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Philippe Lebacque, Alexander Schmidt

► **To cite this version:**

Philippe Lebacque, Alexander Schmidt. The  $K(\pi,1)$ -property for marked curves over finite fields. 2013. hal-00926253v1

**HAL Id: hal-00926253**

**<https://hal.science/hal-00926253v1>**

Preprint submitted on 9 Jan 2014 (v1), last revised 23 Mar 2015 (v2)

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# THE $K(\pi, 1)$ -PROPERTY FOR MARKED CURVES OVER FINITE FIELDS

PHILIPPE LEBACQUE AND ALEXANDER SCHMIDT

ABSTRACT. We investigate the  $K(\pi, 1)$ -property for  $p$  of smooth, marked curves  $(X, T)$  defined over finite fields of characteristic  $p$ . We prove that  $(X, T)$  has the  $K(\pi, 1)$ -property if  $X$  is affine and give positive and negative examples in the proper case. We also consider the unmarked proper case over a finite field of characteristic different to  $p$ .

## 1. INTRODUCTION

In [Sch1],[Sch2],[Sch3], the second author investigated the  $K(\pi, 1)$ -property for  $p$  of arithmetic curves whose function field is of characteristic different to  $p$ . As a result, the Galois group of the maximal unramified outside  $S$  and  $T$ -split pro- $p$ -extension of a global field of characteristic different to  $p$  is often of cohomological dimension less or equal to two. In this paper we consider the case of a smooth curve over a finite field of characteristic  $p$ . We prove that  $(X, T)$  has the  $K(\pi, 1)$ -property if  $X$  is affine and give positive and negative examples in the proper case. We also consider the unmarked proper case over a finite field of characteristic different to  $p$ , which was left out in the earlier papers.

**1.1. The marked étale site and the  $K(\pi, 1)$ -property.** Let  $X$  be a one-dimensional noetherian regular scheme defined over  $\mathbb{F}_q$  (with  $q = p^f$ ) and let  $T$  be a finite set of closed points. In [Sch3], the second author defined the marked site  $(X, T)$  of  $X$  at  $T$  considering finite étale morphisms  $Y \rightarrow X$  inducing isomorphisms  $k(y) \cong k(x)$  on the residue fields for any closed point  $y \in Y$  mapping to  $x \in T$ . Let  $M$  be a  $p$ -torsion sheaf. The resulting cohomology groups are denoted by  $H^i(X, T, M)$  and they satisfy the usual properties we expect from étale cohomology groups. He also proved (see [Sch3] for more details) that these finite marked étale morphisms fit into a Galois theory and (after choosing a base geometric point  $\bar{x} \notin T$ ) we denote by  $\pi_1(X, T)$  the profinite group classifying étale coverings of  $X$  in which the points of  $T$  split completely. We denote by  $(\widetilde{X, T})(p)$  the universal pro- $p$ -covering of  $(X, T)$ . The projection  $(\widetilde{X, T})(p) \rightarrow X$  is Galois with Galois group the maximal pro- $p$ -quotient  $\pi_1(X, T)(p)$  of  $\pi_1(X, T)$ .

Let  $M$  be a discrete  $p$ -torsion  $\pi_1(X, T)(p)$ -module. Consider the Hochschild-Serre spectral sequence:

$$E_2^{ij} = H^i(\pi_1(X, T)(p), H^j((\widetilde{X, T})(p), T, M)) \Rightarrow H^{i+j}(X, T, M).$$

The edge morphisms provide homomorphisms

$$\phi_{i, M} : H^i(\pi_1(X, T)(p), M) \rightarrow H^i(X, T, M).$$

We say that  $(X, T)$  has the  $K(\pi, 1)$ -property for  $p$  if  $\phi_{i, M}$  is an isomorphism for all  $M$  and all  $i \geq 0$ . The following Lemma 1.1 implies in particular, that  $(X, T)$  has the  $K(\pi, 1)$ -property for  $p$  if  $\phi_{i, \mathbb{F}_p}$  is an isomorphism for  $i \geq 2$ .

