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► **To cite this version:**

Diomba Sambou. Quantitative bounds on the discrete spectrum of non self-adjoint quantum magnetic Hamiltonians. 2016. hal-00926769v4

HAL Id: hal-00926769

<https://hal.science/hal-00926769v4>

Preprint submitted on 15 Mar 2016

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QUANTITATIVE BOUNDS ON THE DISCRETE SPECTRUM OF NON SELF-ADJOINT QUANTUM MAGNETIC HAMILTONIANS

DIOMBA SAMBOU

ABSTRACT. We establish Lieb-Thirring type inequalities for non self-adjoint relatively compact perturbations of certain operators of mathematical physics. We apply our results to quantum Hamiltonians of Schrödinger and Pauli with constant magnetic field of strength $b > 0$. In particular, we use these bounds to obtain some information on the distribution of the eigenvalues of the perturbed operators in the neighborhood of their essential spectrum.

1. INTRODUCTION AND AN ABSTRACT RESULT

Recently, a number of results on the spectral properties of the non self-adjoint perturbations of operators of mathematical physics were obtained. We quote the articles by Frank-Laptev-Lieb-Seiringer [5], Borichev-Golinskii-Kupin [1], Demuth-Katriel-Hansmann [3], Hansmann [9], Golinskii-Kupin [8], Pushnitskii-Raikov-Villegas-Blas [18] turned to the study of the discrete spectrum of these perturbations. The purpose of this paper is to announce and to give a brief overview of new results in this direction. The main point is that the first part of this article describes a general construction that applies to a large class of operators containing magnetic Schrödinger, Pauli and Dirac operators of full rank with constant magnetic field, hence generalizing the methods of a recent paper by the author [25].

Let \mathcal{H}_0 be an unbounded self-adjoint operator defined on a dense subset of $L^2(\mathbb{R}^m)$, $m \geq 1$. Suppose that the spectrum $\sigma(\mathcal{H}_0)$ of the operator is given by an infinite sequence of (real) eigenvalues of infinite multiplicity, *i.e.*

$$(1.1) \quad \begin{cases} \sigma(\mathcal{H}_0) = \sigma_{ess}(\mathcal{H}_0) = \bigcup_{j=0}^{\infty} \{\Lambda_j\}, \\ \text{where } \Lambda_0 \geq 0, \Lambda_{j+1} > \Lambda_j, |\Lambda_{j+1} - \Lambda_j| \leq \delta, \delta > 0 \text{ constant.} \end{cases}$$

Concrete examples of operators satisfying these assumptions are Schrödinger operators acting on $L^2(\mathbb{R}^{2d}, \mathbb{C})$, $d \geq 1$, and Pauli operators on $L^2(\mathbb{R}^{2d}, \mathbb{C}^2)$ with constant magnetic field of strength $b > 0$, see Sections 3 and 4, respectively. We can also consider the case of Dirac operators of full rank (see [15]). But, for simplicity, we focus on the two first examples.

On the domain of \mathcal{H}_0 , we consider a (non self-adjoint) relatively compact perturbation V of \mathcal{H}_0 , and the perturbed operator

$$(1.2) \quad \mathcal{H} = \mathcal{H}_0 + V.$$

2010 *Mathematics Subject Classification*. Primary: 35P20; Secondary: 47A75, 47A55.

Key words and phrases. Non self-adjoint relatively compact perturbations of self-adjoint operators, magnetic Schrödinger operators, magnetic Pauli operators, Lieb-Thirring type inequalities.

This means that $\text{dom}(\mathcal{H}_0) \subset \text{dom}(V)$, and $V(\mathcal{H}_0 - \lambda)^{-1}$ is compact for $\lambda \in \rho(\mathcal{H}_0)$, the resolvent set of the operator \mathcal{H}_0 . It is well known (see e.g. [12, Chapter VI]) that under this condition on V , there exists $\mu < 0$ such that

$$(1.3) \quad \sigma(\mathcal{H}) \subset \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq \mu\}.$$

