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COUNTING FUNCTION OF MAGNETIC EIGENVALUES FOR NON-DEFINITE SIGN PERTURBATIONS

DIOMBA SAMBOU

ABSTRACT. We consider the perturbed operator $H(b, V) := H(b, 0) + V$, where $H(b, 0)$ is the 3d Hamiltonian of Pauli with non-constant magnetic field, and V is a *non-definite sign electric potential* decaying exponentially with respect to the variable along the magnetic field. We prove that the only resonances of $H(b, V)$ near the low ground state zero of $H(b, 0)$ are its eigenvalues and are concentrated in the semi axis $(-\infty, 0)$. Further, we establish new asymptotic expansions, upper and lower bounds on their number near zero.

1. INTRODUCTION

In this article, we consider a three-dimensional Pauli operator $H(b, V) = H(b, 0) + V$ acting in $L^2(\mathbb{R}^3) := L^2(\mathbb{R}^3, \mathbb{C}^2)$. It describes a quantum non-relativistic spin- $\frac{1}{2}$ particle, subject to an electric potential V and a non-constant magnetic field $\mathbf{B} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of constant direction. With no loss of generality, we may assume that the magnetic field has the form

$$(1.1) \quad \mathbf{B}(x_1, x_2, x_3) = (0, 0, b(x_1, x_2)).$$

Throughout this paper, $b : \mathbb{R}^2 \rightarrow \mathbb{R}$ will be assumed to be an admissible magnetic field. That is, there exists a constant $b_0 > 0$ satisfying $b(x_1, x_2) = b_0 + \tilde{b}(x_1, x_2)$, where \tilde{b} is a function such that the Poisson equation

$$(1.2) \quad \Delta \tilde{\varphi} = \tilde{b}, \quad \Delta := \partial_1^2 + \partial_2^2,$$

admits a solution $\tilde{\varphi} \in C^2(\mathbb{R}^2)$ verifying $\sup_{(x_1, x_2) \in \mathbb{R}^2} |D^\alpha \tilde{\varphi}(x_1, x_2)| < \infty$, $\alpha \in \mathbb{N}^2$, $|\alpha| \leq 2$, (we refer for instance to [25, section 2.1] for more details on admissible magnetic fields). Notice that $\tilde{b} = 0$ coincides with the constant magnetic field case.

Let $\mathbf{A} = (A_1, A_2, A_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a magnetic potential generating the magnetic field \mathbf{B} . That is,

$$(1.3) \quad \mathbf{B}(X) = \text{curl } \mathbf{A}(X), \quad X = (X_\perp, x_3) \in \mathbb{R}^3, \quad X_\perp = (x_1, x_2) \in \mathbb{R}^2.$$

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The self-adjoint unperturbed Pauli operator $H(b, 0)$ is defined originally on $C_0^\infty(\mathbb{R}^3, \mathbb{C}^2)$ by

$$(1.4) \quad H(b, 0) := \begin{pmatrix} (-i\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-i\nabla - \mathbf{A})^2 + b \end{pmatrix},$$

and then closed in $L^2(\mathbb{R}^3)$. Since b is independent of x_3 , then with no loss of generality, we may assume that A_j , $j = 1, 2$, are independent of x_3 and $A_3 = 0$. Set $\varphi_0(X_\perp) := b_0|X_\perp|^2/4$ and $\varphi := \varphi_0 + \tilde{\varphi}$, so that we have $\Delta\varphi = b$. Introduce the operators

$$(1.5) \quad a = a(b) := -2ie^{-\varphi} \frac{\partial}{\partial \bar{z}} e^\varphi \quad \text{and} \quad a^* = a^*(b) := -2ie^\varphi \frac{\partial}{\partial z} e^{-\varphi},$$

originally defined on $C_0^\infty(\mathbb{R}^2, \mathbb{C})$, where $z := x_1 + ix_2$ and $\bar{z} := x_1 - ix_2$. Define the operators

$$(1.6) \quad H_1(b) := a^*a \quad \text{and} \quad H_2(b) := aa^*.$$

By choosing $A_1 = -\partial_2\varphi$ and $A_2 = \partial_1\varphi$, the operator $H(b, 0)$ can be rewritten in $L^2(\mathbb{R}^3) = L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R})$ as

$$(1.7) \quad H(b, 0) = \begin{pmatrix} H_1(b) \otimes 1 + 1 \otimes (-\partial_3^2) & 0 \\ 0 & H_2(b) \otimes 1 + 1 \otimes (-\partial_3^2) \end{pmatrix} =: \begin{pmatrix} \mathcal{H}_1(b) & 0 \\ 0 & \mathcal{H}_2(b) \end{pmatrix},$$

where $-\partial_3^2$ is originally defined on $C_0^\infty(\mathbb{R}, \mathbb{C})$. From [24, Proposition 1.1], we know that the spectra $\text{sp}(H_j)$ of H_j , $j = 1, 2$, satisfy the following properties:

$$(1.8) \quad \begin{aligned} \text{sp}(H_1) &\subseteq \{0\} \cup [\zeta, +\infty) \text{ with } 0 \text{ an eigenvalue of infinite multiplicity,} \\ \text{sp}(H_2) &\subseteq [\zeta, +\infty), \end{aligned}$$

where

$$(1.9) \quad \zeta := 2b_0e^{-2\text{osc } \tilde{\varphi}} > 0,$$

with $\text{osc } \tilde{\varphi} := \sup_{X_\perp \in \mathbb{R}^2} \tilde{\varphi}(X_\perp) - \inf_{X_\perp \in \mathbb{R}^2} \tilde{\varphi}(X_\perp)$. Since the spectrum of the operator $-\partial_3^2$ coincides with $[0, +\infty)$ and is absolutely continuous, then (1.7) and (1.8) imply that this of $H(b, 0)$ is equal to $[0, +\infty)$ and is absolutely continuous (see [25, Corollary 2.2]).

Remark. It is well known (see e.g. [12]) in the constant magnetic field case, the spectrum of H_1 consists of the Landau levels $2b_0\mathbb{N}$. Further, the multiplicity of each eigenvalue $2b_0q$, $q \in \mathbb{N}$, is infinite. In particular, this implies that the spectrum of H_2 consists of the Landau levels $2b_0\mathbb{N}^*$. Further, $\zeta = 2b_0$.

On the domain of the operator $H(b, 0)$, we introduce the perturbed operator

$$(1.10) \quad H(b, V) = H(b, 0) + V,$$

where we identify V with the multiplication operator by the function V .

