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QUANTITATIVE BOUNDS ON THE DISCRETE SPECTRUM OF NON SELF-ADJOINT QUANTUM MAGNETIC HAMILTONIANS

DIOMBA SAMBOU

ABSTRACT. We establish Lieb-Thirring type inequalities on non self-adjoint unbounded perturbed operators, by a potential that is relatively-compact. Furthermore, we apply our results to quantum Hamiltonians of Schrödinger and Pauli with constant magnetic field of strength $b > 0$. In particular, we use these bounds to derive information on the distribution of eigenvalues around the essential spectrum of these operators.

1. INTRODUCTION

In this present article, we consider two important cases of unbounded self-adjoint operators acting on a separable Hilbert space X . First, let us introduce some conventional definitions we will use in this paper. We denote by $S_p(X)$, $p \in (0, +\infty)$ the Schatten-von Neuman classes of compact linear operators L on X for which the norm $\|L\|_p := (\text{Tr } |L|^p)^{1/p}$ is finite. For an operator \mathcal{H} acting on X , its numerical range is defined by

$$(1.1) \quad N(\mathcal{H}) := \{ \langle \mathcal{H}f, f \rangle : f \in \text{dom}(\mathcal{H}), \|f\|_X = 1 \}.$$

The *discrete spectrum* of \mathcal{H} denoted by $\sigma_{\text{disc}}(\mathcal{H})$ and its *essential spectrum* denoted by $\sigma_{\text{ess}}(\mathcal{H})$ are defined as in [32].

In the first part of this article, we consider unbounded self-adjoint operators H_0 such that $\sigma_{\text{ess}}(H_0) = \sigma(H_0) = E$ is an infinite discrete set of the form

$$(1.2) \quad E := \bigcup_{j=1}^{\infty} \{ \Lambda_j \}, \quad \Lambda_0 \geq 0, \quad \Lambda_{j+1} > \Lambda_j, \quad |\Lambda_{j+1} - \Lambda_j| \leq \delta = \text{constant} \quad \forall \quad j.$$

Concrete examples of operators satisfying (1.2) are Schrödinger operators acting on $L^2(\mathbb{R}^{2d}, \mathbb{C})$, $d \geq 1$, and Pauli operators on $L^2(\mathbb{R}^{2d}, \mathbb{C}^2)$ with constant magnetic field of strength $b > 0$, defined respectively by (3.1) and (4.9). It is well known (see e.g [8]) that their spectrum consists respectively

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to the Landau levels $b(d + 2j)$ and $2bdj$, $j \in \mathbb{N}$, and each Landau level is an eigenvalue of infinite multiplicity.

On the domain of H_0 , we introduce the perturbed operator

$$(1.3) \quad H = H_0 + V,$$

where V is a relatively non self-adjoint compact perturbation with respect to H_0 . This means that $\text{dom}(V) \supset \text{dom}(H_0)$ and for $\lambda \in \rho(H_0)$ the resolvent set of the operator H_0 , the Birman-Schwinger operator $V(H_0 - \lambda)^{-1}$ is compact. Under this assumption on V , it is well known (see e.g. [17, Chap. VI]) that there exists $\mu_1 < 0$ such that

$$(1.4) \quad \sigma(H) \subset \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq \mu_1\}.$$

In the sequel, we assume the same for its numerical range. That is

$$(1.5) \quad \sigma(H) \subset \overline{N(H)} \subset \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq \mu_1\}.$$

Let us fix the constant

$$(1.6) \quad \mu_0 := \mu_1 - 1.$$

Throughout this study we assume that the weighted resolvent $V(H_0 - \mu_0)^{-1} \in S_p(\mathcal{H})$ for any $p > 0$. This means that

$$(1.7) \quad \|V(H_0 - \mu_0)^{-1}\|_p < \infty.$$

In the second part of this paper, we focus on $(2d + 1)$ -dimensional self-adjoint Pauli operators with constant magnetic field, acting on $L^2(\mathbb{R}^{2d+1}) := L^2(\mathbb{R}^{2d+1}, \mathbb{C}^2)$, $d \geq 1$, defined by

$$(1.8) \quad \mathbb{P}_0 := \begin{pmatrix} \mathbb{H}_0 - bd & 0 \\ 0 & \mathbb{H}_0 + bd \end{pmatrix} =: \begin{pmatrix} \mathbb{P}_1 & 0 \\ 0 & \mathbb{P}_2 \end{pmatrix}.$$

Here the constant $b > 0$ is the strength of the magnetic field and for the cartesian coordinates $\mathbf{x} := (x_1, y_1, \dots, x_d, y_d, x) \in \mathbb{R}^{2d+1}$,

$$\mathbb{H}_0 := \sum_{j=1}^d \left\{ \left(D_{x_j} + \frac{1}{2}by_j \right)^2 + \left(D_{y_j} - \frac{1}{2}bx_j \right)^2 \right\} + D_x^2, \quad D_\nu := -i\frac{\partial}{\partial \nu}$$

is the $(2d + 1)$ -dimensional self-adjoint Schrödinger operator with constant magnetic field originally defined on $C_0^\infty(\mathbb{R}^{2d+1}, \mathbb{C})$. It is well known (see e.g [8]) that the spectrum of the operator \mathbb{P}_0 is absolutely continuous, coincides with $[0, +\infty)$ and has an infinite set of thresholds $2bdj$, $j \in \mathbb{N}$, which are called the Landau levels of \mathbb{P}_0 . We shall denote again this set of thresholds by

$$(1.9) \quad E = \{\Lambda_j\}_{j=1}^\infty, \quad \Lambda_j = 2bdj.$$

On the domain of the operator \mathbb{P}_0 , we introduce the perturbed operator

$$(1.10) \quad \mathbb{P} = \mathbb{P}_0 + V,$$

where we identify the perturbation V with the multiplication operator by the matrix-valued function

$$(1.11) \quad V(\mathbf{x}) := \{v_{\ell k}(\mathbf{x})\}_{1 \leq \ell, k \leq 2}.$$