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*Date:* January 9, 2014.

*2010 Mathematics Subject Classification.* 11R34, 11R37.

*Key words and phrases.* Galois cohomology, étale cohomology, restricted ramification.

**Lemma 1.1.** (cf. [Sch3] Lemma 2.2)  $\phi_{i,M}$  is an isomorphism for  $i = 0, 1$  and is a monomorphism for  $i = 2$ . Moreover,  $\phi_{i,M}$  is an isomorphism for all  $i \geq 0$  if and only if

$$\varinjlim_{(Y, T')} H^i(Y, T', M) = 0 \text{ for all } i \geq 1,$$

where the direct limit is taken over all finite intermediate coverings  $(Y, T')$  of  $(\widetilde{X, T})(p) \rightarrow (X, T)$ .

**1.2. Notation.** Unless otherwise stated, we use the following notation:

- $p$  denotes a prime number.
- $\mathbb{F}$  is a finite field,  $\overline{\mathbb{F}}$  an algebraic closure of  $\mathbb{F}$ ,  $\widetilde{\mathbb{F}}$  its maximal pro- $p$ -extension inside  $\overline{\mathbb{F}}$  and  $G_{\mathbb{F}}$  the Galois group of  $\overline{\mathbb{F}}/\mathbb{F}$ .
- $X$  is a smooth projective absolutely irreducible curve defined over  $\mathbb{F}$ .
- $k = \mathbb{F}(X)$  the function field of  $X$ .
- $g_X$  the genus of  $X$ .
- $\overline{X} = X \times_{\mathbb{F}} \overline{\mathbb{F}}$ ,  $\widetilde{X} = X \times_{\mathbb{F}} \widetilde{\mathbb{F}}$ .
- $S, T$  are two disjoint sets (possibly empty) of closed points of  $X$ .
- if  $x$  is a closed point of a  $X$ ,  $X_x$  denotes the henselization of  $X$  at  $x$  and  $T_x = \{x\}$  if  $x \in T$  and  $\emptyset$  otherwise.
- $k_S^T$  denotes the maximal pro- $p$ -extension of  $k$  which is unramified outside  $S$  and in which all places of  $T$  split completely. If empty, we omit  $S$  (or  $T$ ) from the notation.
- $G_S^T(k) = \text{Gal}(k_S^T/k) = \pi_1(X - S, T)(p)$ .
- $H^i(X - S, T)$  denotes the  $i$ -th étale cohomology group  $H_{\text{ét}}^i(X - S, T, \mathbb{F}_p)$  of the marked curve  $(X - S, T)$ .
- for a pro- $p$ -group  $G$  we set  $H^i(G) = H^i(G, \mathbb{F}_p)$ .
- for an abelian group  $A$  and an integer  $m$  we write  $A[m] = \ker(A \xrightarrow{m} A)$

**1.3. New results.** Let  $X$  be a smooth projective absolutely irreducible curve defined over the finite field  $\mathbb{F}$  and let  $k = \mathbb{F}(X)$  be the function field of  $X$ . Let  $S$  and  $T$  be finite disjoint sets of closed points of  $X$ . In this paper, we prove the following result:

**Theorem 1.2.** Assume that  $p = \text{char}(\mathbb{F})$ .

- (i) If  $S \neq \emptyset$ , then  $(X - S, T)$  has the  $K(\pi, 1)$ -property for  $p$  and  $\text{cd } G_S^T(k) = 1$ .
- (ii) If  $T = \emptyset$ , then  $(X - S)$  has the  $K(\pi, 1)$ -property for  $p$  and  $\text{cd } G_S(k) \leq 2$ .

In the remaining cases, we have the following results.