In [29], we investigated the resonances (see Definition 4.1 below) of the operator $H(b, V)$ near zero. We required $V \equiv \{V_{jk}\}_{1 \leq j, k \leq 2}$ to be a hermitian matrix-valued electric potential satisfying

$$(1.11) \quad |V_{jk}(X)| \leq C \langle X_\perp \rangle^{-m_\perp} e^{-2\delta \langle x_3 \rangle}, \quad m_\perp > 0, \quad \delta > 0,$$

where $\langle u \rangle := \sqrt{1 + |u|^2}$, $u \in \mathbb{R}^d$, $d \geq 1$. For V of definite sign, we obtained in [29, Theorem 2.2] an asymptotic expansion of the number of resonances near zero. Further, we showed that they are concentrated in some sector. For V of *non-definite sign*, we obtained in [29, Theorem 2.1] an upper bound of the number of resonances near zero without their localization.

The aim of this paper is to study the same problem by considering the class of anti-diagonal matrix-valued electric potentials

$$(1.12) \quad V(X) := \begin{pmatrix} 0 & \overline{U(X)} \\ U(X) & 0 \end{pmatrix}, \quad X \in \mathbb{R}^3, \quad U(X) \in \mathbb{C},$$

where the function U satisfies the estimate

$$(1.13) \quad |U(X)| \leq C \langle X_\perp \rangle^{-m_\perp} e^{-2\delta \langle x_3 \rangle}, \quad m_\perp > 0, \quad \delta > 0,$$

with $C > 0$ a constant.

Remark. Notice that potentials V satisfying (1.12) are of *non-definite sign*. Indeed, its eigenvalues are $\pm |U(X)|$.

Novelty in this paper is that we prove the only resonances of $H(b, V)$ near zero are its eigenvalues. Further, they are localized in the semi axis $(-\infty, 0)$. We give new estimates on the number of negative eigenvalues $H(b, V)$ near zero. In particular, they show that the behaviour of magnetic eigenvalues for unsigned perturbations is different from that for signed perturbations. The crucial tool is that we exploit the form (1.12) of V in such a way we reduce the analysis of the resonances of $H(b, V)$ near $z = 0$ to that of the semi-effective effective Hamiltonian $\mathcal{H}_1 - \overline{U}(\mathcal{H}_2 - z)^{-1}U$ (see Section 4).

The paper is organized in the following manner. Our main results (Theorems 2.1 and 2.2) are stated in Section 2. In Section 3, we recall auxiliary results on Toeplitz operators and characteristic values of meromorphic operator-valued functions. In Section 4, we reduce the analysis of the resonances near zero to a characteristic value problem. Section 5 is devoted to the proofs of Theorems 2.1 and 2.2.

2. STATEMENT OF THE MAIN RESULTS

In order to formulate our main results, some notations are needed. For T a linear compact self-adjoint operator in a Hilbert space, we denote

$$(2.1) \quad n_+(s, T) := \text{rank } \mathbb{P}_{(s, \infty)}(T), \quad s > 0,$$

where $\mathbb{P}_{(s, \infty)}(T)$ is the orthogonal projection of T in the interval (s, ∞) . The set of negative eigenvalues of the operator $H(b, V)$ is denoted $\text{sp}_{\text{disc}}(H(b, V))$, namely its discrete spectrum. The orthogonal projection onto $\text{Ker } H_1(b)$ defined by (1.6) is denoted $p := p(b)$. The corresponding orthogonal projection in the constant magnetic field case will be denoted

$p_0 := p(b_0)$.

For a bounded operator $\mathcal{B} \in \mathcal{L}(L^2(\mathbb{R}^3))$, we define on $L^2(\mathbb{R}^2)$ the operator $W(\mathcal{B})$ by

$$(2.2) \quad (W(\mathcal{B})f)(X_\perp) := \frac{1}{2} \int_{\mathbb{R}} \overline{U}(X_\perp, x_3) \mathcal{B}(Uf)(X_\perp, x_3) dx_3, \quad X_\perp \in \mathbb{R}^2.$$

Clearly, if I denotes the identity on $L^2(\mathbb{R}^3)$, then $W(I)$ is the multiplication operator by the function

$$(2.3) \quad X_\perp \longmapsto \frac{1}{2} \int_{\mathbb{R}} |U|^2(X_\perp, x_3) dx_3.$$

The function (2.3) will be denoted $W(I)$ again. Let \mathcal{H}_2 be the operator defined by (1.7). If U satisfies (1.13), then [25, Lemma 2.4] implies that the positive self-adjoint operators $pW(I)p$ and $pW(\mathcal{H}_2^{-1})p$ are compact on $L^2(\mathbb{R}^2)$.

We are thus led to our first main result, where the resonances are defined in Definition 4.1 below.

Theorem 2.1. *Assume that (1.12) and (1.13) hold for V and U respectively. Then, there exists a discrete set $\mathcal{E} \subset \mathbb{R}^*$ such that for any $\nu \in \mathbb{R}^* \setminus \mathcal{E}$, the operator $H(b, \nu V)$ has the following properties:*

(i) *Localization: near zero, the resonances are its negative eigenvalues.*

(ii) *Asymptotic: suppose that $n_+(r, pW(\mathcal{H}_2^{-1})p) \rightarrow +\infty$, $r \searrow 0$. Then, there exists a sequence $(r_\ell)_\ell$ tending to 0 such that*

$$(2.4) \quad \#\text{sp}_{\text{disc}}(H(b, \nu V)) \cap (-\infty, -r_\ell^2) = n_+(r_\ell, pW(\mathcal{H}_2^{-1})p) (1 + o(1)), \quad \ell \longrightarrow \infty.$$

(iii) *Upper-bound: let I be the identity on $L^2(\mathbb{R}^3)$. If $W(I) \leq U_\perp$ with U_\perp satisfying the assumptions of Lemma 3.1, then*

$$(2.5) \quad \#\text{sp}_{\text{disc}}(H(b, \nu V)) \cap (-\infty, -r^2) \leq n_+\left(r, \frac{1}{\zeta} pW(I)p\right) (1 + o(1)), \quad r \searrow 0.$$

Remarks. Notice that in virtue of Lemma 3.1, the right hand side of (2.5) implies that the number of negative eigenvalues of $H(b, \nu V)$ near zero is of order $\mathcal{O}(r^{-1/m_\perp})$, $r \searrow 0$. This order is better than the order $\mathcal{O}(r^{-2/m_\perp})$ obtained in [29] for general perturbations V satisfying (1.11). Otherwise, if the function U_\perp is compactly supported, then (2.5) and [25, Lemma 3.4] imply that the number of negative eigenvalues of $H(b, \nu V)$ near zero is of order $\mathcal{O}((\ln |\ln r|)^{-1} |\ln r|)$, $r \searrow 0$, which is similar to that from [29].

