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# **Eighth order Peregrine breather solution of the NLS equation and their deformations with fourteen parameters.**

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## **Abstract**

We construct new families of quasi-rational solutions of the NLS equation of order 8 with 14 real parameters. We obtain new patterns of different types of rogue waves. We recover the triangular configurations as well as isolated rings as found for the lower orders. Moreover, one sees appearing for certain values of the parameters, new configurations of concentric rings.

## **1 Introduction**

From the pioneer work of Zakharov and Shabat in 1972 [14], and the first expressions of the quasi-rational solutions given by Peregrine in 1983 [13], a considerable number of studies were carried out. Akhmediev, Eleonski and Kulagin constructed the first higher order analogue of the Peregrine breather [1, 2] in 1986. Akhmediev et al. [3, 4], constructed other families of order 3 and 4, using Darboux transformations.

In 2010, rational solutions of the NLS equation have been written as a quotient of two Wronskians in [6]. In 2011, an other representation of the solutions of the NLS equation has been constructed in [7], also in terms of a ratio of two Wronskians determinants of order  $2N$ .

In 2012, the solutions of the focusing NLS equation has been written as a ratio of two determinants in [10] using generalized Darboux transform.

Ohta and Yang [12] have given an other representation of solutions of the focusing NLS equation by means of determinants, obtained from Hirota bilinear method.

At the beginning of the year 2012, the author obtained a representation in terms of determinants which does not involve limits [9].

These first two formulations given in [7, 9] did depend in fact only on two parameters; this remark was first pointed out by Matveev. In 2012 Matveev and Dubard obtained for the first time, with the approach defined in [6], the solutions of NLS explicitly, for the order 3 depending on 4 parameters and those for the order 4 depending on 6 parameters[5].

Then the author found for the order  $N$  (for determinants of order  $2N$ ), solutions depending on  $2N - 2$  real parameters.

With this new method, we construct here news deformations at order 8 with 14 real parameters.

## 2 Determinant representation of solutions of NLS equation

### 2.1 Quasi-rational limit solutions of the NLS equation

We recall the results obtained in [7]. We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0. \quad (1)$$

In the following, we consider  $2N$  parameters  $\lambda_\nu$ ,  $\nu = 1, \dots, 2N$  satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N. \quad (2)$$

We define the terms  $\kappa_\nu$ ,  $\delta_\nu$ ,  $\gamma_\nu$  by the following equations,

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad (3)$$

and

$$\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \dots N. \quad (4)$$

The terms  $x_{r,\nu}$  ( $r = 3, 1$ ) are defined by

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N. \quad (5)$$

The parameters  $e_\nu$  are defined by

$$e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \quad 1 \leq j \leq N, \quad (6)$$

where  $a_j$  and  $b_j$ , for  $1 \leq j \leq N$  are arbitrary real numbers.

The terms  $\epsilon_\nu$  are defined by :

$$\begin{aligned} \epsilon_\nu &= 0, & 1 \leq \nu \leq N \\ \epsilon_\nu &= 1, & N+1 \leq \nu \leq 2N. \end{aligned}$$

We use the following notations :

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y - ie_\nu, \quad 1 \leq \nu \leq 2N.$$

We consider the functions

$$\begin{aligned} \phi_{r,\nu}(y) &= \sin \Theta_{r,\nu}, & 1 \leq \nu \leq N, \\ \phi_{r,\nu}(y) &= \cos \Theta_{r,\nu}, & N+1 \leq \nu \leq 2N. \end{aligned} \quad (7)$$

$W_r(y) = W(\phi_{r,1}, \dots, \phi_{r,2N})$  is the Wronskian

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_{r,\nu})_{\nu, \mu \in [1, \dots, 2N]}]. \quad (8)$$

Then we get the following statement [8]