Furthermore, we impose an additional restriction on V allowing us to control the numerical range

$$(1.4) \quad N(\mathcal{H}) := \{\langle \mathcal{H}f, f \rangle : f \in \text{dom}(\mathcal{H}), \|f\|_{L^2(\mathbb{R}^m)} = 1\}$$

of the operator \mathcal{H} . Namely,

$$(1.5) \quad \sigma(\mathcal{H}) \subset \overline{N(\mathcal{H})} \subset \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq \mu_1\}$$

for some $\mu_1 < 0$. For convenience, we put

$$(1.6) \quad \mu_0 = \mu_1 - 1.$$

Recall that a compact operator L defined on a separable Hilbert space belongs to the Schatten-von Neumann class $\mathcal{S}_p, p \geq 1$, if $\|L\|_{\mathcal{S}_p} = (\text{Tr} |L|^p)^{1/p}$ is finite. We also require that

$$V(\mathcal{H}_0 - \lambda)^{-1} \in \mathcal{S}_p,$$

for some $p \geq 1$, which is a stronger condition just saying that the operator $V(\mathcal{H}_0 - \lambda)^{-1}$ is compact.

Since V is a relatively compact perturbation with respect to the self-adjoint operator \mathcal{H}_0 , then the Weyl criterion on the invariance of the essential spectrum implies that $\sigma_{\text{ess}}(\mathcal{H}) = \sigma_{\text{ess}}(\mathcal{H}_0) = \cup_{j=0}^{\infty} \{\Lambda_j\}$. Still, the operator \mathcal{H} can have a (complex) discrete spectrum $\sigma_{\text{disc}}(\mathcal{H})$ accumulating to $\cup_{j=0}^{\infty} \{\Lambda_j\}$, see Gohberg-Goldberg-Kaashoek [6, Theorem 2.1, p. 373], and the coming theorem gives a necessary condition on its distribution. The conclusion of the theorem is written in the form of a relation which is often called a Lieb-Thirring type inequality, see Lieb-Thirring [14] for original work.

Theorem 1.1. *Let \mathcal{H}_0 be a self-adjoint operator with $\sigma(\mathcal{H}_0) = \cup_{j=0}^{\infty} \{\Lambda_j\}$ as above. Consider $\mathcal{H} = \mathcal{H}_0 + V$, and for some $p > 1$ assume that the V satisfies*

$$(1.7) \quad \|V(\mathcal{H}_0 - \mu_0)^{-1}\|_{\mathcal{S}_p}^p \leq K_0,$$

with $K_0 > 0$ constant. Then, we have

$$(1.8) \quad \sum_{\lambda \in \sigma_{\text{disc}}(\mathcal{H})} \frac{\text{dist}(\lambda, \cup_{j=0}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_0 K_0,$$

where $C_0 = C(p, \mu_0, \Lambda_0)$ is a constant depending on p, μ_0 , and Λ_0 .

The proof of this theorem (see Section 2 for more details) is essentially based on a recent theorem of Hansmann [9], and a technical distortion lemma for a conformal mapping coming from complex analysis, see Lemma 2.1.

Applications of this result to magnetic Schrödinger operators on $L^2(\mathbb{R}^{2d}, \mathbb{C})$ and magnetic Pauli operators on $L^2(\mathbb{R}^{2d}, \mathbb{C}^2)$ are given in Theorems 3.1 and

4.1, respectively. In Golinskii-Kupin [8], similar results are obtained for complex perturbations of finite band Schrödinger operators.

Bound (1.8) can be rewritten in a simpler manner for various subsets of $\sigma_{disc}(\mathcal{H})$. For instance, let $\tau > 0$ be fixed. Then, for λ satisfying $|\lambda| \geq \tau$, one has

$$\frac{1}{1+|\lambda|} = \frac{1}{|\lambda|} \frac{1}{1+|\lambda|^{-1}} \geq \frac{1}{|\lambda|} \frac{1}{1+\tau^{-1}},$$

and

$$(1.9) \quad \sum_{\substack{\lambda \in \sigma_{disc}(\mathcal{H}) \\ |\lambda| \geq \tau}} \frac{\text{dist}(\lambda, \cup_{j=0}^{\infty} \{\Lambda_j\})^p}{|\lambda|^{2p}} \leq C_1 \left(1 + \frac{1}{\tau}\right)^{2p} K_0.$$

Furthermore, if $(\lambda_k) \subset \sigma_{disc}(\mathcal{H})$ converges to a point of $\sigma_{ess}(\mathcal{H}) = \cup_{j=0}^{\infty} \{\Lambda_j\}$, one has

$$(1.10) \quad \sum_k \text{dist}(\lambda_k, \cup_{j=0}^{\infty} \{\Lambda_j\})^p < \infty.$$

This means that, *a priori*, the accumulation of the eigenvalues from $\sigma_{disc}(\mathcal{H})$ in a neighborhood of a fixed Λ_j , $j \in \mathbb{N}$, is a monotone function of p .