In the constant magnetic field case $\mathbf{B} = (0, 0, b_0)$, we obtain, in additional, a lower bound of the number of negative eigenvalues near zero.

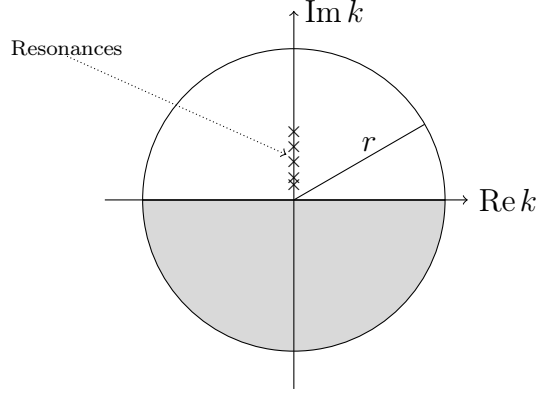


FIGURE 2.1. **Resonances near 0 with respect to the variable k :** For $r \ll 1$, the only resonances $z(k) = k^2$ of $H(b, 0) + V$ near zero are its negative eigenvalues and they satisfy $k \in i]0, +\infty)$.

Before to state our result, some additional notations are needed. If the function U satisfies $U(X_\perp, x_3) = U_\perp(X_\perp)\mathcal{U}(x_3)$, where U_\perp and \mathcal{U} are not necessarily real functions, we define

$$(2.6) \quad K_1 := \frac{\langle (-\partial_3^2 + 2b_0)^{-1} \mathcal{U}, \mathcal{U} \rangle}{2},$$

and

$$(2.7) \quad n_* \left(\left(\frac{r}{K_1} \right)^{\frac{1}{2}}, p_0 U_\perp p_0 \right) := n_+ \left(\frac{r}{K_1}, (p_0 U_\perp p_0)^* p_0 U_\perp p_0 \right).$$

Theorem 2.2 (Lower bound). *Let the magnetic field \mathbf{B} be constant. Assume that (1.12) and (1.13) hold for V and U respectively. Then, there exists a discrete set $\mathcal{E} \subset \mathbb{R}^*$ such that for any $\nu \in \mathbb{R}^* \setminus \mathcal{E}$, the following holds:*

Suppose that $U(X_\perp, x_3) = U_\perp(X_\perp)\mathcal{U}(x_3)$. If we have

$$n_* \left(\left(\frac{r}{K_1} \right)^{\frac{1}{2}}, p_0 U_\perp p_0 \right) = \phi(r)(1 + o(1)), \quad r \searrow 0,$$

where the function $\phi(r)$ is as in Lemma 3.5, then

$$(2.8) \quad \#\text{sp}_{\text{disc}}(H(b, \nu V)) \cap (-\infty, -r^2) \geq n_* \left(\left(\frac{r}{K_1} \right)^{\frac{1}{2}}, p_0 U_\perp p_0 \right) (1 + o(1)), \quad r \searrow 0.$$

In particular, if $U_\perp \geq 0$ and satisfies the assumptions of Lemma 3.1, then

$$(2.9) \quad \#\text{sp}_{\text{disc}}(H(b, \nu V)) \cap (-\infty, -r^2) \geq n_+ \left(\left(\frac{r}{K_1} \right)^{\frac{1}{2}}, p_0 U_\perp p_0 \right) (1 + o(1)), \quad r \searrow 0.$$

Remarks. Notice that estimates (2.9) and (2.5) imply, in the constant magnetic field case, the number of negative eigenvalues of $(H(b, \nu V))$ near 0 is such that

$$(2.10) \quad \begin{aligned} C_{m_\perp} K_1^{1/m_\perp} r^{-1/m_\perp} (1 + o(1)) &\leq \\ \#\text{SP}_{\text{disc}}(H(b, \nu V)) \cap (-\infty, -r^2) & \\ &\leq C_{m_\perp} K_2^{1/m_\perp} r^{-1/m_\perp} (1 + o(1)), \quad r \searrow 0, \end{aligned}$$

where C_{m_\perp} is the constant defined in Lemma 3.1, and

$$(2.11) \quad K_2 := (4b_0)^{-1} \int_{\mathbb{R}} |\mathcal{U}(x_3)|^2 dx_3.$$

It is easy to check that $K_1 < K_2$. On the other hand, the lower bound in (2.10) implies that the negative eigenvalues of $H(b, \nu V)$ accumulate to zero. One can compare (2.10) with the results of [25] on the asymptotic of the counting function of the eigenvalues of $H(b, V)$ near zero, when $V \equiv \{V_{jk}\}_{1 \leq j, k \leq 2}$ has a fixed sign. Indeed, in [25, Corollary 3.6], the author shows that if the coefficients of the potential $V \geq 0$ satisfy

$$|V_{jk}(X)| = \mathcal{O}(\langle X \rangle^{-\nu}), \quad 1 \leq j, k \leq 2,$$

for some $\nu > 3$, then the behaviour near zero of the counting function of the negative eigenvalues of $H(b, V)$ is of order

$$\mathcal{O}(r^{-2/(\nu-1)})(1 + o(1)), \quad r \searrow 0.$$

In particular, this shows that the behaviour of eigenvalues for unsigned perturbations is different from that for signed perturbations.

3. AUXILIARY RESULTS

3.1. Some results on Berezin-Toeplitz operators by Raikov [25], [22]. Consider $U_\perp \in L^\infty(\mathbb{R}^2)$. The asymptotic eigenvalues of the Berezin-Toeplitz operator pU_\perp is the subject of the next lemma. An integrated density of states (IDS) for the operator $H_1 = H_1(b)$ is defined as follows. For $X_\perp \in \mathbb{R}^2$, let χ_{T, X_\perp} be the characteristic function of the square $X_\perp + (-\frac{T}{2}, \frac{T}{2})^2$ with $T > 0$. Denote $\mathbb{P}_I(H_1)$ the spectral projection of H_1 in the interval $I \subset \mathbb{R}$. A non-increasing function $g : \mathbb{R} \rightarrow [0, \infty)$ is called an IDS for H_1 if it satisfies for any $X_\perp \in \mathbb{R}^2$

$$g(t) = \lim_{T \rightarrow \infty} T^{-2} \text{Tr} [\chi_{T, X_\perp} \mathbb{P}_{(-\infty, t)}(H_1) \chi_{T, X_\perp}],$$

for each point t of continuity of g (see e.g. [25]). If the magnetic field is constant, then there exists naturally an IDS for the operator H_1 given by

$$g(t) = \frac{b_0}{2\pi} \sum_{q=0}^{\infty} \chi_{\mathbb{R}_+}(t - 2b_0 q), \quad t \in \mathbb{R},$$

where $\chi_{\mathbb{R}_+}$ is the characteristic function of \mathbb{R}_+ .

