0) & = & q_0, \end{array} \right. \tag{89}$$

where the state $q = (q_1,\ldots,q_n)$ belongs to the complex analytic manifold of dimension n, q_0 is the initial state, the observation f belongs to $\mathbb{C}^{\mathrm{cv}}[\![q_1,\ldots,q_n]\!]$ and $\{A_i\}_{i=0,1}$ is the polysystem defined on (85).

Definition 9. ([27]) *The following power series is called* transport operator *of the polysystem $\{A_i\}_{i=0,1}$ and of the observation f*

$$\mathcal{T} \quad := \quad \sum_{w\in X^*} \alpha_{z_0}^{z}(w)\ \mathcal{A}(w).$$

[21] These singular inputs are not included in the studies of Fliess motivated, in particular, by the renormalization of $y(z)$ at $+\infty$ [20, 22].

By the factorization of the monoid by Lyndon words, we have [27]

$$\mathcal{T} = (\alpha_{z_0}^z \otimes \mathcal{A}) \left(\sum_{w \in X^*} w \otimes w \right) = \prod_{l \in \mathcal{L}ynX} \exp[\alpha_{z_0}^z(S_l)\ \mathcal{A}(P_l)]. \tag{90}$$

The Chen generating series along the path $z_0 \rightsquigarrow z$, associated to ω_0, ω_1 is

$$S_{z_0 \rightsquigarrow z} := \sum_{w \in X^*} \langle S \mid w \rangle\, w \qquad \text{with} \qquad \langle S \mid w \rangle = \alpha_{z_0}^z(w) \tag{91}$$

which solves the differential equation (72) with the initial condition $S_{z_0 \rightsquigarrow z_0} = 1$. Thus, $S_{z_0 \rightsquigarrow z}$ and $\mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ satisfy the same differential equation taking the same value at z_0 and $S_{z_0 \rightsquigarrow z} = \mathrm{L}(z)\mathrm{L}(z_0)^{-1}$.

Any Chen generating series $S_{z_0 \rightsquigarrow z}$ is group like [46] and depends only on the homotopy class of $z_0 \rightsquigarrow z$ [9]. The product of $S_{z_1 \rightsquigarrow z_2}$ and $S_{z_0 \rightsquigarrow z_1}$ is $S_{z_0 \rightsquigarrow z_2} = S_{z_1 \rightsquigarrow z_2} S_{z_0 \rightsquigarrow z_1}$. Let $\varepsilon \in]0,1[$ and $z_i = \varepsilon \exp(\mathrm{i}\beta_i)$, for $i = 0$ or 1. We set $\beta = \beta_1 - \beta_0$. Let $\Gamma_0(\varepsilon, \beta_0)$ (resp. $\Gamma_1(\varepsilon, \beta_1)$) be the path turning around 0 (resp. 1) in the positive direction from z_0 to z_1. By induction on the length of w, one has

$$|\langle S_{\Gamma_i(\varepsilon,\beta)} \mid w \rangle| \quad = \quad (2\varepsilon)^{|w|_{x_i}} \frac{\beta^{|w|}}{|w|!}, \tag{92}$$

where $|w|$ denotes the length of w and $|w|_{x_i}$ denotes the number of occurrences of letter x_i in w, for $i = 0$ or 1. For ε tends to 0^+, these estimations yield $S_{\Gamma_i(\varepsilon,\beta)} = e^{\mathrm{i}\beta x_i} + o(\varepsilon)$. In particular, if $\Gamma_0(\varepsilon)$ (resp. $\Gamma_1(\varepsilon)$) is a circular path of radius ε turning around 0 (resp. 1) in the positive direction, starting at $z = \varepsilon$ (resp. $1 - \varepsilon$), then, by the noncommutative residue theorem [35, 32], we get

$$S_{\Gamma_0(\varepsilon)} = e^{2\mathrm{i}\pi x_0} + o(\varepsilon) \qquad \text{and} \qquad S_{\Gamma_1(\varepsilon)} = e^{-2\mathrm{i}\pi x_1} + o(\varepsilon). \tag{93}$$