Similarly, we can also obtain information on diverging sequences of eigenvalues $(\lambda_k) \subset \sigma_{disc}(\mathcal{H})$. For example, if for some $\tau > 0$ the sequence (λ_k) is such that

$$\text{dist}(\lambda_k, \cup_{j=0}^{\infty} \{\Lambda_j\}) \geq \tau,$$

then one has

$$(1.11) \quad \sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{2p}} < \infty.$$

We shall progress as follows. We give the sketch of the proof of our main abstract result (Theorem 1.1) in Section 2. We apply it to magnetic $2d$ -Schrödinger and $2d$ -Pauli operators in Sections 3 and 4 respectively. In Section 5, we treat the case of magnetic $(2d+1)$ -Pauli operators with constant magnetic field. Here, the essential spectrum of the operator under consideration equals \mathbb{R}_+ , which is rather different from the case of the essential spectrum coinciding with the (discrete) set of "Landau levels" $\cup_{j=0}^{\infty} \{\Lambda_j\}$ (1.1). This requires the use of methods close to those from Sambou [25].

We adopt mathematical physics and spectral analysis notation and terminology from Reed-Simon [24]. As for the classes of compact operators (*i.e.* Schatten-von Neumann ideals), we refer the reader to Simon [26] and Gohberg-Goldberg-Krupnik [7]. Constants are generic, *i.e.* changing from one relation to another. For a real x , $[x]$ denotes its integer part.

Acknowledgements. This work is partially supported by the Chilean Program *Núcleo Milenio de Física Matemática RC120002*.

2. THE ABSTRACT RESULT: SKETCH OF THE PROOF

The following result of Hansmann (see [9, Theorem 1]) is the first crucial point of the proof. Let $B_0 = B_0^*$ be a bounded self-adjoint operator acting on

a separable Hilbert space, and B be a bounded operator satisfying $B - B_0 \in \mathcal{S}_p$, $p > 1$. Then, we have

$$(2.1) \quad \sum_{\lambda \in \sigma_{disc}(B)} \text{dist}(\lambda, \sigma(B_0))^p \leq C \|B - B_0\|_{\mathcal{S}_p}^p,$$

where the constant C is explicit and depends only on p . Note that we cannot apply (2.1) to the unbounded operators \mathcal{H}_0 and \mathcal{H} . To fix this, let us consider bounded the resolvents

$$(2.2) \quad B_0(\mu_0) := (\mathcal{H}_0 - \mu_0)^{-1} \quad \text{and} \quad B(\mu_0) := (\mathcal{H} - \mu_0)^{-1},$$

where μ_0 is the constant defined by (1.6). Furthermore,

$$(\mathcal{H} - \mu_0)^{-1} - (\mathcal{H}_0 - \mu_0)^{-1} = -(\mathcal{H} - \mu_0)^{-1} V (\mathcal{H}_0 - \mu_0)^{-1},$$

and we obtain

$$(2.3) \quad \|B(\mu_0) - B_0(\mu_0)\|_{\mathcal{S}_p}^p \leq \|(\mathcal{H} - \mu_0)^{-1}\|^p \|V(\mathcal{H}_0 - \mu_0)^{-1}\|_{\mathcal{S}_p}^p,$$

where $\|\cdot\|$ stays for the usual operator norm. By (1.5), we have

$$\sigma(\mathcal{H}) \subset \overline{N(\mathcal{H})} \subset \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq \mu_1\}.$$

This implies that $\text{dist}(\mu_0, \overline{N(\mathcal{H})}) \geq 1$, and using [2, Lemma 9.3.14] we get

$$(2.4) \quad \|(\mathcal{H} - \mu_0)^{-1}\| \leq \frac{1}{\text{dist}(\mu_0, \overline{N(\mathcal{H})})} \leq 1.$$

Consequently, by (1.7), (2.3) and (2.4), we obtain

$$(2.5) \quad \|B(\mu_0) - B_0(\mu_0)\|_{\mathcal{S}_p}^p \leq K_0,$$

where K_0 is the constant defined in (1.7). Hence, we obtain

$$(2.6) \quad \sum_{z \in \sigma_{disc}(B(\mu_0))} \text{dist}(z, \sigma(B_0(\mu_0)))^p \leq CK_0$$

by applying Hansmann's theorem (2.1) to the resolvents $B(\mu_0)$ and $B_0(\mu_0)$. Putting $z = \varphi_{\mu_0}(\lambda) = (\lambda - \mu_0)^{-1}$, we have

$$(2.7) \quad z \in \sigma_{disc}(B(\mu_0)) \quad \left(z \in \sigma(B_0(\mu_0)) \right) \iff \lambda \in \sigma_{disc}(\mathcal{H}) \quad \left(\lambda \in \sigma(\mathcal{H}_0) \right).$$