Here we assume that V is a bounded non self-adjoint perturbation such that for any $\mathbf{x} \in \mathbb{R}^{2d+1}$ and $1 \leq \ell, k \leq 2$,

$$(1.12) \quad |v_{\ell k}(\mathbf{x})| \leq CF(\mathbf{x})G(x),$$

where $C > 0$ is a constant, F and G are two positive functions satisfying $F \in L^p(\mathbb{R}^{2d+1})$ for $p \geq 2$, and $G \in (L^2 \cap L^\infty)(\mathbb{R})$. Under this assumption on V , for any $\lambda \in \rho(\mathbb{P}_0)$ the resolvent set of the operator \mathbb{P}_0 , we obtain (see lemma 6.1) that

$$(1.13) \quad \|F(H_0 - \lambda)^{-1}G\|_p < \infty.$$

Once again, this means that V is a relatively compact perturbation. Note that since the potential V is bounded, then the numerical range of the operator \mathbb{P} satisfies

$$(1.14) \quad \sigma(\mathbb{P}) \subseteq \overline{N(\mathbb{P})} \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq -2\|V\|_\infty \text{ and } |\operatorname{Im} \lambda| \leq 2\|V\|_\infty\}.$$

Now for the transparency of the presentation, let us denote \mathcal{H}_0 for the operators H_0 , \mathbb{P}_0 , and \mathcal{H} for the perturbed operators H , \mathbb{P} defined respectively by (1.3) and (1.10). Since V a relatively compact perturbation with respect to the self-adjoint operator \mathcal{H}_0 , then due to a version of the Weyl's criterion concerning the invariance of the essential spectrum [7, Corollary 2.3.3], we have $\sigma_{\text{ess}}(\mathcal{H}) = \sigma_{\text{ess}}(\mathcal{H}_0)$ so that $\sigma_{\text{ess}}(H) = \bigcup_{j=1}^{\infty} \{\Lambda_j\}$ defined by (1.2), and $\sigma_{\text{ess}}(\mathbb{P}) = [0, +\infty)$. However, the perturbation V may generate some discrete spectrum $\sigma_{\text{disc}}(\mathcal{H})$ of eigenvalues. It can accumulate on $\sigma_{\text{ess}}(\mathcal{H})$ only. The aim of the present note is to investigate $\sigma_{\text{disc}}(\mathcal{H})$ with additional assumptions on the perturbation V . We shall use bounds (1.7) and (1.13) to derive quantitative estimates on the discrete spectrum $\sigma_{\text{disc}}(\mathcal{H})$ of the operator \mathcal{H} . In particular, these bounds give *a priori* information on the distribution of eigenvalues in a vicinity of the Landau levels Λ_j , $j \in \mathbb{N}$. A similar recent work by Demuth, Hansmann and Katriel [6] treats the case of $\sigma_{\text{ess}}(\mathcal{H}) = \sigma_{\text{ess}}(\mathcal{H}_0) = [0, \infty)$. They prove their main abstract result by construct first a holomorphic function whose zeros coincide with the discrete eigenvalues of \mathcal{H} . Further they use complex analysis methods and a theorem of Borichev, Golinskii and Kupin [3] to get information on these zeros. Because of the discrete form of the essential spectrum of the operator H_0 , and the existence of infinite thresholds in that of the operator \mathbb{P}_0 , we use different techniques to prove our results. That is, we reason similarly to [12] and [32], using in particular a recent result by Hansmann [13] and a technical distortion lemma (see Lemma 5.1) which leads us to the application of complex analysis methods. The details of proofs can be found in [32] where the particular case of $2d$ and $(2d + 1)$ - dimensional magnetic Schrödinger operators, $d \geq 1$ with bounded non self-adjoint potential V is study.

We shall progress as follows. In section 2, we formulate our main results (*i.e.* Theorem 2.1 and Theorem 2.2), and we discuss some of their immediate corollaries on the discrete eigenvalues of the operators \mathcal{H} . In section 3 and section 4 we apply respectively Theorem 2.1 to Schrödinger operators on $L^2(\mathbb{R}^{2d}, \mathbb{C})$, $d \geq 1$, and Pauli operators on $L^2(\mathbb{R}^{2d}, \mathbb{C}^2)$ with constant magnetic field of strength $b > 0$. In this way, we shall prove that the assumptions appearing in Theorem 2.2 can be concretely satisfied in some cases. In section 5, we give a sketch of proof of Theorem 2.1, and in section 6 a sketch of proof of Theorem 2.2. Section 7 is devoted to the proof of bounds on the discrete spectrum of $2d$ -dimensional Schrödinger operators. In section 8, we prove same estimates on the discrete spectrum of $2d$ -dimensional Pauli operators.

2. STATEMENT OF MAIN RESULTS

2.1. An abstract result. We establish Lieb-Thirring type estimates on the discrete eigenvalues of the operator H defined by (1.3). A sketch of proof of the following theorem is given in section 3.

Theorem 2.1. *Let H_0 be a self-adjoint operator such that $\sigma(H_0) = \bigcup_{j=1}^{\infty} \{\Lambda_j\}$ as in (1.2) and $H = H_0 + V$. Assume that for some $p > 0$, the perturbation V satisfies*

$$(2.1) \quad \|V(H_0 - \mu_0)^{-1}\|_p^p \leq K_0,$$

with K_0 a constant and μ_0 defined by (1.6), and that $N(H)$ satisfies (1.5). Then

$$(2.2) \quad \sum_{\lambda \in \sigma_{\text{disc}}(H)} \frac{\text{dist}(\lambda, \bigcup_{j=1}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_0 K_0,$$

where $C_0 = C(p, \mu_0, \Lambda_0)$ is a constant depending on p , μ_0 , and Λ_0 .