Finally, the asymptotic behaviors of L on (73) give [32, 35]

$$S_{\varepsilon \rightsquigarrow 1-\varepsilon} \quad \widetilde{_{\varepsilon \to 0^+}} \quad e^{-x_1 \log \varepsilon} Z_{\sqcup\!\sqcup}\ e^{-x_0 \log \varepsilon}. \tag{94}$$

In other terms, $Z_{\sqcup\!\sqcup}$ is the regularized Chen generating series $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$ of differential forms ω_0 and ω_1: $Z_{\sqcup\!\sqcup}$ is the noncommutative generating series of the finite parts of the coefficients of the Chen generating series $e^{x_1 \log \varepsilon}\ S_{\varepsilon \rightsquigarrow 1-\varepsilon}\ e^{x_0 \log \varepsilon}$, *i.e.* the concatenation of $e^{x_0 \log \varepsilon}$ and then $S_{\varepsilon \rightsquigarrow 1-\varepsilon}$ and finally, $e^{x_1 \log \varepsilon}$.

3.2.2 Asymptotic behavior of the successive differentiation of solution via extended Fliess fundamental formula

The Fliess' fundamental formula [20] can be then extended as follows

Theorem 9. ([39]) *We have*

$$y(z) = \mathcal{T} \circ f_{|_{q_0}} = \sum_{w \in X^*} \langle S_{z_0 \rightsquigarrow z} \mid w \rangle \langle \mathcal{A}(w) \circ f_{|_{q_0}} \mid w \rangle = \langle \sigma f_{|_{q_0}} \parallel S_{z_0 \rightsquigarrow z} \rangle.$$

By the factorization of the Lie exponential series L, the expansions of the output y of nonlinear dynamical system with singular inputs follow

Corollary 2.

$$\begin{aligned}
y(z) &= \sum_{w\in X^*} g_w(z)\ \mathcal{A}(w)\circ f_{|q_0}\\
&= \sum_{k\ge 0}\sum_{n_1,\ldots,n_k\ge 0} g_{x_0^{n_1}x_1\ldots x_0^{n_k}x_1}(z)\ \mathrm{ad}_{A_0}^{n_1}A_1\ldots\mathrm{ad}_{A_0}^{n_k}A_1 e^{\log zA_0}\circ f_{|q_0}\\
&= \prod_{l\in\mathcal{L}ynX}\exp\Big(g_{S_l}(z)\ \mathcal{A}(P_l)\circ f_{|q_0}\Big)\\
&= \exp\Big(\sum_{w\in X^*} g_w(z)\ \mathcal{A}(\pi_1(w))\circ f_{|q_0}\Big),
\end{aligned}$$

where, for any word w in X^*, g_w belongs to the polylogarithm algebra.

Since $S_{z_0\rightsquigarrow z}=\mathrm{L}(z)\mathrm{L}(z_0)^{-1}$ and since $\sigma f_{|q_0}$ and $\mathrm{L}(z_0)^{-1}$ are invariant by $\partial_z=d/dz$ and by $\theta_0=zd/dz$ then we obtain the n-order differentiation of y :

$$\partial_z^n y(z)=\langle\sigma f_{|q_0}\parallel\partial^n S_{z_0\rightsquigarrow z}\rangle=\langle\sigma f_{|q_0}\parallel\partial_z^n\mathrm{L}(z)\mathrm{L}(z_0)^{-1}\rangle \tag{95}$$

and the n-order differentiation of y with respect to θ_0 :

$$\theta_0^n y(z)=\langle\sigma f_{|q_0}\parallel\theta_0^n S_{z_0\rightsquigarrow z}\rangle=\langle\sigma f_{|q_0}\parallel\theta_0^n\mathrm{L}(z)\mathrm{L}(z_0)^{-1}\rangle. \tag{96}$$

With the notations of Proposition 5, we get respectively

$$\partial_z^n y(z)=\langle\sigma f_{|q_0}\parallel[D_n(z)\mathrm{L}(z)]\mathrm{L}(z_0)^{-1}\rangle=\langle\sigma f_{|q_0}\triangleright D_n(z)\parallel\mathrm{L}(z)\mathrm{L}(z_0)^{-1}\rangle, \tag{97}$$
$$\theta_0^n y(z)=\langle\sigma f_{|q_0}\parallel[E_n(z)\mathrm{L}(z)]\mathrm{L}(z_0)^{-1}\rangle=\langle\sigma f_{|q_0}\triangleright E_n(z)\parallel\mathrm{L}(z)\mathrm{L}(z_0)^{-1}\rangle. \tag{98}$$

