



**HAL**  
open science

## Stable Recovery with Analysis Decomposable Priors

Jalal M. Fadili, Gabriel Peyré, Samuel Vaiter, Charles-Alban Deledalle,  
Joseph Salmon

► **To cite this version:**

Jalal M. Fadili, Gabriel Peyré, Samuel Vaiter, Charles-Alban Deledalle, Joseph Salmon. Stable Recovery with Analysis Decomposable Priors. SPARS 2013, Jul 2013, Lausanne, Switzerland. 1 pp. hal-00926727

**HAL Id: hal-00926727**

**<https://hal.science/hal-00926727v1>**

Submitted on 10 Jan 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Stable Recovery with Analysis Decomposable Priors

Jalal M. Fadili  
GREYC

CNRS-ENSICAEN-Univ. Caen  
Caen, France

Gabriel Peyré and Samuel Vaïter  
CEREMADE

CNRS-Univ. Paris Dauphine  
Paris, France

Charles-Alban Deledalle  
IMB

CNRS-Univ. Bordeaux 1  
Bordeaux, France

Joseph Salmon  
LTCI

CNRS-Télécom ParisTech  
Paris, France

**Abstract**—In this paper, we investigate in a unified way the structural properties of solutions to inverse problems regularized by the generic class of semi-norms defined as a decomposable norm composed with a linear operator, the so-called analysis decomposable prior. This encompasses several well-known analysis-type regularizations such as the discrete total variation, analysis group-Lasso or the nuclear norm. Our main results establish sufficient conditions under which uniqueness and stability to a bounded noise of the regularized solution are guaranteed.

## I. INTRODUCTION

**Problem statement** Suppose we observe

$$y = \Phi x_0 + w, \quad \text{where } \|w\| \leq \varepsilon,$$

where  $\Phi$  is a linear operator from  $\mathbb{R}^N$  to  $\mathbb{R}^M$  that may have a non-trivial kernel. We want to robustly recover an approximation of  $x_0$  by solving the optimization problem

$$x^* \in \underset{x \in \mathbb{R}^N}{\text{Argmin}} \frac{1}{2} \|y - \Phi x\|^2 + \lambda R(x), \quad \text{where } R(x) := \|L^* x\|_{\mathcal{A}}, \quad (1)$$

with  $L : \mathbb{R}^P \rightarrow \mathbb{R}^N$  a linear operator, and  $\|\cdot\|_{\mathcal{A}} : \mathbb{R}^P \rightarrow \mathbb{R}^+$  is a decomposable norm in the sense of [1]. Decomposable regularizers are intended to promote solutions conforming to some notion of simplicity/low complexity that complies with that of  $L^* x_0$ . This motivates the following definition of these norms.

**Definition 1.** A norm  $\|\cdot\|_{\mathcal{A}}$  is decomposable at  $\beta \in \mathbb{R}^P$  if there is a subspace  $T \subseteq \mathbb{R}^P$  and a vector  $e \in T$  such that

$$\partial \|\cdot\|_{\mathcal{A}}(\beta) = \left\{ u \in \mathbb{R}^P : \mathcal{P}_T(u) = e \quad \text{and} \quad \|\mathcal{P}_{T^\perp}(u)\|_{\mathcal{A}}^* \leq 1 \right\}$$

and for any  $z \in T^\perp$ ,  $\|z\|_{\mathcal{A}} = \sup_{v \in T^\perp, \|v\|_{\mathcal{A}}^* \leq 1} \langle v, z \rangle$ , where  $\|\cdot\|_{\mathcal{A}}^*$  is the dual norm of  $\|\cdot\|_{\mathcal{A}}$ ,  $\mathcal{P}_T$  (resp.  $\mathcal{P}_{T^\perp}$ ) is the orthogonal projector on  $T$  (resp. on its orthogonal complement  $T^\perp$ ).

Popular examples covered by decomposable regularizers are the  $\ell_1$ -norm, the  $\ell_1$ - $\ell_2$  group sparsity norm, and the nuclear norm.

**Contributions and relation to prior work** In this paper, we give sufficient conditions under which (1) admits a unique minimizer. Then we develop results guaranteeing that a stable approximation of  $x_0$  can be obtained from the noisy measurements  $y$  by solving (1), with an  $\ell_2$ -error that comes within a factor of the noise level  $\varepsilon$ . This goes beyond [1] which considered identifiability in the noiseless case, with  $L = \text{Id}$  and  $\Phi$  a Gaussian matrix.  $\ell_2$ -stability is also studied in [4] for  $L = \text{Id}$  under stronger sufficient assumptions than ours. Our results generalize the stability guarantee of [3] established when the decomposable norm is  $\ell_1$  and  $L$  is a frame. A general stability result for sublinear  $R$  is given in [2]. The stability is however measured in terms of  $R$ , and  $\ell_2$ -stability can only be obtained if  $R$  is coercive, i.e.,  $L^*$  is injective.