Using the following estimate

$$(2.3) \quad \frac{1}{1 + |\lambda|} = \frac{1}{|\lambda|} \frac{1}{1 + |\lambda|^{-1}} \geq \frac{1}{|\lambda|} \frac{1}{1 + \tau^{-1}},$$

verified for any $\tau > 0$ with $|\lambda| \geq \tau$, we obtain the following corollary.

Corollary 2.1. *Under the assumptions and the notations are of Theorem 2.1, the following bound holds for any $\tau > 0$,*

$$(2.4) \quad \sum_{\substack{\lambda \in \sigma_{\text{disc}}(H) \\ |\lambda| \geq \tau}} \frac{\text{dist}(\lambda, \bigcup_{j=1}^{\infty} \{\Lambda_j\})^p}{|\lambda|^{2p}} \leq C_1 \left(1 + \frac{1}{\tau}\right)^{2p} K_0.$$

For instance, Theorem 2.1 gives immediate corollaries on sequences (λ_k) of isolated eigenvalues that converge to some $\lambda^* \in \sigma_{\text{ess}}(H) = \bigcup_{j=1}^{\infty} \{\Lambda_j\}$. Indeed, by estimate (2.2) we have

$$(2.5) \quad \sum_k \text{dist}(\lambda_k, \bigcup_{j=1}^{\infty} \{\Lambda_j\})^p < \infty.$$

This means *a priori* that the accumulation of eigenvalues of H near the Λ_j is monotone with p . That is, the eigenvalues are getting less densely distributed in a vicinity of the Λ_j with growing p .

By Theorem 2.1, we can also obtain information about sequences of eigenvalues (λ_k) that diverge. For example if for some $\eta > 0$ the sequence (λ_k) satisfies for any k the lower bound

$$(2.6) \quad \text{dist}(\lambda_k, \bigcup_{j=1}^{\infty} \{\Lambda_j\}) \geq \eta,$$

then estimate (2.4) implies that $\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{2p}} < \infty$. This means that $|\lambda_k|$ must converge to infinity sufficiently fast.

2.2. About the magnetic $(2d+1)$ -dimensional Pauli operators, $d \geq 1$.

We establish Lieb-Thirring type inequalities on the discrete eigenvalues of the $(2d+1)$ -dimensional Pauli operator \mathbb{P} defined by (1.10). We give a sketch of proof of the following theorem in section 6; $[r]$ denotes the integer part of $r \in \mathbb{R}$.

Theorem 2.2. *Let $\mathbb{P} = \mathbb{P}_0 + V$ with V satisfying (1.11) and (1.12). Assume that $F \in L^p(\mathbb{R}^{2d+1})$ with $p \geq 2[\frac{d}{2}] + 2$, $d \geq 1$ and $G \in (L^2 \cap L^\infty)(\mathbb{R})$. Define*

$$(2.7) \quad K := \|F\|_{L^p}^p (\|G\|_{L^2} + \|G\|_{L^\infty})^p (1 + \|V\|_\infty)^{d + \frac{p}{2} + \frac{3}{2} + \varepsilon}.$$

for $0 < \varepsilon < 1$. Then we have

$$(2.8) \quad \sum_{\lambda \in \sigma_{\text{disc}}(\mathbb{P})} \frac{\text{dist}(\lambda, [0, +\infty))^{\frac{p}{2} + 1 + \varepsilon} \text{dist}(\lambda, \bigcup_{j=1}^{\infty} \{\Lambda_j\})^{(\frac{p}{4} - 1 + \varepsilon)_+}}{(1 + |\lambda|)^\gamma} \leq C_5 K,$$

where Λ_j are the Landau levels defined by (1.9), $\gamma > d + \frac{3}{2}$ and $C_5 = C(p, b, d, \varepsilon)$ is a constant depending on p, b, d and ε .

Now since for any $\tau > 0$ estimate (2.3) happens, then we get the following corollary.

Corollary 2.2. *Under the assumptions and the notations of Theorem 2.2, the following bound holds for any $\tau > 0$,*

$$(2.9) \quad \sum_{\substack{\lambda \in \sigma_{\text{disc}}(\mathbb{P}) \\ |\lambda| \geq \tau}} \frac{\text{dist}(\lambda, [0, +\infty))^{\frac{p}{2} + 1 + \varepsilon} \text{dist}(\lambda, \bigcup_{j=1}^{\infty} \{\Lambda_j\})^{(\frac{p}{4} - 1 + \varepsilon)_+}}{|\lambda|^\gamma} \leq C_5 \left(1 + \frac{1}{\tau}\right)^\gamma K.$$

Most of known results on the discrete spectrum of Pauli operators deal with self-adjoint perturbations V and study its asymptotic behaviour near the boundary points of its essential spectrum (see e.g. [15], [28]). In this work, we study non-self-adjoint perturbations and, as in [32, Theorem 2.1], it can be verified that Theorem 2.1 above implies that *a priori* the discrete eigenvalues of the perturbed operator \mathbb{P} are getting less densely distributed near the Landau levels than elsewhere near its essential spectrum.

