

Stable Recovery with Analysis Decomposable Priors

Jalal M. Fadili

GREYC

CNRS-ENSICAEN-Univ. Caen
Caen, France

Gabriel Peyré and Samuel Vaïter

CEREMADE

CNRS-Univ. Paris Dauphine
Paris, France

Charles-Alban Deledalle

IMB

CNRS-Univ. Bordeaux 1
Bordeaux, France

Joseph Salmon

LTCI

CNRS-Télécom ParisTech
Paris, France

Abstract—In this paper, we investigate in a unified way the structural properties of solutions to inverse problems regularized by the generic class of semi-norms defined as a decomposable norm composed with a linear operator, the so-called analysis decomposable prior. This encompasses several well-known analysis-type regularizations such as the discrete total variation, analysis group-Lasso or the nuclear norm. Our main results establish sufficient conditions under which uniqueness and stability to a bounded noise of the regularized solution are guaranteed.

I. INTRODUCTION

Problem statement Suppose we observe

$$y = \Phi x_0 + w, \quad \text{where} \quad \|w\| \leq \varepsilon,$$

where Φ is a linear operator from \mathbb{R}^N to \mathbb{R}^M that may have a non-trivial kernel. We want to robustly recover an approximation of x_0 by solving the optimization problem

$$x^* \in \underset{x \in \mathbb{R}^N}{\text{Argmin}} \frac{1}{2} \|y - \Phi x\|^2 + \lambda R(x), \quad \text{where} \quad R(x) := \|L^* x\|_{\mathcal{A}}, \quad (1)$$

with $L : \mathbb{R}^P \rightarrow \mathbb{R}^N$ a linear operator, and $\|\cdot\|_{\mathcal{A}} : \mathbb{R}^P \rightarrow \mathbb{R}^+$ is a decomposable norm in the sense of [1]. Decomposable regularizers are intended to promote solutions conforming to some notion of simplicity/low complexity that complies with that of $L^* x_0$. This motivates the following definition of these norms.

Definition 1. A norm $\|\cdot\|_{\mathcal{A}}$ is decomposable at $\beta \in \mathbb{R}^P$ if there is a subspace $T \subseteq \mathbb{R}^P$ and a vector $e \in T$ such that

$$\partial \|\cdot\|_{\mathcal{A}}(\beta) = \left\{ u \in \mathbb{R}^P : \mathcal{P}_T(u) = e \quad \text{and} \quad \|\mathcal{P}_{T^\perp}(u)\|_{\mathcal{A}}^* \leq 1 \right\}$$

and for any $z \in T^\perp$, $\|z\|_{\mathcal{A}} = \sup_{v \in T^\perp, \|v\|_{\mathcal{A}}^* \leq 1} \langle v, z \rangle$, where $\|\cdot\|_{\mathcal{A}}^*$ is the dual norm of $\|\cdot\|_{\mathcal{A}}$, \mathcal{P}_T (resp. \mathcal{P}_{T^\perp}) is the orthogonal projector on T (resp. on its orthogonal complement T^\perp).

Popular examples covered by decomposable regularizers are the ℓ_1 -norm, the ℓ_1 - ℓ_2 group sparsity norm, and the nuclear norm.

Contributions and relation to prior work In this paper, we give sufficient conditions under which (1) admits a unique minimizer. Then we develop results guaranteeing that a stable approximation of x_0 can be obtained from the noisy measurements y by solving (1), with an ℓ_2 -error that comes within a factor of the noise level ε . This goes beyond [1] which considered identifiability in the noiseless case, with $L = \text{Id}$ and Φ a Gaussian matrix. ℓ_2 -stability is also studied in [4] for $L = \text{Id}$ under stronger sufficient assumptions than ours. Our results generalize the stability guarantee of [3] established when the decomposable norm is ℓ_1 and L is a frame. A general stability result for sublinear R is given in [2]. The stability is however measured in terms of R , and ℓ_2 -stability can only be obtained if R is coercive, i.e., L^* is injective.

II. UNIQUENESS

We first note that traditional coercivity and convexity arguments allow to show that the set of (global) minimizers of (1) is a non-empty compact set if and only if $\ker(\Phi) \cap \ker(L^*) = \{0\}$.

We shall now give a sufficient condition under which problem (1) admits exactly one minimizer. The following assumptions will play a pivotal role in our analysis throughout the paper.

Assumption (SC_x) There exist $\eta \in \mathbb{R}^M$ and $\alpha \in \partial \|\cdot\|_{\mathcal{A}}(L^* x)$ such that the following so-called source condition is verified:

$$\Phi^* \eta = L \alpha \in \partial R(x).$$

Assumption (INJ_T) Let T be the subspace in Definition 1 associated to $L^* x$. Φ is injective on $\ker(\mathcal{P}_{T^\perp} L^*)$.

It is immediate to see that since $\ker(L^*) \subseteq \ker(\mathcal{P}_{T^\perp} L^*)$, (INJ_T) implies that the set of minimizers is indeed non-empty and compact.

Theorem 1. For a minimizer x^* of (1), let T_* and e_* be the subspace and vector in Definition 1 associated to $L^* x^*$. Assume that (SC_{x*}) is verified with $\|\mathcal{P}_{T_*^\perp}(\alpha)\|_{\mathcal{A}}^* < 1$, and that (INJ_{T*}) holds. Then, x^* is the unique minimizer of (1).

III. STABILITY TO NOISE

We are now ready to state our main stability result.

Theorem 2. Let T and e be the subspace and vector in Definition 1 associated to $L^* x_0$. Assume that (SC_{x₀}) is verified with $\|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^* < 1$, and that (INJ_T) holds. Then, for $\lambda = c\varepsilon$

$$\|x^* - x_0\| \leq C\varepsilon,$$

where $C = C_1(2 + c\|\eta\|) + C_2 \frac{(1+c\|\eta\|/2)^2}{c(1-\|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^*)}$, and $C_1 > 0$ and $C_2 < 0$ are constants independent of η and α .

In the following corollary, we provide a stronger sufficient stability condition. It will allow to construct good dual vectors η and α that are computable, which in turn yield explicit constants in the bound. For this, suppose that (INJ_T) is verified, and define $\text{IC}(T, e) = \min_{u \in \ker(L\mathcal{P}_{T^\perp})} \|\Gamma^{[T^\perp]} e + \mathcal{P}_{T^\perp} u\|_{\mathcal{A}}^*$ with $\Gamma^{[T^\perp]} = (L\mathcal{P}_{T^\perp})^+ (\Phi^* \Phi A^{[T^\perp]} - \text{Id}) L \mathcal{P}_T$ and $A^{[T^\perp]} = U (U^* \Phi^* \Phi U)^{-1} U^*$, and U is a matrix whose columns form a basis of $\ker(\mathcal{P}_{T^\perp} L^*)$. Note that $\text{IC}(T, e)$ can be computed by solving a convex program. It also specializes to the criterion developed in [5] for the case of the ℓ_1 analysis prior.

Corollary 1. Assume that $\text{IC}(T, e) < 1$. Then, taking $\eta = \Phi A^{[T^\perp]} L \mathcal{P}_T e$, there exists α such that (SC_{x₀}) is satisfied. Moreover, the bound of Theorem 2 holds true substituting $1 - \text{IC}(T, e)$ for $1 - \|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^*$.

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