So, we come to a distortion lemma for the conformal map $z = \varphi_{\mu_0}(\lambda) = (\lambda - \mu_0)^{-1}$, which is the second important ingredient of the proof of the theorem.

Lemma 2.1. *Let μ_0 be the constant defined by (1.6), and Λ_j , $j \in \mathbb{N}$, be the "Landau levels" defined by (1.1). Then, the following bound holds*

$$(2.8) \quad \text{dist}(\varphi_{\mu_0}(\lambda), \varphi_{\mu_0}(\cup_{j=0}^{\infty} \{\Lambda_j\})) \geq \frac{C \text{dist}(\lambda, \cup_{j=0}^{\infty} \{\Lambda_j\})}{(1 + |\lambda|)^2}, \quad \lambda \in \mathbb{C},$$

where $C = C(\mu_0, \Lambda_0)$ is a constant depending on μ_0 and Λ_0 .

The proof of the lemma goes as [25, Lemma 6.2] and is omitted. Now, combining the above lemma, estimates (2.6) and (2.7), we get

$$\sum_{\lambda \in \sigma_{disc}(\mathcal{H})} \frac{\text{dist}(\lambda, \cup_{j=0}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_0 K_0,$$

where $C_0 = C(p, \mu_0, \Lambda_0)$ is a constant depending on p , μ_0 and Λ_0 . This concludes the proof of Theorem 1.1. \square

3. EXAMPLES: PERTURBATIONS OF MAGNETIC $2d$ -SCHRÖDINGER OPERATORS, $d \geq 1$

Set $X_\perp := (x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d}$, $d \geq 1$, and let $b > 0$ be a constant. We consider

$$(3.1) \quad H_0 := \sum_{j=1}^d \left\{ \left(D_{x_j} + \frac{1}{2} b y_j \right)^2 + \left(D_{y_j} - \frac{1}{2} b x_j \right)^2 \right\},$$

the Schrödinger operator acting on $L^2(\mathbb{R}^{2d}) := L^2(\mathbb{R}^{2d}, \mathbb{C})$ with constant magnetic field of strength $b > 0$. The self-adjoint operator H_0 is originally defined on $C_0^\infty(\mathbb{R}^{2d})$, and then closed in $L^2(\mathbb{R}^{2d})$. It is well known (see e.g. [4]) that its spectrum consists of the increasing sequence of Landau levels

$$(3.2) \quad \Lambda_j = b(d + 2j), \quad j \in \mathbb{N},$$

and the multiplicity of each eigenvalue Λ_j is infinite. As in (1.2), define the perturbed operator

$$(3.3) \quad H = H_0 + V,$$

where we identify the non self-adjoint perturbation V with the multiplication operator by the function $V : \mathbb{R}^{2d} \rightarrow \mathbb{C}$. Most of known results on the discrete spectrum of Schrödinger operators deal with self-adjoint perturbations V , and study its asymptotic behaviour at the edges of its essential spectrum. For V admitting power-like or slower decay at infinity, see for instance the papers [10, Chap. 11-12], [18, 20, 21, 27, 28], and for potentials V decaying at infinity exponentially fast or having a compact support see [23]. For Landau Hamiltonians in exterior domains, see [11, 16, 19].

We shall first consider the class of non self-adjoint electric potentials V satisfying

$$(3.4) \quad (\operatorname{Re}(V)f, f) \geq \mu_1 \|f\|^2$$

for some $\mu_1 < 0$, and the following estimate

$$(3.5) \quad |V(X_\perp)| \leq CF(X_\perp), \quad F \in L^p(\mathbb{R}^{2d}), \quad p \geq 2,$$

where $C > 0$ is a constant and F is a positive function. The definition of the numerical range (1.4) implies that

$$\sigma(H) \subset \overline{N(H)} \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \mu_1\}.$$

Theorem 3.1 is an immediate consequence of the following lemma and Theorem 1.1 with $\mathcal{H}_0 = H_0$, $\mathcal{H} = H$ and $m = 2d$.