Lemma 3.1. [22, Theorem 2.6] Consider $U_\perp \in C^1(\mathbb{R}^2)$ such that

$$0 \leq U_\perp(X_\perp) \leq C_1 \langle X_\perp \rangle^{-\alpha}, \quad |\nabla U_\perp(X_\perp)| \leq C_1 \langle X_\perp \rangle^{-\alpha-1}, \quad X_\perp \in \mathbb{R}^2,$$

where $\alpha > 0$ and $C_1 > 0$. Assume that

- $U_\perp(X_\perp) = u_0(X_\perp/|X_\perp|)|X_\perp|^{-\alpha}(1 + o(1))$ as $|X_\perp| \rightarrow \infty$, where u_0 is a continuous function on \mathbb{S}^1 which does not vanish identically,
- b is an admissible magnetic field,
- there exists an IDS g for the operator $H_1(b)$.

Then we have

$$n_+(s, pU_\perp p) = C_\alpha s^{-2/\alpha}(1 + o(1)), \quad s \searrow 0,$$

where

$$(3.1) \quad C_\alpha := \frac{b_0}{4\pi} \int_{\mathbb{S}^1} u_0(t)^{2/\alpha} dt.$$

3.2. Results on characteristic values by Bony-Bruneau-Raikov [7]. Let \mathcal{H} be a separable Hilbert space. We denote $S_\infty(\mathcal{H})$ (resp. $GL(\mathcal{H})$) the set of compact (resp. invertible) linear operators acting in \mathcal{H} .

Let $D \subseteq \mathbb{C}$ be a connected open set, $Z \subset D$ be a discrete and closed subset, $A : \overline{D} \setminus Z \rightarrow GL(\mathcal{H})$ be a finite meromorphic operator-valued function (see e.g. [7, Definition 2.1]) and Fredholm at each point of Z . The index of A , with respect to a positive oriented contour γ , is defined by

$$(3.2) \quad \text{Ind}_\gamma A := \frac{1}{2i\pi} \text{Tr} \int_\gamma A'(z)A(z)^{-1} dz = \frac{1}{2i\pi} \text{Tr} \int_\gamma A(z)^{-1} A'(z) dz.$$

Here, the operator A does not vanish on the integration contour γ . Let \mathcal{D} be a domain of \mathbb{C} containing 0. Consider a holomorphic operator-valued function $T : \mathcal{D} \rightarrow S_\infty(\mathcal{H})$. For a domain $\Omega \subset \mathcal{D} \setminus \{0\}$, a complex number $z \in \Omega$ is said to be a *characteristic value* of $z \mapsto \mathcal{T}(z) := I - \frac{T(z)}{z}$ if the operator $\mathcal{T}(z)$ is not invertible. The multiplicity of a characteristic value z_0 is defined by

$$(3.3) \quad \text{mult}(z_0) := \text{Ind}_\gamma (I - \mathcal{T}(\cdot)),$$

where γ is a small contour positively oriented, containing z_0 as the unique point z satisfying $\mathcal{T}(z)$ is not invertible.

Define

$$\mathcal{Z}(\Omega) := \left\{ z \in \Omega : I - \frac{T(z)}{z} \text{ is not invertible} \right\}.$$

If there exists $z_0 \in \Omega$ such that $I - \frac{T(z_0)}{z_0}$ is not invertible, then $\mathcal{Z}(\Omega)$ is a discrete set (see e.g. [15, proposition 4.1.4]). So we define

$$\mathcal{N}(\Omega) := \#\mathcal{Z}(\Omega).$$

Assume that $T(0)$ is self-adjoint. Introduce $\Omega \Subset \mathbb{C} \setminus \{0\}$ and the sector

$$(3.4) \quad \mathcal{C}_\alpha(a, b) := \{x + iy \in \mathbb{C} : a \leq x \leq b, -\alpha x \leq y \leq \alpha x\},$$

with $a > 0$ tending to 0 and $b > 0$. Let

$$n(\Lambda) := \text{Tr } \mathbf{1}_\Lambda(T(0))$$

be the number of eigenvalues of the operator $T(0)$ lying in the interval $\Lambda \subset \mathbb{R}^*$, and counted with their multiplicity. Denoted Π_0 the orthogonal projection onto $\text{Ker } T(0)$.

Lemma 3.2. [7, Corollary 3.4] *Let T be as above and $I - T'(0)\Pi_0$ be invertible. Assume that $\Omega \Subset \mathbb{C} \setminus \{0\}$ is a bounded domain with smooth boundary $\partial\Omega$ which is transverse to the real axis at each point of $\partial\Omega \cap \mathbb{R}$.*

(i) *If $\Omega \cap \mathbb{R} = \emptyset$, then $\mathcal{N}(s\Omega) = 0$ for s small enough. This implies that the characteristic values $z \in \mathcal{Z}(\mathcal{D})$ near 0 satisfy $|\text{Im } z| = o(|z|)$.*

(ii) *Moreover, if the operator $T(0)$ has a definite sign ($\pm T(0) \geq 0$), then the characteristic values z near 0 satisfy $\pm \text{Re } z \geq 0$, respectively.*

(iii) *If $T(0)$ is of finite rank, then there are no characteristic values in a pointed neighbourhood of 0. Moreover, if the operator $T(0)\mathbf{1}_{[0, +\infty)}(\pm T(0))$ is of finite rank, then there are no characteristic values in a neighbourhood of 0 intersected with $\{\pm \text{Re } z > 0\}$, respectively.*

Lemma 3.3. [7, Theorem 3.7] *Let T be as above and $I - T'(0)\Pi_0$ be invertible. For $\alpha > 0$ fixed, let $\mathcal{C}_\alpha(r, 1) \subset \mathcal{D}$ be defined as in (3.4). Then, for all $\delta > 0$ small enough, there exists $s(\delta) > 0$ such that, for all $0 < s < s(\delta)$, we have*

$$(3.5) \quad \begin{aligned} \mathcal{N}(\mathcal{C}_\alpha(r, 1)) &= n([r, 1]) (1 + \mathcal{O}(\delta |\ln \delta|^2)) \\ &+ \mathcal{O}(|\ln \delta|^2) n([r(1 - \delta), r(1 + \delta)]) + \mathcal{O}_\delta(1), \end{aligned}$$

where the \mathcal{O} 's are uniform with respect to s, δ but the \mathcal{O}_δ may depend on δ .