**Theorem 2.1** *The function  $v$  defined by*

$$v(x, t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi). \quad (9)$$

*is solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0.$$

To obtain quasi-rational solutions of the NLS equation, we take the limit when the parameters  $\lambda_j \rightarrow 1$  for  $1 \leq j \leq N$  and  $\lambda_j \rightarrow -1$  for  $N+1 \leq j \leq 2N$ . For that, we consider the parameter  $\lambda_j$  written in the form

$$\lambda_j = 1 - 2j^2\epsilon^2, \quad 1 \leq j \leq N. \quad (10)$$

When  $\epsilon$  goes to 0, we obtain quasi-rational solutions of the NLS equation given by :

**Theorem 2.2** *The function  $v$  defined by*

$$v(x, t) = \exp(2it - i\varphi) \lim_{\epsilon \rightarrow 0} \frac{W_3(0)}{W_1(0)}, \quad (11)$$

*is a quasi-rational solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0.$$

## 2.2 Expression of solutions of NLS equation in terms of a ratio of two determinants

We construct the solutions of the NLS equation expressed as a quotient of two determinants which does not involve a passage to the limit.

We use the following notations :

$$\begin{aligned} A_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ B_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2, \end{aligned} \quad (12)$$

for  $1 \leq \nu \leq 2N$ , with  $\kappa_\nu$ ,  $\delta_\nu$ ,  $x_{r,\nu}$  defined in (3), (4) and (5).

The parameters  $e_\nu$  are defined by (6).

Here, the parameters  $a_j$  and  $b_j$ , for  $1 \leq j \leq N$  are chosen in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a}_k \epsilon^{2k+1} j^{2k+1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b}_k \epsilon^{2k+1} j^{2k+1}, \quad 1 \leq j \leq N. \quad (13)$$

We consider the following functions :

$$\begin{aligned} f_{4j+1,k} &= \gamma_k^{4j-1} \sin A_k, & f_{4j+2,k} &= \gamma_k^{4j} \cos A_k, \\ f_{4j+3,k} &= -\gamma_k^{4j+1} \sin A_k, & f_{4j+4,k} &= -\gamma_k^{4j+2} \cos A_k, \end{aligned} \quad (14)$$

for  $1 \leq k \leq N$ , and

$$\begin{aligned} f_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos A_{N+k}, & f_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin A_{N+k}, \\ f_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos A_{N+k}, & f_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin A_{N+k}, \end{aligned} \quad (15)$$

for  $1 \leq k \leq N$ .

We define the functions  $g_{j,k}$  for  $1 \leq j \leq 2N$ ,  $1 \leq k \leq 2N$  in the same way, we replace only the term  $A_k$  by  $B_k$ .

$$\begin{aligned} g_{4j+1,k} &= \gamma_k^{4j-1} \sin B_k, & g_{4j+2,k} &= \gamma_k^{4j} \cos B_k, \\ g_{4j+3,k} &= -\gamma_k^{4j+1} \sin B_k, & g_{4j+4,k} &= -\gamma_k^{4j+2} \cos B_k, \end{aligned} \quad (16)$$

for  $1 \leq k \leq N$ , and

$$\begin{aligned} g_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos B_{N+k}, & g_{4j+2,N+k} &= -\gamma_k^{2N-4j-3} \sin B_{N+k}, \\ g_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos B_{N+k}, & g_{4j+4,N+k} &= \gamma_k^{2N-4j-5} \sin B_{N+k}, \end{aligned} \quad (17)$$

for  $1 \leq k \leq N$ .