Lemma 3.1. [25, Lemma 6.1] *Let Λ_j , $j \in \mathbb{N}$, be the Landau levels defined by (3.2), and consider $\lambda \in \mathbb{C} \setminus \cup_{j=0}^\infty \{\Lambda_j\}$. Assume that $F \in L^p(\mathbb{R}^{2d})$, $d \geq 1$, with $p \geq 2\lceil \frac{d}{2} \rceil + 2$. Then, there exists a constant $C = C(p, b, d)$ such that*

$$(3.6) \quad \|F(H_0 - \lambda)^{-1}\|_{\mathcal{S}_p}^p \leq \frac{C(1 + |\lambda|)^d \|F\|_{L^p}^p}{\operatorname{dist}(\lambda, \cup_{j=0}^\infty \{\Lambda_j\})^p}.$$

Theorem 3.1. *Let $H = H_0 + V$ be the perturbed Schrödinger operator defined by (3.3) with V satisfying conditions (3.4) and (3.5). Assume that $F \in L^p(\mathbb{R}^{2d})$, $d \geq 1$, with $p \geq 2\lceil \frac{d}{2} \rceil + 2$. Then, we have*

$$(3.7) \quad \sum_{\lambda \in \sigma_{disc}(H)} \frac{\text{dist}(\lambda, \cup_{j=0}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_1 \|F\|_{L^p}^p,$$

where the constant $C_1 = C(p, \mu_0, b, d)$ depends on p , $\mu_0 := \mu_1 - 1$, b and d .

Notice that if the electric potential V is bounded, then μ_0 can be eliminated in the constant $C_1 = C(p, \mu_0, b, d)$. The price we pay is the additional factor $(1 + \|V\|_{\infty})^{2p}$ in the RHS of (3.7), see [25, Theorem 2.2].

It goes without saying that we can derive relations similar to (1.9)-(1.11) in the present situation.

It seems appropriate to mention that the assumptions of Theorem 3.1 are typically satisfied by the potentials $V : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ such that

$$(3.8) \quad |V(X_{\perp})| \leq C \langle X_{\perp} \rangle^{-m}, \quad m > 0, \quad pm > 2d, \quad p \geq 2,$$

where $C > 0$ is a constant and $\langle y \rangle := (1 + |y|^2)^{1/2}$, $y \in \mathbb{R}^n$, $n \geq 1$. Certain examples showing the sharpness of our results in the two-dimensional case are discussed in [25, Subsection 2.2].

4. EXAMPLES: PERTURBATIONS OF MAGNETIC $2d$ -PAULI OPERATORS, $d \geq 1$

To simplify, we consider the two-dimensional Pauli operator acting in $L^2(\mathbb{R}^2) := L^2(\mathbb{R}^2, \mathbb{C}^2)$ and describing a quantum non-relativistic $\frac{1}{2}$ -spin-particle subject to a magnetic field of strength b and electric potential V . The general case can be treated in a same manner (see the discussion after Theorem 4.1). The self-adjoint unperturbed Pauli operator H_0 given by

$$(4.1) \quad H_0 := \begin{pmatrix} (-i\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-i\nabla - \mathbf{A})^2 + b \end{pmatrix} =: \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

is defined originally on $C_0^{\infty}(\mathbb{R}^2)$ and then closed in $L^2(\mathbb{R}^2)$. Here, $\mathbf{A} = (A_1, A_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a magnetic potential generating the magnetic field

$$(4.2) \quad b(X_{\perp}) := \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}, \quad X_{\perp} = (x, y) \in \mathbb{R}^2.$$

We focus on the case where $b(X_{\perp}) = b > 0$ is a constant. In this situation, the spectrum $\sigma(H_0)$ of the Pauli operator H_0 (see e.g [4]) is given by

$$(4.3) \quad \sigma(H_0) = \bigcup_{j=0}^{\infty} \{\Lambda_j\}, \quad \Lambda_j = 2bj.$$