3. APPLICATION OF THEOREM 2.1 TO THE MAGNETIC $2d$ -SCHRÖDINGER OPERATORS, $d \geq 1$

Let $X_\perp := (x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^{2d}$, $d \geq 1$ and $b > 0$ a constant. We consider

$$(3.1) \quad H_0 := \sum_{j=1}^d \left\{ \left(D_{x_j} + \frac{1}{2} b y_j \right)^2 + \left(D_{y_j} - \frac{1}{2} b x_j \right)^2 \right\},$$

the Schrödinger operators acting on $L^2(\mathbb{R}^{2d}) := L^2(\mathbb{R}^{2d}, \mathbb{C})$ with constant magnetic field of strength b . The self-adjoint operator H_0 is originally defined on $C_0^\infty(\mathbb{R}^{2d})$, and then closed in $L^2(\mathbb{R}^{2d})$. It is well known (see e.g. [8]) that its spectrum consists of the increasing sequence of Landau levels

$$(3.2) \quad \Lambda_j = b(d + 2j), \quad j \in \mathbb{N},$$

and the multiplicity of each eigenvalue Λ_j is infinite. As in (1.3), define the perturbed operator

$$(3.3) \quad H = H_0 + V,$$

where we identify the non self-adjoint perturbation V with the multiplication operator by the function $V : \mathbb{R}^{2d} \rightarrow \mathbb{C}$. As it was reported in [32], most of known results on the discrete spectrum of Schrödinger operators deal with self-adjoint perturbations V and study its asymptotic behaviour near the boundary points of its essential spectrum. For V admitting power-like or slower decay at infinity see [14, chap. 11-12], [23], [25], [26], [34], [35], and for potentials V decaying at infinity exponentially fast or having a compact support see [29]. For Landau Hamiltonians in exterior domains see [16], [21] and [24].

According to condition (1.5), we first consider the class of non self-adjoint electric potentials V satisfying

$$(3.4) \quad \langle \operatorname{Re}(V)f, f \rangle \geq \mu_1 \|f\|^2$$

and the following estimate

$$(3.5) \quad |V(X_\perp)| \leq CF(X_\perp), \quad F \in L^p(\mathbb{R}^{2d}), \quad p \geq 2,$$

where $C > 0$ is a constant and F a positive function. Under these assumptions on V , we show that condition (1.7) according to (2.1) is in fact naturally satisfied for any $\lambda \in \rho(H_0)$. Namely, Lemma 7.1 says that the p -Schatten

norm of the weighted resolvent $F(H_0 - \lambda)^{-1}$ satisfies some bound depending on the L^p -norm of the function F . Hence we come to the following theorem; a sketch of its proof is given in section 7.

Theorem 3.1. *Let $H = H_0 + V$ be the perturbed Schrödinger operator defined by (3.3) with V satisfying conditions (3.4) and (3.5). Assume that $F \in L^p(\mathbb{R}^{2d})$, $d \geq 1$ with $p \geq 2[\frac{d}{2}] + 2$. Then the following holds*

$$(3.6) \quad \sum_{\lambda \in \sigma_{\text{disc}}(H)} \frac{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_1 \|F\|_{L^p}^p,$$

where Λ_j are the Landau levels defined by (3.2), and $C_1 = C(p, \mu_0, b, d)$ is a constant depending on p , $\mu_0 := \mu_1 - 1$, b and d .

Since for any $\tau > 0$ with $|\lambda| \geq \tau$ estimate (2.3) holds, then we get the following corollary.

Corollary 3.1. *Under the assumptions and the notations of Theorem 3.1, we have the following bound for any $\tau > 0$*

$$(3.7) \quad \sum_{\substack{\lambda \in \sigma_{\text{disc}}(H) \\ |\lambda| \geq \tau}} \frac{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{|\lambda|^{2p}} \leq C_1 \left(1 + \frac{1}{\tau}\right)^{2p} \|F\|_{L^p}^p.$$

By Theorem 3.1, *a priori* the isolated eigenvalues of H are distributed less and less densely in a vicinity of the Landau levels with growing p .

Let us now consider the class of bounded non self-adjoint electric potentials V . In this case, it can be easily checked that condition (3.4), then condition (1.5) are satisfied with $\mu_1 = -\|V\|_{\infty}$. More precisely we have

$$(3.8) \quad \sigma(H) \subset \overline{N(H)} \subset \{\lambda \in \mathbb{C} : \text{Re} \lambda \geq -\|V\|_{\infty} \text{ and } |\text{Im} \lambda| \leq \|V\|_{\infty}\}.$$

So as in Theorem 3.1, we get the following result.

Theorem 3.2. [32, Theorem 2.2] *Let $H = H_0 + V$ be the perturbed Schrödinger operator defined by (3.3) with V bounded and satisfying condition (3.5). Assume that $F \in L^p(\mathbb{R}^{2d})$, $d \geq 1$ with $p \geq 2[\frac{d}{2}] + 2$. Then the following holds*

$$(3.9) \quad \sum_{\lambda \in \sigma_{\text{disc}}(H)} \frac{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_2 \|F\|_{L^p}^p (1 + \|V\|_{\infty})^{2p},$$

where Λ_j are the Landau levels defined by (3.2), and $C_2 = C(p, b, d)$ is a constant depending on p , b and d .

In the spirit of Corollary 3.1, we get the following inequality.

Corollary 3.2. [32, Corollary 2.2] *Under the assumptions and the notations of Theorem 3.2, we have the following bound for any $\tau > 0$*

$$(3.10) \quad \sum_{\substack{\lambda \in \sigma_{\text{disc}}(H) \\ |\lambda| \geq \tau}} \frac{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{|\lambda|^{2p}} \leq C_2 \left(1 + \frac{1}{\tau}\right)^{2p} \|F\|_{L^p}^p (1 + \|V\|_{\infty})^{2p}.$$

It is convenient to mention that the assumptions of theorem 3.1 are typically satisfied by the class of potentials $V : \mathbb{R}^{2d} \rightarrow \mathbb{C}$ such that

$$(3.11) \quad |V(X_{\perp})| \leq C \langle X_{\perp} \rangle^{-m}, \quad m > 0, \quad pm > 2d, \quad p \geq 2,$$

where $C > 0$ is a constant and $\langle y \rangle := (1 + |y|^2)^{1/2}$, $y \in \mathbb{R}^n$, $n \geq 1$. Some examples showing the sharpness of our results in the two-dimensional case are discussed In [32, subsection 2.2].