Then we get the following result :

**Theorem 2.3** *The function  $v$  defined by*

$$v(x, t) = \frac{\det((n_{jk})_{j,k \in [1, 2N]})}{\det((d_{jk})_{j,k \in [1, 2N]})} e^{2it - i\varphi} \quad (18)$$

*is a quasi-rational solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

*depending on  $2N - 2$  parameters  $\tilde{a}_j, \tilde{b}_j, 1 \leq j \leq N - 1$ , where*

$$\begin{aligned} n_{j1} &= f_{j,1}(x, t, 0), & n_{jk} &= \frac{\partial^{2k-2} f_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ n_{jN+1} &= f_{j,N+1}(x, t, 0), & n_{jN+k} &= \frac{\partial^{2k-2} f_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{j1} &= g_{j,1}(x, t, 0), & d_{jk} &= \frac{\partial^{2k-2} g_{j,1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ d_{jN+1} &= g_{j,N+1}(x, t, 0), & d_{jN+k} &= \frac{\partial^{2k-2} g_{j,N+1}}{\partial \epsilon^{2k-2}}(x, t, 0), \\ & & & 2 \leq k \leq N, 1 \leq j \leq 2N \end{aligned} \quad (19)$$

*The functions  $f$  and  $g$  are defined in (14), (15), (16), (17).*

We don't have the space to give the proof in this publication. We will give it in an other forthcoming paper.

The solutions of the NLS equation can also be written in the form :

$$v(x, t) = \exp(2it - i\varphi) \times Q(x, t)$$

where  $Q(x, t)$  is defined by :

$$Q(x, t) := \begin{vmatrix} f_{1,1}[0] & \cdots & f_{1,1}[N-1] & f_{1,N+1}[0] & \cdots & f_{1,N+1}[N-1] \\ f_{2,1}[0] & \cdots & f_{2,1}[N-1] & f_{2,N+1}[0] & \cdots & f_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{2N,1}[0] & \cdots & f_{2N,1}[N-1] & f_{2N,N+1}[0] & \cdots & f_{2N,N+1}[N-1] \\ \hline g_{1,1}[0] & \cdots & g_{1,1}[N-1] & g_{1,N+1}[0] & \cdots & g_{1,N+1}[N-1] \\ g_{2,1}[0] & \cdots & g_{2,1}[N-1] & g_{2,N+1}[0] & \cdots & g_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{2N,1}[0] & \cdots & g_{2N,1}[N-1] & g_{2N,N+1}[0] & \cdots & g_{2N,N+1}[N-1] \end{vmatrix} \quad (20)$$

### 3 Quasi-rational solutions of order 8 with twelve parameters

We have constructed in [7] solutions for the cases from  $N = 1$  until  $N = 7$ , and in [9] with two parameters.

We don't give the analytic expression of the solution of NLS equation with fourteen parameters because of the length of the expression. When all the parameters  $a_j$  and  $b_j$ ,  $1 \leq j \leq 7$  are equal to 0, we get the analogue of the Peregrine breather of order 8. For parameters non equal to 0, we construct figures to show deformations of the eighth Peregrine breather.

We get different types of symmetries in the plots in the  $(x, t)$  plane. We give some examples of this fact in the following.

If we choose  $\tilde{a}_i = \tilde{b}_i = 0$  for  $1 \leq i \leq 7$ , we obtain the classical Peregrine breather :

With other choices of parameters, we obtain all types of configurations : triangles and multiple concentric rings configurations with a maximum of 36 peaks.

In the case of the variation of one parameter, we obtain different types of configurations with a maximum of 36 peaks.

In the cases  $a_1 \neq 0$  or  $b_1 \neq 0$  we obtain triangles with a maximum of 36 peaks; for  $a_2 \neq 0$  or  $b_2 \neq 0$ , we have 5 rings with respectively 7, 7, 8, 8, 5 peaks with in the center one peak. For  $a_3 \neq 0$  or  $b_3 \neq 0$ , we obtain 5 rings with 7 peaks on each of them with a central peak. For  $a_4 \neq 0$  or  $b_4 \neq 0$ , we have 4 rings with 9 peaks on each of them without central peak. For  $a_5 \neq 0$  or  $b_5 \neq 0$ , we have 3 rings of 11 peaks with in the center the Peregrine of order 2. For  $a_6 \neq 0$  or  $b_6 \neq 0$ , we have 2 rings with 13 peaks and in the center the Peregrine breather of order 4. For  $a_7 \neq 0$  or  $b_7 \neq 0$ , we have one ring with 15 peaks and in the center the Peregrine breather of order 6.