For $z_0=\varepsilon\to 0^+$, the asymptotic behavior and the renormalization at $z=1$ of $\partial_z^n y(z)$ and $\theta_0^n y(z)$ (or the asymptotic expansion and the renormalization of its Taylor coefficients at $+\infty$) are deduced from (94) and extend a little bit the results of [39] as follows

Corollary 3. *For any integer n, we have respectively*[22]

[22] We overget, by Note 18, the n-order differentiation of y w.r.t. θ_1 :

$$\theta_1^n y(z)=\langle\sigma f_{|q_0}\parallel\theta_1^n S_{z_0\rightsquigarrow z}\rangle=\langle\sigma f_{|q_0}\parallel\theta_1^n\mathrm{L}(z)\mathrm{L}(z_0)^{-1}\rangle.$$

Hence,

$$\theta_1^n y(z)=\langle\sigma f_{|q_0}\parallel[D_n(z)-E_n(z)]\mathrm{L}(z)\mathrm{L}(z_0)^{-1}\rangle=\langle\sigma f_{|q_0}\triangleright[D_n(z)-E_n(z)]\parallel\mathrm{L}(z)\mathrm{L}(z_0)^{-1}\rangle$$

and then

$$\theta_1^n y(1)\ \underset{\varepsilon\to 0^+}{\widetilde{\quad}}\ \sum_{w\in X^*}\langle\mathcal{A}(w)\circ f_{|q_0}\mid w\rangle\langle[D_n(1-\varepsilon)-E_n(1-\varepsilon)]e^{-x_1\log\varepsilon}\,Z_{\sqcup\!\sqcup}\ e^{-x_0\log\varepsilon}\mid w\rangle.$$

The actions of $\theta_0=u_0(z)^{-1}d/dz$ and $\theta_1=u_1(z)^{-1}d/dz$ over $y(z)$ are equivalent to those of the residuals respectively by x_0 and x_1 over $\sigma f_{|q_0}$ and correspond to *functional* differentiations [21] while $\partial_z=d/dz$ is the ordinary differentiation and is equivalent to the residual by x_0+x_1.

$$\partial_z^n y(1) \underset{\varepsilon\to 0^+}{\widetilde{\quad}} \sum_{w\in X^*} \langle \mathcal{A}(w)\circ f_{|_{q_0}} \mid w\rangle \langle D_n(1-\varepsilon)\, e^{-x_1\log\varepsilon}\, Z_{\sqcup\!\sqcup}\, e^{-x_0\log\varepsilon} \mid w\rangle,$$

$$\theta_0^n y(1) \underset{\varepsilon\to 0^+}{\widetilde{\quad}} \sum_{w\in X^*} \langle \mathcal{A}(w)\circ f_{|_{q_0}} \mid w\rangle \langle E_n(1-\varepsilon)\, e^{-x_1\log\varepsilon}\, Z_{\sqcup\!\sqcup}\, e^{-x_0\log\varepsilon} \mid w\rangle.$$

Corollary 4. *The n-order differentiation of the output y of the system (89) is a $\mathcal{C}$-combination of the elements g belonging to the polylogarithm algebra.*

Moreover, if the ordinary Taylor expansions of $\partial_z^n y$ and $\theta_0^n y$ exist :

$$\partial_z^n y(z) = \sum_{k\geq 0} d_k z^n \quad \text{and} \quad \theta_0^n y(z) = \sum_{k\geq 0} t_k z^k$$

then the coefficients of these expansions belong to the algebra of harmonic sums and there exist algorithmically computable coefficients $a_i, a_i' \in \mathbb{Z}, b_i, b_i' \in \mathbb{N}$ and $c_i, c_i' \in \mathcal{Z}[\gamma]$ such that

$$d_k \underset{k\to\infty}{\widetilde{\quad}} \sum_{i\geq 0} c_i k^{a_i} \log^{b_i} k \quad \text{and} \quad t_k \underset{k\to\infty}{\widetilde{\quad}} \sum_{i\geq 0} c_i' k^{a_i'} \log^{b_i'} k.$$