Lemma 3.4. [7, Corollary 3.9] *Let the assumptions of Lemma 3.3 hold true. Assume that there exists $\gamma > 0$ such that*

$$n([r, 1]) = O(r^{-\gamma}), \quad r \searrow 0,$$

and that $n([r, 1])$ grows unboundedly as $r \searrow 0$. Then there exists a positive sequence $(r_k)_k$ tending to 0 such that

$$(3.6) \quad \mathcal{N}(\mathcal{C}_\alpha(r_k, 1)) = n([r_k, 1]) (1 + o(1)), \quad k \rightarrow \infty.$$

Lemma 3.5. [7, Corollary 3.11] *Let the assumptions of Lemma 3.3 hold true. Suppose that*

$$n([r, 1]) = \Phi(r) (1 + o(1)), \quad r \searrow 0,$$

with $\Phi(r) = r^{-\gamma}$, or $\Phi(r) = |\ln r|^\gamma$, or $\Phi(r) = (\ln |\ln r|)^{-1} |\ln r|$, for some $\gamma > 0$. Then

$$(3.7) \quad \mathcal{N}(\mathcal{C}_\alpha(r, 1)) = \Phi(r) (1 + o(1)), \quad r \searrow 0.$$

4. RESONANCES

From here to the end, we assume that V and U satisfy (1.12) and (1.13) respectively.

4.1. A preliminary property. We establish the main property allowing to reduce the study of the resonances of $H(b, V)$ near $z = 0$ to that of the semi-effective Hamiltonian $\mathcal{H}_1 - \bar{U}(\mathcal{H}_2 - z)^{-1}U$.

Let $z \in \mathbb{C}$ be small enough. We have

$$(4.1) \quad \begin{aligned} (H(b, V) - z) & \begin{pmatrix} 1 & 0 \\ -(\mathcal{H}_2 - z)^{-1}U & (\mathcal{H}_2 - z)^{-1} \end{pmatrix} \\ & = \begin{pmatrix} \mathcal{H}_1 - z - \bar{U}(\mathcal{H}_2 - z)^{-1}U & \bar{U}(\mathcal{H}_2 - z)^{-1} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore,

$$(4.2) \quad H(b, V) - z \text{ is invertible} \Leftrightarrow \mathcal{H}_1 - z - \bar{U}(\mathcal{H}_2 - z)^{-1}U \text{ is invertible.}$$

Further,

$$(4.3) \quad \begin{aligned} (H(b, V) - z)^{-1} & = \begin{pmatrix} 1 & 0 \\ -(\mathcal{H}_2 - z)^{-1}U & (\mathcal{H}_2 - z)^{-1} \end{pmatrix} \\ & \times \begin{pmatrix} (\mathcal{H}_1 - z - \bar{U}(\mathcal{H}_2 - z)^{-1}U)^{-1} & -(\mathcal{H}_1 - z - \bar{U}(\mathcal{H}_2 - z)^{-1}U)^{-1}\bar{U}(\mathcal{H}_2 - z)^{-1} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence, for z small enough, property (4.2) allows to reduce the non-invertibility of the operator $H(b, V) - z$ to that of $\mathcal{H}_1 - z - \bar{U}(\mathcal{H}_2 - z)^{-1}U$.

4.2. Reduction to a semi-effective problem. Consider z lying in the upper half-plane \mathbb{C}^+ . Make the change of variables

$$(4.4) \quad z := z(k) = k^2 \text{ for } k \in \mathbb{C}_{1/2}^+ := \{k \in \mathbb{C}^+ : k^2 \in \mathbb{C}^+\}.$$

Introduce the punctured disk

$$(4.5) \quad D(0, \epsilon)^* := \{k \in \mathbb{C} : 0 < |k| < \epsilon\}, \quad \epsilon < \min(\delta, \sqrt{\zeta}),$$

where the constants δ and ζ are respectively defined by (1.13) and (1.9).

Proposition 4.1. [29, Proposition 4.1]

Let $R(z)$ denote the resolvent of the operator $H(b, V)$. Then, the operator valued-function

$$k \mapsto (R(z(k)) : e^{-\delta\langle x_3 \rangle} L^2(\mathbb{R}^3) \longrightarrow e^{\delta\langle x_3 \rangle} L^2(\mathbb{R}^3)),$$

admits a meromorphic extension from $\mathbb{C}_{1/2}^+ \cap D(0, \epsilon)^*$ to $D(0, \epsilon)^*$. We shall denote this extension $R(z)$ again.

Definition 4.1. We define the resonances of $H(b, V)$ near zero as the poles of the meromorphic extension $R(z)$.

Set $\mathcal{R}(z) := (\mathcal{H}_1 - z - \bar{U}(\mathcal{H}_2 - z)^{-1}U)^{-1}$ and $R_2(z) := (\mathcal{H}_2 - z)^{-1}$. From (4.3) we deduce that

$$(4.6) \quad \begin{aligned} & e^{-\delta\langle x_3 \rangle} R(z) e^{-\delta\langle x_3 \rangle} \\ &= \begin{pmatrix} e^{-\delta\langle x_3 \rangle} \mathcal{R}(z) e^{-\delta\langle x_3 \rangle} & -e^{-\delta\langle x_3 \rangle} \mathcal{R}(z) \bar{U} R_2(z) e^{-\delta\langle x_3 \rangle} \\ -e^{-\delta\langle x_3 \rangle} R_2(z) U \mathcal{R}(z) e^{-\delta\langle x_3 \rangle} & e^{-\delta\langle x_3 \rangle} R_2(z) U \mathcal{R}(z) \bar{U} R_2(z) e^{-\delta\langle x_3 \rangle} + e^{-\delta\langle x_3 \rangle} R_2(z) e^{-\delta\langle x_3 \rangle} \end{pmatrix}. \end{aligned}$$

This together with Proposition 4.1 and assumption (1.13) show that the poles of $R(z)$ coincide with those of $\mathcal{R}(z)$. Then, near $z = 0$, the investigation of the resonances of $H(b, V)$ is reduced to that of the semi-effective Hamiltonian $\mathcal{H}_1 - \bar{U}(\mathcal{H}_2 - z)^{-1}U$.