**Theorem 1.3.** Assume that  $p = \text{char}(\mathbb{F})$ ,  $S = \emptyset$  and  $T \neq \emptyset$ .

- (i) If  $\text{Pic}(X)[p] = 0$ , then  $(X, T)$  has the  $K(\pi, 1)$ -property for  $p$  if and only if  $T = \{x\}$  consists of a single point with  $p \nmid \deg x$ . In this case  $\pi_1(X, T)(p) = 1$ .
- (ii) If  $\text{Pic}(X)[p] \neq 0$  and

$$\sum_{x \in T} \frac{\deg(x)}{(\#\mathbb{F})^{\deg(x)/2} - 1} > g_X - 1,$$

then  $\pi_1(X, T)(p)$  is finite and  $(X, T)$  has not the  $K(\pi, 1)$ -property for  $p$ .

Finally, we consider the unmarked proper case over a finite field of characteristic different to  $p$ , which was left out in the earlier papers.

**Theorem 1.4.** Assume that  $p \neq \text{char}(\mathbb{F})$ . Then  $X$  has the  $K(\pi, 1)$ -property for  $p$  if and only if  $\mu_p(\mathbb{F}) = 0$  or  $\text{Pic}(X)[p] \neq 0$ .

In the remaining case  $\mu_p \subset \mathbb{F}$  and  $\text{Pic}(X)[p] = 0$  we have

$$\pi_1^{\text{ét}}(X)(p) \cong \pi_1^{\text{ét}}(\mathbb{F})(p) \cong \mathbb{Z}_p.$$

In particular,  $H^i(\pi_1^{\text{ét}}(X)(p))$  is always finite and vanishes for  $i > 3$ .

## 2. COMPUTATION OF ÉTALE COHOMOLOGY GROUPS

**Proposition 2.1** (Local computation). *Let  $K$  be a nonarchimedean local (or henselian) field of characteristic  $p$ . Let  $Y = \text{Spec } \mathcal{O}_K$ ,  $y \in Y$  the closed point and let  $T$  be  $\emptyset$  or  $\{y\}$ . Then the local cohomology groups  $H_y^i(Y, T)$  vanish for  $i \neq 2$  and*

$$H_y^2(Y, T) = \begin{cases} H_{/nr}^1(K) & \text{if } T = \emptyset \\ H^1(K) & \text{if } T = \{y\}, \end{cases}$$

where  $H_{/nr}^1 = H^1(K)/H_{nr}^1(K)$ .

*Proof:* We use the excision sequence:

$$\cdots \rightarrow H_y^i(Y, T) \rightarrow H^i(Y, T) \rightarrow H^i(Y - \{y\}) \rightarrow H_y^{i+1}(Y, T) \rightarrow \cdots$$

Since  $Y$  is henselian,  $H^i(Y) \cong H^i(y) = H_{nr}^i(K)$ , hence  $H^i(Y) = 0$  for  $i \geq 2$ . Since  $Y$  is normal,  $H^1(Y, T) \rightarrow H^1(Y - \{y\})$  is injective, hence  $H_y^1(Y, T) = 0$ . Furthermore,  $H^i(Y - \{y\}) = H^i(K)$  and this group vanishes for  $i \geq 2$  since  $cd_p K = 1$  (see [NSW], Cor. 6.1.3). It follows that  $H_y^i(Y, T) \cong H^i(Y, T)$  for  $i \geq 3$ .

For  $T = \emptyset$  we obtain  $H_y^i(Y) = 0$  for  $i \geq 3$  and the short exact sequence

$$0 \rightarrow H^1(Y) \rightarrow H^1(Y - \{y\}) \rightarrow H_y^2(Y) \rightarrow 0$$

implies the result for  $H_y^2(Y)$ .