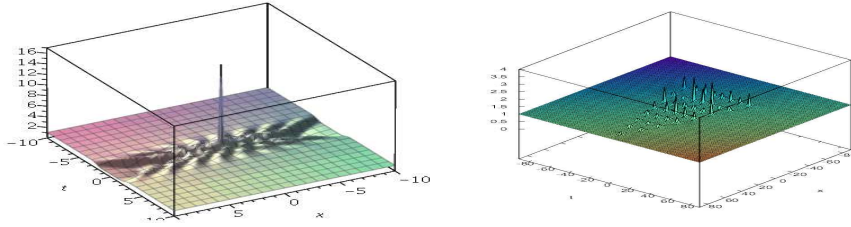


Figure 1: Solution of NLS,  $N=8$ ; left : all parameters equal to 0, the Peregrine  $P_8$ ; right : a regular triangle with 36 peaks.

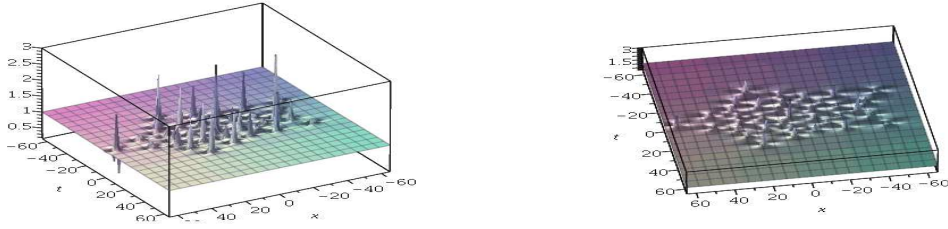


Figure 2: Solution of NLS,  $N=8$ ,  $\tilde{a}_2 = 10^6$  : 5 rings with respectively 7, 7, 8, 8, 5 peaks with in the center one peak; on the right, sight of top.

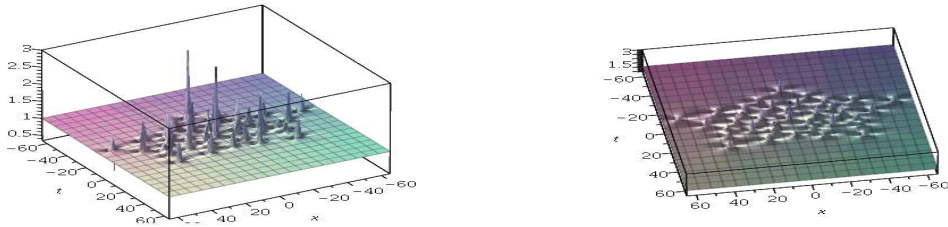


Figure 3: Solution of NLS,  $N=8$ ,  $\tilde{b}_2 = 10^6$  : 5 rings with respectively 7, 7, 8, 8, 5 peaks with in the center one peak; on the right, sight of top.

## 4 Conclusion

In the present paper we have constructed solutions of the NLS equation of order  $N = 8$  with  $2N - 2 = 14$  real parameters. The explicit representation



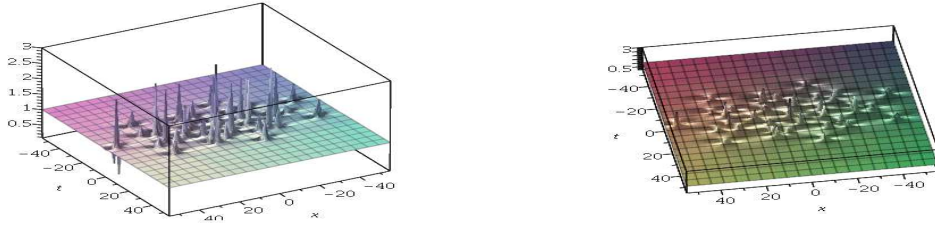


Figure 4: Solution of NLS,  $N=8$ ,  $\tilde{a}_3 = 10^5$  : 5 rings with 7 peaks on each of them with a central peak; on the right, sight of top.