4. APPLICATION OF THEOREM 2.1 TO THE MAGNETIC $2d$ -DIMENSIONAL PAULI OPERATORS, $d \geq 1$

To simplify, we consider the two-dimensional Pauli operator $H = H_0 + V$ acting in $L^2(\mathbb{R}^2) := L^2(\mathbb{R}^2, \mathbb{C}^2)$ and describing a quantum non-relativistic spin- $\frac{1}{2}$ particle subject to a magnetic field of strength b and a electric potential V . The general case can be treated in a same manner (see Remark 4.1 below). The self-adjoint unperturbed Pauli operator H_0 given by

$$(4.1) \quad H_0 := \begin{pmatrix} (-i\nabla - \mathbf{A})^2 - b & 0 \\ 0 & (-i\nabla - \mathbf{A})^2 + b \end{pmatrix} =: \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

is defined originally on $C_0^\infty(\mathbb{R}^2)$ and then closed in $L^2(\mathbb{R}^2)$. Here $\mathbf{A} = (A_1, A_2) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the magnetic potential, and

$$(4.2) \quad b(X_{\perp}) := \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}, \quad X_{\perp} = (x, y) \in \mathbb{R}^2$$

is the magnetic field. We focus on the case where $b > 0$ is a constant. In this situation, the spectrum $\sigma(H_0)$ of the Pauli operator H_0 (see e.g [8]) is given by

$$(4.3) \quad \sigma(H_0) = \bigcup_{j=1}^{\infty} \{\Lambda_j\}, \quad \Lambda_j = 2bj.$$

Now consider the matrix-valued electric potential

$$(4.4) \quad V(X_{\perp}) := \{v_{\ell k}(X_{\perp})\}_{1 \leq \ell, k \leq 2}, \quad X_{\perp} = (x, y) \in \mathbb{R}^2,$$

and introduce the perturbed operator

$$(4.5) \quad H = H_0 + V,$$

where we identify the potential V with the multiplication operator by the matrix-valued function V . As in the case of Schrödingers operators, most

of known results on the discrete spectrum of Pauli operators deal with self-adjoint perturbations V (see e.g. [27]). According to condition (1.5) and as in (3.4), we first consider the class of non self-adjoint electric potentials V satisfying

$$(4.6) \quad \langle \operatorname{Re}(V)f, f \rangle \geq \mu_1 \|f\|^2,$$

and the following estimate

$$(4.7) \quad |v_{\ell k}(\mathbf{x})| \leq CF(\mathbf{x}), \quad 1 \leq \ell, k \leq 2, \quad F \in L^p(\mathbb{R}^{2d}), \quad p \geq 2,$$

where $C > 0$ is a constant and F a positive function. Note that if the potential V is diagonal, *i.e.* $v_{12} = v_{21} = 0$, then assumption (4.6) is satisfied trivially if $\inf(\operatorname{Re}(v_{11}), \operatorname{Re}(v_{22})) \geq \mu_1$. In the case where V is non-diagonal, if $\inf_{\ell, k}(\operatorname{Re}(v_{\ell k})) \geq \omega_0$ for some $\omega_0 < 0$, then it can be checked that assumption (4.6) holds with $\mu_1 = 2\omega_0$. Under these assumptions on the potential V , relation (1.7) according to (2.1) is in fact naturally satisfied for any $\lambda \in \rho(H_0)$ (see Lemma 8.1). More precisely, we obtain that the p -Schatten norm of the weighted resolvent $F(H_0 - \lambda)^{-1}$ is bounded by the L^p -norm of the function F up to a constant depending on λ . The sketch of the proof of the following theorem will be given in section 8.

Theorem 4.1. *Let $H = H_0 + V$ be the perturbed Pauli operator defined by (4.5) with V satisfying conditions (4.6) and (4.7). Assume that $F \in L^p(\mathbb{R}^2)$ with $p \geq 2$. Then the following holds*

$$(4.8) \quad \sum_{\lambda \in \sigma_{\text{disc}}(H)} \frac{\operatorname{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_3 \|F\|_{L^p}^p,$$

where Λ_j are the Landau levels defined by (4.3), and $C_3 = C(p, \mu_0, b, d)$ is a constant depending on p , $\mu_0 := \mu_1 - 1$, b and d .

Remark 4.1. *Theorem 4.1 remains valid if we replace the two-dimensional Pauli operator H_0 by the general 2d-Pauli operators acting on $L^2(\mathbb{R}^{2d}, \mathbb{C}^2)$, $d \geq 1$, defined by*

$$(4.9) \quad H_0 := \begin{pmatrix} \mathbb{H}_{0,\perp} - bd & 0 \\ 0 & \mathbb{H}_{0,\perp} + bd \end{pmatrix} =: \begin{pmatrix} \mathbb{H}_{0,\perp}^- & 0 \\ 0 & \mathbb{H}_{0,\perp}^+ \end{pmatrix},$$

where

$$\mathbb{H}_{0,\perp} := \sum_{j=1}^d \left\{ \left(D_{x_j} + \frac{1}{2}by_j \right)^2 + \left(D_{y_j} - \frac{1}{2}bx_j \right)^2 \right\}$$

is the 2d-Schrödinger operators defined by (3.1). In this case the condition on p for the function $F \in L^p(\mathbb{R}^{2d})$ is $p \geq 2\left[\frac{d}{2}\right] + 2$ and the Landau levels are given by $2bdj$, $j \in \mathbb{N}$.

As above, we have

Corollary 4.1. *Under the assumptions and the notations of Theorem 4.1, we have the following bound for any $\tau > 0$*

$$(4.10) \quad \sum_{\substack{\lambda \in \sigma_{\text{disc}}(H) \\ |\lambda| \geq \tau}} \frac{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{|\lambda|^{2p}} \leq C_3 \left(1 + \frac{1}{\tau}\right)^{2p} \|F\|_{L^p}^p.$$

According to (2.5), by Theorem 4.1, *a priori* the isolated eigenvalues of H are less densely distributed in a vicinity of the Landau levels with growing values of p .