Now consider the matrix-valued electric potential

$$(4.4) \quad V(X_{\perp}) := \{v_{\ell k}(X_{\perp})\}_{1 \leq \ell, k \leq 2}, \quad X_{\perp} = (x, y) \in \mathbb{R}^2,$$

and introduce the perturbed operator

$$(4.5) \quad H = H_0 + V,$$

where we identify the potential V with the multiplication operator by the matrix-valued function V . As in the case of magnetic Schrödinger operators,

most of known results on the discrete spectrum of Pauli operators deal with self-adjoint perturbations V (see e.g. [22]). Let us consider the class of non self-adjoint electric potentials V satisfying

$$(4.6) \quad (\operatorname{Re}(V)f, f) \geq \mu_1 \|f\|^2,$$

and the following estimate

$$(4.7) \quad |v_{\ell k}(X_\perp)| \leq CF(X_\perp), \quad 1 \leq \ell, k \leq 2, \quad F \in L^p(\mathbb{R}^{2d}), \quad p \geq 2,$$

where $C > 0$ is a constant and F a positive function. Note that if the potential V is diagonal, *i.e.* $v_{12} = v_{21} = 0$, then assumption (4.6) is satisfied trivially if $\operatorname{Re}(v_{11}) \geq \mu_1$ and $\operatorname{Re}(v_{22}) \geq \mu_1$. In the case where V is non-diagonal with $\operatorname{Re}(v_{\ell k}) \geq \omega_0$ for some $\omega_0 < 0$, it can be verified that assumption (4.6) holds with $\mu_1 = -2|\omega_0|$. Furthermore, we have the following lemma giving a quantitative bound on the norm $\|V(H_0 - \lambda)^{-1}\|_{\mathcal{S}_p}$ in terms of the L^p -norm of F . Its proof goes along the same lines as the proof of [25, Lemma 6.1].

Lemma 4.1. *Let $d = 1$, Λ_j , $j \in \mathbb{N}$, be the Landau levels defined by (4.3), and consider $\lambda \in \mathbb{C} \setminus \cup_{j=0}^\infty \{\Lambda_j\}$. Assume that $F \in L^p(\mathbb{R}^2)$ with $p \geq 2$. Then, there exists a constant $C = C(p, b, d)$ such that*

$$(4.8) \quad \|F(H_0 - \lambda)^{-1}\|_{\mathcal{S}_p}^p \leq \frac{C(1 + |\lambda|)^d \|F\|_{L^p}^p}{\operatorname{dist}(\lambda, \cup_{j=0}^\infty \{\Lambda_j\})^p}.$$

Setting $\mathcal{H}_0 = H_0$, $\mathcal{H} = H$, $m = 2d = 2$ and recalling Theorem 1.1 readily yields the following result.

Theorem 4.1. *Let $H = H_0 + V$ be the perturbed Pauli operator defined by (4.5) with V satisfying (4.6) and (4.7). Assume that $F \in L^p(\mathbb{R}^2)$ with $p \geq 2$. Then, the following bound holds true*

$$(4.9) \quad \sum_{\lambda \in \sigma_{\text{disc}}(H)} \frac{\operatorname{dist}(\lambda, \cup_{j=0}^\infty \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_3 \|F\|_{L^p}^p,$$

where the constant $C_3 = C(p, \mu_0, b, d)$ depends on p , $\mu_0 := \mu_1 - 1$, b and d .

As above, if the electric potential V is bounded, then μ_0 can be eliminated in $C_3 = C(p, \mu_0, b, d)$ with the additional factor $(1 + \|V\|_\infty)^{2p}$ to pay in counterpart in the RHS of (4.9).

Notice that Theorem 4.1 remains valid if we replace the two-dimensional Pauli operator H_0 by the general $2d$ -Pauli operators acting on $L^2(\mathbb{R}^{2d}, \mathbb{C}^2)$, $d \geq 1$, defined by

$$(4.10) \quad H_0 := \begin{pmatrix} \mathbb{H}_{0,\perp} - bd & 0 \\ 0 & \mathbb{H}_{0,\perp} + bd \end{pmatrix} =: \begin{pmatrix} \mathbb{H}_{0,\perp}^- & 0 \\ 0 & \mathbb{H}_{0,\perp}^+ \end{pmatrix}.$$

Here,

$$\mathbb{H}_{0,\perp} := \sum_{j=1}^d \left\{ \left(D_{x_j} + \frac{1}{2}by_j \right)^2 + \left(D_{y_j} - \frac{1}{2}bx_j \right)^2 \right\}$$

is the $2d$ -Schrödinger operator defined by (3.1). In this case, the set of Landau levels is given by $\cup_{j=0}^\infty \{\Lambda_j\}$ with $\Lambda_j = 2bdj$, and we require that $F \in L^p(\mathbb{R}^{2d})$ with $p \geq 2\lceil \frac{d}{2} \rceil + 2$.