3.3 Differential realization

3.3.1 Differential realization

Definition 10. *The* Lie rank *of a formal power series $S \in \mathbb{K}\langle\langle X\rangle\rangle$ is the dimension of the vector space generated by*

$$\{S \triangleright \Pi \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle\}, \quad \text{or respectively by} \quad \{\Pi \triangleleft S \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle\}.$$

Example 10. *Let $P = x_1x_0x_1 + x_0x_1x_0x_1$ be a polynomial of degree 4 over $X = \{x_0, x_1\}$. A basis up to degree 4 of the Lie algebra $\mathcal{L}ie_{\mathbb{K}}\langle X\rangle$ is following (c.f. Example 4)*

$$\{P_{x_0}, P_{x_1}, P_{x_0x_1}, P_{x_0^2x_1}, P_{x_0x_1^2}, P_{x_0^3x_1}, P_{x_0x_1^3}, P_{x_0^2x_1^2}\}.$$

The Lie-Hankel matrix of P is following

	1_{X^*}	x_1	x_0x_1	$x_1x_0x_1$
P_{x_0}	0	0	0	1
P_{x_1}	0	0	1	0
$P_{x_0x_1}$	0	-1	1	0
$P_{x_0^2x_1}$	0	-2	0	0
$P_{x_0x_1^2}$	-2	0	0	0
$P_{x_0^2x_1^2}$	-2	0	0	0

and its Lie-Hankel rank of P is then 4.

Definition 11. *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$ and let us put*

$$\begin{aligned}\mathrm{Ann}(S) &:= \{\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle \mid S \triangleright \Pi = 0\},\\ \mathrm{Ann}^{\perp}(S) &:= \{Q \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢) \mid Q \triangleright \mathrm{Ann}(S) = 0\}.\end{aligned}$$

It is immediate that $\mathrm{Ann}^{\perp}(S) \ni S$ and it follows that (see [22, 48]).

Lemma 14. *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$. If S is of finite Lie rank, d, then the dimension of $\mathrm{Ann}^{\perp}(S)$ equals d.*

By Lemma 3, the residuals are derivations for shuffle product. Then,

Lemma 15. *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$. Then:*

(1) *For any Q_1 and $Q_2 \in \mathrm{Ann}^{\perp}(S)$, one has $Q_1 ⧢ Q_2 \in \mathrm{Ann}^{\perp}(S)$.*

(2) *For any $P \in \mathbb{K}\langle X\rangle$ and $Q_1 \in \mathrm{Ann}^{\perp}(S)$, one has $P \triangleleft Q_1 \in \mathrm{Ann}^{\perp}(S)$.*

Definition 12. ([22]) *The formal power series $S \in \mathbb{K}\langle\langle X\rangle\rangle$ is* differentially produced *if there exist*

- *an integer d,*
- *a power series $f \in \mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!]$,*
- *a homomorphism $\mathcal{A}$ from X^* maps to the algebra of differential operators generated by*

$$\mathcal{A}(x_i) = \sum_{j=1}^{d} A_i^j(\bar{q}_1, \ldots, \bar{q}_d)\frac{\partial}{\partial \bar{q}_j},$$

where, for $j = 1, \ldots, d$, $A_i^j(\bar{q}_1, \ldots, \bar{q}_d)$ belongs to $\mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!]$ such that

$$\forall w \in X^*, \quad \langle S \mid w\rangle = \mathcal{A}(w) \circ f_{|_0}.$$

The couple $(\mathcal{A}, f)$ is called the differential representation *of S of dimension d.*

Proposition 15. ([48]) *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$. If S is differentially produced then it verifies the growth condition and its Lie rank is finite.*

Proof. Let $(\mathcal{A}, f)$ be a differential representation of S of dimension d. Then, by the notations of Definition 8, we get

$$\sigma f_{|_0} = S = \sum_{w \in X^*} (\mathcal{A}(w) \circ f)_{|_0}\, w.$$

For any $j = 1, \ldots, d$, we put

$$T_j = \sum_{w \in X^*} \frac{\partial(\mathcal{A}(w) \circ f)}{\partial \bar{q}_j}\, w.$$