4.3. Study of the semi-effective problem. With the help of the decomposition

$$(4.7) \quad (\mathcal{H}_2 - z)^{-1} = \mathcal{H}_2^{-1} \left(1 - z\mathcal{H}_2^{-1}\right)^{-1} = \mathcal{H}_2^{-1} \sum_{k \geq 0} z^k \mathcal{H}_2^{-k},$$

z being sufficiently small, we obtain

$$(4.8) \quad (\mathcal{H}_2 - z)^{-1} = \mathcal{H}_2^{-1/2} \left(\mathcal{H}_2^{-1/2} + \mathcal{H}_2^{-1/2} M(z) \right),$$

where

$$(4.9) \quad M(z) := z \sum_{k \geq 0} z^k \mathcal{H}_2^{-k-1}.$$

So, (4.8) implies that

$$(4.10) \quad \bar{U}(\mathcal{H}_2 - z)^{-1}U = \bar{U} \mathcal{H}_2^{-1/2} \left(\mathcal{H}_2^{-1/2}U + \mathcal{H}_2^{-1/2}M(z)U \right).$$

Now define the operator

$$(4.11) \quad \mathbf{w} := \mathcal{H}_2^{-1/2}U.$$

Thus, putting together (4.10) and (4.11) we obtain

$$(4.12) \quad \bar{U}(\mathcal{H}_2 - z)^{-1}U = \mathbf{w}^* (1 + M(z)) \mathbf{w}.$$

We therefore have proved the following

Lemma 4.1. *For z small enough, the operator $\bar{U}(\mathcal{H}_2 - z)^{-1}U$ admits the representation*

$$(4.13) \quad \bar{U}(\mathcal{H}_2 - z)^{-1}U = \mathbf{w}^* (1 + M(z)) \mathbf{w}.$$

Further, the operator-valued function $z \mapsto M(z)$ is analytic near $z = 0$.

Let $R_1(z)$ denote the resolvent of the operator \mathcal{H}_1 . Under the notations of Lemma 4.1, the following lemma holds:

Lemma 4.2. *For z small enough, the operator valued-function*

$$D(0, \epsilon)^* \ni k \mapsto \mathcal{T}_V(z(k)) := \left(1 + M(z(k))\right) \mathbf{w} R_1(z(k)) \mathbf{w}^*,$$

is analytic with values in $S_\infty(L^2(\mathbb{R}^3))$.

Proof. The analyticity of $\mathcal{T}_V(z(k))$ holds since $M(z(k))$ and $R_1(z(k))$ are well defined and analytic for $k \in D(0, \epsilon)^*$.

The compactness of $\mathcal{T}_V(z(k))$ follows from that of $UR_1(z(k))\overline{U}$, using the diamagnetic inequality and [32, Theorem 2.13]. \square

We have the following characterization of the resonances.

Proposition 4.2. *For k near zero, the following assertions are equivalent:*

- (i) $z(k) = k^2$ is a resonance of $H(b, V)$,
- (ii) 1 is an eigenvalue of $\mathcal{T}_V(z(k))$.

Proof. The equivalence follows directly from the identity

$$(4.14) \quad \left(I - (1 + M(z))\mathbf{w}R_1(z)\mathbf{w}^* \right) \left(I + (1 + M(z))\mathbf{w}\mathcal{R}(z)\mathbf{w}^* \right) = I,$$

and the fact that the poles of $R(z)$ coincide with those of $\mathcal{R}(z)$. \square

So, the multiplicity of a resonance $z := z(k)$ is defined by

$$(4.15) \quad \text{mult}(z) := \text{Ind}_\gamma \left(I - \mathcal{T}_V(z(\cdot)) \right),$$

where γ is a small positively oriented contour containing k as the unique point satisfying $z(k)$ is a resonance of $H(b, V)$ (see (3.2)).

Using the terminology of characteristic value recalled in Subsection 3.2, Proposition 4.2 can be formulated as follows:

Proposition 4.3. *For k near zero, the following assertions are equivalent.*

- (i) $z = z(k)$ is a resonance of $H(b, V)$,
- (ii) k is a characteristic value of $I - \mathcal{T}_V(z(\cdot))$.

Further, according to (4.15), the multiplicity of the resonance $z(k)$ coincides with this of the characteristic value k .

5. PROOF OF THE MAIN RESULTS

First, let us introduce some tools. For $p = p(b)$, set $q := I - p$. Define on $L^2(\mathbb{R}^3)$ the projections $P := p \otimes 1$ and $Q := q \otimes 1$. If z lies in the resolvent set the operator \mathcal{H}_1 , we have

$$(5.1) \quad \begin{aligned} (\mathcal{H}_1 - z)^{-1} &= (\mathcal{H}_1 - z)^{-1}P + (\mathcal{H}_1 - z)^{-1}Q \\ &= p \otimes \mathcal{R}(z) + (\mathcal{H}_1 - z)^{-1}Q, \end{aligned}$$

where the resolvent $\mathcal{R}(z) := (-\partial_3^2 - z)^{-1}$ admits the integral kernel

$$(5.2) \quad \mathcal{N}_z(x_3 - x'_3) = ie^{i\sqrt{z}|x_3 - x'_3|} / (2\sqrt{z}), \quad \text{Im } \sqrt{z} > 0.$$

5.1. Proof of Theorem 2.1.

5.1.1. *Preliminary results.* Firstly, we need to split the operator $\mathcal{T}_V(z(k))$ of Lemma 4.2 with the help of (5.1). We get

$$(5.3) \quad \begin{aligned} \mathcal{T}_V(z(k)) &= \mathbf{w}p \otimes \mathcal{R}(k^2)\mathbf{w}^* + M(z(k))\mathbf{w}p \otimes \mathcal{R}(k^2)\mathbf{w}^* \\ &\quad + \left(1 + M(z(k))\right)\mathbf{w}R_1(z(k))Q\mathbf{w}^*. \end{aligned}$$

The operators $M(z(k))$ and $R_1(z(k))Q$ are analytic near zero. Then, it is not difficult to see that the third term of the right hand side of (5.3) is holomorphic near zero, with values in $S_\infty(L^2(\mathbb{R}^3))$. By (5.2), the integral kernel of $N(k) := e^{-\delta\langle x_3 \rangle} \mathcal{R}(k^2) e^{-\delta\langle x_3 \rangle}$ is given by

$$(5.4) \quad e^{-\delta\langle x_3 \rangle} \frac{i e^{ik|x_3-x'_3|}}{2k} e^{-\delta\langle x'_3 \rangle}.$$

This together with (4.9) imply that the second term of the right hand side of (5.3) is analytic in a vicinity of zero, with values in $S_\infty(L^2(\mathbb{R}^3))$.