## II. UNIQUENESS

We first note that traditional coercivity and convexity arguments allow to show that the set of (global) minimizers of (1) is a non-empty compact set if and only if  $\ker(\Phi) \cap \ker(L^*) = \{0\}$ .

We shall now give a sufficient condition under which problem (1) admits exactly one minimizer. The following assumptions will play a pivotal role in our analysis throughout the paper.

**Assumption (SC $_x$ )** There exist  $\eta \in \mathbb{R}^M$  and  $\alpha \in \partial \|\cdot\|_{\mathcal{A}}(L^* x)$  such that the following so-called source condition is verified:

$$\Phi^* \eta = L \alpha \in \partial R(x).$$

**Assumption (INJ $_T$ )** Let  $T$  be the subspace in Definition 1 associated to  $L^* x$ .  $\Phi$  is injective on  $\ker(\mathcal{P}_{T^\perp} L^*)$ .

It is immediate to see that since  $\ker(L^*) \subseteq \ker(\mathcal{P}_{T^\perp} L^*)$ , (INJ $_T$ ) implies that the set of minimizers is indeed non-empty and compact.

**Theorem 1.** For a minimizer  $x^*$  of (1), let  $T_*$  and  $e_*$  be the subspace and vector in Definition 1 associated to  $L^* x^*$ . Assume that (SC $_{x^*}$ ) is verified with  $\|\mathcal{P}_{T_*^\perp}(\alpha)\|_{\mathcal{A}}^* < 1$ , and that (INJ $_{T_*}$ ) holds. Then,  $x^*$  is the unique minimizer of (1).

## III. STABILITY TO NOISE

We are now ready to state our main stability result.

**Theorem 2.** Let  $T$  and  $e$  be the subspace and vector in Definition 1 associated to  $L^* x_0$ . Assume that (SC $_{x_0}$ ) is verified with  $\|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^* < 1$ , and that (INJ $_T$ ) holds. Then, for  $\lambda = c\varepsilon$

$$\|x^* - x_0\| \leq C\varepsilon,$$

where  $C = C_1(2 + c\|\eta\|) + C_2 \frac{(1+c\|\eta\|/2)^2}{c(1-\|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^*)}$ , and  $C_1 > 0$  and  $C_2 < 0$  are constants independent of  $\eta$  and  $\alpha$ .

In the following corollary, we provide a stronger sufficient stability condition. It will allow to construct good dual vectors  $\eta$  and  $\alpha$  that are computable, which in turn yield explicit constants in the bound. For this, suppose that (INJ $_T$ ) is verified, and define  $\text{IC}(T, e) = \min_{u \in \ker(L\mathcal{P}_{T^\perp})} \|\Gamma^{[T^\perp]} e + \mathcal{P}_{T^\perp} u\|_{\mathcal{A}}$  with  $\Gamma^{[T^\perp]} = (L\mathcal{P}_{T^\perp})^+ (\Phi^* \Phi A^{[T^\perp]} - \text{Id}) L\mathcal{P}_T$  and  $A^{[T^\perp]} = U(U^* \Phi^* \Phi U)^{-1} U^*$ , and  $U$  is a matrix whose columns form a basis of  $\ker(\mathcal{P}_{T^\perp} L^*)$ . Note that  $\text{IC}(T, e)$  can be computed by solving a convex program. It also specializes to the criterion developed in [5] for the case of the  $\ell_1$  analysis prior.

**Corollary 1.** Assume that  $\text{IC}(T, e) < 1$ . Then, taking  $\eta = \Phi A^{[T^\perp]} L\mathcal{P}_T e$ , there exists  $\alpha$  such that (SC $_{x_0}$ ) is satisfied. Moreover, the bound of Theorem 2 holds true substituting  $1 - \text{IC}(T, e)$  for  $1 - \|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^*$ .

## REFERENCES

- [1] E. J. Candès and B. Recht. Simple bounds for recovering low-complexity models. *Mathematical Programming*, pages 1–13, 2012.
- [2] M. Grasmair. Linear convergence rates for Tikhonov regularization with positively homogeneous functionals. *Inverse Problems*, 27:075014, 2011.
- [3] M. Haltmeier. Stable signal reconstruction via  $\ell^1$ -minimization in redundant, non-tight frames. *IEEE Trans. on Sig. Proc.*, 2012. to appear.
- [4] S. Negabban, P. Ravikumar, M. J. Wainwright, and B. Yu. A unified framework for high-dimensional analysis of m-estimators with decomposable regularizers. *Statistical Science*, 27(4):538–557, December 2012.
- [5] S. Vaïter, G. Peyré, C. Dossal, and M.J. Fadili. Robust sparse analysis regularization. *To appear in IEEE Trans. Inf. Theo.*, 2011.