Let us now consider the class of bounded non self-adjoint electric potentials V . In this case condition (4.6), then condition (1.5) are satisfied with $\mu_1 = -2\|V\|_{\infty}$. More exactly we have

$$(4.11) \quad \sigma(H) \subset \overline{N(H)} \subset \{\lambda \in \mathbb{C} : \text{Re } \lambda \geq -2\|V\|_{\infty} \text{ and } |\text{Im } \lambda| \leq 2\|V\|_{\infty}\}.$$

So we get the following theorem.

Theorem 4.2. *Let $H = H_0 + V$ be the perturbed Pauli operator defined by (4.5) with V bounded and satisfying (4.7). Assume that $F \in L^p(\mathbb{R}^2)$ with $p \geq 2$. Then the following holds*

$$(4.12) \quad \sum_{\lambda \in \sigma_{\text{disc}}(H)} \frac{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_4 \|F\|_{L^p}^p (1 + \|V\|_{\infty})^{2p},$$

where Λ_j are the Landau levels defined by (4.3), and $C_4 = C(p, b, d)$ is a constant depending on p , b and d .

Remark 4.2. Remark 4.1 remains valid for Theorem 4.2.

Using the lower bound (2.3), we get the following corollary.

Corollary 4.2. *Under the assumptions and the notations of Theorem 4.2, we have the following bound for any $\tau > 0$*

$$(4.13) \quad \sum_{\substack{\lambda \in \sigma_{\text{disc}}(H) \\ |\lambda| \geq \tau}} \frac{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{|\lambda|^{2p}} \leq C_4 \left(1 + \frac{1}{\tau}\right)^{2p} \|F\|_{L^p}^p (1 + \|V\|_{\infty})^{2p}.$$

5. SKETCH OF PROOF OF THE ABSTRACT RESULT

Constants are generic, *i.e.* changing from a relation to another. The following result of Hansmann [13, theorem 1] is the first crucial tool of the proof. Let $B_0 = B_0^*$ be a bounded self-adjoint operator acting on a Hilbert space, B a bounded operator satisfying $B - B_0 \in S_p$, $p > 1$. Then

$$(5.1) \quad \sum_{\lambda \in \sigma_d(B)} \text{dist}(\lambda, \sigma(B_0))^p \leq C \|B - B_0\|_p^p,$$

where the constant C is explicit and depends only on p . Note that we can not apply (5.1) to the unbounded operators H_0 and H . To fix this, let us consider the bounded resolvents

$$(5.2) \quad B_0(\mu_0) := (H_0 - \mu_0)^{-1} \quad \text{and} \quad B(\mu_0) := (H - \mu_0)^{-1},$$

where μ_0 is the constant defined by (1.6). Furthermore,

$$(H - \mu_0)^{-1} - (H_0 - \mu_0)^{-1} = -(H - \mu_0)^{-1}V(H_0 - \mu_0)^{-1},$$

we obtain

$$(5.3) \quad \|B(\mu_0) - B_0(\mu_0)\|_p^p \leq \|(H - \mu_0)^{-1}\|^p \|V(H_0 - \mu_0)^{-1}\|_p^p,$$

Since (1.5) implies that $\text{dist}(\mu_0, \overline{N(H)}) \geq 1$, then using [5, Lemma 9.3.14] we get

$$(5.4) \quad \|(H - \mu_0)^{-1}\| \leq \frac{1}{\text{dist}(\mu_0, \overline{N(H)})} \leq 1.$$

So by assumption (2.1) together with (5.3) and (5.4), we obtain finally

$$(5.5) \quad \|B(\mu_0) - B_0(\mu_0)\|_p^p \leq K_0,$$

where K_0 is the constant defined in (2.1). Thus we see

$$(5.6) \quad \sum_{z \in \sigma_d(B(\mu_0))} \text{dist}(z, \sigma(B_0(\mu_0)))^p \leq CK_0$$

by applying (5.1) to the resolvents $B(\mu_0)$ and $B_0(\mu_0)$. Putting $z = \varphi_{\mu_0}(\lambda) = (\lambda - \mu_0)^{-1}$, we have by the Spectral Mapping theorem

$$(5.7) \quad z \in \sigma_d(B(\mu_0)) \quad \left(z \in \sigma(B_0(\mu_0)) \right) \iff \lambda \in \sigma_d(H) \quad \left(\lambda \in \sigma(H_0) \right).$$

Hence we come to a distortion lemma for the conformal map $z = \varphi_{\mu_0}(\lambda) = (\lambda - \mu_0)^{-1}$, which is the second important tool of the proof of the theorem.

Lemma 5.1. *Let μ_0 be the constant defined by (1.6) and Λ_j the Landau levels defined by (1.2). Then the following bound holds*

$$(5.8) \quad \text{dist}(\varphi_{\mu_0}(\lambda), \varphi_{\mu_0}(\cup_{j=1}^{\infty} \{\Lambda_j\})) \geq \frac{C \text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})}{(1 + |\lambda|)^2}, \quad \lambda \in \mathbb{C},$$

where $C = C(\mu_0, \Lambda_0)$ is a constant depending on μ_0 and Λ_0 .

Proof. Follows as in the proof of Lemma 6.2 of [32]. ■

Now let us turn back to the proof of Theorem 2.1. Combining Lemma 5.1, estimate (5.6) and the identity (5.7), we get

$$\sum_{\lambda \in \sigma_{\text{disc}}(H)} \frac{\text{dist}(\lambda, \cup_{j=1}^{\infty} \{\Lambda_j\})^p}{(1 + |\lambda|)^{2p}} \leq C_0 \|F\|_{L^p}^p,$$

where $C_0 = C(p, \mu_0, \Lambda_0)$ is a constant depending on p , μ_0 and Λ_0 . This concludes the proof of Theorem 2.1.