If  $T = \{y\}$ , the identity of  $(Y, T)$  is cofinal among the covering families of  $(Y, T)$ , hence  $H^i(Y, \{y\}) = 0$  for  $i \geq 1$ . We obtain  $H_y^2(Y, \{y\}) \cong H^1(K)$  and  $H_y^i(Y, \{y\}) = 0$  for  $i \geq 3$ .  $\square$

**Proposition 2.2.** (Global computation) *Let  $X$  be a smooth projective and geometrically irreducible curve over  $\mathbb{F}$ ,  $k = \mathbb{F}(X)$  and  $S$  and  $T$  finite, disjoint sets of closed points of  $X$ .*

*Then  $H^i(X - S, T) = 0$  for  $i \geq 3$  and  $H^2(X - S, T) = 0$  if  $S \neq \emptyset$ . We have an exact sequence*

$$0 \rightarrow H^1(X - S, T) \rightarrow H^1(X - S) \rightarrow \bigoplus_{x \in T} H_{nr}^1(k_x) \rightarrow H^2(X - S, T) \rightarrow H^2(X - S) \rightarrow 0.$$

*Proof:* In the case  $T = \emptyset$  we have  $H^i(X - S) = 0$  for  $i \geq 3$  and  $H^2(X - S) = 0$  if  $S \neq \emptyset$  by [SGA4.3] exp. 10, Thm. 5.1 and Cor. 5.2. Moreover, the sequence is exact for trivial reasons.

Now assume  $T \neq \emptyset$ . Consider the excision sequence for  $(X - S, T)$  and  $(X - (S \cup T))$ :

$$\cdots \rightarrow \bigoplus_{x \in T} H_x^i((X - S)_x, T_x) \rightarrow H^i(X - S, T) \rightarrow H^i(X - (S \cup T)) \rightarrow \cdots$$

Proposition 2.1 shows that  $H^i(X - S, T) \cong H^i(X - (S \cup T)) = 0$  for  $i \geq 3$  and the exactness of the sequence

$$0 \rightarrow H^1(X - S, T) \rightarrow H^1(X - (S \cup T)) \rightarrow \bigoplus_{x \in T} H^1(k_x) \rightarrow H^2(X - S, T) \rightarrow 0. \quad (*)$$

Comparing this with the excision sequence for  $(X - S)$  and  $(X - (S \cup T))$

$$0 \rightarrow H^1(X - S) \rightarrow H^1(X - (S \cup T)) \rightarrow \bigoplus_{x \in T} H_{nr}^1(k_x) \rightarrow H^2(X - S) \rightarrow 0,$$

we obtain the exact sequence of the proposition.

If  $S \neq \emptyset$ , the Strong Approximation Theorem implies that

$$H^1(X - (S \cup T)) \longrightarrow \bigoplus_{x \in T} H^1(k_x)$$

is surjective (see [NSW] Thm. 9.2.5). Using (\*) this shows that  $H^2(X - S, T) = 0$  in this case.  $\square$

**Corollary 2.3.** *If  $G_S^T(k)$  is finite and nontrivial, then  $(X - S, T)$  does not have the  $K(\pi, 1)$ -property for  $p$ .*

*Proof.* In this case we have  $cd G_S^T(k) = \infty$  but  $H^i(X - S, T) = 0$  for  $i \geq 3$ .  $\square$

**Corollary 2.4.** *We have the Euler-Poincaré characteristic formula*

$$\sum_{i=0}^2 (-1)^i \dim_{\mathbb{F}_p} H^i(X, T) = \#T.$$

*Proof.* If  $S = \emptyset$ , all groups in the exact sequence of Proposition 2.2 are finite and we obtain

$$\sum_{i=0}^2 (-1)^i \dim_{\mathbb{F}_p} H^i(X, T) = \#T + \sum_{i=0}^2 (-1)^i \dim_{\mathbb{F}_p} H^i(X).$$

Recall that  $H^1(\bar{X}) = \text{Hom}(\text{Pic}(\bar{X})[p], \mathbb{F}_p)$  (every connected étale covering of  $\bar{X}$  comes by base change from an isogeny of the Jacobian of  $\bar{X}$ ). Hence

$$H^2(X) = H^1(\mathbb{F}, H^1(\bar{X})) = H^1(\mathbb{F}, \text{Hom}(\text{Pic}(\bar{X})[p], \mathbb{F}_p)) = \text{Hom}(\text{Pic}(\bar{X})[p], \mathbb{F}_p)_{G_{\mathbb{F}}}.$$