Now let us focus on the first term $\mathbf{w}p \otimes \mathcal{R}(k^2)\mathbf{w}^*$. According to (5.4), we can write

$$(5.5) \quad N(k) = \frac{1}{k}a + b(k),$$

where $a : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is the rank-one operator defined by

$$(5.6) \quad a(u) := \frac{i}{2} \langle u, e^{-\delta\langle \cdot \rangle} \rangle e^{-\delta\langle x_3 \rangle},$$

and $b(k)$ is the Hilbert-Schmidt operator (for $k \in D(0, \epsilon)^*$) with integral kernel

$$(5.7) \quad e^{-\delta\langle x_3 \rangle} i \frac{e^{ik|x_3-x'_3|} - 1}{2k} e^{-\delta\langle x'_3 \rangle}.$$

Thus,

$$(5.8) \quad \mathbf{w}p \otimes \mathcal{R}(k^2)\mathbf{w}^* = \frac{i}{k} \times \frac{1}{2} \mathbf{w}(p \otimes \tau)\mathbf{w}^* + \mathbf{w}(p \otimes s(k))\mathbf{w}^*,$$

where τ and $s(k)$ are operators acting from $e^{-\delta\langle x_3 \rangle} L^2(\mathbb{R})$ to $e^{\delta\langle x_3 \rangle} L^2(\mathbb{R})$, with integral kernels respectively given by 1 and

$$(5.9) \quad \frac{1 - e^{ik|x_3-x'_3|}}{2ik}.$$

We therefore have proved the following

Proposition 5.1. *Let $k \in D(0, \epsilon)^*$. Then,*

$$(5.10) \quad \mathcal{T}_V(z(k)) = i \frac{\mathbf{w}(p \otimes \tau)\mathbf{w}^*}{2k} + B(k),$$

where

$$(5.11) \quad \begin{aligned} B(k) &:= \mathbf{w}(p \otimes s(k))\mathbf{w}^* + M(z(k))\mathbf{w}p \otimes \mathcal{R}(k^2)\mathbf{w}^* \\ &\quad + \left(1 + M(z(k))\right)\mathbf{w}R_1(z(k))Q\mathbf{w}^*, \end{aligned}$$

is holomorphic in $D(0, \epsilon) := D(0, \epsilon)^* \cup \{0\}$, with values in $S_\infty(L^2(\mathbb{R}^3))$.

Notice that $\mathbf{w}(p \otimes \tau)\mathbf{w}^*$ is a positive self-adjoint compact operator. Indeed, if we define e_\pm as the multiplication operators by the functions $e_\pm : x_3 \mapsto e^{\pm\delta\langle x_3 \rangle}$, it is easy to check that

$$(5.12) \quad \mathbf{w}(p \otimes \tau)\mathbf{w}^* = \mathbf{w}e_+(p \otimes c^*c)e_+\mathbf{w}^* = ((p \otimes c)e_+\mathbf{w}^*)^* ((p \otimes c)e_+\mathbf{w}^*).$$

Here, $c : L^2(\mathbb{R}) \rightarrow \mathbb{C}$ is defined by $c(f) := \langle f, e_- \rangle$, so that $c^* : \mathbb{C} \rightarrow L^2(\mathbb{R})$ is given by $c^*(\lambda) = \lambda e_-$. Now, with the help of (5.12), we deduce that

$$(5.13) \quad n_+ \left(r, \frac{\mathbf{w}(p \otimes \tau)\mathbf{w}^*}{2} \right) = n_+ \left(r, \frac{(p \otimes c)e_+\mathbf{w}^*\mathbf{w}e_+(p \otimes c^*)}{2} \right), \quad r > 0,$$

where the quantity $n_+(r, \cdot)$ is defined by (2.1). By the definition (4.11) of \mathbf{w} , we have $\mathbf{w}^*\mathbf{w} = \overline{U}\mathcal{H}_2^{-1}U$. This together with $\text{sp}(\mathcal{H}_2) \subseteq [\zeta, +\infty)$ imply that

$$(5.14) \quad \frac{(p \otimes c)e_+\mathbf{w}^*\mathbf{w}e_+(p \otimes c^*)}{2} = pW(\mathcal{H}_2^{-1})p \leq \frac{pW(I)p}{\zeta},$$

where for $\mathcal{B} \in \mathcal{L}(L^2(\mathbb{R}^3))$, $W(\mathcal{B})$ is the operator defined by (2.2). Then, by combining (5.13) with (5.14) we obtain

$$(5.15) \quad \begin{aligned} n_+ \left(r, \frac{\mathbf{w}(p \otimes \tau)\mathbf{w}^*}{2} \right) &= n_+ \left(r, pW(\mathcal{H}_2^{-1})p \right) \\ &\leq n_+ \left(r, \frac{pW(I)p}{\zeta} \right), \quad r > 0. \end{aligned}$$

Otherwise, according to Proposition 4.3, the study of the resonances $z(k) = k^2$ of $H(b, \nu V)$ near zero, is reduced to that of the characteristic values of the operator

$$I - \mathcal{T}_{\nu V}(z(k)) = I + \nu^2 \frac{T(ik)}{ik}.$$

Here, taking into account Proposition 5.1, $T(ik) := \frac{\mathbf{w}(p \otimes \tau)\mathbf{w}^*}{2} - ikB(k)$ so that $T(0) = \frac{\mathbf{w}(p \otimes \tau)\mathbf{w}^*}{2}$. Let Π_0 be the orthogonal projection onto $\text{Ker} T(0)$. Since $T'(0)\Pi_0$ is compact, then, there exists a sequence $(\nu_n)_n$ such that $I - \nu T'(0)\Pi_0$ is invertible for any $\nu \in \mathbb{R} \setminus \{\nu_n, n \in \mathbb{N}\}$. Note that we can take $\nu_n = \lambda_n^{-1}$, where $\{\lambda_n, n \in \mathbb{N}\}$ is the set of eigenvalues of the operator $T'(0)\Pi_0$.