Of course, the counterparts of relations (1.9)-(1.11) apply as well to magnetic Pauli operators under consideration.

5. ON LIEB-THIRRING TYPE INEQUALITIES FOR MAGNETIC
(2d + 1)-PAULI OPERATORS, $d \geq 1$

In this section, we focus on $(2d + 1)$ -dimensional self-adjoint Pauli operators with constant magnetic field, acting on $L^2(\mathbb{R}^{2d+1}) := L^2(\mathbb{R}^{2d+1}, \mathbb{C}^2)$, $d \geq 1$, defined by

$$(5.1) \quad \mathbb{P}_0 := \begin{pmatrix} \mathbb{H}_0 - bd & 0 \\ 0 & \mathbb{H}_0 + bd \end{pmatrix} =: \begin{pmatrix} \mathbb{P}_1 & 0 \\ 0 & \mathbb{P}_2 \end{pmatrix}.$$

Here, as usual the constant $b > 0$ is the strength of the magnetic field. And, for the cartesian coordinates $\mathbf{x} := (x_1, y_1, \dots, x_d, y_d, x) \in \mathbb{R}^{2d+1}$,

$$\mathbb{H}_0 := \sum_{j=1}^d \left\{ \left(D_{x_j} + \frac{1}{2}by_j \right)^2 + \left(D_{y_j} - \frac{1}{2}bx_j \right)^2 \right\} + D_x^2, \quad D_\nu := -i\frac{\partial}{\partial \nu},$$

is the $(2d + 1)$ -self-adjoint Schrödinger operator with constant magnetic field originally defined on $C_0^\infty(\mathbb{R}^{2d+1}, \mathbb{C})$. It is well known (see e.g [4]) that the spectrum of the operator \mathbb{P}_0 is absolutely continuous, coincides with $[0, +\infty)$ and has an infinite set of Landau levels

$$(5.2) \quad \Lambda_j = 2bdj, \quad j \in \mathbb{N}.$$

We introduce the perturbed operator on the domain of the operator \mathbb{P}_0

$$(5.3) \quad \mathbb{P} = \mathbb{P}_0 + V,$$

where we identify the perturbation V with the multiplication operator by the matrix-valued function

$$(5.4) \quad V(\mathbf{x}) := \{v_{\ell k}(\mathbf{x})\}_{1 \leq \ell, k \leq 2}.$$

We assume that V is a bounded non self-adjoint perturbation such that for any $\mathbf{x} \in \mathbb{R}^{2d+1}$ and $1 \leq \ell, k \leq 2$,

$$(5.5) \quad |v_{\ell k}(\mathbf{x})| \leq CF(\mathbf{x})G(x),$$

where $C > 0$ is a constant, F and G are two positive functions satisfying $F \in (L^p \cap L^\infty)(\mathbb{R}^{2d+1})$ for $p \geq 2$, and $G \in (L^2 \cap L^\infty)(\mathbb{R})$. Under this assumption on V , we obtain (see Lemma 5.1) that for any $\lambda \in \rho(\mathbb{P}_0)$,

$$(5.6) \quad \|F(\mathbb{P}_0 - \lambda)^{-1}G\|_{\mathcal{S}_p} < \infty.$$

Once again, this implies that V is a relatively compact perturbation.

The first ingredient of the proof is the following lemma obtained by methods similar to [25, Lemma 3.1].

Lemma 5.1. *Let $d \geq 1$ and consider $\lambda \in \mathbb{C} \setminus [0, +\infty)$. Assume that $F \in (L^p \cap L^\infty)(\mathbb{R}^{2d+1})$ with $p \geq 2\lfloor \frac{d}{2} \rfloor + 2$ and $G \in (L^2 \cap L^\infty)(\mathbb{R})$. Then, there exists a constant $C = C(p, b, d)$ such that*

$$(5.7) \quad \|F(\mathbb{P}_0 - \lambda)^{-1}G\|_{\mathcal{S}_p}^p \leq \frac{C(1 + |\lambda|)^{d+\frac{1}{2}}K_1}{\text{dist}(\lambda, [0, +\infty))^{\frac{p}{2}} \text{dist}(\lambda, \cup_{j=0}^\infty \{\Lambda_j\})^{\frac{p}{4}}},$$

where Λ_j , $j \in \mathbb{N}$, are the Landau levels defined by (5.2) and

$$(5.8) \quad K_1 := \|F\|_{L^p}^p (\|G\|_{L^2} + \|G\|_{L^\infty})^p.$$

Note that since the potential V is bounded, then the numerical range of the operator \mathbb{P} satisfies

$$(5.9) \quad \sigma(\mathbb{P}) \subset \overline{N(\mathbb{P})} \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -2\|V\|_\infty \text{ and } |\operatorname{Im} \lambda| \leq 2\|V\|_\infty\}.$$

The Lieb-Thirring type bound for the eigenvalues of the $(2d+1)$ -Pauli operator \mathbb{P} is as follows.