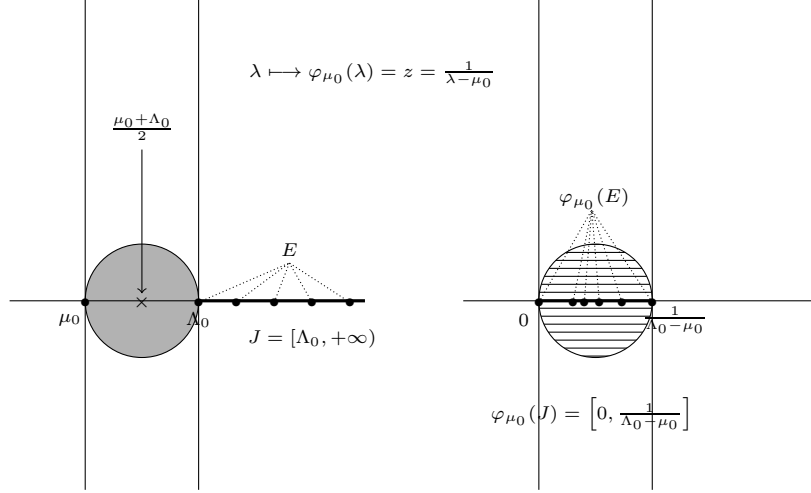


FIGURE 5.1. Image of the complex plane by the conformal map φ_{μ_0} : $\varphi_{\mu_0}(\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < \mu_0\}) = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$; $\varphi_{\mu_0}(\text{the gray disk}) = \{z \in \mathbb{C} : \operatorname{Re} z \geq \frac{1}{\Lambda_0 - \mu_0}\}$; $\varphi_{\mu_0}(\{\mu_0 \leq \operatorname{Re} \lambda \leq \Lambda_0\}) = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\} \cap \text{outside the disk hatched}$; $\varphi_{\mu_0}(\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq \Lambda_0\}) = \text{the disk hatched}$.

6. SKETCH OF PROOF OF THEOREM 2.2

Let \mathbb{P}_0 be the Pauli operator defined by (1.8) acting on $L^2(\mathbb{R}^{2d+1}) = L^2(\mathbb{R}^{2d}) \otimes L^2(\mathbb{R})$, $d \geq 1$. In this section we give a sketch of proof of Theorem 2.2 assuming that the matrix-valued electric potential V satisfies assumptions (1.11) and (1.12).

6.1. Estimate of the weighted resolvents of \mathbb{P}_0 and \mathbb{P} . Recall that the spectrum of the Pauli operator \mathbb{P}_0 consists of Landau levels Λ_j defined by (1.9), $j \in \mathbb{N}$. Let \mathbb{P}_ℓ , $\ell = 1, 2$ be the Schrödinger operators defined in (1.8). Obviously for any λ in the resolvent set of \mathbb{P}_0 we have

$$(6.1) \quad (\mathbb{P}_0 - \lambda)^{-1} = \begin{pmatrix} (\mathbb{P}_1 - \lambda)^{-1} & 0 \\ 0 & (\mathbb{P}_2 - \lambda)^{-1} \end{pmatrix}.$$

Denote by p_j the orthogonal projection onto $\ker(\mathbb{H}_{0,\perp}^- - \Lambda_j)$, where \mathbb{H}_0^\mp are the Schrödinger operators defined by (4.9). Then we have

$$(6.2) \quad \begin{aligned} (\mathbb{P}_1 - \lambda)^{-1} &= \sum_{j \in \mathbb{N}} p_j \otimes (D_x^2 + \Lambda_j - \lambda)^{-1}, \\ (\mathbb{P}_2 - \lambda)^{-1} &= \sum_{j \in \mathbb{N}^*} p_{j-1} \otimes (D_x^2 + \Lambda_j - \lambda)^{-1}. \end{aligned}$$

Consequently we obtain the following lemma which can be proved as in [32, Lemma 3.1].

Lemma 6.1. *Let $d \geq 1$ and $\lambda \in \mathbb{C} \setminus [0, +\infty)$. Assume that $F \in L^p(\mathbb{R}^{2d+1})$ with $p \geq 2\lceil \frac{d}{2} \rceil + 2$ and $G \in (L^2 \cap L^\infty)(\mathbb{R})$. Then there exists a constant $C = C(p, b, d)$ such that*

$$(6.3) \quad \|F(\mathbb{P}_0 - \lambda)^{-1}G\|_p^p \leq \frac{C(1 + |\lambda|)^{d+\frac{1}{2}}K_1}{\text{dist}(\lambda, [0, +\infty))^{\frac{p}{2}} \text{dist}(\lambda, \cup_{j=1}^\infty \{\Lambda_j\})^{\frac{p}{4}}},$$

where Λ_j are the Landau levels defined by (1.9) and

$$(6.4) \quad K_1 := \|F\|_{L^p}^p (\|G\|_{L^2} + \|G\|_{L^\infty})^p.$$

Now let λ_0 be such that

$$(6.5) \quad \min \left(|\text{Im} \lambda_0|, \text{dist}(\lambda_0, \overline{N(\mathbb{P})}) \right) \geq 1 + \|V\|_\infty,$$

where \mathbb{P} is the perturbed operator defined by (1.10). We have the following lemma which can be proved as in [32, Lemma 3.2].