Firstly, by Theorem 8, the generating series σf verifies the growth condition. Secondly, for any $\Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle$ and for any $w \in X^*$, one has

$$\langle \sigma f \triangleright \Pi \mid w\rangle = \langle \sigma f \mid \Pi w\rangle = \mathcal{A}(\Pi w) \circ f = \mathcal{A}(\Pi) \circ (\mathcal{A}(w) \circ f).$$

Since $\mathcal{A}(\Pi)$ is a derivation over $\mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!]$:

$$\begin{aligned} \mathcal{A}(\Pi) &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j) \frac{\partial}{\partial \bar{q}_j}, \\ \Rightarrow \quad \mathcal{A}(\Pi) \circ (\mathcal{A}(w) \circ f) &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j) \frac{\partial(\mathcal{A}(w) \circ f)}{\partial \bar{q}_j} \end{aligned}$$

then we deduce that

$$\begin{aligned} \forall w \in X^*, \quad \langle \sigma f \triangleright \Pi \mid w\rangle &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j) \langle T_j \mid w\rangle, \\ \iff \quad \sigma f \triangleright \Pi &= \sum_{j=1}^{d} (\mathcal{A}(\Pi) \circ \bar{q}_j)\; T_j. \end{aligned}$$

That means $\sigma f \triangleright \Pi$ is $\mathbb{K}$-linear combination of $\{T_j\}_{j=1,\ldots,d}$ and the dimension of the vector space $\mathrm{span}\{\sigma f \triangleright \Pi \mid \Pi \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle\}$ is less than or equal to d. ☐

3.3.2 Fliess' local realization theorem

Proposition 16. ([48]) *Let $S \in \mathbb{K}\langle\langle X\rangle\rangle$ be such that its Lie rank equals d. Then there exists a basis $S_1, \ldots, S_d \in \mathbb{K}\langle\langle X\rangle\rangle$ of $(\mathrm{Ann}^{\perp}(S), ⧢) \cong (\mathbb{K}[\![S_1, \ldots, S_d]\!], ⧢)$ such that the S_i's are proper and for any $R \in \mathrm{Ann}^{\perp}(S)$, one has*

$$R = \sum_{i_1,\ldots,i_d \geq 0} \frac{r_{i_1,\ldots,i_n}}{i_1! \ldots i_d!} S_1^{⧢ i_1} ⧢ \ldots ⧢ S_d^{⧢ i_d},$$

where the coefficients $\{r_{i_1,\ldots,i_d}\}_{i_1,\ldots,i_d \geq 0}$ belong to $\mathbb{K}$ and $r_{0,\ldots,0} = \langle R \mid 1_{X^*}\rangle$.

Proof. By Lemma 14, such a basis exists. More precisely, since the Lie rank of S is d then there exist $P_1, \ldots, P_d \in \mathcal{L}ie_{\mathbb{K}}\langle X\rangle$ such that $S \triangleright P_1, \ldots, S \triangleright P_d \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢)$ are $\mathbb{K}$-linearly independent. By duality, there exist $S_1, \ldots, S_d \in (\mathbb{K}\langle\langle X\rangle\rangle, ⧢)$ such that

$$\forall i, j = 1, \ldots, d, \quad \langle S_i \mid P_j\rangle = \delta_{i,j}, \qquad \text{and} \qquad R = \prod_{i=1}^{d} \exp(S_i\; P_i).$$

Expending this product, one obtains, via PBW theorem, the expected expression for the coefficients $\{r_{i_1,\ldots,i_d} = \langle R \mid P_1^{i_1} \ldots P_d^{i_d}\rangle\}_{i_1,\ldots,i_d \geq 0}$. Hence, $(\mathrm{Ann}^{\perp}(S), ⧢)$ is generated by $S_1, \ldots, S_d$. ☐

With the notations of Proposition 16, one has respectively.

Corollary 5. *If $S \in \mathbb{K}[S_1, \ldots, S_d]$ then, for any $i = 0, 1$ and for any $j = 1, \ldots, d$, one has $x_i \triangleleft S \in \mathrm{Ann}^{\perp}(S) = \mathbb{K}[S_1, \ldots, S_d]$.*

Corollary 6. *The power series S verifies the growth condition if and only if, for any $i = 1, \ldots, d$, S_i also verifies the growth condition.*

Proof. Assume there exists $j \in [1, \ldots, d]$ such that S_j does not verify the growth condition. Since $S \in \mathrm{Ann}^{\perp}(S)$ then using the decomposition of S on $S_1, \ldots, S_d$, one obtains a contradiction with the fact that S verifies the growth condition.