Furthermore, we have an exact sequence

$$0 \rightarrow H^1(\mathbb{F}) \rightarrow H^1(X) \rightarrow \text{Hom}(\text{Pic}(\bar{X})[p], \mathbb{F}_p)^{G_{\mathbb{F}}} \rightarrow 0.$$

Thus Lemma 2.5 below shows  $\sum_{i=0}^2 (-1)^i \dim_{\mathbb{F}_p} H^i(X) = 1 - \dim_{\mathbb{F}_p} H^1(\mathbb{F}) = 0$ .  $\square$

**Lemma 2.5.** *We have  $\text{Pic}(\bar{X})[p]^{G_{\mathbb{F}}} = \text{Pic}(X)[p]$  and*

$$\dim_{\mathbb{F}_p} \text{Pic}(\bar{X})[p]_{G_{\mathbb{F}}} = \dim_{\mathbb{F}_p} \text{Pic}(X)[p].$$

*Proof.* The first equality follows from the Leray spectral sequence

$$E_2^{ij} = H^i(\mathbb{F}, H^j(\bar{X}, \mathbb{G}_m)) \Rightarrow H^{i+j}(X, \mathbb{G}_m)$$

and the vanishing of the Brauer group of a finite field:

$$H^2(\mathbb{F}, H^0(\bar{X}, \mathbb{G}_m)) = H^2(\mathbb{F}, \bar{\mathbb{F}}^\times) = 0.$$

The equality of dimensions follows from the exact sequence of finite-dimensional  $\mathbb{F}_p$ -vector spaces

$$0 \rightarrow \text{Pic}(\bar{X})[p]^{G_{\mathbb{F}}} \rightarrow \text{Pic}(\bar{X})[p] \xrightarrow{1 - \text{Frob}} \text{Pic}(\bar{X})[p] \rightarrow \text{Pic}(\bar{X})[p]_{G_{\mathbb{F}}} \rightarrow 0.$$

$\square$

### 3. PROOF OF THEOREM 1.2

Assume  $S \neq \emptyset$ . From the computations in the last section, we know that  $H^i(X - S, T) = 0$  for  $i \geq 2$ . By Lemma 1.1,  $(X - S, T)$  has the  $K(\pi, 1)$ -property for  $p$  and  $cd G_S^T(k) \leq 1$ . But  $G_S^T(k)$  is nontrivial, which follows from the exact sequence

$$0 \rightarrow H^1(X - S, T) \rightarrow H^1(X - S) \rightarrow \bigoplus_{x \in T} H_{nr}^1(k_x) \rightarrow 0$$

together with the fact that  $\bigoplus_{x \in T} H_{nr}^1(k_x)$  has finite  $\mathbb{F}_p$ -dimension whereas  $H^1(X - S)$  is infinite dimensional.

Now assume that  $S = \emptyset$  and  $T = \emptyset$ . Let  $\tilde{\mathbb{F}}$  be the maximal  $p$ -extension of  $\mathbb{F}$  in  $\bar{\mathbb{F}}$ . Then  $H^2(X_{\tilde{\mathbb{F}}}) = H^2(X_{\tilde{\mathbb{F}}})^{\text{Gal}(\bar{\mathbb{F}}/\tilde{\mathbb{F}})} = 0$ . Hence  $X_{\tilde{\mathbb{F}}}$  is a  $K(\pi, 1)$  for  $p$  and the Hochschild-Serre spectral sequence for  $X_{\tilde{\mathbb{F}}}/X$  shows the same for  $X$ . This finishes the proof of Theorem 1.2.

#### 4. PROOF OF THEOREM 1.3

**Proposition 4.1.** *Assume that  $\text{Pic}(X)[p] = 0$  and  $T \neq \emptyset$  and let  $p^r$  be the maximal  $p$ -power dividing  $\gcd(\deg x, x \in T)$ . Then*

$$G^T(k) = \pi_1(X, T)(p) \cong \text{Gal}(\mathbb{F}'/\mathbb{F}),$$

where  $\mathbb{F}'$  is the unique extension of  $\mathbb{F}$  of degree  $p^r$ .