Lemma 6.2. *Assume that λ_0 satisfies condition (6.5). Under the assumptions of Lemma 6.1, there exists a constant $C = C(p)$ such that*

$$(6.6) \quad \|F(\mathbb{P} - \lambda_0)^{-1}G\|_p^p \leq C(1 + |\lambda_0|)^{d+\frac{1}{2}}K_2,$$

where the constant K_2 is defined by

$$(6.7) \quad K_2 := \|F\|_{L^p}^p \|G\|_{L^\infty}^p.$$

6.2. Proof of Theorem 2.2. Follows as in the proof of [32, Theorem 2.1] with the help of Lemma 6.1 and Lemma 6.2. Since $\sigma_{\text{ess}}(\mathbb{P}) = [0, +\infty)$ with an infinite set of thresholds Λ_j , we shall obtain two types of estimates. First, we bound the sums depending on parts of $\sigma_{\text{disc}}(\mathbb{P})$ concentrated around a Landau level Λ_j using the Schwarz-Christoffel formula (see e.g. [18, Theorem 1, p. 176]), and second, we get global estimates summing up the previous bounds with appropriate weights.

7. ON THE MAGNETIC $2d$ -DIMENSIONAL SCHRÖDINGER OPERATORS, $d \geq 1$

Let H_0 be the Schrödinger operator defined by (3.1) acting on $L^2(\mathbb{R}^{2d}, \mathbb{C})$, $d \geq 1$. In this section we shall give a sketch of proof of Theorem 3.1. We assume that the perturbation V satisfies conditions (3.4) and (3.5).

7.1. Estimate of the weighted resolvent. We recall that the spectrum of the Schrödinger operator H_0 consists of Landau levels Λ_j defined by (3.2), $j \in \mathbb{N}$. Thus for any λ in the resolvent set of H_0 , we have

$$(7.1) \quad (H_0 - \lambda)^{-1} = \sum_{j \in \mathbb{N}} (\Lambda_j - \lambda)^{-1} p_j,$$

where p_j is the orthogonal projection onto $\ker(H_0 - \Lambda_j)$.

Lemma 7.1. [32, Lemma 6.1] *Let Λ_j be the Landau levels defined by (3.2) and $\lambda \in \mathbb{C} \setminus \bigcup_{j=1}^{\infty} \{\Lambda_j\}$. Assume that $F \in L^p(\mathbb{R}^{2d})$, $d \geq 1$ with $p \geq 2[\frac{d}{2}] + 2$. Then there exists a constant $C = C(p, b, d)$ such that*

$$(7.2) \quad \|F(H_0 - \lambda)^{-1}\|_p^p \leq \frac{C(1 + |\lambda|)^d \|F\|_{L^p}^p}{\text{dist}(\lambda, \bigcup_{j=1}^{\infty} \{\Lambda_j\})^p}.$$

7.2. Proof of Theorem 3.1. It suffices to show that (2.1) happens with $K_0 = C_1 \|F\|_{L^p}^p$, where C_1 is the constant defined in (3.6). According to assumption (3.4) on V , the choice of μ_0 defined by (1.6) implies that

$$\text{dist}(\mu_0, E) \geq 1.$$

Otherwise, if V satisfies assumption (3.5), then there exists a bounded operator \mathcal{V} such that for any $X_{\perp} \in \mathbb{R}^{2d}$, $V(X_{\perp}) = \mathcal{V}F(X_{\perp})$. Thus with the help of Lemma 7.1, we conclude in particular that

$$\|V(H_0 - \mu_0)^{-1}\|_p^p \leq C(p, \mu_0, b, d) \|F\|_{L^p}^p.$$

Now, Theorem 3.1 follows immediately using Theorem 2.1.

8. ON THE MAGNETIC $2d$ -DIMENSIONAL PAULI OPERATORS, $d \geq 1$

Let H_0 be the Pauli operator defined by (4.1) acting on $L^2(\mathbb{R}^2, \mathbb{C}^2)$. In this section we give a sketch of proof of Theorem 4.1 and Theorem 4.2. We assume that the matrix-valued electric potential V satisfies assumptions (4.4) and (4.7).

8.1. Estimate of the weighted resolvent. Recall that the spectrum of the Pauli operator H_0 consists of Landau levels Λ_j defined by (4.3), $j \in \mathbb{N}$. Let H_{ℓ} , $\ell = 1, 2$ be the Schrödinger operators defined in (4.1). Obviously for any λ in the resolvent set of H_0 we have

$$(8.1) \quad (H_0 - \lambda)^{-1} = \begin{pmatrix} (H_1 - \lambda)^{-1} & 0 \\ 0 & (H_2 - \lambda)^{-1} \end{pmatrix}.$$

Denote by p_j the orthogonal projection onto $\ker(H_1 - \Lambda_j)$. Then

$$(8.2) \quad (H_1 - \lambda)^{-1} = \sum_{j \in \mathbb{N}} (\Lambda_j - \lambda)^{-1} p_j \quad \text{and} \quad (H_2 - \lambda)^{-1} = \sum_{j \in \mathbb{N}^*} (\Lambda_j - \lambda)^{-1} p_{j-1}$$

So we obtain the following lemma which is the analogue of Lemma 7.1 in the case of Pauli operator with $d = 1$.

Lemma 8.1. *Let Λ_j be the Landau levels defined by (4.3) and $\lambda \in \mathbb{C} \setminus \bigcup_{j=1}^{\infty} \{\Lambda_j\}$. Assume that $F \in L^p(\mathbb{R}^2)$ with $p \geq 2$. Then there exists a constant $C = C(p, b, d)$ such that*

$$(8.3) \quad \|F(H_0 - \lambda)^{-1}\|_p^p \leq \frac{C(1 + |\lambda|)^d \|F\|_{L^p}^p}{\text{dist}(\lambda, \bigcup_{j=1}^{\infty} \{\Lambda_j\})^p}.$$

8.2. Proof of Theorem 4.1. Follows as in the proof of Theorem 3.1.

8.3. Proof of Theorem 4.2. Follows as in the proof of Theorem 2.2 of [32] with the help of Lemma 8.1.

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