Theorem 5.1. *Let $\mathbb{P} = \mathbb{P}_0 + V$ with V satisfying (5.4) and (5.5). Assume that $F \in (L^p \cap L^\infty)(\mathbb{R}^{2d+1})$ with $p \geq 2[\frac{d}{2}] + 2$, $d \geq 1$, and $G \in (L^2 \cap L^\infty)(\mathbb{R})$. Define*

$$(5.10) \quad K := \|F\|_{L^p}^p (\|G\|_{L^2} + \|G\|_{L^\infty})^p (1 + \|V\|_\infty)^{d + \frac{p}{2} + \frac{3}{2} + \varepsilon},$$

for $0 < \varepsilon < 1$. Then, we have

$$(5.11) \quad \sum_{\lambda \in \sigma_{disc}(H)} \frac{\operatorname{dist}(\lambda, [0, +\infty))^{\frac{p}{2} + 1 + \varepsilon} \operatorname{dist}(\lambda, \cup_{j=0}^\infty \{\Lambda_j\})^{(\frac{p}{4} - 1 + \varepsilon)_+}}{(1 + |\lambda|)^\gamma} \leq C_5 K,$$

where Λ_j , $j \in \mathbb{N}$, are the Landau levels defined by (5.2), $\gamma > d + \frac{3}{2}$, and $C_5 = C(p, b, d, \varepsilon)$ is a constant depending on p , b , d and ε .

As usual, $[x]$ denotes the integer part of $x \in \mathbb{R}$, and $x_+ := \max(x, 0)$.

Sketch of the proof of the theorem. The proof goes along the same lines as the proof of [25, Theorem 2.1] with the help of Lemma 5.1. Since $\sigma_{ess}(\mathbb{P}) = [0, +\infty)$ with an infinite set of thresholds Λ_j , $j \in \mathbb{N}$, we obtain two types of estimates.

First, we bound the sums depending on parts of $\sigma_{disc}(\mathbb{P})$ concentrated around a Landau level Λ_j using the Schwarz-Christoffel formula (see e.g. [13, Theorem 1, p. 176]). Namely, if we consider a rectangle

$$\Pi_j := \{\lambda \in \mathbb{C} : |\Lambda_j - \operatorname{Re} \lambda| \leq b \text{ and } |\operatorname{Im} \lambda| \leq \operatorname{Const.}\}$$

around a Landau level Λ_j , then we have

$$\sum_{\lambda \in \sigma_{disc}(H) \cap \Pi_j} \operatorname{dist}(\lambda, [\Lambda_0, +\infty))^{\frac{p}{2} + 1 + \varepsilon} \operatorname{dist}(\lambda, \cup_{j=1}^\infty \{\Lambda_j\})^{(\frac{p}{4} - 1 + \varepsilon)_+} \leq C(p, b, j, \varepsilon) K,$$

with the asymptotic property $C(p, b, j, \varepsilon) \underset{j \rightarrow \infty}{\sim} j^{d + \frac{1}{2}}$.

Second, we get the global bound summing up the previous bounds with appropriate weights as follows:

$$\begin{aligned} \sum_j \frac{1}{(1+j)^\gamma} \sum_{\lambda \in \sigma_{disc}(H) \cap \Pi_j} \operatorname{dist}(\lambda, [\Lambda_0, +\infty))^{\frac{p}{2} + 1 + \varepsilon} \operatorname{dist}(\lambda, \cup_{j=1}^\infty \{\Lambda_j\})^{(\frac{p}{4} - 1 + \varepsilon)_+} \\ \leq \sum_j \frac{C(p, b, d, j, \varepsilon)}{(1+j)^\gamma} K. \end{aligned}$$

Now, choosing γ such that $\gamma > d + \frac{3}{2}$ and using the fact that for any $\lambda \in \Pi_j$ we have $1 + j \simeq 1 + |\lambda|$, we get the global bound (5.11). \square

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