5.1.2. *Back to the proof of Theorem 2.1.* Notations are those from Subsection 3.2.

(i): its follows immediately from Lemma 3.2 with $z = -ik/\nu^2$.

(ii): Theorem 2.1 (i) shows, in particular, for $|k|$ small enough the resonances $z(k) = k^2$ are concentrated in the sector $\{k \in D(0, \epsilon)^* : -ik/\nu^2 \in \mathcal{C}_\alpha(r, r_0)\}$, for any $\alpha > 0$. Hence,

if $\text{Res}(H(b, \nu V))$ denotes the set of resonances of $H(b, \nu V)$, we have

$$(5.16) \quad \begin{aligned} & \#\{z(k) = k^2 \in \text{Res}(H(b, \nu V)) : r < |k| \leq r_0\} \\ &= \#\{z(k) = k^2 \in \text{Res}(H(b, \nu V)) : -ik/\nu^2 \in \mathcal{C}_\alpha(r, r_0)\} + \mathcal{O}(1) \\ &= \mathcal{N}(\mathcal{C}_\alpha(r, r_0)) + \mathcal{O}(1), \quad r \searrow 0. \end{aligned}$$

On the other hand, we have

$$(5.17) \quad n([r, r_0]) = \text{Tr} \mathbf{1}_{[r, r_0]}(T(0)) = n_+ \left(r, \frac{\mathbf{w}(p \otimes \tau) \mathbf{w}^*}{2} \right) + \mathcal{O}(1).$$

This together with the inequality in (5.15) imply that

$$n([r, r_0]) \leq n_+ \left(r, \frac{pW(I)p}{\zeta} \right) + \mathcal{O}(1).$$

Then, Theorem 2.1 (ii) follows from (5.16) together with Lemma 3.4, (5.17) and the equality in (5.15).

(iii): if we have $W(I) \leq U_\perp$, with U_\perp satisfying the assumptions of Lemma 3.1, then

$$(5.18) \quad n_+ \left(r, \frac{pW(I)p}{\zeta} \right) = C_{m_\perp} (\zeta r)^{-1/m_\perp} (1 + o(1)), \quad r \searrow 0,$$

where m_\perp is the constant defined by (1.13). Similarly to the inequality in (5.15), we can show that

$$(5.19) \quad n([r, r_0]) \leq \text{Tr} \mathbf{1}_{[r, r_0]} \left(\frac{pW(I)p}{\zeta} \right) =: \tilde{n}([r, r_0]).$$

Note that due to (5.18),

$$(5.20) \quad \tilde{n}([r, r_0]) = C_{m_\perp} (\zeta r)^{-1/m_\perp} (1 + o(1)), \quad r \searrow 0.$$

Now if $\phi(r) = r^{-\gamma}$, $\gamma > 0$, then $\phi(r(1 \pm \nu)) = r^{-\gamma} (1 \pm \nu)^{-\gamma} = \phi(r) (1 + \mathcal{O}(\nu))$. If $\tilde{n}([r, 1]) = \phi(r) (1 + o(1))$ with $\phi(r(1 \pm \delta)) = \phi(r) (1 + o(1) + \mathcal{O}(\delta))$, then

$$(5.21) \quad \tilde{n}([r(1 - \nu), r(1 + \nu)]) = \tilde{n}([r, 1]) (o(1) + \mathcal{O}(\nu)).$$

Then, Theorem 2.1 (iii) follows from (5.16) together with Lemma 3.3, (5.19), (5.20) and (5.21).

5.2. Proof of Theorem 2.2. To obtain (2.8), it suffices to prove that if the function U satisfies $U(X_\perp, x_3) = U_\perp(X_\perp) \mathcal{U}(x_3)$, then the following operator inequality holds:

$$(5.22) \quad K_1(p_0 U_\perp p_0)^* (p_0 U_\perp p_0) \leq p_0 W(\mathcal{H}_2^{-1}) p_0.$$

Indeed, if (5.22) is true, then with respect to the constant magnetic field, the quantity $n([r, r_0]) = \text{Tr} \mathbf{1}_{[r, r_0]} \left(\frac{\mathbf{w}(p_0 \otimes \tau) \mathbf{w}^*}{2} \right) = \text{Tr} \mathbf{1}_{[r, r_0]} (p_0 W(\mathcal{H}_2^{-1}) p_0)$ satisfies

$$(5.23) \quad n_*([r, r_0]) := \text{Tr} \mathbf{1}_{[r, r_0]} [K_1(p_0 U_\perp p_0)^* (p_0 U_\perp p_0)] \leq n([r, r_0]).$$

Further, if we have

$$\begin{aligned} n_* \left(\left(\frac{r}{K_1} \right)^{\frac{1}{2}}, p_0 U_{\perp} p_0 \right) &:= n_+ \left(\frac{r}{K_1}, (p_0 U_{\perp} p_0)^* p_0 U_{\perp} p_0 \right) \\ &= \phi(r)(1 + o(1)), \quad r \searrow 0, \end{aligned}$$

where the function $\phi(r)$ is as in Lemma 3.5, then

$$(5.24) \quad n_*([r, r_0]) = \phi(r)(1 + o(1)), \quad r \searrow 0.$$

Thus, (2.8) follows by arguing as in the proof of Theorem 2.1 (iii) above.

Now let us proof (5.22). If the magnetic field is constant, then \mathcal{H}_2 satisfies

$$\mathcal{H}_2^{-1} \geq \mathcal{H}_2^{-1} p_0 = p_0 \otimes (-\partial_3^2 + 2b_0)^{-1}.$$

This together with the definition (2.2) of $W(\mathcal{H}_2^{-1})$ imply that, if $U(X_{\perp}, x_3) = U_{\perp}(X_{\perp})\mathcal{U}(x_3)$, then for any $f \in L^2(\mathbb{R}^2)$

$$(5.25) \quad \left\langle W(\mathcal{H}_2^{-1})f, f \right\rangle \geq K_1 \langle \overline{U_{\perp}} p_0 U_{\perp} f, f \rangle.$$

This means that we have the operator inequality

$$W(\mathcal{H}_2^{-1}) \geq K_1 \overline{U_{\perp}} p_0 U_{\perp}.$$

Thus,

$$p_0 W(\mathcal{H}_2^{-1}) p_0 \geq K_1 (p_0 U_{\perp} p_0)^* (p_0 U_{\perp} p_0),$$

which is exactly (5.22). This concludes the proof of Theorem 2.2.

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