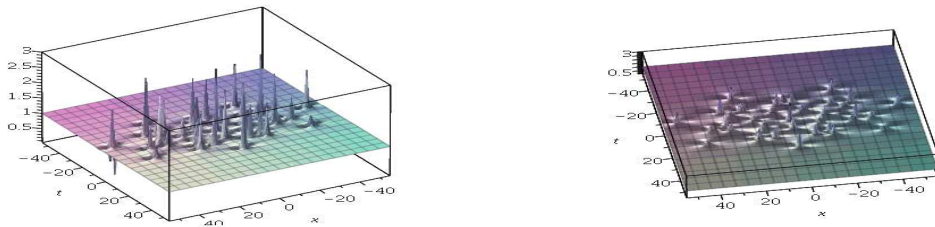


Figure 5: Solution of NLS,  $N=8$ ,  $\tilde{b}_3 = 10^5$  : 5 rings with 7 peaks on each of them with a central peak; on the right, sight of top.



Figure 6: Solution of NLS,  $N=8$ ,  $\tilde{a}_4 = 10^{10}$  : 4 rings with 9 peaks on each of them without central peak; on the right, sight of top.

in terms of polynomials in  $x$  and  $t$  are too monstrous to be published. We can't give in this text.

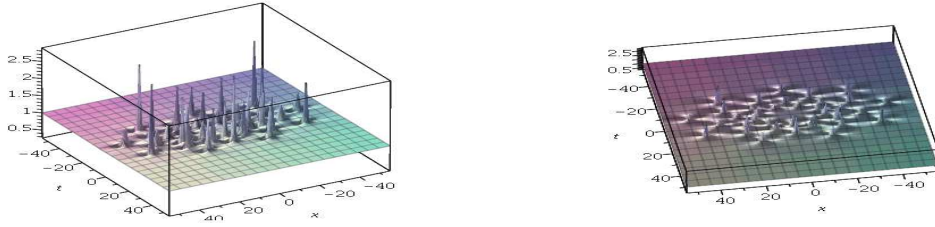


Figure 7: Solution of NLS,  $N=8$ ,  $\tilde{b}_4 = 10^{10}$  : 4 rings with 9 peaks on each of them without central peak; on the right, sight of top.

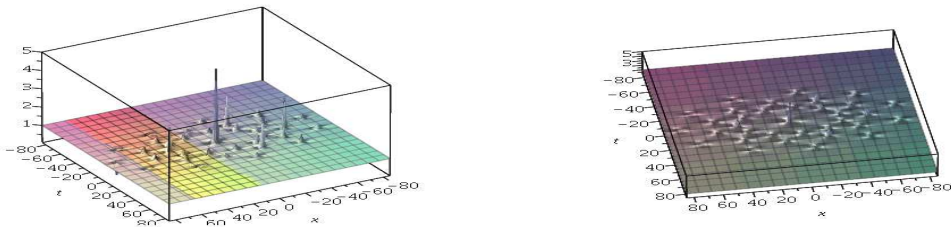


Figure 8: Solution of NLS,  $N=8$ ,  $\tilde{a}_5 = 10^{15}$  : 3 rings of 11 peaks with in the center the Peregrine of order 2; on the right, sight of top.