*Proof.* Let  $\tilde{\mathbb{F}}$  be the maximal  $p$ -extension of  $\mathbb{F}$  in  $\bar{\mathbb{F}}$ . Using Lemma 2.5, we have

$$H^2(X) = H^1(\mathbb{F}, H^1(\bar{X})) \cong \text{Hom}(\text{Pic}(X)[p], \mathbb{F}_p) = 0$$

and Corollary 2.4 shows that  $H^1(X)$  is 1-dimensional. Hence  $\pi_1(X)(p)$  is free of rank 1 and therefore the surjection

$$\pi_1(X)(p) \twoheadrightarrow \text{Gal}(\tilde{\mathbb{F}}/\mathbb{F})$$

is an isomorphism (cf. [NSW], Prop. 1.6.15). The maximal subextension  $\mathbb{F}'/\mathbb{F}$  of  $\tilde{\mathbb{F}}/\mathbb{F}$  such that all points in  $T$  split completely in the base change  $X \otimes_{\mathbb{F}} \mathbb{F}' \rightarrow X$  is exactly the unique extension of degree  $p^r$  of  $\mathbb{F}$ .  $\square$

**Corollary 4.2.** *Assume that  $\text{Pic}(X)[p] = 0$  and  $T \neq \emptyset$ . Then  $(X, T)$  is a  $K(\pi, 1)$  for  $p$  if and only if  $T = \{x\}$  consists of a single point with  $p \nmid \deg x$ . In this case the fundamental group  $\pi_1(X, T)(p)$  is trivial.*

*Proof.* By the Proposition 4.1,  $\pi_1(X, T)(p)$  is finite cyclic. If  $p \mid \gcd(\deg x, x \in T)$ , then  $\pi_1(X, T)(p)$  is nontrivial and  $(X, T)$  is not a  $K(\pi, 1)$  by Corollary 2.3.

Assume  $p \nmid \gcd(\deg x \mid x \in T)$ . Then  $\pi_1(X, T)(p)$  is the trivial group,  $H^1(X, T) = 0$  and  $(X, T)$  is a  $K(\pi, 1)$  if and only if  $H^2(X, T) = 0$ . By Corollary 2.4 this is equivalent to  $\#T = 1$ .  $\square$

**Lemma 4.3.** *Assume that  $\pi_1(X, T)(p)$  is finite and  $\text{Pic}(X)[p] \neq 0$ . Then  $(X, T)$  is not a  $K(\pi, 1)$  for  $p$ .*

*Proof.* By Corollary 2.3,  $(X, T)$  is not a  $K(\pi, 1)$  if  $\pi_1(X, T)(p)$  is nontrivial. Assume that  $\pi_1(X, T)(p) = 1$ . Then  $(X, T)$  is a  $K(\pi, 1)$  if and only if  $H^2(X, T) = 0$ . But by Proposition 2.2,  $H^2(X) \cong \text{Hom}(\text{Pic}(X)[p], \mathbb{F}_p) \neq 0$  is a quotient of  $H^2(X, T)$ .  $\square$

The following theorem is due to Ihara, see [Iha], Thm. 1 (FF).

**Theorem 4.4.** *Assume that  $T \neq \emptyset$  and let  $q = \#\mathbb{F}$ . If*

$$\sum_{x \in T} \frac{\deg(x)}{q^{\deg(x)/2} - 1} > \max(g_X - 1, 0),$$

then  $\pi_1(X, T)$  is finite. In particular,  $\pi_1(X, T)(p)$  is finite.

Summing up, we obtain Theorem 1.3.