Conservely, using Proposition 11, we get the expected results. □

Theorem 10. ([22]) *The formal power series $S \in \mathbb{K}\langle\langle X \rangle\rangle$ is differentially produced if and only if its Lie rank is finite and if it verifies the χ-growth condition.*

Proof. By Proposition 15, one gets a direct proof.

Conversely, since the Lie rank of S equals d then by Proposition 16, by putting $\sigma f_{|0} = S$ and, for any $j = 1, \ldots, d$, $\sigma \bar{q}_i = S_i$,

(1) we choose the observation f as follows

$$f(\bar{q}_1, \ldots, \bar{q}_d) \quad = \sum_{i_1, \ldots, i_d \geq 0} \frac{r_{i_1, \ldots, i_n}}{i_1! \ldots i_d!} \bar{q}_1^{i_1} \ldots \bar{q}_d^{i_d} \quad \in \quad \mathbb{K}[\![\bar{q}_1, \ldots, \bar{q}_d]\!]$$

such that

$$\sigma f_{|0}(\bar{q}_1, \ldots, \bar{q}_d) \quad = \sum_{i_1, \ldots, i_d \geq 0} \frac{r_{i_1, \ldots, i_n}}{i_1! \ldots i_d!} (\sigma \bar{q}_1)^{\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup (\sigma \bar{q}_d)^{\sqcup\!\sqcup\, i_d},$$

(2) it follows that, for $i = 0, 1$ and for $j = 1, \ldots, d$, the residual $x_i \triangleleft \sigma \bar{q}_j$ belongs to $\mathrm{Ann}^{\perp}(\sigma f_{|0})$ (see also Lemma 15),

(3) since σf verifies the χ-growth condition then, by Corollary 6, the generating series $\sigma \bar{q}_j$ and $x_i \triangleleft \sigma \bar{q}_j$ (for $i = 0, 1$ and for $j = 1, \ldots, d$) verify also the growth condition. We then take (see Lemma 12)

$$\forall i = 0, 1, \quad \forall j = 1, \ldots, d, \quad \sigma A_j^i(\bar{q}_1, \ldots, \bar{q}_d) = x_i \triangleleft \sigma \bar{q}_j,$$

by expressing σA_j^i on the basis $\{\sigma \bar{q}_i\}_{i=1,\ldots,d}$ of $\mathrm{Ann}^{\perp}(\sigma f_{|0})$,

(4) the homomorphism $\mathcal{A}$ is then determined as follows

$$\forall i = 0, 1, \quad \mathcal{A}(x_i) \quad = \quad \sum_{j=0}^{d} A_j^i(\bar{q}_1, \ldots, \bar{q}_d) \frac{\partial}{\partial \bar{q}_j},$$

where, by Lemma 9, one has $A_j^i(\bar{q}_1, \ldots, \bar{q}_d) = \mathcal{A}(x_i) \circ \bar{q}_j$.

Thus, $(\mathcal{A}, f)$ provides a differential representation[23] of dimension d of S. □

[23] In [22, 48], the reader can find the discussion on the *minimal* differential representation.

Moreover, one also has the following

Theorem 11. ([22]) *Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$ be a differentially produced formal power series. Let $(\mathcal{A}, f)$ and $(\mathcal{A}', f')$ be two differential representations of dimension n of S. There exist a continuous and convergent automorphism h of $\mathbb{K}$ such that*

$$\forall w \in X^*, \forall g \in \mathbb{K}, \quad h(\mathcal{A}(w) \circ g) = \mathcal{A}'(w) \circ (h(g)) \quad \textit{and} \quad f' = h(f).$$

Since any rational power series verifies the growth condition and its Lie rank is less than or equal to its Hankel rank which is finite [22] then

Corollary 7. *Any rational power series and any polynomial over X with coefficients in $\mathbb{K}$ are differentially produced.*

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