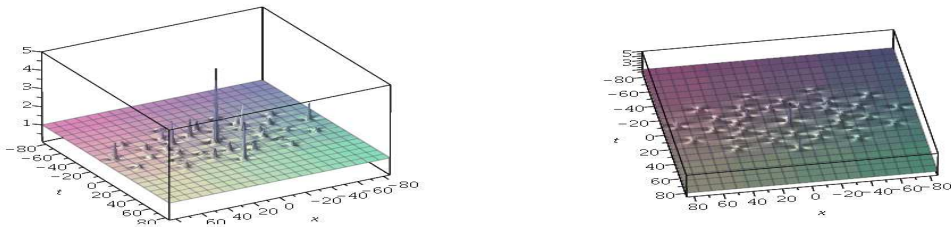


Figure 9: Solution of NLS,  $N=8$ ,  $\tilde{b}_5 = 10^{15}$  : 3 rings of 11 peaks with in the center the Peregrine of order 2; on the right, sight of top.

By different choices of these parameters, we obtained new patterns in the  $(x;t)$  plane; we recognized ring shape as already observed in the case of

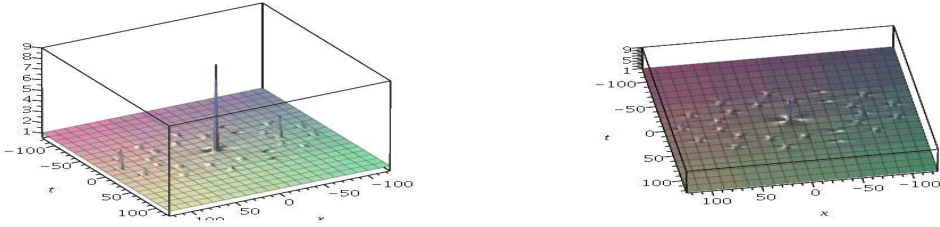


Figure 10: Solution of NLS,  $N=8$ ,  $\tilde{a}_6 = 10^{20}$  : 2 rings with 13 peaks and in the center the Peregrine breather of order 4; on the right, sight of top.

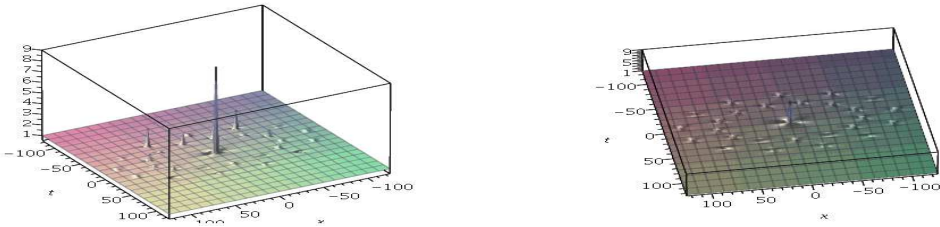


Figure 11: Solution of NLS,  $N=8$ ,  $\tilde{b}_6 = 10^{20}$  : 2 rings with 13 peaks and in the center the Peregrine breather of order 4; on the right, sight of top.

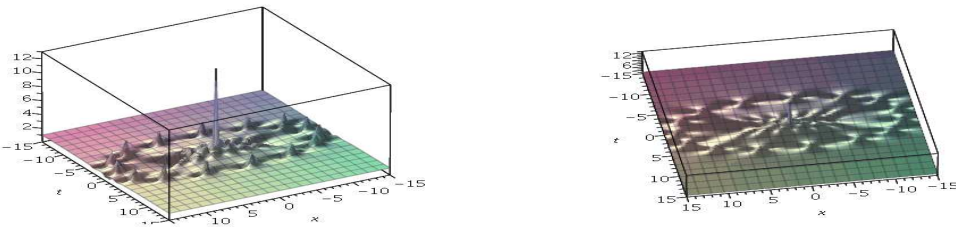


Figure 12: Solution of NLS,  $N=8$ ,  $\tilde{a}_7 = 10^{20}$  : one ring with 15 peaks and in the center the Peregrine breather of order 6; on the right, sight of top.

deformations depending on two parameters [7, 9]. We get new triangular shapes and multiple concentric rings.