#### 5. PROOF OF THEOREM 1.4

Let  $\tilde{\mathbb{F}}$  be the maximal  $p$ -extension of  $\mathbb{F}$  in  $\bar{\mathbb{F}}$  and  $\tilde{X} = X \times_{\mathbb{F}} \tilde{\mathbb{F}}$ . Then

$$X \text{ is a } K(\pi, 1) \text{ for } p \iff \tilde{X} \text{ is a } K(\pi, 1) \text{ for } p$$

and we have

$$H_{et}^i(\tilde{X}) \cong H_{et}^i(\bar{X})^{G(\tilde{\mathbb{F}}/\mathbb{F})}$$

for all  $i$ . Hence  $H_{et}^i(\tilde{X})$  vanishes for  $i \geq 3$  and  $H_{et}^2(\tilde{X}) = \mu_p(\tilde{\mathbb{F}})^* = \mu_p(\mathbb{F})^*$ . Hence  $X$  has the  $K(\pi, 1)$ -property for  $p$  if  $\mu_p(\mathbb{F}) = 0$ . In the following we assume that  $\mathbb{F}$  contains all  $p$ -th roots of unity. For every tower of finite connected étale  $p$ -coverings  $Z \rightarrow Y \rightarrow X$  the natural map

$$\mathbb{Z}/p\mathbb{Z} = H_{et}^2(\tilde{Y}, \mu_p) \longrightarrow H_{et}^2(\tilde{Z}, \mu_p) = \mathbb{Z}/p\mathbb{Z}$$

is multiplication by the degree  $[\tilde{Z} : \tilde{Y}]$ . Hence, by Lemma 1.1,

$$\tilde{X} \text{ is a } K(\pi, 1) \text{ for } p \iff \#(\pi_1^{et}(\tilde{X})(p)) = \infty.$$

Note that

$$\begin{aligned} \pi_1^{ab}(\tilde{X})/p &\cong H_{et}^1(\tilde{X})^* \\ &\cong (H_{et}^1(\tilde{X})^{G(\bar{\mathbb{F}}/\mathbb{F})})^* \\ &\cong \text{Pic}(\tilde{X})[p]_{G(\bar{\mathbb{F}}/\mathbb{F})} \end{aligned}$$

and by Lemma 2.5

$$\text{Pic}(\bar{X})[p]_{G(\bar{\mathbb{F}}/\mathbb{F})} = 0 \iff \text{Pic}(\tilde{X})[p] = 0.$$

Furthermore, since  $G(\bar{\mathbb{F}}/\mathbb{F})$  is a pro- $p$ -group:

$$\text{Pic}(\tilde{X})[p] = 0 \iff \text{Pic}(X)[p] = \text{Pic}(\tilde{X})[p]^{G(\bar{\mathbb{F}}/\mathbb{F})} = 0,$$

and therefore,  $\text{Pic}(X)[p] \neq 0 \iff \pi_1^{ab}(\tilde{X})/p \neq 0$ . Hence it suffices to show the equivalences

$$\#(\pi_1^{et}(\tilde{X})(p)) = \infty \iff \#(\pi_1^{ab}(\tilde{X})(p)) = \infty \iff \#\pi_1^{ab}(\tilde{X})/p \neq 0.$$

Elementary theory of pro- $p$ -groups shows that it remains to show the implication

$$\pi_1^{ab}(\tilde{X})/p \neq 0 \implies \#(\pi_1^{ab}(\tilde{X})(p)) = \infty.$$

Setting  $T := T_p(\tilde{X}) = \pi_1^{ab}(\tilde{X})(p)$ , we can write this implication in the form

$$(T_{G(\bar{\mathbb{F}}/\mathbb{F})})/p \neq 0 \implies \#(T_{G(\bar{\mathbb{F}}/\mathbb{F})}) = \infty.$$

The group  $G(\bar{\mathbb{F}}/\mathbb{F})$  is pro-cyclic of supernatural order prime to  $p$ . Furthermore,  $T \cong \mathbb{Z}_p^{2g}$  and the kernel of the reduction map  $\text{Gl}_{2g}(\mathbb{Z}_p) \rightarrow \text{Gl}_{2g}(\mathbb{F}_p)$  is a pro- $p$ -group. Hence the action of  $G(\bar{\mathbb{F}}/\mathbb{F})$  on  $T$  factors through a finite cyclic group of order prime to  $p$ . We conclude that Theorem 1.4 follows from Lemma 5.1 below.  $\square$

The following Lemma 5.1 and its application in the proof of Theorem 1.4 were proposed to us by J. Stix.