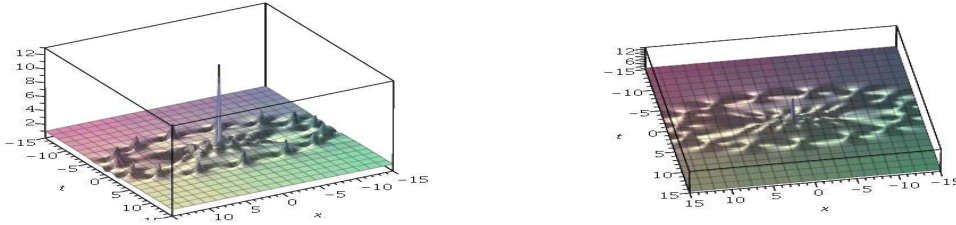


Figure 13: Solution of NLS,  $N=8$ ,  $\tilde{b}_7 = 10^{20}$  : one ring with 15 peaks and in the center the Peregrine breather of order 6; on the right, sight of top.

From the studies already carried out for the order 1 until the order 8, it would be important to obtain a classification for the order  $N$  in general as already done for orders 1 – 6 by Akhmediev et al. in [11]. Research on this subject should be done in the next years.

## References

- [1] N. Akhmediev, V. Eleonsky, N. Kulagin, Generation of periodic trains of picosecond pulses in an optical fiber : exact solutions, *Sov. Phys. J.E.T.P.*, V. **62**, 894-899, (1985).
- [2] N. Akhmediev, V. Eleonskii, N. Kulagin, Exact first order solutions of the nonlinear Schrödinger equation, *Th. Math. Phys.*, V. **72**, N. 2, 183-196, (1987).
- [3] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogue waves and rational solutions of nonlinear Schrödinger equation, *Physical Review E*, V. **80**, N. 026601, (2009).
- [4] N. Akhmediev, A. Ankiewicz, P.A. Clarkson, Rogue waves, rational solutions, the patterns of their zeros and integral relations, *J. Phys. A : Math. Theor.*, V. **43**, 122002, 1-9, (2010).
- [5] P. Dubard, V.B. Matveev, preprint RIMS-1777, 1-38, to appear in *Nonlinearity* **26/12** (2013).

- [6] P. Dubard, P. Gaillard, C. Klein, V.B. Matveev, On multi-rogue waves solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Special Topics*, V. **185**, 247-258, (2010).
- [7] P. Gaillard, Families of quasi-rational solutions of the NLS equation and multi-rogue waves, *J. Phys. A : Meth. Theor.*, V. **44**, 1-15, 2011
- [8] P. Gaillard, Wronskian representation of solutions of the NLS equation and higher Peregrine breathers, *Scientific Advances*, V. **13**, N. 2, 71-153, 2012
- [9] P. Gaillard, Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves, *Jour. Of Math. Phys.*, V. **54**, 013504-1-32, 2013
- [10] B. Guo, L. Ling, Q.P. Liu, Nonlinear Schrodinger equation: Generalized Darboux transformation and rogue wave solutions, *Phys. Rev. E*, V. **85**, 026607, (2012).
- [11] D. J. Kedziora, A. Ankiewicz, N. Akhmediev, *Phys. Review E* **88** (2013) 013207-1-12 Classifying the hierarchy of the nonlinear Schrödinger equation rogue waves solutions, *Phys. Review E*, V. 88, 013207-1-12, 2013
- [12] Y Ohta, J. Yang, General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation, *arXiv : 1110.5873 [nlin.SI]* 26 Oct. 2011.
- [13] D. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, *J. Austral. Math. Soc. Ser. B*, V. **25**, 16-43, (1983).
- [14] V. E. Zakharov, A.B. Shabat Exact theory of two dimensional self focusing and one dimensional self modulation of waves in nonlinear media, *Sov. Phys. JETP*, V. **34**, 62-69, (1972)