**Lemma 5.1.** *Let  $G = \langle g \rangle$  be a finite cyclic group of order  $n$ . Let  $p$  be a prime number with  $p \nmid n$  and let  $T$  be a finitely generated free  $\mathbb{Z}_p$ -module with a  $G$ -action. Then*

$$\#T_G = \infty \iff (T/p)_G \neq 0.$$

*Proof.* Since  $(T/p)_G = T_G/p$ , the implication  $\implies$  follows from Nakayama's Lemma.

We consider  $V = T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  as a  $\mathbb{Q}_p[X]$ -module with the action  $X \cdot v = g(v)$ . Then  $X^n$  acts identically and the main theorem on finitely generated modules over principal ideal domains implies that

$$V \cong \mathbb{Q}_p[X]/(f_1^{e_1}) \oplus \cdots \oplus \mathbb{Q}_p[X]/(f_r^{e_r}),$$

where  $f_i \in \mathbb{Z}_p[X]$  are irreducible monic polynomials dividing  $X^n - 1$ . Put

$$T' = \mathbb{Z}_p[X]/(f_1^{e_1}) \oplus \cdots \oplus \mathbb{Z}_p[X]/(f_r^{e_r}) \subset V$$

and let  $\bar{f}_i \in \mathbb{F}_p[X]$  be the reductions of the  $f_i$ . As divisors of  $X^n - 1$ , the  $\bar{f}_i$  are separable and Hensel's Lemma shows that they are irreducible.

$T'_G$  is infinite if and only if one of the  $f_i$  is equal to  $X - 1$ , which is equivalent to one of the  $\bar{f}_i$  being equal to  $X - 1$ . This again is equivalent to the fact that  $\mathbb{Z}/p\mathbb{Z}$  (with trivial action) occurs as a Jordan-Hölder quotient of  $T'/p$ . By [CR] (16.16)  $T/p$  and  $T'/p$  (being reductions of lattices in the same  $\mathbb{Q}_p[G]$ -module  $V$ ) have the same Jordan-Hölder quotients. Therefore we obtain the following chain of implications:

$$\begin{aligned} (T/p)_G \neq 0 &\implies \mathbb{Z}/p\mathbb{Z} \text{ is a J-H quotient of } T/p \\ &\implies \mathbb{Z}/p\mathbb{Z} \text{ is a J-H quotient of } T'/p \\ &\implies \#T'_G = \infty. \end{aligned}$$

Finally, since  $G$  is cyclic, we have the exact sequence

$$0 \rightarrow T^G \rightarrow (T + T')^G \rightarrow (T + T'/T)^G \rightarrow T_G \rightarrow (T + T')_G \rightarrow (T + T'/T)_G \rightarrow 0.$$

As  $(T + T'/T)$  is finite,  $T_G$  and  $(T + T')_G$  have the same  $\mathbb{Z}_p$ -rank. The same argument shows that  $T'_G$  and  $(T + T')_G$  have the same  $\mathbb{Z}_p$ -rank. Since  $T'_G$  is infinite,  $T_G$  is infinite. This finishes the proof.  $\square$

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LABORATOIRE DE MATHÉMATIQUES DE BESANÇON, 16 ROUTE DE GRAY, 25030 BESANÇON, FRANCE  
*E-mail address:* philippe.lebacque@univ-fcomte.fr

UNIVERSITÄT HEIDELBERG, MATHEMATISCHES INSTITUT, IM NEUENHEIMER FELD 288, D-69120  
 HEIDELBERG, DEUTSCHLAND  
*E-mail address:* schmidt@mathi.uni-heidelberg.de